On the arithmetic of some division algebras

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Für die Zusendung Ihrer neuen
Arbeit (Crelle) danke ich Ihnen bestens.
Hoffentlich haben Sie auch einige Separata
von Urysohn und mir, die ich vor
kurzem Ihnen geschickt habe, richtig
erhalten.

30 Murray Place, Princeton, New Jersey.
20.IV.1928.

Lieber und hochverehrter Herr Hausdorff!

Herzlichen Dank für Ihren Brief vom 12.IV! Er enthält wieder einmal einen wohlberechtigten Vorwurf ("wenn Sie – worauf ich eigentlich gehofft hatte – ... ein Lebenszeichen gegeben hätten"!), auf den ich nur mit einem stummen Erröten antworten konnte – in der Hoffnung, daß Sie indessen meine Schuld vergeben haben! Ich freute mich aber wirklich sehr über Ihren Brief.

Der Beweis, den Sie für den (von Niemytzki und Tychonoff erst im Winter 1926-27 bewiesenen) Satz, daß ein in jeder Metrik vollständiger Raum notwendig kompakt ist, bereits in 1925 erbracht haben, finde ich insbesondere deswegen interessant, daß Sie ihn durch die Lösung eines (in diesem Falle ja ganz natürlichen) Erweiterungsproblem erhalten.

Dabei hat der Niemytzki-Tychonoffsche Beweis (trotz seiner Kürze – kaum mehr als 1 $\frac{1}{2}$ – 2 Seiten) den wesentlichen methodischen Mangel, insofern "transzendente" Hilfsmittel zu benutzen, als er wesentlich auf einer Anwendung des von Urysohn und mir bewiesenen ("allgemeinen") Metrisationssatzes (Comptes Rendus, 177) beruht. Wenn Sie mir meine diesbezügliche Meinung zu äußern erlauben, würde ich mich dringend für die Veröffentlichung Ihres Beweises aussprechen: ich halte den Satz für prinzipiell interessant und für das Verständnis des Wesens der Kompaktheit durchaus wichtig; schon das allein würde genügen, einen (von jeglichen Metrisationssätzen unabhängigen) Beweis als sehr wünschenswert zu betrachten. Ihr Beweis bietet aber noch das hohe Interesse eines sehr eigenartigen Erweiteungssatzes; ich würde einen solchen Satz bestimmt

On the arithmetic of some division algebras

Ernst–Ulrich Gekeler*

Introduction

As is well known since Deuring's pioneering work [5], there is a close relationship between the theory of elliptic curves in positive characteristic p, and the arithmetic of the definite quaternion algebra H(p) over $\mathbb{Q}$ ramified at p. Deuring's results relied heavily on Eichler's class number formula for H(p) [8], proved shortly before by analytical means. A more geometrical interpretation (and independent proof) of these results has later been given by Igusa [16], and in particular by Deligne and Rapoport [2]. The main feature is that supersingular elliptic curves (i.e., special points on a certain modular scheme) are in 1–1 correspondence with the set of left ideal classes in a maximal order of H(p). That correspondence may be used to derive properties of the modular scheme from those of H(p), but also vice versa.

Now the question arises whether the same type of relationship holds if one replaces "elliptic curves" by objects that in many other respects behave similarly, namely by "Drinfeld modules". In this case, instead of H(p), one considers division algebras $D = D(r,\mathfrak{p},\infty)$ of dimension r^2 over their center K (a global field of positive characteristic), and that ramify at precisely two places $\mathfrak{p}$, ∞ of K, with invariants $1/r$, $-1/r$, respectively.

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Let A be the subring of K of elements regular away from ∞ . It turns out that (definitions to be given below) "supersingular Drinfeld A–modules of rank r in characteristic $\mathfrak{p}$"

(i) have maximal A–orders B in D as their endomorphism rings;

(ii) their isomorphism classes correspond to the left ideal classes of a fixed A–order B .

In some cases, enough is known about the modular schemes for Drinfeld modules to be able to count the number of supersingular points. This way, we arrive at class number formulas for $D(r,\mathfrak{p},\infty)$ that could not be obtained otherwise. This is notably the case if $r = 2$ [14] , or if K is a rational function field $\mathbb{F}_q(T)$ and " ∞ " is the usual place at infinity. The latter case will be treated in detail. In particular, we shall describe the associated modular scheme and its supersingular locus. The principal result, Theorem 5.13 , is an explicit expression for the number of ideal classes with a fixed weight. We also obtain the Mass formula 5.11 , which generalizes Deuring's formula

$$\sum 1/\#\,(\mathrm{Aut}(E)) = (p-1)/24 \ .$$

Recall that the sum on the left hand side is over the supersingular classes of elliptic curves in characteristic p , and $(p-1)/24$ is one half the value of the Riemann zeta function at -1 , deprived from its Euler factor at p .

Besides the relationship with Drinfeld modules mentioned above, our proof relies on

(a) the transfer principle (3.5);

(b) the reducedness of the supersingular locus (4.3);

(c) some calculations (see section 6) special to the case of a polynomial ring A .

From (b) and (c) we derive the Mass formula, which, combined with (a), yields the theorem. But note that both (a) and (b) do not depend on specific assumptions on A .

In principle, our Drinfeld module interpretation of the division algebra D should also allow to determine its type number (= number of conjugacy classes of maximal orders). At least, Proposition 7.5 yields a geometrical description of the set of types. As an example, we present the case $r = 2$, which is quite analogous with the elliptic curve case. However, for $r > 2$, further research is needed for a numerical evaluation through zeta values and commutative class numbers.

Some of the results of this paper (e.g. Thm. 5.13) have been announced in the C.R. note [13].

1. Notations

Throughout, K will denote a function field in one variable over the finite field $\mathbb{F}_q$ qith q elements, of characteristic p . We assume that $\mathbb{F}_q$ is the exact constant field of K . We fix, once for all, a place " ∞ " of K , and let A be the subring of elements of K regular away from ∞ . The places of K different from ∞ are in 1–1 correspondence with the maximal ideals ("primes") of A . We will not distinguish between the two concepts; for $\mathfrak{p}$ such a prime, $\mathbb{F}_{\mathfrak{p}}$ is the field $A/\mathfrak{p}$. Associated with ∞ , we have the normalized absolute value " $|?|$ " and the degree function " deg " on K defined by $\deg x = \log_q |x|$. The basic example is given by

(1.1) $K = \mathbb{F}_q(T)$ a rational function field,

∞ the usual place at infinity, and

$A = \mathbb{F}_q[T]$ the polynomial ring.

Then $\deg x$ agrees with the degree of the polynomial $x \in A$.

For any $r \in \mathbb{N}$ and prime $\mathfrak{p}$, the central division algebra $D = D(r,\mathfrak{p})$ over K is determined up to isomorphism by the following data:

(1.2) (i) $\dim_K(D) = r^2$

(ii) $\mathrm{inv}_{\mathfrak{p}}(D) = 1/r$

(iii) $\mathrm{inv}_{\infty}(D) = -1/r$

(iv) $\mathrm{inv}_v(D) = 0$, if $v \neq \mathfrak{p},\infty$.

(" inv_v " is the local invariant at the place v of K, cf. [17].) We call these algebras <u>of Drinfeld type</u>; as we will see, their ideal theory is related to Drinfeld A–modules.

(1.3) An <u>order</u> in D will be a maximal A–order in D, i.e., a subring B of D that (i) contains A; (ii) is finitely generated as an A–module; (iii) satisfies $KB = D$, and is maximal with these properties. A <u>left ideal</u> of B is an A–lattice $0 \neq \mathfrak{I} \subset D$ that satisfies $B\mathfrak{I} \subset \mathfrak{I}$. Two left ideals $\mathfrak{I}, \mathfrak{I}'$ are in the same class if there exists $f \in D^*$ such that $\mathfrak{I}' = \mathfrak{I}f$. The <u>right order</u> of $\mathfrak{I}$ is $B^{\mathfrak{I}} = \{f \in D \mid \mathfrak{I}f \subset \mathfrak{I}\}$. For the convenience of the reader, we collect the most important properties (which are well known and hold in much greater generality):

(1.4) (i) the <u>type number</u> $t(D)$ of conjugacy classes of orders in D is finite.

(ii) Fix an order B in D. The number of left ideal classes of B is finite and independent of B, therefore an invariant of D. It is called the <u>class number</u> $h(D)$.

(iii) Each order B' in D is the right order of $B^{\mathcal{I}}$ of some left ideal $\mathcal{I}$ of B. In particular, $t(D) \leq h(D)$.

All the orders B' in D contain the unit group $\mathbb{F}_q^*$ of A, which "generically" is the full unit group of B'. We define the weight $w(\mathcal{I})$ of a left ideal $\mathcal{I}$ of B as

$$w(\mathcal{I}) = \#((B^{\mathcal{I}})^*)/(q-1) \ .$$

(1.5) Finally, we let $\zeta_K(s)$ be the zeta function of K [18]. It is a rational function $P(q^{-s})/(1-q^{-s})(1-q^{1-s})$ in q^{-s}. The polynomial $P(X)$ has integral coefficients, degree $2g$ (where g is the genus of K), and satisfies $P(1) =$ class number of K. For a finite set S of places of K, we put

$$\zeta_{K,S}(s) = \prod_{v \in S} (1-q^{-(\deg v)s})\zeta_K(s) \ .$$

In practice, S will be $\{\mathfrak{p},\infty\}$, so

$$\zeta_{K,S}(s) = P(q^{-s}) \frac{(1-q^{-ds})(1-q^{-d_\infty s})}{(1-q^{-s})(1-q^{1-s})} \ , \tag{1.6}$$

where d and d_∞ are the degrees of $\mathfrak{p}$ and ∞, respectively. Thus for the example (1.1) and S as above,

$$\zeta_{K,S}(s) = (1-q^{-ds})/(1-q^{1-s}) \ .$$

2. Review of Drinfeld modules

In positive characteristic p, the additive group scheme G_a has non–trivial module structures, due to the existence of non–scalar endomorphisms. More precisely, let τ_p be the Frobenius endomorphism $x \longmapsto x^p$. For any field L of characteristic p, the ring of L–endomorphisms $\mathrm{End}_L(G_a)$ is the non–commutative polynomial ring $L\{\tau_p\}$ with the commutator rule $\tau_p x = x^p \tau_p$ for constants $x \in L$. We put $\tau = \tau_q = \tau_p^f$, if $q = p^f$. Let now L be equipped with an A–structure $\gamma : A \longrightarrow L$, i.e., L is an extension of K or of some $\mathbb{F}_{\mathfrak{p}}$.

(2.1) A Drinfeld A–module of rank r over L is a structure of A–module on $G_a | L$, given by a ring homomorphism

$$\phi : A \longrightarrow \mathrm{End}_L(G_a) = L\{\tau_p\}$$

$$n \longmapsto \phi_n$$

(necessarily taking its values in $L\{\tau\}$), such that for $0 \neq n \in A$, $\phi_n = \sum g_i(\phi,n)\tau^i$, the following conditions are satisfied:

(i) $g_0(\phi,n) = \gamma(n)$

(ii) $\deg_\tau \phi_n = r \cdot \deg n$.

The characteristic of ϕ (or of L) is $\mathfrak{p}$, if $\mathbb{F}_{\mathfrak{p}} \subset L$, and ∞ if $K \subset L$. Morphisms of Drinfeld modules are morphisms of group schemes compatible with the A–actions. We put $\mathrm{End}_L(\phi)$ for the ring of L–endomorphisms of ϕ, i.e., for the centralizer of $\phi(A)$ in

$L\{\tau\}$. Further, for $0 \neq n \in A$, we denote by ${}_{n}\phi$ the scheme in A–modules $\ker \phi_n$. Conditions (i) and (ii) imply that it is flat and finite of degree $|n|^r$. For an ideal $\mathfrak{n}$ of A , we put ${}_{\mathfrak{n}}\phi = \cap \, {}_{n}\phi$, n running though $\mathfrak{n}$. It is étale if and only if $\mathfrak{n}$ is relatively prime with the characteristic of L . In this latter case, the abstract A–module of points of ${}_{\mathfrak{n}}\phi$ over the algebraic closure $\bar{L}$ of L is isomorphic with $(A/\mathfrak{n})^r$. Also, one may define Drinfeld modules, morphisms, the schemes ${}_{\mathfrak{n}}\phi$ ... over arbitrary A–schemes. Thus one has level structures, modular schemes ... for Drinfeld modules. For all of this, see [6], [11], [1].

<u>2.2. Example</u>: If $A = \mathbb{F}_q[T]$ as in (1.1), a rank r Drinfeld A–module ϕ is determined by ϕ_T , which must have the form

$$\phi_T = \gamma(T) + \lambda_1\tau + \ldots + \lambda_r\tau^r \qquad (\lambda_i \in L)$$

with the single condition $\lambda_r \neq 0$.

(2.3) Let now $\mathfrak{p}$ be a prime of A . A Drinfeld module ϕ in characteristic $\mathfrak{p}$ is called <u>supersingular</u> if ${}_{\mathfrak{p}}\phi$ is local, or equivalently, ${}_{\mathfrak{p}}\phi(\bar{L}) = 0$. This is also equivalent with $\mathrm{End}(\phi)$ being projective of rank r^2 as an A–module [7] [12]. Therefore, rank one Drinfeld modules are always supersingular. Also, in the situation of (2.2), the module ϕ in characteristic (T) determined by $\phi_T = \lambda_r\tau^r$ is s.s.. All the s.s. Drinfeld modules in characteristic $\mathfrak{p}$ may be defined over some finite extension L of the "prime field" $\mathbb{F}_{\mathfrak{p}}$. If m is the order of $\mathfrak{p}$ in the class group Pic A of A , one may actually take the extension L of $\mathbb{F}_{\mathfrak{p}}$ of degree $m' = m \cdot r$ ([12], Prop. 4.2). In particular, the set $\Sigma(r,\mathfrak{p})$ of $\bar{\mathbb{F}}_{\mathfrak{p}}$–isomorphism classes of supersingular Drinfeld modules of rank r is finite.

The connection with $D(r,\mathfrak{p})$ is through the next theorem, which is similar to Deuring's theorem on elliptic curves:

2.4. Theorem ([12], Theorem 4.3): Let ϕ be a supersingular Drinfeld module of rank r over the A–field L of characteristic $\mathfrak{p}$, and suppose that L is large enough such that $\mathrm{End}_L(\phi) = \mathrm{End}_{\bar{L}}(\phi)$.

(i) The K–algebra $\mathrm{End}(\phi) \otimes K$ is isomorphic with $D(r, \mathfrak{p})$.

(ii) $B := \mathrm{End}(\phi)$ is an order (i.e., maximal) in $\mathrm{End}(\phi) \otimes K$.

(iii) There is a canonical bijection from the set LI(B) of left ideal classes of B to $\Sigma(r, \mathfrak{p})$.

We briefly describe the bijection: For $b \in B$, let ${}_b\phi$ be the subscheme ker b of G_a, and for a left ideal $\mathcal{I} \subset B$, ${}_{\mathcal{I}}\phi = \cap\, {}_b\phi$ $(b \in \mathcal{I})$. The latter is the kernel of some morphism $\phi \longrightarrow \phi^{\mathcal{I}}$ of Drinfeld modules, where $\phi^{\mathcal{I}}$ is uniquely determined up to isomorphism. Moreover, $\phi^{\mathcal{I}}$ is supersingular, its class depends only on the left ideal class of $\mathcal{I}$, and the induced map $(\mathcal{I}) \longmapsto (\phi^{\mathcal{I}})$ from LI(B) to $\Sigma(r, \mathfrak{p})$ is bijective.

Thus LI(B) may be described through $\Sigma(r, \mathfrak{p})$. Our strategy will be to identify $\Sigma(r, \mathfrak{p})$ with a certain set of geometric points on a suitable modular scheme. Classical geometric arguments will then lead to the determination of its cardinality. The corresponding modular schemes are sufficiently well known for that purpose in the cases (at least):

(a) $r = 2$,

(b) $A = \mathbb{F}_q[T]$.

In case (a), i.e., if D is the quaternion algebra ramified in $\mathfrak{p}$ and ∞, all the ingredients for a discussion à la Deligne–Rapoport are available [10], [11]:

- Drinfeld module analogues $M_0(\mathfrak{p})$ of Hecke modular curves with conductor $\mathfrak{p}$;
- structure of the special fiber (only ordinary double points on $M_0(\mathfrak{p}) \times \mathbb{F}_{\mathfrak{p}}$, and these agree with the supersingular points);
- calculation of the genus of $M_0(\mathfrak{p})$.

Some complications arise, however, from the existence of non–principal ideals in A. Using this approach, one can prove the following result (for details, see [14]):

2.5. Theorem: Let B an order in $D = D(2,\mathfrak{p})$.

(i) The weight $w(\mathcal{I})$ of a left ideal $\mathcal{I}$ of B is 1 or $q+1$. Let h_1 (h_2) be the number of ideal classes ($\mathcal{I}$) with $w(\mathcal{I}) = 1$ ($w(\mathcal{I}) = q+1$), respectively.

(ii) If at least one of the degrees d of $\mathfrak{p}$ and d_∞ of ∞ is even, we have

$$h_1 = d_\infty P(1)P(q)Q \text{ and } h_2 = 0 \ .$$

If d and d_∞ are odd, we have

$$h_1 = d_\infty P(1)\,[P(q)Q - P(-1)/(q+1)] \text{ and } h_2 = d_\infty P(1)P(-1) \ .$$

Here,

$$Q = \frac{(q^d-1)(q^{d_\infty}-1)}{(q-1)(q^2-1)} \ .$$

(iii) In any case, the mass formula holds:

$$\sum_{\mathcal{I}\in LI(B)} w(\mathcal{I})^{-1} = d_\infty P(1)P(q)Q \ .$$

Note that by (1.5) and (1.6), $d_\infty P(1)$ is the order of the class group Pic A, whereas $P(q)Q = \zeta_{K,S}(-1)$. Our mass formula therefore "agrees" with Deuring's (see also (5.11)).

In case (b), if the rank r is strictly greater than two, these arguments do not apply. In what follows, we will develop what is needed to handle that case.

3. Transfer principle

In this section, all the Drinfeld modules ϕ are defined over the A–field $L = \mathbb{F}_{\mathfrak{p}}$, and $\mathrm{End}(\phi) = \mathrm{End}_L(\phi)$. The automorphism group $\mathrm{Aut}(\phi)$ is the finite subgroup of elements of L^* that commute with all the operators ϕ_n, $n \in A$. As is easily seen, this is the multiplicative group of some extension of $\mathbb{F}_q$ of degree s, say. We call $s = s(\phi)$ the size and $w = w(\phi) = (q^s-1)/(q-1)$ the weight of ϕ. Since $\mathrm{Aut}(\phi)$ generates a commutative subfield of $\mathrm{End}(\phi)$, it follows that

(3.1) $s(\phi)$ is a divisor of $r = \mathrm{rank}(\phi)$.

(3.2) The map $\mathcal{I} \longmapsto \phi^{\mathcal{I}}$ of (2.4) induces an isomorphism of the right order $B^{\mathcal{I}}$ of $\mathcal{I}$ with $\mathrm{End}(\phi^{\mathcal{I}})$ [12, 3.8]. In particular, the unit group $(B^{\mathcal{I}})^*$ is isomorphic with $\mathrm{Aut}(\phi^{\mathcal{I}})$. Therefore, $w(\mathcal{I})$ as defined in (1.4) agrees with $w(\phi^{\mathcal{I}})$.

3.3. Lemma: The size of a supersingular Drinfeld module ϕ over L is always relatively prime with d and d_∞. (Recall that d and d_∞ are the degrees of $\mathfrak{p}$ and ∞, respectively.)

Proof: By assumption, the constant field extension K_s of K of degree $s = s(\phi)$ embeds into $D = \mathrm{End}(\phi) \otimes K$. As is well known [17], this means that the ramified places $\mathfrak{p}$ and ∞ extend uniquely to K_s. This in turn implies $(s,d) = 1 = (s,d_\infty)$.

In what follows, t will be a divisor of $s = \mathrm{size}(\phi)$. Let A_t and $K_t = \mathrm{Quot}(A_t)$ be the constant field extensions of degree t of A and K, respectively. As stated above, there are unique extensions to K_t of $\mathfrak{p}$ and ∞, denoted by $\mathfrak{p}_t$ and ∞. Having closen an embedding of $\mathbb{F}_{\mathfrak{p}_t} = A_t/\mathfrak{p}_t$ into L, the rank r Drinfeld A–module $\phi : A \longrightarrow L\{\tau\}$ has a unique extension to a Drinfeld A_t–module $\phi' : A_t \longrightarrow L\{\tau\}$. Since $\phi_n = \phi'_n$ $(n \in A)$, and $\mathbb{F}_{q^t}$ is the exact constant field of K_t,

(3.4) (i) $r' = \mathrm{rank}(\phi') = r/t$

(ii) $s' = \mathrm{size}(\phi') = s/t$

(iii) ϕ' is supersingular if and only if ϕ is.

We put $\Sigma(r,\mathfrak{p},q,s)$ for the set of L–isomorphism classes of supersingular Drinfeld A–modules of rank r and size s over $L = \mathbb{F}_{\mathfrak{p}}$. Thus $\Sigma(r,\mathfrak{p})$ is the disjoint union of the $\Sigma(r,\mathfrak{p},q,s)$, s running through the divisors of r coprime with d and d_∞. Furthermore, the lift $\phi \longmapsto \phi'$ defines a map

$$\ell_t : \Sigma(r,\mathfrak{p},q,s) \longrightarrow \Sigma(r/t,\mathfrak{p}_t,q^t,s/t) .$$

<u>3.5. Proposition</u>: ℓ_t is bijective.

<u>Proof</u>: The inverse of ℓ_t is given by restricting ϕ' to A.

Clearly, the decomposition according to sizes and the above "transfer principle" also apply to the study of ideal classes of $D(r,\mathfrak{p})$. It would be interesting to know to what extent this generalizes to division algebras not necessarily of Drinfeld type. In our case, considering simultaneously A and all its extensions A_t, we will use an induction procedure to calculate

(3.6) $$\sigma(r,\mathfrak{p},q,s) = \# \Sigma(r,\mathfrak{p},q,s) \ .$$

4. The supersingular locus

Let M^r be the coarse modular scheme for rank r Drinfeld modules in characteristic $\mathfrak{p}$ [6] [1] . It is the fiber product with $L = \mathbb{F}_{\mathfrak{p}}$ of the A–scheme called $M^r(1)$ in [11]. The L–valued points of M^r correspond bijectively to the L–classes of rank r Drinfeld modules. Recall that d is the degree and m the order of $\mathfrak{p}$ in Pic A , i.e., $\mathfrak{p}^m = (f)$ with $f \in A$. For $i = 1...r–1$, let

(4.1) $H_i(\phi)$ = coefficient of τ^{idm} in the polynomial ϕ_f in τ .

This is a modular form of weight $q^{idm}–1$ ([15], [11]), the *i–th Hasse invariant*. (Clearly, it depends on the choice of the generator f of $\mathfrak{p}^m$, but this doesn't matter). We have the trivial equivalences

(4.2) ϕ supersingular $\Leftrightarrow$ ϕ_f a monomial const.τ^{rdm} in $\tau \Leftrightarrow H_i(\phi) = 0$,
$i = 1,...,r–1$.

Therefore, we define the *supersingular locus* $\Sigma = \Sigma(r,\mathfrak{p})$ in M^r as the zero locus of the $r–1$ forms $H_i(\phi)$. It is a finite subscheme of M^r with the set $\Sigma(r,\mathfrak{p})$ of (2.3) as its L–valued points. The double use of the symbol $\Sigma(r,\mathfrak{p})$ is justified by the next proposition.

4.3. Proposition: The scheme $\Sigma(r,\mathfrak{p})$ is reduced.

Proof: This is more or less a restatement of (a special case of) the results given in [6], sections 4 and 5. We show how (4.3) follows from loc. cit., using the terminology given there. Also, the next few references are with respect to loc. cit.. Let ϕ correspond to $x \in \Sigma(r,\mathfrak{p})(L)$, and let $\tilde{\phi}$ be an infinitesimal deformation, i.e., a Drinfeld module over the dual numbers $L[\epsilon]$, where $\epsilon^2 = 0$. Now the deformation theory of ϕ agrees with that of its $\mathfrak{p}$–divisible module (Prop. 5.4), and, since ϕ is supersingular, with that of its formal $A_{\mathfrak{p}}$–module (Prop. 4.5, $A_{\mathfrak{p}}$ = completion of A at $\mathfrak{p}$). Let $\pi \in A$ be a prime element, and write ϕ_π , $\tilde{\phi}_\pi$ for the corresponding formal module operators derived from ϕ , $\tilde{\phi}$, respectively. The supersingularity condition translates to $\phi_\pi = \text{const.}\ \tau^{rd}$ + higher terms. Proposition 4.2 implies that $\tilde{\phi}$ is isomorphic with some formal module ψ given by

$$\psi_\pi = \phi_\pi + \epsilon \sum_{1\le i\le r-1} t_i \tau^{id} \qquad (t_i \in L) \ .$$

Let $\tilde{\phi}$ correspond to the $L[\epsilon]$–valued point $\tilde{x}$ of M^r , and suppose that

(*) $\tilde{x}$ factors through $\Sigma(r,\mathfrak{p})$

(i.e., $\tilde{\phi}$ supersingular, too). If $m = 1$, we may take $\pi = f$, and (*) says $t_i = 0$, $i = 1...r-1$. It is not hard to see that also for $m > 1$, (*) implies the vanishing of the t_i . That means, each deformation $\tilde{x}$ of x in $\Sigma(r,\mathfrak{p})$ is constant, which gives the assertion.

<u>4.4. Remark</u>: The analogous result in the elliptic curve case states that Deuring's polynomial

$$H_p(\lambda) = \sum_{0\le i\le s} \begin{bmatrix} s \\ i \end{bmatrix}^2 \lambda^i \qquad (p \neq 2 \text{ prime},\ s = (p-1)/2)$$

has only simple roots (see [16]). It is also equivalent with the fact that the two irreducible components of the Hecke modular curve $X_0(p) \times \mathbb{F}_p$ intersect transversally in supersingular points [2].

5. The case of a polynomial ring

From now on, we assume that A is the polynomial ring $\mathbb{F}_q[T]$. Let the prime ideal $\not{p}$ be generated by the monic irreducible polynomial p of degree d. ($p = \operatorname{char} \mathbb{F}_q$ will not further be used.) All our Drinfeld modules will be defined over $L = \mathbb{F}_{\not{p}}$. Two such, ϕ, ϕ', given by the coeffcients λ_i, λ_i' of ϕ_T, ϕ_T', respectively (compare (2.2)), are isomorphic if and only if there exists $c \in L^*$ such that

$$(5.1) \qquad \lambda_i' = c^{q^i-1}\lambda_i \quad (i = 1...r = \operatorname{rank}(\phi) = \operatorname{rank}(\phi')) \ .$$

Now consider λ_i as an indeterminate of weight $e_i = (q^i-1)/(q-1)$. Let $\overline{M} = \overline{M}^r$ be the scheme $\operatorname{Proj} L[\lambda_1,...,\lambda_r]$ and $M = M^r \hookrightarrow \overline{M}$ the open subscheme defined by $\lambda_r \neq 0$. From (5.1) it follows that M is the modular scheme considered in the last section. (The "natural" weight for the indeterminate λ_i would be q^i-1. Dividing through the gcd $q-1$ doesn't of course change the resulting M.) Later on, we will need the following observation:

(5.2) For natural numbers i, j, we have

$$i \mid j \Leftrightarrow (q^i-1) \mid (q^j-1) \Leftrightarrow e_i \mid e_j \ .$$

This implies that the greatest common divisor of e_i and e_j is e_k, where $k = \gcd(i,j)$. Next, we specify the supersingular locus $\Sigma = \Sigma(r,\mathfrak{p})$ on M. If ϕ is given by $\underline{\lambda} = (\lambda_1,\dots,\lambda_r) \in L^r$, i.e.,

$$\phi_T = \gamma(T) + \sum \lambda_i \tau^i \ ,$$

ϕ_p may be written

$$\phi_p = \sum_{1 \leq i \leq rd} g_i(\underline{\lambda}) \tau^i \ ,$$

where $g_i(\underline{\lambda})$ depends polynomially on $\underline{\lambda}$. More precisely, $g_i(\underline{\lambda})$ is an isobaric polynomial of weight e_i, and

$$(5.3) \qquad H_i(\phi) = H_i(\underline{\lambda}) = g_{id}(\underline{\lambda}) \ ,$$

of weight $f_i = (q^{id}-1)/(q-1)$, is the i–th Hasse invariant.

<u>5.4. Lemma</u>: The $\overline{M}$–locus $V_{\overline{M}}(H_1,\dots,H_{r-1})$ of $H_1,\dots,H_{r-1}$ is contained in M.

<u>Proof</u>: Let $\underline{\lambda} = (\lambda_1,\dots,\lambda_i,0,\dots,0) \in L^r$ with $\lambda_i \neq 0$, $0 < i < r$. Then $H_i(\underline{\lambda})$ is the leading coefficient of ϕ_p, where ϕ is the rank i Drinfeld module defined by $\underline{\lambda}$. Therefore, $H_i(\underline{\lambda}) \neq 0$, i.e., $V_{\overline{M}}(H_1,\dots,H_{r-1},\lambda_r) = \emptyset$.

Let $\overline{N}$ be the projective $(r-1)$–space over L with projective coordinates $\ell_1,\dots,\ell_r$, and let $\pi : \overline{N} \longrightarrow \overline{M}$ be defined by $\pi(\ell_1 : \dots : \ell_r) = (\lambda_1 : \dots : \lambda_r)$, where $\lambda_i = \ell_i^{e_i}$. We further let $\mu(e_i)$ be the group of e_i–th roots of unity in L,

$$G^* = \prod_{1\leq i<r} \mu(e_i) \ , \quad G = G^* \times \mu(e_r) \ .$$

G acts effectively on $\overline{N}$ through $(..c_i..)(..:e_i:..) = (..:c_ie_i:..)$, and π is the associated quotient morphism. If $N = \operatorname{Spec} L[\ell_1,...,\ell_{r-1}]$ denotes the complement of $V_{\overline{N}}(\ell_r)$ in $\overline{N}$, the quotient $N^* = N/G^*$ is the affine space $\operatorname{Spec} L[\lambda_1,...,\lambda_{r-1}]$, and

$$M = N/G = N^*/\mu(e_r) \ ,$$

where $c \in \mu(e_r)$ acts on N^* by $c(..,\lambda_i,..) = (..,c^{-e_i}\lambda_i,..)$. Define the schemes Σ^* and $\tilde{\Sigma}$ as the fiber products

$$\Sigma^* = \Sigma \underset{M}{\times} N^* \ , \ \tilde{\Sigma} = \Sigma^*_{red} \underset{N^*}{\times} N \ ,$$

respectively. Hence in the diagram

$$(5.5) \qquad \begin{tikzcd}
\tilde{\Sigma} \arrow[rr, hook] \arrow[d] & & N \arrow[r, hook] \arrow[d] & \overline{N} \arrow[d, "\pi_1"] \arrow[r, phantom, "\left.\right\} G^*"] & {} \\
\Sigma^*_{red} \arrow[r, hook] & \Sigma^* \arrow[r, hook] \arrow[d] & N^* \arrow[r, hook] \arrow[d] & \overline{N}/G^* \arrow[d, "\pi_2"] \arrow[r, phantom, "\left.\right\} \mu(e_r)"] & {} \\
 & \Sigma \arrow[r, hook] & M \arrow[r, hook] & \overline{M} \ ,
\end{tikzcd}$$

all the rectangles are cartesian, where the upper (lower) vertical arrows are quotients by G^* $(\mu(e_r))$, respectively. In what follows, "points" of these schemes are points over $L = \mathbb{F}_p$.

5.6. Lemma: Let $x \in N^*(L)$ and ϕ be the Drinfeld module associated with $\pi_2(x)$. The stabilizer of x in $\mu(e_r)$ has order $w(\phi)$.

Proof: Let $x = (\lambda_1,\dots,\lambda_{r-1})$. Then $w(\phi) = (q^s-1)/(q-1)$ with $s = s(\phi) = \max\{t \mid \lambda_i \neq 0 \Rightarrow t \mid i,\ \ i = 1\dots r-1\}$. The stabilizer is the subgroup $\{c \in \mu(e_r) \mid \lambda_i \neq 0 \Rightarrow c^{e_i} = 1,\ i = 1\dots r\}$, which has order $\gcd(\{e_i \mid \lambda_i \neq 0\} \cup \{e_r\})$. The latter equals $w(\phi)$, as follows from (5.2).

In particular, Σ^* is in general not reduced; from (4.3) and the above we see that its points occur with multiplicity $w(x) = w(\phi)$. Next, for $i = 1\dots r-1$, we define the functions H_i^* on N^* by

$$H_i^*(\lambda_1,\dots,\lambda_{r-1}) = H_i(\lambda_1,\dots,\lambda_{r-1},1) \ .$$

It is clear that their common zero locus $X = V_{N^*}(H_1^*,\dots,H_{r-1}^*)$ is contained in Σ^* and agrees set–theoretically with Σ^*.

5.7. Proposition: X is the reduced scheme Σ_{red}^* underlying Σ^*.

It has to be shown that X is reduced. Since the proof is somewhat technical and doesn't connect with the present material, it will be given in the next section. Note however that the reducedness in points of size 1 results directly from (5.6).

Finally, we define the polynomial $\tilde{H}_i$ $(i = 1\dots r-1)$ by

$$\tilde{H}_i(\ell_1,\dots,\ell_r) = H_i(\lambda_1,\dots,\lambda_r) \ ,$$

where $\lambda_j = \ell_j^{e_j}$. Then $\tilde{H}_i$ is homogeneous of degree $f_i = (q^{id}-1)/(q-1)$, and from (5.4) and (5.7),

$$\tilde{\Sigma} = V_{\overline{N}}(\tilde{H}_1,\dots,\tilde{H}_{r-1}) \ .$$

Its degree (number of points counted with multiplicity) is therefore given by

$$\deg(\tilde{\Sigma}) = \prod_{1 \leq i < r} f_i \ . \tag{5.8}$$

On the other hand, (5.6) implies that the multiplicity of $y \in \tilde{\Sigma}$ in the fiber $\Sigma \underset{M}{\times} N = \Sigma^* \underset{N^*}{\times} N$ is $w(\pi(y))$ times its multiplicity in $\tilde{\Sigma}$. Together with (4.3), this yields

$$\deg(\tilde{\Sigma}) = \deg(\pi) \sum_{x \in \Sigma} w(x)^{-1} \tag{5.9}$$

with

$$\deg(\pi) = \prod_{1 \leq i \leq r} e_i \ .$$

Let r_1 be the largest divisor of r coprime with d, so the possible sizes of supersingular rank r Drinfeld modules over L are the divisors of r_1. Putting $\Theta = \Theta(r,\mathfrak{p},q)$ for the <u>measure</u> of $\Sigma = \Sigma(r,\mathfrak{p})$, (5.10) $$\Theta = (q-1)^{-1} \sum_{x\in\Sigma} w(x)^{-1} = \sum_{s\,|\,r_1} \frac{\sigma(r,\mathfrak{p},q,s)}{q^s-1}\ ,$$

and comparing with (1.6), we arrive at the

<u>5.11. Mass formula</u>:

$$\Theta(r,\mathfrak{p},q) = (q-1)^{-1} \prod_{1\leq i<r} (q^{id}-1)/(q^{i+1}-1) = (q-1)^{-1} \prod_{1\leq i<r} \zeta_{K,S}(-i)\ .$$

Since this depends only on $d = \deg \mathfrak{p}$, we will also denote it by $\Theta(r,d,q)$.

<u>5.12. Remark</u>: The number Θ is in fact the Haar measure of a certain adelic double coset associated with the algebra D [3]. The word "mass" is an erroneous but commonly used translation of the german word "Maß" = "measure" [9].

Now it is easy to calulate the class number $\sigma(r,\mathfrak{p},q,s)$. Recall it is the number of classes of supersingular Drinfeld A–modules of rank r and size s in characteristic $\mathfrak{p}$, or, equivalently, the number of left ideal classes of size s in a maximal order B in $D(r,\mathfrak{p})$.

Let $\mu(i)$ be the Möbius function: $\mu(i) = (-1)^n$ if n is a product of n different prime factors, and $\mu(i) = 0$ if a square divides i .

<u>5.13. Theorem</u>: $\sigma(r,\mathfrak{p},q,s) = \sigma(r,d,q,s)$ depends only on the degree d of $\mathfrak{p}$. It is given by

$$\sigma(r,d,q,s) = \frac{q^s-1}{q^r-1} \sum_{i\,|\,(r_1/s)} \mu(i) \prod_{\substack{0<j<r\\ j\equiv 0\,(is)}} \frac{q^{jd}-1}{q^j-1}\ .$$

Proof: First note that in the situation of (3.5), we have

$$(*)\qquad \sigma(r,\wp,q,s) = \sigma(r/t,\wp_t,q^t,s/t)\ .$$

If $r_1 = 1$, only $s = 1$ contributes to the measure in (5.10), so (5.11) gives the result. Now let $r_1 > 1$. For $s > 1$, the inversion formula reads

$$1 = -\sum_{1 \neq i \mid s} \mu(i)\ .$$

Therefore,

$$\begin{aligned}
\Theta(r,d,q) - \sigma(r,\wp,q,1)/(q-1) &= \sum_{1 \neq s \mid r_1} \frac{\sigma(r,\wp,q,s)}{q^s-1} \\
&= -\sum_{1 \neq s \mid r_1} \frac{\sigma(r,\wp,q,s)}{q^s-1} \sum_{1 \neq i \mid s} \mu(i) \\
&= -\sum_{1 \neq i \mid r_1} \mu(i) \sum_{\substack{s \mid r_1 \\ s \equiv 0(i)}} \frac{\sigma(r/i,\wp_i,q^i,s/i)}{q^{i(s/i)}-1} \\
&= -\sum_{1 \neq i \mid r_1} \mu(i)\Theta(r/i,\wp_i,q^i)\ , \text{ i.e.,}
\end{aligned}$$

$$\sigma(r,\wp,q,1) = (q-1)\sum_{i \mid r_1} \mu(i)\Theta(r/i,d,q^i)\ .$$

The right hand side depends only on the degree d of $\mathfrak{p}$. Hence (5.11) yields the wanted formula for $s = 1$ and, together with (*), for general s.

6. Proof of (5.7)

Let ϕ be the Drinfeld module over $L = \mathbb{F}_{\mathfrak{p}}$ defined by

$$\phi_T = \sum_{0 \leq i \leq r} \lambda_i \tau^i \ ,$$

where $\lambda_0 = \gamma(T)$ and $\lambda_r = 1$. Write

$$\phi_p = \sum_{0 \leq i \leq rd} g_i \tau^i \ .$$

Then $g_0 = \gamma(p) = 0$, $g_{rd} = 1$ and $g_{id} = H_i$ $(i = 1 \ldots r-1)$. We will show the nonsingularity of the functional matrix

$$\left[\frac{\partial H_i}{\partial \lambda_j}\right]_{i,j = 1 \ldots r-1} \quad \text{in supersingular points } \underline{\lambda} = (\lambda_1, \ldots, \lambda_{r-1}, 1) \ .$$

First, we have $\phi_p \circ \phi_T = \phi_T \circ \phi_p$ in $L\{\tau\}$. Equating the τ^k–coefficients yields

$$[k] g_k + \sum_{n<k} \left[g_n \lambda_{k-n}^{q^n} - g_n^{q^{k-n}} \lambda_{k-n}\right] = 0 \ . \tag{6.1}$$

Here, k is any non–negative integer, [k] is the residue of $T^{q^k}-T$ in $\mathbb{F}_p \subset L$ and $\lambda_i = 0$ if $i \notin \{0,\dots,r\}$, $g_i = 0$ if $i \notin \{0,\dots,rd\}$, respectively. Note that

(6.2) $$[k] = 0 \text{ if and only if } k \text{ is divisible by } d .$$

We abbreviate $\partial g_k/\partial\lambda_j$ by $a_{k,j}$. Applying $\partial/\partial\lambda_j$ to (6.1) gives

$$[k]\,a_{k,j} + \sum_{n<k} a_{n,j}\lambda_{k-n}^{q^n} - g_{k-j}^{q^j} = 0$$

since $g_0 = 0$. Now, if $\underline{\lambda}$ is as above and supersingular, $g_{k-j} = 1$ if $k-j = rd$ and $g_{k-j} = 0$ otherwise, i.e.,

(6.3) $$[k]\,a_{k,j} + \sum_{k-r\le n<k} a_{n,j}\,\lambda_{k-n}^{q^n} = \begin{matrix} 0 & (k \ne rd+j) \\ 1 & (k = rd+j) \end{matrix} .$$

Put for the moment $h_i = g_{rd+i-r}$.

6.4. Lemma: Let $0 < i,j < r$. Then

$$\left[\frac{\partial h_i}{\partial\lambda_j}\right](\lambda) = \begin{matrix} 0 & (j < i) \\ 1 & (j = i) \end{matrix} .$$

In particular, the matrix is nonsingular.

Proof: Since $\lambda_r = 1$, (6.3) gives a linear recursion for $a_{k-r,j}$ in terms of $a_{n,j}$ with $n > k-r$. This shows that $a_{k-r,j} = 0$ as long as $k > rd+j$, and $a_{k-r,j} = 1$ for

$k = rd+j$. Putting $k = rd+i$ gives the assertion.

6.5. Lemma: Consider λ_j as indeterminate and the g_k as elements of the polynomial ring $L[\lambda_1,\dots,\lambda_{r-1}]$. If $id \leq k < (i+1)d$, g_k lies in the ideal generated by $H_1,\dots,H_i$.

Proof: If $k = id$, $g_k = H_i$. If $k > id$, $[k] \neq 0$ by (6.2). Now use (6.1) and induction.

End of the proof of (5.7): By the above, the functions h_i may be written $h_i = \sum\limits_k u_{i,k}H_k$ with some $(r-1,r-1)$–matrix $(u_{i,k})$ in $L[\lambda_1,\dots,\lambda_{r-1}]$. Thus

$$\frac{\partial h_i}{\partial \lambda_j} = \sum_k \left[\frac{\partial u_{i,k}}{\partial \lambda_j} H_k + u_{i,k} \frac{\partial H_k}{\partial \lambda_j} \right] .$$

Evaluating at a supersingular $\underline{\lambda}$ (i.e., where the H_k vanish) shows that the nonsingular matrix $(\partial h_i/\partial \lambda_j)(\underline{\lambda})$ is the product of $(u_{i,k})(\lambda)$ and $(\partial \lambda_j)(\lambda)$. Hence the latter is also nonsingular.

Again, the result generalizes the squarefreeness of Deuring's polynomial (see (4.4)).

7. Examples and Complements

Recall that $\Sigma(r,\mathfrak{p},q,s)$ corresponds bijectively to the subset of those left ideal classes $(\mathcal{I})$ of a fixed order B in $D = D(r,\mathfrak{p})$ for which $w(\mathcal{I}) = (q^s-1)/(q-1)$, or, equivalently, for which the order $B^{\mathcal{I}}$ has a unit group isomorphic with $(\mathbb{F}_{q^s})^*$. In such situations, one usually doesn't know which unit groups actually occur. In our case, the answer is given by

7.1. Corollary: Let $d = \deg \mathfrak{p} > 1$. Then for each divisor s of r_1, there exists a (maximal) A–order B in $D(r,\mathfrak{p})$ whose unit group is isomorphic with $(\mathbb{F}_{q^s})^*$.

Proof: From the above, we have to show that $\sigma(r,\mathfrak{p},q,s)$ is positive. This follows by an easy estimate from (5.13).

7.2. Example: In the missing case $d = 1$, our formula gives $\sigma(r,\mathfrak{p},q,s) = 0$ if $s < r$ and 1 if $s = r$, so the class number $h(D)$ is one. Of course, this can be seen directly, using a well known construction. Assume, without restriction, that $\mathfrak{p}$ is the ideal (T). Then D may be constructed as the full quotient ring $B \otimes K$ of $B = L\{\tau\}$, where L is the extension of degree r of $\mathbb{F}_{\mathfrak{p}} = \mathbb{F}_q$. A is embedded in B by mapping T to τ^r, which makes B into a projective A–module (left or right) of rank r^2. Moreover, B is a maximal A–order in $B \otimes K$. Since $L\{\tau\}$ is left euclidean, its class number is one.

7.3. Example: Let $r_1 = 1$, i.e., each prime divisor of r divides d. Then $h(D(r,\mathfrak{p})) = (q-1)\Theta(r,\mathfrak{p},q) = \prod_{1 \leq i < r} \zeta_{K,S}(-i)$.

7.4. Example (see also [4]): If r is prime then

$$h(D(r,\mathfrak{p})) = \begin{matrix} (q-1)\Theta(r,\mathfrak{p},q) & (d \equiv 0(r)) \\ (q-1)\Theta(r,\mathfrak{p},q)+(q^r-q)/(q^r-1) & (d \not\equiv 0(r)) \end{matrix} .$$

In principle, the Drinfeld module description of $D(r,\mathfrak{p})$ also allows the determination of the type number. Namely

7.5. Proposition ([12], Prop. 4.6): (i) Each element of $\Sigma(r,\mathfrak{p})$ is isomorphic to one defined over the extension L of degree r of $\mathbb{F}_{\mathfrak{p}}$. (ii) The bijection of (2.4) (iii) induces a bijection of the set of conjugacy classes of maximal orders in D with the set of orbits of $\Sigma(r,\mathfrak{p})$ under $\mathrm{Gal}(L|\mathbb{F}_{\mathfrak{p}})$.

This latter set may be studied geometrically, using the description given in the last section. Its cardinality is related to class numbers of certain abelian extensions of K. We limit ourselves to give the result in the least complicated case where $r = 2$ and $\mathrm{char}(\mathbb{F}_q) \neq 2$. Here, $D(r,\mathfrak{p})$ is the quaternion algebra over K ramified in $\mathfrak{p}$ and ∞.

7.6. Theorem ([12], see also [10]): Let the characteristic be different from 2. The type number of $D = D(2,\mathfrak{p})$ is given by

$$t(D) = \begin{array}{ll} \frac{1}{2}\left[\frac{q^d-q}{q^2-1} + 1 + \frac{1}{2}(h_1+h_2)\right] & (d \text{ odd}) \\ \frac{1}{2}\left[\frac{q^d-q}{q^2-1} + \frac{1}{2}h\right] & (d \text{ even}) . \end{array}$$

Here, h_1, h_2, h are the class numbers of the rings of A–integers in the quadratic field extensions of K, namely:

h_1, h_2: the two extensions ramified in $\mathfrak{p}$ and ∞;

h: the unique extension ramified in $\mathfrak{p}$ and inert at ∞.

Note that t(D) is less stable than h(D) in that it depends effectively on $\mathfrak{p}$ and not only on its degree d.

(7.7) In determining the class number of D, our basic ingredients were the transfer principle 3.5 and the mass formula 5.11 (or 2.5. (iii)). It seems possible that one can prove similar mass formulas in the general case, where A is any ring as described in section 2, i.e., a function ring with one place at infinity. Having both ingredients available, the proof scheme of Thm. 5.13 could be applied. Also, the transfer principle might turn out to hold for a larger class of algebras D than those of Drinfeld type. Together with the properties of the zeta function of D [3], [4], this would yield a method to attack the class number problem for that larger class.

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