GEOMETRIC QUANTIZATION IN THE FRAMEWORK OF ALGEBRAIC LAGRANGIAN GEOMETRY

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Abstract. This is a short version of the author's habilitation thesis. The main results have been published but many details are developed and clarified. As well some new results are included: we additionally discuss here quasi classical limit of ALG(a) - quantization, mention some topological properties of the moduli space of Bohr - Sommerfeld Lagrangian cycles of fixed volume and investigate some properties of the Kaehler structure on it.

Introduction

The main theme of this paper is quantization of the classical mechanical systems in terms of algebraic geometry. Thus here one relates the questions of theoretical physics and mathematics. To start with let us recall briefly the main problems and methods which turn us to study a new subject.

Quantization itself is the main topic of the theoretical physics. Necessity of its introduction and development was dictated by the creators of the quantum theory. According to the Copenhagen philosophy, the physical predictions of a quantum theory must be formulated in terms of classical concepts (the first phrase of [33]; here we quote the beginning of this survey). So in addition to the usual structures (Hilbert space, unitary transformations, selfadjoint operators...) any sensitive quantum theory has to admit an appropriate passage to a classical limit such that the quantum observables are transfered to the classical ones. However as it was pointed out by Dirac at the beginning of the quantum age the correspondence between quantum theory and classical theory has to be based not only on numerical coincidences taking place in the limit $h \to \infty$ but on an analogy between their mathematical structures. Classical theory does approximate the quantum theory but it does do even more — it supplies a frame to some interpretation of the quantum theory. Using this idea we can understand quantization procedure in general as a correspondence between classical theories and quantum theories. In this sense quantization of the classical mechanical systems is the moving in one direction while taking quasi classical limit we go in the opposite direction of this correspondence. More abstractly: the moduli space of the quantum theories is a n - covering of the moduli space of the classical ones (one supposes that n equals 2), and quantization is the structure of this covering.

Quantization at all is a very popular subject. There are a number of different approaches to this problem. But one of them is honored as the first one in theoretical physics and it is named as canonical quantization. In simple cases the

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correspondence comes with some choice of fixed coordinates. If classical observable is represented by a function $f(p_a, q^b)$ in these coordinates then the corresponding quantum observable equals to the operator

$$f(-\imath h\frac{\partial}{\partial q^a}, q^a).$$

The canonical quantization of the harmonic oscillator is a standard of the theoretical physics: any alternative approach should be compared with it and if the answer is sufficiently different from the classical one then this approach is rejected. However this formal substitution (when one puts some differential operators instead of coordinates p_a) introduces a lot of problems. Indeed, beyond of the simple cases during this process the result of the quantization depends on the order of p and q in the expression for the classical observable f and moreover the result strongly depends on the coordinate choice and it is not invariant under generic canonical transformations. Nevertheless this canonical quantization supplied by some physical intuition together with its various generalization takes the central part of the modern theoretical physics.

One way to develop the canonical method and avoid the difficulty is provided by geometric quantization. The geometric quantization (which is discussed in the text) has two slightly different meanings as a term. One could understand this one either as a concrete construction (see f.e. [13], [14], [21], [33], etc) well known as Souriau - Kostant quantization or as a general approach to the problem based on geometry. Nowadays the problem of quantization is used to be solved by quite different methods: algebraic approach includes deformation quantization, formal geometry, non commutative geometry, quantum groups; analytical approach consists of the theory of integral Fourier operators, Toeplitz structures and other ones. All the methods mentioned above have one mutual marking point — using these we almost completely forget about the structure of the given system (and the Dirac suggestion mentioned above) and the "homecoming" turns to be absolutely impossible. At the same time going in the geometric quantization direction one at least tries to keep (at least in mind) the original system. The corresponding symplectic manifold remains to be basic for all the constructions and takes real part in the definition of all auxiliary geometrical objects which give us the result of the quantization. At the same time the starting point of the geometric quantization is to avoid the choice of any coordinates and this basic wish gives a possibility to deal with the complicated systems which do not admit any global coordinates at all. But starting with a given classical phase space geometric quantization should give us a result which has to be comparable with the canonical one for the simple systems. Thus in any case geometric quantization is a generalization of the canonical quantization. To keep the relationship one usually pays a cost loosing generality of the construction: from the whole space of classical observables one takes only a subclass of "quantizable" functions, and this subclass is sufficiently small. To separate such quantizable objects one should choose a polarization of given symplectic manifold (see [20], [33]) then these objects are distinguished by the condition that their Hamiltonian vector fields preserve the polarization. The separation necessity is dictated by a classical von Hove result (see f.e. [13]) which ensures us that it is impossible to realize so called "ideal quantization" over given manifold in terms of usual geometry of this manifold (functions, differential operators etc). Let us recall that one understands such ideal quantization of classical mechanical system corresponding to given symplectic manifold (M, ω) as a procedure giving us some appropriate Hilbert space $\mathcal{H}$ together with an irreducible representation

$$q : C^{\infty}(M, \mathbb{R}) \to O(\mathcal{H})$$

(where the first space one takes as a Lie algebra endowed with the Poisson bracket and the second one is the space of self adjoint operators on $\mathcal{H}$) such that it satisfies a correspondence principle, namely

$$i\hbar[q(f), q(g)] = q(\{f, g\})$$

for every f, g (and, of course, one is interested in the cases when the operators on the right hand side satisfy a number of properties which makes the corresponding quantum mechanical system meaningful and "computable") so q has to be a representation of the Poisson algebra. The requirement for q to be irreducible led to the separation mentioned above of subclasses of "quantizable functions" in the "classical" construction which is known as Souriau - Kostant geometric quantization. As the result of this construction one gets some representation of whole space $C^{\infty}(M, \mathbb{R})$ which is manifestly reducible being the direct sum of two isomorphic representations. This imposes consideration and quantization of a half part of the classical observable space.

The known schemes of geometric quantization are unified by the fact that usually they takes the space of regular sections of a prequantization bundle as the Hilbert space (and again one imposes some additional conditions on these sections to be regular in our sense). In original Souriau - Kostant construction one takes all smooth sections with bounded L^2 - norm (with respect to a given hermitian structure on the fibers of the prequantization bundle weighted by the Liouville form). Further specializations come in different ways: Rawnsley - Berezin method (see [16]) uses only the sections which are holomorphic with respect to a complex polarization (= fixed complex structure on M) as well as in Toeplitz - Berezin approach (see [4]) while in the real polarization case one collects only such sections (weighted by half weights) which are invariant with respect to infinitesimal transformations tangent to the fibers of a real polarization.

One should say that the introduction of an additional structure — complex polarization — turns the subject of geometric quantization to the most developed region of modern mathematics namely to algebraic geometry. As it was mentioned above a number of methods uses complex polarization. It imposes the additional condition that our symplectic manifold (M, ω) admits a Kaehler structure: there exists some complex structure J compatible with ω which is integrable. Together these two structures ω, J give us the corresponding riemannian metric g such that complex manifold M, J is endowed with a hermitian metric, and since ω is closed by the definition it gives us a Kaehler structure on M. Moreover, one has usual for any quantization method requirement for ω to have integer cohomology class:

$$[\omega] \in H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R})$$

(the charge integrality condition). This implies that the Kaehler metric described above is of the Hodge type and therefore the Kaehler manifold is an algebraic variety. So one can quantize a symplectic manifold if it admits the algebraic geometrical structure! It is not so surprising if we take in mind so called geometric formulation of quantum mechanics. The basic idea is to replace the algebraic methods of quantum mechanics by algebro geometrical methods. The author found all these ones in [3], [18] but of course the original sources exist as one thinks since the birth of the quantum theory itself. Anyway [18] contains the history of the question. Roughly speaking, the starting point is that usually in quantum mechanics one deals with a Hilbert space but the quantum states are represented by rays in the space since two vectors ψ_1, ψ_2 represent the same state iff they are proportional. Thus it is natural to consider the projectivization $\mathbb{P}(\mathcal{H})$ instead of $\mathcal{H}$ as the space of quantum states. This finite - or infinite - dimensional complex manifold is automatically endowed with a hermitian metric (Fubini - Study) so one can regard it as a real manifold with Kaehler structure. This real manifold (finite or infinite dimensional) is endowed automatically with symplectic structure and riemannian metric. Quantum states are represented just by points of this manifold. Quantum observables are represented by smooth real functions of special type which one should call Berezin symbols. The specialty means that the desired functions are distinguished by the following condition:

function f is a Berezin symbol iff its Hamiltonian vector field preserves the riemannian metric as well as the symplectic form.

Further, instead of the commutator of two operators one has just the usual Poisson bracket for the corresponding symbols. Eigenstate means critical point for the symbols, eigenvalue equals to the critical value. Dynamics of the quantum system is described in terms of the classical Hamiltonian dynamics. Probability amplitudes are given by the geodesic distances so the riemannian metric on the space responds to the probability aspects of the theory([3], [18]).

Applying these ideas to geometric quantization let us reformulate and slightly generalize the main subject. First of all we give the following

Definition. *Let $\mathcal{K}$ is a Kaehler manifold endowed with a Kaehler triple (Ω, I, G). Then real function f is called quasi symbol iff its Hamiltonian vector field preserve the riemannian metric:*

$$Lie_{X_f} G = 0.$$

For any Kaehler manifold $\mathcal{K}$ the set of quasi symbols is a Lie subalgebra in the Poisson algebra. Let us denote it as $C^\infty_q(\mathcal{K}, \mathbb{R})$. If $\mathcal{K}$ is a projective space then the notion is equivalent to the notion of Berezin symbol.

Now we are ready to formulate what we will call

Algebro geometric quantization (AGQ). *Let (M, ω) be a symplectic manifold corresponding to a classical mechanical system. Then one says that AGQ results with an appropriate algebraic variety $\mathcal{K}$ together with a correspondence*

$$q : C^\infty(M, \mathbb{R}) \to C^\infty_q(\mathcal{K}, \mathbb{R}),$$

satisfying the following conditions:

1) $q(f+g) = q(f) + q(g)$;
2) $q(cf) = cq(f)$ where c is a real constant;
3) $q(c) = c$ where c is a real constant;
4) q is irreducible;
5) the correspondence principle in the form

$$\{q(f), q(g)\}_\Omega = q(\{f,g\}_\omega)$$

holds for every functions f, g.

In this list we need to explain what the irreducibility (item 4) means. One knows what does it mean for usual linear representations. But since we start with the case of the projective spaces one sees that during the projectivization the usual irreducibility condition is translated to the following double:

4a) q has trivial kernel;

4b) the distribution $Vect_\Omega(Imq)$ over $\mathcal{K}$ either is non integrable or spans whole the tangent bundle of $\mathcal{K}$,

where $Vect_\Omega(Imq) \subset Vect(\mathcal{K})$ is the Lie subalgebra which consists of the Hamiltonian vector fields of the functions from the image of q. We will show in Section 1 that these two conditions form an analogy of the usual "linear" irreducibility condition.

Thus the statement of the quantization problem is slightly changed. For a given symplectic manifold we are looking for an appropriate algebraic manifold (variety) together with an inclusion q. Following the authors of [3] we require for the construction of this algebraic manifold to avoid as an intermideate step the introduction of Hilbert spaces known from usual methods of geometric quantization.

The main aim of this text is to present an example of succesfull algebro geometric quantization for compact simply connected symplectic manifolds. We call this method ALG(a) - quantization. Of course, it is an abbreviation. To decode this one we need to recall some basic facts belonging to a new subject which was invented just on the border between algebraic and symplectic geometries (if this border does exist).

One could say that different subjects are mixed in modern mathematics. For example, in connection with the mirror symmetry conjecture one accepts the idea that algebraic geometry of a manifold X corresponds to symplectic geometry of its mirror partner X'. The ingredients of algberiac geometry over X (bundles, sheaves, divisors ...) are compared with some derivations of symplectic geometry (Lagrangian submanifolds of special types). F.e. in so called homological mirror symmetry one compares two categories came from algebraic geometry and symplectic geometry respectively and in some particular cases (elliptic curve) this approach gives the desired result. On the other hand one has a number of moduli spaces generated in the framework of algebraic geometry over X and another way is to find a number of moduli spaces in the framework of symplectic geometry. The development of this idea comes in different ways and even now one could report about a number of promised results and ideas clarifying the original one (see, f.e., [22], [24]). But these results are sufficiently far to be complete and to cover all the problems. But the main idea which proclaims to create some new synthetic (or at least synergetic) geometry unifying algebraic and symplectic ones remains to be very attractive and seems to be true.

One step in this way was done in 1999 when the moduli space of half weighted Bohr - Sommerfeld Lagrangian cycles of fixed volume and topological type was proposed in [25] and constructed in [10]. Starting with a simply connected compact symplectic manifold with integer symplectic form (read "classical mechanical system with compact simply connected phase space which satisfies the Dirac condition") the authors construct a set of infinite dimensional moduli spaces which are infinite dimensional algebraic manifolds in dependence on the choice of some topological fixing and a real number — the volume of the half weighted cycles. Lagrangian geometry is mixed in the construction with algebraic geometry and this construction itself belongs to some new synthetic geometry. The authors called it ALAG — abelian Lagrangian algebraic geometry (so it is wrong to think that they took their initials and made a mistake). It was created as a step in the approach to mirror symmetry conjecture generalizing some notions from standard geometric quantization (prequantization data, Bohr - Sommerfeld condition, etc.) so it is not quite surprising that this construction should play an important role in geometric quantization. Namely, it was shown in [28] and [30] that these moduli spaces of half weighted Bohr - Sommerfeld Lagrangian subcycles of fixed volume solve the problem of algebro geometric quantization stated above for simply connected compact symplectic manifolds. Briefly, if (M, ω) is the symplectic manifold with integer symplectic form then one can construct the moduli space $\mathcal{B}_S^{hw,r}$ of fixed volume r consists of pairs (S, θ) where S is a Bohr - Sommerfeld Lagrangian cycle and θ is a half weight over it such that the volume $\int_S \theta^2$ equals to r (see Section 2). For every smooth function $f \in C^\infty(M, \mathbb{R})$ (= classical observable) one has a smooth function F_f on the moduli space which is given by the map

$$\mathcal{F}_\tau : C^\infty(M, \mathbb{R}) \to C^\infty(\mathcal{B}_S^{hw,r}, \mathbb{R}),$$
$$\mathcal{F}_\tau(f)(S, \theta) = F_f(S, \theta) = \tau \cdot \int_S f|_S \theta^2.$$

It's easy to see that $\mathcal{F}_\tau$ maps constants to constants:

$$f \equiv s \implies F_f \equiv \tau \cdot r \cdot c.$$

Moreover, we prove that this $\mathcal{F}_\tau$ is a homomorphism of Lie algebras namely

$$\{F_f, F_g\}_\Omega = 2\tau F_{\{f,g\}_\omega}$$

for any $f, g \in C^\infty(M, \mathbb{R})$ (we prove it in Section 2 by direct computations). Moreover, in Section 3 we prove that for every $f \in C^\infty(M, \mathbb{R})$ the corresponding function F_f is a quasi symbol (= quantum observable). Moreover, we prove that $\mathcal{F}_\tau$ is irreducible.

Thus, summing up all the facts one ensures that the pair $(\mathcal{B}_S^{hw,r}, \mathcal{F}_\tau)$ satisfies all conditions of algebro geometric quantization if one takes

$$\tau = \tfrac{1}{2},$$
$$r = 2.$$

So we have that

$$\mathcal{K} = \mathcal{B}_S^{hw,2},$$
$$q = \mathcal{F}_{\frac{1}{2}}$$

is a solution for AGQ problem.

This method, proved in [28], [30], was called ALG(a) - quantization. This new method gives new results which are nevertheless quite consistent with the old ones if an appropriate polarization on (M, ω) is chosen (see [30]). In Section 4 we show that if (M, ω) admits a real polarization then one can derive the result of the standard geometric quantization described f.e. in [20] from the result of ALG(a) - quantization. The same is true for the complex case. It means that ALG(a) - quantization is a natural generalization of the known methods of geometric quantization.

At the same time we can change our parameters r and τ. Usually in geometric quantization one has an additional integer parameter: if (L, a) is the prequantization line bundle with prequantization $U(1)$ - connection deriving the Bohr - Sommerfeld subcycles from the space of all Lagrangian subcycles then one can take its tensor power (L^k, a_k) in dependence with integer parameter k. Then one understands Planck constant in this construction, following F. Berezin, as

$$\hbar = \frac{1}{k},$$

and it is not hard to see that while h tends to zero one gets a dense subset of the space of all Lagrangian subcycles which consists of Bohr - Sommerfeld subcycles of any levels. The point is that dealing with the parameters one gets the following limit: any Lagrangian subcycle plus zero half weight. Hence the space of the quasi classical states is the space of all unweighted (or unhalfweighted) Lagrangian subcycles of fixed topological type. We show that this space is endowed with say canonical dynamical properties while it hasn't any canonical riemannian metric to define the corresponding probability aspects. Really if we choose a Hamiltonian function H on the given symplectic manifold then we automatically get an induced vector field on the space of all Lagrangian subcycles. It means that we have dynamics on the space of Lagrangian subcycles which is induced by the classical dynamics of the given system and is compatible with it. At this point the story comes beyond of usual geometry. Namely the thing which is generated by a smooth function on the given manifold is not just a vector field on the space of Lagrangian submanifolds. If f is constant on a Lagrangian submanifold S then it defines a number which is evedentely equals to this constant. This means that every f generates a "thing" which is neither vector field no function but it belongs to another geometry. Usually it is called super geometry. The "thing" should be called super function but of some special type. If $\mathcal{L}_S$ is the space of all Lagrangian subcycles of fixed topological type then one takes odd supersymplectic manifold $\Pi T^* \mathcal{L}_S$. Every vector field over $\mathcal{L}_S$ now can be regarded as an odd super function on $\Pi T^* \mathcal{L}_S$ so it has parity 1 while any numerical function has parity 0 as a super function. "Super selection rules" usually forbid to consider nonhomogineous super functions with different parity in different points but nevertheless we've got such special nonhomogineous super functions. So the "thing" induced by any real function f is a vector field in Lagrangian subcycles if it's not constant at these cycles and it is a number function on the cycles which belong to the level set of this function. Of course, these super functions have very special type but nevertheless the fact takes place and the further investigation of the subject should keep this way as well as the other possible ways. While the mentioned observation belongs to odd supersymplectic geometry we have another point enforcing one to consider even super symplectic case. One should notice that the definition of Bohr - Sommerfeld Lagrangian cycles has very natural and simple reformulation on the language of even super symplectic manifolds. For this let us translate to the language the usual setup for geometric quantization. One has a real symplectic manifold (M, ω) and a prequantization bundle (L, a) together with the corresponding prequantization $U(1)$ - connection. Then one could consider this picture (see [17]) as follows: we have an even super symplectic manifold. At each point of the total space

$$E = L \to M$$

one has a decomposition of the tangent bundle TE with respect to connection a and the super symplectic form equals to usual ω being restricted to the horizontal part and to the real part of the hermitian metric at the vertical part. The fact that a is hermitian implies that the Jacobi identity holds for the induced super Poisson bracket (Butten bracket). After this setup is understood one can take Lagrangian submanifolds of the even super symplectic manifold E having in mind the usual sense: S is Lagrangian if it is isotropic with respect to the super symplectic form and has maximal dimension. The point is that such Lagrangian submanifolds project exactly to our Bohr - Sommerfeld submanifolds. And the quantization condition belongs exactly to the super geometry. It gives a hint how the abelian case could be extended to non abelian cases.

The text is organazied as follows.

The first section briefly recalls and discusses two sources of the new method: the geometric formulation of quantum mechanics and geometric quantization itself. We start with the first one, almost copying appropriate parts of [3] so we would like to strongly recommend this paper as complete and perfect. Further we touch known methods of geometric quantization; at the same time we add some realizations of the methods in terms of the geometric formulation of quantum mechanics.

In Section 2 we introduce a natural map from the space of smooth functions on a given simply connected compact symplectic manifold to the space of smooth functions over the corresponding moduli space of half weighted Bohr - Sommerfeld Lagrangian cycles and prove that this map is a homomorphism of the Poisson algebras. To do this first of all we recall the construction of the moduli spaces (and again we recommend to use the original source [10] or its preliminary version [9]), describing the local geometry of the moduli space needed to the further computation. Then we define the map

$$\mathcal{F}_\tau : C^\infty(M, \mathbb{R}) \to C^\infty(\mathcal{B}_S^{hw,r}, \mathbb{R})$$

and prove by the direct computation that this map preserves both the Poisson structures. First time it was done in preprint [7] and then published as [28].

In Section 3 we present the main technical result: we prove that the image of $\mathcal{F}_\tau$ consists of quasi symbols. The proof is based on what we call "dynamical correspondence". This means briefly speaking that for every $f \in C^\infty(M, \mathbb{R})$ the differential of $\mathcal{F}_\tau$ maps the Hamiltonian vector field X_f to the Hamiltonian vector field X_{F_f} (up to a constant). It means that the correspondence is "dynamical": it is compatible with classical dynamics of the system. The results of this section were established in preprint [29] and published as [30].

In Section 4 we apply the mathematical results to introduce ALG(a) - quantization of classical mechanical systems. As the first two steps we show the reduction of this method in presence of either complex or real polarization. Technically it requires the consideration of critical points of the quasi symbols. Further we discuss the corresponding quasi classical limit.

In the last small section we add some observations linking ALG(a) - quantization with super geometry. This section is the smallest one but it looks like quite profitable in a future. While the previous sections have been finished (moreless) the last one consists of remarks and observations which could be exploited in the further investigations.

Asknolegements. The author's work on this theme was started in 1999 at Max - Planck - Institute for Mathematics (Bonn) when, just by a chance, two preprints ([3] and [9]) were collected by him. The main results of the text were established in MPI (Bonn), Korean Institute of Advanced Study (Seoul) and Joint Institute for Nuclear Research (Dubna) so first of all the author would like to thank all the people from the institutions for the hospitality, the friendel attention and very good working conditions. Personally I would like to thank Prof. I.R. Shafarevich as the leader of the scientific Al school (to which the author belongs) and as an attentive listener whose remarks were extremely important and useful for the work. I would like to thank Prof. A.G. Sergeev and all the participants of his seminar in Steklov Mathematical Institute (Moscow) for the constant attention to the work during these years and valuable discussions and remarks. I would like to thank Prof. A.S. Tikhomirov for the interest to the subject and a number of important remarks. I would like to cordially thank Professors Vik. Kulikov, A. Gorodentsev, V. Pidstrigach, B. Kim, S. Kuleshov, D. Orlov, B. Karpov, A. Isaev, and especially H. Khudaverdyan and P. Pyatov. Additionally I thank my family for the hard assistance which I get all the time. Thus the double appreciation (and the first one) should be expressed to Prof. A.N. Tyurin.

Quantum states of a quantum mechanical system are represented by rays in the corresponding Hilbert space (the space of wave functions, see f.e. [1]). The space of the rays is a projective space endowed with a Kaehler structure. Turning to the projective geometry language one gets the postulates of quantum mechanics in pure geometrical terms. Applications of this idea in quantization problem are quite well known. Here we follow [3], [18] as modern and complete texts on the subject. The aim is to formulate the quantum mechanics postulates without references on Hilbert spaces and linear structures. The basic notions of symplectic geometry can be found f.e. in [2]. Further in this section we recall three main constructions in geometric quantization together with basic notions of it. We begin with the definition of the prequantization data and then remind the basic Souriau - Kostant construction. It needs an additional specification since it gives a reducible representation of the Poisson algebra. This specification comes with an additional choice of an appropriate polarization (if the given manifold admits this choice). We list then two "extremal" cases when it admits complex or real polarization. Recalling these known constructions we place some additional colors coming when one takes in mind the geometric reformulation.

1.1. Hilbert space as a Kaehler manifold.

Let us consider a Hilbert space $\mathcal{H}$ as a real space endowed with a complex structure J. It is a real linear operator such that

$$J^2 = -id. \tag{1.1.1}$$

At the same time the hermitian pairing can be decomposed on the real and imaginary parts:

$$<\Phi, \Psi> = \frac{1}{2h} G(\Phi, \Psi) + \frac{i}{2h} \Omega(\Phi, \Psi). \tag{1.1.2}$$

The standard properties of hermitian products imply that G is a symmetric positive pairing while Ω is skew symmetric and both the pairing are related by the complex operator J. The compatibility condition reads as

$$G(\Phi, \Psi) = \Omega(\Phi, J\Psi) \tag{1.1.3}$$

thus one has a Kaehler structure on the real space $\mathcal{H}$. Therefore any Hilbert space can be described as a real space endowed with a Kaehler triple (G, J, Ω). So this space is simulteniously a symplectic space and a complex space. As a symplectic space it has the corresponding Poisson bracket for the functions. In classical mechanics the observables are smooth real functions which are defined (up to constant) by the Hamiltonian vector fields (we are dealing with connected manifolds only). In the quantum case one could represent the observables by vector fields as well: for any linear self adjoint operator $\hat{F}$ (a quantum observable) let us call the vector field

$$Y_{\hat{F}}(\Psi) = \frac{1}{h} J\hat{F}\Psi \tag{1.1.4}$$

as Schroedinger. The point is that since in the real setup the Schroedinger equation reads as

$$\dot{\Psi} = -\frac{1}{h} J\hat{H}\Psi \tag{1.1.5}$$

for a Hamiltonian operator $\mathcal{H}$ then it can be shortly rewritten due to the given notation. It's not hard to establish some appropriate properties of the Schroedinger fields. First, we know that quantum observable $\hat{F}$ generates one - parameter family of unitary transformations of the Hilbert space. Since $Y_{\hat{F}}$ is the generator of the family this implies that the Schroedinger vector field preserves all three structures which the Kaehler triple comprises. Therefore field $Y_{\hat{F}}$ is locally Hamiltonian and taking in mind linearity of $\mathcal{H}$ it is globally Hamiltonian as well. The corresponding function is very well known in the standard quantum mechanics. It is the expectation value given by the formula

$$F(\Psi) = <\Psi, \hat{F}\Psi> \tag{1.1.6}$$

for operator $\hat{F}$. Then if η is a tangent vector at point Ψ we have

$$\begin{aligned} dF_\eta(\Psi) &= \frac{d}{dt} < \Psi + t\eta; \hat{F}(\Psi + t\eta) > |_{t=0} \\ &= <\Psi, \hat{F}\eta> + <\eta, \hat{F}\Psi> = \frac{1}{h} G(\hat{F}\Psi, \eta) \\ &= \Omega(Y_{\hat{F}}(\Psi), \eta) = \imath_{Y_{\hat{F}}(\Psi)}\Omega(\eta), \end{aligned} \tag{1.1.7}$$

using self duality of $\hat{F}$ together with the definitions of G, Ω and $Y_{\hat{F}}$. This gives us a well known fact that the evolution in time of any quantum mechanical system can be described in terms of Hamiltonian mechanics; the corresponding Hamiltonian function equals to the expectation value of the Hamiltonian operator.

Second, let $\hat{F}, \hat{K}$ be two quantum observables with expectation values F, K. A short computation gives

$$\{F, K\}_\Omega = \Omega(X_F, X_K) = < \frac{1}{ih}[\hat{F}, \hat{K}] >, \tag{1.1.8}$$

where we use brackets $< \quad >$ for the expectation value (1.1.6). This means that the correspondence

$$\hat{F} \mapsto F \tag{1.1.9}$$

is a homomorphism of Lie algebras. Its kernel is trivial thus it is an isomorphism of the Lie algebra of self adjoint operators and a subalgebra of the Poisson algebra over the Kaehler space. The condition on real functions distinguishing this subalgebra can be derived evidently what we'll do in the next subsection.

Third, one has an additional ingredient in the picture — the riemannian metric G. It defines a real pairing of Hamiltonian vector fields which corresponds to the Jordan product in the standard quantum mechanics

$$\{F, K\}_+ = \frac{h}{2} G(X_F, X_K) =< \tfrac{1}{2}[\hat{F}, \hat{K}] > . \tag{1.1.10}$$

One could take the first equality in (1.1.10) as the definition of the riemannian bracket for F and K. This riemannian bracket has not any classical analog being an ingredient of pure quantum theory. F.e., it could be exploited to define what one calls "uncertainty" of a quantum observable in a quantum state with unit norm namely

$$(\Delta \hat{F})^2 =< \hat{F}^2 > - < \hat{F} >^2 = \{F, F\}_+ - F^2. \tag{1.1.11}$$

And the famous Heisenberg uncertainty relation looks like

$$(\Delta \hat{F})^2 (\Delta \hat{K})^2 \geq (\frac{h}{2} \{F, K\}_\Omega)^2 + (\{F, K\}_+ - FK)^2 \tag{1.1.12}$$

in these terms.

1.2. The projectivization.

As it was already mentioned the real space of quantum states is represented by rays in the corresponding Hilbert space. Collecting the rays one get the projective space $\mathcal{P}$ which is a Kaehler manifold itself. The procedure giving us the desired structure over $\mathcal{P}$ is usually called either "the Dirac theory of constraints" or "the Kaehler reduction" (it depends on the context). One has the natural $U(1)$ - action (phase rotations) on $\mathcal{H}$, preserving all the structures. As well it preserves the function

$$C(\Psi) =< \Psi, \Psi > -1, \tag{1.2.1}$$

which is usually called either "the Dirac constraint function" or "momentum map" since it is an equivariant map acting from the Hilbert space to the Lie co - algebra of $U(1)$. After this is understood we just apply the standard procedure of Kaehler reduction getting as the result the projective space endowed with the corresponding Kaehler structure. In other words since the time evolution preserves the levels of the constraint function we get a first constraint system; f.e., unit sphere $S \in \mathcal{H}$ (the level $C = 0$) is preserved by the motion generated by C itself:

$$L_{X_C} C = \{C, C\}_\Omega = 0. \tag{1.2.2}$$

For a first constraint system there is a gauge freedom and the corresponding gauge transformations are generated by the Hamiltonian flow of function C. In our case these gauge directions is defined by the Schroedinger vector field

$$X_C(\Psi) = -\frac{1}{h} J\Psi. \tag{1.2.3}$$

Thus the vector field

$$\mathcal{J}_C = h X_C|_S \tag{1.2.4}$$

generates the phase rotations. Factorizing by the phase rotations one gets the real space of quantum states (sometimes called as "reduced phase space"). To emphasize its physical role and geometrical structure we would like to call it quantum phase space.

To define the symplectic structure on $\mathcal{P}$ consider the pair of maps

$$\begin{aligned} \imath &: S \to \mathcal{H}, \\ \pi &: S \to \mathcal{P}, \end{aligned} \tag{1.2.5}$$

where the first one is the inclusion and the second — the projection. On the unit sphere S we have closed 2- form $\imath^*\Omega$ which degenerates exactly in the rotation directions. Moreover this 2 - form is constant along the directions (since these directions are given by the Hamiltonian vector field of the constraint function). Therefore there exists a closed non degenerated 2 - form $\pi_*\imath^*\Omega$ on $\mathcal{P}$ (and moreover it is symplectic in strong sense for the infinite dimensional case). Let us remark that we've described nothing but the mechanism of the symplectic reduction valid in the infinite dimensional case (see f.e. [8]).

Now what happens with quantum observables during this procedure? If $\hat{F}$ is a bounded self adjoint operator then the corresponding expectation value is a good smooth function being restricted to the unit sphere. First, the restriction $\imath^*F$ is evedentely invariant under the gauge transformations therefore there exists a smooth function f on $\mathcal{P}$ such that

$$\pi^* f = \imath^* F. \tag{1.2.6}$$

Second, the relation between F and f is "dynamical" — Hamiltonian vector field X_F projects exactly to Hamiltonian vector field X_f. We'll come back to this dynamical correspondence later on, presenting now the following

Definition. *Let $f : \mathcal{P} \to \mathbb{R}$ be a smooth real function on the projectivization of a Hilbert space $\mathcal{H}$. Then we call it a symbol iff there exists a bounded self - adjoint operator $\hat{F}$ on $\mathcal{H}$ such that $f = \pi_*\imath^* < \hat{F} >$.*

Remark. This notion was introduced by F. Berezin so we'd like to follow him in the terminology.

Now coming back to the Hamiltonian vector fields let us note that any $\hat{F}$ commutes with Id thus any expectation value F commutes with the constraint function so

$$L_{X_C} F = 0. \tag{1.2.7}$$

Therefore X_F is constant along the integral curves of $\mathcal{J}$ and one can push it down to $\mathcal{P}$, getting exactly X_f. Going further let's take the Poisson bracket induced by the reduced symplectic form Ω_p. For any expectation values $F, K : \mathcal{H} \to \mathbb{R}$ one has

$$\begin{aligned} \pi^*\{f,k\}_{\Omega_p} &= \pi^*(\Omega_p(X_f, X_k)) \\ = \Omega_p(\pi_* X_F, \pi_* X_K) &= \Omega(X_F, X_K)|_S = \{F,K\}|_S \end{aligned} \tag{1.2.8}$$

by the projection formula where $f, k : \mathcal{P} \to \mathbb{R}$ are the corresponding reduced functions.

Thus for any quantum observable there exists the corresponding symbol. Moreover, the correspondence is one - to - one being an isomorphism of Lie algebras. At the same time we call this correspondence "dynamical" because it transfers the Schroedinger dynamics exactly to the Hamiltonian dynamics on the reduced quantum phase space.

1.3. Riemannian geometry and the measurement.

As we have seen above the quantum phase space carries an additional structure which is unusual in classical mechanics. The riemannian metric responds to the probability aspects of the quantum theory.

The riemannian metric over $\mathcal{P}$ is given during the same Kaehler reduction procedure. The restriction $\imath^* G$ to S is a non degenerate riemannian metric. Since every Schroedinger vector field preserves whole hermitian structure then $\mathcal{J}$ is a Killing vector field for G over S:

$$L_{\mathcal{J}}(\imath^* G) = 0. \tag{1.3.1}$$

Thus $\imath^* G$ is constant along the integral curves of $\mathcal{J}$ being non degenerated. Taking

$$\tilde{G}_p = (G - \frac{1}{2h}(\Psi \otimes \Psi + \mathcal{J} \otimes \mathcal{J}))|_S \tag{1.3.2}$$

we get a symmetric tensor equals to $\imath^* G$ on the transversal to $\mathcal{J}$ subspaces and which degenerates exactly in the directions of $\mathcal{J}$. Therefore $\tilde{G}_p$ can be factorized by the phase transformations which gives us the corresponding riemannian metric over $\mathcal{P}$. Now we get the Kaehler structure over $\mathcal{P}$. The question is how one can recognize the symbols over $\mathcal{P}$? One has the following

Proposition 1.1 ([3]). *Function $f : \mathcal{P} \to \mathbb{R}$ is a symbol iff its Hamiltonian vector field preserves the riemannian metric:*

$$L_{X_f} G = 0 \tag{1.3.3}$$

(thus at the same time it is a Killing vector field).

This means that as in classical dynamics the space of observables in the quantum theory consists of the smooth functions whose Hamiltonian vector fields give the infinitesimal symmetry of the given structure. At the same time one sees that in contrast with the classical case the space of quantum observables is small: f.e., in finite dimension one gets a finite dimensional space of quantum observables. Necessity of (1.3.3) has been explained so it remains to construct the corresponding operator on $\mathcal{H}$ for every symbol f on $\mathcal{P}$. The correspondence comes with Berezin - Rawnsley quantization scheme (see [16]) namely let's take holomorphic line bundle $\mathcal{O}(1)$ over $\mathcal{P}$. The space of its holomorphic sections is dual to $\mathcal{H}$. Taking any appropriate hermitian structure on $\mathcal{O}(1)$, compatible with the given riemannian metric, one gets:

a self adjoint operator Q_f as the result of Berezin - Rawnsley quantization;

an isomorphism between the dual spaces.

Using this isomorphism one transfers Q_f from $H^0(\mathcal{O}(1), \mathcal{P})$ to $\mathcal{H}$ getting what one needs. It's not hard to see that the result doesn't depend on the choice of the hermitian structure. We'll recall the Berezin - Ransley quantization method further in subsection 1.5.

The notion of uncertainty can be transfered in natural way to the projective space. Let's define riemannian bracket of the form

$$(f,k) = \frac{h}{2} G_p(X_f, X_k). \tag{1.3.4}$$

Since these f,k correspond to expectation values $F,K : \mathcal{H} \to \mathbb{R}$ then (see (1.3.2))

$$\begin{aligned}\{F,K\}_+ = \frac{h}{2} G(X_F, X_K) = \frac{h}{2} \tilde{G}_p(X_F, X_K) + \\ \frac{1}{4} G(\mathcal{J}, X_F) G(\mathcal{J}, X_K) = \pi^*((f,k) + fk).\end{aligned} \tag{1.3.5}$$

Therefore the symbol which corresponds to the Jordan product of $\hat{F}$ and $\hat{K}$ is

$$\{f,k\}_+ = (f,k) + fk. \tag{1.3.6}$$

It's reasonable to call this expression as "symmetric bracket" of f and k. Note that the riemannian bracket of two symbols is not in general a symbol but it carries a natural physical sense: it is exactly the function of quantum covariance. In particular, $(f,f)(p)$ is the square of the uncertainty of observable f at state p:

$$(\Delta f)^2(p) = (\Delta \hat{F})^2(\pi^{-1} p) = (f,f)(p). \tag{1.3.7}$$

Uncertainty relation then reads as

$$(\Delta f)(\Delta k) \geq (\frac{h}{2}\{f,k\}_{\Omega_p})^2 + (f,k)^2. \tag{1.3.8}$$

One gets its standard form

$$(\Delta f)(\Delta k) \geq (\frac{h}{2}\{f,k\}_{\Omega_p})^2, \tag{1.3.9}$$

at the states where the Hamiltonian vector fields X_f, X_k are conjugated up to sign by the complex structure on $\mathcal{P}$. Therefore this quantum covariance $(f,k)(p)$ measures the "coherence" of state p with respect to observables f and k.

Consider the probabilistic aspects. Let Ψ_0 be a vector with unit norm in the original Hilbert space $\mathcal{H}$ and $p_0 = \pi(\Psi) \in \mathcal{P}$. Then one defines a function d_Ψ on S in natural way:

$$d_{\Psi_0}(\Psi) = |<\Psi_0, \Psi>|^2 \tag{1.3.10}$$

for every Ψ with unit norm. Since d_{Ψ_0} doesn't depend on the gauge transformations it can be push down to $\mathcal{P}$ which gives us a function d_{p_0} on the projective space defined by

$$d_{p_0}(p) = d_{\Psi_0}(\pi^1(p)). \tag{1.3.11}$$

If the quantum mechanical system stays at the state p_0 then the probabilistic distribution is given by d_{p_0}. It's well known that this function has an expression in terms of geodesic lines, namely

Proposition 1.2 ([3]). *For any $p_0, p \in \mathcal{P}$ there exists a closed geodesic line passing from p_0 to p such that*

$$d_{p_0}(p) = cos^2(\frac{1}{\sqrt{2h}}\sigma(p_0, p)), \tag{1.3.12}$$

where σ is the corresponding geodesic distance.

To prove it in the finite dimensional case it's sufficient to recall what the Fubini - Study metric on a projective space is (see f.e. [11]). To reduce the infinite dimensional case to the previous one it's sufficient to note that for any points $p_0, p \in \mathcal{P}$ there exists a projective plane $\mathbb{P}^2 \subset \mathcal{P}$ which contains both of them. Then it's clear that the desired geodesic line in any case lies in this $\mathbb{P}^2$.

The translation of all the measurement aspects to the projective language needs two more steps. First,

Proposition 1.3 ([3]). *Vector Ψ is an eigenvector of quantum observable $\hat{F}$ with eigenvalue λ iff the corresponding point $p = \pi(\Psi)$ is a critical one of the corresponding symbol f with critical value λ.*

The proof is evident. Second, let $\hat{F}$ be a quantum observable with an arbitrary spectrum. To define the spectrum of the corresponding symbol f we use (1.3.4) and (1.3.6), getting the following

Definition. *Spectrum $sp(f)$ of symbol f consists of real numbers such that the function*

$$n_\lambda : \mathcal{P} \to \mathbb{R} \cup \{\infty\},$$
$$n_\lambda : p \mapsto ((\Delta f)^2(p) + (f(p) - \lambda)^2)^{-1}$$

is not bounded.

The point λ where $n_\lambda = \infty$ is a critical value of f. Now turn to the spectral projectors. If Λ is a continuous component of $sp(f)$ and $P_{\hat{F},\Lambda}$ is the projector which corresponds to $\hat{F}$ and Λ then let's denote as $\mathcal{E}_{f,\Lambda}$ the following projectivization

$$\mathcal{E}_{f,\Lambda} = \mathbb{P}(ImP_{\hat{F},\Lambda}) \subset \mathcal{P}.$$

This submanifold corresponds to the critical set with critical volume 1 of the symbol, induced by projector $P_{\hat{F},\Lambda}$ (notice that every such projector is a bounded self adjoint operator). With submanifold $\mathcal{E}_{f,\Lambda}$ we can take the geodesic projection instead of the spectral projector. In this terms projector $P_{f,\Lambda}$ maps point $p \in \mathcal{P}$ to the point of $\mathcal{E}_{f,\Lambda}$ which is the nearest one with respect to the geodesic distances.

1.4. Postulates of quantum mechanics.

Summing up we get the following picture. One has a projective space $\mathcal{P}$ which is a Kaehler manifold equipped with the corresponding Kaehler structure. Thus $\mathcal{P}$ has a fixed symplectic structure which defines the corresponding Lie algebra structure on the function space and governs the evolution of the system. But there are two major differences between our picture and the classical mechanics case. First, the quantum phase space has very special nature as a Kaehler manifold being a projective space (we will speak about possible generalizations later). Second, as a Kaehler manifold, it is equipped with a riemannian metric and this last one governs the measurement process. This ingredient was absent in the classical theory — in quantum theory it responds for such notions as uncertainty, state reduction and so on.

We list below short vocabulary just to summarize the translation.

Physical states. Physical states of quantum system correspond to point of an appropriate Kaehler manifold (which is in the basic example a projective space).

Kaehler evolution. The time evolution of physical system is defined by a flow over $\mathcal{P}$ which preserves whole the Kaehler structure. This flow is generated by a vector field which is dense everywhere on $\mathcal{P}$.

Observables. Physical observables are given by special real smooth functions over $\mathcal{P}$ whose Hamiltonian vector fields preserve the Kaehler structure. In other words physical observables are presented by symbols.

Probability aspects. Let $\Lambda \subset \mathbb{R}$ is a closed subset of $sp(f)$ and the system stays at a state represented by point $p \in \mathcal{P}$. The probability to get the result after the measurement process which belongs to Λ is given by the formula

$$\delta_p(\Lambda) = cos^2(\frac{\sigma(p, P_{f,\Lambda}(p))}{\sqrt{2h}},$$

where $P_{f,\Lambda}(p)$ is the nearest to p point of $\mathcal{E}_{f,La}$.

Reduction. The choice of an arbitrary closed subset Λ in $sp(f)$ defines the ideal measurement which can be performed. This measurement answers to the question whether or not the volume f belongs to Λ. After the measurement process is performed the state of the system is defined by either $P_{f,\Lambda}(p)$ or $P_{f,\Lambda^c}(p)$ in dependence on the measurement result.

Now we need two direct citation from [3] to clarify the ways of possible generalizations. First, let's note that the postulates of quantum mechanics can be formulated on pure geometrical language without any references to Hilbert spaces. Of course the standard Hilbert space considerations and related algerbaic machinery equip one with a good setup for concrete computations. But mathematically the situation is very similar to the usual geometric consideration of compact manifolds with nontrivial topology: in practice it's convenient at the starting point of investigations to include this manifold to an appropriate ambient space (Euclidean, projective, ...) and then study it. But the inclusion is needed only for convenience; one could derive all what one needs directly from the geometry of this manifold. Second, the quantum mechanics linearity could be an analogy of the inertial systems in special relativity and the geometric formulation of quantum mechanics may be equivalent to Minkowski formulation of special relativity and as the last one showed the way to general relativity the geometric formulation would lead us to a new theory. One way generalizing quantum mechanics can be presented immideately after the geometric formulation is understood. One can assume that there are exist some other Kaehler manifolds (not only projective spaces) which could carry quantum mechanical systems. The dynamical properties could be easily satisfied (since classical mechanics allows one to consider some other symplectic manifolds than only projective spaces). The first question arising in this way is the question of observables. Since there are exist the Kaehler manifolds which doesn't admit any real functions whose Hamiltonian vector field would keep the Kaehler structure the question is really of the first range. F.e., one could require for the Kaehler manifold to admit the maximal possibility — it corresponds to the Kaehler manifolds of constant holomorphic sectional curvature (see [3]). It is well known that in finite dimension this condition is satisfied only by the projective spaces. In the infinite dimensional case this problem is still open so one could conjecture that there are infinite dimensional Kaehler manifolds satisfying the condition and different from the projective spaces. We would say here that this requirement is too strong: it is sufficient to impose the condition that the space of quantum observables allowed over a tested Kaehler manifold *is sufficiently large*. So the Kaehler structure over this manifold has to supply us with a good reserve, which we could use in our investigations. So during the text we will keep in mind this conception. Now we continue with the natural

Definition. *For a Kaehler manifold $\mathcal{K}$ let's call a real smooth function f "quasi symbol" if its Hamiltonian vector field preserves whole the Kaehler structure.*

Let's denote the space of all quasi symbols as $\mathcal{C}^\infty_q(\mathcal{K}, \mathbb{R})$. A manifest property of such functions comes immideately

Proposition 1.4. *For any Kaehler manifold $\mathcal{K}$ the space $C^\infty(\mathcal{K}, \mathbb{R})$ is a Lie subalgebra of the Poisson algebra.*

To prove it one takes the Poisson bracket of any two quasi symbols, ensures that the corresponding Hamiltonian vector field is proportional to the commutator of the Hamiltonian vector field of the given functions and then differentiates the given riemannian metric in the commutator direction. The answer is obvious.

Further we will construct a nonlinear generalization of quantum mechanics. The nonlinearity means that our exploited Kaehler manifold is not a projective space. But in Section 4 we will show that this Kaehler manifold admits a large space of quasi symbols sufficient to include all smooth functions from some given symplectic manifold. Since we would like to speak concretely about the quantization of classical mechanical systems it is quite enough for us. Therefore we omit in this text any discussions about pure generalizations of quantum mechanics which should be discovered in this way. So from this moment we speak about quantization of classical mechanical systems.

1.5. Souriau - Kostant quantization.

Let's remind first of all what the problem is. If one considers a classical mechanical system with finite degrees of freedom then the phase space is represented by a symplectic manifold M with symplectic form ω. Classical observables are given by smooth real functions over M and the corresponding Poisson bracket $\{,\}_\omega$ defines the Lie algerba structure on the space of observables. One understands then the ideal quantization as a procedure which results a Hilbert space Q_M together with a correspondence

$$q : C^\infty(M, \mathbb{R}) \to Op(Q_M), \tag{1.5.1}$$

satisfying the usual properties (the terminology can vary: while one takes the complete list of Dirac conditions then it's called "quantization"; if one relaxes the requirements leaving the irreducibility condition then it is often called "prequantization" but we'd like to reserve the last word for the notion "prequantization data" reminded below). According to a celebrated van Hove theorem ([13]) one can't realize the ideal quantization using the geometry of (M, ω) itself (since as we'll see usually one takes the space of functions or the space of sections of some appropriate bundle over M as the Hilbert space). As we will see the known examples of quantization leave unsatisfied one or two conditions from the complete Dirac list. The constructions begin with so called "prequantization data" (see [10]). First of all, the cohomology class of ω has to be integer (some times one calls this condition "the Dirac condition of the charge integrality"):

$$[\omega] \in H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R}). \tag{1.5.2}$$

If this condition holds then there exists a topological complex vector bundle L of rank 1 over M characterized uniquely by the condition $c_1(L) = [\omega]$ where c_1 is as usual the first Chern class. One can choose and fix any appropriate hermitian structure on L, getting a $U(1)$ - bundle. Then one gets another prequantization datum — a hermitian connection $a \in \mathcal{A}_h(L)$ satisfying the following natural condition

$$F_a = 2\pi i \omega, \tag{1.5.3}$$

where F_a is the curvature form. If our given symplectic manifold M is simply connected then this prequantization connection is unique up to the gauge transformations. In other case the space of the equivalence classes of solutions is isomorphic to $H^1(M, \mathbb{R})$. In this text we are interested mostly in the simply connected case so in the rest of the paper we'll work with compact simply connected symplectic manifolds only. All remarks about non simply connected case are more than welcome.

Whatever the case we are trawling through there is an uncertainty in the choice of the prequantization data: even if M is simply connected then we are free to take any hermitian structure on L and this implies that the prequantization connections can be different. Recall briefly how two different hermitian structures could be compared. For any pair h_1, h_2 where h_i is a hermitian structure on L one has a real positive function $e^{\phi(x)}$ satisfying the following property

$$\forall v_1, v_2 \in \Gamma(M, L) \qquad < v_1, v_2 >_{h_1} = e^{\phi(x)} < v_1, v_2 >_{h_2} \tag{1.5.4}$$

(since M is simply connected!), and in terms of this function we can relate hermitian connections compatible with h_1 and h_2. Consider the spaces $\mathcal{A}_{h_1}(L)$ and $\mathcal{A}_{h_2}(L)$ consist of all hermitian connections compatible with h_1 or h_2 respectively. These are two affine subspaces in the ambient space of all C^* - connections on L which we denote as $\mathcal{A}(L)$. One has the following

Proposition 1.5. *1) The affine spaces $\mathcal{A}_{h_i}(L) \subset \mathcal{A}(L)$ either do not intersect each other or coincide;*

2) two connections $a_i \in \mathcal{A}_{h_i}(L)$ differ by a complex 1 - form $i\rho + \frac{1}{2} d\phi$, where function ϕ is defined in (1.5.4).

Really, connection a is compatible with hermitian structure h if for any two sections $s_1, s_2 \in \Gamma(L)$ the following identity holds:

$$d < s_1, s_2 >_h = < d_a s_1, s_2 >_h + < s_1, d_a s_2 >_h \tag{1.5.5}$$

(see, f.e., [8]). If connection a_0 is compatible simultaneously with h_1 and h_2:

$$a_0 \in \mathcal{A}_{h_1}(L) \cap \mathcal{A}_{h_2}(L),$$

then using (1.5.5) one gets

$$d < s_1, s_2 >_{h_1} = d < s_1, s_2 >_{h_2} \tag{1.5.6}$$

for any sections $s_1, s_2 \in \Gamma(L)$. But it implies that

$$de^{\phi} < s_1, s_2 >_{h_1} = 0 \tag{1.5.7}$$

for any sections $s_1, s_2 \in \Gamma(L)$. It means that $d\phi$ vanishes and $\phi = const$. Coming back to (1.5.5) one sees that in this case $\mathcal{A}_{h_1}(L) = \mathcal{A}_{h_2}(L)$. This gives the first statement of the proposition.

Further, using once more (1.5.5), we have

$$\begin{aligned} d < s_1, s_2 >_{h_2} &= de^{\phi} < s_1, s_2 >_{h_1} \\ = e^{\phi} d\phi < s_1, s_2 >_{h_1} &+ e^{\phi} d < s_1, s_2 >_{h_1} \\ = e^{\phi} < s_1, s_2 >_{h_1} d\phi &+ e^{\phi} < d_{a_1} s_1, s_2 >_{h_1} \\ + e^{\phi} < s_1, d_{a_1} s_2 >_{h_1} &= e^{\phi} (< s_1, s_2 >_{h_2} d\phi \\ < d_{a_1} s_1, s_2 >_{h_2} &+ < s_1, d_{a_1} s_2 >_{h_2}). \end{aligned} \tag{1.5.8.}$$

Comparing the last expression with (1.5.5) for $i = 2$, we get

$$< \delta s_1, s_2 >_{h_2} + < s_1, \delta s_2 >_{h_2} = < s_1, s_2 >_{h_2} d\phi, \tag{1.5.9}$$

where $\delta = d_{a_1} - d_{a_2}$ is a complex 1 - form. Its real part is fixed by (1.5.9) while the imaginary part is arbitrary. The second part of the proof is over.

Therefore if one takes two affine subspaces in $\mathcal{A}(L)$ corresponding to hermitian structures h_1, h_2 on L then there exists a natural affine map

$$\begin{aligned} \mathcal{A}_{h_1}(L) &\to \mathcal{A}_{h_2}(L) \\ a_1 &\mapsto a_2 = a_1 + \tfrac{1}{2} d\phi, \end{aligned} \tag{1.5.10}$$

and this is true only in the simply connected case. It reflects the fact that in absence of additional structures over an arbitrary (not necessary simply connected) manifold X there exists unique cohomology class in $H^1(X, \mathbb{R})$ which has a distinguished representation by closed 1 -form. This class is [0] and the distinguished closed form is zero 1- form. On the other hand, if we fix any C^* - connection which solves equation (1.5.3) then there exists a hermitian structure on L such that this connection is compatible with this structure (in our simply connected case). Thus one can estimate the uncertainty in the choice of the prequantization data. So let us choose an appropriate structure (or connection). In Souriau - Kostant quantization one takes the space $L^2(M,L)$ as the space of wave functions: the fixed hermitian structure on L together with the Liouville volume form defines hermitian structure

$$< s_1, s_2 >_q = \int_M < s_1, s_2 >_h d\mu_L \tag{1.5.11}$$

on the space of sections $\Gamma(L)$, and one takes the completion of the space with respect to this hermitian structure q. Then

$$\mathcal{H} = \overline{\{s \in \Gamma(L) | \int_M < s, s >_h d\mu_L < \infty\}}. \tag{1.5.11'}$$

Every function f induces the following operator

$$Q_f : \mathcal{H} \to \mathcal{H} \qquad | \quad Q_f s = \nabla_{X_f} s + 2\pi i f \cdot s \tag{1.5.12}$$

on the wave function space $\mathcal{H}$. The commutator of two such operators can be directly computed:

$$\begin{aligned} [Q_f, Q_g] = (\nabla_{X_f} + 2\pi i f)(\nabla_{X_g} + 2\pi i g) \\ -(\nabla_{X_g} + 2\pi i g)(\nabla_{X_f} + 2\pi i f) = \\ \nabla_{X_f}\nabla_{X_g} - \nabla X_g \nabla_{X_f} + 2\pi i \nabla_{X_f} g - 2\pi i \nabla_{X_g} f \\ = \nabla_{X_f}\nabla_{X_g} - \nabla X_g \nabla_{X_f} + 4\pi i \{f, g\} \\ = \nabla_{[X_f, X_g]} + R(X_f, X_g) + 4\pi i \{f, g\} \\ = \nabla_{X_{\{f,g\}}} - 2\pi i \{f, g\} + 4\pi i \{f, g\} \\ = \nabla_{X_{\{f,g\}}} + 2\pi i \{f, g\}. \end{aligned} \tag{1.5.13}$$

This means that such operators could be exploited for the quantization. The point is that such Q_f is not self adjoint — conversely, it is a unitary operator. Really, for any two wave functions we have

$$\begin{aligned} < Q_f s_1, s_2 >_q = \int_M < \nabla_{X_f} s_1 + 2\pi i f s_1, s_2 >_h d\mu_L = \\ \int_M < d_a s_1, s_2 >_h (X_f) d\mu_L - \int_M < s_1, s\pi i s_2 >_h d\mu_L = \\ \int_M d < s_1, s_2 >_h (X_f) d\mu_L - \int_M < s_1, d_a s_2 > (X_f) d\mu_L - \\ \int_M < s_1, 2\pi i s_2 >_h d\mu_L = \int_M \{f, g\} d\mu_L - \int_M < s_1, \nabla_{X_f} s_2 >_h d\mu_L \\ - \int_M < s_1, 2\pi i s_2 >_h d\mu_L = - \int_M < s_1, Q_f s_2 >_h d\mu_L = - < s_1, Q_f s_2 >_q, \end{aligned} \tag{1.5.14}$$

where

$$g = < s_1, s_2 >_h \in C^\infty(M, \mathbb{R}).$$

To correct the picture let's take the dual bundle L^* with connection a' whose curvature is equal to $-2\pi i\omega$. Then it changes the sign in (1.5.13) so

$$[Q_f, Q_g] = -Q_{\{f,g\}}.$$

Now if we consider

$$\hat{Q}_f = iQ_f, \tag{1.5.12'}$$

then these operators should be self adjoint and the correspondence principle remains to be satisfied. This way approaches us to the Souriau - Kostant quantization (see f.e. [13], [14], [21], [33]).

What we could derive from this construction taking in mind some dynamical properties of the given system? Really smooth functions on (M, ω) correspond (up to constant) to infinitesimal deformations of the symplectic manifold. Add now the prequantization data: let P be the principle $U(1)$ - bundle, associated with line bundle L, and $A \in \Omega^1_P(i\mathbb{R})$ — the corresponding connection. Its differential equals just to $i\omega$. For any smooth function f consider the lifting

$$Y_f = X_f + g \cdot \frac{\partial}{\partial t} \tag{1.5.15}$$

of the Hamiltonian vector field to a vector field which preserves our hermitian connection A. There g is a smooth function on M and

$$\frac{\partial}{\partial t} = \theta_t$$

is the generator of the canonical circle action along the fibers of $\pi : P \to M$. The condition that the lifting Y_f preserves A means that

$$Lie_{Y_f} A = 0,$$

thus one gets

$$\begin{aligned} Lie_{Y_f} A = d(A(Y_f)) + \imath_{Y_f} dA = \\ \pi^* id(g) + \pi^* \imath_{X_f}(i\omega) = 0. \end{aligned} \tag{1.5.16}$$

Therefore one can relate f and g, since (1.5.16) implies

$$dg = -\imath_{X_f} \omega, \tag{1.5.17}$$

and consequently g coincides with f up to constant. This means that every Hamiltonian vector field can be lifted to P almost canonically. So any symplectomorphism of the base (M, ω) can be lifted to an automorphism of the principal bundle P, preserving A, up to canonical $U(1)$ - transformations. The constant which would be add to g in (1.5.17) just responds to an infinitesimal canonical transformation of P. To avoid the uncertainty let's take the projectivization of the wave function space

$$\mathcal{P} = \mathbb{P}(\mathcal{H}).$$

Then any Hamiltonian vector field X_f can be lifted uniquely which gives a vector field on $\mathcal{P}$. Denote this unique vector field $\tilde{Y}_f$ and consider what are the properties distinguishing this one. The main property is that this field is defined *dynamically.* It means the following: since any symplectomorphism of (M, ω) preserves the projective space $\mathcal{P}$ then its infinitesimal part corresponds to the infinitesimal part of the corresponding automorphism of $\mathcal{P}$ which is a vector field. Really such vector field can be defined for any Hamiltonian vector field (again, we are working with compact simply connected symplectic manifolds!). Then one has a correspondence between Hamiltonian vector fields on (M, ω) and some special vector fields on $\mathcal{P}$. This specialty means that since any automorphism of $\mathcal{P}$ induced by a symplectomorphism of (M, ω) by the definition keeps whole the Kaehler structure on $\mathcal{P}$ then the induced vector fields preserve the Kaehler structure as well. Therefore for any f the dynamically correspondent vector field $\tilde{Y}_f$ is Hamiltonian for the symplectic structure on $\mathcal{P}$ and is a Killing field for the riemannian metric. Since our projective space is simply connected one could reconstruct a function which Hamiltonian vector field equals to $\tilde{Y}_f$. Denote this function as Q_f. By the definition this Q_f is a symbol (in the original Berezin sense). Of course, this Q_f is defined only up to constant. But for any such function we can construct a self adjoint operator on $\mathcal{H}$. It's clear that a normalization rule is included in (1.5.12') and the operator defined in this formula coincides up to constant with the one coming from the symbol Q_f. Therefore one should say that the good success of the Souriau - Kostant construction is based on the dynamical property, mentioned above.

So at this stage the idea of dynamical correspondence comes to geometric quantization. Let's consider the Souriau - Kostant construction as the following procedure. For (M, ω) as above one takes the prequantization data (L, a). Then one takes the projectivization

$$\mathcal{P} = \mathbb{P}(\mathcal{H})$$

of the completed space of sections with finite norm. Then one has the dynamical correspondence

$$\Theta^p_{DC} : Vect_\omega(M) \to Vect_K(\mathcal{P}) \tag{1.5.18}$$

such that the image consists of the fields which preserve the Kaehler structure on $\mathcal{P}$. Dynamical properties of θ_{DC} dictates the following "correspondence principle":

$$\Theta^p_{DC}([X_f, X_g]) = [\Theta^p_{DC}(X_f), \Theta^p_{DC}(X_g)] \tag{1.5.19}$$

So the quantization question is the following: can we lift this correspondence to a correspondence between smooth functions on both the spaces? More rigouresly we will say that the dynamical correspondence is quantizable iff there exists a linear map

$$F : C^\infty(M, \mathbb{R}) \to C^\infty(\mathcal{P}, \mathbb{R})$$

such that the following diagram

$$\begin{array}{ccc} C^\infty(M, \mathbb{R}) & \xrightarrow{F} & C^\infty(\mathcal{P}, \mathbb{R}) \\ \downarrow & & \downarrow \\ Vect_\omega(M) & \xrightarrow{\Theta^p_{DC}} & Vect_K(\mathcal{P}) \end{array} \tag{1.5.20}$$

commutes. And one can prove that in the Souriau - Kostant case the dynamical correspondence given by (1.5.18) is quantizable. (Hint: if one takes unit sphere $S \subset \mathcal{H}$ which consists of

$$S = \{s \quad | \int_M < s, s > d\mu_L = r\}$$

then for every real function $f \in C^\infty(M, \mathbb{R})$ one has

$$\tilde{F}_f(s) = \tau \int_M f \cdot < s, s > d\mu_L \tag{1.5.21}$$

which is a real function on S. It's clear that this function is invariant with respect to the phase rotations therefore one gets the corresponding function F_f on the projective space $\mathcal{P}$. This function depends on two real parameters — r and τ. Choosing appropriate values one gets the desired function which satisfies (1.5.20) and moreover maps the constant function $f \equiv 1$ to the constant function $F_f \equiv 1$.)

The Souriau - Kostant method was recognized as universal but not quite successful since it gives a reducible representation of the Poisson algebra $C^\infty(M, \mathbb{R})$. This well known fact (see, f.e., [13]) we are going to explain in terms of geometric reformulation. So the question is: what "reducibility" and "irreducibility" mean when we are speaking on the projectivization language? Usually they mean that some invariant subspaces either exist or do not exist. After the projectivization one either gets some special submanifold of the projective space or doesn't get it. This submanifold should possess the following property: for every smooth function $f \in C^\infty(M, \mathbb{R})$ the Hamiltonian vector field of the corresponding symbol preserves this submanifold. In other words it is parallel to the submanifold. On the other hand, we know that the image of q (see the definition of AGQ on pp. 4 and 5 above) should give a Lie subalgebra in the Lie algebra of vector fields over the quantum phase space therefore if the submanifold exists it were that the vector fields form an integrable distribution. F.e., using the function map, given by (1.5.21) one can easily find the invariant submanifold for the Souriau - Kostant quantization. But even more easily one could find that the resulting Hamiltonian vector fields form an integrable distribution on the reduced phase space of the quantization. This is an explanation why we imposed condition 4b) in the definition of AGQ. Condition 4a) was introduced to get some complete picture (to be honest, it was imposed since we can make it satisfied in what follows).

More generally, let us formulate the dynamical principle as a universal one in symplectic geometry. Namely, if we associate any object with a given symplectic manifold (M, ω) in invariant way (so if it depends only on (M, ω) itself) then every symplectomorphism of (M, ω) generates an automorphism of this associated object. Moreover, the generated automorphism should preserve the structures on the associated object if they were defined in invariant way. Turning to the infinitesimal level it gives us a correspondence between Hamiltonian vector fields and some special fields on the associated object such that the induced vector fields preserve the structures. Moreover the identity (1.5.19) holds for the dynamical correspondence. Then if the associated object is a symplectic manifold then the dynamical correspondence maps Hamiltonian vector fields on the source manifold to Hamiltonian vector fields on the target manifold and one can ask whether or not this correspondence is quantizable so does there exist a map from the space of smooth functions on the given manifold to the space of smooth functions on the associated manifold such that the diagram (1.5.20) commutes. If the map exists then one can represent the Poisson algebra of the given manifold in terms of the Poisson algebra of the associated symplectic manifold.

Now we leave dynamics for a while to continue the recalling of the relevant constructions.

1.6. Real and complex polarizations.

In this subsection we discuss what happens in known quantization picture in presence of an additional structure — a polarization. Here we focus only on the simplest examples when this polarization is "pure" — real or complex. Moreover we simplify the story avoiding the introduction of metaplectic structure and metaplectic correction since in the rest of this text we will not meet these notions (and unfortunately we are still not in position to introduce this correction in ALG(a) - quantization).

A complex polarization is a fixed integrable complex structure I over (M, ω), compatible with ω. Thus our M_I is a Kaehler manifold. The choice of I defines a riemannian metric g on M. Moreover, since our symplectic form is integer then M_I is an algebraic manifold (more precisely — an algebraic variety). The prequantization data (L, a) over (M, ω) can be specified in this complex case as follows. Since F_a is proportional to ω it has type $(1,1)$ with respect to the fixed complex structure. This means that (L, a) is a holomorphic line bundle: the holomorphic structure is defined by operator $\overline{\partial}_a$. This operator defines the space of holomorphic sections $H^0(M_I, L)$ which is naturally included in $\mathcal{H}$ (our M is compact). The celebrated Kodaira theorem ensures us that there exists a power of L such that the corresponding linear system defines an embedding of M_I to a projective space. This means that for some appropriate $k \in \mathbb{N}$ the space of holomorphic sections $H^0(M_I, L^k)$ hasn't based components; in other words for any point $x \in M_I$ there exists a holomorphic section $s \in H^0(M_I, L^k)$ non vanishing at the point

$$s(x) \neq 0.$$

Moreover, the subspace $H_x \subset H^0(M_I, L^k)$ consists of vanishing at x holomorphic sections of L^k has codimension 1 for any x. Therefore the following embedding is correctly defined

$$\begin{aligned} \psi :& M_I \hookrightarrow \mathbb{P}H^0(M_I, L^k)^*, \\ \psi :& x \mapsto (\mathbb{P}H_x)^* \in \mathbb{P}H^0(M_I, L^k)^* \end{aligned} \tag{1.6.1}$$

which is the embedding by complete linear system $|kD|$ where D is the Poincare dual divisor.

Usually one considers the space of holomorphic sections as the wave function space. The desired self adjoint operators can be defined in different ways.

Berezin - Rawnsley quantization. In [16] (it has subtitle "geometric interpretation of Berezin quantization") one finds an approach to quantize this Kaehler case. First of all, one restricts to a subclass of smooth functions which are called "qunatizable": smooth function f is quantizable over M_I iff its Hamiltonian vector field preserves the complex structure. In our notations it means that f is a quasi symbol over the Kaehler manifold. For such a function the corresponding Souriau - Kostant operator $\hat{Q}_f$ (see subsection 1.5) maps the holomorphic sections to the holomorphic sections thus it is an automorphism of the space $H^0(M_I, L)$. The correspondence principle is satisfied since we just restrict the full picture over $\mathcal{H}$ to the corresponding subspace. We need not to explain what is geometric reformulation of this method because it is geometric itself: all the paper [16] is done in this style and we could not add anything to it. A Berezin idea to consider on the projectivized space the symbols instead of self adjoint operators is exactly the same geometric formulation of quantum mechanics applying to geometric quantization.

Berezin - Toeplitz quantization. In [4] one can find another way defining self adjoint operators on the same Hilbert space. Let us denote

$$\mathcal{H}_k = H^0(M_I, L^k)$$

and call k the level of quantization. For any level we have the orthogonal projection

$$S_k : \mathcal{H} \to \mathcal{H}_k \tag{1.6.3}$$

which is called Szoege projector. Then every smooth function f gives a self adjoint operator on any level

$$\hat{A}_f : \mathcal{H}_k \to H_k$$

coming as the composition of the multiplication by f and the corresponding Szoege projector:

$$\hat{A}_f(s) = S_k(f \cdot s) \in \mathcal{H}_k \tag{1.6.4}$$

where s belongs to $\mathcal{H}_k$. It's clear that for every f the corresponding Toeplitz operator is self adjoint. The choice of the holomorphic section spaces as the subspaces to project on is an appropriate one: in [4] one proves that in this case the correspondence principle is satisfied asymptotically.

Apply the geometric reformulation scheme to the Berezin - Toeplitz method over compact symplectic manifolds (see [28]). Instead of finite dimensional Hilbert space $\mathcal{H}_k$ we take the corresponding projective space $\mathbb{P}_k$. Any Toepltiz operator $\hat{A}_f$ can be replaced by its symbol Q_f. Therefore one gets a linear map:

$$A : C^\infty(M, \mathbb{R}) \to C^\infty_q(\mathbb{P}_k, \mathbb{R}). \tag{1.6.5}$$

The last space is finite dimensional thus this map A has a huge kernel. The question about its cokernel can be simplified by the introduction of the Fourier - Berezin transformation. Let M_I as above be a Kaehler compact manifold, (L, a) be the prequantization data read as a holomorphic line bundle equipped with a hermitian structure. Realize as above the projective space as the factorization of unit sphere $S_k \subset \mathcal{H}_k$ by the gauge transformations. Then one has a real non negative smooth function $u_k(x,p)$ on the direct product $M_I \times \mathbb{P}_k$ defined as follows. If section $s \in S_k$ represents point $p \in \mathbb{P}_k$ then

$$u_k(x,p) = < s(x), s(x) >_h \in \mathbb{R}. \tag{1.6.6}$$

This expression again is preserved by the gauge transformation so we get a correctly defined smooth function on the direct product. On the other hand this function strongly depends on the hermitian structure choice made before.

Proposition 2.3. *For any smooth function f on M the corresponding Toeplitz symbol A_f is given by the formula*

$$A_f(p) = \int_M f \cdot u(x,p) d\mu_L. \tag{1.6.7}$$

This integral operator with kernel $u_k(x,p)$ performs the Fourier - Berezin transform (of level k) mentioned above. At the same time there is a distinguished element in the space of classical observables over M. Smooth function

$$\lambda_k(x) = \int_{\mathbb{P}_k} u_k(x,p) d\mu, \tag{1.6.8}$$

where $d\mu$ is the Liouville volume form on $\mathbb{P}_k$, depends, of course, on the choice of the prequantization hermitian structure: when we vary the hermitian structure then both the integrand and the volume form change in the right hand side of (1.6.8). This function λ_k is an analogy of the Rawnsley θ - function (or ε - function) (see [16]). To solve the cokernel problem for the map (1.6.5) one can choose such a prequantization hermitian structure on L that λ_k would have the best shape (and the bestest case is when λ_k is constant, but it is possible only if M is a projective space itself, see [16]). We don't touch it here but an interesting unsolved problem is to find these distinguished functions just in the simplest cases (f.e., for complete intersections) for algebraic manifolds in the framework of algebraic geometry. As well in symplectic geometry one should ask the questions about a subspace in $C^\infty(M,\mathbb{R})$ consists of all possible λ: how large this subspace is and what is the geometrical meaning of this subspace (note that this subspace is defined absolutely canonically without any additional choices over (M,ω) with integer ω).

To prove the statement it's convenient to choose an orthonormal basis in the Hilbert space $\mathcal{H}_k$. Assume that an appropriate one is fixed, denoting it as $\{s_0, ..., s_d\}$. Thus

$$\int_M < s_i, s_j >_h d\mu_L = \delta_{ij}. \tag{1.6.9}$$

Then the Toeplitz operator $\hat{A}_f$ is represented in this basis by a matrix whose elements are given by the formula

$$(\hat{A}_f)_{i,j} =< fs_i, s_j >_q= \int_M < fs_i, s_j >_h d\mu_L. \tag{1.6.10}$$

A unit vector $s \in S_k \subset \mathcal{H}_k$ can be decomposed

$$s = \sum_i \alpha_i s_i.$$

If this vector represents point $p \in \mathbb{P}_k$ then under the $\hat{A}_f$ - action it comes to

$$\hat{A}_f s = \sum_j (\sum_i \alpha_i \int_M < fs_i, s_j >_h d\mu_L) s_j. \tag{1.6.11}$$

The expectation value can be computed in the form

$$< \hat{A}_f s, s >_q = \sum_j \alpha_j (\sum_i \alpha_i \int_M < f s_i, s_j >_h d\mu_L). \tag{1.6.12}$$

It gives us

$$< \hat{A}_f s, s >_q = \int_M f < s, s >_h d\mu_L. \tag{1.6.13}$$

Taking in mind the definition of $u_k(x, p)$ given in (1.6.6) we understand formula (1.6.13) the same one as given in the statement.

One should remark that formula (1.6.7) looks very similar to formula (1.5.21). Really it means that F_f defined by (1.5.21) and A_f defined by (1.6.7) are the same function but the first one is defined on the ambient space while the second one equals to the restriction of the first one to a finite dimensional subspace. The question arises: why in the first case the correspondence principle is satisfied completely while in the second it is satisfied only asymptotically? The point is that if a function f on M generates the Hamiltonian vector field preserving the complex structure I as well as given symplectic structure then the Hamiltonian vector field X_{F_f} on $\mathcal{P}$ is tangent to the projective subspace $\mathbb{P}_k \subset \mathcal{P}$. For such functions ("qunatizable" in terms of Berezin - Rawnsley method) the results of two different methods are the same (it was shown f.e. by G. Tuynman in [23]). At the same time if f is arbitrary then X_{F_f} is not tangent to $\mathbb{P}_k$ and using the corresponding projections for two such functions f and g we get that $A_{\{f,g\}}$ differs from $\{A_f, A_g\}$. But taking the limit $k \to \infty$ we make this difference smaller and smaller and this means that in the Berezin - Toeplitz method the correspondence principle is satisfied asymptotically.

On the other side of the spectrum one has real polarizations. While the complex case as we've seen belongs to algebraic geometry and the basic examples are given by the projective spaces, the real polarization case has the cotangent bundles as the main examples. It was proposed exactly as an additional structure on a symplectic manifold which could mimicry the case when one has two different kinds of variables: position and momentum ones. Therefore the best example of real polarized manifold is given by cotangent bundle: if any symplectic manifold is globally symplectomorphic to a cotangent bundle then one can divide the variables in any local chart into two sets such that the first set can be called position variables while the second one — momentum variables. One can see that the method reminded below is specified exactly to that case. At the same time we will see that this method has no choice to be well defined in the compact case — but anyway it can be retranslated in this case as we'll show in Section 4.

Avoiding the discussion of Lagrangian distribution we directly say that one can understand real polarization as the case when over (M, ω) there are exist n smooth real functions f_i (where $2n$ is the dimension of M) in involution:

$$\{f_i, f_j\} = 0 \qquad \forall \quad i, j = 1, ..., n, \tag{1.6.14}$$

which define a Lagrangian fibration

$$\pi : M \to \Delta \subset \mathbb{R}^n, \tag{1.6.15}$$

where Δ is a convex polytope in $\mathbb{R}^n$, namely

$$\pi^{-1}(t_1, ..., t_n) = \cap_i \{f_i = const = t_i\}. \tag{1.6.16}$$

The Hamiltonian vector fields X_{f_i} at any point of any fiber $\pi^{-1}(t)$ form a complete basis of the tangent to the fiber space. As usual one fixes a prequantization data (L, a) and gets the corresponding big Hilbert space $\mathcal{H}$. Further, following the same policy as in the complex case, one takes only the sections which satisfy some compatibility condition with respect to the fixed real polarization. Namely these sections must be invariant under the flows generated by $\{f_i\}$. Again one introduces the notion of level k quantization taking all powers of L. Thus the Hilbert space of wave functions consists of the following sections:

$$\mathcal{H}_1 = \{s \in \mathcal{H} \quad | \qquad \nabla_{X_{f_i}} s \equiv 0 \quad \forall 1 \leq i \leq n\}.$$

So the wave function is constant along the fibers and it varies only along the base (thus it depends on just a half of the variables). But the based manifold Δ wasn't equipped with any canonical measure or volume form since the basic Liouville form could not give any help when we turn to Δ. At the same time in general non compact case the fibers could be noncompact and then any section constant along the fibers should have unbounded norm with respect to the Liouville norm. To avoid this difficulty in the noncompact case (note that everybody takes in mind a cotangent bundle working in this way) one introduces the following correction

$$L' = L \otimes \sqrt{\Lambda^n F} \tag{1.6.17}$$

where F is the subbundle of TM tangent to the fibers of (1.6.15) at every point. In the real case we are talking about this additional term is topologically trivial moreover it carries a canonical connection gauge equivalent to ordinary d. Thus from the topological point of view this additional term doesn't change a choice on the base which we'll mention in a moment. But now this term which is known as the bundle on half weights allows one to define some natural pairing on $\mathcal{H}'_1$ which is defined in the same vein as $\mathcal{H}_1$. For any pair of the sections one takes the hermitian product for L - parts and just the tensor product for the half weight parts. This give a weight on the base Δ which can be integrated, giving a number. Now let us discuss what is the relationship between these special sections from $\mathcal{H}_1$ and some data inside the based polytope. Really if a section s is constant along a fiber $S = \pi^{-1}(t)$ then either it vanishes at S or this Lagrangian submanifold S must satisfy some special condition. We will discuss this condition (which is called here the Bohr - Sommerfeld condition) in Section 2 but here one can formulate it easily: if we take the restriction of the prequantization data

$$(L, a)|_S$$

to our Lagrangian fiber S then it should admit a covarinatly constant section. It's clear that by the definition of the prequantization data this restriction must be special, consisting of trivial line bundle plus a flat connection. And it is true for *any* Lagrangian submanifold. But this flat connection could be in general non equivalent to the ordinary one if our S has non trivial fundamental group. Thus in this case the requirement for the restriction to be trivial is called the Bohr - Sommerfeld condition. The confusion point is that there is a number of conditions having the same name; f.e. the integrality condition for the symplectic form is often called by the same name. Generally speaking any Bohr - Sommerfeld condition is just an appropriate quantization condition so f.e. in our main compact case the integrality condition is not sufficient to perform the quantization. Therefore we have to deepen and specify this condition to get a real Bohr - Sommerfeld condition for the quantization.

Now, it's clear that if S is Bohr - Sommerfeld with respect to (L, a) then it remains to be Bohr - Sommerfeld with respect to (L', a'). Let us collect all the fibers which satisfy the Bohr - Sommerfeld condition getting a submanifold $B \subset \Delta$ which is called again the Bohr - Sommerfeld submanifold. Outside of $\pi^{-1}(B) \subset M$ all sections from $\mathcal{H}_1$ have to vanish. The structure of the wave function space is the following: for any connected component S_α of $\pi^{-1}(B)$ there is a subspace $\mathcal{H}_\alpha$ in $\mathcal{H}_1$ which consists of the sections whose support is exactly S_α. Then the whole space is decomposed

$$\mathcal{H}_1 = \sum_\alpha \mathcal{H}_\alpha. \tag{1.6.18}$$

(We will show in Section 3 that if M is compact as well as the fibers then the Bohr - Sommerfeld manifold B either consists of a finite number of isolated points on Δ or coincides with Δ. Therefore in the first case the wave function space is spanned on a discrete set of δ - sections.) Again as in the complex case one should distinguish some set of functions in $C^\infty(M, \mathbb{R})$ which could be quantized. These functions are quite rare: in some extremal cases only our based f_i are quantizable. For any such f_i the corresponding self adjoint operator is diagonal: it acts by multiplication

$$\hat{Q}_{f_i}(s) = f_i \cdot s.$$

Since f_i is constant along the fibers it's clear that $f_i \cdot s$ remains to be covariantly constant along the fibers so

$$\hat{Q}_i : \mathcal{H}_1 \to \mathcal{H}_1$$

is an automorphism of the quantum space. In the compact case when the sections from $\mathcal{H}_1$ have bounded usual Liouville norms one could perform the same procedure as in Berezin - Toeplitz method, defining the corresponding orthogonal projector to subspace $\mathcal{H}_1 \subset \mathcal{H}$. Then a priori for every function $f \in C^\infty(M, \mathbb{R})$ there exists a direct analog to the Toeplitz operators discussed above. At this point the main difference between $\mathcal{H}_1$ and $\mathcal{H}'_1$ comes: the quantum space $\mathcal{H}'_1$ doesn't belong to $\mathcal{H}$ and moreover can't be included there. It carries manifestly different hermitian structure. Thus one must use some additional technique to define quantum operators for a wider set of smooth functions. Recall briefly, what one could do here, following [20]. Let F_1, F_2 be two real polarizations with transversal fibers. Then on the direct product of the corresponding quantum Hilbert spaces $\mathcal{H}^1_1$ and $\mathcal{H}^2_1$ there is a sesquilinear pairing

$$K : \mathcal{H}^1_1 \times \mathcal{H}^2_1 \to \mathbb{C} \tag{1.6.19}$$

which is called the Blattner - Kostant - Strenberg kernel ([20]). The hermitian structure on L plays the role in the definition of this pairing as well as the half weight parts. For reducible sections

$$s'_i = s_i \otimes \theta_i \quad | \quad s_i \in \mathcal{H}_1^i, \theta_i \in \Gamma(\sqrt{\Lambda^n F_i}),$$

where $i = 1, 2$, one defines

$$< s'_1, s'_2 >_\theta = \int_M < s_1, s_2 >_h \cdot \theta_1^2 \wedge \theta_2^2 \in \mathbb{C}. \qquad (1.6.20)$$

Indeed, since F_1, F_2 are transversal the wedge product of two elements from $\Lambda^n F_1$ and $\Lambda^n F_2$ is just a volume form on M. This sesquilinear pairing together with given hermitian structures on $\mathcal{H}_1^i$ defines a $\mathbb{C}$ - linear map

$$U_{21} : \mathcal{H}_1^2 \to \mathcal{H}_1^1. \qquad (1.6.21)$$

Assume now that a smooth function f (which doesn't belong to the algebraic span of the set $\{f_i\}$) generates the flow which deforms the given real polarization to a polarization transversal to the given one. Then the quantum operator is defined as the composition of the infinitesimal moving of $\mathcal{H}_1$ in the direction of X_f with the map defined in (1.6.21). One can see that in compact case there is no such a function at all since the definition means that for the function its Hamiltonian vector field should be nonvanishing everywhere (otherwise we get a point where these two real polarizations would be tangent). On the other hand even in non compact case it's an open problem because this derived operator is not in general self dual (the detailed story can be found f.e. in [20], [33]).

Thus in the compact case we are speaking about the subject in geometric formulation since as we've seen the real geometric information is carried by the set of the Bohr - Sommerfeld fibers.

2. The correspondence principle in the algebraic Lagrangian geometry

In this Section we construct a special representation of the Poisson algebra over a simply connected compact symplectic manifold with integer symplectic form. During this procedure the Poisson algebra is represented in the space of smooth functions over an appropriate moduli space which is itself a Kaehler manifold. This moduli space was constructed in [10], [25] and called the moduli space of half weighted Bohr - Sommerfeld Lagrangian cycles of fixed topological type and volume. To establish that this is a faithful representation we prove following [28] that the corresponding map is a homomorphism of the Poisson algebras. This representation could be called canonical since it doesn't require any additional choices for the definition. At the beginning we recall briefly the basic notions and constructions from ALAG.

2.1. Bohr - Sommerfeld condition.

Let (M, ω) be again a compact simply connected symplectic manifold of dimension $2n$ with integer symplectic form. Consider again the prequantization data (L, a) where L is defined by the condition

$$c_1(L) = [\omega]$$

and $F_a = 2\pi i \omega$. For an appropriate smooth oriented connected n - dimensional manifold S consider the space of smooth Lagrangian embeddings of fixed topological type so such smooth maps

$$\phi : S \to M \tag{2.1.1}$$

that

$$\phi^* \omega \equiv 0 \tag{2.1.2}$$

and the images represent the same homology class $[S] \in H_n(M, \mathbb{Z})$. The choice of the prequantization data make it possible to impose an extra condition on the Lagrangian embeddings which has been mentioned in the previous section. So we call an embedding Bohr - Sommerfeld if the restriction of the prequantization data to the image admits covariant constant sections. In other words flat connection $\phi^* a$ on trivial bundle $\phi^* L$ should have trivial periods with respect to the fundamental group of S. The condition has more elegant version recovering its geometrical meaning more distinct what will be discussed in Section 5. If one takes the corresponding $U(1)$ - principle bundle with the corresponding connection 1 - form A then it is an example of a contact manifold (about these manifolds see, f.e., [2]). Connection A, retwisted by i, satisfies the standard condition

$$\alpha \wedge (d\alpha)^n = d\mu,$$

where $d\mu$ is a volume form on P. In terms of the principle bundle one can easily reformulate the Bohr - Sommerfeld condition. Lagrangian submanifold satisfies the Bohr - Sommerfeld condition iff it can be lifted to P along the fibers of the canonical projection $P \to M$. Our connection A decomposes the tangent to P space at every point as the direct sum of the horizontal and the vertical parts and a map

$$\tilde{\phi} : S \to P$$

is called planckian if $T(\tilde{\phi})$ belongs to the horizontal part at every point of $\tilde{\phi}$ and $\tilde{\phi}^* \pi^* \omega \equiv 0$ where π is the standard projection.

Define now the notions of Bohr - Sommerfeld and planckian cycles. Let $\tilde{\mathcal{B}}_S$ is the space of all Bohr - Sommerfeld Lagrangian embeddings (2.1.1) of fixed topological type. Then the moduli space of Bohr - Sommerfeld Lagrangian cycles is given by the factorization

$$\mathcal{B}_S = \tilde{\mathcal{B}}_S / Diff_0 S, \tag{2.1.3}$$

where $Diff_0 S$ is the identity component in the diffeomorphism group of S. Recall that S is oriented thus $Diff_0 S$ could be understood as the parameterization group of S. Points of the moduli space are called Bohr - Sommerfeld cycles of fixed topological type. The moduli space of planckian cycles has almost the same definition: one just starts with the space of all planckian embeddings of S to M described above. We denote it as $\mathcal{P}_S$, following [10], [25]. Every planckian cycle is represented by a covariant constant lifting of a Bohr - Sommerfeld cycle therefore the natural map

$$\pi : \mathcal{P}_S \to \mathcal{B}_S \tag{2.1.4}$$

gives a principle $U(1)$ - bundle structure on $\mathcal{P}_S$ such that the canonical $U(1)$ - action is generated by the canonical $U(1)$ - action on P. This principle bundle is called Barry bundle.

It's quite natural to include the notion of level k here: so if one takes the corresponding tensor power (L^k, a_k) then one could define the notions with respect to this power as well as in the original case. This gives us a set of moduli spaces parameterized by k:

$$\mathcal{B}_S = \mathcal{B}_{S,1}, ..., \mathcal{B}_{S,k}, ...$$
$$\mathcal{P}_S = \mathcal{P}_{S,1}, ..., \mathcal{P}_{S,k}, ...$$

In Sections 2 and 3 we formulate the main facts in terms of level 1, but all of these statements can be easily generalized to the case of any level. We just have to keep in mind that if one starts with the symplectic manifold $(M, k\omega)$ then the pair (L^k, a_k) is precisely a prequantization data for it. And two major differences come: the Poisson bracket for $k\omega$ is slightly different from the original one and the Liouville volume form for $k\omega$ is slightly different to. All of these remarks will be valuable, important and meaningful in Section 4 in the discussion of the quasi classical limit.

Our first aim is to describe smooth structures on the moduli spaces $\mathcal{B}_S, \mathcal{P}_S$. The first one is given by the following

Proposition 2.1.([10], [25]). *The tangent space $T_S \mathcal{B}_S$ at any point $S \in \mathcal{B}_S$ is isomorphic to $C^\infty(M, \mathbb{R})$ modulo constant functions.*

The proof can be found in [10]. Briefly, let S be a regular point of the moduli space $\mathcal{B}_S$. We identify it in our discussion with the image of the corresponding class of maps so we understand S as an oriented smooth Bohr - Sommerfeld Lagrangian submanifold of M. According to the Darboux - Weinstein theorem (see [32]) there exists a tubular neighborhood $N(S)$ such that it is symplectomorphic to an ε - neighborhood of the zero section of the cotangent bundle:

$$\psi : N(S) \to N_\varepsilon(T^*S), \tag{2.1.5}$$

where the last one is equipped with the restriction of the canonical symplectic form on the cotangent bundle. Thus the work is reduced to the canonical case. Recall that T^*S is endowed with a natural 1 - form η which is called by the natural reason the canonical 1 - form. Any 1 - form α on the based manifold S can be regarded from two points of view: as a 1 - form on S itself and as a section Γ_α of the cotangent bundle:

$$\Gamma_\alpha : S \hookrightarrow T^*S \to S, \tag{2.1.6}$$

where the last arrow is the canonical projection. Then the canonical 1 - form on T^*S is correctly defined by the following property satisfied for any 1 - form α:

$$\Gamma^*_\alpha \eta = \alpha. \tag{2.1.7}$$

The differential of this form $d\eta$ is a non degenerated everywhere closed 2 - form defining the canonical symplectic structure over T^*S. Therefore one could understand submanifolds of M which belong to the neighborhood $N(S)$ as submanifolds of T^*S sufficiently close to S. If we are talking about lagrangian submanifolds then all of them are described by sufficiently "small" closed 1 - forms over S. It's clear that section $\beta : S \hookrightarrow T^*S$ is lagrangian iff

$$\omega|_{\Gamma_\beta} = 0 = d\eta|_{\Gamma_\beta} = d\beta. \tag{2.1.8}$$

The prequantization data for the cotangent bundle are given by

$$L = T^*S \times \mathbb{C}, \qquad a = d + 2\pi i \eta, \tag{2.1.9}$$

where d is the ordinary differential. The restriction of this flat connection on any Lagrangian graph $\Gamma_\beta \subset T^*S$ corresponding to $\beta \in \Omega^1_S$ has the periods along the group elements from $\pi_1(S)$ equal to the periods of the given 1 - form β (it follows from the definition of η). The last periods are trivial iff the form is exact. Then the Bohr - Sommerfeld condition is equivalent to the exactness of the corresponding 1 - forms. And this gives us

$$T_S \mathcal{B}_S = B^1(S) = \{df\}. \tag{2.1.10.}$$

Turning to the principal bundle (2.1.4) it's easy to see that the lifting corresponds to the employing of the constant functions so one gets

$$T_{\tilde{S}} \mathcal{P}_S = C^\infty(M, \mathbb{R}),$$

where $\tilde{S}$ is the corresponding planckian cycle. Moreover, the Darboux - Weinstein theorem ensures that the representations for the tangent spaces are integrable so one has distinct local coordinate systems on both the moduli spaces. The set of the Darboux - Weinstein neighborhoods gives atlases of the smooth structures. The fact stated in the theorem underlies of all our constructions. One could say even more: this fact shows that Lagrangian submanifolds of symplectic manifold look like points of symplectic manifold. Really it's well known (from the original Darboux lemma) that locally all symplectic manifolds are the same (in contrast, f.e., with riemannian manifolds which have local invariants distinguishing each from others). Thus all the points of a given symplectic manifold are indistinguishable having the same local structure. And according to the generalization of the old classical result the same is almost true for Lagrangian submanifolds: they differ only by the topological type. Therefore one sees that "points" and "Lagrangian submanifolds" have quite similar behavior from the kinematical point of view. We'll see that the same should be said about the dynamical behavior.

More generally, if one defines the moduli space of all Lagrangian cycles in the same fashion as it was done for Bohr - Sommerfeld cycles (applying the Darboux - Weinstein theorem and getting again an atlas for the smooth structure and "canonical" local coordinate) then there is a natural map

$$\chi : \mathcal{L}_S \to J_S = H^1(S, \mathbb{R})/H^1(S, \mathbb{Z}), \tag{2.1.11}$$

where $\mathcal{L}_S$ is the moduli space of all Lagrangian cycles of fixed topological type. J_S is the Jacobian of S understood as the set of flat connection classes modulo gauge transformations. Obviously the preimage of zero class is exactly $\mathcal{B}_S$. On the other hand one can take any fiber $\chi^{-1}(pt)$ as a moduli space and this space would have the same description of the tangent bundle as in Proposition 2.1. The main difference is that one couldn't define for this generic fiber the notion of the planckian lifting. Thus one couldn't extend the description from $C^\infty(S, \mathbb{R})/const$ (as for $\mathcal{B}_S$) to $C^\infty(S, \mathbb{R})$ (as for $\mathcal{P}_S$). Therefore our choice of the fiber is based on the possibility of the lifting. At the same time as we'll see one can get almost the same fibration as (2.1.11) in absolutely general symplectic situation.

The description of the tangent bundles for $\mathcal{B}_S$ and $\mathcal{P}_S$ in terms of the smooth functions on S has quite important consequences. Before the story is coming to the Kaehler setup we present here four remarks.

1. If S has trivial fundamental group (or even trivial first homology group) then each Lagrangian cycle is Bohr - Sommerfeld.

2. The linearization of the Bohr - Sommerfeld condition is exactly the same as the so - called isodractic (or Hamiltonian) deformations. Indeed, any smooth function f on S can be extended to a smooth function $\tilde{f}$ over M. Then the Hamiltonian vector field $X_{\tilde{f}}$ generates some deformation of S. This infinitesimal deformation preserves the Lagrangian condition. Moreover it preserves the Bohr - Sommerfeld condition and the linearly deformed cycle is exactly that one in the neighborhood of S given by df. It means that the Bohr - Sommerfeld condition is a classical dynamical condition over symplectic manifolds. Therefore we can introduce a kind of fibration (2.1.11) in the case when ω is not integer at all. Namely this analogy is given by the flows of all complete Hamiltonian vector fields over M. But in the integer case we can first of all avoid the questions about the completeness of the Hamiltonian vector fields and define the fibration on the "kinematical" level. The same remark can be addressed to the planckian cycles as well.

3. Let's show some formulas to illustrate how we'll work in the setup. If you choose any function f on $S \in \mathcal{B}_S$ and extend it arbitrary to M, getting a smooth function $\tilde{f}$ then one can decompose the corresponding Hamiltonian vector field $X_{\tilde{f}}$ on the horizontal and vertical components with respect to TS and ω and this decomposition is absolutely canonical. So we have

$$X_{\tilde{f}} = X_{ver} + X_{hor}, \tag{2.1.12}$$

where X_{hor} belongs to TS while X_{ver} can be identified with a section of the normal to S bundle

$$N_S = TM|_S/TS. \tag{2.1.13}$$

It's clear that X_{hor} corresponds to the part of deformation which preserves the cycle S (its flow generates some motion on S). Thus the deformation of S depends only on X_{ver}. Let's use the isomorphism

$$\omega : TM \to T^*M, \tag{2.1.14}$$

getting the formula

$$X_{ver} = \omega^{-1}(d\tilde{f}|_S) = \omega^{-1}(d(\tilde{f}|_S)) = \omega^{-1}(df). \tag{2.1.15}$$

Therefore the deformation depends only on the restriction to S.

4. As we've seen there is a natural infinitesimal action of the Hamiltonian vector fields on the moduli space of Bohr - Sommerfeld Lagrangian cycles. Indeed, every Hamiltonian vector field gives an infinitesimal deformation of the based manifold so generates a vector field on the moduli space of Bohr - Sommerfeld Lagrangian cycles (dynamical correspondence) since the definition is posed in invariant terms. The same is true for any fiber of (2.1.11). The point is that the dynamical correspondent vector field A_f for any (global) smooth function f on whole $\mathcal{L}_S$ is given by the following very simple formula

$$A_f(S) = d(f|_S) \in T_S\mathcal{L}_S.$$

Thus this "quantum" vector field preserves the fibers of (2.1.11). The corresponding foliation is integrable and we get (2.1.11) in another fashion, which has been hinted.

2.2. Doubling circuit: Kaehler structure.

In this subsection we complexify the moduli space $\mathcal{B}_S$, following [10], [25]. At the first step we take the moduli space of planckian cycles $\mathcal{P}_S$. The source manifold S is equipped with a space of half weights (see [10], [25]). Since S is orientable the determinant line bundle

$$detT^*S = \Lambda^n T^*S \tag{2.2.1}$$

is trivial. Roughly speaking, a half weight is almost the same as a half form without zeros (at list in our case when S is endowed with a fixed orientation we will understand it so). For any pair of half weights there are two derivations:

$$\int_S \theta_1 \cdot \theta_2 \in \mathbb{R} \tag{2.2.2}$$

and

$$\frac{\theta_1}{\theta_2} \in C^\infty(S, \mathbb{R}). \tag{2.2.3}$$

Moreover, the space of half weights admits a canonical involution which could be written in the half form representation just as the multiplication by -1. The tangent space to the set of half weights over each point is modeled by $C^\infty(S, \mathbb{R})$ ([10], [25]) and we consider the moduli space of half weighted planckian cycles ([10], [25]) consists of pairs

$$(\tilde{S}, \theta) \in \mathcal{P}_S^{hw}, \tag{2.2.4}$$

where $\tilde{S}$ is a planckian cycle and θ is a half weight on the first element which one understands as the image of the corresponding half weight on the source manifold. The volume of this pair is given by

$$\int_S \theta^2 \in \mathbb{R}. \tag{2.2.5}$$

By the definition the moduli space of half weighted moduli space is fibered over the old one

$$\begin{aligned} \pi_{un} &: \mathcal{P}_S^{hw} \to \mathcal{P}_S, \\ \pi_{un} &: (\tilde{S}, \theta) \mapsto \tilde{S}. \end{aligned} \tag{2.2.6}$$

Moreover, there is another natural fibration

$$\pi_c : \mathcal{P}_S^{hw} \to \mathcal{B}_S \tag{2.2.7}$$

equals to the composition of (2.1.4) and (2.2.6). Therefore the moduli space of half weighted planckian cycles inherits a $U(1)$ - principle bundle structure coming from (2.1.4). And it was already mentioned that $\mathcal{P}_S^{hw}$ carries the canonical volume function

$$\begin{aligned} \mu &: \mathcal{P}_S^{hw} \to \mathbb{R}, \\ \mu(\tilde{S}, \theta) &= \int_{\tilde{S}} \theta^2 \end{aligned} \tag{2.2.8}$$

which is obviously invariant under the $U(1)$ - action. The last remark is going to be very important after the following fact is established:

Proposition 2.3 ([10], [25]). *The moduli space of half weighted planckian cycles $\mathcal{P}_S^{hw}$ admits a Kaehler structure invariant under the $U(1)$ - action.*

The idea of the proof is to exploit the specialty of the tangent spaces to the moduli space $\mathcal{P}_S^{hw}$. Over a point it is the direct sum

$$T_{(\tilde{S},\theta)}\mathcal{P}_S^{hw} = C^\infty(S, \mathbb{R}) \oplus C^\infty(S, \mathbb{R}), \tag{2.2.9}$$

and both the summands are identified canonically. Moreover, as one has canonical Darboux - Weinstein local coordinates for the planckian "unweighted" cycles as well there are canonical *complex* Darboux - Weinstein local coordinates for the moduli space of half weighted planckian cycles (see [10]). These coordinates were introduced in [10]. Thus in an arbitrary point $(\tilde{S}_0, \theta_0)$ belongs to the moduli space the canonical local coordinates are given by a pair of real smooth functions

$$(\psi_1, \psi_2), \qquad \psi_i \in \mathbb{C}^\infty(S, \mathbb{R}),$$

where the first function responds to the deformations of the planckian cycle while the second one reflects the deformations of the half weight part. One can easily express in these coordinates two natural tensors "living" on the moduli space. The first one has type (1,1) being a linear operator:

$$I|_{(\tilde{S}_0, \theta_0)}(\psi_1, \psi_2) = (-\psi_2, \psi_1). \tag{2.2.10}$$

The next one has type (2,0) being a skew symmetric 2 - form:

$$\Omega_{(\tilde{S}_0, \theta_0)}(v_1, v_2) = \int_{\tilde{S}_0} [\psi_1 \phi_2 - \psi_2 \phi_1] \theta_0^2, \tag{2.2.11}$$

where $v_1 = (\psi_1, \psi_2), v_2 = (\phi_1, \phi_2)$ are tangent vectors. One could easily check that this 2 - form is nondegenerated everywhere and that Ω is compatible with I. The corresponding riemannian metric has the form

$$G_{(\tilde{S}_0, \theta_0)}(v_1, v_2) = \int_{\tilde{S}_0} [\phi_1 \psi_1 + \phi_2 \psi_2] \theta_0^2. \tag{2.2.12}$$

One can check directly that the form is closed and that the complex structure is integrable. But the authors use in [10], [25] the following elegant trick to establish the results. The point is that for any smooth manifold X its cotangent bundle admits a canonical symplectic structure while its tangent bundle admits a canonical complex structure. Thus it's sufficient to consider two natural maps (see [10]): the first one is a local isomorphism of $\mathcal{P}_S^{hw}$ and $T\mathcal{P}_S$ and the second one is a global double covering $\mathcal{P}_S^{hw} \to T^*\mathcal{P}_S$ without ramification. Both the map are defined canonically so they do not require any additional choices than were made in the beginning of the story. Moreover, as it was checked the structures (2.2.10) and (2.2.11) are isomorphic to the canonical ones. Moreover, since the canonical symplectic structure on $T^*\mathcal{P}_S$ is "strong" then the same is true for (2.2.10). Therefore $\mathcal{P}_S^{hw}$ is an infinite dimensional Kaehler manifold and one should mention that the Kaehler structure was constructed canonically without any additional choices. As well one can easily check that this Kaehler structure is invariant under the action of $U(1)$ described above. The function μ defined in (2.2.8) is a moment map for this action (see [10]) thus one can produce a new Kaehler manifold using the standard mechanism of Kaehler reduction. To get this new manifold we choose a regular value of the moment map function

$$\mu(\tilde{S}, \theta) = \int_{\tilde{S}} \theta^2 = r \in \mathbb{R}. \tag{2.2.13}$$

The reduced Kaehler manifold is called the moduli space of Bohr - Sommerfeld Lagrangian cycles of fixed volume and denoted as $\mathcal{B}_s^{hw,r}$. Thus the real parameter r shows the volume of the weighted cycles. This moduli space is fibered over $\mathcal{B}_S$ so as a symplectic manifold it admits a canonical real polarization. At the same time it admits a canonical complex polarization being a Kaehler manifold. Moreover, it is algebraic since the Kaehler metric is of the so - called Hodge type (the Berry bundle is related with the Kaehler class, see [10]). So this algebraic manifold is ready to be quantized. But the main topic of our text is to show that this moduli space is the quantum phase space for a system which is strongly related to the original classical mechanical system, described by our given manifold (M, ω). Having this fact as the target we shall omit all the related discussion (which could be found in [10], [25]).

2.3. Induced functions on the moduli space.

We will perform the computations using the following short local description of the moduli space of Bohr - Sommerfeld Lagrangian cycles of fixed volume 1. So it is an infinite dimensional Kaehler manifold which consists of pairs (S, θ) where S is a Bohr - Sommerfeld Lagrangian cycle in M and θ is a half weight on it such that for any pair $(S, \theta) \in \mathcal{B}_S^{hw,1}$ the volume is fixed:

$$\int_S \theta^2 = 1. \tag{2.3.1}$$

The computation is done for the moduli space $\mathcal{B}_S^{hw,1}$ but it can be easily rearranged for any other positive volume r or any higher level k. The tangent space to the moduli space $\mathcal{B}_S^{hw,1}$ in a point (S, θ) is represented by pairs (ψ_1, ψ_2) such that

$$\int_S \psi_i \theta^2 = 0, \qquad \psi_i \in C^\infty(S, \mathbb{R}). \tag{2.3.2}$$

For any two tangent vectors $v_1 = (\psi_1, \psi_2), v_2 = (\phi_1, \phi_2)$ at the point (S, θ) the symplectic form Ω has the form

$$\int_S [\psi_1 \phi_2 - \psi_2 \phi_1] \theta^2. \tag{2.3.3}$$

We've spoke about the dynamical correspondences which could be either quantizable or not. But we'd like to keep the order of the investigated facts so here in this section we will not say anything about the dynamical correspondence and just directly introduce some functions over the moduli space induced by smooth functions from $C^\infty(M, \mathbb{R})$, following [28]. For any $f \in C^\infty(M, \mathbb{R})$ one has

$$F_f \in C^\infty(\mathcal{B}_S^{hw,1}, \mathbb{R}), \tag{2.3.4}$$

defined in absolutely natural way. Indeed, at each point (S, θ) it is given by

$$F_f(S, \theta) = \tau \int_S f|_S \theta^2 \in \mathbb{R}, \tag{2.3.5}$$

where τ is a real parameter. Formula (2.3.5) gives a map

$$\mathcal{F}_\tau : C^\infty(M, \mathbb{R}) \to C^\infty(\mathcal{B}_S^{hw,1}, \mathbb{R}) \tag{2.3.6}$$

which is obviously linear. The main aim of the rest of the present section is to show that this linear map is a homomorphism of Lie algebras. The original symplectic structure ω defines the Poisson bracket on the source space while the constructed symplectic structure (2.3.3) defines the quantum Poisson bracket on the target space in (2.3.6). And as we'll see the map $\mathcal{F}_\tau$ transforms the classical bracket to the quantum one up to a constant which depends on our real parameter τ. Here we follow [28] keeping the notations.

Let $f \in C^\infty(M, \mathbb{R})$ is an arbitrary smooth function on our given symplectic manifold M. Then its differential *df* gives a tangent vector being restricted to any Bohr - Sommerfeld cycle S. This gives a vector field A_f on the moduli space $\mathcal{B}_S^{hw,1}$ which doesn't depend on the second coordinate. Therefore this A_f is constant along the fibers

$$\mathcal{B}_S^{hw,1} \to \mathcal{B}_S. \tag{2.3.7}$$

It's not hard to see that the singular points of A_f are given by the pairs (S, θ) such that S belongs to a level set of f. As well our smooth function f generates naturally a 1 - form on $\mathcal{B}_S^{hw,1}$ which we denote as B^f. In point (S_0, θ_0) this 1 - form is given by

$$B^f_{(S_0,\theta_0)}(\psi_1, \psi_2) = \int_{S_0} f \cdot \psi_2 \theta_0^2. \tag{2.3.8}$$

The direct substitution to the formula (2.3.3) ensures that these objects are related, namely:

$$B^f = \Omega^{-1}(A_f). \tag{2.3.9}$$

Indeed, for any vector field

$$v = (\psi_1(S,\theta), \psi_2(S,\theta))$$

one has

$$\Omega_{(S,\theta)}(A_f, v) = \int_S [f\psi_2 - \psi_1 \cdot 0]\theta^2 = \int_S f\psi_2\theta^2 = B^f(v),$$

since our vector field in the local coordinates looks as $(f, 0)$. Now we are in position to state and prove

Proposition 2.4.([28]). *For any smooth functions $f, g \in C^\infty(M, \mathbb{R})$ the following identity holds*

$$\{F_f, F_g\}_\Omega = 2\tau F_{\{f,g\}_\omega} \tag{2.3.10}$$

where F_f, F_g are the images of f, g under $\mathcal{F}_\tau$.

Let's remark first of all that the map $\mathcal{F}_\tau$ doesn't preserve the standard algebraic structure on $C^\infty(M, \mathbb{R})$, defined by usual pointwise multiplication. It follows from the same property of the integral: the integration of the product $f \cdot g$ is not usually the same as the product of two integrations of f and g respectively. It means that

$$F_f \cdot F_g \neq F_{f \cdot g}.$$

At the same time according to Proposition 2.4. the image

$$Im \mathcal{F}_\tau \subset C^\infty(\mathcal{B}_S^{hw,1}, \mathbb{R})$$

is a Lie subalgebra. Assume that the given classical mechanical system is integrable hence (M, ω) admits a set of n smooth functions in involution which are algebraically independent. Since

$$\{f_i, f_j\}_\omega = 0$$

the induced functions $F_{f_1}, ..., F_{f_n}$ commute with respect to the quantum Poisson bracket over the moduli space $\mathcal{B}_S^{hw,1}$. But the same is true for the set consists of the functions of the following shape

$$F_{f_1^{r_1} \cdot ... \cdot f_n^{r_n}}, \qquad r_i \in \mathbb{Z}.$$

The corresponding preimages, of course, lie in the algebraic span of $\{f_i\}$. But according to our remark the last function doesn't belong to the algebraic span of $F_{f_1}, ..., F_{f_n}$. It means that for any integrable classical system the corresponding moduli space (read: quantum system) is integrable to. Now a question arises: if our given classical system was completely integrable (so $dimM = 2n$) is the same true for the quantum system? Roughly the space of commuting functions over $\mathcal{B}_S^{hw,1}$ has dimension $\mathbb{Z}^n$ while the moduli space itself has dimension $2 \cdot C^\infty(S, \mathbb{R}) - 2$ thus it seems that in general the question is not quite obvious.

Now we start the proof, performing the computation for the case when $\tau = 1$ for simplicity since the general case follows immideately. For a function $f \in C^\infty(M, \mathbb{R})$ we take the corresponding induced function F_f on the moduli space and compute its differential. Perturbing the arguments one gets in the local coordinates that

$$dF_f(S_0, \theta_0)(\psi_1, \psi_2) = \int_{S_0} 2f\psi_2\theta_0^2 + \int_{S_0} d\psi_1(\omega^{-1}(df)|_{S_0})\theta_0^2, \tag{2.3.11}$$

where the first 1 - form on the right hand side is constant with respect to the first coordinate while the second 1 - form doesn't depend on the second coordinate. It's easy to recognize our 1 - from B^f (2.3.8), multiplied by 2, under the first summand in (2.3.11). As it was already pointed out (see (2.3.9)) the symplectically dual to B^f vector field is represented exactly by A_f. Therefore the Hamiltonian vector field for our induced function F_f has the following form

$$X_{F_f} = \Omega^{-1}(dF_f) = 2A_f + C_f,$$

where C_f is the vector field symplectically dual to the second summand in (2.3.11):

$$(C_f)^{sd} = \int_{S_0} d\psi_1(\omega^{-1}(df)|_{S_0})\theta_0^2. \tag{2.3.12}$$

We don't need to compute any explicit formula for C_f taking in mind the following argument. It was mentioned that vector field A_f is constant along the second coordinate while the symplectically dual 1 - form B^f maps to zero any vector field which depends only on the first coordinate. The vector field C_f and its symplectically dual 1 - form possess the inverse property. Our "quantum" symplectic form is divided in the variables hence one has

$$\begin{aligned} \{F_f, F_g\}_\Omega = \Omega(X_{F_g}, X_{F_f}) &= \Omega(2A_g + C_g, 2A_f + C_f) = \\ &= 2\Omega(A_g, C_f) - 2\Omega(A_f, C_g), \end{aligned} \tag{2.3.13}$$

since it follows from (2.3.3) that

$$\Omega(A_f, A_g) = \Omega(C_f, C_g) = 0$$

for any $f, g \in C^\infty(M, \mathbb{R})$. Further, from (2.3.13) one gets

$$\{F_f, F_g\}_\Omega = 2(C_g)^{sd}(A_f) - 2(C_f)^{sd}(A_g), \tag{2.3.14}$$

where $(C_f)^{sd}$ is given by (2.3.12). According to (2.3.14) we can compute the bracket, avoiding to write down any expilcite formula for C_f. Substituting the expressions for the vector fields and 1 - forms in (2.3.14) we get

$$\begin{aligned} \{F_f, F_g\}_\Omega = 2\int_{S_0} df|_{S_0}(\omega^{-1}(dg)|_{S_0})\theta_0^2 \\ -2\int_{S_0} dg|_{S_0}(\omega^{-1}(df)|_{S_0})\theta_0^2. \end{aligned} \tag{2.3.15}$$

We claim that the total integrand

$$2[df|_{S_0}(\omega^{-1}(dg)|_{S_0}) - dg|_{S_0}(\omega^{-1}(df)|_{S_0})] \tag{2.3.16}$$

is the restriction of a smooth function to S_0. We'll see in a moment that this smooth function is exactly the Poisson bracket of f and g multiplied by 2. To check this coincidence we take the Poisson bracket $\{f, g\}_\omega$. It's easy to see that

$$\begin{aligned} 2\{f, g\}_\omega = 2df(\omega^{-1}(dg)) \\ = -2dg(\omega^{-1}(df)) = df(\omega^{-1}(dg)) - dg(\omega^{-1}(df)). \end{aligned} \tag{2.3.17}$$

For simplicity let's choose any compatible riemannian metric g on M getting the corresponding almost complex structure I. We are interested in the induced by this choice local decomposition of the tangent bundle TM over our Bohr - Sommerfeld Lagrangian cycle S_0. At each point $s \in S_0$ one has

$$T_sM = T_sS_0 \oplus I(T_sS_0). \tag{2.3.18}$$

First of all, expression (2.3.16) differs from (2.3.17) by the restrictions of the differentials and the vector fields to S_0. Let's manage the corresponding decompositions in (2.3.17). We denote as "hor" - components the restrictions to the tangent directions to S_0 and as "vert" components all what belong to the transversal spaces (thanks to the riemannian metric). Rearrange the first summand in the right hand side of (2.3.17) as follows

$$\begin{aligned} (df_{ver}((Ig^{-1}(dg))_{ver}) + df_{ver}((Ig^{-1}(dg))_{hor}) + \\ df_{hor}((Ig^{-1}(dg))_{ver}) + df_{hor}((Ig^{-1}(dg))_{hor})). \end{aligned} \tag{2.3.19}$$

In (2.3.19) only two summands are non trivial — namely the first and the fourth ones ("ver - ver" and "hor - hor"). Analogiuosly, one has to the second summand in (2.3.17) that it equals to

$$-(dg_{ver}((Ig^{-1}(df))_{ver}) + dg_{hor}((Ig^{-1}(df))_{hor})), \tag{2.3.20}$$

thus again we have only two summands. However from the compatibility condition of g, ω and I we have that

$$df_{ver}((Ig^{-1}(dg))_{ver}) = -dg_{hor}((Ig^{-1}(df))_{hor}) \tag{2.3.21}$$

and

$$df_{hor}((Ig^{-1}(dg))_{hor}) = -dg_{ver}((Ig^{-1}(df))_{ver}). \tag{2.3.22}$$

Therefore we can rewrite the expression for the Poisson bracket $\{f, g\}_\omega$ restricted to S_0 in terms of "hor" - components only:

$$\{f, g\}_\omega|_{S_0} = df_{hor}((\omega^{-1}(dg))_{hor}) - dg_{hor}((\omega^{-1}(df))_{hor}). \tag{2.3.23}$$

It remains to mention that "hor" - components are exactly what one gets taking restrictions to S_0 thus

$$\{f, g\}_\omega|_{S_0} = df|_{S_0}(\omega^{-1}(dg)|_{S_0}) - dg|_{S_0}(\omega^{-1}(df)|_{S_0}) \tag{2.3.24}$$

and the result doesn't depend on the choice of any compatible riemannian metric. Comparing (2.3.16) and (2.3.24) we get the desired identity.

2.4. Adjunct: integer and real parameters.

In this small subsection we'd like to mention how the identity, stated in Proposition 2.4, changes when we vary integer and real parameters, contained in the picture. Recall, that there are two real continuous parameters r and τ and one integer parameter k. We start with level $k = 1$.

The first level. In this case one has that the Possion brackets are proportional with coefficient 2τ (see (2.3.10). It's clear that this coefficient doesn't depend on the volume of cycles. On the other hand, $\mathcal{F}_\tau$ maps

$$f \equiv const = c \qquad \Longrightarrow F_f \equiv const = \tau \cdot r \cdot c. \tag{2.4.1}$$

Therefore if one wants to establish the situation when all numerical quantization requirements from the Dirac list hold (it means that

$$\tau \cdot r = 1,$$
$$2\tau = 1)$$

it would be necessary to take

$$\tau = \tfrac{1}{2},$$
$$r = 2.$$

At this step we see that anyway the product $\tau \cdot r$ should equal to 1 while 2τ can vary with respect to the question about the Planck constant. We understand the Planck constant just as a proportionality coefficient. From the Berezin point of view the Planck constant depends only on the level of quantization. Let's see what happens when we change the last one.

General level. One could perform the same constructions for any generic level. Let's fix any $k \in \mathbb{N}$ and construct the moduli space $\mathcal{B}^{hw,r}_{S,k}$ in the same manner as $\mathcal{B}^{hw,r}_{S}$ starting with slightly different prequantization data (L^k, a_k). Then one has a natural embedding

$$\mathcal{B}^{hw,r}_{S} \to \mathcal{B}^{hw,r}_{S,k} \tag{2.4.2}$$

(see [10]). The Kaehler structures on both the moduli spaces in (2.3.26) are slightly different; this means that f.e. the symplectic form Ω on the level 1 moduli doesn't coincide with the restriction of the symplectic form Ω_k defined on the level k: here it's a crucial point which comes from the difference between canonical Darboux - Weinstein coordinates for $\mathcal{B}^{hw,r}_{S}$ and $\mathcal{B}^{hw,r}_{S,k}$. Indeed, if (S, θ) is originally Bohr - Sommerfeld Lagrangian cycle with respect to (L, a) then it's clear that it remains to be Bohr - Sommerfeld with respect to (L^k, a_k). But the canonical complex Darboux - Weinstein coordinates for the level 1 moduli space are given by $\omega^{-1}(df) \oplus df$ where f lives on S while for the level k moduli space they are given by $(k\omega)^{-1}(df) \oplus df = k^{-1}\omega^{-1}(df) \oplus df$. This means that one rescales one half of the first coordinate system to get the second. Locally the difference can be recognized as follows. Let T^*S is as usual the tangent bundle to S. Then this tangent bundle admits not only one symplectic structure (the canonical one) but a family of symplectic structures with respect to real parameter λ. Indeed, in the basic definition of the canonical 1 - form (see (2.1.7)) one could slightly change the right hand side

$$\forall \alpha \in \Omega^1_S \qquad \Gamma^*_\alpha \eta_\lambda = \lambda \alpha. \tag{2.4.3}$$

Thus for any $\lambda \in \mathbb{R}$ one gets an "almost" canonical 1 - form η_λ which is nondegenerated and gives a nondegenerated 2 - form $\omega_\lambda = d\eta_\lambda$. This "almost" canonical symplectic form looks like the canonical one; it's not hard to find an appropriate symplectomorphism

$$\Psi_\lambda : (T^*S, d\eta) \to (T^*S, d\eta_\lambda), \tag{2.4.4}$$

it just multiplies every fiber of the canonical projection by λ. The family $d\eta_\lambda$ is a possible degeneration of the canonical symplectic structure on T^*S. At the same time an interesting effect appears: due to the canonical form of symplectic structure in the canonical coordinates $d\eta$ and $d\eta_\lambda$ are proportional while the same is not true for the canonical coordinates. The proportionality coefficient is just λ. Turning to the canonical Poisson brackets one sees that the corresponding skew symmetrical pairings on the function space are proportional to; the ratio is λ^{-1}. Coming back to the moduli spaces we get that Ω and $\Omega_k|_{\mathcal{B}^{hw,r}_S}$ are proportional with coefficient k. Therefore if one defines induced functions on $\mathcal{B}^{hw,r}_{S,k}$ constructing the similiur map

$$\mathcal{F}^k_\tau : C^\infty(M, \mathbb{R}) \to C^\infty(\mathcal{B}^{hw,r}_{S,k}, \mathbb{R}), \tag{2.4.3}$$

given by the same formula

$$F_f(S, \theta) = \tau \int_S f|_S \theta^2 \in \mathbb{R} \tag{2.4.4}$$

then

$$\{F_f, F_g\}_{\Omega_k}|_{\mathcal{B}_S^{hw,r}} = \frac{1}{k}\{F_f, F_g\}_{\Omega}. \tag{2.4.5}$$

But it doesn't lead to a contradiction since

$$\{f, g\}_{k\omega} = \frac{1}{k}\{f, g\}_{\omega}.$$

And totally it gives

$$F_{\{f,g\}_\omega} = \frac{k}{2\tau}\{F_f, F_g\}_{\Omega_k} \tag{2.4.6}$$

over $\mathcal{B}_{S,k}^{hw,r}$. Now if we want to satisfy the Dirac conditions one needs

$$\begin{aligned} \frac{2\tau}{k} &= \hbar, \\ \tau \cdot r &= 1. \end{aligned} \tag{2.4.7}$$

Of course we could say that 2τ has to equal 2 and r equals $\frac{1}{2}$ as above. On the other hand we can fix the ratio

$$\frac{2\tau}{k} = const,$$

then it implies that

$$\tau \to \infty \qquad \Longrightarrow r \to 0$$

and in the limit one gets the moduli space of unweighted Lagrangian cycles. Really, if k tends to ∞ then the moduli space of unweighted Bohr - Sommerfeld cycles covers the moduli space of all Lagrangian cycles as a dense set (as rational points cover $b_1(M)$ - torus which is the Jacobian). At the same time the weights are going to zero (since $r \to 0$) so in the limit one could forget about the second components in the pairs (S, θ). Anyway one couldn't define, say, a Poisson structure on the moduli space of Lagrangian cycles as the limit of the symplectic structures on the moduli spaces of different levels since as we've seen the symplectic structure on the moduli space $\mathcal{B}_{S,k}^{hw,r}$ degenerates when $k \to \infty$.

We'll come back to the discussion in Section 4.

3. Dynamical correspondence in the algebraic Lagrangian geometry

In this section we essentially develop the result of the previous one. We show that the functions given by map (2.3.6) on the moduli space are quasi symbols for any $f \in C^\infty(M, \mathbb{R})$. To prove this main fact we use a dynamical correspondence showing that this one is quantizable in the sense of subsection 1.5, (1.5.20). Here we follow [29], [30].

3.1. Quasi symbols over Kaehler manifolds.

Let $\mathcal{K}$ is a Kaehler manifold. It could be considered from two different points of view: in the setup of complex geometry it is a complex manifold equipped with a positive polarization while in the framework of symplectic geometry it is a symplectic manifold equipped with a complex polarization. If we say that it is a real manifold equipped with a Kaehler triple it were a boundary (or the most complete) setup. Anyway we understand a Kaehler manifold in this way: one has a Kaehler triple (G, J, Ω) over the based real manifold and each element in the triple is of the same importance. Here G is a riemannian metric, J is a complex structure and Ω is a symplectic form such that the usual compatibility conditions hold. One could recover each element of any triple from the other two. The elements in Kaehler triples play their own individual roles: the riemannian metric G responds to some measurement process (distances, volumes, etc.), the symplectic form generates some dynamical properties while the complex structure imposes a different kind of geometry (holomorphic vector bundles, complex submanifolds, etc.) growing up over the real one. Moreover, one could consider the Kaehler triples over any real manifold as solutions of some natural $Diff$ - invariant equations on the space of all possible hermitian structures over the base (see [26]). (As well one has to mention here that symplectic geometry itself admits generalizations of classical results from the algebro geometrical setup, f.e., S. Donaldson extended the famous Kodaira result to the symplectic case, see [7], and we will exploit this generalization below).

We remind here the following definition from the previous part of the text:

Definition. *Smooth real function over Kaehler manifold $\mathcal{K}$ with Kaehler structure (G, J, Ω) is called quasi symbol iff its Hamiltonian vector field preserves the riemannian metric:*

$$Lie_{X_f} G \equiv 0. \tag{3.1.1}$$

Notice that the constant functions satisfy the requirement above so we will mention that they are quasi symbols. For any Kaehler manifold $\mathcal{K}$ we denote the space of quasi symbols as $C^\infty_q(\mathcal{K}, \mathbb{R})$. It's obvious (see Proposition 1.3.) that $C^\infty_q(\mathcal{K}, \mathbb{R})$ is a Lie subalgebra of the Poisson algebra over $\mathcal{K}$. The dimensions of this subalgebra (as a vector space and as a finitely generated subalgebra) are two integer characteristics of the Kaehler manifold. It's not hard to roughly estimate the first number in the framework of complex or algebraic geometry. First of all it's easy to see that the low boundary always equals to 1. As well we have

Proposition 3.1. *For any compact Kaehler manifold $dimC_q^\infty(\mathcal{K},\mathbb{R})$ is finite.*

To prove it we just remark that if Hamiltonian vector field X_f preserves the riemannian metric then it preserves as well the complex structure. It means that X_f is the real part of a holomorphic vector field. Thus one has a map

$$\sigma : C_q^\infty(\mathcal{K},\mathbb{R}) \to H^0(M_J, T^{1,0}M_J). \tag{3.1.2}$$

This map is $\mathbb{R}$ - linear. It's clear that its kernel contains only constant functions. Indeed, every holomorphic vector field can be reconstructed from its real part uniquely. Since real holomorphic vector field doesn't exist the map is an inclusion modulo constant. On the other hand, if a holomorphic vector field $v_h \in H^0(M_J, T^{1,0})$ has real part Rev_h equals to a Hamiltonian vector field

$$X_f = Rev_h$$

then Imv_h is not Hamiltonian. Really, if Rev_f is Hamiltonian then Imv_f is proportional to the gradient vector field for f with respect to G. But a gradient vector field never coincides with a Hamiltonian one. Therefore one complex dimension in $H^0(M_J, T)$ can give at least one real dimension in $C^\infty(\mathcal{K},\mathbb{R})$. Thus

$$1 \leq dimC^\infty(\mathcal{K},\mathbb{R}) \leq h^0(M_J, T) + 1. \tag{3.1.3}$$

Since $\mathcal{K}$ is compact the right hand side in (3.1.3) is finite (and can be estimated by, say, the Riemann - Roch formula) so the same is true for the "middle hand side".

Of course, the bounds (3.1.3) are too rough. Even in the case when $\mathcal{K}$ as a complex manifold admits a lot of infinitesimal automorphisms these holomorphic vector fields need not possess the property that their real parts are represented by some Hamiltonian functions with respect to some symplectic structure. From the point of view of holomorphic geometry the question is quite meaningful and useful: if X is a complex manifold what is the Kaehler metric on it which admits the maximal symmetry? This classical question from the complex analysis has as we've seen a projection to the problem of quantization. On the other hand, in the set of Kaehler manifolds one could distinguish a subset consists of such submanifolds for which the question can be reformulated. We mean the Kaehler metrics of Hodge type. For a metric of Hodge type the corresponding Kaehler class is integer and we turn to the framework of algebraic geometry. Here we meet first of all the most "algebraic" manifold — the projective space. This algebraic manifold admits holomorphic vector fields and moreover every complex dimension in the space of these fields can be realized as a complexified real dimension in the space of quasisymbols with respect to the Fubini - Study metric. This metric is the result of the projectivization procedure (discussed in Section 1) starting from a Hilbert space. Every holomorphic vector field on $\mathbb{CP}^n$ comes from a linear operator on $\mathbb{C}^{n+1}$. If a hermitian structure on $\mathbb{C}^{n+1}$ is fixed then the space of linear operators splits into the direct sum of real and imaginary parts (self adjoint and skew symmetric operators). Every real part (= self adjoint operator) gives a real function on the projective space which is a quasi symbol. Thus the real part of any holomorphic vector field is just the Hamiltonian vector field of the quasi symbol given by the real part of the linear operator which induced our holomorphic vector field. In this case one has

$$(n+1)^2$$

quasi symbols (real dimension, of course) while $(n+1)^2-1$ holomorphic vector fields (complex dimension) — we count the identical linear operator in the first set while we don't count it in the second one since it gives zero vector field.

Thus the projective spaces admit some maximality property with respect to the question how many quantum symmetries one Kaehler manifold carries. This fact is included in complex analysis in the following way: for hermitian metrics there is a notion of sectional holomorphic curvature (and it is discussed in [3] from the point of view, appropriate for us) and the projective spaces possess the property that the Fubini - Study metrics are of the constant sectional holomorphic curvature. Thus the condition of the constant sectional holomorphic curvature can be retranslated to the language of quantum theory: a Kaehler manifold $\mathcal{K}$ is of the constant sectional holomorphic curvature if this quantum space carries maximal quantum symmetries. It's well known fact that in finite dimension there is unique such a Kaehler manifold — the projective space. In the infinite dimensional case it is an open question in complex analysis. Therefore if one constructs an infinite dimensional Kaehler manifold which admits maximal quantum symmetries (and which is not a projective space) then it were a candidate to the solution of this classical problem in complex differential geometry.

Now, if we have any Kaehler manifold with integer Kaehler class then we can emdedd it to a projective space (by the complete linear system of some power of the Kaehler class). So the question can be reduced to the case when Q is a submanifold in $\mathbb{CP}^n$. Of course, it's just a first step in the study of the question but we should skeep this discussion leaving some additional details (available, f.e., from [16]) outside of this text to keep our major theme.

Notice, that the arguments, applied above to prove Proposition 3.1, give us an additional statement which will be useful in subsection 3.4. Hence we place here

Lemma. *If $f \in C^\infty(\mathcal{K}, \mathbb{R})$ is a quasi symbol over some Kaehler manifold $\mathcal{K}$ then the critical set*

$$Crit(f) = \{x \in \mathcal{K} \quad | \quad df(x) = 0\}$$

is a complex submanifold of $\mathcal{K}$.

Really, for any $f \in C^\infty_q(\mathcal{K}, \mathbb{R})$ it should be a holomorphic vector field $\sigma(f)$, given by (3.1.2), such that by the definition

$$Crit(f) \equiv (\sigma(f))_0 \subset \mathcal{K},$$

where $(\sigma(f))_0$ is the vanishing set of $\sigma(f)$. This immediately gives us the statement since $(\sigma(f))_0$ has to be a complex submanifold.

Let us emphasize that the last statement is true in the infinite dimensional case as well as in the previous one when we estimate the dimension. But we have to go to the infinite dimensional case since it's clear that the finite dimensional is exhausted.

In the previous section it was constructed a map from the space of smooth functions over M to the space of smooth functions over the moduli space of half weighted Bohr - Sommerfeld cycles of fixed volume. But as we've seen in Section 1 one is interested only on smooth functions of special type in the quantization framework; quantum observables have to possess the property which requires for their Hamiltonian vector field to preserve all the kinematical data on the quantum phase space. Thus to continue the story one has to show that all these induced smooth functions are quasi symbols over the moduli space. We formulate the main result of this section in the following

Theorem 1 ([30]). *Let (M, ω) be a simply connected compact symplectic manifold with integer symplectic class, $[S]$ be a homological class of middle dimension and $\mathcal{B}_S^{hw,r}$ be the corresponding moduli space of half weighted Bohr - Sommerfeld Lagrangian cycles of fixed volume equipped with the corresponding Kaehler triple (G, I, Ω). Then for the linear map $\mathcal{F}_\tau$ defined in (2.3.5) one has that*

1) for any $f \in C^\infty(M, \mathbb{R})$ the induced function F_f is a quasisymbol over the moduli space;

2) the correspondence principle takes place in the form

$$\{F_f, F_g\}_\Omega = 2\tau F_{\{f,g\}_\omega};$$

3) the map

$$\mathcal{F}_\tau : C^\infty(M, \mathbb{R}) \to C_q^\infty(\mathcal{B}_S^{hw,r}, \mathbb{R})$$

gives an irreducible representation of the Poisson algebra.

The second item from the list is already known from the previous section but during the construction which is given by a dynamical correspondence we'll get the identity "charge-free".

3.2. Dynamical correspondence.

We've discussed dynamical correspondences in subsection 1.5. The root idea is very natural in the symplectic setup. The point is that symplectic geometry was created (and understood) as a strong and convenient mathematical language describing Hamiltonian mechanical systems. So dynamics should be imposed to any investigation in the symplectic setup. At the same time one should emphasize that the Lagrangian condition

$$\omega|_S \equiv 0$$

looks like local and static while the Bohr - Sommerfeld condition is dynamical: local Bohr - Sommerfeld deformations (see subsection 2.1) precisely correspond to Hamiltonian deformations induced by the Hamiltonian dynamics. Thus the dynamical property of the system defines a correspondence between Hamiltonian vector fields on the based manifold and some special vector fields on the moduli space. Let us discuss this point more concretely.

Let (M, ω) be as usual a simply connected compact symplectic manifold of dimension $2n$ with integer symplectic form. Then for a fixed homology class $[S] \in H_n(M, \mathbb{Z})$ one has an infinite dimensional moduli space $\mathcal{B}_S^{hw,r}$ of half weighted Bohr - Sommerfeld cycles of fixed volume, endowed with a canonical Kaehler structure, described in Section 2. The moduli space consists of pairs (S, θ) where the first element is a Bohr - Sommerfeld Lagrangian cycle and θ is a half weight over it such that

$$\int_S \theta^2 = 1.$$

Of course, this moduli space could be empty for some classes from $H_n(M, \mathbb{Z})$ and some source manifold S, but we reject these vanishing cases (f.e. every symplectic manifold carries the moduli space corresponds to the trivial homology class in $H_n(M, \mathbb{Z})$ and an appropriate topological type of S). Since our M is simply connected we identify the space of Hamiltonian vector fields with the smooth function space modulo constant. One has a map

$$\Theta_{DC} : Vect_\omega(M) \equiv C^\infty(M, \mathbb{R})/const \to Vect(\mathcal{B}_S^{hw,r}), \tag{3.2.1}$$

which is called a dynamical correspondence. It can be constructed as follows. For any function f one has the corresponding dynamics, induced by the Hamiltonian vector field on M. This dynamics preserves our symplectic manifold (M, ω) and moreover if we choose a prequantization data it could be lifted to this setup almost uniquely (up to the canonical gauge transformations). Thus this dynamics preserves whole the data hence it defines a germ of automorphism of the moduli space $\mathcal{B}_S^{hw,r}$. This just gives us a vector field on the moduli space which admits some additional properties. The specialty reflects the fact that this vector field preserve the Kaehler structure since it was defined in invariant way. Since M is compact every function f defines a germ of symplectomorphism so every Hamiltonian vector field gives an infinitesimal transformation of the moduli space. Generalizing over the space of all Hamiltonian vector fields we get the map, which is obviously linear. Moreover, by the construction one gets

Proposition 3.2([30]). *The image*

$$Im\Theta_{DC} \subset Vect_K(\mathcal{B}_S^{hw,1}),$$

where the last space consists of the fields which preserve the Kaehler structure on the moduli space.

To make the story even more concrete we should compute the coordinates of any such special field which is generated by some smooth function $f \in C^\infty(M, \mathbb{R})$. This function first of all gives us the corresponding vector field X_f over M. Let $(S, \theta) \in \mathcal{B}_S^{hw,r}$ be a half weighted Bohr - Sommerfeld cycle. Over the support S the vector field X_f can be decomposed into the inner and the outer parts:

$$X_f = X_{ex} + X_{in}, \tag{3.2.2}$$

where

$$X_{in} \in TS$$

is the tangent component. Of course, we've met this decomposition in subsection 2.3 where we denoted it as "ver - hor". However here we'd like to change the notations emphasizing that decomposition (3.2.2) doesn't depend on any additional choices. Indeed, it's a remarkable property of symplectic geometry that over any point of a Lagrangian submanifold the tangent space canonically splits with respect to the symplectic form. Namely over $p \in S$ one has T_pS and $\omega^{-1}(T_p^*S)$ as two direct summands decomposing T_pM:

$$T_pM = T_pS \oplus \omega^{-1}(T_p^*S).$$

And then $X_{in} \in TS$ while X_{ex} belongs to $\omega^{-1}(T^*S)$. Thus X_{ex} gives a deformation of the cycle S itself while the inner part X_{in} deforms θ (and preserves S being tangent to it). Notice that if we reject the half weight parts then it leads to some degeneration of the picture: in this case the deformation is defined only by the restriction of the source function to S but in the half weighted case two functions with the same restriction to S give different deformations of the pair (S, θ) if they differs in a small neighborhood of S.

Therefore we can express the corresponding deformation given by X_f in the canonical complex Darboux - Weinstein coordinates (see subsection 2.2) as follows:

Proposition 3.3 ([30]). *The dynamically correspondent vector field, induced by X_f, has at a point (S, θ) coordinates (ψ_1, ψ_2) such that*

$$\begin{aligned} \psi_1 &= f|_S - \int_S f|S\theta^2, \\ \psi_2 &= \frac{Lie_{X_{in}}\theta}{\theta}. \end{aligned} \tag{3.2.3}$$

The second equality in (3.2.3) is moreless evident, so let's ensure that the first one takes place. The external part of the Hamiltonian vector field can be identified with a section of the normal bundle $N_S = TM|_S/TS$. On the other hand, we identified in subsection 2.1 the deformations of S with the function space over it modulo constant. Our symplectic form ω can be regarded as an isomorphism

$$\omega : TM \to T^*M. \tag{3.2.4}$$

Over S both the spaces are decomposed in the direct sums

$$TM|_S = TS \oplus V, \qquad T^*M|_S = T * S \oplus V^*. \tag{3.2.5}$$

The map

$$\omega : V \to T^*S \tag{3.2.5}$$

(which has been used a moment ago) is an isomorphism, giving

$$\omega(X_{ex}) = df|_S = d(f|_S). \tag{3.2.7}$$

The last equality gives the first equality in Proposition 3.3. Thus we see that the dynamically correspondent to X_f vector field over the moduli space has very natural expression in terms of the canonical "dynamical" coordinates. Dynamical properties of the given system give us another condition satisfied by the vector fields from $Im\Theta_{DC}$ namely

Proposition 3.4 ([30]). *For any pair of smooth functions f, g the following identity holds*

$$\Theta_{DC}([X_f, X_g]) = [\Theta_{DC}(X_f), \Theta_{DC}(X_g)],$$

where in the right hand side one takes the standard commutator over the moduli space.

Now the natural question arises: does the dynamical correspondence Θ_{DC} described above admit a lift to the function level? In subsection 1.5 we called this as qunatability property hence in that terms one asks whether or not Θ_{DC} is quantizable. We find the answer to the question comparing for any smooth function $f \in C^{\infty}(M, \mathbb{R})$ two vector fields over the moduli space, namely $\Theta_{DC}(X_f)$ and the Hamiltonian vector field for F_f, derived from (2.3.6). This matching gives us the following

Proposition 3.5 ([30]). *For any smooth function $f \in C^{\infty}(M, \mathbb{R})$ one has*

$$X_{F_f} = 2\tau \Theta_{DC}(X_f).$$

The proof of this key statement is contained in the next subsection (we check this relationship just by direct computations). At the rest of the present one we explain how the statements of Theorem 1 follow from the propositions listed above.

The first statement of the theorem follows from the definition of quasi symbols and Propositions 3.2 and 3.5: for any function f the induced function F_f generates the Hamiltonian vector field proportional to a vector field which belongs to $Im\Theta_{DC}$. It means that X_{F_f} preserves whole the Kaehler structure over the moduli space hence F_f is a quasi symbol.

The second item follows from Propositions 3.4 and 3.5: indeed, one continues the equality of Proposition 3.4 in both directions, substituting the equality of Proposition 3.5

$$\frac{1}{2\tau} X_{F_{\{g,f\}_\omega}} = \Theta_{DC}([X_f, X_g]) =$$
$$[\Theta_{DC}(X_f), \Theta_{DC}(X_g)] = \frac{1}{4\tau^2}[X_{F_f}, X_{F_g}]$$

and the last term has the standard representation as a Hamiltonian vector field. This gives the correspondence principle in the familiar form

$$\{F_f, F_g\}_\Omega = 2\tau F_{\{f,g\}_\omega}.$$

The third statement is very important for us. As it was explained in subsection 1.5 the irreducibility condition in our non linear algebro geometrical setup contains two items: the first one says that $\mathcal{F}_\tau$ has trivial kernel and the second says that there is no any smooth proper submanifold in the moduli space such that for every f the corresponding Hamiltonian vector field X_{F_f} is tangent to the submanifold. We begin with the second requirement and we check it even in much more stronger form. Namely we show that for every point $(S, \theta) \in \mathcal{B}_S^{hw,r}$ and every tangent vector $v = (\psi_1, \psi_2) \in T_{(S,\theta)}\mathcal{B}_S^{hw,r}$ there exists a smooth real function $f \in C^{\infty}(M, \mathbb{R})$ such that the Hamiltonian vector field of the induced function F_f gives this vector at this point:

$$X_{F_f}(S,\theta) = v.$$

Of course, this stronger condition could be exploited in the discussion on properties of our Kaehler metric over the moduli space (f.e., is it a metric of constant holomorphic sectional curvature or not) but these questions come outside of the main theme of our text. One verifies this condition using Propositions 3.3 and 3.5 namely for any pair of smooth functions $\psi_1, \psi_2 \in C^\infty(S, \mathbb{R})$ one has to find a smooth function f over whole M such that

$$\begin{aligned} \psi_1 &= f|_S - const, \\ \psi_2 &= \frac{Lie_{X_{in}}\theta}{\theta}, \end{aligned} \tag{3.2.8}$$

(see (3.2.3)). The first equation in (3.2.8) can be easily solved taking any extension of ψ_1 over M as the function f. The second one is more delicate: this question extracts from the set of possible extensions those which are appropriate and a priori one could not say whether or not such extensions exist. We reduce the question to the following

Lemma. *Let S be any smooth compact real oriented manifold and η a volume form. Then for any real smooth function $\psi \in C^\infty(S, \mathbb{R})$ with zero integral:*

$$\int_S \psi\eta = 0,$$

there exists a vector field Y such that

$$\psi = \frac{Lie_Y \eta}{\eta}. \tag{3.2.9}$$

Indeed, since

$$\begin{aligned} \int_S \psi\eta &= 0, \\ d(\psi\eta) &= 0, \end{aligned}$$

by the hypothesis of the lemma our n - form $\psi\eta$ belongs to the trivial cohomology class. It means that there exists a $n-1$ form ζ such that

$$d\zeta = \psi\eta. \tag{3.2.10}$$

On the other hand, decoding $Lie_Y\eta$ one gets

$$Lie_Y\eta = d(\imath_Y\eta), \tag{3.2.11}$$

hence it remains to compare (3.2.9), (3.2.10) and (3.2.11) taking in mind that η has no zeros. The desired vector field thus can be constructed locally to solve the equation

$$\imath_Y\eta = \zeta$$

which is obviously possible (notice again that η is nonvanishing everywhere).

Now one can apply this lemma in our context using the following trick (which will be very useful in the next subsection as well). Since our S is oriented (see the first step in the construction of the moduli space $\mathcal{B}_S^{hw,r}$) we can consider the corresponding volume form η instead of the square θ^2. The Lie derivatives of θ and η are related by the identity

$$\frac{Lie_Y \theta}{\theta} = \frac{1}{2} \frac{Lie_Y \eta}{\eta} \tag{3.2.12}$$

(see the next subsection), thus the lemma above ensures us that for every ψ_2 there exists a vector field Y on the Bohr - Sommerfeld cycle S such that the second equation in (3.2.8) is satisfied. Now it remains to construct such an extension of ψ_1 to a neighborhood of S in M that the corresponding Hamiltonian vector field should give us our Y as its X_{in}. Since the consideration are local we will construct such extension for a small neighborhood of zero section in T^*S. The desired function $\tilde{f}$ has the form

$$\tilde{f}(x,p) = \psi_1(x) + p_x(Y_x), \tag{3.2.13}$$

where x is the S - coordinate, p is the coordinate along the fiber, identified simultaneously with the corresponding cotangent vector p_x, and Y is the vector field on S defined by ψ_2 which exists due to the lemma above. The standard isomorphism of a neighborhood of S in M and the neighborhood of zero section in T^*S maps $\tilde{f}$ to a function which we denote as f, claiming that it possesses the properties one needs to solve (3.2.8).

Thus we can deform any fixed point (S, θ) in any direction along the moduli space, acting by an appropriate induced quasi symbol. On the other hand, here we'd like to enforce the statement which has been proved in [30] for homogeneous symplectic manifolds. We mean that the map (2.3.6) is an inclusion. Here we establish it in full generality just slightly extending the arguments of [30]. Namely, it's not hard to see that $\mathcal{F}_\tau$ could have a kernel if it were a point $x \in M$ with a neighborhood $\mathcal{O}(x)$ such that for every Bohr - Sommerfeld Lagrangian cycle $S \in \mathcal{B}_S$ one has

$$S \cap \mathcal{O}(x) = \emptyset.$$

This means that if one takes a bump smooth function concentrated in $\mathcal{O}(x)$ then it has trivial restriction on any Bohr - Sommerfeld cycle and therefore it belongs to the kernel. But it is not the case if $\mathcal{B}_S^{hw,r}$ is nonempty. Indeed, for any point x we can arrange a Bohr - Sommerfeld Lagrangian cycle passing any fixed neighborhood of x. If x is any fixed point and S is a Bohr - Sommerfeld cycle (which does exist since we assume that $\mathcal{B}_S^{hw,r}$ is nonempty) then it's not hard to construct a smooth function f such that the corresponding flow generated by the Hamiltonian vector field X_f moves S to the place of x. Due to the dynamical property of the Bohr - Sommerfeld condition (see subsection 2.1) the image of the Bohr - Sommerfeld cycle should be Bohr - Sommerfeld again, thus for any compact smooth symplectic manifold if $\mathcal{B}_S^{hw,r}$ is nonempty then its "points" cover whole the based manifold. This remark is extremely important if we are going to exploit a "universal cycle" induced by the construction (and we are really going to do that in a future), — this remark hints that such a cycle does exist. Now, after this fact is understood, let us note that if for every smooth function $f \in C^\infty(M, \mathbb{R})$ there exists a Bohr - Sommerfeld Lagrangian cycle S such that the restriction $f|_S$ is non trivial then one can choose a half weight θ over S such that

$$\int_S f|_S \theta^2 \neq 0.$$

It's not hard to see that one can do the choice.

Thus we've completed the proof of Theorem 1 modulo the statement of Proposition 3.5. The next subsection contains direct verification of the desired identity.

3.3. Computations for Proposition 3.5.

First of all we remind again that the tangent space to the moduli space $\mathcal{B}_S^{hw,r}$ at point (S_0, θ_0) is represented by pairs of smooth functions (ψ_1, ψ_2) over S_0 satisfying the norm condition

$$\int_{S_0} \psi_i \theta_0^2 = 0. \tag{3.3.1}$$

Consider infinitesimal action of the Hamiltonian vector field X_f generated by a smooth function $f \in C^\infty(M, \mathbb{R})$. Again we mention the possibility to split the tangent space TM at the points of S_0 into two parts:

$$X_f = V_f + W_f$$

where

$$V_f = \omega^{-1}(d(f|_{S_0})), \tag{3.3.2}$$

and the second part

$$W_f = X_f - V_f \tag{3.3.3}$$

is tangent to S_0 (it contains in TS_0 at points of S_0). We've made an agreement to understand V_f as the "outer" part of the Hamiltonian vector field with respect to S_0 while W_f is the "inner" part, which preserves S_0 itself. Thus the inner part acts on the objects which live over the fixed base S_0. The infinitesimal deformation of (S_0, θ_0) under X_f is reflected by the coordinates of the dynamically correspondent field $\Theta_{DC}(X_f)$. This deformation splits as well into two parts: the inner part and the outer part. As we've seen (see Proposition 3.3) the outer part represented by vector field V_f precisely corresponds to

$$\psi_1 = f|_{S_0} - const, \tag{3.3.4}$$

where the constant is fixed by (3.3.1) so

$$const = \int_{S_0} f|_{S_0} \theta_0^2.$$

The second, inner, component $\Theta_{DC}(X_f)$ has the shape

$$\psi_2 = \frac{Lie_{W_f} \theta_0}{\theta_0}, \tag{3.3.5}$$

where the Lie derivative is defined by the inner part of the Hamiltonian vector field and is applied to a inner object - our half weight θ_0. To perform our computations successfully let us note that since the source manifold S from the definition of the moduli space (see 2.1) is oriented we can consider S_0 as equipped with the corresponding orientation. In this case the square θ_0^2 represents a volume form over S_0 and we consider this volume form instead of θ_0^2. We denote it as η_0. Then one has the following smooth real function

$$L_f = \frac{Lie_{W_f} \eta_0}{\eta_0},$$

corresponding to the logarithmic Lie derivative. At the same time

$$Lie_{W_f}(\theta_0^2) = 2\theta_0 \cdot Lie_{W_f} \theta_0,$$

hence

$$L_f = 2 \frac{Lie_{W_f} \theta_0}{\theta_0} = 2\psi_2.$$

Therefore we reformulate the statement of Proposition 3.3 substituting

$$\psi_2 = \tfrac{1}{2} L_f.$$

Now let's compute the coordinates of the second vector field induced by f over the moduli space. So we take the corresponding induced function F_f and write down the coordinates of its Hamiltonian vector field. Its differential dF_f has been computed in Section 2, namely

$$dF_f(\alpha_1, \alpha_2) = \tau \int_{S_0} f|_{S_0} 2\alpha_2 \theta_0^2 + \tau \int_{S_0} d\alpha_1(W_f)\theta_0^2 \tag{3.3.6}$$

over point (S_0, θ_0) for tangent vector $v = (\alpha_1, \alpha_2)$. Due to the simplicity of the expression for the symplectic form Ω (see Section 2, (2.3.3)) one can easily convert the first summand in (3.3.6), getting the first coordinate of the Hamiltonian vector field:

$$\psi_1' = 2\tau f|_{S_0} - const', \tag{3.3.7}$$

where the constant is defined by the same ruling condition. The second summand requires a little bit more time. First of all,

$$\begin{gathered}\int_{S_0} d\alpha_1(W_f)\theta_0^2 = \int_{S_0} d\alpha_1 \wedge \imath_{W_f} \eta_0 = \\ -\int_{S_0} \alpha_1 \cdot d(\imath_{W_f} \eta_0) = -\int_{S_0} \alpha_1 \frac{Lie_{W_f} \eta_0}{\eta_0} \theta_0^2,\end{gathered} \tag{3.3.8}$$

where at the second step we use the integration by parts. Substituting this result to the expression (2.3.3) for the symplectic form we see that the second coordinate of the Hamiltonian vector field is the following

$$\psi_2' = \tau \frac{Lie_{W_f} \eta_0}{\eta_0} = 2\tau L_f. \tag{3.3.9}$$

It remains to compare (3.3.4) and (3.3.7), (3.3.5) and (3.3.9) and as the result of the matching one gets the statement of Proposition 3.5.

3.4. Critical points of F_f.

The next question we'd like to discuss is the problem of critical points and critical values for the induced quasi symbols over the moduli space. This question so quite important for the quantum theory. Really, to perform the measurement process the quantized observables should have enough eigenstates (= critical points, see Section 1). As well in any quantum mechanical model one needs some requirement for the corresponding spectrum. As usual the case when the spectrum is discrete, simple, etc. is understood as a good case. In our case there are two ways to arrange what one needs: first, one could find an appropriate compactification of the moduli space; second, one can establish that the moduli space itself carries the property that for sufficiently generic smooth function on the based manifold the corresponding induced quasi symbol has a number of good critical points. The situation when one has these two alternative ways are rather simuliar, say, to the crossroad which one meets in the Donaldson theory: from the first view point there exists a good compactification of the moduli space of instantons, constructed by K. Ulenbeck, but S. Donaldson computes the numbers, taking such appropriate representatives for μ - classes which belong to the moduli space itself (see [8]). Of course, this analogy is too rough and pretentious but nevertheless we have to make a choice coming to the consideration of the critical points. In this text we'll study the second way (although the first one is even more interesting). To begin with we assume that for a sufficiently good function $f \in C^\infty(M, \mathbb{R})$ (we'll deal with Morse functions) the corresponding quasisimbol F_f has critical points. Then we'll list a number of good facts about these critical points. At the end of this subsection we discuss the most important part — the existence theorem.

So the first fact about the critical points is rather trivial:

Proposition 3.6. *Pair $(S, \theta) \in \mathcal{B}_S^{hw,r}$ is a critical point for function F_f iff the Hamiltonian vector field of the original function f preserves this pair or in other words*

$$\begin{aligned} f|_S &= const, \\ Lie_{W_f}\theta &\equiv 0. \end{aligned} \tag{3.4.1}$$

The association "in other words" above has very important geometrical sense:

Lemma. *Hamiltonian vector field X_f preserves Lagrangian submanifold S if and only if S lies on a level set of f.*

Thus in any case one should look for critical points on the level sets of a given f. On the other hand, for any Morse function $f \in C^\infty(M, \mathbb{R})$ one has the following

Proposition 3.7. *Let f be a sufficiently generic function over M. Then for each generic value the corresponding critical set for F_f is descrite.*

Here we assume that $\tau = \frac{1}{r}$ such that

$$f \equiv const \implies F_f \equiv const$$

(see subsection 2.4). Thus we need to prove that if $(S, \theta) \in \mathcal{B}_S^{hw,r}$ is a critical point for F_f with critical value v outside of the set of some special values (we will specialize what these values are in a moment) then it is isolated. The specialty of values means, firstly, that v doesn't belong to the set of critical values of f. Further, we divide the verification into two steps. Firsts, let's show that the support S is isolated. Indeed, if S_δ lies sufficinely close to S then it comes to some Darboux - Weinstein neighborhood of S. But all Bohr - Sommerfeld Lagrangian cycles in this neighborhood have to intersect our given S. These cycles are represented by exact 1 - forms but every exact 1 - form over a compact manifold has zeroes (at least two since every real smooth function over a compact manifold has minimum and maximum which give at least two points where the differential of this function vanishes). Therefore S_δ intersects S. At the same time S and S_δ belong to level sets of our function f being supports of critical points. If these level sets are different then we come to the contradiction since two submanifolds from different level sets never intersect each other. If S and S_δ belong to the same level set $f^{-1}(v)$ then let's note that

1) X_f is a nonvanishing vector field being restricted to non critical level set $f^{-1}(v)$;

2) X_f is tangent to both S and S_δ;

and we establish, comparing items 1) and 2), that the intersection

$$S \cap S_\delta$$

either is trivial or its dimension is at least 1. If it is trivial then applying again the arguments above we show that S and S_δ are distinct and separated. If there is an intersection of positive dimension then the level where S and S_δ is special: it contains a limit cycles. Indeed, the intersection should be invariant under the flow given by X_f. The theory of limit cycles predicts that in general the number of these limit cycles is finite. Thus if we reject the level sets with these special values then for the remaining generic value the level set could carry only a discrete set of the Bohr - Sommerfeld Lagrangian cycles. On the other hand if S is not isolated one gets even more than the existence of a limit cycle. Namely one should have a family of deformations of S inside of the level set; this would mean that for our generic function f and for the fixed level set there is a Bott integral (or may be even more - there is an integral for the dynamical system defined by (M, ω) and Hamiltonian f). Here we would like to decode prefix "sufficiently" in the statement of Proposition 3.7: f is "sufficiently generic" if the corresponding system is non integrable. It was already mentioned in [30] that the continuous deformations of a stable cycle S with respect to f comes from the functions which commute with f. Of course, if g satisfies

$$\{f, g\}_\omega = 0$$

it doesn't imply that g gives a deformation of S since it can happen that

$$g|_S = const.$$

But in general the converse takes place: if we can deform a stable cycle on a generic level set that we can derive from this deformation some integral for the system. But one should say that integrable system is not the general case in classical mechanics.

The second step is very simple: we just need to apply Lemma from subsection 3.1 above together with Theorem 1. Indeed, if the support is fixed then the half weight part could not vary continuously. Otherwise one gets that the critical set of quasi symbol F_f (since Theorem 1 ensures that it is) is Lagrangian at some point while according to the lemma it has to be a complex submanifold.

Remark. From the proof we derive first constraints on the topology of S. Indeed, as we've seen if (S, θ) is a critical point with critical value v which doesn't belong to the critical value set of our given f then the euler characteristic of S should vanishes. Really, in this case S admits a nonvanishing everywhere vector field (namely W_f which coincides with X_f if S belongs to the level set). Thus we get a first "vanishing" result: if the euler characteristic of S is non trivial then for every Morse function f one has

$$Spec F_f \subset Spec f \subset \mathbb{R},$$

where $Spec$ is the set of critical values. On the other hand, one can easily deduce that either one could apply the same argument as above to the case of the critical levels or one gets that there are no "critical" smooth Bohr - Sommerfeld cycles belonging to the critical levels of our Morse function f. Indeed, let (S, θ) be a critical point of F_f such that $S \subset f^{-1}(v_i)$ where $v_i \in Spec f$. Let $p_1, ..., p_d \in f^{-1}(v_i)$ are the critical points. Then it appears two possibilities: either

$$S \cap (p_1, ..., p_d) = \emptyset$$

or S contains some of p_i's. The first case is reduced to the case of noncritical levels since X_f vanishes only at $(p_1, ..., p_d)$. The second one is impossible in general since it means that S is not smooth. Therefore if we speak about simply connected S (say, n - dimensional sphere) then we have to insert on the discussion the question of an appropriate completion (or compactification) of the moduli space. Here the simplest step should be the following: one has to consider the maps

$$\phi : S \to M$$

which are smooth outside of finite sets of points and continues everywhere together with pure smooth maps (as it was in the definition of the moduli space, see subsection 2.1). But we'd like to postpone the discussion on the completion problem taking in mind that a priori the case when one takes S with zero euler characteristic is much more attractive for us. Indeed, n - torus is the classical (and primordial) example of compact Lagrangian submanifold. So it seems that in this case we can avoid any completion procedure keeping the moduli space itself as the central geometric object. At the end of the discussion we just mention that it remains a global problem in our considerations — the existence problem. This problem should be stated first of all for the moduli space itself: for any compact simply connected symplectic manifold with integer form one asks whether or not some middle homological class can be realized as a Lagrangian Bohr - Sommerfeld cycle of a fixed topological type. If the answer is the affirmative one then one asks for a sufficiently wide class of smooth functions over M are there sufficiently many critical points. Some preliminary result in this way is given by the following construction.

Let (M, ω) is a simply connected compact symplectic manifold with integer symplectic form. Then according to the Donaldson result (see [7]) there exists a level

k such that the power of the prequantization line bundle L^k has a smooth section s such that its zero set $M_1 \subset M$ is a smooth symplectic submanifold. Thus the pair $(M_1, \omega_1 = \omega|_{M_1})$ is again a symplectic manifold with integer symplectic form. Hence we can apply again the argument. Since M is finite dimensional one can choose such level k that starting with

$$M_0 = M, \omega_0 = k\omega$$

one constructs a flag of symplectic submanifolds

$$pt = M_n \subset M_{n-1} \subset ... \subset M_0$$

such that every M_i is presented by zeros of the corresponding section s_{i-1} of the prequantization line bundle L_{i-1} over

$$(M_{i-1}, \omega_{i-1} = \omega_0|_{M_{i-1}}).$$

Let f be a generic smooth function over M such that pt belongs to a non critical level set. Fix this level set and denote it as $N \subset M$. The intersection of N with M_{n-1} is a smooth circle. Notice, that the symplectic setup is much more flexible in contrast with the holomorphic case so we can deform slightly every stepping stone in our pyramid — f.e., the second one — to get the desired property. Namely if the intersection $N \cap M_{n-1}$ is not smooth we can slightly deform M_{n-1} to get some smooth intersection. This smooth intersection is a set of loops and we just choose one single circle to be our first constructed pip. It is Lagrangian in (M_{n-1}, ω_{n-1}) just by the dimensional property. We denote it as N_1. Further, consider the next stepping stone, including the picture to the symplectic manifold (M_{n-2}, ω_{n-2}). Notice that by the definition of the flag the normal bundle of M_{n-1} in M_{n-2} is trivial. This means that there exists a neighborhood $\mathcal{O}_{n-1}$ of M_{n-1} in M_{n-2} which is presented topologically as the direct product $M_{n-1} \times S^2$. Then (perhaps, after small deformation) the intersection $N_2 = N \cap M_{n-2}$ is smooth submanifold of M_{n-2} fibered over N_1 (we take, if it's necessary, just an appropriate smooth component of the intersection). The point is that in general N_2 is Lagrangian in (M_{n-2}, ω_{n-2}). Indeed, the restriction of ω_{n-2} to $\mathcal{O}_{n-1}$ splits as the direct product of ω_{n-1} and a symplectic form on S^2. Since the intersection of any fiber S^2 with N has dimension 1 it should be Lagrangian. Thus whole the intersection N_2 is Lagrangian, being the product of two Lagrangian cycles. Let's remark that topologically N_2 is 2 - torus which is topologically trivial. Repeating the procedure toward the top of the pyramid one gets n - torus N_n in $(M, k\omega)$ such that:

it represents trivial homology class;

it belongs to a level set of our function f.

Now, we didn't say anything about some connection on the prequantization bundle. At the same time the ingredient could be included in the pyramid such that the resulting Lagrangian cycle will be Bohr - Sommerfeld. It remains to construct an appropriate half weight, invariant under the flow of W_f in the terminology of subsection 3.3. To get such a half weight we can take any half weight θ_0 on the constructed $S = N_n$ and then average it switching on the flow of W_f. Since our S is compact and W_f is regular (without zeros) the resulting half weight is correctly defined. By the construction it should be invariant under the flow.

4. ALG(a) - quantization

In this section we use all the results which were established above to perform algebro geometric quantization of simply connected compact symplectic manifold with integer symplectic form. We call this method "ALG(a) - quantization" since it is a quantization in the framework of algebraic Lagrangian geometry (abelian case). In subsection 4.1 we project all the results to the subject of the first part of Section 1 and show that our construction looks like a quantization in that terms. However the result of the quantization is quite consistent with the results of known geometric quantization constructions: in subsections 4.2 and 4.3 we study how ALG(a) - quantization can be reduced in the case when the based manifold M is equipped with an appropriate polarization. We begin with the real case in 4.2 and then consider the case of complex polarization in subsection 4.3. The rest subsection is devoted to some applied questions: we discuss a natural quasi classical limit of ALG(a) - quantization.

4.1. The geometry of quantization.

Let's come back to Section 1. There we postulate that symplectic manifolds can be quantized in terms of geometric formulation of quantum mechanics. Thus we ask for any symplectic manifold about an appropriate algebraic manifold which should play the role of the corresponding quantum phase space. This quantum phase space should admit sufficiently many functions whose Hamiltonian vector fields would generate its symmetries: we call these functions quasi symbols. Moreover, we have to define an appropriate map from the space of classical observables to the space of quantum observables and this map has to be an irreducible representation of the given Poisson structure. The irreducibility condition has been discussed there; here we just remind that irreducibility means that the kernel of this map is trivial and there is no a smooth submanifold of the quantum phase space which is invariant under the flow generated by quantized observables unless whole the quantum space. After all this reformulation is understood let's turn to Algebraic Lagrangian Geometry, proposed in [10], [25]. For any simply connected compact symplectic manifold with integer symplectic form this programme gives series of infinite dimensional algebraic manifolds which are called the moduli spaces of half weighted Bohr - Sommerfeld Lagrangian cycles of fixed volume and topological type. It looked like a very natural proposal to consider such moduli space as the quantum phase space for some algebro geometric quantization. One gets the same feeling just comparing the materials of Section 1 and Section 2. Further, in Section 2 we construct a map

$$\mathcal{F}_\tau : C^\infty(M, \mathbb{R}) \to C^\infty(\mathcal{B}_S^{hw,r}, \mathbb{R})$$

and prove that it is a homomorphism of Lie algebras (it maps Poisson bracket to Poisson bracket). Moreover, in Section 3 we prove that the image of $\mathcal{F}_\tau$ belongs to the subalgebra of quasi symbols and that $\mathcal{F}_\tau$ is irreducible. On the other hand, Theorem 1 shows that $\mathcal{B}_S^{hw,r}$ is a good candidate to be the quantum phase space of some quantum mechanical system. Indeed, we prove (see there) that for every point $(S,\theta) \in \mathcal{B}_S^{hw,r}$ and every tangent vector at this point $v \in T_{(S,\theta)}\mathcal{B}_S^{hw,r}$ there exists a quantum observable (moreover, a quantized classical observable) whose Hamiltonian vector field coincides at this point with this vector. It means roughly that this quantum phase space possesses maximal quantum symmetries.

Now to perform some additional construction, making the quantization more familiar for the audience, we need the fact which has been mentioned many times above:

Proposition 4.1 ([10], [25]). *The symplectic form on the moduli space $\mathcal{B}_S^{hw,r}$ has integer cohomology type.*

To prove the statement it sufficient to remind the definitions from Section 2. The symplectic structure on the moduli space of half weighted planckian cycles is a double cover of the canonical symplectic structure on a cotangent bundle (see subsection 2.2) thus it is integer (however it corresponds to the trivial topological line bundle). Then we apply the symplectic reduction getting our moduli space $\mathcal{B}_S^{hw,r}$ but during the procedure the structure remains to be integer. From these defining constructions one gets the following

Proposition 4.2 ([10], [25]). *The prequantization line bundle corresponding to symplectic form (2.3.3) is exactly the complex line bundle, associated to $U(1)$ - principle Berry bundle, defined in (2.1.4).*

Notice that an integer symplectic manifold, given by our moduli space $\mathcal{B}_S^{hw,r}$ is almost ready to be quantized: it is canonically equipped with a real polarization

$$\pi : \mathcal{B}_S^{hw,r} \to \mathcal{B}_S$$

and a complex polarization, given by the complex structure. It remains to choose an appropriate prequantization connection and then start a known programme. But we would like to mention here an additional aspect coming when one applies Berezin - Rawnsley method. Namely if $F \in C^\infty(\mathcal{B}_S^{hw,r}, \mathbb{R})$ is a quantized function by Berezin - Rawnsley (and it is a quasi symbol in our terminology) then it gives a self adjoint operator on the space of holomorphic sections of the Berry bundle. Indeed, if one fixes an appropriate prequantization connection then one could perform the Berezin - Rawnsley construction getting desired correspondence. But we can restrict the investigations to the subset in $C^\infty_q(\mathcal{B}_S^{hw,r}, \mathbb{R})$ which consists of the images of F_τ. Then one has the following composition:

$$\begin{aligned} C^\infty(M,\mathbb{R}) \hookrightarrow & C^\infty_q(\mathcal{B}_S^{hw,r}, \mathbb{R}) \to Op(\mathcal{H}), \\ f \mapsto & F_f \mapsto \hat{Q}_{F_f}, \end{aligned} \tag{4.1.1}$$

where $\mathcal{H}$ is the space of holomorphic sections of the Berry bundle and $\hat{Q}_f$ is the corresponding self adjoint operator, given by the Berezin - Rawnsley procedure.

The possibility to construct this composition is the reason why we emphasized everywhere that the Kaehler manifold desired in AGQ should be algebraic. In this case there is a superstructure which makes it possible to come back, getting some geometric quantization procedure. Indeed, the composed correspondence given by (4.1.1) is an example of a representation of the Poisson algebra in some Hilbert space. It satisfies a number requirements from the Dirac list; f.e., it's obviously linear; at the same time we can manage that constants would come to the same constants; it is a homomorphism of the Lie algebra (since both the maps in the first row of (4.1.1) possess the property). As well one could prove that it is irreducible, but it is another story and we will return to it in a future.

Now we are going to discuss how ALG(a) - quantization is related to the known methods, recalled in Section 1.

4.2. Real polarization.

Assume now that our symplectic manifold (M, ω) admits an appropriate real polarization. It means that (M, ω) can be equipped with a Lagrangian distribution that is a field of Lagrangian subspaces in the complexified tangent bundle which is integrable. The complex and the real case are differ by the nature of these Lagrangian subspaces: in the real case all of them are real while in the complex one all of them are pure complex. We've recalled both the notions in Section 1. Turning to that we understand the real polarization case as that one when on M there are a set of smooth functions $f_1, ..., f_n$ such that

$$\{f_i, f_j\}_\omega = 0, \qquad \forall i, j, \tag{4.2.1}$$

defining some Lagrangian fibration

$$\pi : M \to \Delta, \tag{4.2.2}$$

where $\Delta \subset \mathbb{R}^n$ is a convex polytope. For each inner point

$$(t_1, ..., t_n) \in \Delta \backslash \partial \Delta$$

the corresponding fiber

$$\pi^{-1}(t_1, ..., t_n) = f_1^{-1}(t_1) \cap ... \cap f_n^{-1}(t_n) \tag{4.2.3}$$

is a smooth Lagrangian cycle. The degenerations at the faces of Δ are regular so the inner part of each $n-k$ - face corresponds to $n-k$ - dimensional isotropic submanifolds. Usually (see [20]) the quantization scheme in this case distinguishes some special fibers of (4.2.2) namely Bohr - Sommerfeld fibers (we've discussed this subject in Section 1). However as we prove below there are finitely many such fibers for the compact case hence one has just a finite set $S_1, ..., S_l$. Then the Hilbert space of the quantization is

$$\mathcal{H} = \sum_{i=1}^{l} \mathbb{C} < S_i >, \tag{4.2.4}$$

and S_i play the role of a canonical basis. The functions $\{f_i\}$ are represented then by diagonal operators; for some other functions one can define operators which are not in general unitary, see Section 1.

Now we wish to apply ALG(a) - programme to our completely integrable system taking the homology class of the fiber as $[S]$. Then one gets the corresponding moduli space $\mathcal{B}_S^{hw,1}$ where

$$[S] = [\pi^{-1}(pt)] \in H_n(M, \mathbb{Z})$$

and we take volume 1 just for simplicity. The set $(f_1, ..., f_n)$ defines quasi symbols $F_{f_1}, ..., F_{f_n}$ which are again in involution (and moreover, one can manage an infinite set of functions in involution taking all finite products of $f_1, ..., f_n$ and then mapping them by $\mathcal{F}_\tau$). Denote as $Crit(F_{f_i})$ the set of critical points of F_{f_i}. Consider the following intersection

$$P = Crit(F_{f_1}) \cap ... \cap Crit(F_{f_n}) \subset \mathcal{B}_S^{hw,1} \tag{4.2.5}$$

so the mutual critical set. One has

Proposition 4.3 ([30]). *The set P is a double cover of the set $\{S_i\}$ consists of the Bohr - Sommerfeld fibers.*

Thus in general we can recover the well known method from [20]: one just takes the mutual critical set for the distinguished functions which preserve given real polarization and the supports of these critical points correspond to the basis in $\mathcal{H}$. Moreover, the proposition is true in general non compact case (but we do not consider it here) so one could try to expolite this correspondence in some different context. On the other hand, one can deduce what happens for some other function f_0 which doesn't belong to the algebraic span of $\{f_1, ..., f_n\}$. Say, one can deform the given system $\{f_1, ..., f_n\}$ in involution using such a function if the set

$$g_i = \{f_0, f_i\}_\omega$$

consists of functions in involution. Then this set defines some other real polarization which is given by the action of the flow of X_{f_0} on the original one. Then if we are lucky one can define an operator, corresponding to f_0 in terms of $\mathcal{H}$. But here we would like to discuss only general facts so at this step we know how to quantize only the functions which preserve the polarization.

The proof of Proposition 4.3 uses the equations for critical points of quasisymbols given in subsection 3.4. Since for every Bohr - Sommerfeld Lagrangian fiber S_i all f_i are constant along it then the first equation from (3.4.1) is satisfied automatically. To complete the proof we need to find an appropriate half weight θ_i on S_i such that the pairs $(S_i, \pm\theta_i)$ should be invariant under all Hamiltonian vector fields X_{f_j}. Note, that since S_i lies on a level set for any f_j then

$$X_{f_j} \equiv W_{F_j}$$

at the points of S_i. Really we are looking for an appropriate volume form on S_i using again the same argument as before. Now take over S_i the set of differentials df_i and combine n - form

$$\tilde{\eta} = df_1 \wedge ... \wedge df_n. \tag{4.2.7}$$

From the properties of the completely integrable system one sees that this n - form is non degenerated everywhere over S_i (the order of f_i's is fixed by the given orientation, see again the definition of the Bohr - Sommerfeld cycles). Then there exists such n - form η that

$$d\mu_L = \eta' \wedge \tilde{\eta} \tag{4.2.8}$$

over S_i, where $d\mu_L$ is as above the Liouville form. Of course, this η' defined by (4.2.8) is not unique but its restriction

$$\eta = \eta'|_{S_i} \tag{4.2.9}$$

is defined uniquely. By the construction this volume form is invariant under the action of every X_{f_i}. Indeed, both the Liouville form and the intermideate $\tilde{\eta}$ are invariant under the flows thus the last η possesses the same property. It remains to normalize η comparing given volume r with the volume

$$\int_{S_i} \eta_i.$$

Then one takes the square roots

$$\pm\theta_i$$

such that

$$\theta_i^2 \equiv \eta_i.$$

Let show that there are no any other invariant half weights over S_i. Indeed, let θ'_i is another half weight over S_i which is invariant under each X_{f_j}. Then the ratio

$$\psi = \frac{\theta'_i}{\theta_i}$$

is a smooth function over S_i, satisfying

$$Lie_{W_{f_j}} \psi = 0 \qquad \forall j = 1, ..., n. \tag{4.2.10}$$

But the set $\{X_{f_j}\}$ form some local basis of the tangent space at each point of S_i thus ψ has to be constant. The normalizing condition implies that this constant is either plus or minus 1 hence it remains only one pair $\pm\theta_i$ as possible solutions.

Conversely, let (S_0, θ_0) be a mutual critical point for all F_{f_j}'s. Then, again applying (3.4.1) we get that all functions f_j's are constant over S_0. Forgetting about the second component one gets

$$f_j|_S = const = t_j \qquad \forall j = 1, ..., n,$$

consequently S is a Bohr - Sommerfeld fiber:

$$S = \pi^{-1}(t_1, ..., t_n).$$

It completes the proof of Proposition 4.3.

Further, we've mentioned that in the compact case the set of Bohr - Sommerfeld fibers is finite. Although it doesn't lie on the mainstream of our consideration we present here

Theorem 2 ([30]). *Let (M,ω) be a symplectic manifold with integer symplectic form which admits Lagrangian fibration*

$$\pi : M \to \Delta$$

with compact fibers. Then the set of smooth Bohr - Sommerfeld fibers is discrete.

The proof is very short in the case when (M,ω) is a completely integrable system: the statement follows from Proposition 3.7 and Proposition 4.3. In our more general case one uses the argument which has been exploited in the proof of Proposition 3.7: if S_0 is a Bohr - Sommerfeld fiber so

$$S_0 = \pi^{-1}(p_0), \qquad p_0 \in \Delta$$

then there exists a neighborhood $\mathcal{O}(p_0)$ of the point in Δ such that $\pi^{-1}(\mathcal{O}(p_0)$ is a Darboux - Weinstein neighborhood of S_0 in M. Then if we suppose that there is another Bohr - Sommerfeld fiber S projecting to $p \subset \mathcal{O}(p_0) \subset \Delta$ then it should be a smooth function $\psi \in C^\infty(S_0,\mathbb{R})$ such that S coincides with the graph of $d\psi$ in this Darboux - Weinstein neighborhood. Since S_0 and S have zero intersection being two different fibers the differential $d\psi$ has to be nonvanishing everywhere. But any smooth function on a compact set has at least two extremum points: the minimal and the maximal ones. This means that $d\psi$ has to vanish somewhere which leads to the contradiction. Therefore if our S_0 is Bohr - Sommerfeld then there exists a neighborhood of $\pi(S_0) = p_0$ in Δ such that $p_0 \in \mathcal{O}(p_0)$ is unique "Bohr - Sommerfeld point" in this neighborhood. Thus, globally, every Bohr - Sommerfeld fiber of π is separated by such a neighborhood and hence the set of Bohr - Sommerfeld fibers is discrete. Moreover if M is compact it follows that the set is finite.

Remark. It's quite natural and reasonable to continue here the observation given at the end of subsection 2.3 above. There we spoke about the case of completely integrable systems: in this case one could construct, starting with the given set of first integrals $\{f_1, ..., f_n\}$, an infinite set of commuting quantum observables over the moduli space $\mathcal{B}_S^{hw,r}$. Indeed, one just takes

$$\{F_{f_j^k}\}, \qquad j = 1, ..., n, k > 0 \tag{4.2.11}$$

and for every pair from this set the quantum Poisson bracket vanishes (see subsection 2.3). However it's clear that the mutual critical set P is the same for every degrees k and any combinations of the first integrals (while the corresponding quantum observables are not longer algebraically dependent). Really, the conditions

$$f_j|_S = const$$

and

$$f_j^k|_S = const$$

are absolutely equivalent (since our functions are real and smooth). And despite of the fact that quasisymbols of type (4.2.11) are algebraically independent, their critical values *are* algebraically dependent in mutual critical points. Indeed, every first integral f_j gives the following critical values (via the powers of f_j):

$$c = f_j|_S, c^2, ..., c^k, ...,$$

and it's clear that this set is algebraically dependent. Hence we could not derive some additional geometric information for the completely integrable systems (at least in the present discussion) using out method.

4.3. Complex polarization.

This case belongs exactly to algebraic geometry. Complex polarization is a choice on the symplectic manifold (M,ω) of any compatible with ω integrable complex structure I (so one suggests that such a structure exists), transforming our M to an algebraic manifold. While any symplectic manifold admits an infinite set of almost complex structure the possibility to choose in this set some integrable one makes the horizon of the examples much less wider. This condition implies that (M_I,ω) is a Kaehler manifold and the integrability condition for ω ensures that the Kaehler metric has the Hodge type hence (M_I,ω) is an algebraic variety.

Known quantization methods have been discussed in Section 1. It was mentioned there that they are based on some reductions of the basic Souriau - Kostant method. Thus we'll use here both the reductions: Berezin - Rawnsley method is more appropriate to claim some dynamical coherence while Berezin - Toeplitz method is described by some explicit formulas, see Proposition 2.3. In any case the corresponding Hilbert space is the same — the space of holomorphis sections of the prequantization line bundle with respect to the prequantization connection. We porjectivize the space following the strategy of Section 1. The first step is to relate the quantum phase spaces of the known method and ALG(a) - quantization. The desired relationship is given by so-called BPU - map ("BPU" means the first letters of the authors names, see [5]). The moduli space $\mathcal{B}_S^{hw,r}$ is fibered over the projective space:

$$BPU : \mathcal{B}_S^{hw,r} \to \mathbb{P}H^0(M_I,L). \tag{4.3.1}$$

Recall the construction following [10], [25]. Let $s \in H^0(M_I,L)$ be a holomorphic section of the prequantization bundle. Restrict it to on any half weighted Bohr - Sommerfeld cycle $S,\theta)$. The restriction is represented by a smooth complex function. Indeed, it is true for any smooth section of L: the pair of prequantization data (L,a) is restricted to S as trivial bundle with flat connection admitting a covarinatly constant trivialization by the definition hence every section over S is presented by a covarinatly constant section multiplied by a smooth complex function. Since the trivialization over S is defined up to scale one should lift the consideration to the set of half weighted planckian cycles to kill the ambiguity. Then the restriction of any section to $\tilde{S}$ is exactly a complex function. Then we define a map

$$\mathcal{P}_S^{hw} \to H^0(M_I,L) \tag{4.3.2}$$

by the following condition

$$(\tilde{S}_0,\theta_0) \mapsto s_0$$

iff

$$\int_{\tilde{S}_0} s|_{\tilde{S}_0}\theta_0^2 = \int_M < s, s_0 > d\mu_L \tag{4.3.3}$$

for any holomorphic $s \in H^0(M_I,L)$. Since $H^0(M_I,L)$ is finite dimensional (we work over a compact symplectic manifold) for any $(\tilde{S}_0,\theta_0)$ such holomorphic section exists. Indeed, every $(\tilde{S}_0,\theta_0)$ defines a linear functional on $H^0(M_I,L)$ given by the left hand side of (4.3.3) hence one has a fibration

$$\mathcal{P}_S^{hw} \to (H^0(M_I,L))^* \tag{4.3.2'}$$

and then one can get (4.3.2) applying a standard identification since $H^0(M_I, L)$ is equipped with "quantum" hermitian form $<,>_q$ (see Section 1).

It's clear that the canonical $U(1)$ - actions on the source and on the target in (4.3.2) are compatible. Hence one could take the corresponding Kaehler reductions of the both spaces: then we get a map from $\mathcal{B}_S^{hw,r}$ to the projective space $\mathbb{P}H^0(M_I, L)$ which is the result of the Kaehler reduction of $H^0(M_I, L)$. This is BPU - map (4.3.1) which gives a reduction of the quantum phase space of ALG(a) - quantization and the one of the known method.

Further, let smooth function f be a quantizable observable (as in Section 1) with respect to the given complex polarization. Then the Hamiltonian vector field X_f preserves the complex structure I hence the dynamically correspondent vector field on $\mathbb{P}(\Gamma(M, L))$ preserves the finite dimensional piece $\mathbb{P}(H^0(M_I, L))$. Moreover, the field $\Theta^p_{DC}(X_f) \in Vect(\mathbb{P}(H^0))$ preserves whole the Kaehler structure and corresponds (see Section 1) to some smooth function Q_f (let's emphasize again that it is true since f is quantizable). On the other hand, the Hamiltonian vector field defines infinitesimal transformations on both the ingredients of the BPU - map. For each quantizable function the corresponding dynamical actions on the source space and the target space have to be compatible. Thus for any quantizable function (in the sense of Rawnsley - Berezin method) one has:

1) a pair of quasisymbols F_f and Q_f on the source and the target spaces respectively;

2) a pair of dynamically correspondent vector fields $\Theta_{DC}(f)$ and $\Theta^p_{DC}(f)$ on the source and the target spaces respectively.

Taking into account the dynamical arguments we get that the differential of BPU - map translates our special vector field $\Theta_{DC}(f)$ to the special vector field $\Theta^p_{DC}(f)$. Propositions 1.6. and 3.5. ensure that there is

Proposition 4.4 ([30]). *For any quantizable function f the Hamiltonian vector fields of quasisymbols F_f and Q_f are related as follows*

$$dBPU(X_{F_f}) = c \cdot X_{Q_f}, \tag{4.3.4}$$

where c is a real constant.

Thus one reduces ALG(a) - quantization to a well known method in the complex polarization case. As a consequence of Proposition 4.4 we get how to find the eigenstates of quantum observable Q_f having the eigenstates of F_f. The relationsheep is very similar to the answer in the previous real case:

Corollary 4.5 ([30]). *BPU - map projects the set of eigenstates of the quantum observable F_f to the set of eigenstates of Q_f.*

Note that we get the statement of Proposition 4.4 just directly from the dynamical arguments. It's quite hard to compute explicitly the differential, but it's clear that the definition of BPU - map and the definition of our quasisymbols over the moduli space are sufficiently close. Moreover, one could see that as one defines a function F_f over the moduli space starting with a smooth function on the given symplectic manifold one can define a section of a vector bundle on the moduli space starting with any section of the prequantization bundle. We've almost seen what the bundle over the moduli space is: for every section $s \in \Gamma(L)$ there is a submanifold $K_s \subset \mathcal{B}_S^{hw,r}$. The topological type of the submanifold is obviously fixed. Thus the Chern classes of the bundle could be defined using the topology of these submanifolds together with their intersection theory. But one could define the corresponding bundle taking into account just local considerations in the Darboux - Weinstein neighborhoods. We just hint here some topological relationships given in the picture: namely for any symplectic manifold X with integer symplectic form there two distinguished 2 - cohomology classes, the class of the symplectic form and the associated canonical class. What are the classes for our moduli space $\mathcal{B}_S^{hw,r}$? The symplectic class is represented by the Berry bundle. At the same time, one hints that the associated canonical bundle is given by our prequantization bundle L. Then from BPU - map one could relate the Berry bundle with the canonical bundle over given M associated with given symplectic form. It gives an interesting duality: during the process the canonical bundle turns to be a prequantization bundle while the prequantization bundle turns to be the canonical bundle over the moduli space.

The construction of BPU - map outlines a way how to understand some properties of the moduli space which we derived above. Namely, let us consider instead of the ruling map (4.3.2) the following one

$$\mathcal{P}_S^{hw} \to \mathcal{H}, \tag{4.3.5}$$

where $\mathcal{H}$ is the Souriau - Kostant space of wave functions (see subsection 1.5). Indeed, apply the same scheme: every planckian cycle gives a linear map from $\mathcal{H}$ to complex numbers. Thus one gets

$$\mathcal{P}_S^{hw} \hookrightarrow \mathcal{H}^*, \tag{4.3.6}$$

and the question is whether or not one can use here the corresponding duality property to get (4.3.5). If it the case then we again perform the Kaehler reduction and get a fibration

$$gBPU : \mathcal{B}_S^{hw,r} \to \mathbb{P}(\mathcal{H}). \tag{4.3.7}$$

Then one expects that

1) this map is a double cover;

2) this map is α - holomorphic, so the differential maps holomorphic directions to directions with constant Kaehler angle;

3) for every smooth function f the quasi symbols F_f and Q_f on the corresponding quantum phase spaces are related by

$$d(gBPU)(X_{F_f}) = c \cdot X_{Q_f}.$$

This result would give very good compactification of the moduli space. This compactification would possess very good property: over it any induced quasi symbol should have a lot of eigenstates. And at last it would explain three things: why the moduli space carries a hermitian metric of constant holomorphic sectional curvature; why the Souriau - Kostant quantization is reducible; and why the moduli space is a good candidate to solve AGQ - problem.

At the same time the complex polarization case has another face: the point is that for any choice of a compatible (almost) complex structure over our given symplectic manifold gives simultenously the corresponding riemannian metric, compatible with ω. This gives us the following specialization for some half weighted Bohr - Sommerfeld Lagrangian cycles: we denote as $\mathcal{B}^g_S$ the subset in the moduli space which consists of such pairs $(S, \theta) \in \mathcal{B}^{hw,r}_S$ that

$$d\mu(g, S) \equiv \theta^2 \tag{4.3.8}$$

where $d\mu(g, S)$ is the volume form, given over S by the restriction of our riemannian metric g to our cycle S. Turning to the moduli space of "unweighted" Bohr - Sommerfeld Lagrangian cycles one can see that in the presence of g the moduli space is fibered over real numbers:

$$\begin{aligned} W_g &: \mathcal{B}_S \to \mathbb{R}_+, \\ W_g(S) &= Vol_g S \end{aligned} \tag{4.3.9}$$

(since S is oriented). Then one has the following simple

Lemma. *The subset $\mathcal{B}^g_S \subset \mathcal{B}^{hw,r}_S$ is represented as the double cover of $W_g^{-1}(r)$.*

Indeed, (S, θ) belongs to $\mathcal{B}^g_S$ if and only if (4.3.8) holds. But if $S \in W_g^{-1}(r)$ then there are exist exactly two half weights such that (4.3.8) takes place. And if S doesn't belong to $W_g^{-1}(r)$ then the equality (4.3.8) is impossible for any half weight over it. Thus one gets the picture

$$\mathcal{B}^g_S \to W_g^{-1}(r) \subset \mathcal{B}_S.$$

Then for this subset $\mathcal{B}^g_S \subset \mathcal{B}^{hw,r}_S$ one could formulate a number of valuable statements and include it to the quantization picture.

4.4. Quasi classical limit of ALG(a) - quantization.

Here we list some remarks on the limit of our method when an appropriate parameter goes to infinity. We discuss how the picture changes with respect to the level k in subsection 2.4. Here we translate the mathematical dependences on the parameters to quasi classical background.

Following Berezin we think that level k is inversely proportional to the Planck constant. We distinguish some dependence above (see subsection 2.4) introducing parameter τ in the definition of $\mathcal{F}_\tau$. Now the time is to recover that our parameter τ has to be proportional to k. Then the formulation of the Dirac principle (see Propositions 2.4) turns to be more familiar in the framework of quantization. At the same time one has that during the limit the volume of Bohr - Sommerfeld Lagrangian cycles decreases. Really, according to condition (2.4.7) r tends to zero while $\tau \to \infty$. Since for every half weight θ the square θ^2 is "positive" it means that during the limit one kills the half weight part. Hence in the limit one gets the moduli space of unweighted cycles. At the same time the result is based on not only Bohr - Sommerfeld Lagrangian cycles. The point is that when level k tends to infinity the moduli space $\mathcal{B}_{S,k}$ covers whole the set of Lagrangian cycles. Let explain it more carefully.

Recall that since a prequantization data are fixed then one has the following map

$$\chi : \mathcal{L}_S \to J_S. \tag{4.4.1}$$

Indeed, the restriction of the prequantization line bundle equipped with a prequantization connection gives a pair (trivial line bundle, flat connection) on the embedded Lagrangian cycle S. The flat connections modulo gauge transformations are described by so - called Jacobian of S:

$$J_S = H^1(S, \mathbb{R})/H^1(S, \mathbb{Z}).$$

It can be seen more explicitly if we fix a basis in $H_1(S, \mathbb{Z})$. Then for any flat connection one has a set of numbers which are just the result of the integration of our flat connection in any trivialization along the basic submanifolds. The changing of the trivialization gives nothing since we are working modulo integer points of the lattice $H^1(S, \mathbb{Z})$. Thus the map

$$\chi : \mathcal{L}_S \to J_S \tag{4.4.2}$$

is defined. And the moduli space of Bohr - Sommerfeld Lagrangian cycles is just the preimage

$$\mathcal{B}_S = \chi^{-1}(0).$$

Further, what happens when we go up to the higher levels? Then it's clear that for any level k one gets

$$\mathcal{B}_{S,k} = \chi^{-1}(0) \cup \chi^{-1}(p_1) \cup ... \cup \chi^{-1}(p_{k^{b_1}-1}), \tag{4.4.3}$$

(here it is a formal expression since $B_{S,k}$ is connected) where p_i's are the points of order k on the Jacobian (and there are k^{b_1} such points including zero) and b_1 is the first Betty number of S. It's clear that when k tends to infinity the order k point set covers densely the torus J_S and consequently when we arrange the procedure the moduli space $\mathcal{B}_{S,k}$ covers densely the moduli space of Lagrangian cycles.

Now the question is: what are the limits of the quantum observables F_f during the process? And what about the limiting Poisson bracket on the moduli space $\mathcal{L}_S$? One gets the answer to both the questions considering the following objects over the moduli space of Lagrangian cycles. Namely each smooth function $f \in C^\infty(M, \mathbb{R})$ generates a special object on $\mathcal{L}_S$ which possesses two different natures. From the first viewpoint one has just a vector field Y_f defined by the restrictions of f to the Lagrangian cycles (we denoted the same field as A_f in subsection 2.3). Let's remind, the restriction of f to $S \in \mathcal{L}_S$ gives the corresponding Hamiltonian (isodrastic) deformation of S (given by $d(f|_S)$ in the Darboux - Weinstein neighborhood); hence Y_f at the point just equals to this tangent vector. On the other hand, Y_f is not a single vector field: the point is that at the points where Y_f vanishes as a vector field one has some numerical values. Indeed, Y_f vanishes as a vector field at S if and only if our function f is constant being restricted to S. But it means that it gives some number equals to this constant. Thus the induced object Y_f is described by a pair

$$Y_f = (Y_f^0, Y_f^1), \tag{4.4.4}$$

where Y_f^0 is a real function (sufficiently singular, of course) and Y_f^1 is a vector field (absolutely smooth, of course). Let us denote the set of all such objects (given by smooth functions on M) as $C^q(\mathcal{L}_S)$. Then one has the following

Proposition 4.5. *The set $C^q(\mathcal{L}_S)$ is a Lie algebra.*

To ensure that the fact takes place we note firstly that the correspondence

$$f \mapsto Y_f$$

is obviously linear. The operation $[,]$ is given just by the formula

$$[Y_f, Y_g] = Y_{\{f,g\}_\omega}. \tag{4.4.5}$$

The Jacobi identity is satisfied just by the definition.

On the other hand, we should emphasize that:

Proposition 4.6.

1) For every f the corresponding object Y_f is the natural result of the limiting procedure, applying to quasisymbol F_f;

2) the Lie bracket (4.4.5) is the natural result of the limiting procedure, applying to the quantum Poisson bracket $\{,\}_\Omega$.

Further, we see that the system based on $\mathcal{L}_S$ is equipped with some dynamical properties coming from the classical dynamics of the given classical mechanical system. Indeed, if one choose a Hamiltonian $H \in C^\infty(M, \mathbb{R})$ then it generates a dynamics on $\mathcal{L}_S$, preserving the Lie bracket on the space of objects over the moduli space. It's just a simple exercise in the computational technique which we adopt during whole the text. Therefore we understand the process as an appropriate quasi classical limit of ALG(a) - quantization method: during the limiting procedure we lose the measurement aspects but we keep the dynamical properties compatible (more then compatible, we'd say) with dynamics of the given system.

At the end of the story we have to mention that we just present here some introductory part of a new quantization method, skeeping a lot of additional questions and details which will be clarified and established (if any) in a future. Overing the discussion we claim that we set up the problem having in mind some (may be accidental) coincidences listed above but we are pretty sure that the study of algebraic Lagrangian geometry, introduced in [10], [25], will lead to new and interesting results.

Addendum: supergeometry

In this small additional part we mention that some geometrical notions and objects from the main text can be naturally understood in terms of supergeometry. Here we use the supergeometry just as a convenient language which allows us to simplify the definitions but at the same time to outline some ways how one could generalize the constructions.

Bohr - Sommerfeld condition. We start with the basic for the method condition. As it was pointed out it is a dynamical condition compatible with Hamiltonian deformations from the "classical" position. At the same time it could be understood in terms of even supersymplectic manifold. Again, let (M, ω) is a symplectic manifold with integer symplectic form. Again, let's take some prequantization data (L, a) (see Section 1) such that

$$c_1(L) = [\omega]$$

and

$$F_a = 2\pi i \omega.$$

Consider the associated principle $U(1)$ - bundle

$$\pi : P \to M;$$

it is equipped with the same connection, represented now by a pure imaginary 1 - form A. This principle bundle equipped with the connection (compatible with the fixed hermitian structure on L) is an example of even supersymplectic manifold. The supersymplectic form is divided at each point into two parts: the connection defines a splitting

$$T_p P = T_{hor} \oplus T_{ver}$$

of the tangent space and then the supersymplectic form Ω is defined as the direct sum of the usual symplectic pairing on the horizontal part plus the natural symmetric $U(1)$ - invariant pairing defined by the natural metric on the vertical component. It can be checked directly that the Jacoby identity for the corresponding super bracket is implied just by the compatibility condition on the hermitian structure and our prequantization connection. To be more familiar the picture can be drawn over the prequantization bundle L which is considered as a real rank 2 bundle. Then the riemannian metric in the fibers is given just by the real part of the hermitian structure, and it's clear that the prequantization connection keeps this real part as well as whole the hermitian structure. More generally, Rothstein theorem (see [17]) distinguishes the case as the main one (for Batchelor trivial manifolds). Let us consider the prequantization principle $U(1)$ - bundle as an even supersymplectic manifold (P, Ω_s). The following statement gives an elegant reformulation of the Bohr - Sommerfeld condition:

Proposition A. *A cycle $S \subset M$ is Bohr - Sommerfeld Lagrangian if and only if it is the projection of a Lagrangian cycle of even supersymplectic manifold (P, Ω_s).*

The proof is rather routine: if S is a Bohr - Sommerfeld Lagrangian cycle then it can be lifted to P (we called the lifting planckian cycles) such that the resulting $\tilde{S}$ is horizontal at each point. Hence the restriction of Ω_s to $\tilde{S}$ is trivial. Conversely, if a submanifold $K \in P$ is Lagrangian with respect to Ω_s (so it is isotropical and has maximal dimension) then

1) it has to be horizontal at each point; otherwise the restriction of Ω_s to K should be nontrivial,

and

2) it has to give a Lagrangian submanifold being projected to the base (M, ω); otherwise the restriction should be nontrivial.

Here (to be honest) we mention that the simplification in the definition could be rewritten in terms of contact geometry: the principle bundle P with connection A (rescaled to be real) gives an example of contact manifold with contact form A. Indeed, dA by the definition is projected to ω thus the wedge product

$$A \wedge (dA)^{\wedge n}$$

is a volume form on P which is (locally) the product of the Liouville volume form lifted from M and our nonvanishing vertical 1- form A. Then

Proposition A'. *A cycle $S \subset M$ is Bohr - Sommerfeld Lagrangian iff it is the projection of a maximal isotropical cycle $\tilde{S} \subset P$ such that*

$$A|_{\tilde{S}} = dA|_{\tilde{S}} \equiv 0.$$

But we can go further, where the contact geometry doesn't appear: we mean applications of supersymplectic settings in the nonabelian cases. In the main text we discuss ALG(a) - programme but what about any ALG(n) - generalization; "ALG(n)" means that we would like to consider some nonabelian algebraic Lagrangian geometry. The subject of ALG(n) now exists just as a number of suggestions and examples. First steps in this way could be done using the superproperties of the Bohr - Sommerfeld condition. Namely, let $(\mathcal{M}, \Omega_s)$ is an even supersymplectic manifold which is projected by some appropriated Batchelor trivialization to a "classical" symplectic manifold (M, ω). Let $\mathcal{L}_S$ is the moduli space of usual Lagrangian cycles on M of a fixed topological type. It's natural to consider the set of super Lagrangian cycles in $(\mathcal{M}, \Omega_s)$ so the cycles which satisfy

$$\Omega_s|_{\tilde{S}} \equiv 0.$$

The local geometry of these cycles is not well understood (one has some version of the Darboux theorem in the super case, see f.e. [12], but what about the Darboux - Weinstein theorem?) but it can be done in appropriate cases. The projections of these super Lagrangian cycles would lie in $\mathcal{L}_S$ and should be analogous to the Bohr - Sommerfeld Lagrangian cycles in the abelian case. To project $\tilde{S}$ to M one uses some Batchelor trivialization (in general) therefore (in general) there is some dependence on $H^2(M, \mathbb{Z})$ (since every two Bachelor trivialzations differ by a line bundle, see [17]): the homology class of projected cycles depends on the Batchelor trivialization choice. To illustrate the generalization way as a sufficiently new method we study the following simple

Example. We discuss a classical construction relating riemannian and holomorphic geometries namely the twistor construction (see f.e. [19]). We take a symplectic 4 - dimensional manifold (M, ω) and to perform the construction we choose a riemannian metric g, compatible with the symplectic form. Remark that in contrast with the material of subsections 1.6, 4.3 we don't impose the integrability condition on the corresponding almost complex structure. Consider the twistor space, defined by the conformal class of the riemannian metric. It is a 6 - dimensional real manifold denoted as Y, which is fibered over X with fiber $\mathbb{CP}^1$. One could construct it as follows: let W^- be the spinor bundle defined by our metric, canonically equipped with a hermitian structure (see, f.e. [26]). The adjoint $SO(3)$ - bundle $adW^- \to M$ is equipped with:

1) fiberwise standard riemannian metric

and

2) a connection, induced by the Levi - Civita connection, compatible with the riemannian metrics.

Topologically over a point $x \in M$ the fiber adW^-_x is 3 - sphere, fibered over the twistor 2 - sphere $\mathbb{CP}^1_x$ as in the Hopf bundle picture. (It can be seen as the construction of subsection 1.2: we construct the projective line, factorizing unit sphere S^3 in $\mathbb{C}^2 = W^-_x$.) The total space of the adjoint bundle is the twistor space Y.

Further, our riemannian metric g defines over the twistor space a hermitian triple (G, J, Ω), see [26] (notice, that the triple is defined by the riemannian metric, not by its conformal class). Here J is usual twistor complex structure, which depends only on the conformal class, but G and Ω feel the deformations inside of a fixed class. Suppose that Ω is closed (so our g is a special, satisfying some appropriate condition) and that the class of Ω is integer. Thus one can take the pair (Y, Ω) and perform a few first steps of ALAG - programme, fixing an appropriate topological class in $H_3(Y, \mathbb{Z})$. Then we get the moduli space $\mathcal{B}^Y_S$ of Bohr - Sommerfeld Lagrangian cycles in Y with respect to some prequantization data. It's easy to see that every Bohr - Sommerfeld Lagrangian cycle $\tilde{S} \in \mathcal{B}^Y_S$ admits smooth projection to the based manifold M. But it's not longer true that the projection is a Lagrangian submanifold in M since our symplectic form Ω is twisted along the fibers as well as the classical twistor complex structure J. But since we started with a symplectic manifold one has in the picture some additional ingredient: namely, the corresponding to ω and J almost complex structure I defines a smooth embedding (see [26]):

$$i_I : M \to Y,$$

and we can separate a component from the moduli space $\mathcal{B}^Y_S$ consists of the Bohr - Sommerfeld cycles intersecting the image of i_I in maximal dimension 2. Then these intersections are transformed by i^*_I back to M and it's clear that the resulting submanifolds are Lagrangian with respect to our given symplectic form ω. Remind that we can perform this construction if the given symplectic manifold admits a compatible riemannian metric which possesses the property that the corresponding induced 2- form Ω is closed. But in any case the total space adW^- for any riemannian metric can be considered as an even supersymplectic manifold, endowed with the corresponding to ω and g supersymplectic even form (g here defines simultaneously the bundle W^-, the hermitian structure and the desired connection), and hence one could consider superlagrangian submanifolds of it. The answers are different for these two constructions: indeed, the first case is twisted while the second is "constant" along the fibers. The first construction belongs to riemannian geometry (f.e. one can simply remove the symplectic structure and consider the picture just in the riemannian framework; f.e. such a construction can be (and has been) done for S^4 with standard self dual metric) but the second lives in symplectic geometry. In [6] it was shown that these "pseudolagrangian" submanifolds for S^4 (which doesn't admit at all any symplectic structure) just are minimal in the corresponding topological classes. (However in [19] one finds an interesting way to quantize in terms of the twistor spaces.) But in the second case we get something new. For instance it isn't quite clear what is the dimension of the projected superlagrangian cycles in M. For some more general cases it could be that the dimension is less then half of the dimension of the based manifold. Thus one can introduce to the picture considerations of any rank bundles and some special isotropic submnaifolds, mixing the algebraic Lagrangian geometry for both the abelian and nonabelian cases. All together they give us a completion of the quantization picture, presented in the main text.

Special superfunctions.

From the geometrical point of view (see [3]) classical and quantum mechanical systems are distinguished by the presence in the last one of some additional kinematical ingredient which was unknown in the classical case. The quantum phase space in contrast with the classical one is endowed by a riemannian metric, compatible with given symplectic structure such that the corresponding complex structure is integrable. However in geometric formulation of quantum mechanics, which underlies of all our quantization programme, a natural question about the measurement process arises. Indeed, in the formulation any quantum observable is a special function (quasisymbol), which has exact numerical values in the points of the quantum phase space (and this value is precisely the same as the expectation value of the original operator, see Section 1). Of course, the presence of the riemannian metric gives us the corresponding measurement process, but why we should perform it when we have these exact values? From this point of view it would be natural to look for a realization of the quantization programme where some geometrical objects, corresponding to the classical observables, carry more "uncertant" properties than to be just smooth functions. At the same time we've seen in subsection 4.4 that in quasi classical limit one gets some strange objects Y_f which are neither pure functions no pure vector fields. Thus another question is: what is Y_f?

Coming back to the basic constructions we see that these two questions are related if we restrict the discussion of subsection 4.4 to the moduli space of unweighted Bohr - Sommerfeld Lagrangian cycles. Then for every smooth function f one has over the moduli space $\mathcal{B}_S$ the object Y_f. One could say that Y_f is a super function in some super symplectic setting. Recall, that as T^*M is a "classical" symplectic manifold for any smooth manifold M, one has at the same time an odd super symplectic manifold ΠT^*M (see f.e. [12]). The manifold ΠT^*M is given by the reversing of the parity of the fibers of T^*M (of course, there is a huge list of references for the "super" subject, but I would like to refer here survey [12] since my interest to the subject was intended by its author), and the corresponding odd bracket is called Butten bracket. Now what is a superfunction for $\Pi T^* M$? It is a sum of polyvector fields of different degrees on the given manifold M (f.e. pure numerical function has degree zero) and the Butten bracket could be reduced to the standard Schouten bracket for multivector fields. Therefore one can regard our object Y_f as a superfunction on odd supersymplectic manifold $\Pi T^* \mathcal{B}_S$. This function is rather special: it has different degrees in different points and it isn't smooth. But the representation of the classical observables by such type superfunctions looks very natural and interesting. This representation is dynamical (we discussed it above). One has a Lie bracket on the set of such special superfunction hence it would be natural to suppose that the Butten bracket, defined by smooth functions only, can be extended to a more general class of superfunctions and this extension gives us our Lie bracket. Remark, that while our quaisymbols F_f's have exact numerical values at the quantum states our special superfunctions do not have exact values in all the points — only in some "eigenstates", where f is constant. Thus from some point of view (expressed at the beginning of this last remark) a way to use these supersetting is more attractive: here we have some intriguing uncertainty as one accustomed to have. Here we have an appropriate dynamical property but we don't have any invariant measurement process. If we find this desired ingredient (say, in this supersetting) it were an important movement in the geometric quantization theory.

References

[1] R. Abraham, J. Marsden, *Foundation of mechanics*, 2 end ed, Benjamin, Mass, 1978.

[2] V.I. Arnol'd, A.B. Givental', *Symplectic geometry (in Russian)*, Current problems in math., Fund. directions, Vol. 4 (1985), 5 - 139.

[3] A. Ashtekar, T. Schilling, *Geometrical formulation of quantum mechanics*, arXiv: gr - qc/ 9706069.

[4] M. Bordemann, E. Meinreken, M. Schlichenmair, *Toeplitz quantization of Kaehler manifolds*, Comm. Math. Phys., 165 (1994), 281 - 296.

[5] D. Borthwick, T. Paul, A. Uribe, *Legendrian distributions with applications to relative Poincare series*, Invent. math., 122 (1995), 359 - 402.

[6] R. Bryant, *Conformal and minimal immersions of compact surfaces in the 4 - sphere*, J. Differ. Geom., 17 (1982), 455 - 473.

[7] S.K. Donaldson, *Symplectic submanifolds and almost complex geometry*, J. Diff. Geom., 44, No. 4 (1996), 666 - 705.

[8] S.K. Donaldson, P.B. Kronheimer, *The geometry of 4 - manifolds*, Oxford, University press, 1990.

[9] A.L. Gorodentsev, A.N. Tyurin, *ALAG*, Preprint MPI(Bonn) N 7 (2000), 1 - 34.

[10] A.L. Gorodentsev, A.N. Tyurin, *Abelian Lagrangian algebraic geometry*, Izvestiya: Math., 65: 3 (2001), 437 - 467.

[11] P. Griffits, J. Harris, *Principles of algebraic geometry*, Wiley, NY (1978).

[12] H. (O.M.) Hudaverdyan, *Semidensities on odd supersymplectic supermanifolds*, arXiv: DG/ 0012256.

[13] N. Hurt, *Geometric quantization in action*, Reidel, 1983.

[14] B. Kostant, *Quantization and unitary representation*, Springer, Berlin, Lect. Notes in Math., 170 (1970), 87 - 208.

[15] L. Landau, E. Lifschitz, *Quntenmechanik*, Academie - Verlag, Berlin (1985).

[16] J. Rawnsley, M. Cahen, S. Gutt, *Quantization of Kaehler manifolds*, JGP, Vol. 7 (1) (1990), 45 - 62.

[17] M. Rothstein, *The structure of supersymplectic supermanifolds*, Lect. Notes in Phys., 375 (1991), Springer, Berlin, 331 - 343.
[18] T. Schilling, *The geometry of quantum mechanics*, Ph.D. thesis, Penn.State Univ., 1996.
[19] A.G. Sergeev, *Twistor quantization of loop spaces and general Kaehler manifolds*, Contemp. Math., 212 (1998), 221 - 228.
[20] J. Sniatycki, *Quantization and quantum mechanics*, Springer, Berlin (1987).
[21] J.M. Souriau, *Structure des systems dynamiques*, Paris (1970).
[22] R. Thomas, P. Seidel, *Braid group actions on derived categories of coherent sheaves*, arXiv:AG/0001043.
[23] G.M. Tuynman, *Quantization: towards a comparison between methods*, J. Math. Phys., 28(12) (1987), 2829 - 2840.
[24] A.N. Tyurin, *Special Lagrangian geometry as slightly deformation of algebraic geometry*, Izvestiya: Math., 64 : 2 (2000), 363 - 437.
[25] A.N. Tyurin, *Complexification of Bohr - Sommerfeld conditions*, Preprint Inst. of Math., Oslo Univ., N 15 (1999).
[26] N. A. Tyurin, *The space of hermitian triples and the Seiberg - Witten equations*, Izvestiya: Math., 65: 1 (2001), 181 - 205.
[27] N.A. Tyurin, *Hamiltonian dynamics on the moduli spaces of half weighted Bohr - Sommerfeld Lagrangian cycles of fixed volume*, Preprint MPI (Bonn), N 106 (2000).
[28] *The correspondence principle in abelian Lagrangian geometry*, Izvestiya: Math., 65: 4 (2001), 823 - 834.
[29] N. Tyurin, *ALAG-quantization*, Preprint KIAS (Seoul) M010002 (2001); arXiv:SG/0106004.
[30] N.A. Tyurin, *Dynamical correspondence in algebraic Lagrangian geometry*, Izvestiya: Math., 66: 3 (2002), 611 - 629.
[31] N.A. Tyurin, *Instantons and monopoles*, Russian Math. Survey, 57: 2 (2002), 305 - 360.
[32] A. Weinstein, *Lagrangian submanifolds and Hamiltonian systems*, Ann. of Math., 98 (1973), 377 - 410.
[33] N. Woodhouse, *Geometric quantization*, Oxford Univ. Press (1980).

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