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ELLIPTIC CURVE CONFIGURATIONS ON FANO SURFACES

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Abstract. We give the classification of the configurations of the elliptic curves on the Fano surface of a smooth cubic threefold. That means that we give the number of such curves, their intersections and a plane model. This classification is linked to the classification of the automorphism groups of theses surfaces. The Fano surface of the Fermat cubic of $\mathbb{P}^4$ is the only one to contain 30 elliptic curves and we study its properties in detail.

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Introduction.

Context of the study.
Let S be a minimal complex surface of general type, Ω_S its cotangent sheaf, ω_S the canonical sheaf and $m>0$ be an integer. To the invertible sheaf $\omega_S^{\otimes m}$ is associated a rational map

$$\phi_m : S \rightarrow \mathbb{P}(H^o(S,\omega_S^{\otimes m})^*)$$

called the m-th pluricanonical map. We know the importance of the pluricanonical maps ϕ_m for the classification of the surfaces of general type. Since the work of E. Bombieri [6] for $m \geq 2$ and A. Beauville [2] for $m=1$, the behaviour of the pluricanonical map ϕ_m is controlled by the values of m, by the first Chern number and by the particular geometry of S. The following results are classical [1]:

Theorem 0.1. *A) The canonical sheaf of S is ample if and only if the surface does not contain a (-2)-curve (i.e. a smooth curve C of genus 0 such that $C^2=-2$).*
B) If $m \geq 5$, the rational map ϕ_m is a morphism and if $D=\sum_{i=1}^{i=n} C_i$ is a 1 dimensional fibre of ϕ_m, then its irreducible components C_i are (-2)-curves. Moreover, we know the configuration of these (-2)-curves: their dual graph is one of the graphs A_n, $(n\geq 1)$, D_n, $(n \geq 4)$ or E_n, $(n\in\{6,7,8\})$.

However, we are still far from having a complete classification for surfaces of general type. It is expected that a finer classification will result from the study of the subtler invariant Ω_S rather the study of the sheaf $\omega_S=\wedge^2\Omega_S$

and we were lead in [19] to introduce the notion of "cotangent map of S". The definition of this map requires the following assumption:

Hypothesis 0.2. The surface S is a smooth complex surface of general type. The cotangent sheaf Ω_S of S is generated by its space $H^o(\Omega_S)$ of global sections and the irregularity $q = \dim H^o(\Omega_S)$ satisfies $q > 3$.

Let us denote T_S the tangent sheaf of S,

$$\pi : \mathbb{P}(T_S) \to S$$

the projection onto S of the projectivized tangent sheaf, and $\mathcal{O}_{\mathbb{P}(T_S)}(1)$ the invertible sheaf on $\mathbb{P}(T_S)$ such that:

$$\pi_*(\mathcal{O}_{\mathbb{P}(T_S)}(1)) \simeq \Omega_S.$$

We identify the space of global sections of $\mathcal{O}_{\mathbb{P}(T_S)}(1)$ with $H^o(\Omega_S)$. Since the morphism $H^o(\Omega_S) \otimes \mathcal{O}_S \to \Omega_S$ is surjective, the morphism $H^o(\Omega_S) \otimes \mathcal{O}_{\mathbb{P}(T_S)} \to \mathcal{O}_{\mathbb{P}(T_S)}(1)$ is also surjective and defines a morphism:

$$\psi : \mathbb{P}(T_S) \to \mathbb{P}(H^o(\Omega_S)^*) = \mathbb{P}^{q-1}.$$

Definition. The map ψ is called the cotangent map of S.

The image under ψ of a fibre $\pi^{-1}(s) \simeq \mathbb{P}^1$ is a line in $\mathbb{P}^{q-1}$. The cotangent sheaf is said to be ample if the sheaf $\mathcal{O}_{\mathbb{P}(T_S)}(1)$ is ample, and this is so if and only if the fibres of the cotangent map ψ are finite. If p is a point of $\mathbb{P}^{q-1}$ such that the fibre $\psi^{-1}(p)$ is not finite, then p is the vertex of a cone in the image of ψ and $\psi^{-1}(p)$ is 1 dimensional [19]. In this case, a 1 dimensional irreducible component of $\pi(\psi^{-1}(p))$ is called a non-ample curve of S.

The morphism ψ is the analogue, for the cotangent sheaf, of the morphism ϕ_1 for the canonical sheaf. The non-ample curves play the same role for the cotangent sheaf as the (-2)-curves do for the canonical sheaf. It is thus natural to study these curves and their configurations.

The cotangent map ψ is the object of a forthcoming paper [19]. In this paper, we study the cotangent map and the configurations of the non-ample curves on Fano surfaces.

These surfaces were discovered by G. Fano and interest in them has been stimulated by the work of H. Clemens, P. Griffiths [8] and A. Tyurin [22], [23] in 1971.
By definition, a Fano surface is the Hilbert scheme of lines of a smooth cubic threefold $F \hookrightarrow \mathbb{P}^4$. This scheme is a surface S that verifies Hypothesis 0.2 and has irregularity $q = 5$.
By the Tangent Bundle Theorem 12.37 of [8], the image of the cotangent map $\psi : \mathbb{P}(T_S) \to \mathbb{P}(H^o(\Omega_S)^*)$ of S is a hypersurface F' of $\mathbb{P}(H^o(\Omega_S)^*) \simeq \mathbb{P}^4$ that is isomorphic to F. Moreover, when we identify F and F', the triple $(\mathbb{P}(T_S), \pi, \psi)$ is the universal familly of lines on F.

We will prove that for the Fano surfaces, a non-ample curve is a smooth curve of genus 1. The main results presented here, which are similar to the classical results A) and B) of Theorem 0.1 are:

Theorem 0.3. *A') The cotangent sheaf of a Fano surface is ample if and only if this surface does not contain smooth curves of genus* 1.
B') There are only 10 *configurations of smooth genus* 1 *curves on the Fano surfaces. We know a plane model of each curve, the number of these curves and their intersection numbers.*
Furthermore, we prove that the classification of the elliptic curve configurations on Fano surfaces is equivalent to the classification of the Fano surfaces with respect to a particular subgroup of their automorphism group.

In this study of the geometry of Fano surfaces, the elliptic curves will play the the role that is devoted to the rational curves for classical examples of surfaces e.g. the (-1)-curves on del Pezzo surfaces or the lines on hypersurfaces. The counterpart is that the techniques used for these surfaces cannot be applied here. It is the theoretic framework of the cotangent map that raises the right questions and enables us to obtain well understood examples of irregular surfaces of general type, lying in a 10 dimensional familly.

Statement of the results.
We have divided this work into three sections:

Section 1. In the first section, we start by recalling known facts about Fano surfaces and the cotangent map and we introduce preliminary materials.

Let S be a Fano surface and let $F \hookrightarrow \mathbb{P}^4$ be the image of the cotangent map ψ. As this image is smooth, the general theory of the cotangent map implies that a curve $E \hookrightarrow S$ is non-ample if and only if E is smooth and of genus 1. Moreover, if E is such a curve, the cone $\psi_*\pi^*E$ is a hyperplane section of F.

We then prove that the automorphism group of the surface S and that of F are isomorphic and that the Néron-Severi group of S is generated, up to index 2, by the divisors coming from its Albanese variety.

Section 2. In the second section, we establish the classification of configurations of smooth curves of genus 1 on Fano surfaces.
Let S be a Fano surface, $\psi : \mathbb{P}(T_S) \to \mathbb{P}^4$ be its cotangent map and F be the image of ψ. If s is a point on S, we denote by L_s the line $\psi(\pi^{-1}(s)) \hookrightarrow F$.

An important tool used to classify the configurations of elliptic curves is the observation that we may associate to each elliptic curve $E \hookrightarrow S$, an involutive automorphism $\sigma_E : S \to S$ and a fibration $\gamma_E : S \to E$. The classification of the groups generated by these involutions then gives the classification of the elliptic curve configurations.
The construction of the morphisms σ_E and γ_E is as follows:
Let us denote by p_E the vertex of the cone $\psi_*\pi^*E$. For a generic point s in S, the plane X_s which contains the point p_E and the line L_s, cuts the cubic F along three lines :
1) the line L_s,
2) the line $L_{\gamma_E s}$ (in the cone $\psi_*\pi^*E$) passing through the vertex p_E and the intersection point of the line L_s and the cone $\psi_*\pi^*E$,

3) the residual line $L_{\sigma_E s}$ such that :

$$X_sF = L_s + L_{\gamma_E s} + L_{\sigma_E s}.$$

The morphisms σ_E and γ_E satisfy the following proposition:

Proposition 0.4. *Let E, E' be two different elliptic curves on S. The curve E verifies $E^2 = -3$, the intersection number of E and E' equals 0 or 1 and:*

$$(\sigma_E \sigma_{E'})^{3-EE'} = 1.$$

The curve E' is contained in a fibre of γ_E if and only if $EE' = 1$. If $EE' = 0$, then E' is a section of γ_E and $E'' = \sigma_E(E') = \sigma_{E'}(E)$ is a third elliptic curve on S such that $EE'' = E'E'' = 0$.

This Proposition implies that the dual graph of the elliptic curves $E, E', ..$ gives the relations between two involutions $\sigma_E, \sigma_{E'}$ as a Coxeter-Dynkin diagram. This fact will be used to determine the group generated by these involutions.

Let $E \hookrightarrow S$ be an elliptic curve. The automorphism σ_E acts on the space $H^o(\Omega_S)$ and we denote by $M_{\sigma_E} \in GL(H^o(\Omega_S)^*)$ its dual representation. The order 2 automorphism $-M_{\sigma_E}$ is an order 2 reflection of $H^o(\Omega_S)^*$. Let $\mathcal{E}$ be the set of smooth curves of genus 1 on S. We denote by G_S the complex reflection group generated by the morphisms:

$$-M_{\sigma_E},\ (E \in \mathcal{E}).$$

Let m, $n > 0$ be integers and let p be an integer dividing m. We denote by $\Sigma_n \subset GL_n(\mathbb{C})$ the group of permutation matrices and $A(m,p,n) \subset GL_n(\mathbb{C})$ the group of diagonal matrix D of order m such that $\det(D)^{\frac{m}{p}} = 1$. Let $G(m,p,n)$ be the complex reflection group generated by $A(m,p,n)$ and Σ_n and let $[\,]^2$ be the reflection group of order 2.
The following theorem gives the configuration classification of elliptic curves on Fano surfaces:

Theorem 0.5. *(Classification Theorem). Let n_S be the number of elliptic curves on S. If the group G_S is irreducible, then it is isomorphic to one of the following groups:*

Group G_S	$\{1\}$	$[\,]^2$	$G(3,3,2)$	Σ_4	Σ_5	$G(3,3,5)$
n_S	*0*	*1*	*3*	*6*	*10*	*30*
Order of G_S	*1*	*2*	*6*	*24*	*120*	*9720*

Else, G_S is isomorphic to one of the following groups:

$[\,]^2 \times [\,]^2$	$G(3,3,2) \times [\,]^2$	$G(3,3,2) \times G(3,3,2)$	$G(3,3,3) \times G(3,3,2)$
2	*4*	*6*	*12*
4	*12*	*36*	*324*

Let E, E' be in $\mathcal{E}$. We know the intersection number EE' and we give a plane model of E.
These configurations of elliptic curves and these automorphism groups acting on a Fano surface do actually occur in an effective way.

To prove this Theorem, we use the classification of complex reflection groups established by G. Shephard and J. Todd [20].

We give the following example to show the effectiveness of the Classification Theorem 0.5: we not only have the dual graph of the configuration, but we can also construct a Fano surface that contains a prescribed elliptic curve.

Example 0.6. Suppose that the reflection group G_S is isomorphic to Σ_5. Then there is a scalar $\mu \notin \{1, \frac{1}{9}, \frac{1}{25}\}$ such that the cubic F is isomorphic to

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 - \mu(x_1 + x_2 + x_3 + x_4 + x_5)^3 = 0.$$

The Fano surface of F has ten copies $\{E_{ij},\, 1 \leq i < j \leq 5\}$ of the elliptic curve

$$x^3 + y^3 + z^3 - \mu(x + y + z)^3 = 0.$$

We have $E_{ij}E_{st} = -3$ if $\{i, j\} = \{s, t\}$, $E_{ij}E_{st} = 1$ if $\{i, j\} \cap \{s, t\} = \emptyset$ and otherwise $E_{ij}E_{st} = 0$. The dual graph of this configuration is the Petersen graph.

Let us now discuss some consequences of our Classification Theorem:

Corollary 0.7. *We can build an infinite number of Fano surfaces S that are "singular" in the Shioda sense : their Picard number achieve the upper bound* $25 = \dim H^1(S, \Omega_S)$.

We remark that the classical examples of singular surfaces of general type contain (-2)-curves in the case of Horikawa surfaces and double coverings of the plane [18]. In our case, a Fano surface is embedded in its Albanese variety and hence contains no rational curves.
Moreover, a Fano surface S of the above Corollary 0.7, has the following property:

Corollary 0.8. *The Albanese variety A of S is isomorphic to a product of elliptic curves.*

The relevance of this fact is that A carries a principal polarisation Θ and that one of the main results of [8] is that (A, Θ) cannot be isomorphic (as a principally polarised Abelian variety) to a product of Jacobians.

Recall that among the K3 surfaces, the Kummer surfaces are recognized as those K3 having 16 disjoint (-2)-curves. The sum of these 16 curves is divisible by 2 and the associated degree 2 cyclic cover is the blow-up of an Abelian surface [17]. We have an analogous result for Fano surfaces with 12 elliptic curves as follows:
Let E be an elliptic curve, let $\Delta \hookrightarrow E \times E$ be the diagonal and define:

$$\begin{aligned} T_1 &= \{x + 2y = 0\} \hookrightarrow E \times E \\ T_2 &= \{2x + y = 0\} \hookrightarrow E \times E. \end{aligned}$$

Let Z be the blow-up of $E \times E$ at the points of 3-torsion of Δ and let D be the proper transform of $\Delta + T_2 + T_2$ in Z. The divisor D is divisible by

3 and for each of the 81 invertible sheaves $\mathcal{L}$ such that $\mathcal{L}^{\otimes 3} = \mathcal{O}_Z(D)$, we denote by $S_{\mathcal{L}} \to Z$ the associated degree 3 cyclic cover of Z branched over D.

Corollary 0.9. *There exists a unique invertible sheaf $\mathcal{L}$ such that $S_{\mathcal{L}}$ is a Fano surface with 12 elliptic curves. By this construction, we obtain all Fano surfaces with 12 elliptic curves. If E has no complex multiplication, the Néron-Severi group of $S_{\mathcal{L}}$ has rank 12 and is rationally generated by its 12 elliptic curves.*

By the Classification Theorem 0.5, the number n_S of elliptic curves verifies $0 \leq n_S \leq 30$ and we prove that the surfaces for which $n_S > 0$ constitute a 7 dimensional space in the 10 dimensional moduli space of Fano surfaces.
The Picard number ρ_S of a Fano surface S satisfies $1 \leq \rho_S \leq 25$ and we prove that it is 1 for S generic. In fact, the number of elliptic curves is linked with the Picard number ρ_S: we have $\rho_S \geq n_S$ unless $n_S = 30$, in which case $\rho_S = 25$. The number n_S is also the number of 1 dimensional fibres of the cotangent map ψ. This shows that the geometric properties of ψ vary non-trivialy with the Fano surface.

Section 3. In the third section, we study the Fano surface S of the Fermat cubic $F \hookrightarrow \mathbb{P}^4 = \mathbb{P}(H^o(\Omega_S)^*)$:

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0.$$

The computation of the the rank of the Néron-Severi group of a given surface is generally a difficult problem that is related to arithmetic questions such as the Tate Conjecture [21]. A fortiori, to find a $\mathbb{Z}$-basis also is a more difficult question that may be treated for some cases such that the K3 surfaces or hypersurfaces [7]. Knowing the image of the cotangent map of the Fano surface S, we gain insight into its geometry and we compute its Néron-Severi group.
Let μ_3 be the group of third roots of unity and let $\alpha \in \mu_3$ be a primitive root. For s a point of S, we denote by C_s the incidence divisor that parametrises the lines in F that cut the line $L_s = \psi(\pi^{-1}(s))$.

Theorem 0.10. *The surface S is the unique Fano surface that contains 30 smooth curves of genus 1. These curves are numbered:*

$$E_{ij}^{\beta},\ 1 \leq i < j \leq 5,\ \beta \in \mu_3.$$

1) Each smooth curve of genus 1 is isomorphic to the plane Fermat cubic

$$\mathbb{E} : x^3 + y^3 + z^3 = 0.$$

2) Let E_{ij}^{γ} and E_{st}^{β} be two such curves, then:

$$E_{ij}^{\beta} E_{st}^{\gamma} = \begin{cases} 1 & \text{if } \{i,j\} \cap \{s,t\} = \emptyset \\ -3 & \text{if } E_{ij}^{\beta} = E_{st}^{\gamma} \\ 0 & \text{else.} \end{cases}$$

3) The Néron-Severi group $\mathrm{NS}(S)$ *of* S *has rank :* $25 = \dim H^1(S, \Omega_S)$ *and discriminant* 3^{18}. *These* 30 *elliptic curves generates an index* 3 *sub-lattice of* $\mathrm{NS}(S)$ *and with the class of an incidence divisor* C_s $(s \in S)$*, they generate the Néron-Severi group.*
4) Let i, j, r, s, t *be integers such that* $\{i, j, r, s, t\} = \{1, 2, 3, 4, 5\}$. *There exists a fibration* $\gamma_i : S \to \mathbb{E}$ *such that the* 3 *singular fibers of* γ_i *are*

$$B_{jr} + B_{st},\ B_{js} + B_{rt},\ B_{jt} + B_{rs}$$

where $B_{kl} = B_{lk} = \sum_{\beta \in \mu_3} E_{kl}^{\beta}$ *for* $1 \leq k < l \leq 5$.
5) The relations between the 30 *elliptic curves in* $\mathrm{NS}(S)$ *are generated by the relations :*

$$B_{jr} + B_{st} = B_{js} + B_{rt} = B_{jt} + B_{rs}$$

for all $\{i, j, r, s, t\} = \{1, 2, 3, 4, 5\}$.

Given a smooth curve of low genus and with a sufficiently large automorphism group, it is possible to calculate the period matrix of its Jacobian [5]. In this paper, we calculate the period lattice of the Albanese variety of the 2 dimensional variety S. This computation is used to determine the Néron-Severi group of S.

Notations.

Let S be a surface, we denote by ω_S the canonical sheaf, K the canonical divisor, Ω_S the cotangent sheaf and $H^o(\Omega_S)$ its space of global sections. We identify locally free sheaves and vector bundles over S. We identify elliptic curves and smooth curves of genus 1: the invariants that we compute are independent of the choice of a neutral element.
- $c_1 \in H^2(S, \mathbb{Z})$ and $c_2 \in \mathbb{Z}$ are Chern classes of S.
- If D is a divisor, $g(D)$ is its arithmetic genus $g(D) = \frac{1}{2}(D^2 + KD) + 1$.
- $e_1, .., e_5$ is the canonical basis of $\mathbb{C}^5$, we also consider it as a basis of the tangent space of the Albanese variety of a Fano surface (see next).
- x_1, x_2, x_3, x_4, x_5 denotes the dual basis of $e_1, .., e_5$, we also consider it as a basis of $H^o(\Omega_S)$ where S is a Fano surface.
- $x = (x_1, x_2, x_3, x_4, x_5) = \sum_{i=1}^{i=5} x_i e_i$ denotes a point of $\mathbb{C}^5$.
- $\mathbb{P}^4$ is the variety of lines of $\mathbb{C}^5$.
- $\mathbb{C}x$ (where $x \in \mathbb{C}^5 \setminus \{0\}$) is the vector space generated by the vector v or the point of $\mathbb{P}^4$ which corresponds to this space ; we will specify as need be.
- $F \hookrightarrow \mathbb{P}^4$ denotes a smooth cubic in $\mathbb{P}^4$.
- S is the Fano surface of the cubic F.
- $T_{F,p} \hookrightarrow \mathbb{P}^4$ denotes the tangent projective hyperplane at a point p of F.
- $\langle V_1, V_2 \rangle$ the linear hull of two sub-varieties V_1 and V_2 of $\mathbb{P}^4$.
- μ_3 denotes the group of third roots of unity.
- α is a generator of μ_3.

1. Main properties of Fano surfaces.

1.1. **Definition and properties of the cotangent map.**

1.1.1. *General properties of the cotangent map.* Let $S_{/\mathbb{C}}$ be a surface which verifies the hypothesis 0.2. In the introduction, we defined the cotangent map

$$\psi : \mathbb{P}(T_S) \to \mathbb{P}(H^o(\Omega_S)^*) = \mathbb{P}^{q-1}$$

by the surjective morphism $H^o(\Omega_S) \otimes O_{\mathbb{P}(T_S)} \to O_{\mathbb{P}(T_S)}(1)$. Here, we state general results about this morphism which will be used in the sequel.
Let A be the Albanese variety of S and let $\vartheta : S \to A$ be an Albanese morphism.

Lemma 1.1. *([13] p.331.) The differential $d\vartheta : T_S \to \vartheta^* T_A \simeq S \times H^o(\Omega_S)^*$ of the morphism ϑ is the dual morphism of the evaluation map $H^o(\Omega_S) \otimes O_S \to \Omega_S$.*

Let $p_r : S \times H^o(\Omega_S)^* \to H^o(\Omega_S)^*$ be the projection on the second factor. Lemma 1.1 implies:

Corollary 1.2. *The cotangent map is the projectivization of the morphism:*

$$p_r \circ d\vartheta : T_S \to H^o(\Omega_S)^*.$$

Let us recall that $\pi : \mathbb{P}(T_S) \to S$ is the projection. Let s be a point of S. The restriction of the invertible sheaf $\mathcal{O}_{\mathbb{P}(T_S)}(1)$ to the fibre $\pi^{-1}(s) \simeq \mathbb{P}^1$ is the degree 1 invertible sheaf. Hence the image under ψ of the fibre $\pi^{-1}(s)$ is a line $L_s \hookrightarrow \mathbb{P}^{q-1}$.
Let $G(2,q) = G(2, H^o(\Omega_S)^*)$ be the Grassmannian of 2 dimensional subspaces of $H^o(\Omega_S)^*$. This Grassmannian also parametrises the projective lines in $\mathbb{P}^{q-1}$.

Definition 1.3. The surjection $H^o(\Omega_S) \otimes O_S \to \Omega_S$ defines a map

$$\mathcal{G} : S \to G(2,q)$$

called the Gauss map of S.

By construction, the point $\mathcal{G}(s)$ represents the projective line L_s, in other words:

Corollary 1.4. *The point $\mathcal{G}(s)$ represents the plane:*

$$p_r \circ d\vartheta_s(T_{S,s}) \subset H^o(\Omega_S)^*.$$

For the 3 following propositions, we refer to [19].

Lemma 1.5. *Let E be an elliptic curve on S. The curve E is a non-ample curve and the image under ψ of $\pi^* E$ is a cone.*
Let p_E be the vertex of this cone and let s be a point of E. The underlying space of the point p_E is the space:

$$pr \circ d\vartheta_s(T_{E,s}) \hookrightarrow H^o(\Omega_S)^*$$

where $T_{E,s} \hookrightarrow T_{S,s}$ is the tangent space of E at s.

In particular, the space $pr \circ d\vartheta_s(T_{E,s})$ is independent of the choice of the Albanese morphism ϑ (and of $s \in S$).

Remark 1.6. This interpretation of the underlying space of p_E is an important key of the paper.

In [19], we give a classification of the non-amples curves C according to the sign of C^2 and we provide example of smooth non-ample curves of arbitrary genus. We prove that a curve $C \hookrightarrow S$ is non-ample and satisfies $C^2 < 0$ if and only if C is a smooth curve of genus 1. This enables us to prove the following proposition:

Proposition 1.7. *Suppose that the image F of the cotangent map is smooth and $q = 5$. A curve $C \hookrightarrow S$ is non-ample if and only if C is a smooth curve of genus 1. In that case, the cone $\psi(\pi^{-1}(C))$ is the hyperplane section of F by the projective tangent space to the vertex of the cone.*

The following proposition links the geometry of S to the geometry of the image of the cotangent map:

Proposition 1.8. *Let $C \hookrightarrow S$ be a curve and let K be a canonical divisor of S. The degree of the cycle $\psi_*\pi^*C$ equals KC.*

1.1.2. *Main properties of Fano surfaces.* Let F be a smooth cubic hypersurface of $\mathbb{P}^4$. The Hilbert scheme of lines in F is a smooth surface called the Fano surface of F.
This surface S verifies Hypothesis 0.2 and has irregularity $q = 5$. It has the following important property ([8] or [22]):

Theorem 1.9. *(Cotangent bundle Theorem). The image of the cotangent map of S is a cubic hypersurface $F' \hookrightarrow \mathbb{P}(H^o(\Omega_S)^*) \simeq \mathbb{P}^4$ isomorphic to the cubic $F \hookrightarrow \mathbb{P}^4$.*

We fix an isomorphism between the spaces $H^o(\Omega_S)^*$ and $H^o(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$ that allows us to identify the cubics F' and F.

The Chern numbers of a Fano surface verify : $c_1^2 = 45$, $c_2 = 27$. The cotangent map has degree 6: there are 6 lines through a generic point of F.
Let $\vartheta : S \to A$ be an Albanese morphism.

Lemma 1.10. *The Gauss morphism and the morphism ϑ are embeddings.*

Let s be a point of S and let C_s be the closure of all points t such that the lines L_t intersects the line L_s.

Proposition 1.11. *Let s be a point of S. The incidence divisor C_s is ample, reduced, has self-intersection $C_s^2 = 5$ and genus 11. The divisor $3C_s$ is numerically equivalent to a canonical divisor of S.*

1.1.3. *Propreties of a non-ample curve on a Fano surface.* Let S be a Fano surface and let $F \hookrightarrow \mathbb{P}^4$ be the image of its cotangent map ψ. For a point p of F, we denote by $T_{F,p} \hookrightarrow \mathbb{P}^4$ the projective tangent hyperplane to F

at p. If $E \hookrightarrow S$ is a non-ample curve, we denote by p_E the vertex of the cone $\psi(\pi^{-1}(E))$. As the Gauss map is an embedding, we have $\psi(\pi^{-1}(E)) = \psi_*\pi^*E$.
By Proposition 1.7, a curve $E \hookrightarrow S$ is non-ample if and only if E is a smooth curve of genus 1. The fact that the 1 dimensional fibers of ψ are smooth curves of genus 1 is known [8], [22] but it seems interesting to explain that by the general theory of the cotangent map.

Proposition 1.12. *Let E be an elliptic curve on the Fano surface S. The cone $\psi_*\pi^*E$ is the section of F by the hyperplane T_{F,p_E} where p_E is the vertex of the cone, moreover:*

$$E^2 = -3,\ C_sE = 1.$$

Proof. By Proposition 1.7, the cone $\psi_*\pi^*E$ is the hyperplane section of F by T_{F,p_E}, hence this cycle has degree 3. Let K be a canonical divisor of S. By Proposition 1.8, we have $KE = 3$ hence $E^2 = -3$ and since K is numerically equivalent to $3C_s$, we obtain: $EC_s = 1$. □

The following proposition is [8], §8 & §10.

Proposition 1.13. *The Fano surface S contains at most* 30 *smooth curves of genus* 1. *If S contains* 30 *elliptic curves, then the sum of these* 30 *curves is a bicanonical divisor.*

1.2. **The automorphism groups of the cubic and of the Fano surface, Theta polarisation.**

1.2.1. *Study of the automorphism group of S and F.* Let us denote by $\mathrm{Aut}(X)$ the automorphisms group of a variety X. Let S and F be a Fano surface and its cubic. We prove here that $\mathrm{Aut}(S)$ and $\mathrm{Aut}(F)$ are isomorphic.
An automorphism $h \in \mathrm{Aut}(F)$ is the restriction of an element of $PGL(H^o(\Omega_S)^*)$ and we denote by $\rho(h) : S \to S$ the automorphism of S such that:

$$L_{\rho(h)(s)} = h(L_s)$$

for all $s \in S$. Let A be the Albanese variety and let $\vartheta : S \to A$ be a fixed Albanese morphism. The tangent space of A at 0 is the space $H^o(\Omega_S)^*$. Let $\tau \in \mathrm{Aut}(S)$ be an automorphism. There exists an unique automorphism τ' of A such that the following diagram

$$\begin{array}{ccc} S & \xrightarrow{\vartheta} & A \\ \downarrow \tau & & \downarrow \tau' \\ S & \xrightarrow{\vartheta} & A \end{array}$$

is commutative. Let $M_\tau \in GL(H^o(\Omega_S)^*)$ be the differential at 0 of the linear part of τ'. The group $\mathrm{Aut}(S)$ acts also on the space $H^o(\Omega_S)$, and the morphism $\tau \to M_\tau$ is the dual representation of this action. Let us denote the quotient morphism by

$$q' : GL(H^o(\Omega_S)^*) \to PGL(H^o(\Omega_S)^*).$$

The following diagram sums up the used notations

$$\begin{array}{ccccccc} \mathrm{Aut}(F) & \to & \mathrm{Aut}(S) & \to & \mathrm{Aut}(A) & \to & PGL(H^o(\Omega_S)^*). \\ h & \to & \rho(h) & & \tau' & \to & q'(M_\tau) \\ & & \tau & \to & \tau' & & \end{array}$$

We have:

Proposition 1.14. *For all $h \in \mathrm{Aut}(F) \subset PGL(H^o(\Omega_S)^*)$, the morphism $M_{\rho(h)}$ raises h:*

$$q'(M_{\rho(h)}) = h.$$

The morphism ρ is an isomorphism, and its inverse is the morphism defined by :

$$\begin{array}{ccc} \mathrm{Aut}(S) & \to & \mathrm{Aut}(F) \\ \tau & \to & q'(M_\tau). \end{array}$$

Once we have fixed the notations and with the help of the following Lemma, it is easy to verify Proposition 1.14 ; we omit it.

Lemma 1.15. *Let τ be an automorphism of S and let s be a point of S. The following diagram is commutative :*

$$\begin{array}{ccc} T_{S,s} & \overset{p_r \circ d\vartheta_s}{\to} & H^o(\Omega_S)^* \\ \downarrow d\tau_s & & \downarrow M_\tau \\ T_{S,\tau(s)} & \overset{p_r \circ d\vartheta_{\tau(s)}}{\to} & H^o(\Omega_S)^* \end{array}$$

Proof. The morphisms $p_r \circ d\vartheta_t$ (t a point of S) are defined in Corollary 1.4. This results from the equality $\vartheta \circ \tau = \tau' \circ \vartheta$. □

1.2.2. *Theta polarisation.* Let S be a Fano surface, let A be its Albanese variety and let $\vartheta : S \hookrightarrow A$ be an Albanese morphism. By [8], Theorem 13.4, the image Θ of $S \times S$ under the morphism $(s_1, s_2) \to \vartheta(s_1) - \vartheta(s_2)$ is a principal polarisation of A. Let τ be an automorphism of S and let τ' be the automorphism of A induced by τ. Let (s_1, s_2) be a point of $S \times S$, then : $\tau'(\vartheta(s_1) - \vartheta(s_2)) = \vartheta(\tau(s_1)) - \vartheta(\tau(s_2))$. Thus:

Lemma 1.16. *The automorphism τ' preserves the polarisation : $\tau'^*\Theta = \Theta$.*

For a variety X, we denote by $H^2(X, \mathbb{Z})_f$ the group $H^2(X, \mathbb{Z})$ modulo torsion. We denote by $\mathrm{NS}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})_f$ its Néron-Severi group and by ρ_X its Picard number. For a divisor D in X, we denote its Chern class by $c_1(D)$.

The author wishes to thank Bert Van Geemen for a useful discussion on the following Theorem:

Theorem 1.17. *a) If D and D' are two divisors of A, then:*

$$\vartheta^*(D)\vartheta^*(D') = \int_A \frac{1}{3!} \wedge^3 c_1(\Theta) \wedge c_1(D) \wedge c_1(D).$$

b) The following sequence is exact:

$$0 \to \mathrm{NS}(A) \xrightarrow{\vartheta^*} \mathrm{NS}(S) \to \mathbb{Z}/2\mathbb{Z} \to 0$$

c) The Néron-Severi group of S is generated by $\vartheta^*\mathrm{NS}(A)$ and by the class of an incidence divisor C_s ($s \in S$). The class of $2C_s$ is equal to $\vartheta^*(\Theta)$.
d) We have $\rho_A = \rho_S \leq 25 = \dim H^1(S, \Omega_S)$ and $\rho_S = 1$ for S generic.

Proof. The morphism ϑ is an embedding and the homological class of $\vartheta(S)$ is equal to $\frac{1}{3!}\Theta^3$ ([4] proposition 7), this proves a).
Since Θ is a polarisation, the bilinear symetric form

$$Q_\Theta : H^2(A, \mathbb{C}) \times H^2(A, \mathbb{C}) \to \mathbb{C}$$

defined by

$$Q_\Theta(\eta_1, \eta_2) = \int_A \frac{1}{3!} \wedge^3 c_1(\Theta) \wedge \eta_1 \wedge \eta_2$$

is non-degenerate (Hodge-Riemann bilinear relations, section 7 chapter 0 of [13]). As S and A have the same second Betti number [12] (2), this implies that the morphism

$$H^2(A, \mathbb{C}) \xrightarrow{\vartheta^*} H^2(S, \mathbb{C})$$

is an isomorphism. It follows moreover, that the morphism

$$H_2(S) \xrightarrow{\vartheta_*} H_2(A, \mathbb{Z})$$

is injective (where $H_2(S)$ is $H_2(S, \mathbb{Z})$ modulo torsion).
By [10], 2.3.5.1, we have the following exact sequence:

$$H_2(S) \xrightarrow{\vartheta_*} H_2(A, \mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z} \to 0$$

and we know that this sequence is also exact on the left. By duality, this yields:

$$0 \to H^2(A, \mathbb{Z}) \xrightarrow{\vartheta^*} H^2(S) \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

This implies that the sequence:

$$0 \to \mathrm{NS}(A) \xrightarrow{\vartheta^*} \mathrm{NS}(S) \to \mathbb{Z}/2\mathbb{Z} \to 0$$

is exact. The fact that $\vartheta^*\Theta = 2C_s$ follows from Lemma 11.27 of [8]. Since Θ is a principal polarisation, it is not divisible by 2, hence the class of C_s and the image of ϑ^* generate $\mathrm{NS}(S)$, thus c).
By [12] (2), the space $H^1(S, \Omega_S)$ is 25 dimensional. Let us call (improperly) "intermediate Jacobian" the Albanese variety of a Fano surface. By [10], any Jacobian of a hyperelliptic curve of genus 5 is a limit of intermediate Jacobians. By [15], the endomorphism ring of a Jacobian of a generic hyperelliptic curve is isomorphic to $\mathbb{Z}$. If a generic intermediate Jacobian were not simple, then also its limit would be non-simple. This is a contradiction, hence the Picard number of a generic intermediate Jacobian is 1. $\square$

2. Configurations of the elliptic curves.

2.1. **Configurations of 2 or 3 elliptic curves.** Let S be a Fano surface and F be the image of its cotangent map. Let $E \hookrightarrow S$ be an elliptic curve and p_E be the vertex of the cone $\psi_*\pi^*E$.
For sub-varieties V_1, V_2 of $\mathbb{P}^4$, let us denote by $\langle V_1, V_2\rangle$ their linear hull. Let s be a point of S outside the curve E. The line L_s is not inside the cone $\psi_*\pi^*E$ and the plane $X_s := \langle L_s, p\rangle$ cuts the cubic F in three lines:
1) the line L_s,
2) the line $L_{\gamma_E s}$ (on the cone) through the vertex p and the intersection point of L_s and the hyperplane $T_{F,p}$,
3) the residual line $L_{\sigma_E s}$ such that :

$$X_s F = L_s + L_{\gamma_E s} + L_{\sigma_E s}.$$

The rational map $\gamma_E : S \to E$ onto the elliptic curve is a morphism ; the rational map σ_E verifies $\sigma_E^2 = 1$ and is an automorphism of S because S is contained in its Albanese variety.
Let s, t be two points of the curve $E \hookrightarrow S$. The line L_s cuts the line L_t at the vertex of the cone $\psi_*\pi^*E$: hence s is a point of the incidence divisor C_t and there exists an effective divisor R_t such that:

$$C_t = E + R_t.$$

Theorem 2.1. *a) Let t be a point of $E \hookrightarrow S$. The divisor R_t is the fibre at t of γ_E and has genus 7. The morphisms γ_E and σ_E verify $\gamma_E\sigma_E = \gamma_E$.*
b) The automorphism $-M_{\sigma_E} \in GL(H^o(\Omega_S)^*)$ is a complex order 2 reflection. The underlying space of the vertex $p_E \in \mathbb{P}^4$ is the eigenspace of M_{σ_E} with eigenvalue 1.
c) Let $E' \hookrightarrow S$ be second elliptic curve with $E \neq E'$, then $0 \leq EE' \leq 1$.
d) The automorphisms σ_E and $\sigma_{E'}$ verify:

$$(\sigma_E\sigma_{E'})^{3-EE'} = 1.$$

e) If $EE' = 1$, then the fibration γ_E contracts E'.
f) If $EE' = 0$, then there exists a third elliptic curve E'' such that

$$\sigma_E^*(E') = \sigma_{E'}^*(E) = E''.$$

Moreover the curves E' and E'' are sections of γ_E.

Let us prove Theorem 2.1. Let t be a point of $E \hookrightarrow S$. If s is a generic point of $R_t = C_t - E$, then the line L_s cuts the line $L_t \hookrightarrow \psi_*\pi^*E$. By definition, $\gamma_E s = t$ and s is a point of $\gamma_E^* t$. This proves that R_t is a component of $\gamma_E^* t$. Conversely, we have $R_t^2 = (C_t - E)^2 = 5 - 2 + (-3) = 0$, hence R_t is the fibre at t of γ_E.
A canonical divisor K is numerically equivalent to $3C_t$, hence $KR_t = K(C_t - E) = 12$ and R_t has genus $\frac{12+0}{2} + 1$.
The plane X_s ($s \in S$) is equal to the plane $X_{\sigma_E s}$ hence $\gamma_E s = \gamma_E \sigma_E s$.
Let $E' \hookrightarrow S$ an elliptic curve. Since R_t is a fibre, we have:

$$R_t E' = (C_t - E)E' = 1 - EE' \geq 0.$$

If $E = E'$, we see that γ_E has degree 4 on E. Suppose now that $E \neq E$, then $EE' \geq 0$ and hence $0 \leq EE' \leq 1$. This proves a) and c).

If $EE' = 1$, then $R_t E = 0$ and E' is contained on a fibre of γ_E, hence e). If $EE' = 0$, then $R_t E' = 1$ and E' is a section of γ_E.

Let us describe the behavior of σ_E and γ_E on E. If s is a point of S not in E, we have:

$$X_s = \langle L_s, L_{\gamma_E s} \rangle \,.$$

The restriction of γ_E to E has degree 4, hence there are a finite number of points such that $\gamma_E s = s$ and the relation $X_s = \langle L_s, L_{\gamma_E s} \rangle$ extends the definition of the rational map $s \to X_s$ outside these points.

Let $Y \hookrightarrow T_{F,p}$ be an plane that does not contain p_E. The section of the cone $\psi_* \pi^* E = FT_{F,p}$ by Y is a plane model of E. Let us denote by $+$ the law of $E \hookrightarrow Y$ defined by chords and tangents and by the choice of an inflection point of $E \hookrightarrow Y$. We have a translation of the equality:

$$X_s F = L_s + L_{\gamma_E s} + L_{\sigma_E s}$$

by:

$$s + \sigma_E s = -\gamma_E s$$

for all points s of E such that $s \neq \gamma_E s$. Since σ_E and γ_E are regular morphisms, this equality is true for all points of E.

Let us prove part b). Let $\vartheta : S \to A$ be an Albanese morphism.

Lemma 2.2. *([8] 11.9) There is a point u_o on A such that for all points s_1, s_2, s_3 on S such that the lines L_{s_1}, L_{s_2}, L_{s_3} are coplanar, we have:*

$$\vartheta(s_1) + \vartheta(s_2) + \vartheta(s_3) = u_o.$$

For all $s \in S$, the lines L_s, $L_{\gamma_E s}$, $L_{\sigma_E s}$ are coplanar, hence the morphism $s \to \vartheta(s) + \vartheta(\sigma_E s) + \vartheta(\gamma_E s)$ is constant. Let I_A be the identity of A and let

$$\Gamma_E : A \to E$$

be the morphism such that $\Gamma_E \circ \vartheta = \gamma_E$. Let I be the identity of $H^o(\Omega_S)^*$. The morphism $I_A + \sigma'_E + \vartheta \circ \Gamma_E$ is constant, hence:

Lemma 2.3. *The differential of the linear part of $\vartheta \circ \Gamma_E : A \to A$ is the endomorphism $N_E \in End(H^o(\Omega_S)^*)$ such that :*

$$I + M_{\sigma_E} + N_E = 0.$$

The image of N_E is the 1 dimensional tangent space of the curve $\vartheta(E)$ translated to 0. Since M_{σ_E} has order 2, we see that the eigenvalues of M_{σ_E} are -1 with multiplicity 4 and 1 with multiplicity 1. Hence $-M_{\sigma_E}$ is an order 2 reflection.

For s a point of E, we have $s + \sigma_E s = -\gamma_E s$. Since the fibration has degree 4 on E, there exists an order 2 point b of E such that $\sigma_E s = s + b$ for all s in E. In particular, the differential M_{σ_E} is the identity on the tangent space of the curve $\vartheta(E) \hookrightarrow A$ translated to 0. The underlying space of the point p_E corresponds to the eigenspace of M_{σ_E} of eigenvalue 1. This proves b).

Let $E' \neq E$ a second elliptic curve on S. Let us prove that $(\sigma_E \sigma_{E'})^{3-EE'} = 1$.

Case $EE' = 1$**.** Suppose that $EE' = 1$. Let t be the intersection point of E and E'. With this neutral element, the curves E and E' are elliptic curves. Let s be a point of E' different from t, then:

$$X_{E,s}F = L_s + L_{\gamma_E s} + L_{\sigma_E s}.$$

Since the line L_s cuts the line L_t at the vertex $p_{E'}$, the point t is one of the three points s, $\gamma_E s$, $\sigma_E s$ and we see that $\gamma_E s = t$. thus E' is a component of $\gamma_E^* t$ and $\sigma_E s = -s$ for all points s of E'.

Definition 2.4. Let us define $\widetilde{\sigma_E} := q'(M_{\sigma_E}) \in \mathrm{Aut}(F)$. This automorphism fix a point and a hyperplane of $\mathbb{P}^4$. It is called a homology.

Remark 2.5. Since $\sigma_E(s) = -s$ for all point s of E', the endomorphism M_{σ_E} is the morphism of multiplication by -1 on the tangent space to the curve $\vartheta(E')$ (translated to 0). Hence we have $\widetilde{\sigma_E}(p_{E'}) = p_{E'}$ and $p_{E'}$ is the intersection point of the line L_t and the hyperplane of fixed points of $\widetilde{\sigma_E}$. This implies that the points p_E and $p_{E'}$ are the only vertices of cone on the line L_t.

Let s be a generic point of S, then

$$\widetilde{\sigma_{E'}} X_{E,s} = \widetilde{\sigma_{E'}} \langle p_E, L_s \rangle = \langle p_E, L_{\sigma_{E'} s} \rangle = X_{E, \sigma_{E'} s},$$

hence:

$$\widetilde{\sigma_{E'}} X_{E,s} F = L_{\sigma_{E'} s} + L_{\gamma_E \sigma_{E'} s} + L_{\sigma_E \sigma_{E'} s}.$$

But $X_{E,s}F = L_s + L_{\gamma_E s} + L_{\sigma_E s}$ hence:

$$\widetilde{\sigma_{E'}} X_{E,s} F = L_{\sigma_{E'} s} + L_{\sigma_{E'} \gamma_E s} + L_{\sigma_{E'} \sigma_E s}.$$

Since $\gamma_E \sigma_{E'} s$ and $\sigma_{E'} \gamma_E s$ are points of E, we see that $\sigma_E \sigma_{E'} s = \sigma_{E'} \sigma_E s$, thus $(\sigma_E \sigma_{E'})^2 = 1$.

Case $EE' = 0$**.** Suppose now that $EE' = 0$. Let s be a point of E'. By definition $X_{E,s} = \langle p_E, L_s \rangle$ and:

$$X_{E,s}F = L_s + L_{\gamma_E s} + L_{\sigma_E s}.$$

The point $\sigma_E s$ is not a point of E', otherwise the plane $X_{E,s}$ would cut the hyperplane section $\psi_* \pi^* E'$ into two lines, and thus the cone $\psi_* \pi^* E'$ would contain the third and $\gamma_E s$ would be a point of E'. The automorphism σ_E preserves $S \setminus E$ and so the point $\sigma_E s$ is not a point of E. This proves that the surface S contains a third smooth curve $E'' = \sigma_E(E')$ of genus 1 and that $EE'' = E'E'' = 0$.

For a point s of E', the plane $X_{E,s} = \langle p_E, L_s \rangle$ contains the line $L_{\gamma_E s}$ and the point $p_{E'} \in L_s$, hence:

$$X_{E,s} = \langle p_E, L_s \rangle = \langle p_{E'}, L_{\gamma_E s} \rangle = X_{E', \gamma_E s}.$$

But we have

$$X_{E,s}F = L_s + L_{\gamma_E s} + L_{\sigma_E s}$$

and
$$X_{E',\gamma_E s}F = L_{\gamma_E s} + L_{\gamma_{E'}\gamma_E s} + L_{\sigma_{E'}\gamma_E s}.$$
Since the points s, $\gamma_E s$ and $\sigma_E s$ are respectively points of E', E and E'', we see that for all points s of E':
$$\sigma_{E'}\gamma_E s = \sigma_E s.$$
Hence the restriction of $\sigma_{E'}$ to E is a morphism from E to E'' and $\sigma_{E'}(E) = \sigma_E(E') = E''$, this proves f).
Let s be a point of E'. The lines $L_s, L_{\gamma_E s}$ and $L_{\sigma_E s}$ contain respectively the vertices $p_{E'}, p_E$, and $p_{E''}$. As s varies in E', the plane $X_{E,s}$ varies and the linear hull of the points p_E, $p_{E'}$ and $p_{E''}$ cannot be a plane. Moreover:

Remark 2.6. The line ℓ which contains the vertices $p_E, p_{E'}, p_{E''}$ lies outside the cubic F, otherwise the curves E and E' would have a common point. The line ℓ cuts the cubic F at these three points.

Let s be a point of S. The homology $\widetilde{\sigma}_{E'} = q'(M_{\sigma_{E'}})$ verifies : $\widetilde{\sigma}_{E'}(L_s) = L_{\sigma_{E'}s}$, and furthermore:
$$\widetilde{\sigma}_{E'}X_{E,\sigma_{E'}s} = \widetilde{\sigma}_{E'}\left\langle p_E, L_{\sigma_{E'}s}\right\rangle = \langle p_{E''}, L_s\rangle = X_{E'',s} \quad (1).$$
We have $FX_{E,\sigma_{E'}s} = L_{\sigma_{E'}s} + L_{\gamma_E\sigma_{E'}s} + L_{\sigma_E\sigma_{E'}s}$. Hence
$$\widetilde{\sigma}_{E'}FX_{E,\sigma_{E'}s} = L_{\sigma^2_{E'}s} + L_{\sigma_{E'}\gamma_E\sigma_{E'}s} + L_{\sigma_{E'}\sigma_E\sigma_{E'}s},$$
but by (1):
$$\widetilde{\sigma}_{E'}FX_{E,\sigma_{E'}s} = FX_{E'',s}.$$
Since $FX_{E'',s} = L_s + L_{\gamma_{E''}s} + L_{\sigma_{E''}s}$, we see that $\sigma_{E'}\sigma_E\sigma_{E'} = \sigma_{E''}$. So the group generated by $\sigma_E, \sigma_{E'}, \sigma_{E''}$ is isomorphic to Σ_3 and $(\sigma_E\sigma_{E'})^3 = 1$.
This ends the proof of Theorem 2.1. □
Let us now study the configuration of 3 elliptic curves.

Proposition 2.7. *Let E_1, E_2 and E be three elliptic curves on S such that $E_1E = E_2E = 1$. We have $E_1E_2 = 0$ and the curve $E_3 = \sigma^*_{E_1}(E_2)$ verifies $E_3E = 1$.*

Proof. Suppose $E_1E_2 = 1$. This implies that the line through p_{E_1} and p_{E_2} lies on the cone $\psi_*\pi^*E$ and hence goes through p_E. But by remark 2.5, the three points p_E, p_{E_1} and p_{E_2} cannot be on a line. Hence $E_1E_2 = 0$.
Since $EE_1 = 1$, we have $\sigma^*_{E_1}(E) = E$ and $E_3E = \sigma^*_{E_1}(E_2)\sigma^*_{E_1}(E) = E_2E = 1$. □

2.2. **The graph of the configuration of the vertices of cones.** Let $\mathcal{P}$ be the set of vertices of cones on the cubic. Let us consider the following graph $\mathbb{G}$: the set of vertices of $\mathbb{G}$ is $\mathcal{P}$ and an edge links $p \in \mathcal{P}$ to $p' \in \mathcal{P}$ if and only if the line through p and p' lies outside F. let p be the vertex of the cone that corresponds to the elliptic curve $E \hookrightarrow S$. To simplify the notations, we denote by $t_p \in \mathrm{Aut}(F)$ the automorphism $q(M_{\sigma_E})$.
Let $\mathbb{W}$ be the sub-group of $\mathrm{Aut}(F)$ generated by the automorphisms t_p, $p \in$

$\mathcal{P}$. We have the three following relations between the generators:
a) for all $p \in \mathcal{P}$, t_p verifies $t_p^2 = 1$,
b) an edge links p and p' if and only if $(t_p t_{p'})^3 = 1$,
c) otherwise $(t_p t_{p'})^2 = 1$.
The following corollary is a consequence of Proposition 2.7.

Corollary 2.8. *Let p_1, p_2 and p_3 be three elements of $\mathcal{P}$. At least one edge links two of the three vertices p_1, p_2, p_3 of the graph $\mathbb{G}$.*
There is no sub-group of $\mathbb{W}$ generated by some elements t_p which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$.

Proof. Let E_i be the curve which corresponds to the vertex p_i. If there are no edge between vertices p_1 and p_3 and between the vertices p_2 and p_3, then $E_1E_3 = E_2E_3 = 1$ and the Proposition 2.7 implies that $E_1E_2 = 0$. Thus an edge links the vertices p_1 and p_2. □

This Corollary of Proposition 2.7 may be reformulated as follows:

Corollary 2.9. *The graph $\mathbb{G}$ is connected or $\mathbb{G}$ has two connected components $\mathbb{G}_1, \mathbb{G}_2$ such that two different vertices of a component $\mathbb{G}_i$ are linked by an edge.*

Let us remark that if the graph $\mathbb{G}$ may be broken into two connected components, then the group $\mathbb{W}$ is the direct product of two sub-groups.

Let $\mathcal{P}'$ be a sub-set of $\mathcal{P}$. Let $\mathbb{G}'$ be the graph whose set of vertices is $\mathcal{P}'$ and such that an edge links two points of $\mathcal{P}'$ if and only if these vertices are linked by an edge in $\mathbb{G}$. Suppose that the three relations a), b) and c) above are the only ones between the elements t_p, $p \in \mathcal{P}'$. Let $\mathbb{W}'$ be the Weyl group generated by the automorphisms t_p, $p \in \mathcal{P}'$.

Corollary 2.10. *If the graph $\mathbb{G}'$ is connected and has n vertices, then $1 \leq n \leq 4$ and the group $\mathbb{W}'$ is isomorphic to the permutation group of the set of $n+1$ elements.*

Proof. By the classification of the Weyl groups, the graph $\mathbb{G}'$ must be isomorphic to one of the graphs A_n, $1 \leq n \leq 4$. The Weyl group $W(A_n)$ associated to A_n is the permutation group of $n+1$ elements. □

2.3. **The classification of automorphism groups.**

2.3.1. *Restrictions on the complex reflection groups.* Let $\mathcal{P}$ be the set of vertices of the cones in the cubic F. Let us denote by G_S the group generated by the morphisms $-M_{\sigma_E}$ where $E \hookrightarrow S$ is an elliptic curve. It is a complex reflection group. For the basic definition of reflexion groups see [20], [9] or [11].

Definition 2.11. Let G_1 and G_2 be two reflection groups acting on spaces V_1 and V_2, we say (improperly) that G_1 is a reflection sub-group of G_2, if there exists an injective morphism $G_1 \to G_2$ of complex reflection groups. In this case, we denote the elements of G_1 and G_2 by the same letters.

The list of the 37 types of irreducible reflection groups was compiled by G. Sheppard and J. Todd [20]. The type 2 reflection groups are the groups $G(m,p,n)$ (with $m>0, n>0$ integers and p dividing m) described in the introduction. The type 3 groups constitute the familly $[\,]^n$ for $n \in \mathbb{N} \setminus \{0,1\}$ where the $[\,]^n$ is the group of morphisms $x \in \mathbb{C} \to \xi x$, $\xi^n = 1$.

Theorem 2.12. *An irreducible sub-group of G_S generated reflections of order 2 is isomorphic to one of the following groups:*

$$\{1\},\ [\,]^2,\ G(3,3,n),\ G(1,1,n) = \Sigma_n,\ 2 \leq n \leq 5.$$

Proof. Let $E \hookrightarrow S$ be an elliptic curve, by Proposition 1.14 and Theorem 2.1, the automorphism $q'(M_{\sigma_E})$ fixes pointwise a hyperplane and the vertex p_E. By the Remarks 2.5 and 2.6, a projective line of $\mathbb{P}^4$ contains at most 3 vertices of a cone, hence a 2 dimensional reflection sub-group of G_S contains at most 3 reflections of order 2. The groups of type 4 to 22 are 2 dimensional irreducible groups. They either have no or at least 6 reflections of order 2 ([9] table p.395): none of these groups can be a reflection sub-group of G_S.

The groups numbered $25, 29, 31, 32, 33, 34$ have either no or at least 40 reflexions of order 2 ([9] p.412). Since reflections of order 2 of G_S are in bijection with elliptic curves on S and a Fano surface contains at most 30 elliptic curves (Proposition 1.13), none of these groups is a sub-group of G_S.

The groups $23, 28, 30, 35, 36, 37$ are real reflection groups ([20] p.299) which have been excluded by the Corollary 2.10.

The group 26 has a reflection sub-group isomorphic to the group number 4 already excluded ([20] p.302).

The groups 24 and 27 have two reflections R_1 and R_2 of order 2 such that $(R_1R_2)^4 = 1$ ([20] p.299) and hence they cannot be sub-groups of G_S.

It thus remains to study the reflection groups of types 1, 2 and 3. Let M be an irreducible sub-group of G_S generated by order 2 reflections.

Type 3. Fact: there exists $n > 1$ such that $[\,]^n \simeq M$ if and only if $n = 2$.

Type 2 and 1. The group $G(m,p,n)$ (where p divides $m \in \mathbb{N}^*$ and $n > 1$) is an n dimensional irreducible reflection group if and only if $m > 1$ and $(m,p,n) \neq (2,2,2)$. The representation $G(1,1,n) = \Sigma_n$ of the permutation group breaks up into a 1 dimensional trivial representation and an $n-1$ dimensional irreducible representation $W(A_{n-1})$ called the standard representation. The groups:

$$W(A_n),\ n \in \mathbb{N}^*$$

constitute the number 1 reflection type in the Shephard-Todd classification.

Let m, p, n be integers such that $m > 0$, p divides m and $n > 1$. Suppose that M is the group $G(m,p,n)$. Theorem 2.1 implies that a group generated by two reflections of order 2 of G_S is the diedral group of order 4 or 6. For $m > 1$, the group $G(m,p,n)$ has a diedral sub-group of order $2m$ generated by two reflections of order 2. Thus the integer m is an element of $\{1,2,3\}$; moreover $n \leq 5$.

The group $G(2,2,2)$ is not irreducible and is thus excluded. The groups

$G(2,1,n)$ with $n \geq 3$ and the groups $G(2,2,n)$ with $n \geq 4$ contain sub-groups isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$ generated by reflections of order 2. By Corollary 2.9, such groups cannot be reflection sub-groups of G_S.
The reflection group Σ_4 is isomorphic to the group $G(2,2,3)$ plus the trivial representation. So $G(2,2,3)$ is implicitly in the list of Theorem 2.12.
The group $G(3,1,n)$ cannot be isomorphic to M because its order 2 reflections generate the sub-group $G(3,3,n) \neq G(3,1,n)$.
By Corollary 2.10, if $W(A_n)$ is a reflection sub-group of G_S, then $n \leq 4$. The group Σ_n is isomorphic to the reflection group $W(A_{n-1})$ plus the trivial représentation.

Hence, we have proven that M is isomorphic to one of the groups $\{1\}$, $[\,]^2$, $G(3,3,n)$, $G(1,1,n) = \Sigma_n$, $2 \leq n \leq 5$. □

2.3.2. *Classification of Fano surfaces.* Here we classify the Fano surfaces according to the configuration of their elliptic curves.

We need some notations and preliminary materials.
The order of $G(m,p,n)$ is $\frac{m}{p}m^{n-1}n!$ and the number of its order 2 reflections is $m\frac{n(n-1)}{2}$. The group $G(m,m,n)$ acts on the polynomial space of $\mathbb{C}^n$. The algebra of invariant polynomials is generated by the polynomials:

$$\sum_{i=1}^{i=n} x_i^{mk},\ k \in \{1,...,n-1\}$$

and by $x_1 x_2 .... x_n$ (see [20]).
Let S be a Fano surface such that the group $G(m,m,n)$ ($m \in \{1,3\}, 2 \leq n \leq 5$) is a reflection sub-group of G_S. Let F_{eq} be an equation of the image of the cotangent map F. There exists a morphism:

$$\chi : G_S \to \mathbb{C}^*$$

such that $F_{eq} \circ N = \chi(N) F_{eq}$ for all $N \in G(m,m,n)$.
Let $m \in \{1,3\}$ and $n > 1$. We easily check that the only non-trivial morphism from $G(m,m,n)$ to $\mathbb{C}^*$ is the determinant. We call a polynomial P an anti-invariant of $G(m,m,n)$ if $P \circ N = (\det N)P$ for all $N \in G(m,m,n)$.
We verify that:

Lemma 2.13. *The only reflection groups $G(m,m,n)$ with $m \in \{1,3\}$ and $n \geq 2$ that possess an anti-invariant polynomial of degree ≤ 3 are $G(1,1,2)$, $G(3,3,2)$ and $G(1,1,3)$.*

For $\lambda^3 - 1 \neq 0$, we denote by E_λ the smooth plane cubic:

$$x^3 + y^3 + z^3 - 3\lambda xyz = 0.$$

Its modular invariant is : $j(E_\lambda) = \frac{\lambda^3(\lambda^3+8)^3}{\lambda^3-1}$ [1] p.36.

We denote by A the Albanese variety of S, by $\vartheta : S \to A$ an Albanese morphism, by $e_1,..,e_5$ the dual basis of the basis $x_1,...,x_5 \in H^o(\Omega_S)$. If v is a non zero vector, $\mathbb{C}v$ is the vector space generated by v or the point of

$\mathbb{P}^4$ corresponding to this space, we specify as need be. We denote by μ_3 the group of third roots of unity.

Let $E \hookrightarrow S$ be an elliptic curve. Let p be the vertex of the cone $\psi_* \pi^* E$ in the cubic $F \hookrightarrow \mathbb{P}^4$. This cone is the intersection of F and the projective hyperplane $T_{F,p}$. A plane model of the curve E is the intersection of this cone with the hyperplane of fixed points of the homology $q'(M_{\sigma_E})$ (see definition 2.4).

In the sequel, we proceed to the classification according to the dimension of the studied irreducible reflection sub-group of G_S.

- 1 dimensional reflection sub-groups of G_S.

■ If the group $[]^2$ is a reflection sub-group of G_S, there exist a basis $x_1, .., x_5$ of $H^o(\Omega_S)$ such that the image of the cotangent map is

$$F = \{x_1^2 x_2 + G(x_2, ..., x_5) = 0\}$$

where G is a cubic form. Let $E \hookrightarrow S$ be the elliptic curve that corresponds to the cone of vertex $p = \mathbb{C}e_1$. It is easy to check that the intersection of the cubic F with the hyperplane $\{x_1 = 0\}$ of fixed points of $q'(M_{\sigma_E})$ is necessarily a smooth cubic surface, hence:

Corollary 2.14. *The fixed locus of the involution $\sigma_E : S \to S$ is the union of the curve E and 27 fixed isolated points.*

Conversely, if $G = G(x_2, x_3, x_4, x_5)$ is a cubic form such that the surface $X = \{G = x_1 = 0\}$ and the plane cubic $E = \{G = x_1 = x_2 = 0\}$ are smooth, then the cubic $\{x_1^2 x_2 + G(x_2, ..., x_5) = 0\}$ is a smooth threefold. The moduli space such pairs (X, E) is 7 dimensional. In other words: the moduli space of pairs (S, E) where S is a Fano surface and $E \hookrightarrow S$ is a smooth curve of genus 1 forms a 7 dimensional locus in the 10 dimensional moduli space of Fano surfaces.

- 2 dimensional reflection sub-groups of G_S.

■ The anti-invariant polynomials of the reflection group $G(3,3,2)$ yield singular cubics. The invariants of the group $G(3,3,2)$ are generated by the polynomials:

$$x_1^3 + x_2^3,\ x_1 x_2,\ x_3, x_4, x_5.$$

Up to a change of variables, the cubic F is:

$$F = \{x_1^3 + x_2^3 + 3x_1 x_2 l(x_3, x_4, x_5) + x_3^3 + x_4^3 + x_5^3 - 3\lambda x_3 x_4 x_5 = 0\}$$

where l is a linear form and $\lambda \in \mathbb{C}$.

The Fano surface has three skew elliptic curves E_{12}^{β}, $\beta \in \mu_3$, where the tangent space to the curve $\vartheta(E_{12}^{\beta})$ (translated to 0) in A is $\mathbb{C}(e_1 - \beta e_2) \subset H^o(\Omega_S)^*$. The three elliptic curves have the following plane model:

$$x_3^3 + x_4^3 + x_5^3 - 3\lambda x_3 x_4 x_5 + l^3 = 0.$$

Note that we also have studied the reflection group $G(1,1,3) = \Sigma_3$ because its representation decomposes into the trivial one plus the standard representation $W(A_2)$ which is equal to $G(3,3,2)$. The dual graph of this configuration is $3A_1$.

- 3 dimensional reflection sub-groups of G_S.

■ Suppose that $G(3,3,3)$ is a reflection sub-group of G_S. There exist coordinates $x_1,..,x_5$ and $\lambda^3 \neq 1$ such that:

$$F = \{x_1^3 + x_2^3 + x_3^3 - 3\lambda x_1x_2x_3 + x_4^3 + x_5^3 = 0\}.$$

The group $G(3,3,3)\times G(3,3,2)$ is a reflection sub-group of G_S. The 9 points $\mathbb{C}(e_i - \beta e_j)$, $1 \leq i < j \leq 3$, $\beta \in \mu_3$ are vertices of cones. The corresponding elliptic curves :

$$E_{ij}^{\beta},\ 1 \leq i < j \leq 3,\ \beta \in \mu_3$$

are isomorphic to the curve E_0 and are skew.
The 3 points $\mathbb{C}(e_4 - \beta e_5)$, $\beta \in \mu_3$ are vertices of cones. The corresponding 3 elliptic curves E_{45}^{β}, $\beta \in \mu_3$ are skew and isomorphic to the curve E_λ. For all $1 \leq i < j \leq 3$, $(\beta,\gamma) \in \mu_3^2$, we have:

$$E_{ij}^{\beta}E_{45}^{\gamma} = 1.$$

The Fano surface has 12 elliptic curves ; they constitute an abstract configuration $(9_3, 3_9)$.

■ The invariants of degree less than or equal to 3 of Σ_4 are generated by:

$$x_5^k,\ x_1^k + x_2^k + x_3^k + x_4^k,\ k \in \{1,2,3\}.$$

Let F be a smooth cubic defined by an element of this space. For $1 \leq i < j \leq 4$, the point $\mathbb{C}(e_i - e_j)$ of $\mathbb{P}^4$ is the vertex of a cone ; let us denote by:

$$E_{ij} \hookrightarrow S$$

the corresponding elliptic curve. Let E_{ij}, E_{st} be two such elliptic curves, then $E_{ij}E_{st} = 1$ if and only if $\{i,j\} \cap \{s,t\} = \emptyset$. The dual graph of this configuration is the graph $3A_2$.

- 4 dimensional reflection sub-groups of G_S.

■ Suppose that Σ_5 is a reflection sub-group of G_S. There exist $\lambda,\mu \in \mathbb{C}$ such that the image of the cotangent map of S is:

$$F_{\lambda,\mu} = \{\sum_{i=1}^{i=5} x_i^3 + \lambda(\sum_{i=1}^{i=5} x_i)(\sum_{i=1}^{i=5} x_i^2) - \mu(\sum_{i=1}^{i=5} x_i)^3 = 0\}.$$

For $1 \leq i < j \leq 5$, the point

$$p_{ij} = \mathbb{C}(e_i - e_j)$$

is the vertex of a cone and we denote by $E_{ij} \hookrightarrow S$ the corresponding elliptic curve. Let E_{ij}, E_{st} be two such curves. We have $E_{ij}E_{st} = 1$ if and only if $\{i,j\}\cap\{s,t\} = \emptyset$. Let D be the sum $\sum_{i<j} E_{ij}$. It verifies $D^2 = 0$ and we can prove that D is a fibre of a fibration of S (same construction as Corollary 3.10).

By an appropriate change of variables, we can prove that the family $F_{\lambda,\mu}$, $\lambda,\mu \in \mathbb{C}$ is 1 dimensional in moduli space of cubic threefolds and that any cubic which has 10 cones can be written as in the introduction. We can further see that the Fermat cubic $F_{0,0}$ and the cubic $F_{-\frac{3}{5},0}$ are (up to isomorphism) the only cubics of this family for which the group $q'(G_S)$ is stricter in the automorphism group. The cubic $F_{-\frac{3}{5},0}$ possesses an order 3 homology which commutes with the elements of $q'(G_S)$. The dual graph of this configuration of these 10 curves is trivalent Petersen graph.

- 5 dimensional reflection sub-group of G_S.

■ The Fermat cubic:

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0$$

is the only cubic stable under $G(3,3,5)$. We study it in the next section. We did not study the 4 dimensional group $G(3,3,4)$. Its only invariants of degree less than or equal 3 are

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 \text{ and } x_5^3,$$

hence if $G(3,3,4)$ is a reflection sub-group of G_S, the cubic F is isomorphic to the Fermat cubic.

Next, we have to study the case where the reflection group G_S is not irreducible. Corollary 2.9 proves that :

Lemma 2.15. *If the reflection group G_S is not irreducible, it is the direct product of two irreducible reflection groups $\mathbb{W}_1$ and $\mathbb{W}_2$ such that if R_1 and R_2 are two different reflections of order 2 of the group $\mathbb{W}_i$, we have :*

$$(R_1R_2)^3 = 1.$$

The groups with this last property and listed in Theorem 2.12 are:

$$[\,]^2,\ G(3,3,2) \text{ or } G(3,3,3).$$

■ The case where one of the groups $\mathbb{W}_i$ ($i \in \{1,2\}$) of Lemma 2.15 is equal to $G(3,3,3)$ has already been studied. In that case

$$\mathbb{W}_1 \times \mathbb{W}_2 \simeq G(3,3,3) \times G(3,3,2)$$

is a reflection sub-group of G_S.

■ If $[\,]^2 \times [\,]^2$ is a reflection sub-group of G_S, there exist coordinates such that :

$$F = \{x_1^2 l_1(x_3,x_4,x_5) + x_2^2 l_2(x_3,x_4,x_5) + x_3^3 + x_4^3 + x_5^3 - 3\lambda x_3x_4x_5 = 0\}$$

where l_1 and l_2 are two linearly independent forms. The Fano surface S contains two elliptic curves E, E' such that $EE' = 1$. The dual graph of this configuration is the graph A_2.

■ If $G(3,3,2) \times [\,]^2$ is a reflection sub-group of G_S, then there exist coordinates such that:

$$F = \{x_1^3 + x_2^3 - 3\lambda x_1x_2 l_1(x_4,x_5) + x_3^2 l_2(x_4,x_5) + x_4^3 + x_5^3 = 0\},$$

where l_1 are l_2 linear forms. The Fano surface has three skew elliptic curves that cut another elliptic curve. The dual graph of this configuration is the graph D_4.

■ The last case is the group $G(3,3,2)\times G(3,3,2)$ for which the cubic is:

$$F=\{x_1^3+x_2^3+3ax_1x_2x_5+x_3^3+x_4^3+3bx_3x_4x_5+x_5^3=0\},$$

$(a,b\in\mathbb{C})$ and the Fano surface contains 3 skew elliptic curves E_{12}^{β}, $\beta\in\mu_3$ isomorphic to the plane cubic

$$x_3^3+x_4^3+(1+a^3)x_5^3+3bx_3x_4x_5=0,$$

and three others skew elliptic curves E_{34}^{α}, $\alpha\in\mu_3$ isomorphic to the cubic:

$$x_1^3+x_2^3+(1+b^3)x_5^3+3ax_1x_2x_5=0.$$

These curves verify $E_{12}^{\beta}E_{34}^{\gamma}=1$ $(\beta,\gamma\in\mu_3)$. We define D by $D=\sum_{\beta}E_{12}^{\beta}+E_{34}^{\beta}$, it verifies $D^2=0$ and we can prove that D is a fibre of a fibration of S (same construction as Corollary 3.9). The Fano surface has 6 elliptic curves ; they constitute an abstract configuration (3_3).

2.3.3. *Consequences of the classification, examples.* Let $\lambda\in\mathbb{C}$, $\lambda^3\neq 1$. The Fano surface S_λ of the cubic

$$F_\lambda=\{x_1^3+x_2^3+x_3^3-3\lambda x_1x_2x_3+x_4^3+x_5^3\}\hookrightarrow\mathbb{P}^4$$

possesses 12 smooth curves of genus 1 for which we use the notations of the previous paragraph. Let E_λ be the elliptic curve

$$x_1^3+x_2^3+x_3^3-3\lambda x_1x_2x_3=0$$

with neutral element $(1:-1:0)$. Let Y be the surface $Y=E_\lambda\times E_\lambda$, let us denote by Δ the diagonal and define

$$\begin{array}{l} T_1=\{x+2y=0\}\hookrightarrow Y\\ T_2=\{2x+y=0\}\hookrightarrow Y.\end{array}$$

Any 2 of the 3 curves T_1,T_2,Δ meet transversally at the 9 points of 3-torsion of Δ. We denote by Z the blow-up of Y at these 9 points.

Proposition 2.16. *The Fano surface S_λ is a triple cyclic cover of Z branched along the proper transform of $\Delta+T_1+T_2$ in Z. The divisor*

$$K_{45}=\sum_{\beta\in\mu_3}2E_{45}^{\beta}+E_{12}^{\beta}+E_{13}^{\beta}+E_{23}^{\beta}$$

is a canonical divisor of S_λ.

Proof. Let $\alpha\in\mu_3$ be a primitive root. The order 3 automorphism

$$f:x\to(x_1:x_2:\alpha x_3:\alpha x_4:\alpha x_5)$$

acts on F_λ. We denote by $\tau=\rho(f)$ the induced action on S_λ. The fixed locus of τ is the smooth divisor $E_{45}^1+E_{45}^{\alpha}+E_{45}^{\alpha^2}$. The quotient of S_λ by τ is a smooth surface Z' with Chern numbers $c_1^2=-9$ and $c_2=9$ and the degree 3 quotient map $\eta:S_\lambda\to Z'$ is ramified over $E_{45}^1+E_{45}^{\alpha}+E_{45}^{\alpha^2}$.
The morphism

$$g=(\gamma_{E_{45}^{\alpha}},\gamma_{E_{45}^{\alpha^2}}):S_\lambda\to Y$$

has degree $3=(C_s-E_{45}^{\alpha})(C_s-E_{45}^{\alpha^2})$. It is left invariant by τ, hence there is a birational morphism:

$$h: Z'\to Y$$

such that $g=h\circ\eta$.
Let t be the intersection point of E_{12}^1 and E_{45}^1 and let $\vartheta: S_\lambda\to A_\lambda$ be the Albanese morphism such that $\vartheta(t)=0$. It is an embedding and we consider S_λ as a subvariety of A_λ. Let $e_1,..,e_5$ be the dual basis of $x_1,..,x_5$. The tangent space of the curve $E_{ij}^{\beta}\hookrightarrow A_\lambda$ (translated to 0) is $V_\beta=\mathbb{C}(\beta e_i-\beta^2 e_j)$. The tangent space of $E_{45}^{\alpha}\times E_{45}^{\alpha^2}$ is $V_\alpha\oplus V_{\alpha^2}$. With the help of Lemma 2.3, it is easily checked that the images under g of the curves $E_{45}^1, E_{45}^{\alpha}, E_{45}^{\alpha^2}$ are respectively Δ, T_1 and T_2.
Moreover, the morphism g has degree 1 on these 3 elliptic curves and contracts the 9 elliptic curves $E_{ij}, 1\leq i<j\leq 3$. This implies that the image under g of $E_{45}^1+E_{45}^{\alpha}+E_{45}^{\alpha^2}$ is $T_1+T_2+\Delta$ and Z' is isomorphic to Z.
Let us remark that Corollary 0.9 of the introduction is equivalent to Proposition 2.16. The fact that K_{45} is a canonical divisor is proven in [1], Lemma 17.1. □

Let E_{ij}^{β} be an elliptic curve on S_λ. We denote by $\Gamma_{ij}^{\beta}: A_\lambda\to E_{ij}^{\beta}$ the morphism such that $\Gamma_{ij}^{\beta}\circ\vartheta=\gamma_{E_{ij}^{\beta}}$.

Lemma 2.17. *The degree of the morphism*

$$\Gamma=(\Gamma_{12}^1,\Gamma_{23}^1,\Gamma_{12}^{\alpha},\Gamma_{45}^1,\Gamma_{45}^{\alpha}): A_\lambda\to E_{12}^1\times E_{23}^1\times E_{12}^{\alpha}\times E_{45}^1\times E_{45}^{\alpha}$$

divides 81.

Proof. Let Υ be the morphism Γ composed with the morphism $E_{12}^1\times E_{23}^1\times E_{12}^{\alpha}\times E_{45}^1\times E_{45}^{\alpha}\to A_\lambda$. By Lemma 2.3, we can compute the differential $d\Upsilon$ of Υ and we find $|\det(d\Upsilon)|^2=81$, hence the assertion. □

Let α be a third root of unity.

Proposition 2.18. *A) Let $\lambda\in\mathbb{C}$ be such that E_λ has no complex multiplication. The Néron-Severi group of S_λ has rank* 12. *The sub-lattice generated by the elliptic curves and the class of an incidence divisor C_s has rank* 12 *and discriminant* 2.3^{10}.
B) Let $\lambda\in\mathbb{C}$ be such that E_λ has complex multiplication by $\mathbb{Q}(\alpha)$.
The Abelian variety A_λ is isomorphic to a product of elliptic curves and the Picard number of S_λ is equal to $25=\dim H^1(S,\Omega_S)$.

Proof. If E_λ has no complex multiplication, Lemma 2.17 implies that the Néron-Severi group of A has rank 12 and Theorem 1.17 implies that the Picard number of S_λ is 12.
Suppose now that E_λ as complex multiplication by $\mathbb{Q}(\alpha)$. Then Lemma 2.17 and [5] 5.6 (10) imply that the Abelian variety A_λ is isomorphic to a product of elliptic curves and its Picard number is equal to 25. Theorem 1.17 implies that the Picard number of S_λ is equal to 25. □

3. Fano surface of the Fermat cubic.

3.1. **Elliptic curve configuration of the Fano surface of the Fermat cubic.** Let S be the Fano surface of the Fermat cubic $F \hookrightarrow \mathbb{P}^4 = \mathbb{P}(H^o(\Omega_S)^*)$:

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0.$$

Let $e_1, .., e_5 \in H^o(\Omega_S)^*$ be the dual basis of basis $x_1, ..., x_5$. The reflection group G_S is equal to $G(3,3,5)$, this group possesses 30 order 2 reflections. Let μ_3 be the group of third roots of unity, let $1 \leq i < j \leq 5$ and let $\beta \in \mu_3$. The point:

$$p_{ij}^\beta = \mathbb{C}(e_i - \beta e_j) \in \mathbb{P}^4$$

is the vertex of a cone on the cubic F. We denote by $E_{ij}^\beta \hookrightarrow S$ the elliptic curve which corresponds to the cone of vertex p_{ij}^β.

Proposition 3.1. *The Fano surface of the Fermat cubic possesses* 30 *smooth curves of genus* 1 *numbered :*

$$E_{ij}^\beta, 1 \leq i < j \leq 5,\ \beta \in \mu_3.$$

1) Each smooth genus 1 *curve of the Fano surface is isomorphic to the Fermat plane cubic* $\mathbb{E} : x^3 + y^3 + z^3 = 0$.
2) Let E_{ij}^γ *and* E_{st}^β *be two smooth curves of genus* 1. *We have :*

$$E_{ij}^\beta E_{st}^\gamma = \begin{cases} 1 & \textit{if } \{i,j\} \cap \{s,t\} = \emptyset \\ -3 & \textit{if } E_{ij}^\beta = E_{st}^\gamma \\ 0 & \textit{else.} \end{cases}$$

3) Let $E \hookrightarrow S$ *be a smooth curve of genus* 1. *The fibration* γ_E *has* 20 *sections and contracts* 9 *elliptic curves.*
4) The automorphism group of S *is isomorphic to* $G(3,3,5)$.

Proof. It is easy to check assertions 1) and 2). The assertion 3) results from Theorem 2.1. For 4), we use the fact that an automorphism of F must preserve the $(30_{10}, 100_3)$ configuration of the 30 vertices of cones and the 100 lines that contain 3 such vertices. □

3.2. **The Albanese variety of the Fano surface of the Fermat cubic.** Let S be the Fano surface of the Fermat cubic F. Our main aim is to compute the full Néron-Severi group of S: this will be done in the next paragraph. We first need to study the Albanese variety A of S.

3.2.1. *Construction of fibrations.* In order to know the period lattice of the Albanese variety of S, we construct morphisms of the Fano surface onto an elliptic curve and we study their properties.

Let $\vartheta : S \to A$ be a fixed Albanese morphism. It is an embedding and we consider S to be a sub-variety of A. Recall that if τ is an automorphism of S, we denote by $\tau' \in \mathrm{Aut}(A)$ the unique automorphism such that $\tau' \circ \vartheta = \vartheta \circ \tau$. The automorphism group $G(3,3,5)$ (in the basis $e_1, .., e_5$ of $H^o(\Omega_S)^*$) is the

analytic representation of the automorphisms τ', ($\tau \in \mathrm{Aut}(S)$). The ring $\mathbb{Z}[G(3,3,5)] \subset \mathrm{End}(H^o(\Omega_S)^*)$ is then the analytic representation of a subring of endomorphisms of the Abelian variety A.
Let us denote by Λ_A^* the rank 5 sub-$\mathbb{Z}[\alpha]$-module of $H^o(\Omega_S)$ generated by the forms:

$$x_i - \beta x_j \, (i < j, \, \beta \in \mu_3).$$

Let ℓ be an element of Λ_A^*. The endomorphism of $H^o(\Omega_S)^*$ defined by $x \to \ell(x)(e_1 - e_2)$ is an element of $\mathbb{Z}[G(3,3,5)]$. Let us denote by $\Gamma_\ell : A \to \mathbb{E}$ the corresponding morphism of Abelian varieties where $\mathbb{E} \hookrightarrow A$ is the elliptic curve with tangent space $(e_1 - e_2)$. We denote by $\gamma_\ell : S \to \mathbb{E}$ the morphism $\Gamma_\ell \circ \vartheta$.

For $1 \leq i < j \leq 5$ and $\beta \in \mu_3$, the space:

$$\mathbb{C}(e_i - \beta e_j) \subset H^o(\Omega_S)^*$$

is the tangent space to the elliptic curve $E_{ij}^\beta \hookrightarrow A$ translated to 0.
Let $H_1(A, \mathbb{Z}) \subset H^o(\Omega_S)^*$ be the period lattice of A. The elliptic curve $\mathbb{E}$ has complex multiplication by the principal ideal domain $\mathbb{Z}[\alpha]$ (where $\alpha \in \mu_3$ is a primitive root). There exists $c \in \mathbb{C}^*$ such that :

$$H_1(A, \mathbb{Z}) \cap \mathbb{C}(e_1 - e_2) = \mathbb{Z}[\alpha]c(e_1 - e_2).$$

Up to the basis change of $e_1, .., e_5$ by $ce_1, ..., ce_5$, we may suppose $c = 1$. Since $G(3,3,5)$ acts transitively on the 30 spaces $\mathbb{C}(e_i - \beta e_j)$, we have:

$$H_1(A, \mathbb{Z}) \cap \mathbb{C}(e_i - \beta e_j) = \mathbb{Z}[\alpha](e_i - \beta e_j).$$

We define the Hermitian product of two forms $\ell, \ell' \in \Lambda_A^*$ by :

$$\langle \ell, \ell' \rangle := \sum_{k=1}^{k=5} \ell(e_k)\overline{\ell'(e_k)},$$

and the norm of ℓ by: $\|\ell\| = \sqrt{\langle \ell, \ell \rangle}$. Let C_s be an incidence divisor.

Theorem 3.2. *Let ℓ be a non zero element of Λ_A^* and let F_ℓ be a fibre of γ_ℓ.*
1) The intersection number of F_ℓ and $E_{ij}^\beta \hookrightarrow S$ is equal to:

$$E_{ij}^\beta F_\ell = |\ell(e_i - \beta e_j)|^2.$$

2) We have $F_\ell C_s = 2\|\ell\|^2$ and the fibre F_ℓ has genus:

$$g(F_\ell) = 1 + 3\|\ell\|^2.$$

3) Let ℓ and ℓ' be two linearly independent elements of $\Lambda_A^* \subset H^o(\Omega_S)$. The morphism $\tau_{\ell,\ell'} = (\gamma_\ell, \gamma_{\ell'}) : S \to \mathbb{E} \times \mathbb{E}$ has degree equal to $F_\ell F_{\ell'}$ and :

$$F_\ell F_{\ell'} = \|\ell\|^2 \|\ell'\|^2 - \langle \ell, \ell' \rangle \langle \ell', \ell \rangle.$$

Remark 3.3. The known intersection numbers $F_\ell E_{ij}^\beta$ and $F_\ell C_s$ enable us to write the numerical equivalence class of a fibre in a $\mathbb{Z}$-basis given Theorem 3.12 below.

Let us prove Theorem 3.2. The part 1) is easy:
For $1 \leq i < j \leq 5$ and $\beta \in \mu_3$, the intersection $E_{ij}^{\beta}F_\ell$ is the degree of the restriction of γ_ℓ to $E_{ij}^{\beta} \hookrightarrow S$. It is also the degree of the restriction of Γ_ℓ to $E_{ij}^{\beta} \hookrightarrow A$. The degree of the morphism Γ_ℓ on E_{ij}^{β} is then equal to $|\ell(e_i - \beta e_j)|^2$. Thus $F_\ell E_{ij}^{\beta} = |\ell(e_i - \beta e_j)|^2$.

Let us study the genus of F_ℓ:

Lemma 3.4. *The fibre of F_ℓ has genus $1 + 3\|\ell\|^2$ and $C_sF_\ell = 2\|\ell\|^2$ (where s is a point of S and C_s the incidence divisor).*

Proof. The sum Σ of the 30 elliptic curves of S is a bicanonical divisor (Proposition 1.13). We have

$$\Sigma F_\ell = \sum_{i,j,\beta} F_\ell E_{ij}^{\beta} = \sum_{i,j,\beta} |\ell(e_i - \beta e_j)|^2 = 12\|\ell\|^2 .$$

Since F_ℓ is a fibre, we have $F_\ell^2 = 0$ and F_ℓ has genus $1 + \frac{0+6\|\ell\|^2}{2}$.
The divisor $3C_s$ (a point s of S) is numerically equivalent to a canonical divisor (Proposition 1.11). Thus: $C_sF_\ell = 2\|\ell\|^2$. $\square$

We identify the Chern class of a divisor of the Abelian variety A with an alternating form on the tangent space $H^o(\Omega_S)^*$ of A ([5], Theorem 2.12). Let Θ be the principal polarisation defined in paragraph 1.2.2.

Lemma 3.5. *The Chern Class of Θ is equal to:*

$$a\frac{i}{\sqrt{3}}\sum_{j=1}^{5} dx_j \wedge d\bar{x}_j$$

where a is a scalar and $i^2 = -1$.

Proof. Let H be the matrix (in the basis $e_1, .., e_5$) of the Hermitian form associated to $c_1(\Theta)$ (see [5], Lemma 2.17). The automorphisms τ', ($\tau \in \mathrm{Aut}(S)$) preserves the polarisation Θ (Lemma 1.16). This implies that for all $M = (m_{jk})_{1\leq j,k\leq 5} \in G(3,3,5)$, we have :

$${}^tMH\bar{M} = H$$

(where $\bar{M}$ is the matrix $\bar{M} = (\bar{m}_{jk})_{1\leq j,k\leq 5}$) and this proves that

$$H = \frac{2}{\sqrt{3}}aI_5$$

where I_5 is the identity matrix and $a \in \mathbb{C}$. Hence: $c_1(\Theta) = a\frac{i}{\sqrt{3}}\sum_{j=1}^{5} dx_j \wedge d\bar{x}_j$. $\square$

Since $H_1(A,\mathbb{Z}) \cap \mathbb{C}(e_1 - e_2) = \mathbb{Z}[\alpha](e_1 - e_2)$, the Néron-Severi group of the elliptic curve $\mathbb{E}$ is the $\mathbb{Z}$-module generated by

$$\eta = \frac{i}{\sqrt{3}}dz \wedge d\bar{z}$$

where z is the coordinate on the space $\mathbb{C}(e_1-e_2)$.
Let $\ell = a_1x_1 + .. + a_5x_5$ be an element of Λ_A^*. The pull back of the form η by the morphism $\Gamma_\ell : A \to \mathbb{E}$ is:

$$\Gamma_\ell^*\eta = \frac{i}{\sqrt{3}} d\ell \wedge d\bar{\ell}.$$

The form $\Gamma_\ell^*\eta$ is the Chern class of the divisor $\Gamma_\ell^* 0$ and $\gamma_\ell^*\eta = \vartheta^*\Gamma_\ell^*\eta$ is the Chern class of the divisor F_ℓ.

Lemma 3.6. *Let ℓ and ℓ' be two elements of Λ_A^*, then:*

$$F_\ell F_{\ell'} = \|\ell\|^2 \left\|\ell'\right\|^2 - \left\langle \ell, \ell' \right\rangle \left\langle \ell', \ell \right\rangle$$

and $c_1(\Theta) = \frac{i}{\sqrt{3}} \sum_{i=1}^{i=5} dx_i \wedge d\bar{x}_i$.

Proof. By the Theorem 1.17, $\vartheta^* c_1(\Theta)$ is the Chern class of the divisor $2C_s$ ($s \in S$) and:

$$2C_sF_\ell = \vartheta^* c_1(\Theta)\vartheta^*\Gamma_\ell^*\eta = \int_A \frac{1}{3!} \wedge^4 c_1(\Theta) \wedge \Gamma_\ell^*\eta$$

hence:

$$2C_sF_\ell = (\frac{i}{\sqrt{3}})^5 \int_A (\sum a_j dx_j) \wedge (\sum \bar{a}_j d\bar{x}_j) \wedge 4a^4 \sum_{1 \leq k \leq 5} (\wedge_{j\neq k}(dx_j \wedge d\bar{x}_j))$$

and:

$$2C_sF_\ell = (\frac{4}{a} \sum_{k=1}^{k=5} a_k\bar{a}_k) \frac{1}{5!} \int_A \wedge^5 c_1(\Theta).$$

Since Θ is a principal polarisation, we have $\frac{1}{5!}\int_A \wedge^5 c_1(\Theta) = 1$, hence: $2C_sF_\ell = \frac{4}{a}\|\ell\|^2$. We have seen in Lemma 3.4 that $C_sF_\ell = 2\|\ell\|^2$. Thus we deduce that $a = 1$.
By Theorem 1.17, for $\ell = a_1x_1 + .. + a_5x_5$ and $\ell' = b_1x_1 + .. + b_5x_5 \in \Lambda_A^*$,

$$F_\ell F_{\ell'} = \int_A \frac{1}{3!} \wedge^3 c_1(\Theta) \wedge \Gamma_\ell^*\eta \wedge \Gamma_{\ell'}^*\eta.$$

Since

$$\frac{1}{3!}(\frac{i}{\sqrt{3}})^2 d\ell \wedge d\bar{\ell} \wedge d\ell' \wedge d\overline{\ell'} \wedge (\wedge^3 c_1(\Theta)) = (\sum_{k\neq j} a_k\bar{a}_k b_j\bar{b}_j - a_k\bar{a}_j b_j\bar{b}_k)\frac{1}{5!} \wedge^5 c_1(\Theta),$$

the result follows. □

Let ℓ and ℓ' be two linearly independent elements of Λ_A^*. The degree of the morphism $\tau_{\ell,\ell'} = (\gamma_\ell, \gamma_{\ell'})$ is equal to $F_\ell F_{\ell'}$ because $\tau_{\ell,\ell'}^*(\mathbb{E} \times \{0\}) = F_{\ell'} \in \mathrm{NS}(S)$, $\tau_{\ell,\ell'}^*(\{0\} \times \mathbb{E}) = F_\ell \in \mathrm{NS}(S)$ and the intersection number of the divisors $\{0\} \times \mathbb{E}$ and $\mathbb{E} \times \{0\}$ is equal to 1.

This completes the proof of Theorem 3.2. □

3.2.2. *Period lattice of A.* We compute here the period lattice of the Albanese variety A in the basis $e_1,..,e_5$.

Theorem 3.7. *The lattice $H_1(A,\mathbb{Z})$ is equal to:*

$$\begin{array}{rl}\mathbb{Z}[\alpha](e_1-e_5) & +\mathbb{Z}[\alpha](e_2-e_5)+\mathbb{Z}[\alpha](e_3-e_5)+\mathbb{Z}[\alpha](e_4-e_5)\\ & +\frac{1+\alpha}{1-\alpha}\mathbb{Z}[3\alpha](\alpha^2e_1+\alpha^2e_2+\alpha e_3+\alpha e_4+e_5).\end{array}$$

The image of the morphism $\vartheta^*:\mathrm{NS}(A)\to\mathrm{NS}(S)$ is the sub-lattice of rank 25 *and discriminant* $2^2 3^{18}$ *generated by the divisors :*

$$F_{x_i-\beta^2x_j}=C_s-E_{ij}^{\beta},\ 1\leq i<j\leq 5,\ \beta\in\mu_3 \text{ and } \sum_{i<j}E_{ij}^1.$$

Proof. By the preceding paragraph, we have :

$$H_1(A,\mathbb{Z})\cap\mathbb{C}(e_i-\beta e_j)=\mathbb{Z}[\alpha](e_i-\beta e_j),$$

hence $H_1(A,\mathbb{Z})$ contains the lattice $\Lambda_0=\sum_{i\leq j,\beta\in\mu_3}\mathbb{Z}[\alpha](e_i-\beta e_j)$.
For $1\leq i<j\leq 5$, $\beta\in\mu_3$, the differential of $\Gamma_{x_i-\beta x_j}$ is the morphism $x\to(x_i-\beta x_j)(e_1-e_2)$. Thus:

$$\forall\lambda=(\lambda_1,..,\lambda_5)\in H_1(A,\mathbb{Z}),\ \lambda_i-\beta\lambda_j\in\mathbb{Z}[\alpha].$$

Let us define

$$\Lambda=\{x=(x_1,..,x_5)\in\mathbb{C}^5/x_i-\beta x_j\in\mathbb{Z}[\alpha],1\leq i<j\leq 5,\beta\in\mu_3\}.$$

This lattice Λ contains $H_1(A,\mathbb{Z})$ and is equal to

$$\mathbb{Z}[\alpha]e_1\oplus..\oplus\mathbb{Z}[\alpha]e_4\oplus\frac{1}{\alpha-1}\mathbb{Z}[\alpha]w,$$

where $w=e_1+..+e_5$. Let $\phi:\Lambda\to\Lambda/\Lambda_0$ be the quotient map. The group Λ/Λ_0 is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$ and contains 6 sub-groups. The reciprocal images of these groups are the lattices

$$\begin{array}{rlrl}\Lambda_0= & \phi^{-1}(0) & \Lambda_{\alpha^2}= & \Lambda_0+\frac{\alpha^2}{\alpha-1}\mathbb{Z}w\\ \Lambda_1= & \Lambda_0+\frac{1}{\alpha-1}\mathbb{Z}w & \Lambda_{\alpha-1}= & \Lambda_0+\mathbb{Z}w\\ \Lambda_\alpha= & \Lambda_0+\frac{\alpha}{\alpha-1}\mathbb{Z}w & \Lambda= & \Lambda_0+\frac{1}{\alpha-1}\mathbb{Z}[\alpha]w.\end{array}$$

These are the 6 lattices Λ' which verify $\Lambda_0\subset\Lambda'\subset\Lambda$, hence the lattice $H_1(A,\mathbb{Z})$ is equal to one of these.

Let ω be the alternating form $\omega=\frac{i}{\sqrt{3}}\sum_{k=1}^{k=5}dx_k\wedge d\bar{x}_k$ (see Lemma 3.6). We have

$$\frac{1}{\alpha-1}w,\frac{\alpha}{\alpha-1}w\in\Lambda.$$

However

$$\omega(\frac{1}{\alpha-1}w,\frac{\alpha}{\alpha-1}w)=-\frac{5}{3}$$

is not an integer, hence Λ is different from $H_1(A,\mathbb{Z})$.
The Pfaffian of $c_1(\Theta)$ relative to $H_1(A,\mathbb{Z})$ is equal to 1 because Θ is a principal polarisation. The Pfaffian of ω relative to the lattice Λ_0 is equal to 9, hence Λ_0 is different from $H_1(A,\mathbb{Z})$.

We have $\Lambda_{1-\alpha} = \oplus\mathbb{Z}[\alpha]e_i$ and the principally polarised Abelian variety $(\mathbb{C}^5/\Lambda_{1-\alpha}, \omega)$ is isomorphic to a product of Jacobians. Since $(A, c_1(\Theta))$ cannot be isomorphic to a product of Jacobians ([8], 0.12), $H_1(A,\mathbb{Z}) \neq \Lambda_{1-\alpha}$.
The lattice Λ_{α^i} is equal to:

$$\begin{aligned}\mathbb{Z}[\alpha](e_1-e_5) \quad &+\mathbb{Z}[\alpha](e_2-e_5)+\mathbb{Z}[\alpha](e_3-e_5)+\mathbb{Z}[\alpha](e_4-e_5)\\ &+\tfrac{\alpha^i}{1-\alpha}\mathbb{Z}[3\alpha](\alpha^2e_1+\alpha^2e_2+\alpha e_3+\alpha e_4+e_5).\end{aligned}$$

The lattices Λ_1 and Λ_α depend upon the choice of α such that $\alpha^2+\alpha+1=0$, hence the lattice $H_1(A,\mathbb{Z})$ is equal to Λ_{α^2}.

Since $H_1(A,\mathbb{Z})$ is known, it is easy to calculate a basis of the Néron-Severi group of A ([5] chapter 5) and the image of the morphism $\vartheta^* : \mathrm{NS}(A) \to \mathrm{NS}(S)$. $\square$

3.2.3. *Study of some fibrations, remarks.* Let X be a surface, C a smooth curve, $\gamma : X \to C$ a fibration with connected fibres. A point of X is called a critical point of γ if it is a zero of the differential:

$$d\gamma : T_X \to \gamma^* T_C.$$

A fibre of γ is singular at a point if and only if this point is a critical point ([1] Chapter III, section 8).
Let us suppose that C is an elliptic curve. The critical points of γ are then the zeros of the form $\gamma^*\omega \in H^o(X,\Omega_X)$ where ω is a generator of the trivial sheaf Ω_C.
Let us assume further that X verifies the hypothesis 0.2. A point x of X is a critical point if and only if the line $L_x = \psi(\pi^{-1}(x))$ (ψ cotangent map, π the projection) lies in the hyperplane

$$\{\gamma^*\omega = 0\} \hookrightarrow \mathbb{P}(H^o(\Omega_S)^*).$$

Let S be the Fano surface of the Fermat cubic F. We give here examples of fibrations which are of particular interest.

Notation 3.8. For $1 \leq i < j \leq 5$, we define $B_{ij} = B_{ji} = \sum_{\beta\in\mu_3} E_{ij}^\beta$.

■ Let $1 \leq i \leq 5$ and $j < r < s < t$ be such that $\{i,j,r,s,t\} = \{1,2,3,4,5\}$. We define $\ell_i = (1-\alpha)x_i \in \Lambda_A^*$.

Corollary 3.9. *The fibration $\gamma_{\ell_i} : S \to \mathbb{E}$ is stable, has connected fibres of genus* 10 *and its only singular fibres are given by:*

$$B_{jr}+B_{st},\ B_{js}+B_{rt},\ B_{jt}+B_{rs}$$

The 27 *intersection points of the curves E_{jr}^β and E_{st}^τ ($\beta,\gamma \in \mu_3$) constitute the set of critical points of this fibration.*

Proof. By Theorem 3.2, a fibre of γ_{ℓ_i} has genus $1+3|1-\alpha|^2 = 10$.
Let $\beta \in \mu_3$, $h,k \in \{j,r,s,t\}$, $h<k$. The form ℓ_i is zero on the space $\mathbb{C}(e_h - \beta e_k)$, hence E_{hk}^β is contracted to a point and is a component of the fibre of γ_{ℓ_i}.
The divisor $D_1 = B_{jr}+B_{st}$ is connected, satisfies $(B_{jr}+B_{st})^2 = 0$ and has

genus 10. Its irreducible components are contracted by γ_{ℓ_i}. Hence, it is a fibre and γ_{ℓ_i} has connected fibres. Likewise, the divisors $D_2 = B_{js} + B_{rt}$ and $D_3 = B_{jt} + B_{rs}$ are fibres of γ_{ℓ_i}.
The 27 lines inside the intersection of the Fermat cubic and the hyperplane $\{\ell_i = 0\}$ correspond to the 27 intersection points of the curves E^{β}_{hk} and E^{γ}_{lm} such that : $h < k$, $l < m$ and $\{i, h, k, l, m\} = \{1, 2, 3, 4, 5\}$. These 27 critical points lie in the fibres D_1, D_2, D_3. These 3 fibres are thus the only singular fibres of γ_{ℓ_i}.
The singularities of D_1, D_2 and D_3 are double ordinary and the surface possesses no rational curve, the fibration is thus stable. □

■ Let $(a_1, ..., a_5) \in \mu_3^5$ be such that $a_1...a_5 = 1$ and let

$$\ell = (1 - \alpha)(a_1x_1 + .. + a_5x_5) \in \Lambda_A^*.$$

Corollary 3.10. *The divisor*

$$D = \sum_{1 \le i < j \le 5} E^{a_i/a_j}_{ij}$$

is a singular fibre of the Stein factorization of γ_ℓ.

Proof. The connected divisor D satisfies $D^2 = 0$, has genus 16 and by Theorem 3.2, the irreducible components of D are contracted by γ_ℓ.
Let $w \in H^o(\Omega_S)^*$ be : $w = e_1 + .. + e_5$. We have

$$H^1(A, \mathbb{Z}) \cap \mathbb{C}w = \frac{\alpha^2}{1 - \alpha}\mathbb{Z}[3\alpha]w.$$

The morphism $x \to (a_1x_1 + .. + a_5x_5)w \in End(H^o(\Omega_S)^*)$ is an element of $\mathbb{Z}[G(3, 3, 5)]$. It is the differential of a morphism $\Gamma'_\ell : A \to \mathbb{E}'$ where $\mathbb{E}' = (\mathbb{C}/\frac{\alpha^2}{1-\alpha}\mathbb{Z}[3\alpha])w$.
The morphism Γ_ℓ has a factorization by Γ'_ℓ and a degree 3 isogeny between $\mathbb{E}'$ and $\mathbb{E}$. The divisor D is a connected fibre of the morphism $\vartheta \circ \Gamma'_\ell$, the Stein factorization of γ_ℓ. □

■ The curve $E^{\beta^2}_{ij}$ is the closed set of critical points of the fibration $\gamma_{(1-\alpha)(x_i+\beta x_j)}$. This fibration has only one singular fiber and this fiber is not reducted.

■ We can construct an infinite number of fibrations with 9 sections and which contract 9 elliptic curves. Let us take $a \in \mathbb{Z}[\alpha]$ and

$$\ell = x_1 - (1 + (1 - \alpha)a)x_2 \in \Lambda_A^*.$$

Corollary 3.11. *The 9 curves $E^{\beta}_{13}, E^{\beta}_{14}$ and E^{β}_{15} ($\beta \in \mu_3$) are sections of γ_ℓ. The curves $E^{\beta}_{34}, E^{\beta}_{35}$ and E^{β}_{45} ($\beta \in \mu_3$) are contracted.*

Proof. This follows from Theorem 3.2 and the fact that

$$|\ell(e_1 - \beta e_3)| = |\ell(e_1 - \beta e_4)| = |\ell(e_1 - \beta e_5)| = 1, \quad (\beta \in \mu_3)$$
$$|\ell(e_3 - \beta e_4)| = |\ell(e_3 - \beta e_5)| = |\ell(e_4 - \beta e_5)| = 0.$$

The fibration γ_ℓ has connected fibres. □

3.3. **The Néron-Severi group of the Fano surface of the Fermat cubic.**

Let S be the Fano surface of the Fermat cubic and $\mathrm{NS}(S)$ the Néron-Severi group of S.

Theorem 3.12. *The Néron-Severi group of S has rank $25 = \dim H^1(S, \Omega_S)$. The 30 elliptic curves generate an index 3 sub-lattice of $\mathrm{NS}(S)$.*
The group $NS(S)$ is generated by these 30 curves and the class of an incidence divisor C_s ($s \in S$), it has discriminant 3^{18}.
The relations between the 30 elliptic curves in $\mathrm{NS}(S)$ are generated by the relations :

$$B_{jr} + B_{st} = B_{js} + B_{rt} = B_{jt} + B_{rs},$$

for indices such that $1 \leq j < r < s < t \leq 5$.

The Corollary 3.9 gives a geometric interpretation of the numerical equivalence relations of this Theorem.

Proof. By Theorem 3.1, we know the intersection matrix $\mathcal{I}$ of the 30 elliptic curves. As we can verify, the matrix $\mathcal{I}$ has rank 25. The intersection matrix of the 25 elliptic curves different to the 5 curves

$$E_{13}^{\alpha}, E_{15}^{\alpha}, E_{24}^{\alpha}, E_{34}^{\alpha}, E_{45}^{\alpha}$$

has determinant equal to 3^{20}. Moreover, this 25 curves form a $\mathbb{Z}$-basis of the lattice generated by the 30 elliptic curves.
By Theorem 3.7, the image of the morphism

$$\mathrm{NS}(A) \xrightarrow{\vartheta^*} \mathrm{NS}(S)$$

is a lattice of discriminant $2^2 3^{18}$ generated by the class of $C_s - E_{ij}^{\beta}$ and by $\sum_{1 \leq i < j \leq 5} E_{ij}^1$. Theorem 1.17 implies that $\mathrm{NS}(S)$ is generated by these classes and the class of an incidence divisor C_s. This lattice is also generated by the classes of the 30 elliptic curves and C_s. $\square$

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