THE GROUP OF UNITS AND ABELIAN TOTALLY RAMIFIED EXTENSIONS OF A COMPLETE DISCRETE VALUATION FIELD

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We consider a description of abelian totally ramified p-extensions of a complete discrete valuation field with residue field of characterisitic $p > 0$. The exposition shows that there exists a certain theory (Theorem (1.6), Proposition (1.8)) very similar to the classical one. The main result is that there is a reciprocity isomorphism between the group $(\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\sim} = \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\widetilde{F}/F), \mathrm{Gal}(L/F)^{\mathrm{ab}})$ of continuous homorphisms from the profinite group $\mathrm{Gal}(\widetilde{F}/F)$ of the maximal unramified abelian p-extension to the abelian part of the Galois group $\mathrm{Gal}(L/F)$ of a totally ramified p-extension L/F considered as a discrete group and the subquotient group $U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}} / N_{L/F} U_{1,L}$ of the group of principal units of F. Although we consider in this paper p-extensions, the whole theory can be extended on arbitrary totally ramified extensions.

Comparing this theory with higher dimensional class field theories, one can briefly say that the latter ones cover all extensions including nonseparable residue field extensions but only for a specific type of residue fields; the former one covers the general case of residue fields but describes only totally ramified extensions. It seems more convenient to work with subquotients of the group of principal units than with topological K-groups.

Another specific feature of the theory is that, in fact, one can develop many class field theories for a given complete discrete valuation field depending on the choice of an unramified extension (see Remark 2 in (1.6) and Proposition (1.9)).

The second section deals with extensions for which a fixed prime element is a norm. We deduce from the theory of section 1 that the compositum of two such extensions is a totally ramified extension (in the case of perfect residue fields the prime element even is a norm for the compositum). Then we prove that for a complete discrete valuation field F with non-algebraically-p-closed residue field norm groups $N_{L/F}L^*$ in F^* are in one-to-one correspondence with abelian totally ramified p-extensions L/F.

The third section contains discussions on the existence theorem. In the general case of imperfect residue field with respect to the perfect residue field case one needs

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additional information about the structure of norm subgroups (i.e. Eisenstein polynomials). The existence theorem in general form is established only for fields of positive characteristic and fields with small absolute ramification index ($< p-1$). It implies a connection between Witt vectors and cyclic p-extensions which was discovered earlier by Kurihara [K] by employing a very different approach.

When the first and second sections of this work had been completed, I found a twenty-years-old work of Miki [M] (see Remark 1 in (1.6)) several results of which may be considered as predecessors of this theory.

1. Reciprocity map

Let F be a complete (or Henselian) discrete valuation field with a residue field $\overline{F}$ of characteristic $p > 0$. It will be assumed that $\overline{F}$ has a nontrivial separable p-extension. (If $\overline{F}$ is separably p-closed, then class field theory of F is the limit of the theories for subfields F_α with non-separably-p-closed residue fields when F_α tends to F). Denote by $\widetilde{F}$ the maximal unramified abelian p-extension of F, i.e. the unramified extension corresponding to the maximal abelian p-extension $\overline{F}^{\mathrm{abp}}$ of the residue field $\overline{F}$. It is known that $\mathrm{Gal}(\overline{F}^{\mathrm{abp}}/\overline{F})$ is a free abelian profinite p-group on $\kappa = \dim_{\mathbf{F}_p} \overline{F}/\wp(\overline{F})$ generators, where $\wp(X) = X^p - X$. Then there is a non-canonical isomorphism $\mathrm{Gal}(\widetilde{F}/F) \simeq \prod_\kappa \mathbf{Z}_p$. Let U_F be the group of units of the ring of integers of F and let $U_{i,F}$ denote the subgroup of principal units $\equiv 1 \mod \pi_F^i$ with a prime element π_F of F.

1.1. Let L/F be a Galois totally ramified p-extension. Then $\mathrm{Gal}(L/F)$ can be identified with $\mathrm{Gal}(\widetilde{L}/\widetilde{F})$, and $\mathrm{Gal}(\widetilde{L}/F)$ is isomorphic with $\mathrm{Gal}(\widetilde{L}/\widetilde{F}) \times \mathrm{Gal}(\widetilde{L}/L)$. Let $\mathrm{Gal}(L/F)^\sim = \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Gal}(\widetilde{F}/F), \mathrm{Gal}(L/F))$ denote the group of continuous homomorphisms from the profinite group $\mathrm{Gal}(\widetilde{F}/F)$ which is a $\mathbf{Z}_p$-module ($a \cdot \sigma = \sigma^a$, $a \in \mathbf{Z}_p$) to the discrete $\mathbf{Z}_p$-module $\mathrm{Gal}(L/F)$. This group is isomorphic (non-canonically) with $\oplus_\kappa \mathrm{Gal}(L/F)$.

Now let L/F be of finite degree. Let $\chi \in \mathrm{Gal}(L/F)^\sim$ and Σ_χ be the fixed field of all $\tau_\varphi \in \mathrm{Gal}(\widetilde{L}/F)$, where $\tau_\varphi|_{\widetilde{F}} = \varphi, \tau_\varphi|_L = \chi(\varphi)$ and φ runs a topological $\mathbf{Z}_p$-basis of $\mathrm{Gal}(\widetilde{F}/F)$. Then $\widetilde{L}/\Sigma_\chi$ is unramified and Σ_χ/F is a totally ramified p-extension.

For a prime element π_χ of Σ_χ put

$$\Upsilon_{L/F}(\chi) = N_{\Sigma_\chi/F}\pi_\chi N_{L/F}\pi_L^{-1} \mod N_{L/F}U_L,$$

where π_L is a prime element in L. $\Upsilon_{L/F}$ is called to be a generalized Neukirch's map (as a generalization of constructions in [N1]).

1.2. Lemma. *The map* $\Upsilon_{L/F}\colon \mathrm{Gal}(L/F)^\sim \to U_F/N_{L/F}U_L$ *is well defined.*

Proof. $\Upsilon_{L/F}$ does not depend on the choice of π_L: let M be the compositum of Σ_χ and L. Then M/Σ_χ is unramified and any prime element in Σ_χ can be written as $\pi_\chi N_{M/\Sigma_\chi}\varepsilon$ for a suitable $\varepsilon \in U_M$. Then $N_{M/F}\varepsilon = N_{L/F}(N_{M/L}\varepsilon) \in N_{L/F}U_L$. □

Note that if $\varepsilon = N_{\widetilde{L}/\widetilde{F}}\beta$ with $\beta \in U_{\widetilde{L}}$, then one can write $\beta = \theta\eta$ with $\theta \in U_L, \eta \in U_{1,\widetilde{L}}$ and then $\varepsilon' = N_{\widetilde{L}/\widetilde{F}}\eta \in U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}}$ is uniquely defined mod $N_{L/F}U_{1,L}$. Thus, the quotient group $U_F \cap N_{\widetilde{L}/\widetilde{F}}U_{\widetilde{L}}/N_{L/F}U_L$ is mapped isomorphically onto $U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}}/N_{L/F}U_{1,L}$ by $\varepsilon \to \varepsilon'$. Put

$$U_{L/F} = U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}}/N_{L/F}U_{1,L}$$

and denote the map $\mathrm{Gal}(L/F)^{\sim} \to U_{L/F}$ by the same notation $\Upsilon_{L/F}$.

1.3. Proposition.

(1) *Let L/F, L_1/F_1 be totally ramified galois p-extensions, and F_1/F, L_1/L be totally ramified. Then the diagram*

$$\begin{array}{ccc} \mathrm{Gal}(L_1/F_1)^{\sim} & \xrightarrow{\Upsilon_{L_1/F_1}} & U_{L_1/F_1} \\ \downarrow & & \downarrow{\scriptstyle N_{F_1/F}} \\ \mathrm{Gal}(L/F)^{\sim} & \xrightarrow{\Upsilon_{L/F}} & U_{L/F} \end{array}$$

is commutative, where the left vertical homomorphism is induced by the natural restrictions $\mathrm{Gal}(L_1/F_1) \to \mathrm{Gal}(L/F)$ and $\mathrm{Gal}(\widetilde{F}_1/F_1) \xrightarrow{\sim} \mathrm{Gal}(\widetilde{F}/F)$.

(2) *Let L/F be a totally ramified galois p-extension, and let σ be an automorphism. Then the diagram*

$$\begin{array}{ccc} \mathrm{Gal}(L/F)^{\sim} & \xrightarrow{\Upsilon_{L/F}} & U_{L/F} \\ {\scriptstyle \sigma^{\sim}}\downarrow & & \downarrow \\ \mathrm{Gal}(\sigma L/\sigma F)^{\sim} & \xrightarrow{\Upsilon_{\sigma L/\sigma F}} & U_{\sigma L/\sigma F} \end{array}$$

is commutative, where $(\sigma^{\sim}\chi)(\sigma\varphi\sigma^{-1}) = \sigma\chi(\varphi)\sigma^{-1}$.

Proof. Apply the arguements of the proof of Proposition (1.8) of [F1] together with the following commutative diagram

$$\begin{array}{ccc} U_{F_1} \cap N_{\widetilde{L}_1/\widetilde{F}_1}U_{\widetilde{L}_1}/N_{L_1/F_1}U_{L_1} & \longrightarrow & U_{L_1/F_1} \\ \downarrow{\scriptstyle N_{F_1/F}} & & \downarrow{\scriptstyle N_{F_1/F}} \\ U_F \cap N_{\widetilde{L}/\widetilde{F}}U_{\widetilde{L}}/N_{L/F}U_L & \longrightarrow & U_{L/F} \end{array}$$

□

1.4. Recall the behavior of the norm map. Let L/F be a cyclic totally ramified extension of degree p. Let π_L be a prime element in L. Then $\pi_F = N_{L/F}\pi_L$ is prime in F. Let σ be a generator of $\mathrm{Gal}(L/F)$,

$$\frac{\sigma\pi_L}{\pi_L} = 1 + \theta_0\pi_L^s + \dots$$

with $\theta_0 \in U_F$, $s = s(L|F) > 0$. Then it is well known that

$$\begin{aligned} N_{L/F}(1+\theta\pi_L^i) &= 1+\theta^p\pi_F^i+\dots && \text{for } i<s,\ \theta\in U_F \\ N_{L/F}(1+\theta\pi_L^s) &= 1+(\theta^p-\theta_0^{p-1}\theta)\pi_F^s+\dots && \text{for } \theta\in U_F \\ N_{L/F}(1+\theta\pi_L^{s+pi}) &= 1-\theta_0^{p-1}\theta\pi_F^{s+i}+\dots && \text{for } i>0,\ \theta\in U_F. \end{aligned}$$

From this it follows that in a fixed Galois totally ramified p-extension M/F for any index i there exists an integer $r = r(i, M/F) \geqslant 0$ such that $1+\theta^{p^r}\pi_F^i \in U_{i+1,\widetilde{F}}N_{\widetilde{M}/\widetilde{F}}U_{1,\widetilde{M}}$ for any $\theta \in U_F$ and $1+\theta^{p^{r-1}}\pi_F^i \notin U_{i+1,\widetilde{F}}N_{\widetilde{M}/\widetilde{F}}U_{1,\widetilde{M}}$ for any $\theta \in U_F$ with $\overline{\theta} \notin \overline{F}^p$.

1.5. Denote by $\mathcal{F}$ an extension of F such that $e(\mathcal{F}/F) = 1$ and the residue field of $\mathcal{F}$ is the perfection $\overline{\widetilde{F}}^{\text{perf}}$ of the residue field of $\widetilde{F}$, i.e. $= \cup_n \overline{\widetilde{F}}^{p^{-n}}$ ($\mathcal{F}$ isn't uniquely defined). Let L/F be a finite totally ramified Galois p-extension. For $\sigma \in \mathrm{Gal}(L/F)$ put

$$c(\sigma) = {\pi_L}^{-1}\sigma\pi_L \mod I(L|F),$$

where π_L is a prime element in L, and $I(L|F)$ is the subgroup of $U_{1,\widetilde{L}}$ generated by the elements $\varepsilon^{-1}\sigma(\varepsilon)$ with $\varepsilon \in U_{1,L\mathcal{F}}$, $\sigma \in \mathrm{Gal}(L/F)$. Then the sequence

$$1 \to \mathrm{Gal}(L/F)^{\mathrm{ab}} \xrightarrow{c} U_{1,\widetilde{L}}/I(L|F) \xrightarrow{N_{\widetilde{L}/\widetilde{F}}} N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}} \to 1$$

is exact (this follows from the case of a perfect residue field, see [F1], [H, section 4], [I, (2.2)]).

Now we introduce the map inverse to $\Upsilon_{L/F}$. Let $\varepsilon \in U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}}$ and $\varphi \in \mathrm{Gal}(\widetilde{F}/F)$. Let $\eta \in U_{1,\widetilde{L}}$ be such that $N_{\widetilde{L}/\widetilde{F}}\eta = \varepsilon$. Since $N_{\widetilde{L}/\widetilde{F}}(\eta^{-1}\tilde{\varphi}(\eta)) = 1$ for an extension $\tilde{\varphi} \in \mathrm{Gal}(\widetilde{L}/F)$ of φ, it follows that $\eta^{-1}\tilde{\varphi}(\eta) \equiv \pi_L\sigma(\pi_L^{-1}) \mod I(L|F)$ for a suitable $\sigma \in \mathrm{Gal}(L/F)^{\mathrm{ab}}$, where π_L is a prime element in L. Set $\chi(\varphi) = \sigma$. Then it is easy to verify that $\chi(\varphi_1\varphi_2) = \sigma_1\sigma_2$. This means $\chi \in (\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\sim}$. Put $\Psi_{L/F}(\varepsilon) = \chi$.

Lemma. *The map* $\Psi_{L/F}\colon U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}}/N_{L/F}U_{1,L} \to (\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\sim}$ *is well defined and a homomorphism.*

Proof. If $N_{\widetilde{L}/\widetilde{F}}\rho = \varepsilon$, then for $\mu = \eta^{-1}\rho$ the element $\mu^{-1}\varphi(\mu)$ belongs to $I(L|F)$. If $\varepsilon = \varepsilon_1\varepsilon_2$, then one may assume $\eta = \eta_1\eta_2$, consequently $\sigma = \sigma_1\sigma_2$ in $\mathrm{Gal}(L/F)^{\mathrm{ab}}$. Thus, $\Psi_{L/F}(\varepsilon_1\varepsilon_2) = \Psi_{L/F}(\varepsilon_1)\Psi_{L/F}(\varepsilon_2)$. □

$\Psi_{L/F}$ is called to be a generalized Hazewinkel's homomorphism (as a generalization of constructions in [H]).

1.6. Theorem. *Let L/F be a Galois totally ramified p-extension. The map $\Upsilon_{L/F}$ induces an isomorphism $\Upsilon^{\mathrm{ab}}_{L/F}\colon (\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\sim} \to U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}} / N_{L/F} U_{1,L}$ and the map $\Psi_{L/F}$ is the inverse one.*

Proof. First we verify that $\Psi_{L/F} \circ \Upsilon^{\mathrm{ab}}_{L/F} = \mathrm{id}$. Indeed, let $\pi_\chi = \pi_L \eta$ with $\eta \in U_{\widetilde{L}}$. Let $\varphi = \tilde{\varphi}|_{\widetilde{F}} \in \mathrm{Gal}(\widetilde{F}/F)$ with $\tilde{\varphi} \in \mathrm{Gal}(\widetilde{L}/L)$ and $\tau_\varphi \in \mathrm{Gal}(\widetilde{L}/F)$ be such that $\tau_\varphi|_{\widetilde{F}} = \varphi$, $\tau_\varphi|_L = \sigma = \chi(\varphi)$. Put $\eta = \theta\eta_1$ with $\theta \in U_F$, $\eta_1 \in U_{1,\widetilde{L}}$. Then

$$\pi_L^{1-\sigma} = \eta^{\tau_\varphi - 1} \equiv \eta_1^{\tilde{\varphi}-1} \mod I(L|F)$$

and $N_{\widetilde{L}/\widetilde{F}}\eta_1 = N_{\Sigma_\chi/F}\pi_\chi N_{L/F}(\theta\pi_L)^{-1}$. Therefore, $\chi = \Psi_{L/F}(\Upsilon^{\mathrm{ab}}_{L/F}(\chi))$. Thus, the homomorphism $\Psi_{L/F}$ is surjective and the map $\Upsilon^{\mathrm{ab}}_{L/F}$ is injective.

According to Proposition (1.3) the following diagram is commutative:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Gal}(L/M)^{\sim} & \longrightarrow & \mathrm{Gal}(L/F)^{\sim} & \longrightarrow & \mathrm{Gal}(M/F)^{\sim} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \Upsilon_{L/M}} & & \downarrow{\scriptstyle \Upsilon_{L/F}} & & \downarrow{\scriptstyle \Upsilon_{M/F}} & & \\
 & & U_{L/M} & \xrightarrow{N_{M/F}} & U_{L/F} & \longrightarrow & U_{M/F} & \longrightarrow & 1
\end{array}$$

where M/F is a cyclic subextension of degree p in L/F. The low sequence is exact by the first part of the proof of Proposition (1.7) below (it doesn't depend on (1.6)). Now we are going to verify that $\Psi_{L/F}$ is injective for a cyclic extension of degree p. Then this immediately implies that $\Upsilon_{L/F}$ is surjective for a cyclic extension of degree p. Since $\mathrm{Gal}(L/F)$ is solvable, one deduces using the diagram that $\Upsilon_{L/F}$ is surjective for an arbitrary totally ramified p-extension. Then $\Psi_{L/F}$ is injective.

In order to show that $\Psi_{L/F}$ is injective for a cyclic extension of degree p it suffices to show keeping in mind (1.4) that if $\varepsilon = N_{\widetilde{L}/\widetilde{F}}\eta \equiv 1 + \theta\pi_L^s + \dots$ with $\overline{\theta} \notin \overline{\theta}_0^p \wp(\overline{F})$ (that is $\varepsilon \notin N_{L/F} U_{1,L}$), where s, θ_0 are as in (1.4), then $\Psi_{L/F}(\varepsilon) \neq 1$. As $\eta = 1 + \theta'\pi_L^s + \dots$ with $\overline{\theta'}^p - \overline{\theta}_0^{p-1}\overline{\theta'} = \overline{\theta}$, one deduces that the condition $\Psi_{L/F}(\varepsilon) = 1$ implies that $\theta' \in U_L U_{s+1,\widetilde{L}}$ and $\overline{\theta} \in \overline{\theta}_0^p \wp(\overline{F})$. $\square$

Remark 1. *In* [M] *Miki has shown without explicit introduction of reciprocity maps that for a totally ramified cyclic extension F'/F of degree m and for a finite abelian unramified extension E/F of exponent m the group*

$$F \cap N_{EF'/E} U_{EF'} / N_{F'/F} U_{F'}$$

is canonically isomorphic to the character group of $\mathrm{Gal}(E/F)$.

Remark 2. *In our description of the Galois group of a totally ramified p-extension L/F one can take instead the maximal unramified abelian p-extension $\widetilde{F}/F$ any its infinite profinite subextension $\hat{F}/F$: then $\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\hat{F}/F), \mathrm{Gal}(L/F))$ is not so large, $N_{\hat{L}/\hat{F}} U_{1,\hat{L}}$ isn't the possible largest subgroup and isn't too far from $N_{L/F} U_{1,L}$.*

Remark 3. *A corollary of the theorem: the pairing*

$$U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}} / N_{L/F} U_{1,L} \times \mathrm{Gal}(\widetilde{F}/F) \to \mathrm{Gal}(L/F)^{\mathrm{ab}}, \qquad (\varepsilon, \varphi) \to \Psi_{L/F}(\varepsilon)(\varphi)$$

is nondegenerate.

Remark 4. *There is a functorial property of $\Upsilon_{L/F}$ additional to those of (1.3). Let L/F be a Galois totally ramified p-extension and M/F be its subextension. Then the diagram*

$$\begin{array}{ccc} (\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\sim} & \longrightarrow & U_{L/F} \\ {\scriptstyle \mathrm{Ver}^{\sim}} \downarrow & & \downarrow \\ (\mathrm{Gal}(L/M)^{\mathrm{ab}})^{\sim} & \longrightarrow & U_{L/M} \end{array}$$

is commutative, where $\mathrm{Ver}^{\sim}$ is induced by $\mathrm{Ver}\colon \mathrm{Gal}(L/F)^{\mathrm{ab}} \to \mathrm{Gal}(L/M)^{\mathrm{ab}}$. A proof is the same as in the case of perfect residue field: Let $\varepsilon = N_{\widetilde{L}/\widetilde{F}} \eta$ and $\eta^{\varphi - 1} = \pi_L^{1-\sigma} \gamma$ for a prime element π_L in L, $\sigma \in \mathrm{Gal}(L/F)$, $\gamma \in I(L|F)$. Then $\sigma = \chi(\varphi)$, $\chi = \Psi_{L/F}(\varepsilon)$. Let $\tau_i \in \mathrm{Gal}(\widetilde{L}/\widetilde{F})$ be a set of representatives of $\mathrm{Gal}(\widetilde{L}/\widetilde{F})$ over $\mathrm{Gal}(\widetilde{L}/\widetilde{M})$. Then $\varepsilon = N_{\widetilde{L}/\widetilde{M}} \eta_1$ with $\eta_1 = \prod \eta^{\tau_i}$ and $\eta_1^{\varphi-1} = \prod \pi_L^{(1-\sigma)\tau_i} \prod \gamma^{\tau_i}$. Let $\sigma\tau_i = \tau_{i'} h_i(\sigma)$ with $h_i(\sigma) \in \mathrm{Gal}(\widetilde{L}/\widetilde{M})$. Now one deduces

$$\prod \pi_L^{(1-\sigma)\tau_i} = \prod \pi_L^{\tau_{i'}(1-h_i(\sigma))} \equiv \pi_L^{\prod(1-h_i(\sigma))} = \pi_L^{1-\mathrm{Ver}(\sigma)} \mod I(L|M).$$

Since $\prod \gamma^{\tau_i} \in I(L|M)$, one concludes that $\eta_1^{\varphi-1} \equiv \pi_L^{1-\mathrm{Ver}(\sigma)} \mod I(L|M)$, as desired.

1.7. Proposition. *Let L/F be a Galois totally ramified p-extension, M/F be its abelian subextension. Then*

$$N_{M/F}(U_{1,M} \cap N_{\widetilde{L}/\widetilde{M}} U_{1,\widetilde{L}}) = N_{M/F} U_{1,M} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}}.$$

Proof. One needs to verify the inclusion of the right hand side expression into the left hand side.

First the case of $|M : F| = p$ will be considered. Let $\alpha \in N_{M/F} U_{1,M} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}}$ and $\alpha = N_{M/F} \beta$ with $\beta = \delta N_{\widetilde{L}/\widetilde{M}} \gamma$, $\delta = \rho^{\sigma-1}$, where σ is a generator of $\mathrm{Gal}(M/F)$. One can assume without loss of generality that $\rho = 1 + \theta' \pi_M^i + \dots$ with i prime to p. If $\theta' \in U_{\widetilde{M}} \setminus U_M$ and $\rho^{\sigma-1} = 1 + i\theta'\theta_0 \pi_M^{i+s} + \dots \notin N_{\widetilde{L}/\widetilde{M}} U_{1,\widetilde{L}}$, where θ_0, s are as in (1.4), then denote by F_1 the unramified extension of F such that $\overline{F_1} = \overline{F}(\overline{\theta'})$; let $\varphi \in \mathrm{Gal}(\widetilde{L}/L)$ be such that $\overline{\varphi}(\overline{\theta'}) \neq \overline{\theta'}$. Since $(\rho^{\sigma-1})^{\varphi-1} = N_{\widetilde{L}/\widetilde{M}}(\gamma^{1-\varphi})$ one deduces using (1.4) that $\overline{\theta_0}(\overline{\varphi}(\overline{\theta'}) - \overline{\theta'}) = \overline{\xi}^{p^r} \in \overline{F_1}^{p^r}$ and $\overline{\theta'\theta_0} \notin \overline{F_1}^{p^r}$ for $r = r(i+s, L/M)$. Then $\mathrm{Tr}_{\overline{F_1}/\overline{F_2}} \overline{\xi}^{p^r} = (\mathrm{Tr}_{\overline{F_1}/\overline{F_2}} \overline{\xi})^{p^r} = 0$ and $\overline{\xi} = \overline{\varphi}(\overline{\lambda}) - \overline{\lambda}$ for some $\overline{\lambda} \in \overline{F_1}$, where $\overline{F_2}$ is the fixed field of $\overline{\varphi}$ in $\overline{F_1}$. Therefore $\overline{\theta'\theta_0} \in \overline{F_1}^{p^r} + \overline{F_2}$, and continue in this way we obtain $\overline{\theta'\theta_0} \in \overline{F_1}^{p^r} + \overline{F}$. This implies that $\delta = \rho_1^{\sigma-1}\rho_2^{\sigma-1}\rho_3^{\sigma-1}$ with $\rho_1^{\sigma-1} \in \ker N_{M/F}$, $\rho_2^{\sigma-1} = N_{\widetilde{L}/\widetilde{M}}\gamma_1$, $\rho_3^{\sigma-1} \in U_{i+1,\widetilde{M}}$. Thus, $\alpha = N_{M/F}\beta_1$ with $\beta_1 = \delta_1 N_{\widetilde{L}/\widetilde{M}}(\gamma\gamma_1)$ and $\delta_1 = \rho_3^{\sigma-1}$. Since $U_{j,\widetilde{M}} \subset N_{\widetilde{L}/\widetilde{M}}U_{1,\widetilde{L}}$ for a sufficiently large j, we conclude $\alpha \in N_{M/F}(U_{1,M} \cap N_{\widetilde{L}/\widetilde{M}}U_{1,\widetilde{L}})$.

Now the assertion of the proposition in the general case will be proved by induction on the degree of L/F. Let E/F be a Galois subextension in L/F of degree p, $E \subset M$. Let $\alpha = N_{M/F}\beta \in N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}}$. Then $\alpha \in N_{E/F}U_{1,E} \cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}}$ and by the previous part of the proof $N_{M/F}\beta = N_{E/F}\gamma$ for some $\gamma \in U_{1,E} \cap N_{\widetilde{L}/\widetilde{E}}U_{1,\widetilde{L}}$. Then $\gamma N_{M/E}\beta^{-1} \in \ker N_{E/F} \cap N_{\widetilde{M}/\widetilde{E}}U_{1,\widetilde{M}}$. Applying part (1) of Proposition (1.3) and Theorem (1.6) one deduces that $\gamma N_{M/E}\beta^{-1} = N_{M/E}\delta$ and $\gamma = N_{M/E}(\beta\delta) \in N_{M/E}U_{1,M} \cap N_{\widetilde{L}/\widetilde{E}}U_{1,\widetilde{L}}$. The inductional assumption implies that $\gamma \in N_{M/E}(U_{1,M} \cap N_{\widetilde{L}/\widetilde{M}}U_{1,\widetilde{L}})$. Thus, $\alpha \in N_{M/F}(U_{1,M} \cap N_{\widetilde{L}/\widetilde{M}}U_{1,\widetilde{L}})$. □

1.8. Proposition. *Let L_1/F, L_2/F, L_1L_2/F be abelian totally ramified p-extensions. Put $L_3 = L_1L_2$, $L_4 = L_1 \cap L_2$. Then*

$$N_{L_3/F}U_{1,L_3} = N_{L_1/F}U_{1,L_1} \cap N_{L_2/F}U_{1,L_2} \cap N_{\widetilde{L}_3/\widetilde{F}}U_{1,\widetilde{L}_3},$$
$$N_{L_4/F}U_{1,L_4} \cap N_{\widetilde{L}_3/\widetilde{F}}U_{1,\widetilde{L}_3} = (N_{L_1/F}U_{1,L_1}N_{L_2/F}U_{1,L_2}) \cap N_{\widetilde{L}_3/\widetilde{F}}U_{1,\widetilde{L}_3},$$
$$N_{L_1/F}U_{1,L_1} \subset N_{L_2/F}U_{1,L_2} \quad \text{if and only if} \quad L_1 \supset L_2.$$

Let M/F be the maximal abelian subextension in a Galois totally ramified p-extension L/F. Then

$$N_{M/F}U_{1,M} \cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}} = N_{L/F}U_{1,L}.$$

Proof. Put $H_i = \mathrm{Gal}(L_3/L_i)$, $i = 1, 2$. Then by Proposition (1.3), Theorem (1.6) and Proposition (1.7)

$$\begin{aligned} N_{L_3/F}U_{1,L_3} &= \Psi_{L_3/F}^{-1}(H_1 \cap H_2) = \Psi_{L_3/F}^{-1}(H_1) \cap \Psi_{L_3/F}^{-1}(H_2) \\ &= N_{L_1/F}(U_{1,L_1} \cap N_{\widetilde{L}_3/\widetilde{L}_1}U_{1,\widetilde{L}_3}) \cap N_{L_2/F}(U_{1,L_2} \cap N_{\widetilde{L}_3/\widetilde{L}_2}U_{1,\widetilde{L}_3}) \\ &= N_{L_1/F}U_{1,L_1} \cap N_{L_2/F}U_{1,L_2} \cap N_{\widetilde{L}_3/\widetilde{F}}U_{1,\widetilde{L}_3}; \end{aligned}$$

and similarly

$$\begin{aligned} &N_{L_4/F}U_{1,L_4} \cap N_{\widetilde{L}_3/\widetilde{F}}U_{1,\widetilde{L}_3} \\ &= (N_{L_1/F}U_{1,L_1} \cap N_{\widetilde{L}_3/\widetilde{F}}U_{1,\widetilde{L}_3})(N_{L_2/F}U_{1,L_2} \cap N_{\widetilde{L}_3/\widetilde{F}}U_{1,\widetilde{L}_3}) \\ &\subset (N_{L_1/F}U_{1,L_1}N_{L_2/F}U_{1,L_2}) \cap N_{\widetilde{L}_3/\widetilde{F}}U_{1,\widetilde{L}_3}, \end{aligned}$$

and the inverse inclusion is obvious.

Further, if $N_{L_1/F}U_{1,L_1} \subset N_{L_2/F}U_{1,L_2}$, then

$$N_{L_1/F}U_{1,L_1} \cap N_{\widetilde{L}_3/\widetilde{F}}U_{1,\widetilde{L}_3} = N_{L_3/F}U_{1,L_3}.$$

Now by Proposition (1.3) and Theorem (1.6) one deduces that $L_2 \subset L_1$.

Finally, the last assertion of the proposition follows from Proposition (1.3) and Theorem (1.6) □

Remark. *If the residue field $\overline{F}$ is perfect, then one can omit $N_{\widetilde{L}_3/\widetilde{F}}U_{1,\widetilde{L}_3}$ and $N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}}$ in the assertions of the proposition (see* [F1, (1.8)]*). In the general case this can't be made: let, for instance, L_1/F, L_2/F be totally ramified Galois extensions of degree p such that $p < s(L_1|F) \leqslant s(L_2|F)$. Assume that L_1L_2/F is totally ramified and $\pi \in N_{L_1L_2/F}(L_1L_2)^*$ is a prime element of F. Then $(1+\theta\pi)^p = 1+\theta^p\pi^p+\dots$ belongs to $N_{L_1/F}U_{1,L_1}$ and $N_{L_2/F}U_{1,L_2}$ and for $\overline{\theta} \notin \overline{F}^p$ doesn't belong to $N_{L_1L_2/F}U_{1,L_1L_2}$, since $r(p, L_1L_2/F) = p^2$ in terms of (1.4).*

1.9. Assume that the residue field $\overline{F}$ is a formal power series field of $n-1$ indeterminates over a perfect field k which is not algebraically p-closed. Denote a lifting in F of a system of local parameters of $\overline{F}$ by $t_{n-1}, \dots, t_1$. Then $\pi, t_{n-1}, \dots, t_1$ form a system of local parameters of F as of an n-dimensional local field over k. In what follows notations of [F3] will be used.

Proposition. *Let L/F be a Galois totally ramified p-extension with respect to the discrete valuation of rank 1. Then the following diagram is commutative*

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\widetilde{F}/F), \mathrm{Gal}(L/F)) & \xrightarrow{\Upsilon_{L/F}} & U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}}/N_{L/F}U_{1,L} \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\hat{F}/F), \mathrm{Gal}(L/F)) & \xrightarrow{\mathcal{Y}_{L/F}} & U_1K_n^{\mathrm{top}}(F)/N_{L/F}U_1K_n^{\mathrm{top}}(L)
\end{array}$$

where $U_1K_n^{\mathrm{top}}(F)$ is the subgroup of the topological K-group $K_n^{\mathrm{top}}(F)$ generated by $U_{1,F} \in K_1^{\mathrm{top}}(F)$, $N_{L/F}$ at the right bottom corner is the norm map on the topological K-groups. The homomorphism $\mathcal{Y}_{L/F}$ is the inverse to the reciprocity map for the extension L/F of n-dimensional local fields, the left vertical map is induced by the surjection $\mathrm{Gal}(\widetilde{F}/F) \to \mathrm{Gal}(\hat{F}/F)$ given by

$$\mathrm{Gal}\left(k((\overline{t_{n-1}}))\dots((\overline{t_1}))^{\mathrm{abp}}/k((\overline{t_{n-1}}))\dots((\overline{t_1}))\right) \to$$
$$\mathrm{Gal}\left(k^{\mathrm{abp}}((\overline{t_{n-1}}))\dots((\overline{t_1}))/k((\overline{t_{n-1}}))\dots((\overline{t_1}))\right),$$

the right one is induced by

$$\varepsilon \to \{\varepsilon, t_{n-1}, \dots, t_1\}.$$

In addition, the following diagram is commutative

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\hat{F}/F), \mathrm{Gal}(L/F)) & \xrightarrow{\hat{\Upsilon}_{L/F}} & U_{1,F} \cap N_{\hat{L}/\hat{F}}U_{1,\hat{L}}/N_{L/F}U_{1,L} \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\hat{F}/F), \mathrm{Gal}(L/F)) & \xrightarrow{\mathcal{Y}_{L/F}} & U_{0,\dots,1}K_n^{\mathrm{top}}(F)/N_{L/F}U_1K_n^{\mathrm{top}}(L)
\end{array}$$

where $\hat{\Upsilon}_{L/F}$ is taken for the extension $\hat{F}/F$ (see Remark 2 of (1.6)). In this case the left and the right vertical homomorphisms are isomorphisms.

Proof. Follows immediately from the description of the reciprocity map in sections 1 and 3 of [F3], and Remark 2 of (1.6). $\square$

2. Extensions with a fixed prime element as a norm

2.1. Proposition. *Let F be a complete discrete valuation field with arbitrary residue field of characteristic p. Let L_1/F, L_2/F be abelian totally ramified p-extensions and $\pi \in N_{L_1/F}L_1^* \cap N_{L_2/F}L_2^*$ for a prime element π of F. Then L_1L_2/F is a totally ramified extension.*

Proof. If $\overline{F}$ is algebraically p-closed, then the assertion is evident. If $\overline{F}$ is perfect and not algebraically p-closed, then L_1L_2/F is totally ramified by (3.3) of [F1]. If $\overline{F}$ is imperfect, then one can construct a henselian discrete valuation field $\mathfrak{F} = \varinjlim F'$ for which $e(\mathfrak{F}/F) = 1$, and the residue field of which is $\cup_n \overline{F}^{p^{-n}}$. In the case of positive characteristic of F the field $\mathfrak{F}$ can be chosen as a purely inseparable extension of F, and then $L_1\mathfrak{F} \cap L_2\mathfrak{F} = (L_1 \cap L_2)\mathfrak{F}$. If F is of characteristic 0, then for any $\overline{\theta} \in \overline{F}' \setminus \overline{F'}^p$ there exists $\alpha \in U_{F'}$ such that $\overline{\alpha} = \overline{\theta}$ and $L_1L_2F'(\alpha_1) \neq L_1L_2F'$ with $\alpha_1^p = \alpha$. Indeed, otherwise one would deduce that $U_{1,F'} \subset (L_1L_2F')^p$ which is impossible, since L_1L_2F'/F' is of finite degree and $U_{1,F'}/U_{1,F'}^p$ is of infinite order. Assume that $L_1F' \cap L_2F' = (L_1 \cap L_2)F'$, then if $L_1F'(\alpha_1) \cap L_2F'(\alpha_1) \neq (L_1F' \cap L_2F')(\alpha_1)$, one would have $N_1(\alpha_1) = N_2(\alpha_1)$ for a suitable extensions $N_1/(L_1 \cap L_2)F'$ in $L_1F'/(L_1 \cap L_2)F'$ and $N_2/(L_1 \cap L_2)F'$ in $L_2F'/(L_1 \cap L_2)F'$ of degree p. Then it would be $N_1(\alpha_1) = N_1N_2 \subset L_1L_2F'$, contradiction. Thus, proceeding in this way, one can construct $\mathfrak{F}/F$ with the property $L_1\mathfrak{F} \cap L_2\mathfrak{F} = (L_1 \cap L_2)\mathfrak{F}$.

Now, $L_1L_2\mathfrak{F}/\mathfrak{F}$ is totally ramified (as the residue field of $\mathfrak{F}$ is perfect) of degree

$$
\begin{aligned}
&|L_1L_2\mathfrak{F} : (L_1 \cap L_2)\mathfrak{F}|\,|(L_1 \cap L_2)\mathfrak{F} : \mathfrak{F}| \\
&= |L_1\mathfrak{F} : (L_1 \cap L_2)\mathfrak{F}|\,|L_2\mathfrak{F} : (L_1 \cap L_2)\mathfrak{F}|\,|(L_1 \cap L_2)\mathfrak{F} : \mathfrak{F}| \\
&= |L_1 : L_1 \cap L_2|\,|L_2 : L_1 \cap L_2|\,|L_1 \cap L_2 : F| = |L_1L_2 : F|.
\end{aligned}
$$

Therefore L_1L_2/F is totally ramified. $\square$

Remark. *If the residue field of F is perfect, then under assumptions of the proposition $\pi \in N_{L_1L_2/F}U_{1,L_1L_2}$ (see* [F1, section 3]*). When the residue field is imperfect this doesn't hold in general. Indeed, in terms of Remark (1.8) the prime element $\pi(1+\theta\pi)^p \in N_{L_1/F}L_1^* \cap N_{L_2/F}L_2^*$, and $\notin N_{L_1L_2/F}L_1L_2^*$.*

2.2. Theorem. *Let F be a complete discrete valuation field with a residue field of characterisitic p which isn't separably p-closed. Let L_1/F, L_2/F be totally ramified abelian p-extensions. Then $N_{L_1/F}L_1^* = N_{L_2/F}L_2^*$ if and only if $L_1 = L_2$.*

Proof. According to the previous proposition L_1L_2/F is totally ramified. Proposition (1.8), (3) implies now that $L_1 = L_2$. □

Remark. *A weaker assertion (for the case when the residue field if contained in an extension of fields of type $k((t_1))...((t_n))/k$ with a perfect not-p-closed field k) has been proved in* [F2] *by using higher local class field theory. Note that if one replaces the words "totally ramified abelian p-extensions" by either "totally ramified abelian extensions", or by "abelian p-extensions", or by "totally ramified p-extensions", then the assertion of the theorem doesn't hold in general (see* [F2]*).*

3. On existence theorem

Let F be a complete discrete valuation field with residue field of characteristic p.

3.1. In the general case of imperfect residue field it seems that to describe norm subgroups of totally ramified p-extensions of F isn't easy (for the perfect case see [F1, section 3]).

For example, in the case of a Galois totally ramified extension L/F of degree p take a prime element π_L of L and $\pi_F = N_{L/F}\pi_L$, and let s, θ_0 be as in (1.4). Let v_F be the discrete valuation of F. Let $e_i(X) = X^p + a_{p-1}X^{p-1} + \cdots + a_0$ be an irreducible polynomial of π_L^i over F for i prime to p. After some calculations one deduces that $m(i) = \min_{0 \leqslant t < p} v_F(a_t) = v_F(a_j)$ with $ij \equiv -s \mod p$ and $a_j = is^{-1}\theta_0^{p-1}\pi_F^{i+s-(s+ij)/p} + \dots$. Moreover, one can show that there exists an element $\alpha = \pi_L^i + \cdots \in L$ satisfying the equation $g_i(\alpha) = 0$, where $g_i(X) = X^p + b_jX^j + b_0$, $v_F(b_j) = v_F(a_j)$, $v_F(b_0) = i$ (see, for instance, [A]). This implies that

$$N_{L/F}(1 - \theta\alpha) = 1 + b_j\theta^{p-j} + b_0\theta^p, \quad v_F(b_j) = m(i), \quad v_F(b_0) = i$$

for θ in the ring of integers of F. In the case of the perfect residue field these formulas show that there is a polynomial $g_i(X)$ such that

$$1 + \theta\pi_F^i + g_i(\theta)\pi_F^s \in N_{L/F}U_{1,L}$$

for any θ in the ring of integers of F (see sect. 3 Chap. V of [FV]).

If the absolute ramification index of F is $\geqslant p - 1$, this isn't the case for imperfect residue field: one can't expect that there is a polynomial $g_i(X)$ such that for all θ

$$1 + \theta^p\pi_F^i + g_i(\theta)\pi_F^s \in N_{L/F}U_{1,L}.$$

Certainly, instead of this one can take an expression of the form

$$1 + \theta^{p^{n(i)}}\pi_F^i + h_i(\theta)\pi_F^s \in N_{L/F}U_{1,L}$$

with some $n(i)$ (even with $n(i) \leqslant 2$). As a direct generalization of the description of norm subgroups in the perfect residue field case (see (3.1) of [F1]), one would have expected the following: let π be a prime element of F. Then subgroups $\mathcal{N}$ in $U_{1,F}$ which are norm groups of cyclic totally ramified extension L/F of degree p with $\pi \in N_{L/F}L^*$ are characterized as

(1) $\mathcal{N}$ is open;
(2) for any $i > 0$ there exists a polynomial $f_i(X)$ with coefficients in the ring of integers $\mathcal{O}_F$ of F such that its residue $\overline{f}_i$ is non-zero $\overline{F}$-decomposable and $1 + f_i(\theta^{p^{n(i)}})\pi^i \subset \mathcal{N}$ for $\theta \in \mathcal{O}_F$;
(3) for any $i > 0$ the image of $(U_{i,F} \cap \mathcal{N})U_{i+1,F}$ under the projection
$$U_{i,F} \to U_{i,F}/U_{i+1,F} \xrightarrow{\sim} \overline{F}, \qquad 1 + \theta\pi^i \to \overline{\theta},$$
is equal to $p_i(\overline{F})$, where $p_i(X) = X^p$ for $i < s$, $p_i(X) = X$ for $i > s$, and $p_s(X) = X^p - \overline{\theta}_0^{p-1} X$.

Unfortunately, there exist subgroups $\mathcal{N}$ satisfying these properties which *are not* norm subgroups. For example, for $e = 3$, $p = 3$, $s = 4$, and an imperfect residue field the subgroup $\mathcal{N} \in U_{1,F}$ determined by relations
$$\begin{aligned} 1 + \theta^p\pi^i &\in \mathcal{N}, \quad \text{for} \quad i < s, \quad \theta \in \mathcal{O}_F \\ 1 + (\theta^p - \theta_0^{p-1}\theta)\pi^s &\in \mathcal{N}, \quad \text{for} \quad i = s, \quad \theta \in \mathcal{O}_F \\ 1 + \theta\pi^i &\in \mathcal{N}, \quad \text{for} \quad i > s, \quad \theta \in \mathcal{O}_F \end{aligned}$$
isn't a norm subgroup of any extension L/F with $\pi \in N_{L/F}L^*$ (in this case $g_1(X) = X^3 + \pi'^3 X^2 + \pi'$, $v_F(\pi') = 1$).

The conclusion is that an additional information arising from Eisenstein polynomials should be involved in the description of the norm subgroups – this is the imperfect phenomenon which is hidden in the perfect residue field case.

Note that if F is as in (1.9), then one can show that $\varepsilon \in N_{L/F}U_{1,L}$ is equivalent to $\{\varepsilon, t_{n-1}, \dots, t_1\} \in N_{L/F}U_1 K_n^{\mathrm{top}}(L)$ for any system of local parameters $t_{n-1}, \dots, t_1$. Since the description of $N_{L/F}U_1 K_n^{\mathrm{top}}(L)$ is known, one can try to apply this to describe norm subgroups in $U_{1,F}$ – however, this is too unexplicit.

3.2. There is a complete description of the norm subgroups of cyclic totally ramified p-extensions when $p > 2$ and the absolute ramification index $e(F)$ is $< p - 1$ or $= +\infty$.

Introduce a function
$$\mathcal{E}_{n,\pi_F} : \underbrace{W_n(\overline{F}) \oplus \cdots \oplus W_n(\overline{F})}_{e_F \text{ times}} \to U_{1,F}/U_{1,F}^{p^n}$$
by the formula
$$\mathcal{E}_{n,\pi_F}\left((a_{0,j}, \dots, a_{n-1,j})_{1 \leqslant j \leqslant e_F}\right) = \prod_{0 \leqslant i \leqslant n-1,\ 1 \leqslant j \leqslant e_F} E(\tilde{a}_{i,j}^{p^{n-1-i}} \pi_F^j)^{p^i},$$
where $E(X) = \exp(X + X^p/p + X^{p^2}/p^2 + \dots)$ is the Artin–Hasse function. $\tilde{a}_{i,j}$ is a lifting of $a_{i,j} \in \overline{F}$ in the ring of integers of an inertia subfield F_0 of F ($e_{F_0} = 1$, $\overline{F}_0 = \overline{F}$) in the case of $\mathrm{char}(F) = 0$ and $\tilde{a}_{i,j} = a_{i,j}$ in the case of $\mathrm{char}(F) = p$.

Theorem. *Let F be a complete discrete valuation field with residue field of characteristic p. Cyclic totally ramified extension L/F of degree p^n, such that $\pi_F \in N_{L/F}L^*$ are in one-to-one correspondence with subgroups*

$$\mathcal{E}_{n,\pi_F}(\mathbf{P}W_n(\overline{F}) \oplus \dots \oplus (a_{0,j}^p, \dots, a_{n-1,j}^p)\wp W_n(\overline{F}) \oplus W_n(\overline{F}) \oplus \dots)U_{1,F}^{p^n}$$

in $U_{1,F}$, where $1 \leqslant j \leqslant e_F$, $(a_{0,j}, \dots, a_{n-1,j})$ is invertible in $W_n(\overline{F})$, $\wp = \mathbf{P} - 1$, and $\mathbf{P}(a_0, \dots, a_{n-1}) = (a_0^p, \dots, a_{n-1}^p)$.

Proof. First, let L/F be a cyclic totally ramified extension of degree p^n. Let $\pi_F = N_{L/F}\pi_L$ with a prime element π_L of L. For a generator σ of $\mathrm{Gal}(L/F)$ put $s_l = v_L(\sigma^{p^l}(\pi_L)/\pi_L - 1)$, $0 \leqslant l \leqslant n-1$. If $\mathrm{char}(F) = 0$, then the Eisenstein polynomial $e_i(X) = X^{p^n} + a_{p^n-1}X^{p^n-1} + \dots + a_0$ of π_L^i for $(i,p) = 1$ satisfies the property: $v_F(a_t) \geqslant s_1 + (n - 1 - v_p(t))e_F$, where v_p is the p-adic valuation. This implies that $N_{\tilde{L}/\tilde{F}}U_{1,\tilde{L}}$ coincides with

$$\mathcal{E}_{n,\pi_F}(\mathbf{P}W_n(\overline{F}^{\mathrm{alp}}) \oplus \dots \oplus W_n(\overline{F}^{\mathrm{alp}}) \oplus \dots),$$

where $W_n(\overline{F}^{\mathrm{alp}})$ stand at the places starting from the s_1th one. A similar observation in the case of $\mathrm{char}(F) = p$ shows that the same assertion holds there. Now as $N_{\tilde{L}/\tilde{F}}U_{1,\tilde{L}} \cap U_{1,F}/N_{L/F}U_{1,L}$ is isomorphic to $\oplus_\kappa \mathrm{Gal}(L/F)$ according to Theorem (1.6) one obtains that $N_{L/F}U_{1,L}$ is of the type described in the assertion of the theorem.

Second, for the case of perfect residue field $\overline{F}$ it follows from the existence theorem (Theorem (3.5) of [F1]) that any subgroup of $U_{1,F}$ of the type indicated in the assertion of the theorem is $N_{L/F}U_{1,L}$ for some cyclic totally ramified p-extension L/F with $\pi_F \in N_{L/F}L^*$.

Thus, it remains to treat the case of imperfect residue field. Denote by N the subgroup in $U_{1,F}$ indicated in the assertion of the theorem. For an extension E/F with $e(E|F) = 1$ denote by N_E the subgroup in the group of principal units of E of the form

$$\mathcal{E}_{n,\pi_F}(\mathbf{P}W_n(\overline{E}) \oplus \dots \oplus (a_{0,j}^p, \dots, a_{n-1,j}^p)\wp W_n(\overline{E}) \oplus W_n(\overline{E}) \oplus \dots)U_{1,E}^{p^n}.$$

According to the previous considerations there exists a totally ramified p-extension $\mathcal{F}'/\mathcal{F}$ (with $\mathcal{F}$ as in (1.5) — the residue field of $\mathcal{F}$ is the perfection of $\overline{F}$) such that $N_{\mathcal{F}'/\mathcal{F}}U_{1,\mathcal{F}'} = N_{\mathcal{F}}$ and $\pi_F \in N_{\mathcal{F}'/\mathcal{F}}\mathcal{F}'^*$. In fact this extension $\mathcal{F}'/\mathcal{F}$ is defined over a finite extension of F, so it is sufficient to treat without loss of generality the case when $\mathcal{F}' = E'\mathcal{F}$, E'/E is a totally ramified cyclic extension of degree p and $E = F(\theta), \theta^p = \theta_0 \in F$, $E \subset \mathcal{F}$.

One has $N_{E'/E}U_{1,E'} \subset N_{\mathcal{F}} \cap U_{1,E}$, and the description of $N_{\mathcal{F}}$ together with injectivity of the homomorphism $U_{1,E}/U_{1,E}^{p^n} \to U_{1,\mathcal{F}}/U_{1,\mathcal{F}}^{p^n}$ imply that $N_{\mathcal{F}} \cap U_{1,E} = N_E$. Therefore $N_{E'/E}U_{1,E'} = N_E$ by using Theorem (1.6).

Let $\mathrm{char}(F) = p$. Then there is an abelian totally ramified extension F'/F such that $E' = EF'$. Then by the previous arguements $N = N_{F'/F}U_{1,F'}$.

Let $\mathrm{char}(F) = 0$. Denote the degree of the extension $F(\zeta_p)/F$ by l.

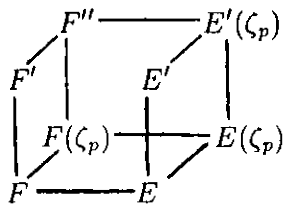
There exists a prime element of $F(\zeta_p)$ such that $\pi^l_{F(\zeta_p)} = \pi_F$. Then $\pi_{F(\zeta_p)} \in N_{E'(\zeta_p)/E(\zeta_p)}E'(\zeta_p)^*$. In addition, $\pi_{F(\zeta_p)} \in N_{\sigma E'(\zeta_p)/E(\zeta_p)}(\sigma E'(\zeta_p))^*$ for any imbedding σ of $E'(\zeta_p)$ in $E'(\zeta_p)^{\mathrm{alg}}$ over $F(\zeta_p)$. This permits one to conclude referring to Proposition (2.1) that $E'(\zeta_p)\sigma E'(\zeta_p)/E(\zeta_p)$ is a totally ramified extension. Since the extension $E'(\zeta_p)/E$ is abelian, and the norm map $N_{E(\zeta_p)/E}$ maps $U_{1,E(\zeta_p)}$ onto $U_{1,E}$, it follows from (1.3) and (1.6) that $N_{E'(\zeta_p)/E(\zeta_p)}U_{1,E'(\zeta_p)} = N^{-1}_{E(\zeta_p)/E}(N_E)$. Keeping in mind specific structure of N_E one obtains $\varepsilon^{\sigma-1} \in N_{E'(\zeta_p)/E(\zeta_p)}U_{1,E'(\zeta_p)}$ for any $\varepsilon \in N_{\widetilde{E'(\zeta_p)}/\widetilde{E(\zeta_p)}}U_{1,\widetilde{E'(\zeta_p)}} \cap U_{1,E(\zeta_p)}$. The last conclusion together with (1.3), (1.6) imply that $E'(\zeta_p)/F(\zeta_p)$ is an abelian extension. It isn't cyclic, since otherwise easy calculations show that

$$\pi_F^{\sigma-1} = (\pi^l_{F(\zeta_p)})^{\sigma-1} = N_{E'(\zeta_p)/E(\zeta_p)}(\pi^{\sigma-1})^l \neq 1,$$

(where $\pi_{F(\zeta_p)} = N_{E'(\zeta_p)/E(\zeta_p)}(\pi))$ for a generator σ of $\mathrm{Gal}(E'(\zeta_p)/F(\zeta_p)$ which is impossible. Hence, there exists a cyclic totally ramified extension $F''/F(\zeta_p)$ such that $F''E(\zeta_p) = E'(\zeta_p)$. One gets also

$$N_{F''/F(\zeta_p)}U_{1,F''} \subset N_{E'(\zeta_p)/E(\zeta_p)}U_{1,E'(\zeta_p)} \cap F(\zeta_p) = N^{-1}_{F(\zeta_p)/F}(N),$$

and the inclusion can be replaced by equality.



Again, by (1.3) and (1.6) one deduces that F''/F is an abelian extension, and there exists a cyclic totally ramified extensions F'/F such that $F'' = F'(\zeta_p)$. Finally $N_{F'/F}U_{1,F'} = N$, $\pi_F \in N_{F'/F}F'^*$ as desired. $\square$

3.3. The correspondence between Witt vectors of length n and cyclic totally ramified extension of degree p^n for the case $e_F = 1$ has been established by Kurihara ([K]): there exists an exact sequence (here there is a canonical prime element $\pi_F = p$)

$$1 \to H^1(F, \mathbf{Z}/p^n)_{nr} \to H^1(F, \mathbf{Z}/p^n) \to W_n(\overline{F}) \to 1$$

with nice functorial properties. This approach is based on study of the sheaf of the etale vanishing cycles on the special fiber of a smooth scheme over the ring of integers of F and of filtrations on Milnor's K-groups of local rings.

One can ask, generalizing a question established in [K], what is an explicit description of the extension L/F corresponding to

$$\mathcal{E}_{n,\pi_F}(\mathbf{P}W_n(\overline{F}) \oplus \cdots \oplus (a_{0,j}^p, \ldots, a_{n-1,j}^p)\wp W_n(\overline{F}) \oplus W_n(\overline{F}) \oplus \ldots).$$

The answer is known for $n = 1$:

$$L = F(\alpha) \quad \text{with} \quad \wp(\alpha) = \alpha^p - \alpha = (\tilde{a}_{0,j}^p \pi_F^j)^{-1}$$

(see e.g. sect.2 Chapt.III [FV]).

There is a nice answer in the case of $\operatorname{char}(F) = p$ for arbitrary n:

$$L = F(\wp^{-1}\left((a_{0,j}^{p^n}, \ldots, a_{n-1,j}^{p^n})^{-1}(\pi_F^{-j}, 0, \ldots)\right)),$$

where $(a_{0,j}^{p^n}, \ldots, a_{n-1,j}^{p^n})^{-1}(\pi_F^{-j}, 0, \ldots)$ is an element of $W_n(F)$. This has been proved for the case of quasi-finite residue field by Sekiguchi [S], the same arguements and p-class field theory of [F1] provide the proof for the case of perfect residue field. Finally, the arguements in the fourth paragraph of the proof of the theorem show that in the case of imperfect residue field the situation is the same.

Now let $\operatorname{char}(F) = 0$ and $n > 1$. Then, first of all, it isn't true that L/F can be defined as a Witt extension. Some information can be produced by using the theory of fields of norms due to Fontaine and Wintenberger (see [FW], [W], or sect. 5 Chap. III of [FV]):

Consider a tower of fields $F_i = F_{i-1}(\pi_i)$ with $\pi_i^p = \pi_{i-1}$, $\pi_0 = \pi_F$. Let M be the union of all F_i. Then $\mathcal{M}/\mathcal{F}$ for $\mathcal{F}$ as in (1.5) and $\mathcal{M} = M\mathcal{F}$ is an arithmetically profinite extension (see [FW]).

Denote by $\mathbf{M}$ the corresponding field of norms. The preimage $\mathbf{N}$ in $U_{1,\mathbf{M}}$ of N is equal to

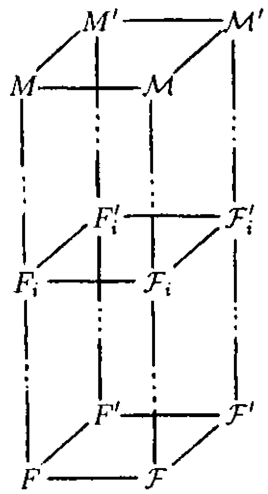
$$\mathcal{E}_{n,\pi_{\mathbf{M}}}(\mathbf{P}W_n(\overline{F}^{\mathrm{abp}}) \oplus \cdots \oplus (a_{0,j}^p, \ldots, a_{n-1,j}^p)\wp W_n(\overline{F}^{\mathrm{abp}}) \oplus W_n(\overline{F}^{\mathrm{abp}}) \oplus \ldots)U_{1,\mathbf{M}}^{p^n},$$

where $\pi_{\mathbf{M}} = (\pi_i)$ is a prime element of $\mathbf{M}$. By the previous considerations $\mathbf{N} = N_{\mathbf{M}'/\mathbf{M}}U_{1,\mathbf{M}'}$ for a cyclic extension

$$\mathbf{M}' = \mathbf{M}(\wp^{-1}\left((a_{0,j}^{p^n}, \ldots, a_{n-1,j}^{p^n})^{-1}(\pi_{\mathbf{M}}^{-j}, 0, \ldots)\right)).$$

This cyclic extension corresponds to a cyclic extension $\mathcal{M}'/\mathcal{M}$ of the same degree according to the general theory of fields of norms. Even more, by the theory of field of norms (e.g. see the proof of Theorem 5.7 in Chapter III [FV]) it originates from an extension F_i'/F_i for a sufficiently large i.

The preimage of $N_{F_i'/F_i}U_{1,F_i'}$ in $U_{1,\mathbf{M}}$ coincides with $\mathbf{N}$, since class field theory is compatible with the theory of fields of norms (for the case of a finite residue field see [L] or sect. 6 Chapt. IV [FV]). Hence $N_{F_i'/F_i}U_{1,F_i'} = N_{F_i/F}^{-1}(N)$. By using similar arguements with ones of the proof of the theorem, one can deduce that F_i'/F_i originates from a cyclic extension F'/F and $N_{F'/F}U_{1,F'}$ coincides with N.



Note, that $F_i' = F_i(\wp^{-1}\left((\tilde{a}_{0,j}^{p^n}, \ldots, \tilde{a}_{n-1,j}^{p^n})^{-1}(\pi_i^{-j}, 0, \ldots)\right))$. In other words, extensions F_i'/F_i are Witt extensions for $i \geqslant i(n)$. For instance, $i(1) = 0$ (Artin–Schreier extensions in characteristic 0), and $i(2) = 1$ (see section 3 of [VZ]).

By using the previous description one can develop an analogue of Witt duality for complete discrete valuation fields of characteristic 0.

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