ON LINEAR FORMS WITH COEFFICIENTS IN $\mathbb{N}\zeta(1+\mathbb{N})$

The detailed version, **Part 2**

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In memory of Professor Alexander Buchstab

Content

§ 0. Foreword

This is the second part of the detailed version of my work "On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$" (see [9] for the first part). I continue the proof of the following theorem.

Theorem. *Let*

$$\phi_i(x_1,x_2)=\sum_{k=1}^{2}(2ik-i-k+2)(\zeta(i+k+1))x_k, \quad i=1,2,$$

for any $d\in\mathbb{R}$, let $\|d\|$ be the distance between d and $\mathbb{Z}$; let $\gamma=43,464412$. There is a positive constant c such that

$$\|\phi_1(x_1,x_2)\|+\|\phi_2(x_1,x_2)\|\geq c(|x_1|+|x_2|)^{-\gamma},$$

where $x_1\in\mathbb{Z}, x_2\in\mathbb{Z}, |x_1|+|x_2|>0$.

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§ 1. Some relations for general Mejer's functions

In this section, I continue to study the general functions of C.S.Mejer, which are defined as follows ([10], ch. 5). Let

$$G_{p,q}^{(m,n)}\left(z\middle|\begin{matrix}a_1, & \dots & a_p\\ a_1, & \dots & a_p\end{matrix}\right)=\frac{1}{2\pi i}\int\limits_L g(s)ds, \quad \text{where} \tag{2.1.1}$$

(2.1.2)

$$g=g(s)=g_{p,q}^{(m,n)}\left(z\middle|\begin{matrix}a_1, & \dots & a_p\\ a_1, & \dots & a_p\end{matrix}\right)(s)=\frac{z^s\prod\limits_{j=1}^{m}\Gamma(b_j-s)\prod\limits_{j=1}^{n}\Gamma(1-a_j+s)}{\prod\limits_{j=1+m}^{q}\Gamma(1-b_j+s)\prod\limits_{j=n+1}^{p}\Gamma(a_j-s)},$$

an empty product is, by definition, equal to 1, $0\le m\le q$, $0\le n\le p$; the parameters $a_j\in\mathbb{C}$, $j=1,\dots,p$, and $b_k\in\mathbb{C}$, $k=1,\dots,q$, are chosen in such a way that, if $P_r(g)$ denotes the set of all poles of the functions $\Gamma(b_k-s)$, $k=1,\dots,m$, and $P_l(g)$ denotes the set of all poles of the functions $\Gamma(1-a_j+s)$, $j=1,\dots,n$, then $P_l(g)\cap P_r(g)=\oslash$. There are 3 possibilities to choose the curve L.

(A) First, the curve $L=L_0$ may be chosen to pass from $-i\infty$ to $+i\infty$ in such a way that the set $P_r(g)$ lies to the right of it and the set $P_l(g)$ lies to the left of it. The integral (2.1.1) is convergent in either of the following two cases:

$(A1)$ $|\arg(z)|<(m+n-p/2-q/2)\pi$;

$(A2)$ $|\arg(z)|\le(m+n-p/2-q/2)\pi$ and $(p-q)/2+\mathrm{Re}\,(\Delta^*(g))<-1$, where

$$\Delta^*(g)=\sum_{k=1}^{q}b_k-\sum_{j=1}^{p}a_j. \tag{2.1.3}$$

(B) Second, the curve $L=L_1$ may be chosen to pass from $+i\infty$ to $+i\infty$, in such a way that the set $P_r(g)$ lies to the right of it and the set $P_l(g)$ lies to the left of it. The integral (2.1.1) is convergent in each of the following three cases:

$(B1)$ $p<q$;

$(B2)$ $1\le p\le q$ and $|z|<1$;

$(B3)$ $1\le p\le q$, $|z|\le 1$ and $\mathrm{Re}\,(\Delta^*(g))<-1$.

(C) Third, the curve $L=L_2$ may be chosen to pass from $-\infty$ to $-\infty$, in such a way that the set $P_r(g)$ lies to the right of it and the set $P_l(g)$ lies to the left of it. The integral (2.1.1) is convergent in each of the following 3 cases:

$(C1)$ $q<p$;

$(C2)$ $1\le q\le p$ and $|z|>1$;

$(C3)$ $1\le q\le p$, $|z|\ge 1$ and $\mathrm{Re}\,(\Delta^*(g))<-1$.

If both conditions (A) and (B) are satisfied, then each of these conditions leads to the same result. If both conditions (A) and (C) are satisfied, then each of these conditions leads to the same result.

Let G denote the integral (2.1.1) with $L = L_k$, where $k = 1, 2$, and let S_k be the set of all the unremovable singularities of g encircled by L_k. If for $k = 1$ one of conditions (B) is satisfied or for $k = 2$ one of conditions (C) is satisfied, then

$$G = (-1)^k \sum_{s \in S_k} \operatorname{Res}(g; s), \tag{2.1.4}$$

All these assertions has been proved in [9].

Remark 2.1. In each of the three cases (A), (B) and (C) the curve of integration L is chosen in such a way that the set $P_r(g)$ lies to the right of it and the set $P_l(g)$ lies to the left of it.

One may write $g = g_5 g_6$, where

$$g_5(s) = z^s \prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s), \tag{2.1.5}$$

$$g_6(s) = \frac{1}{\prod\limits_{j=1+m}^{q} \Gamma(1 - b_j + s) \prod\limits_{j=n+1}^{p} \Gamma(a_j - s)}. \tag{2.1.6}$$

Remark 2.2. In view of the K.Weierstrass representation of the gamma-function [12], ch. 12:

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left\{ (1 + \frac{z}{n}) e^{-\frac{z}{n}} \right\},$$

if $s = a$ is the pole of order r of the function $g_5(s)$, then $s = a$ is either a removable singularity of the function $g(s)$ or a pole of order not bigger than r.

Let the operator $\nabla_{b;k}$, with $k = 1, \dots, q$ (respectively $\nabla_{a;j}$, with $j = 1, \dots, p$) when acting on the function g replaces parameter b_k by $b_k + 1$ (respectively a_j by $a_j + 1$), and let δ denote the operator $z\frac{\partial}{\partial z}$.

It is clear that

$$P_r(\nabla_{b;k} g) \subset P_r(g) \tag{2.1.7}$$

for $k = 1, \dots, q$, and

$$P_l((\nabla_{a;j})^{-1} g) \subset P_l(g) \tag{2.1.8}$$

for $j = 1, \dots, p$.

In view (2.1.7) and (2.1.8),

$$\nabla_{b;k} G = \frac{1}{2\pi i} \int_L (\nabla_{b;k} g)(s) ds, \tag{2.1.9}$$

where $k = 1, \dots, q$, and

$$(\nabla_{a;j})^{-1} G = \frac{1}{2\pi i} \int_L ((\nabla_{a;j})^{-1} g)(s) ds, \tag{2.1.10}$$

where $j = 1, \dots, p$, assuming, of course, that one of the conditions (A), (B), (C) of the absolute convergence of the integral (2.1.9) (respectively (2.1.10)) is satisfied; here the contour of integration L is taken to be the same as in (2.1.1).

Lemma 2.1.1. *Let, as before, $g = g(s) = g(z, s)$ be the integrand of the integral (2.1.1). Let, further, $\varepsilon_1(k) = -1$ if $k = 1, \dots, m$, and $\varepsilon_1(k) = 1$ if $k = m+1, \dots, q$. Let finally $\varepsilon_2(j) = 1$ if $j = 1, \dots, n$, and $\varepsilon_2(j) = -1$, if $j = n+1, \dots, p$. If one of the conditions of the absolute convergence (A), (B), (C) is satisfied for either of the integrals (2.1.9) and (2.1.10), then the corresponding condition of the absolute convergence (A), (B), (C) is satisfied also for the integral (2.1.1). Moreover, if one of the conditions (B1), (B2), (C1), (C2) is satisfied for the integral (2.1.1), then the corresponding condition of the absolute convergence is satisfied for either of the integrals (2.1.9) and (2.1.10),*

$$\varepsilon_1(k) \nabla_{b;k} G = (\delta - b_k) G, \tag{2.1.11}$$

where $k = 1, \dots, q$, and

$$\varepsilon_2(j) (\nabla_{a;j})^{-1} G = (\delta + 1 - a_j) G, \tag{2.1.12}$$

where $j = 1, \dots, p$.

Proof. If one of the conditions (B1), (B2), (C1), (C2) is satisfied, then the assertion concerning the absolute convergence is obvious because in such a case it does not dependent on the values of the parameters. It is clear that

$$\Delta^*(\nabla_{b;k} g) = \Delta^*(g) + 1, \tag{2.1.13}$$

where $k = 1, \dots, q$, and

$$\Delta^*((\nabla_{a;j})^{-1} g) = \Delta^*(g) + 1, \tag{2.1.14}$$

where $j = 1, \dots, p$. Therefore if one of the conditions of the absolute convergence (A), (B), (C) is satisfied for either of the integrals (2.1.9) and (2.1.10), then the corresponding condition of the absolute convergence (A), (B), (C) is satisfied also for the integral (2.1.1). It follows from the following equalities

$$-\Gamma(1+b_k-s) = (s-b_k)\Gamma(b_k-s), \quad k = 1, \dots, m,$$

$$\frac{1}{\Gamma(-b_k+s)} = \frac{(s-b_k)}{\Gamma(1-b_k+s)}, \quad k = m+1, \dots, q,$$

$$\Gamma(s+1-(a_j-1)) = (s+1-a_j)\Gamma(s+1-a_j), \quad j = 1, \dots, n,$$

$$-\frac{1}{\Gamma(a_j-1-s)} = \frac{s+1-a_j}{\Gamma(a_j-s)}, \quad j = n+1, \dots, p,$$

that

$$\varepsilon_1(k) \bigtriangledown_{b;k} g + b_k g = sg = \delta g \tag{2.1.15}$$

where $k = 1, \dots, q$,

$$\varepsilon_2(j)(\bigtriangledown_{a;j})^{-1} g + (1-a_j)g = sg = \delta g, \tag{2.1.16}$$

where $j = 1, \dots, p$.

Suppose that one of the conditions (A), (B), (C) is satisfied for the integral (2.1.1) and that the contour of integration L is chosen in accordance with the description given in [9]. Then, in view of the relations (2.1.15) and (2.1.16), it follows that

$$\varepsilon_1(k) \bigtriangledown_{b;k} G + b_k G = \frac{1}{2\pi i}\int\limits_L sg(z,s)ds, \tag{2.1.17}$$

where $k = 1, \dots, q$, and

$$\varepsilon_2(j)(\bigtriangledown_{a;j})^{-1} G + (1-a_j)G = \frac{1}{2\pi i}\int\limits_L sg(z,s)ds, \tag{2.1.18}$$

where $j = 1, \dots, p$.

Suppose now that one of the conditions (B1), (C1) is satisfied and that the contour of integration L is chosen in accordance with the description given in [9], as before. Then, according to the relations (1.19) and (1.20) in [9], on the rays $|\tau| = \tau_0$ the equality

$$sg(z,s) = O(1)|\sigma|^{-2} \tag{2.1.19}$$

holds uniformly with respect to z. Therefore the integrals in the right hand side of the equalities (2.1.13), (2.1.14) are uniformly convergent with respect to z and, moreover,

$$\frac{1}{2\pi i}\int\limits_{L} sg(z,s)ds = \delta G. \tag{2.1.20}$$

Thus if one of the conditions (B1), (C1) is satisfied, then the lemma is proved.

Suppose that one of the conditions (B2), (C2) is satisfied and the contour of integration L is chosen in accordance with description given in [9]. Then $0 < ||z|-1|$. Let $\delta = \frac{1}{2}\min(1, ||z|-1|)$. Then, for w in the circle $|w-z| < \delta$, we have the inequalities

$$|z| - \delta < |w| < |z| + \delta. \tag{2.1.21}$$

If $|z| > 1$, then the left of the inequalities (2.1.21) implies the inequalities

$$|w| > 1 + (|z|-1) - \delta > 1 + \delta, |\log(|w|)| > \log(1+\delta). \tag{2.1.22}$$

If $|z| < 1$, then the right of the inequalities (2.1.21) implies the inequalities

$$|w| < 1 - (1-|z|) + \delta < 1 - \delta, |\log(|w|)| > -\log(1-\delta) > \log(1+\delta). \tag{2.1.23}$$

Consequently, according to the relation (1.21) in [9], on the rays $|\tau| = \tau_0$ the equality

$$sg(w,s) = O(1)|\sigma|^{O(1)}e^{-|\sigma|\log(1+\delta)} \tag{2.1.24}$$

holds uniformly with respect to w in the circle $|w-z| < \delta$. Therefore the integral

$$\frac{1}{2\pi i}\int\limits_{L} sg(w,s)ds$$

is uniformly convergent in the circle $|w-z| < \delta$ and it follows that

$$\frac{1}{2\pi i}\int\limits_{L} sg(w,s)ds = w\frac{\partial}{\partial w}G(w)$$

In particular, if we take $w = z$, then it follows that the equality (2.1.20) holds again. The lemma is proved.

Corollary. *If one of conditions, (B1), (B2), (C1), (C2) is satisfied, then the function G given by the integral (2.1.1) is regular in the corresponding domain and, for each $d \in \mathbb{N}$, the equalities*

$$(\nabla_{b;k})^d G = (\varepsilon_1(k))^d \Big(\prod_{\kappa=0}^{d-1}(\delta - b_k - \kappa)\Big) G, \tag{2.1.25}$$

where $k = 1, \dots, q$, and

$$(\nabla_{a;j})^{-d} G = (\varepsilon_2(j))^d \Big(\prod_{\kappa=0}^{d-1}(\delta - a_j + 1 + \kappa)\Big) G, \tag{2.1.26}$$

where $j = 1, \dots, p$, hold true.

Proof. To prove (2.1.25) and (2.1.26) we use induction on d. For $d = 1$ the equality (2.1.25) (respectively (2.1.26)) coincides with (2.1.11) (respectively (2.1.12)).

Suppose (2.1.25) holds for some $d \in \mathbb{N}$, and let $G^* = (\nabla_{b;k})^d G$. Further, let $b^*_\kappa, \kappa = 1, \dots, q$, and $a^*_j,\ j = 1, \dots, p$, be the parameters of G^*. Then

$$b^*_\kappa = b_\kappa,\ \kappa \in \{1, \dots, q\}\backslash\{k\}, \quad \text{and} \quad b^*_k = b_k + d,\ a^*_j = a_j,\ j = 1, \dots, p.$$

Therefore it follows from lemma 2.1.1 that

$$(\nabla_{b;k})^{d+1} G = \nabla_{b;k} G^* = \varepsilon_1(k)(\delta - b^*_k) G^* =$$

$$= \varepsilon_1(k)(\delta - b^*_k - d)(\varepsilon_1(k))^d \Big(\prod_{\kappa=0}^{d-1}(\delta - b_k - \kappa)\Big) G = (\varepsilon_1(k))^{d+1}\Big(\prod_{\kappa=0}^{d}(\delta - b_k - \kappa)\Big) G.$$

Thus (2.1.25) holds for every $d \in \mathbb{N}$.

Suppose (2.1.26) holds for some $d \in \mathbb{N}$, and let $\tilde{G} = (\nabla_{a;j})^{-d} G$. Further, let $\tilde{b}_k,\ k = 1, \dots, q$, and $\tilde{a}_\kappa,\ \kappa = 1, \dots, p$, be the parameters of $\tilde{G}$. Then

$$\tilde{b}_k = b_k,\ k = 1, \dots, q, \quad \tilde{a}_j = a_j - d, \quad \text{and} \quad \tilde{a}_\kappa = a_\kappa,\ \kappa \in \{1, \dots, p\}\backslash\{j\}.$$

Therefore it follows from lemma 2.1.1 that

$$(\nabla_{a;j})^{-d-1} G = (\nabla_{a;j})^{-1} \tilde{G} = \varepsilon_2(j)(\delta + 1 - \tilde{a}_j)\tilde{G} =$$

$$= \varepsilon_2(j)(\delta+1-a_j+d)(\varepsilon_2(j))^d\Big(\prod_{\kappa=0}^{d-1}(\delta+1-a_j+\kappa)\Big) G = (\varepsilon_2(j))^{d+1}\Big(\prod_{\kappa=0}^{d}(\delta+1-a_j+\kappa)\Big) G.$$

Thus (2.1.26) holds for every $d \in \mathbb{N}$. The corollary is proved.

Let $\bigtriangledown$ denote the operator replacing each of the parameters b_k, with $k = 1, \dots, q$, by $b_k + 1$ and each of the parameters a_j, with $j = 1, \dots, p$, by $a_j + 1$. As in [8], for each $A \subset \mathbb{C}$ and each $c \in \mathbb{C}$, the set $\{c + a : a \in A\}$ is denoted by $A + c$. It is clear that

$$P_r(g) \neq P_r(\bigtriangledown g) = P_r(g) + 1 \subset P_r(g) \tag{2.1.27}$$

as in (2.1.7). But

$$P_l(g) \subset P_l(g) + 1 = P_l(\bigtriangledown g) \neq P_l(g) \tag{2.1.28.}$$

The first inequality in (2.1.27) holds because the set $P_r(g)$ includes some (possibly not unique) number β with the smallest real part; then $\beta \notin P_l(g) + 1$. The last inequality in (2.1.28) holds because the set $P_r(g) + 1$ includes some (possibly not unique) number α with the biggest real part; then $\alpha \notin P_l(g)$. Nevertheless

$$(P_l(g) + 1) \bigcap (P_r(g) + 1) = P_l(g) \bigcap P_r(g) = \oslash. \tag{2.1.29}$$

Therefore for the same contour of integration L, which is used in the integral (2.1.1), we let

$$(\bigtriangledown G)(z) = \frac{1}{2\pi i} \int\limits_{L+1} (\bigtriangledown g)(z, s) ds, \tag{2.1.30}$$

assuming, of course, that one of the conditions (A), (B), (C) of the absolute convergence of the integral (2.1.30) is satisfied.

Lemma 2.1.2. *If, for the integral (2.1.1), one of the conditions (B1), (B2), (C1), (C2) is satisfied, then the corresponding condition of the absolute convergence of the integral (2.1.30) is also satisfied, and*

$$(\bigtriangledown G)(z) = zG(z). \tag{2.1.31}$$

Proof. If one of the conditions (B1), (B2), (C1), (C2) is satisfied, then the assertion concerning the absolute convergence is obvious, because it is independent from the values of parameters in such a case. When s ranges over the points of the curve $L + 1$, then $\tilde{s} = s - 1$ ranges over the points of the curve L. Now, the assertion of the lemma follows immediately from the equalities

$$(\bigtriangledown g)(z, s) = zg(z, s - 1) = zg(z, \tilde{s}),$$

$$(\bigtriangledown G)(z) = \frac{1}{2\pi i}\int\limits_{L+1}(\bigtriangledown g)(z,s)ds = \frac{1}{2\pi i}\int\limits_{L+1} zg(z,s-1)ds = \frac{1}{2\pi i}\int\limits_{L} zg(z,\tilde{s})d\tilde{s} = zG(z).$$

The lemma is proved.

Remark 2.3. In view of (2.1.28), if not only $P_l(g)$ but also $P_l(g)+1$ lies to the left of the contour of integration L and one of the conditions (B1), (B2), (C1), (C2) is satisfied, then

$$(\bigtriangledown G)(z) = \frac{1}{2\pi i}\int\limits_{L}(\bigtriangledown g)(z,s)ds. \tag{2.1.32}$$

This remark is important since the location of the curve L must be taken into account. We shall make use of this remark when we apply both Lemma 2.1.1 and Lemma 2.1.2 simultaneously.

Lemma 2.1.3. *If the conditions mentioned in Remark 2.3 are satisfied, then*

$$(-1)^{m+p-n}z((\prod_{j=1}^{p}(\delta+1-a_j))G)(z) = ((\prod_{k=1}^{q}(\delta-b_k))G)(z). \tag{2.1.33}$$

Proof. It follows from the relations (2.1.31) and (2.1.32) that

$$zG(z) = (\bigtriangledown G)(z) = (\prod_{k=1}^{q}\bigtriangledown_{b;k}\prod_{j=1}^{p}\bigtriangledown_{a;j}G)(z),$$

therefore

$$z(\prod_{j=1}^{p}(\bigtriangledown_{a;j})^{-1}G)(z) = (\prod_{k=1}^{q}\bigtriangledown_{b;k}G)(z).$$

Applying now Lemma 2.1.1 we obtain the equality

$$(\prod_{k=1}^{q}\varepsilon_1(k)\prod_{j=1}^{p}\varepsilon_2(j)z(\prod_{j=1}^{p}(\delta+1-a_j))G)(z) = (\prod_{k=1}^{q}(\delta-b_k)G)(z).$$

Since $\prod\limits_{k=1}^{q}\varepsilon_1(k)\prod\limits_{j=1}^{p}\varepsilon_2(j) = (-1)^{m+p-n}$, the equality (2.1.33) and the lemma are proved.

In what follows, we identify the Riemann surface $\mathfrak{F}$ of the function $Log(z)$, with the direct product of the multiplicative group $\mathbb{R}_+^* = \{r\in\mathbb{R}|\ r>0\}$ of positive real numbers with the operation $\times$, not to be written down explicitly as usual, and the additive group $\mathbb{R}$ of all the real numbers, so that

$$z_1 z_2 = (r_1 r_2, \phi_1 + \phi_2)$$

for any two points $z_1 = (r_1, \phi_1)$ and $z_2 = (r_2, \phi_2)$ on this Riemann surface. Let $\theta_0(z) = re^{i\phi}$ for every $z = (r, \phi) \in \mathbb{R}_+^* \times \mathbb{R}$. It is clear that θ_0 is a surjective homomorphism of $\mathbb{R}_+^* \times \mathbb{R}$ on the multiplicative group of the field $\mathbb{C}$. Let us fix a domain D in $\mathfrak{F}$. For a complex-valued function $f(z)$ on D, let $f(z) = f^\wedge(r, \phi)$. It is well known that $f(z)$ is holomorphic in D if the complex-valued function $f^\wedge(r, \phi)$ of two real variables r and ϕ has continuous partial derivatives in D and, for every point $z = (r, \phi) \in D$, the Cauchy-Riemann conditions

$$e^{-i\phi}(\frac{\partial}{\partial r} f^\wedge)(r, \phi) = e^{-i\phi}\frac{1}{ir}(\frac{\partial}{\partial \phi} f^\wedge)(r, \phi) = (\frac{\partial}{\partial z} f)(z)$$

are satisfied. Let us introduce an operator δ on the vector space of all the holomorphic in D functions, defined as follows:

$$(\delta f)(z) = \theta_0(z)(\frac{\partial}{\partial z} f)(z)$$

for any $f(z)$ in this vector space.

Lemma 2.1.4. *Let H stand for either $\mathbb{C}$ or $\mathbb{R}_+^* \times \mathbb{R}$, let D be a domain in H, and let $f(z)$ be a function holomorphic in D. Given any $a \in H$, satisfying the condition $a \neq 0$ if $H = \mathbb{C}$, let $\eta_a(z) = az$ for $z \in H$. Then*

$$(\delta(f \circ \eta_a))(z) = ((\delta f) \circ \eta_a)(z)$$

for $z \in a^{-1}D$ and the function $\delta(f \circ \eta_a)$ (and therefore $(\delta f) \circ \eta_a$) is holomorphic in the domain $a^{-1}D$.

Proof. The assertion of the lemma is obvious if $H = \mathbb{C}$, because in this case if $z \in a^{-1}D$, then $az \in D$ and according to the rules of differentiation it follows that

$$(\delta(f \circ \eta_a)(z) = z(\frac{\partial}{\partial z}(f \circ \eta_a)(z) = z((\frac{\partial}{\partial z} f)(az))(\frac{\partial}{\partial z}\eta_a)(z) =$$

$$z((\frac{\partial}{\partial z} f)(az))a = (\delta f)(az) = ((\delta f) \circ \eta_a)(z).$$

Let $H = \mathbb{R}_+^* \times \mathbb{R}$, and let f(z) be a function holomorphic in a domain $D \subset \mathbb{R}_+^* \times \mathbb{R}$. Then $f(z) = f^\wedge(r, \phi)$, where the complex-valued function $f^\wedge(r, \phi)$ of two real variables r and ϕ has continuous partial derivatives in D and, for any point $z = (r, \phi) \in D$, the Cauchy-Riemann conditions

$$e^{-i\phi}(\frac{\partial}{\partial r}f^\wedge)(r,\phi) = e^{-i\phi}\frac{1}{ir}(\frac{\partial}{\partial \phi}f^\wedge)(r,\phi) = (\frac{\partial}{\partial z}f)(z), \tag{2.1.34}$$

hold; moreover,

$$(\delta f)(z) = \theta_0(z)(\frac{\partial}{\partial z}f)(z) \tag{2.1.35}$$

Let $a = (r_0, \phi_0)$, then

$$((f \circ \theta_a)^\wedge(r,\phi) = (f)^\wedge(rr_0, \phi + \phi_0)$$

for each point $z = (r, \phi) \in a^{-1}D = \{a^{-1}z |\ z \in D\}$. Since $az \in D$, it follows from (2.1.34) that

$$e^{-i(\phi_0+\phi)}(\frac{\partial}{\partial r}f^\wedge)(r_0 r, \phi_0 + \phi) = e^{-i(\phi_0+\phi)}\frac{1}{ir_0 r}(\frac{\partial}{\partial \phi}f^\wedge)(r_0 r, \phi_0 + \phi) = (\frac{\partial}{\partial z}f)(az),$$

and therefore

$$e^{-i\phi}(\frac{\partial}{\partial r}(f \circ \eta_a)^\wedge)(r,\phi) = r_0 e^{-i\phi}(\frac{\partial}{\partial r}f^\wedge)(rr_0, \phi_0 + \phi) =$$

$$e^{-i\phi}\frac{1}{ir}(\frac{\partial}{\partial \phi}f^\wedge)'(r_0 r, \phi + \pi) = r_0 e^{i\phi_0}(\frac{\partial}{\partial z}f)(az)$$

for each point $z = (r, \phi) \in a^{-1}D$. Consequently, the function $(f \circ \eta_a)(z)$ is holomorphic in the domain $a^{-1}D$ and

$$(\frac{\partial}{\partial z}(f \circ \eta_a))(z) = \theta_0(a)(\frac{\partial}{\partial z}f)(az),$$

for any point $z = (r, \phi) \in a^{-1}D$. It follows now that

$$(\delta(f \circ \eta_a))(z) = \theta_0(z)\frac{\partial}{\partial z}(f \circ \eta_a)(z) = \theta_0(z)\theta_0(a)(\frac{\partial}{\partial z}f)(az) = \theta_0(az)(\frac{\partial}{\partial z}f)(az) =$$

$$= \theta_0(az)(\frac{\partial}{\partial z}f)(az) = (\delta f)(az) = ((\delta f) \circ \eta_a)(z)$$

for any point $z = (r, \phi) \in a^{-1}D$. The lemma is proved.

Corollary 1. *Let Ω be a domain in H, and let $a \in H$ satisfy the condition $a \neq 0$ if $H = \mathbb{C}$. If the function $f(z)$ is holomorphic in the domain $\eta_a\Omega = \{az : z \in \Omega\}$, then*

$$\delta(f \circ \eta_a))(z) = ((\delta f) \circ \eta_a)(z)$$

and the function $\delta(f \circ \eta_a)$ (and therefore $(\delta f) \circ \eta_a$) is holomorphic in Ω.

Proof. The assertion follows from Lemma 2.1.4 with $D = \eta_a\Omega$.

In [8], the half-open interval $(\frac{-3\pi}{2}, \frac{\pi}{2}]$ has been chosen as the domain of the main values $arg(z)$ of $Arg(z)$; therefore, for each $z = (r, \phi) \in \mathbb{R}_+^* \times \mathbb{R}$, we denote $(r, \phi - \pi)$ by $-z$. (Let us remark that $-(-z) \neq z :$ $\mathbb{R}_+^* \times \mathbb{R}$ is just a multiplicative group and has no additive structure).

Corollary 2. *Let Ω be a domain in H. If the function $f(z)$ is holomorphic in the domain $-\Omega = \{-z : z \in \Omega\}$, then*

$$(\delta(f \circ \theta_1))(z) = ((\delta f) \circ \theta_1)(z) = (\delta f)(-z)$$

and the function $\delta(f \circ \theta_1)$ (and therefore $(\delta f) \circ \theta_1$) is holomorphic in Ω.

Proof. The assertion follows from Corollary 1 on taking $a = (1, -\pi)$.

It has been proved in the first part of this work, [9], that the function $G(z)$ is holomorphic in the domain $\Omega = \mathbb{R}_+^* \times \mathbb{R}$ if one of the conditions (B1) or (C1) is satisfied, that it is holomorphic in the domain $\Omega = \{z = (r, \phi) : (r, \phi) \in \mathbb{R}_+^* \times \mathbb{R},\ r < 1\}$ if condition (B2) is satisfied, and that it is holomorphic in the domain $\Omega = \{z = (r, \phi) : (r, \phi) \in \mathbb{R}_+^* \times \mathbb{R},\ r > 1\}$ if condition (C2) is satisfied. In each of these cases $\eta_a\Omega = \Omega$ for any $a \in SO(2) = \{a = (r_0, \phi_0) : r_0 = 1\}$.

Lemma 2.1.5. *Let $d \in \mathbb{N}$, $a \in SO(2) = \{a = (r_0, \phi_0) : r_0 = 1\}$ and suppose that one of the conditions (B1), (C1), (B2), (C2) is satisfied. Then*

$$(\nabla_{b;k})^d (G \circ \eta_a) = (\varepsilon_1(k))^d (\prod_{\kappa=0}^{d-1} (\delta - b_k - \kappa))(G \circ \eta_a), \tag{2.1.36}$$

where $k = 1, \dots, q$,

$$(\nabla_{a;j})^{-d} (G \circ \eta_a) = (\varepsilon_2(j))^d (\prod_{\kappa=0}^{d-1} (\delta - a_j + 1 + \kappa))(G \circ \eta_a), \tag{2.1.37}$$

where $j = 1, \dots, p$, and

$$(\nabla(G \circ \eta_a))(z) = \theta_0(a)\theta_0(z)(G \circ \eta_a)(z). \tag{2.1.38}$$

If the conditions of Remark 2.3 are satisfied, then

$$(-1)^{m+p-n}\theta_0(a)\theta_0(z)((\prod_{j=1}^{p} (\delta + 1 - a_j))(G \circ \eta_a))(z) = ((\prod_{k=1}^{q} (\delta - b_k))(G \circ \eta_a))(z). \tag{2.1.39}$$

Proof. It follows from the corollary to Lemma 2.1.1 that

$$((\bigtriangledown_{b;k})^d G)\circ\eta_a = (\varepsilon_1(k))^d((\prod_{\kappa=0}^{d-1}(\delta - b_k - \kappa))G)\circ\eta_a, \tag{2.1.40}$$

where $k = 1, \dots, q$, and that

$$((\bigtriangledown_{a;j})^{-d} G)\circ\eta_a = (\varepsilon_2(j))^d((\prod_{\kappa=0}^{d-1}(\delta - a_j + 1 + \kappa))G)\circ\eta_a, \tag{2.1.41}$$

where $j = 1, \dots, p$. It is clear that

$$(\bigtriangledown_{b;k})^d(G\circ\eta_a) = ((\bigtriangledown_{b;k})^d G)\circ\eta_a, \tag{2.1.42}$$

where $k = 1, \dots, q$, and that

$$(\bigtriangledown_{a;j})^{-d}(G\circ\eta_a) = ((\bigtriangledown_{a;j})^{-d} G)\circ\eta_a, \tag{2.1.43}$$

where $j = 1, \dots, p$. In view of Corollary 1 to Lemma 2.1.4,

$$((\prod_{\kappa=0}^{d-1}(\delta - b_k - \kappa))G)\circ\eta_a = (\prod_{\kappa=0}^{d-1}(\delta - b_k - \kappa))(G\circ\eta_a), \tag{2.1.44}$$

where $k = 1, \dots, q$, and

$$((\prod_{\kappa=0}^{d-1}(\delta - a_j + 1 + \kappa))G)\circ\eta_a = (\prod_{\kappa=0}^{d-1}(\delta - a_j + 1 + \kappa))(G\circ\eta_a), \tag{2.1.45}$$

where $j = 1, \dots, p$. Equality (2.1.36) follows from the equalities (2.1.40), (2.1.42), and (2.1.44); equality (2.1.37) follows from the equalities (2.1.41), (2.1.43) and (2.1.45). Clearly,

$$(\bigtriangledown(G\circ\eta_a))(z) = ((\bigtriangledown G)\circ\eta_a)(z) \tag{2.1.46}$$

for any $z \in \Omega$. It follows from Lemma 2.1.2 that

$$((\bigtriangledown G)\circ\eta_a)(z) = \theta_0(a)\theta_0(z)(G\circ\eta_a)(z) \tag{2.1.47}$$

for any $z \in \Omega$. Relation (2.1.38) follows from (2.1.46) and (2.1.47). Furthermore, if the conditions of Remark 2.3 are satisfied, then it follows from Lemma 2.1.3 that

$$(-1)^{m+p-n}\theta_0(a)\theta_0(z)((\prod_{j=1}^{p}(\delta+1-a_j)G)\circ\eta_a)(z) =$$

(2.1.48)

$$(-1)^{m+p-n}\theta_0(a)\theta_0(z)(((\prod_{j=1}^{p}(\delta+1-a_j))G)\circ\eta_a)(z) = (((\prod_{k=1}^{q}(\delta-b_k))G)\circ\eta_a)(z)$$

for any $z \in \Omega$. It follows from Corollary 1 to Lemma 2.1.4 that

$$(((\prod_{j=1}^{p}(\delta+1-a_j))G)\circ\eta_a)(z) = ((\prod_{j=1}^{p}(\delta+1-a_j))(G\circ\eta_a))(z), \tag{2.1.49}$$

for any $z \in \Omega$ and

$$(((\prod_{k=1}^{q}(\delta-b_k))G)\circ\eta_a)(z) = ((\prod_{k=1}^{q}(\delta-b_k))(G\circ\eta_a))(z) \tag{2.1.50}$$

for any $z \in \Omega$. Relation (2.1.39) follows from (2.1.48), (2.1.49) and (2.1.50). The lemma is proved.

Corollary. *Let $d \in \mathbb{N}$ and suppose that one of the conditions (B1), (C1), (B2), and (C2) is satisfied. Then*

$$(\triangledown_{b;k})^d(G\circ\theta_1) = (\varepsilon_1(k))^d(\prod_{\kappa=0}^{d-1}(\delta-b_k-\kappa))(G\circ\theta_1), \tag{2.1.51}$$

where $k = 1, \dots, q$,

$$(\triangledown_{a;j})^{-d}(G\circ\theta_1) = (\varepsilon_2(j))^d(\prod_{\kappa=0}^{d-1}(\delta-a_j+1+\kappa))(G\circ\theta_1), \tag{2.1.52}$$

where $j = 1, \dots, p$, and

$$(\triangledown(G\circ\theta_1))(z) = -\theta_0(z)(G\circ\theta_1)(z). \tag{2.1.53}$$

Moreover, if the conditions of Remark 2.3 are satisfied, then

$$(-1)^{m+p-n+1}\theta_0(z)((\prod_{j=1}^{p}(\delta+1-a_j))(G\circ\theta_1))(z) = \tag{2.1.54}$$

$$((\prod_{k=1}^{q}(\delta-b_k))(G\circ\theta_1))(z).$$

Proof. To establish the assertions of the corollary it suffices to let $a = (1, -\pi)$ in Lemma 2.1.5.

§ 2. Some relations for my auxiliary functions

Let Ω^* stand for the set

$$\{z \in \mathbb{C}\colon\ |z| \geq 1,\ -3\pi/2 < \arg(z) \leq \pi/2\}.$$

Let further $\Delta \in \mathbb{N}$, $1 < \Delta$, $\delta_0 = 1/\Delta$, $\gamma_1 = (1-\delta_0)/(1+\delta_0)$, and $d_l = \Delta + (-1)^l$, $\quad l = 1, 2$. Let finally

$$\Omega^{(h)} = \{z \in \mathbb{C}\colon\ |z| \leq 1, -3\pi/2 < \phi = \arg(z) \leq \pi/2\}.$$

Clearly, $\Omega^{(h)}$ contains the following domain

$$\Omega = \Omega_1^{(h)} = \{z \in \mathbb{C}\colon\ |z| < 1,\ -3\pi/2 < \phi = \arg(z) < \pi/2\}.$$

My first auxiliary function $f_1(z,\nu)$, ranging over the set $\Omega^{(h)}$, is defined in terms of the Mejer's function as follows:

$$(2.2.1) \qquad f_1(z,\nu) =$$

$$= -(-1)^{\nu(\Delta+1)} G_{6,6}^{(1,3)}\left(z\Big| \begin{matrix} -\nu d_1, & -\nu d_1, & -\nu d_1, & 1+\nu d_2, & 1+\nu d_2, & 1+\nu d_2 \\ 0, & 0, & 0, & \nu, & \nu, & \nu \end{matrix}\right) =$$

$$= -(-1)^{\nu(\Delta+1)} \frac{1}{2\pi i} \int\limits_{L_1} g_{6,6}^{(1,3)}(s)\,ds,$$

where $\nu \in \mathbb{N}$,

$$g_{6,6}^{(1,3)}(s) = (z)^s \Gamma(-s)(\Gamma(1+s))^{-2}(\Gamma(1+\nu d_1+s)/(\Gamma(1-\nu+s)\Gamma(1+\nu d_2-s)))^3,$$

and the curve L_1 passes from $+\infty$ to $+\infty$ in the negative direction in such a way that the set $\mathbb{N}-1$ lies to the right of this curve and the set $-\mathbb{N}$ lies to the left of it. Here $p = q = 6$, $m = 1$, $n = 3$, $a_1 = a_2 = a_3 = -\nu d_1$, $a_4 = a_5 = a_6 = 1 + \nu d_2$, $b_1 = b_2 = b_3 = 0$, $b_4 = b_5 = b_6 = \nu$, $\Delta^* = -3\nu - 3$, and, since we take $|z| \leq 1$, the convergence condition (B3) is satisfied; moreover, for $z \in \Omega_1^h$, the convergence condition (B2) also holds. Since $d_1 \in \mathbb{N}$, it follows that

$$P_l(g_{6,6}^{(1,3)}) + 1 = 1 - \nu d_1 - \mathbb{N} \subset -\mathbb{N};$$

consequently, the condition of Remark 2.3 is satisfied. Therefore it follows from Lemma 2.1.3 that

$$(2.2.2) \qquad (-1)^{1+6-3} z((\delta+1+\nu d_1)^3(\delta-\nu d_2)^3 f_1)(z,\nu) = (\delta^3(\delta-\nu)^3 f_1)(z,\nu)$$

for $z \in \Omega_1^{(h)}$. Let

$$G = G_{6,6}^{(1,3)}\left(z\Big|\begin{matrix} -\nu d_1, & -\nu d_1, & -\nu d_1, & 1+\nu d_2, & 1+\nu d_2, & 1+\nu d_2 \\ 0, & 0, & 0, & \nu, & \nu, & \nu \end{matrix}\right), \tag{2.2.3}$$

and let

$$G_1 = (\prod_{k=3}^{6} \nabla_{b;k})(\prod_{j=1}^{3}(\nabla_{a;j})^{-d_1})(\prod_{j=4}^{6}(\nabla_{a;j})^{d_2})G.$$

Clearly,

$$f_1(z, \nu+1) = -(-1)^{(\nu+1)(\Delta+1)}G_1 \tag{2.2.4}$$

and

$$(\prod_{j=4}^{6}(\nabla_{a;j})^{-d_2})G_1 = (\prod_{k=3}^{6} \nabla_{b;k})(\prod_{j=1}^{3}(\nabla_{a;j})^{-d_1})G. \tag{2.2.5}$$

In view of Lemma 2.1.1,

$$(\prod_{j=4}^{6}(\nabla_{a;j})^{-d_2})G_1 = (-1)^{3d_2}(\prod_{\kappa=0}^{d_2-1}(\delta-(\nu+1)d_2+\kappa)^3)G_1 \tag{2.2.6}$$

and

$$(\prod_{k=3}^{6} \nabla_{b;k})(\prod_{j=1}^{3}(\nabla_{a;j})^{-d_1})G = (\delta-\nu)^3(\prod_{\kappa=0}^{d_1-1}(\delta+1+\nu d_1+\kappa)^3)G. \tag{2.2.7}$$

It follows from the relations (2.2.1), (2.2.3), (2.2.4), (2.2.5), (2.2.6), and (2.2.7) that

$$(\prod_{\kappa=1}^{d_2}(\delta-\nu d_2-\kappa)f_1)(z,\nu+1) = (\delta-\nu)^3(\prod_{\kappa=1}^{d_1}(\delta+\nu d_1+\kappa)^3 f_1)(z,\nu). \tag{2.2.8}$$

The equalities (2.2.2) and (2.2.8) hold in $\Omega_1^{(h)}$; by [9, (2.4)], however, $f_1(z,\nu)$ is a polynomial, therefore these relations hold in the whole complex plane according to the principle of analytical continuation. Let $\Omega_0^* = \{z \in \Omega^* : \mathrm{Re}\,(z) > 0\}$; clearly, Ω_0^* contains the following domain

$$\Omega_1^* = \{z \in \Omega_0^* : |z| > 1\}.$$

My second auxiliary function ranges over the set Ω_0^*; it can be expressed in terms of the Mejer's functions as follows

$$f_2(z,\nu) = \tag{2.2.9}$$

$$-(-1)^{\nu\Delta}G_{6,6}^{(4,3)}\left(-z\Big|\begin{matrix}-\nu d_1, & -\nu d_1, & -\nu d_1, & 1+\nu d_2, & 1+\nu d_2, & 1+\nu d_2\\ 0, & 0, & 0, & \nu, & \nu, & \nu\end{matrix}\right) =$$

$$-(-1)^{\nu\Delta}\frac{1}{2\pi i}\int\limits_{L_2} g_{6,6}^{(4,3)}(s)ds,$$

where $z \in \Omega_0^*$, $\nu \in \mathbb{N}$,

$$g_{6,6}^{(4,3)} = g_{6,6}^{(4,3)}(s) =$$

$$(-z)^s(\Gamma(-s))^3\Gamma(-s+\nu)(\Gamma(1-\nu+s))^{-2}(\Gamma(1+\nu d_1+s))^3(\Gamma(1+\nu d_2-s))^{-3},$$

and the curve L_2 passes from $-\infty$ to $-\infty$ in the negative direction in such a way that the set $\mathbb{N}-1$ lies to the right from this curve and the set $-\mathbb{N}$ lies to the left from it. Here $p = q = 6$, $m = 4$, $n = 3$, $a_1 = a_2 = a_3 = -\nu d_1$, $a_4 = a_5 = a_6 = 1+\nu d_2$, $b_1 = b_2 = b_3 = 0$, $b_4 = b_5 = b_6 = \nu$, $\Delta^* = -3\nu - 3$. Since now $|-z| \geq 1$, the convergence condition (C3) is satisfied; moreover, for $z \in \Omega_1^*$ convergence condition (C2) also holds. Since $d_1 \in \mathbb{N}$, it follows that

$$P_l(g_{6,6}^{(1,3)}) + 1 = 1 - \nu d_1 - \mathbb{N} \subset -\mathbb{N}$$

and the condition of Remark 2.3 is satisfied. Therefore it follows from the corollary to Lemma 2.1.5 that

$$(-1)^{4+6-3+1}z((\delta+1+\nu d_1)^3(\delta-\nu d_2)^3 f_2)(z,\nu) = (\delta^3(\delta-\nu)^3 f_2)(z,\nu) \tag{2.2.10}$$

for each $z \in \Omega_1^*$. Let

$$G = G_{6,6}^{(4,3)}\left(-z\Big|\begin{matrix}-\nu d_1, & -\nu d_1, & -\nu d_1, & 1+\nu d_2, & 1+\nu d_2, & 1+\nu d_2\\ 0, & 0, & 0, & \nu, & \nu, & \nu\end{matrix}\right) \tag{2.2.11}$$

and let

$$G_1 = (\prod_{k=3}^{6}\nabla_{b;k})(\prod_{j=1}^{3}(\nabla_{a;j})^{-d_1}(\prod_{j=4}^{6}(\nabla_{a;j})^{d_2})G.$$

Clearly,

$$f_2(z,\nu+1) = -(-1)^{(\nu+1)\Delta}G_1, \tag{2.2.12}$$

and

$$(\prod_{j=4}^{6}(\nabla_{a;j})^{-d_2})G_1 = (\prod_{k=3}^{6}\nabla_{b;k})(\prod_{j=1}^{3}(\nabla_{a;j})^{-d_1})G. \tag{2.2.13}$$

It follows from the corollary to Lemma 2.1.5 that

$$(\prod_{j=4}^{6}(\nabla_{a;j})^{-d_2})G_1 = (-1)^{3d_2}(\prod_{\kappa=0}^{d_2-1}(\delta-(\nu+1)d_2+\kappa)^3)G_1, \tag{2.2.14}$$

and

$$(\prod_{k=3}^{6}\nabla_{b;k})(\prod_{j=1}^{3}(\nabla_{a;j})^{-d_1})G = -(\delta-\nu)^3(\prod_{\kappa=0}^{d_1-1}(\delta+1+\nu d_1+\kappa)^3)G. \tag{2.2.15}$$

In view of the relations (2.2.9), (2.2.11), (2.2.12), (2.2.13), (2.2.14), and (2.2.15) one concludes that

$$((\prod_{\kappa=1}^{d_2}(\delta-\nu d_2-\kappa))f_2)(z,\nu+1) = ((\delta-\nu)^3(\prod_{\kappa=1}^{d_1}(\delta+\nu d_1+\kappa)^3)f_2)(z,\nu). \tag{2.2.16}$$

My third auxiliary function $f_3(z,\nu)$ ranges over the set Ω_0^*; it can be expressed in terms of the Mejer's functions as follows

$$f_3(z,\nu) = \tag{2.2.17}$$

$$(-1)^{\nu(\Delta+1)}G_{6,6}^{(5,3)}\left(z\middle| \begin{matrix} -\nu d_1, & -\nu d_1, & -\nu d_1, & 1+\nu d_2, & 1+\nu d_2, & 1+\nu d_2 \\ 0, & 0, & 0, & \nu, & \nu, & \nu \end{matrix}\right) =$$

$$(-1)^{\nu(\Delta+1)}\frac{1}{2\pi i}\int\limits_{L_2} g_{6,6}^{(5,3)}(s)ds,$$

where $z\in\Omega_0^*, \nu\in\mathbb{N}$,

$$g_{6,6}^{(5,3)} = g_{6,6}^{(5,3)}(s) =$$

$$(z)^s(\Gamma(-s))^3(\Gamma(-s+\nu))^2(\Gamma(1-\nu+s))^{-1}(\Gamma(1+\nu d_1+s))^3(\Gamma(1+\nu d_2-s))^{-3}.$$

Here $p=q=6,\ m=5,\ n=3,\ a_1=a_2=a_3=-\nu d_1,\ a_4=a_5=a_6=1+\nu d_2,\ b_1=b_2=b_3=0,\ b_4=b_5=b_6=\nu,\ \Delta^*=-3\nu-3$. Since now $|z|\geq 1$, the convergence condition (C3) is satisfied; moreover, for $z \in \Omega_1^*$ the convergence condition (C2) also holds. Once again

$$P_l(g_{6,6}^{(1,3)}) + 1 = 1 - \nu d_1 - \mathbb{N} \subset -\mathbb{N}$$

and the condition of Remark 2.3 holds. Therefore it follows from Lemma 2.1.3 that

$$(-1)^{5+6-3} z((\delta+1+\nu d_1)^3(\delta-\nu d_2)^3 f_3)(z,\nu) = (\delta^3(\delta-\nu)^3 f_3)(z,\nu). \tag{2.2.18}$$

Let

$$G = G_{6,6}^{(5,3)}\left(z\middle|\begin{matrix} -\nu d_1, & -\nu d_1, & -\nu d_1, & 1+\nu d_2, & 1+\nu d_2, & 1+\nu d_2 \\ 0, & 0, & 0, & \nu, & \nu, & \nu \end{matrix}\right) \tag{2.2.19}$$

and let

$$G_1 = (\prod_{k=3}^{6} \nabla_{b;k})(\prod_{j=1}^{3} (\nabla_{a;j})^{-d_1})(\prod_{j=4}^{6} (\nabla_{a;j})^{d_2}) G.$$

Clearly,

$$f_1(z,\nu+1) = -(-1)^{(\nu+1)(\Delta+1)} G_1 \tag{2.2.20}$$

and

$$(\prod_{j=4}^{6} (\nabla_{a;j})^{-d_2}) G_1 = (\prod_{k=3}^{6} \nabla_{b;k})(\prod_{j=1}^{3} (\nabla_{a;j})^{-d_1}) G. \tag{2.2.21}$$

It follows from Lemma 2.1.1 that

$$(\prod_{j=4}^{6} (\nabla_{a;j})^{-d_2}) G_1 = (-1)^{3d_2} (\prod_{\kappa=0}^{d_2-1} (\delta-(\nu+1)d_2+\kappa)^3) G_1 \tag{2.2.22}$$

and

$$(\prod_{k=3}^{6} \nabla_{b;k})(\prod_{j=1}^{3} (\nabla_{a;j})^{-d_1}) G = (-1)^2 (\delta-\nu)^3 (\prod_{\kappa=0}^{d_1-1} (\delta+1+\nu d_1+\kappa)^3) G. \tag{2.2.23}$$

In view of the relations (2.2.17), (2.2.19), (2.2.20), (2.2.21), (2.2.22), and (2.2.23) one concludes that

$$((\prod_{\kappa=1}^{d_2} (\delta-\nu d_2-\kappa)) f_3)(z,\nu+1) = ((\delta-\nu)^3 (\prod_{\kappa=1}^{d_1} (\delta+\nu d_1+\kappa)^3) f_3)(z,\nu). \tag{2.2.24}$$

Finally, my auxiliary function $f_5^{\vee}(z,\nu)$ ranges over the set Ω_0^*; it can be expressed in terms of the Mejer's functions as follows:

$$f_5^{\vee}(z,\nu) = \tag{2.2.25}$$

$$-(-1)^{\nu\Delta}G^{(6,3)}_{6,6}\left(-z\Big| \begin{matrix} -\nu d_1, & -\nu d_1, & -\nu d_1, & 1+\nu d_2, & 1+\nu d_2, & 1+\nu d_2 \\ 0, & 0, & 0, & \nu, & \nu, & \nu \end{matrix}\right) =$$

$$-(-1)^{\nu\Delta}\frac{1}{2\pi i}\int\limits_{L_2} g^{(6,3)}_{6,6}(s)ds,$$

where $z \in \Omega_0^*$, $\nu \in \mathbb{N}$, and

$$g^{(6,3)}_{6,6} = g^{6,3)}_{6,6}(s) =$$

$$(-z)^s(\Gamma(-s))^3\Gamma(-s+\nu))^3(\Gamma(1+\nu d_1+s))^3(\Gamma(1+\nu d_2-s))^{-3}.$$

Here $p = q = 6$, $m = 6$, $n = 3$, $a_1 = a_2 = a_3 = -\nu d_1$, $a_4 = a_5 = a_6 = 1+\nu d_2$, $b_1 = b_2 = b_3 = 0$, $b_4 = b_5 = b_6 = \nu$, $\Delta^* = -3\nu - 3$. Since $|-z| \geq 1$, the convergence condition (C3) is now satisfied; moreover, for $z \in \Omega_1^*$ the convergence condition (C2) holds also. Since $d_1 \in \mathbb{N}$, it follows that

$$P_l(g^{(1,3)}_{6,6}) + 1 = 1 - \nu d_1 - \mathbb{N} \subset -\mathbb{N}$$

and that the condition of Remark 2.3 is satisfied. Therefore it follows from the corollary to Lemma 2.1.5 that

$$(-1)^{6+6-3+1}z((\delta+1+\nu d_1)^3(\delta-\nu d_2)^3 f_5^{\vee})(z,\nu) = (\delta^3(\delta-\nu)^3 f_5^{\vee})(z,\nu) \tag{2.2.26}$$

for each $z \in \Omega_1^*$. Let

$$G = G^{(6,3)}_{6,6}\left(-z\Big| \begin{matrix} -\nu d_1, & -\nu d_1, & -\nu d_1, & 1+\nu d_2, & 1+\nu d_2, & 1+\nu d_2 \\ 0, & 0, & 0, & \nu, & \nu, & \nu \end{matrix}\right) \tag{2.2.27}$$

and let

$$G_1 = (\prod_{k=3}^{6} \nabla_{b;k})(\prod_{j=1}^{3}(\nabla_{a;j})^{-d_1})(\prod_{j=4}^{6}(\nabla_{a;j})^{d_2})G.$$

Clearly,

$$f_5^{\vee}(z,\nu+1) = -(-1)^{(\nu+1)\Delta}G_1 \tag{2.2.28}$$

and

$$(\prod_{j=4}^{6}(\nabla_{a;j})^{-d_2})G_1 = (\prod_{k=3}^{6} \nabla_{b;k})(\prod_{j=1}^{3}(\nabla_{a;j})^{-d_1})G. \tag{2.2.29}$$

It follows from the corollary to Lemma 2.1.5 that

$$(\prod_{j=4}^{6}(\nabla_{a;j})^{-d_2})G_1 = (-1)^{3d_2}(\prod_{\kappa=0}^{d_2-1}(\delta-(\nu+1)d_2+\kappa)^3)G_1 \tag{2.2.30}$$

and

$$(\prod_{k=3}^{6}\nabla_{b;k})(\prod_{j=1}^{3}(\nabla_{a;j})^{-d_1}G = (-1)^3(\delta-\nu)^3\prod_{\kappa=0}^{d_1-1}(\delta+1+\nu d_1+\kappa)^3)G. \tag{2.2.31}$$

In view of the relations (2.2.25), (2.2.27), (2.2.28), (2.2.29), (2.2.30) and (2.2.31) it follows that

$$((\prod_{\kappa=1}^{d_2}(\delta-\nu d_2-\kappa))f_5^{\vee})(z,\nu+1) = \tag{2.2.32}$$

$$((\delta-\nu)^3(\prod_{\kappa=1}^{d_1}(\delta+\nu d_1+\kappa)^3)f_5^{\vee})(z,\nu).$$

Let $f(z;\nu)$ stand for one of the functions $f_1(z;\nu), f_2(z;\nu), f_3(z;\nu), f_5^{\vee}(z;\nu)$. Then it follows from (2.2.2), (2.2.10), (2.2.18), (2.2.26), (2.2.8), (2.2.16), (2.2.24) and (2.2.32) that

$$((\delta+1+\nu d_1)^3(\delta-\nu d_2)^3 f)(z,\nu) = (\delta^3(\delta-\nu)^3 f)(z,\nu) \tag{2.2.33}$$

and

$$((\prod_{\kappa=1}^{d_2}(\delta-\nu d_2-\kappa))f)(z,\nu+1) = ((\delta-\nu)^3\prod_{\kappa=1}^{d_1}(\delta+\nu d_1+\kappa)^3)f)(z,\nu). \tag{2.2.34}$$

As in [9], let

$$D^{\wedge}\ (z,\nu,w) = z(w+1+\nu d_1)^3(w-\nu d_2)^3 - w^3(w-3\nu)^3, \delta = zd/dz, \tag{2.2.35}$$

where w is an independent variable. Then relation (2.2.33) may by rewritten as follows:

$$D^{\wedge}\ (z,\nu,\delta)f = 0. \tag{2.2.36}$$

In view [9, (2.14)], relations (2.2.36) and (2.2.34) hold also for $f(z,\nu) = f_5(z,\nu) = f_5^{\vee}(z,\nu) + i\pi f_3(z,\nu)$. As in [9, (2.15)], let

$$f_j^*(z,\nu) = (\frac{\nu\Delta)!}{(\nu d_1)!})^3 f_j(z,\nu), \quad j = 1,2,3,5; \tag{2.2.37}$$

the functions $f(z,\nu) = f_j^*(z,\nu)$, $j = 1,2,3,5$, satisfy equation (2.2.36). It follows from (2.2.37) that

$$f_j(z,\nu) = (\frac{(\nu d_1)!}{(\nu\Delta)!})^3 f_j^*(z,\nu), \quad j = 1,2,3,5. \tag{2.2.38}$$

Let us substitute relation (2.2.38) in (2.2.34); this gives

$$(\frac{((\nu+1)d_1)!}{((\nu+1)\Delta)!})^3 \prod_{\kappa=1}^{d_2}(\delta - \nu d_2 - \kappa) f_j^*(z,\nu+1) =$$

$$(\frac{(\nu d_1)!}{(\nu\Delta)!})^3 (\delta-\nu)^3 \prod_{\kappa=1}^{d_1}(\delta + \nu d_1 + \kappa)^3 f_j^*(z,\nu),\ j = 1,2,3,5.$$

It follows from the last equation that

$$\prod_{\kappa=1}^{\Delta-1}(\nu(\Delta-1)+\kappa)^3 \prod_{\kappa=1}^{d_2}(\delta - \nu d_2 - \kappa) f_j^*(z,\nu+1) = \tag{2.2.39}$$

$$(\prod_{\kappa=1}^{\Delta}(\nu\Delta+\kappa)^3)(\delta-\nu)^3 \prod_{\kappa=1}^{d_1}(\delta + \nu d_1 + \kappa)^3 f_j^*(z,\nu),$$

where $j = 1,2,3,5$.

§ 3. The first bars of the main theme

As in [9], let ν^{-1} be an independent variable taking its values in $\mathbb{C}$ including 0; let, for $i = 0,1$,

$$D_i^*(z,\nu^{-1},w) = \nu^{-6} D^{\wedge}(z,\nu+i,\nu w) = \tag{2.3.1}$$

$$= b_{i,0}^{\vee}(z,\nu^{-1})w^0 + \ldots + b_{i,5}^{\vee}(z,\nu^{-1})w^5 + (z-1)w^6,$$

$$P_i^* = P_i^*(\nu^{-1},w) = \tag{2.3.2}$$

$$(w-1)^{3-3i}\Big(\prod_{k=1}^{\Delta-i}(\Delta-i+k\nu^{-1})^3\Big)\prod_{k=1}^{\Delta-1+2i}(w+(-1)^i(d_{1+i}+k\nu^{-1})^3)=$$

$$p_{i,0}(\nu^{-1})w^0+\ldots+p_{i,3\Delta+3i}(\nu^{-1})w^{3\Delta+3i}\in\mathbb{Z}[\nu^{-1},w],$$

and set $P_i^{\sim}(w)=P_i^*(0,w)$. Then

$$D_i^*(z,\nu^{-1},w)\in Q[z,\nu^{-1},w], D_0^*(z,0,w)=D_1^*(z,0,w).$$

It follows from (2.2.37) that

$$D_i^*(z,\nu^{-1},\nu^{-1}\delta)f_k^*(z,\nu+i)=0 \tag{2.3.3}$$

for $\nu\in\mathbb{N}, z\in\Omega_1^*, k=1,2,3,5$; furthermore, it follows from (2.2.39) that

$$(P_1^*(\nu^{-1},\nu^{-1}\delta)f_k^*)(z,\nu+1)=(P_0^*(\nu^{-1},\nu^{-1}\delta)f_k^*)(z,\nu) \tag{2.3.4}$$

for $\nu\in\mathbb{N}, z\in\Omega_1^*, k=1,2,3,5)$. For $k=1,2,3,5$, let $X_k(z,\nu)$ denote the column consisting of the six elements $((\nu^{-1}\delta)^{j-1}f_k^*)(z,\nu)$, $j=1,\ldots,6$. For $i=0,1$, let $B_i(z,\nu^{-1})=(a_{kj})$ be a 6×6-matrix defined as follows:

$$a_{6j}=b^{\vee}_{i,j-1}(z,\nu^{-1})/(1-z),\ j=1,\ldots,6,\quad a_{12}=a_{23}=a_{34}=a_{45}=a_{56}=1,$$

and each of the other entries a_{kj} is equal to 0. Following the well known procedure of representation of a differential equation in matrix form, we deduce from the relation (2.3.3) that

$$(\nu^{-1}\delta)X_k(z,\nu+i)=B_i(z,\nu^{-1})X_k(z,\nu+i) \tag{2.3.5}$$

for $i=0,1$, $k=1,2,3,5$, $\nu\in\mathbb{N}$, and $z\in\Omega_1^*$. According to (2.3.2), the operator $\nu^{-1}\delta$ commutes with the operator $P_i^*(\nu^{-1},\nu^{-1}\delta)$ for $i=0,1$. Therefore it follows from the relation (2.3.5) that

$$P_1^*(\nu^{-1},\nu^{-1}\delta)X_k(z,\nu+1)=P_0^*(\nu^{-1},\nu^{-1}\delta)X_k(z,\nu) \tag{2.3.6}$$

for $i=1,2$, $k=1,2,3,5$, $\nu\in\mathbb{N}$, and $z\in\Omega_1^*$.

§ 4. Development of the main theme

Let $\mathrm{Mat}_n(K)$, for $n\in\mathbb{N}$, denote the set of all the $n\times n$-matrices with entries in the subset K of a given ring. Let

$$P^*(\nu^{-1},w)=P_0^*(\nu^{-1},w), Q^*(\nu^{-1},w)=P_1^*(\nu^{-1},w). \tag{2.4.1}$$

Relation (2.3.4) may be rewritten as follows:

$$Q^*(\nu^{-1},\nu^{-1}\delta)X_k(z,\nu+1)=P^*(\nu^{-1},\nu^{-1}\delta)X_k(z,\nu), \tag{2.4.2}$$

(cf. relation (2) in [8]). Let further

$$D^{\sim}(z,w)=D_0^*(z,0,w)=D_1^*(z,0,w)=z(w+d_1)^3(w-d_2)^3-w^3(w-1)^3, \tag{2.4.3}$$

$$P^{\sim}(w)=P^*(0,w)=\Delta^{3\Delta}(w+d_1)^{3d_1}, \tag{2.4.4}$$

$$Q^{\sim}(w)=Q^*(0,w)=(d_1)^{3d_1}(w-d_2)^{3d_2}, \tag{2.4.5}$$

and

$$B^{\sim}(z)=B_0(z,0)=B_1(z,0). \tag{2.4.6}$$

Regarded as a polynomial of w, the polynomial

$$D_1^{\sim}(z,w)=(1/(z-1))D^{\sim}(z,w)=(\nu^{-1}w)^6-\sum_{k=1}^{6}b_k^{\sim}(z)(w)^{k-1}, \tag{2.4.7}$$

coincides with the characteristic polynomial of the matrix $B^{\sim}(z)$, so that

$$\det(B^{\sim}(z)-d_2E)=D_1^{\sim}(z,d_2)=(1/(z-1))(\Delta(\Delta+1))^3. \tag{2.4.8}$$

It follows that $(B^{\sim}(z)-d_2E)^{-1}\in \mathrm{Mat}_6(\mathbb{Q}[z])$; moreover, the last column of this matrix consists of the elements of the ideal $(z-1)\mathbb{Q}[z]$ in the ring $\mathbb{Q}[z]$. Let

$$K_m=(z-1)^{-m}\mathbb{Q}[z]\cap\mathbb{Q}[(z-1)^{-1}], K_m^*=(z-1)^{-m}\mathbb{Q}[z,\nu^{-1}]\cap\mathbb{Q}[(z-1)^{-1},\nu^{-1}]$$

where $m\in\mathbb{N}-1$. Clearly, $K_0=\mathbb{Q}$, $K_0^*=\mathbb{Q}[\nu^{-1}]$, and

$$B_i(z,\nu^{-1})-B^{\sim}(z)\in\nu^{-1}\,\mathrm{Mat}_6(K_1^*)$$

for $i=0,1$.

Lemma 2.4.1. *For $i=0,1$ and $m\in\mathbb{N}$, suppose that*

$$H_{i,m-1}^{\sim}\in\mathrm{Mat}_6(K_{m-1}),\quad H_{i,m-1}^*\in\mathrm{Mat}_6(K_{m-1}^*),$$

$$H_{i,m-1}=H_{i,m-1}^{\sim}+\nu^{-1}H_{i,m-1}^*,\quad b_{i,m}\in\mathbb{Q},\ c_{i,m}\in\mathbb{Q}.$$

Then, for $i = 0, 1,$ *there exists* $H^*_{i,m}(z, \nu^{-1}) \in \operatorname{Mat}_6(K^*_m)$ *such that*

$$(\nu^{-1}\delta + b_{i,m} + c_{i,m}\nu^{-1})H_{i,m-1}X_k(z, \nu + i) = \tag{2.4.9}$$

$$(H^{\sim}_{i,m-1}(B^{\sim}(z) + b_{i,m}E) + \nu^{-1}H^*_{i,m}(z, \nu^{-1}))X_k(z, \nu + i),$$

where $k = 1, 2, 3, 5,$ $\nu \in \mathbb{N},$ *and* $z \in \Omega^*_1$.

Proof. Since

$$B_i(z, \nu^{-1}) - B^{\sim}(z) \in \nu^{-1} \operatorname{Mat}_6(K^*_1)$$

we can let

$$H^*_{i,m}(z, \nu^{-1}) = (\delta + c_{i,m})H_{i,m-1} + H^*_{i,m-1}(B^{\sim}(z) + b_{i,m-1}E) +$$

$$+ H_{i,m}(B_i(z, \nu^{-1}) - B^{\sim}(z)).$$

The lemma is proved.

Corollary. *Let* $b_{i,s} \in \mathbb{Q}$ *and* $c_{i,s} \in \mathbb{Q}$ *be given, for* $i = 0, 1$ *and* $s = 1, \dots, m$ *with* $m \in \mathbb{N}$. *Then, for each* $i = 0, 1,$ *there exists a matrix* $H^*_{i,m} \in \operatorname{Mat}_6(K^*_m)$ *depending on* $b_{i,1}, \dots, b_{i,m}, c_{i,1}, \dots, c_{i,m}$ *and such that*

$$(\prod_{s=1}^{m}(\nu^{-1}\delta + b_{i,m+1-s} + \nu^{-1}c_{i,m+1-s}))X_k(z, \nu + i) =$$

$$= (\nu^{-1}H^*_{i,m} + \prod_{s=1}^{m}(B^{\sim}(z) + b_{i,m+1-s}E))X_k(z, \nu + i),$$

where E *is the unit* 6×6*-matrix and* $k = 1, 2, 3, 5$.

Proof. The proof is by induction on m. For $m = 1$, the assertion of the lemma follows from Lemma 2.4.1 with $m = 1$ on letting $H^{\sim}_{i,0} = E$ and $H^*_{i,0}$ equal to the zero 6×6-matrix. Let $m \geq 2$ and suppose the assertion of the lemma holds for $m - 1$. Then the assertion of the lemma for m follows from Lemma 2.4.1 on letting

$$H^{\sim}_{i,m-1} = \prod_{s=1}^{m-1}(B^{\sim}(z) + b_{i,m-s}E).$$

The Corollary is proved.

Lemma 2.4.2. *Let*

$$\mathfrak{W}_i = \operatorname{Mat}_6(K^*_{3\Delta+3i})$$

for $i = 0, 1$. *Then, for* $i = 0, 1,$ *there is a matrix* $U_i(z, \nu^{-1})$ *satisfying the following conditions:*

$$U_i(z, \nu^{-1}) \in \mathfrak{W}_i,$$

$$(U_0(z,\nu^{-1}) - P^{\sim}(B^{\sim}(z)) \in \nu^{-1}\mathfrak{W}_0, \tag{2.4.10}$$

$$U_1(z,\nu^{-1}) - Q^{\sim}(B^{\sim}(z)) \in \nu^{-1}\mathfrak{W}_1, \tag{2.4.11}$$

$$P^*(\kappa^{-1},\kappa^{-1}\delta)X_k(z,\kappa) = U_0(z,\kappa^{-1})X_k(z,\kappa), \tag{2.4.12}$$

and

$$Q^*(\kappa^{-1},\kappa^{-1}\delta)X_k(z,\kappa+1) = U_1(z,\kappa^{-1})X_k(z,\kappa+1) \tag{2.4.13}$$

for $k = 1,2,3,5,\ \kappa \in \mathbb{N}$.

Proof. Let us use relations (2.3.2) and (2.4.1). Let

$$\Pi_i^{\vee}(\nu^{-1}) = \prod_{k=1}^{\Delta-i}(\Delta - i + k\nu^{-1})^3,$$

where $i = 0,1$. Then

$$\Pi_i^{\vee}(\nu^{-1}) \in K_0^*, \Pi_i^{\vee}(0) = (\Delta-i)^{3\Delta-3i}, \quad \Pi_i^{\vee}(\nu^{-1}) - (\Delta-i)^{3\Delta-3i} \in \nu^{-1}K_0^*$$

for $i = 0,1$. In the corollary to Lemma 2.4.1, for $i = 0,1$ let $m = 3\Delta + 3i$ and

$$b_{i,3k+l} = (-1)^i d_{1+i}, c_{i,3k+l} = (-1)^i k,$$

with $l = 1,2,3,\ k = 0,\dots,\Delta-2+2i$. Let further in this corollary

$$b_{i,3\Delta+3i-l+1} = -1, \quad c_{3\Delta+3i-l+1} = 0,$$

for $i = 1$ and $l = 1,2,3$; it follows then that

$$\Pi_i^{\vee}(\nu^{-1})\Big((\nu^{-1}\delta-1)^{3-3i}\prod_{k=1}^{\Delta-1+2i}(\nu^{-1}\delta + (-1)^i(d_{1+i}+k\nu^{-1})^3)\Big)X_k(z,\nu+i) =$$

$$\Pi_i^{\vee}(\nu^{-1})\Big(\nu^{-1}H^*_{i,3\Delta+3i} + (B^{\sim}(z)-E)^3\prod_{k=1}^{\Delta-1+2i}(B^{\sim}(z)+(-1)^i(d_{1+i}E)^3)\Big)X_k(z,\nu+i),$$

where $k = 1,2,3,5$. Let

$$U_i(z,\nu^{-1}) = \Pi_i^{\vee}(\nu^{-1})\Big(\nu^{-1}H^*_{i,3\Delta+3i} + (B^{\sim}(z)-E)^3\prod_{k=1}^{\Delta-1+2i}(B^{\sim}(z)+(-1)^i(d_{1+i}E)^3)\Big);$$

it follows that $U_i(z,\nu^{-1}) \in \mathfrak{W}_i$ and

$$(U_i(z,\nu^{-1}) - (\Delta-i)^{3\Delta-3i})(B^{\sim}(z)-E)^3) \prod_{k=1}^{\Delta-1+2i} (B^{\sim}(z)+(-1)^i(d_{1+i}E)^3) =$$

$$= \Pi_i^{\vee}(\nu^{-1})\nu^{-1}H^*_{i,3\Delta+3i} +$$

$$(\Pi_i^{\vee}(\nu^{-1} - (\Delta-i)^{3\Delta-3i})(B^{\sim}(z)-E)^3 \prod_{k=1}^{\Delta-1+2i} (B^{\sim}(z)+(-1)^i(d_{1+i}E)^3 \in \nu^{-1}\mathfrak{W}_i,$$

for $i=0,1$. The lemma is proved.

Remark 2.4. The assertions of Lemmata 2.4.1 and 2.4.2 become obvious if one observes that the operator $\nu^{-1}\delta$ maps the ring $\mathbb{Q}[(z-1)^{-1},\nu^{-1}]$ (respectively $\mathbb{Q}[z,\nu^{-1}]$-module $(z-1)^{-m}\mathbb{Q}[z,\nu^{-1}]$) into its ideal $\nu^{-1}\mathbb{Q}[(z-1)^{-1},\nu^{-1}]$ (respectively into $\mathbb{Q}[z,\nu^{-1}]$-module $\nu^{-1}(z-1)^{-m-1}\mathbb{Q}[z,\nu^{-1}]$) and that

$$\nu^{-1}(z-1)^{-m-1}\mathbb{Q}[z,\nu^{-1}] \supset (z-1)^{-m}\mathbb{Q}[z,\nu^{-1}] \times \nu^{-1}(z-1)^{-1}\mathbb{Q}[z,\nu^{-1}].$$

In view of the relation (2.4.8) and the argument preceding Lemma 2.4.1, it follows from the relation (2.4.5) that

$$\det(Q^{\sim}(B^{\sim}(z)) = ((-1/(z-1))(\Delta(\Delta+1))^3)^{3d_2}(d_1)^{3d_1}, \tag{2.4.14}$$

that

$$Q^{\sim}((B^{\sim}(z))^{-1} \in \mathrm{Mat}_6(\mathbb{Q}[z]),$$

that each of the elements of the last column of that matrix lies in the ideal $(z-1)\mathbb{Q}[z]$ of the ring $Q[z]$, and that

$$A^{\sim}(z) = (Q^{\sim}(B^{\sim}(z)))^{-1}P^{\sim}(B^{\sim}(z)) \in \mathrm{Mat}_6(\mathbb{Q}[z,1/(z-1)]). \tag{2.4.15}$$

Let

$$A(z,\nu^{-1}) = (U_1(z,\nu^{-1}))^{-1}U_0(z,\nu^{-1}). \tag{2.4.16}$$

Clearly,

$$A(z,\nu^{-1}) \in \mathrm{Mat}_6(\mathbb{Q}(z,\nu^{-1})).$$

A rational function $R(x) \in k(x), x := (x_1,\dots,x_n)$, is said to be well defined in a point $a \in \bar{k}^n$ if $R(x) = P(x)Q(x)^{-1}$ with $P(x) \in k[x]$, $Q(x) \in k[x]$, $Q(a) \neq 0$, where $\bar{k}$ stands for the algebraic closure of the field k. For $\varepsilon > 0$, let $\mathfrak{N}_0(\varepsilon) = \{\nu^{-1} \in \mathbb{C}\colon\ |\nu^{-1}| \leq \varepsilon\}$. Given a compact subset F of the complex plane $\mathbb{C}$, let

$\mathfrak{H}(F,\varepsilon)$ stand for the subset of the field $\mathbb{Q}(z,\nu^{-1})$ consisting of those elements of that field, which are well defined in every point of the set $F\times\mathfrak{N}_0(\varepsilon)$, and let $\mathfrak{H}_0(F)$ be the subring of the field $\mathbb{Q}(z)$ consisting of those rational functions in that field, which are well defined in every point $z\in F$.

Lemma 2.4.3. *Let F be a closed bounded subset of the set $\mathbb{C}\diagdown\{1\}$. There exists an $\varepsilon_0=\varepsilon_0(F)\in(0,1)$ such that*

$$A(z,\nu^{-1})\in \mathrm{Mat}_6(\mathfrak{H}(F,\varepsilon_0));$$

moreover, $A(z,0)=A^{\sim}(z)$ for each $z\in F$.

Proof. Since $1\notin F$, it follows that

$$U_i(z,\nu^{-1})\in \mathrm{Mat}_6(\mathfrak{H}(F,1)),\ i=0,1.$$

Therefore $\det(U_i(z,\nu^{-1}))\in\mathfrak{H}(F,1)$. By Lemma 2,

$$U_0(z,0)=P^{\sim}(B^{\sim}(z))\in\mathfrak{H}_0(F), U_1(z,0)=Q^{\sim}(B^{\sim}(z))\in\mathfrak{H}_0(F).$$

Let $\mu=\min\{|\det(Q^{\sim}(B^{\sim}(z)))|: z\in F\}$. In view of (2.4.14), $\mu>0$. Since F is compact, there is an $\varepsilon_0=\varepsilon_0(F)\in(0,1)$ such that

$$|\det(U_2(z,\nu^{-1}))-\det(U_2(z,0))|<\frac{\mu}{2},$$

where $z\in F$ and $|\nu^{-1}|\leq\varepsilon_0$. This proves the lemma.

It follows from the relations (2.4.2), (2.4.12), (2.4.13), (2.4.15), and Lemma 3 that the following equality

$$X_k(z,\nu+1)=A(z,\nu^{-1})X_k(z,\nu), \tag{2.4.17}$$

where $k=1,2,3,5, z\in F,\nu\in\mathbb{N}+[1/\varepsilon_0(F)]$, holds for any closed bounded subset F of the domain Ω_1^*.

In what follows A^B stands for the set of all maps from an non-empty set B to an non-empty set A.

Lemma 2.4.4. *As above, let F be a compact subset of the complex plane $\mathbb{C}$ (in particular, F may be an one-point set). Let $\mathfrak{H}_0(F)$ and $\mathfrak{H}(F,\varepsilon_0)$ be the subrings of the fields $\mathbb{Q}(z)$ and $\mathbb{Q}(z,\nu^{-1})$ defined above respectively; here $\varepsilon_0\in(0,1)$. For $n\in\mathbb{N}$, $z\in F$, $\nu\in\mathbb{N}+[1/\varepsilon_0]$, let*

$$A(z,\nu^{-1})\in\mathrm{Mat}_n(\mathfrak{H}(F,\varepsilon_0)) \tag{2.4.18}$$

and let the sequence $X_\nu\in(\mathbb{C}^F)^n$ satisfy the following relation

$$X_{\nu+1}(z)=A(z,\nu^{-1})X_\nu(z). \tag{2.4.19}$$

Moreover, suppose that, for each $z \in F$, the characteristic polynomial

$$\det(\lambda E - A(z,0)) \tag{2.4.20}$$

has only simple roots, that none of the elements of the first row of the matrix

$$C = C(z) \in M_n(\mathbb{C}^F) \tag{2.4.21}$$

vanishes, that $\det(C(z)) \neq 0$, *and that the matrix* $(C(z))^{-1}A(z,0)C(z)$ *is diagonal. Then there is* $\varepsilon_2 \in (0, \varepsilon_0)$ *such that, for any* $z \in F, \nu \in \mathbb{N}+[1/\varepsilon_2]$, *the first element* $x_\nu(z)$ *of the column* $X_\nu(z)$ *satisfies the equation*

$$x_{\nu+n}(z) + \sum_{j=0}^{n-1} q_j^*(z, \nu^{-1}) x_{\nu+j}(z) = 0, \tag{2.4.22}$$

moreover,

$$q_j^*(z, \nu^{-1}) \in \mathfrak{H}(F, \varepsilon_2), \tag{2.4.23}$$

for $j = 0, \dots, n-1$, *and the polynomial*

$$\lambda^n + \sum_{j=0}^{n-1} q_j^*(z, 0) \lambda^{\nu+j}(z). \tag{2.4.24}$$

coincides with the polynomial 2.4.20.

Proof. Let

$$S(z, \nu^{-1}, 0) = E, \tag{2.4.25}$$

where E stands for the unit $n \times n$ - matrix, and let

$$S(z, \nu^{-1}, \kappa) = A(z, \nu^{-1}(1 + (\kappa - 1)\nu^{-1})^{-1}) S(z, \nu^{-1}, \kappa - 1), \tag{2.4.26}$$

for $z \in F, |\nu^{-1}| \leq \varepsilon_0$, $\kappa = 1, \dots, n$. It follows from (2.4.18) that

$$S(z, \nu^{-1}, \kappa) \in \mathrm{Mat}_{\,n}(\mathfrak{H}(F, \varepsilon)) \tag{2.4.27}$$

for $\kappa = 0, \dots, n$. Clearly,

$$B(z, 0, \kappa) = (A(z, 0))^\kappa, \tag{2.4.28}$$

for $\kappa = 0, \ldots, n$. Let $s(z, \nu^{-1}, \kappa)$ be the first row of the matrix (2.4.27). Then $s(z, 0, \kappa)$ coincides with the first row of the matrix (2.4.28). Let $V(z, \nu^{-1})$ be the $(n+1) \times n$ - matrix, whose l-th row, for $l = 1, \ldots, n+1$, coincides with the row $s(z, \nu^{-1}, l-1)$. Let $W = (w_l)$ be a column of the $n+1$ variables w_l, $l = 1, \ldots, n+1$. For $j = 0, \ldots, n$, let $Q_j(z, \nu^{-1})$ be the cofactor of the element w_{j+1} in the matrix $(V(z, \nu^{-1}), W)$. Clearly,

$$\sum_{j=0}^{k} Q_j(z, \nu^{-1}))s(z, \nu^{-1}, j) = 0. \tag{2.4.29}$$

It follows from the relation (2.4.19) that

$$s(z, \nu^{-1}, j)X_\nu(z) = x_{\nu+j}(z) \tag{2.4.30}$$

for $z \in F$, $\nu \in \mathbb{N} + [1/\varepsilon]$, $j = 0, \ldots, n$. On multiplying both sides of the relation (2.4.29) by $X_\nu(z)$ from the right, one deduces from (2.4.30) that, for $z \in F$, $\nu \in \mathbb{N} + [1/\varepsilon]$, $j = 0, \ldots, n$, the following equation

$$\sum_{j=0}^{n} Q_j(z, \nu^{-1})x_{\nu+j}(z) = 0, \tag{2.4.31}$$

is satisfied. For $j = 0, \ldots, n$, let $a(z, j)$ be the first row of the matrix $A(z, 0)^j$; for $k = 1, \ldots, n$ and $j = 0, \ldots, n$, let $a_k(z, j)$ and $s_k(z, \nu^{-1}, j)$ be the k-th elements of the rows $a(z, j)$ and $s(z, \nu^{-1}, j)$ of the same matrix respectively. Then

$$s(z, 0, j) = a(z, j), \tag{2.4.32}$$

for $j = 0, \ldots, n$. Let the polynomial (2.4.20) be written in the form

$$\sum_{j=0}^{n} d_j(z)\lambda^j; \tag{2.4.33}$$

then $d_n(z) = 1$, and $d_j(z) \in H_0(F)$ for $j = 0, \ldots, n-1$. By Cayley-Hamilton's theorem,

$$\sum_{j=0}^{n} d_j(z)a(z, j) = 0. \tag{2.4.34}$$

It follows from the relations (2.4.20) and (2.4.32) that

$$\sum_{j=0}^{n} Q_j(z, 0)a(z, j) = 0. \tag{2.4.35}$$

Therefore the following system of linear equations

$$\sum_{j=0}^{n} y_j a_k(z,j) = 0, \quad k = 1, \dots, n, \tag{2.4.36}$$

has two solutions: $y_j = d_j(z)$ and $y_j = Q_j(z,0)$, in both cases $j = 0, \dots, n$. We shall prove that, for any $z \in F$, the second solution does not vanish and that the space of solutions of the system (2.4.36) is one - dimensional. The following relation

$$Q_n(z,0) \neq 0 \tag{2.4.37}$$

for each $z \in F$ would imply either of these assertions. Let $V^\wedge(z,\nu)$ denote the matrix obtained by removing from the matrix $V(z,\nu^{-1})$ its last row, let $c(z)$ be the first row of the matrix $C(z)$, and let $J(z) = (C(z))^{-1}A(z,0)C(z)$. Let further $C^\wedge(z)$ be the diagonal $n \times n$-matrix having, for $l = 1, \dots, n$, as its l-th diagonal element the l-th element of the row $c(z)$. Finally let $W_0(z)$ be the $n \times n$-matrix defined by the following conditions: all elements of its first row are equal to 1 and, when $n \geq 2$, $l = 2, \dots, n$, $s = 1, \dots, n$, having in its l-th row and s-th column the s-th diagonal element of the matrix $(J(z))^{l-1}$. Then

$$Q_n(z,\nu^{-1}) = \det(V^\wedge(z,\nu^{-1})) \tag{2.4.38}$$

and, for $l = 1, \dots, n$, the row $s(z,\nu^{-1},l-1)$ coincides with the l-th row of the matrix $V^\wedge(z,\nu^{-1})$. Moreover, it follows that

$$(J(z))^{l-1} = (C(z))^{-1}(A(z,0))^{l-1}C(z) \quad \text{and} \quad C(z)(J(z))^{l-1} = (A(z,0))^{l-1}C(z)$$

for $l \in \mathbb{N}$, that

$$c(z)(J(z))^{l-1} = a(z,l-1)C(z)$$

for $l = 1, \dots, n$, that

$$W_0(z)C^\wedge(z) = V^\wedge(z,\nu^{-1})C(z),$$

and that

$$Q_n(z,0) = \det(V^\wedge(z,0)) = (\det(W_0(z))(\det C^\wedge(z))/\det(C(z)).$$

For each $z \in F$, the Vandermonde determinant $\det(W_0(z))$ does not vanish because the polynomial (2.4.10) has only simple roots; furthermore, $\det(C^\wedge(z)) \neq 0$

because, for $z \in F$, no element of the first row of the matrix $C(z)$ vanishes . The inequality (2.4.37) is proved. Consequently,

$$Q_j(z,0)/Q_n(z,0) = d_j(z) \tag{2.4.39}$$

for $z \in F, j = 1, \dots, n$. Since

$$Q_j(z,\nu^{-1}) \in \mathfrak{H}(F,\varepsilon_0) \tag{2.4.40}$$

for $j = 0, \dots, n$ and the set F is compact, it follows that

$$m = \inf\{|Q_n(z,0)| : z \in F\} > 0; \tag{2.4.41}$$

thus there is an $\varepsilon_2 \in (0, \varepsilon_0)$ such that

$$|Q_n(z,\nu^{-1}) - Q_n(z,0)| \leq m/2, \tag{2.4.42}$$

for $z \in F$, $|\nu^{-1}| < \varepsilon_2$. It follows from the relations (2.4.41) and (2.4.42) that

$$|Q_n(z,\nu^{-1})| \geq m/2, \tag{2.4.43}$$

for $z \in F, |\nu^{-1}| < \varepsilon_2$.

Let

$$q_j^*(z,\nu^{-1}) = Q_j(z,\nu^{-1})/Q_n(z,\nu^{-1}),\ j = 1, \dots, n;$$

it follows from the relations (2.4.31), (2.4.39) and (2.4.43) that the relations (2.4.22) and (2.4.23) are satisfied and the polynomial (2.4.24) coincides with the polynomial (2.4.33). The lemma is proved.

Lemma 2.4.5. *Let us consider a compact subset F of $\mathbb{C}$ (in particular, F may be an one-point set). For $n \in \mathbb{N}$ and $\nu \in \mathbb{N}$, let $X_\nu(z) \in (\mathbb{C}^F)^n$ and let $\mathfrak{H}_0(F)$ be the subring of those functions in $\mathbb{Q}(z)$, which are well defined for every $z \in F$. For $\varepsilon_0 \in (0,1)$, let $\mathfrak{H}(F,\varepsilon_0)$ be the subring of those functions in $\mathbb{Q}(z,\nu^{-1})$, which are well defined for every (z,ν^{-1}) with $z \in F, |\nu^{-1}| \leq \varepsilon_0$. Let further $b_j^\sim(z) \in H_0(F)$, $j = 0, \dots, n$, and let $b_n^\sim(z) = 1$. Let*

$$A(z,\nu^{-1}) \in \mathrm{Mat}_n((\mathfrak{H}(F,\varepsilon_0)). \tag{2.4.44}$$

Suppose that, for each $z \in F$, all the roots of the polynomial

$$D^\sim(z,\lambda) = \sum_{j=0}^{n} b_j(z)\lambda^j \tag{2.4.45}$$

are simple. Let $B^{\sim}(z) = (b_{ik})$ *be an* $n \times n$ *- matrix defined as follows:* $b_{nl} = -b^{\sim}_{l-1}(z)$, $l = 1, \dots, n$; *moreover, if* $n > 1$, *then* $b_{i1} = 0$, $i = 1, \dots, n-1$, *and the* $n-1 \times n-1$ *- submatrix of the matrix* $B^{\sim}(z)$ *formed by its first* $n-1$ *rows and its last* $n-1$ *columns is the unit matrix. Let further* $P^{\sim}(z,\lambda) \in \mathfrak{H}_0(F)[\lambda]$ *and* $Q^{\sim}(z,\lambda) \in \mathfrak{H}_0(F)[\lambda])$, *and suppose that, for any* $z \in F$, *the polynomial* $Q^{\sim}(z,\lambda)$ *does not vanish on the set of all the roots of the polynomial* $D^{\sim}(z,\lambda)$ *and that, on the latter set, the map*

$$\lambda \to P^{\sim}(z,\lambda)/Q^{\sim}(z,\lambda) \tag{2.4.46}$$

is injective. Let finally

$$Q^{\sim}(z, B^{\sim}(z))A(z,0) = P^{\sim}(z, B^{\sim}(z)) \tag{2.4.47}$$

for $z \in F$.

Then, for every $z \in F$, *all the roots of the polynomial (2.4.20) are simple, and there exists a matrix* $C = C(z) \in M_n(\mathbb{C}^F)$ *satisfying the following conditions:*

1) the relations (2.4.21) are fulfilled;

2) for every $z \in F$, *the matrix* $(C(z))^{-1}A(z,0)C(z)$ *is diagonal;*

3) for every $z \in F$, *no element in the first row of the the matrix* $(C(z))$ *is equal to zero.*

Proof. For $z \in F$, let

$$\lambda_i = \lambda_i(z),\ i = 1, \dots, n, \tag{2.4.48}$$

be the list of all the roots of the polynomial $D^{\sim}(z,\lambda)$. Let further $J^{\sim}(z)$ be the diagonal $n \times n$ - matrix having λ_l, where $l = 1, \dots, n$, as its $l-$th diagonal element, and let $C(z)$ be the $n \times n-$ matrix having $(\lambda_s(z))^{l-1}$ in its $l-$th row and $s-th$ column, if $n \geq 2$, $l = 2, \dots, n$ and $s = 1, \dots, n$, and having all the other elements equal to 1. The Vandermonde determinant $\det(C(z))$ is not equal to zero because no two of the numbers (2.4.48) coincide. It is clear that

$$B^{\sim}(z)C(z) = C(z)J^{\sim}(z) \tag{2.4.49}$$

for $z \in F$. On multiplying both sides of the equation (2.4.49) by $(C(z))^{-1}$ from the left and by $C(z)$ from the right, one obtains the following relation

$$Q^{\sim}(z, \tilde{J(z)})(C(z))^{-1}A(z,0)C(z) = P^{\sim}(z, J^{\sim}(z))) \tag{2.4.50}$$

for each $z \in F$.

Let $z \in F$. By assumption, the polynomial $Q^{\sim}(z, \lambda)$ does not vanish on the set of all the roots of the polynomial $D^{\sim}(z, \lambda)$, therefore the matrix $Q^{\sim}(z, J^{\sim}(z))$ is invertible. Furthermore the matrix $P^{\sim}(z, J^{\sim}(z))$ and the matrix $Q^{\sim}(z, J^{\sim}(z))$ are diagonal, therefore the matrix $(Q^{\sim}(z, J^{\sim}(z)))^{-1}$ is also diagonal. Consequently the matrix

$$(C(z))^{-1}A(z,0)C(z) = (Q^{\sim}(z, J_0(z))^{-1}(P^{\sim}(z, J_0(z))$$

is diagonal. By the assumption on the set of all the roots of the polynomial $D^{\sim}(z, \lambda)$ the map (2.4.46) is injective, therefore, for any $z \in F$, the polynomial (2.4.20) has no multiple roots. Moreover, by construction, each of the elements in the first row of the matrix $C(z)$ is equal to 1. The lemma is proved.

The substitution

$$w = (\eta d_2 + d_1)/(\eta - 1) \tag{2.4.51}$$

transforms the polynomials (2.4.4) – (2.4.6) and the rational function $\dfrac{P^{\sim}(w)}{Q^{\sim}(w)}$, respectively, in

$$D^{\sim}(z, w) = -(\eta-1)^{-6}\ \Delta^3(\Delta+1)^3((\eta\ +\ 1)^3(\eta\ +\gamma_1)^3 - 2^3(1+\ \gamma_1)^3 z\eta^3), \tag{2.4.52}$$

$$P^{\sim}(w) = P_1^{\sim}(\eta) = \Delta^3(\eta - 1)^{-3\Delta}(d_1)^{-3d_1}((\eta\ +\ 1)^3(2\Delta\eta)^{3d_1}, \tag{2.4.53}$$

$$Q^{\sim}(w) = Q_1^{\sim}(\eta) = (\eta - 1)^{-3d_2}(\Delta)^{-3\Delta}\ (2\Delta)^{3d_2}, \tag{2.4.54}$$

and

$$\begin{aligned} P^{\sim}(w)/Q^{\sim}(w) \ &= \ h^{\sim}(\eta) = \\ &= P_1^{\sim}(\eta)/Q_1^{\sim}(\eta) = (\eta - 1)^3(1 - \delta_0)^{-3d_1}(\eta + 1)^3 2^{-6}\ \eta^{3d_1}, \end{aligned} \tag{2.4.55}$$

where $\delta_0 = 1/\Delta$, $\gamma_1 = (1 - \delta_0)/(1 + \delta_0)$. Let

$$D^{\wedge}(z, \eta) = (\eta + 1)^3(\eta + \gamma_1)^3 - 2^3(1 + \gamma_1)^3 z\eta^3. \tag{2.4.56}$$

The substitution (2.4.51) and the inverse substitution

$$\eta = (d_1 + w)/(w - d_2). \tag{2.4.57}$$

relate the roots w of the polynomial $D^{\sim}(z, w)$ with the roots η of the polynomial $D^{\wedge}(z, \eta)$. To be able to make use of Lemmata 2.4.4 and 2.4.5, it is necessary to study the roots of the polynomial (2.4.56). I shall start this investigation at the beginning of the next part of this paper.

§ 5. A list of corrections to [8] and [9]

In the course of preparation of this paper we have found the following misprints and omissions in the "short version" [8].

P. 3, line 13:
read

$$((\delta-\nu)\prod_{l=1}^{d_1}(\nu\ d_1+l)^{-1}(\delta+\nu\ d_1+l))^3 f_k^*(z,\nu)=$$

$$\prod_{l=1}^{\Delta}(\nu\Delta+l)^{-3}(\prod_{l=1}^{d_2}(\delta-1-(\nu+1)d_2+l)^3 f_k^*(z,\nu+1)$$

instead of the formula printed there.

P. 3, line 11 from the bottom:
read "If $\nu\in\mathbb{N},\mathrm{Re}\,(z)>0$..."
instead of "$\nu\in\mathbb{N},\mathrm{Im}\,(z)>0$..."

P. 4, the first line:
Insert the formula

$$P^*(\nu^{-1},w)=P_0^*(\nu^{-1},w),Q^*(\nu^{-1},w)=P_1^*(\nu^{-1},w)$$

just before (2).

P. 4, the second line:
Insert before Lemma 1 "Let
$K_m=(z-1)^{-m}\mathbb{Q}[z]\cap\mathbb{Q}[(z-1)^{-1}],K_m^*=(z-1)^{-m}\mathbb{Q}[z,\nu^{-1}]\cap\mathbb{Q}[(z-1)^{-1},\nu^{-1}]$
where $m\in\mathbb{N}-1$. Clearly, $K_0=\mathbb{Q},K_0^*=\mathbb{Q}[\nu^{-1}]$."

The formulation of Lemma 1 should be changed as follows:

"**Lemma 1.** *Suppose that*
$H^{\sim}\in\mathrm{Mat}_6(K_m),\ H^*\in\mathrm{Mat}_6(K_m^*),\ H=H^{\sim}+\nu^{-1}H^*,\ a_0\in\mathbb{Q},\ a_1\in\mathbb{Q},$
for $m\in\mathbb{N}$. *Then there exists a matrix* $R_i(z,\nu^{-1})\in\nu^{-1}(z-1)^{-1}\mathrm{Mat}_6(K_m^*)$ *such that*
$(\nu^{-1}\delta+a_0+a_1\nu^{-1})HX_k(z,\nu+i)=(H^{\sim}(B^{\sim}(z)+a_0E)+R_i(z,\nu^{-1}))X_k(z,\nu+i),$
where $i=0,1,\ k=1,2,3,5,\ \nu\in\mathbb{N},\ \mathrm{Re}\,(z)>0,\ |z|>1.$"

P. 4, line 10:
In the formulation of Lemma 2, read
"Let $\mathfrak{W}_i=\mathrm{Mat}_6(K_{3\Delta+3i}^*)$ for $i=0,1$. "
instead of
"Let $\mathfrak{W}_i=\mathrm{Mat}_6(\mathbb{Q}[z,\nu^{-1}](z-1)^{-3\Delta-3i}]\cap Q[(z-1)^{-1},\nu^{-1}])(i=0,1)$."

In the first part of this paper [9] on p.0, the title and address page:
read "Trekhsvyatitelskij"
instead of "Ttretjakovskij."

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