THE GEOMETRIC COMPLEX OF A MORSE–BOTT–SMALE PAIR AND AN EXTENSION OF A THEOREM BY BISMUT–ZHANG

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ABSTRACT. We define the geometric complex associated to a Morse–Bott–Smale vector field, cf. [AB95], and its associated spectral sequence. We prove an extension of the Bismut–Zhang theorem to Morse–Bott–Smale functions, see Theorem 2 below. The proof is based on the Bismut–Zhang theorem for Morse–Smale functions, see [BZ92].

CONTENTS

1. INTRODUCTION

Let M be a closed connected manifold, g a Riemannian metric on M, E a flat vector bundle over M and μ a fiber metric on E. This permits to define the formal adjoint of the deRham differential on $\Omega^*(M;E)$ and

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the Laplacian Δ^q acting on $\Omega^q(M;E)$. In this situation the Ray–Singer torsion [RS71] is defined as:

$$\log T_{\mathrm{an}}^M(E,g,\mu) := \frac{1}{2}\sum_q (-)^{q+1} q \log \det{}'\Delta^q$$

where $\det{}'\Delta^q$ denotes the zeta regularized determinant of Δ^q with zero modes discarded. The main motivation of this paper is the calculation of the Ray–Singer torsion. This will be done with the help of the geometric complex associated to a Morse–Bott–Smale vector field described below, which is an object of independent interest by itself.

Suppose X is a Morse–Bott–Smale vector field on M, i.e. $X = -\operatorname{grad}_{\tilde g}(f)$ where $(f,\tilde g)$ is a Morse–Bott–Smale pair, cf. Definition 3 in section 2.4. Let Σ denote the critical manifold of X and $N^- \to \Sigma$ the negative normal bundle. For every connected component $S \subseteq \Sigma$ we have a Morse index $\operatorname{ind}(S) \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ which coincides with the rank of N^- over S. Every connected component $S \subseteq \Sigma$ inherits a Riemannian metric g_S from g. Let $\mathcal{O}_{N^-}$ denote the orientation bundle of N^-. The flat vector bundle $E_S := (E|_\Sigma \otimes \mathcal{O}_{N^-})|_S$ over S inherits a fiber metric μ_S from the fiber metric $\mu|_S$ on $E|_S$ and the canonic parallel fiber metric on $\mathcal{O}_{N^-}$.[1] So we have Ray–Singer torsions $\log T_{\mathrm{an}}^S(E_S, g_S, \mu_S)$ for every connected component $S \subseteq \Sigma$.

The Morse–Bott–Smale vector field X gives rise to a geometric complex $C^*(X;E)$ whose underlying graded vector space is

$$C^*(X;E) := \bigoplus_{S\subseteq\Sigma} \Omega^{*-\operatorname{ind}(S)}(S;E_S)$$

where the sum is over all connected components $S \subseteq \Sigma$. Fiber integration over the unstable manifolds provides a homomorphism of complexes $\operatorname{Int} : \Omega^*(M;E) \to C^*(X;E)$, see section 2 below. The Morse index provides a decreasing finite filtration

$$C_p^*(X;E) := \bigoplus_{\operatorname{ind}(S)\geq p} \Omega^{*-\operatorname{ind}(S)}(S;E_S)$$

on the geometric complex. This gives rise to a spectral sequence $(E_kC^*(X;E), \delta_k)$, see for instance [BT82].

Theorem 1.

[1] The frame bundle of N^- induces a principal $\mathbb{Z}_2$–covering of Σ. We think of the orientation bundle $\mathcal{O}_{N^-}$ as the flat line bundle associated to this principal covering with respect to the standard non-trivial $\mathbb{Z}_2$–action on $\mathbb{R}$. It thus inherits a canonic parallel fiber metric from the standard Euclidean inner product on $\mathbb{R}$.

(i) *The integration map induces an isomorphism in cohomology* $\mathrm{Int} : H^*(M; E) = HC^*(X; E)$.

(ii) *The spectral sequence* $(E_kC^*(X; E), \delta_k)$ *converges to* $HC^*(X; E)$. *Moreover* $E_1C^*(X; E) = \bigoplus_{S\subseteq\Sigma} H^{*-\mathrm{ind}(S)}(S; E_S)$. *Particularly* $E_kC^*(X; E)$ *is finite dimensional for all* $k \geq 1$ *and we have* $E_kC^*(X; E) = E_\infty C^*(X; E)$ *for sufficiently large* k.

The above theorem was first stated in [AB95] but known to experts. Unfortunately, despite its usefulness and simplicity there is no complete (satisfactory) treatment of the geometric complex in the existing literature. Since we need the geometric complex for our main result we will complete the discussion of [AB95] and provide a complete treatment of the geometric complex and its basic properties.

Remark 1. If X is a Morse–Smale vector field the geometric complex is finite dimensional, $(E_1C^*(X; E), \delta_1) = (C^*(X; E), \delta)$ is the familiar Morse complex, and $\delta_k = 0$ for $k \geq 2$. In general $(E_1C^*(X; E), \delta_1)$ has a very simple form, see section 3.4.

Remark 2. If $\pi : M \to N$ is a smooth bundle with M and N closed manifolds, and X_N is a Morse–Smale vector field on N then one can construct a Morse–Bott–Smale vector field on X_M on M with $d_x\pi(X_M(x)) = X_N(\pi(x))$ for all $x \in M$. The spectral sequence associated with the geometric complex of X_M identifies with the Leray–Serre spectral sequence of the bundle π provided by the cell structure on N induced by X_N, cf. Propositions 14 and 15 in section 6.1.

The Riemannian metric g and the fiber metric μ induce inductively scalar products on every E_k–term of the spectral sequence. Indeed, the restriction of g and μ to Σ provide a scalar product on the deRham complex $\Omega^*(\Sigma; E_\Sigma)$. Its restriction to harmonic forms provides the scalar product on $E_1C^*(X; E)$, cf. Theorem 1(ii). Now suppose inductively that $E_{k-1}C^*(X; E)$ is equipped with a scalar product. Then $E_kC^*(X; E) = H(E_{k-1}C^*(X; E), \delta_{k-1})$ inherits a scalar product as a subquotient of the finite dimensional Euclidean vector space. Equivalently, one can describe this scalar product via finite dimensional Hodge theory on the complex $(E_{k-1}C^*(X; E), \delta_k)$.

The scalar products on $E_kC^*(X; E)$ permit to define the adjoint of the differential δ_k on $E_kC^*(X; E)$ and a Laplacian Δ_k^q acting on the finite dimensional $E_kC^q(X; E)$, $k \geq 1$. Define the combinatorial torsion by:

$$\log T^M_{\mathrm{comb}}(E, X, g, \mu) := \sum_{k\geq 1} \frac{1}{2} \sum_q (-)^{q+1} q \log \det{}' \Delta_k^q$$

The deRham cohomology inherits a scalar product via Hodge decomposition which provides a scalar product on the one dimensional real vector space $\det H^*(M;E)$ referred to as the *Hodge scalar product.*[2] The canonical identification $\det HC^*(X;E) = \det E_\infty C^*(X;E)$ and the scalar product on $E_\infty C^*(X;E) = E_k C^*(X;E)$ for k large, define a scalar product on $\det HC^*(X;E)$ referred to as the *geometric scalar product.* The isomorphism in cohomology induced by integration provides an isomorphism $\det \mathrm{Int} : \det H^*(M;E) \to \det HC^*(X;E)$. Define[3]

$$\log T^M_{\mathrm{met}}(E,X,g,\mu) := \log \mathrm{vol}(\det \mathrm{Int}).$$

The fiber metric μ on E provides a closed one form $\theta \in \Omega^1(M;\mathbb{R})$ which measures to what extend the induced length one section on $\det E \otimes \mathcal{O}_E$ is not parallel, see section 4.2. Define

$$\mathcal{R}(\theta,X,g) := \int_{M\setminus\Sigma} \theta \wedge X^*\Psi_g$$

where $\Psi_g \in \Omega^{\dim M-1}(TM\setminus M;\mathcal{O}_M)$ denotes the global angular form, cf. [BT82], also called Mathai–Quillen form in [BZ92]. The integral on the right hand side need not be convergent but can be regularized, see section 4.1.

The main theorem of this paper is the following extension of a result by Bismut–Zhang [BZ92], Cheeger [C77], [C79] and Müller [M78].

Theorem 2. *Let (M,g) be a closed Riemannian manifold, E a flat vector bundle with fiber metric μ over M, and let X be a Morse–Bott–Smale vector field on M, see Definition 3 below. Then*

$$\begin{aligned}\log T^M_{\mathrm{an}}(E,g,\mu) &= \sum_{S\subseteq\Sigma} (-)^{\mathrm{ind}(S)} \log T^S_{\mathrm{an}}(E_S,g_S,\mu_S)\\ &\quad + \log T^M_{\mathrm{comb}}(E,X,g,\mu) + \log T^M_{\mathrm{met}}(E,X,g,\mu) + \mathcal{R}(\theta,-X,g)\end{aligned}$$

where the sum is over all connected components $S \subseteq \Sigma$.

One can regard Σ as a graded manifold where the degree of the component S is $\mathrm{ind}(S)$ and write $\log T^\Sigma_{\mathrm{an}} := \sum_{S\subseteq\Sigma}(-)^{\mathrm{ind}(S)} \log T^S_{\mathrm{an}}$. Then one can view the theorem above as a localization theorem stating that

$$\log T^M_{\mathrm{an}} = \log T^\Sigma_{\mathrm{an}} + \log T_{\mathrm{comb}} + \log T_{\mathrm{met}} + \mathcal{R}(\theta,-X,g)$$

[2]Here we use the notation $\det V = \Lambda^{\dim(V^{\mathrm{even}})} V^{\mathrm{even}} \otimes \Lambda^{\dim(V^{\mathrm{odd}})}(V^{\mathrm{odd}})^*$ for a finite dimensional $\mathbb{Z}_2$–graded vector space. See appendix B for more details on determinant lines.

[3]Recall that for an isomorphism $\alpha : V \to W$ between Euclidean vector spaces one defines $\mathrm{vol}\,\alpha := \sqrt{\det(\alpha^*\alpha)}$.

with the term $\mathcal{R}(\theta, -X, g)$ computable in terms of differential geometry; the term $\log T_{\mathrm{comb}}$ computable in terms of finite dimensional linear algebra; and, in case $H^*(M;E) = 0$, vanishing term $\log T_{\mathrm{met}}$.

Theorem 2 can be extended to the case E is a flat bundle of Hilbertian $\mathcal{A}$–modules of finite type where $\mathcal{A}$ is a finite von Neuman algebra using the extension of the Bismut–Zhang theorem to this case as formulated in [BFK01]. The proof and the results are exactly the same.

Theorem 2 can be extended to G–manifolds where G is a finite group and the torsions replaced by the G–equivariant torsions which are class functions on G with positive real values. This will use the corresponding extension of the Bismut–Zhang theorem cf. [BZ94]. The particular case $G = \mathbb{Z}_2$ when applied to the double of a manifold with boundary resp. to a bordism will imply an extension of Theorem 2 to compact manifolds with boundary resp. to bordisms and, as application, will provide a very simple derivation of all known glueing formulae for analytic torsion cf. [BFK99]. This is contained in [BH04].

Theorem 2 permits to recover and generalize results of Lück–Schick–Thielmann cf. [LST98] about the torsion of smooth bundles, cf. section 6. In a forthcoming paper we will use the above results and reduce the calculation of G–equivariant torsion for G a compact Lie group to finite dimensional linear algebra.

There is a more conceptual way to formulate Theorem 2 and understand the extend to which the result is a localization theorem for torsion. To explain this consider M a closed manifold, E a flat vector bundle over M, $x_0 \in M$ a base point, X a Morse–Bott–Smale vector field which gives rise to the geometric complex $C^*(X;E)$, and $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M)$ an Euler structure based at x_0, see section 4.1. The analytic torsion $\tau_{\mathrm{an}}^{E,\mathfrak{e}}$ can be regarded as an element defined up to multiplication by ± 1 in the one dimensional vector space[4]

$$\mathrm{Det}_{x_0}(M;E) := \det H^*(M;E) \otimes (\det E_{x_0})^{-\chi(M)}$$

where $H^*(M;E)$ denotes the deRham cohomology with coefficients in E and $\det H^*(M;E)$ its determinant line in the graded sense, see appendix B. The geometric torsion $\tau_{\mathrm{geom}}^{E,\mathfrak{e},X}$ is an element well defined up to multiplication by ± 1 in the one dimensional vector space

$$\mathrm{Det}_{x_0}(X;E) := \det HC^*(X;E) \otimes (\det E_{x_0})^{-\chi(M)}$$

[4]Here we use the following convention. For a line L, i.e. a one dimensional vector space, we write $L^k := L \otimes \cdots \otimes L$, $L^{-k} = L^* \otimes \cdots \otimes L^*$ and we write $V^0 := \mathbb{R}$. Note that there is a canonic isomorphism $L^{k_1+k_2} = L^{k_1} \otimes L^{k_2}$ for all k_1 and k_2.

where $HC^*(X;E)$ denotes the cohomology of the geometric complex, see section 3 for definition. The integration induces an isomorphism

$$\det \mathrm{Int} : \mathrm{Det}_{x_0}(M;E) \to \mathrm{Det}_{x_0}(X;E)$$

and Theorem 2 is equivalent to the statement that $\det \mathrm{Int}(\tau_{\mathrm{an}}^{E,\mathfrak{e}}) = \tau_{\mathrm{geom}}^{E,\mathfrak{e},X}$, cf. Theorem 5 below.

The geometric torsion $\tau_{\mathrm{geom}}^{E,\mathfrak{e},X}$ is obtained from the analytic torsions $\tau_{\mathrm{an}}^{E_S,\mathfrak{e}_S}$ for a collection of Euler structures $\mathfrak{e}_S \in \mathfrak{Eul}_{x_S}(S)$, one for every connected component $S \subseteq \Sigma$. Precisely, the geometric complex $C^*(X;E)$ provides an isomorphism:

$$I_1 : \det HC^*(X;E) \to \bigotimes_{S\subseteq\Sigma} \big(\det H^*(S;E_S)\big)^{(-)^{\mathrm{ind}(S)}}$$

The Euler structure $\mathfrak{e}$ and the Euler structures $\mathfrak{e}_S$ provide a one chain $c \in C_1(M;\mathbb{R})$ which one writes as $c = c_0 + \sum_{S\subseteq\Sigma}(-)^{\mathrm{ind}(S)}\chi(S)c_S$ where c_S is a path from x_0 to x_S and $c_0 \in C_1(M;\mathbb{R})$ which then satisfies $\partial c_0 = 0$. Parallel transport along the c_S and proper scaling determined by c_0 provides an isomorphism well defined up to sign:

$$I_2 : (\det E_{x_0})^{-\chi(M)} \to \bigotimes_{S\subseteq\Sigma} \big((\det E_S)_{x_S}^{-\chi(S)}\big)^{(-)^{\mathrm{ind}(S)}}$$

The geometric torsion is then defined by:

$$\tau_{\mathrm{geom}}^{E,\mathfrak{e},X} := (I_1 \otimes I_2)^{-1}\Big(\bigotimes_{S\subseteq\Sigma}\big(\tau_{\mathrm{an}}^{E_S,\mathfrak{e}_S}\big)^{(-)^{\mathrm{ind}(S)}}\Big)$$

This explains the localization aspect of the result.

The paper contains a number of intermediate results of independent interest such as: the canonic compactification of the unstable sets and the space of trajectories of a Morse–Bott–Smale vector field, see Propositions 2 and 3 and Theorems 3 and 4; the invariant $\mathcal{R}$ and the discussion of Euler structures.

2. Analysis of the ODE

2.1. **Morse–Bott functions.** Let M be a closed manifold, i.e. M is compact and without boundary. Let $f : M \to \mathbb{R}$ be a smooth function and let $\Sigma := \{x \in M \mid df_x = 0\}$ denote the *critical set of* f. Suppose x is a critical point of f and $Y, Z \in T_xM$. Then $H(Y,Z) := Y \cdot df(Z) = Z \cdot df(Y)$ for df is closed and vanishes at x. So we obtain a fiber wise symmetric bilinear form H on the vector bundle $TM|_\Sigma$

referred to as the *Hessian of* f.[5] If $\Sigma \subseteq M$ happens to be a closed submanifold the Hessian will provide a smooth fiber wise bilinear form on the smooth vector bundle $TM|_\Sigma$. Moreover we obviously have $T\Sigma \subseteq \ker H$. Therefore the Hessian induces a fiber wise bilinear form on the normal bundle $N := TM|_\Sigma/T\Sigma$ of Σ which we will denote by H too.

Definition 1 (Morse–Bott functions)**.** A function $f : M \to \mathbb{R}$ is called *Morse–Bott* if its critical set Σ is a closed submanifold of M and the Hessian considered as fiber wise bilinear form on the normal bundle N of Σ is non-degenerate. In this case Σ will also be referred to as the *critical manifold* or the *rest manifold*. The index, i.e. the number of negative eigen values, of the Hessian provides a locally constant $\mathrm{ind} : \Sigma \to \mathbb{N}_0$ called the *Morse index of* f. Finally, define the *critical manifold of index* q by $\Sigma_q := \bigsqcup_{\mathrm{ind}(S)=q} S$ where the disjoint union is over all connected components $S \subseteq \Sigma$ of index q.

The following well known result totally clarifies the local behavior of f near its critical submanifold.

Proposition 1 (Morse lemma)**.** *Let f be a Morse–Bott function on M, let $p_N : N \to \Sigma$ denote the normal bundle of its critical manifold Σ and let H denote the Hessian. Define a smooth function $\frac{1}{2}H^2 : N \to \mathbb{R}$ on the total space of N by $\frac{1}{2}H^2(Y) := \frac{1}{2}H(Y,Y)$. Then there exists a tubular neighborhood $\varphi : N \to M$ such that $\varphi^* df = d(\frac{1}{2}H^2)$ holds in a neighborhood of the zero section $\Sigma \subseteq N$. Particularly $\varphi^* f = \frac{1}{2}H^2 + p_N^*(f|_\Sigma)$ with $f|_\Sigma$ locally constant. A tubular neighborhood like this is called a* Morse chart *for f.*

2.2. **Compatible Riemannian metrics.** Suppose f is a Morse–Bott function on a close manifold M. It is easy to see that there are H–orthogonal subbundles $N^+ \oplus N^- = N$ such that $g_\pm := \pm H|_{N^\pm} > 0$. Certainly $H = g_+ \ominus g_-$ with respect to this splitting. The bundle N^+ is called the *positive normal bundle* and the bundle N^- is called the *negative normal bundle* of f. Note however, that these bundles are well defined only up to isomorphisms of vector bundles. For the rank of N^- we have $\mathrm{rank}(N^-) = \mathrm{ind} : \Sigma \to \mathbb{N}_0$.

Suppose we have a Riemannian metric g_Σ on Σ and linear connections $\nabla^\pm$ on $N^\pm$ such that $\nabla^\pm g_\pm = 0$. Define a fiber metric $g_N := g_+ \oplus g_-$ and a linear connection $\nabla^N := \nabla^+ \oplus \nabla^-$ on N. Then $\nabla^N g_N = 0 =$

[5] This bilinear form can be extended (in a non-canonical way) to a smooth bilinear form on TM as follows. Pick any linear torsion free connection ∇ on TM and set $H(Y,Z) := (\nabla df)(Y,Z)$. This will be symmetric for df is a closed one form and ∇ was supposed to be torsion free. It will coincide with our previous definition over Σ since df vanishes along Σ.

$\nabla^N H$. Define a Riemannian metric $\tilde{g}$ on the total space of N using g_N, g_Σ and ∇^N in the most obvious way. Clearly the restriction of $\tilde{g}$ to Σ coincides with g_Σ. On the normal bundle of $\Sigma \subseteq N$, which canonically identifies to $N \to \Sigma$, the Riemannian metric $\tilde{g}$ induces a fiber metric and a linear connection which coincide with g_N and ∇^N. The splitting of N into positive and negative H–eigen spaces using g_N coincides with $N^+ \oplus N^- = N$. The second fundamental form of $\Sigma \subseteq N$ vanishes. The endomorphism $-(\mathrm{id}_{N^+} \ominus \mathrm{id}_{N^-})$ of N can be considered as a vertical vector field on the total space of N which coincides with $X := -\operatorname{grad}_{\tilde{g}}(\frac{1}{2}H^2)$. The unstable set of Σ, i.e. the points in N whose X trajectory departs at Σ, coincides with $N^- \subseteq N$ hence is a smooth submanifold of N. Similarly the stable set of Σ coincides with the smooth submanifold $N^+ \subseteq N$.

Definition 2 (Compatible Riemannian metrics)**.** Let f be a Morse–Bott function on M. A Riemannian metric g on M is called *compatible with* f if there exists a Morse chart $\varphi : N \to M$ such that in addition to $\varphi^* df = d(\frac{1}{2}H^2)$ we have $\varphi^* g = \tilde{g}$ on a neighborhood of $\Sigma \subseteq N$. Here $\tilde{g}$ is a Riemannian metric on the total space of N constructed from an H–orthogonal splitting $N^+ \oplus N^- = N$ for which $\pm H|_{N^\pm} > 0$, a Riemannian metric on Σ and linear connections on $N^\pm$ for which the restrictions of H are parallel as explained in the previous paragraph.

Note that a Riemannian metric g on M induces a Riemannian metric g_Σ on Σ, a fiber metric g_N on the normal bundle $N = (T\Sigma)^\perp$, an orthogonal splitting $N^+ \oplus N^- = N$ and a smooth map (the exponential map) $\exp : N \to M$ from the total space N of the normal bundle of Σ to M. If g is compatible then the parallel transport along a path in Σ leaves N and $N^\pm$ invariant and therefore define the connection ∇^N, and $\nabla^\pm$. The Riemannian metric g_Σ, the fiber metric g_N and the connection ∇^N induce a Riemannian metric on N. If g is compatible then exp is an isometry when restricted to a small neighborhood of the zero section of the normal bundle. This can also be taken as an equivalent definition of compatible metric with respect to f.

Note that given an arbitrary metric g and an arbitrary neighborhood U of Σ one can produce a Riemannian metric g' which is compatible for f and agrees with g on Σ and on $M \setminus U$.

2.3. **Stable and unstable manifold.** Let f be a Morse–Bott function on a closed manifold M and let g be a compatible Riemannian metric on M, see Definition 2. Define a vector field $X := -\operatorname{grad}_g(f)$. For a connected component $S \subseteq \Sigma$ define its *stable* resp. *unstable set* by

$$W_S^\pm := \left\{ x \in M \;\middle|\; \lim_{t \to \pm\infty} \phi_t(x) \in S \right\}$$

where ϕ_t denotes the flow of the vector field X at time t. $W^\pm_S$ is a smooth boundary-less submanifold of M and we let $i^S_\pm : W^\pm_S \to M$ denote the inclusions. Moreover we get smooth fiber bundles $p^S_\pm : W^\pm_S \to S$ diffeomorphic to $N^\pm_S \to S$. Define $W^\pm := \bigsqcup_{S\subseteq\Sigma} W^\pm_S$ where the disjoint union is over all connected components S of Σ. The $i^S_\pm$ give rise to a smooth immersion $i_\pm : W^\pm \to M$ and the $p^S_\pm$ give rise to a smooth fiber bundle $p_\pm : W^\pm \to \Sigma$ diffeomorphic to $N^\pm \to \Sigma$.

Let $V_{W^\pm} \to W^\pm$ denote the vertical (vector) bundle of $p_\pm : W^\pm \to \Sigma$. For its rank we find $\operatorname{rank}(V_{W^-}) = p^*_- \operatorname{ind} : W^- \to \mathbb{N}_0$. Here we write $p^*_- \operatorname{ind}$ for the composition $\operatorname{ind} \circ p_- : W^- \to \Sigma \to \mathbb{N}_0$. The Lie transport along the normalized $\frac{X}{|X|}$ provides an isomorphism of vector bundles $V_{W^\pm} = p^*_\pm N^\pm$ and thus a canonic isomorphism of orientation bundles:

$$\mathcal{O}_{V_{W^\pm}} = p^*_\pm \, \mathcal{O}_{N^\pm} \tag{1}$$

Moreover, if E is a flat vector bundle over M then parallel transport along X provides a canonic isomorphism of flat vector bundles:

$$i^*_\pm E = p^*_\pm E|_\Sigma \tag{2}$$

2.4. **The space of trajectories.** Let f be a Morse–Bott function on a closed manifold M and let g be a compatible Riemannian metric on M, cf. Definition 2. For $x \in \Sigma$ let $i^x_- : W^-_x \to M$ denote the restriction of $i_- : W^- \to M$ to the fiber W^-_x of $p_- : W^- \to \Sigma$ over x.

Definition 3 (Morse–Bott–Smale vector fields)**.** A pair (f, g) consisting of a Morse–Bott function f and a compatible Riemannian metric g on a closed manifold M is called *Morse–Bott–Smale pair* if the mappings $i_+ : W^+ \to M$ and $i^x_- : W^-_x \to M$ are transversal for all $x \in \Sigma$.[6] A vector field X is called Morse–Bott–Smale if there exists a Morse–Bott–Smale pair (g, f) such that $X = -\operatorname{grad}_g(f)$.

[6]Given a Morse–Bott function f one can ask if there exists a Riemannian metric g so that the pair (f, g) is a Morse–Bott–Smale pair. Such a function can be called 'good' Morse–Bott function. All Morse functions are good Morse–Bott functions but, as expected, not all Morse–Bott functions are good. To decide when a Morse–Bott function is good is an interesting problem. Here are a few examples:

1) If $f : N \to \mathbb{R}$ is a Morse function and $\pi : M \to N$ is a smooth bundle with compact fibers then $f \circ \pi$ is a good Morse–Bott function.

2) If M is a compact smooth G–manifold with G a compact Lie group, any normal G–Morse function $f : M \to \mathbb{R}$ (i.e. for any critical point the negative isotropy representation is trivial) is a good Morse–Bott function cf. [BH04]. It is known that normal G–Morse functions are C^0–dense in the space of all smooth G–invariant functions in C^0–topology.

3) If f is a Morse–Bott function so is $-f$ but it is possible that f is good and $-f$ is not.

Suppose X is a Morse–Bott–Smale vector field on M. Let $S \neq S'$ be two different connected components of Σ. Then $\mathcal{M}(S,S') := W^-_S \cap W^+_{S'}$ is a smooth submanifold of M. Define $\mathcal{M} := \bigsqcup_{S\neq S'\subseteq\Sigma}\mathcal{M}(S,S')$ where the disjoint union is over all pairs of different connected components S and S' of Σ. Let $i : \mathcal{M} \to M$ denote the obvious smooth immersion, $p_+ : \mathcal{M} \to \Sigma$ the obvious smooth mapping and let $p_- : \mathcal{M} \to \Sigma$ denote the obvious smooth fiber bundle.

Let $V_\mathcal{M} \to \mathcal{M}$ denote the vertical bundle of $p_- : \mathcal{M} \to \Sigma$. We have a short exact sequence of vector bundles:

$$0 \to V_\mathcal{M} \to V_{W^-}|_\mathcal{M} \to TM|_\mathcal{M}/TW^+|_\mathcal{M} \to 0$$

As in section 2.3, Lie transport along $\frac{X}{|X|}$ provides canonic isomorphisms $V_{W^-}|_\mathcal{M} = p_-^*N^-$ and $TM|_\mathcal{M}/TW^+|_\mathcal{M} = p_+^*N^-$. So we obtain a short exact sequence of vector bundles:

$$0 \to V_\mathcal{M} \to p_-^*N^- \to p_+^*N^- \to 0$$

Particularly $\operatorname{rank}(V_\mathcal{M}) = p_-^*\operatorname{ind} - p_+^*\operatorname{ind} : \mathcal{M} \to \mathbb{N}_0$. Moreover, this short exact sequence provides us with a canonic isomorphism of orientation bundles:[7]

$$(-)^{\operatorname{rank}(V_\mathcal{M})\cdot\operatorname{rank}(p_+^*N^-)} : \mathcal{O}_{V_\mathcal{M}} \otimes p_+^*\mathcal{O}_{N^-} = p_-^*\mathcal{O}_{N^-} \qquad (3)$$

If E is a flat vector bundle over M we get canonic isomorphisms $p_-^*E|_\Sigma = i^*E = p_+^*E|_\Sigma$ using parallel transport along X.

The flow ϕ_t restricts to a smooth action of $\mathbb{R}$ on $\mathcal{M}$. This action is free and we have a smooth orbit space $\mathcal{T} := \mathcal{M}/\mathbb{R}$. The mapping $p_+ : \mathcal{M} \to \Sigma$ factors to a smooth map $\pi_+ : \mathcal{T} \to \Sigma$. The fiber bundle $p_- : \mathcal{M} \to \Sigma$ factors to a smooth fiber bundle $\pi_- : \mathcal{T} \to \Sigma$. Let $q : \mathcal{M} \to \mathcal{T}$ denote the projection and let $[X] \subseteq V_\mathcal{M}$ denote the line bundle over $\mathcal{M}$ generated by the vector field X. We have a short exact sequence of vector bundles:

$$0 \to [X] \to V_\mathcal{M} \to q^*V_\mathcal{T} \to 0$$

So the rank of the vertical bundle $V_\mathcal{T}$ is $\operatorname{rank}(V_\mathcal{T}) = \pi_-^*\operatorname{ind} - \pi_+^*\operatorname{ind} - 1 : \mathcal{T} \to \mathbb{N}_0$. Since the line bundle $[X]$ is oriented via X we obtain an isomorphism of orientation bundles $\mathcal{O}_{V_\mathcal{M}} = q^*\mathcal{O}_{V_\mathcal{T}}$, that is we use the X last convention here. Using the isomorphism (3) we get an isomorphism $q^*(\mathcal{O}_{V_\mathcal{T}} \otimes \pi_+^*\mathcal{O}_{N^-}) = q^*\pi_-^*\mathcal{O}_{N^-}$ which descends to a canonic

[7] Throughout the paper we use the following convention. If $0 \to E \to F \to G \to 0$ is a short exact sequence of vector bundles we use an orientation of G_x followed by an orientation of E_x to yield the compatible orientation of F_x, x in the common base space. We refer to this as the G_x first convention. The notation in (3) should indicate that we deviate from this convention and use the $V_\mathcal{M}$ first convention here.

isomorphism of orientation bundles:

$$\mathcal{O}_{V_{\mathcal{T}}} \otimes \pi_+^* \mathcal{O}_{N^-} = \pi_-^* \mathcal{O}_{N^-} \tag{4}$$

If E is a flat vector bundle over M then the isomorphism $p_-^*E|_\Sigma = p_+^*E|_\Sigma$ descends to a canonic isomorphism of flat vector bundles:

$$\pi_-^*E|_\Sigma = \pi_+^*E|_\Sigma \tag{5}$$

For two connected components S and S' of Σ set $\mathcal{T}(S,S') := \pi_-^{-1}(S) \cap \pi_+^{-1}(S') \subseteq \mathcal{T}$. Clearly this is the disjoint union over several connected components of $\mathcal{T}$. The rank of its vertical bundle $V_{\mathcal{T}(S,S')}$ is constant and $\operatorname{rank}(V_{\mathcal{T}(S,S')}) = \operatorname{ind}(S) - \operatorname{ind}(S') - 1$. Note that this implies $\mathcal{T}(S,S') = \emptyset$ unless $\operatorname{ind}(S) > \operatorname{ind}(S')$, a consequence of the strong transversality condition. Also note, that we have $\mathcal{T}(S,S) = \emptyset$ by our very definitions.

2.5. **Compactification of $\mathcal{T}$.** Let X be a Morse–Bott–Smale vector field on M. For $k \geq 0$ define the manifold of *unparameterized k–times broken trajectories* by:

$$\hat{\mathcal{T}}_k := \underbrace{\mathcal{T} \times_\Sigma \mathcal{T} \times_\Sigma \cdots \times_\Sigma \mathcal{T}}_{k+1 \text{ copies}}$$

Here $\mathcal{T} \times_\Sigma \mathcal{T} := \{(x,y) \in \mathcal{T} \times \mathcal{T} \mid \pi_+(x) = \pi_-(y)\}$, and similarly for more factors. Define a smooth map $(\hat{\pi}_+)_k : \hat{\mathcal{T}}_k \to \Sigma$ by composing the projection to the last factor with $\pi_+ : \mathcal{T} \to \Sigma$. Define a smooth fiber bundle $(\hat{\pi}_-)_k : \hat{\mathcal{T}}_k \to \Sigma$ by composing the projection to the first factor with $\pi_- : \mathcal{T} \to \Sigma$. Note that by definition $\hat{\mathcal{T}}_0 = \mathcal{T}$, $(\hat{\pi}_+)_0 = \pi_+$ and $(\hat{\pi}_-)_0 = \pi_-$.

Let $V_{\hat{\mathcal{T}}_k}$ denote the vertical bundle of $(\hat{\pi}_-)_k : \hat{\mathcal{T}}_k \to \Sigma$. Define an isomorphism of orientation bundles

$$(-)^{p_1^* \operatorname{rank}(V_{\mathcal{T}})} : \mathcal{O}_{V_{\hat{\mathcal{T}}_1}} \otimes (\hat{\pi}_+)_1^* \mathcal{O}_{N^-} = (\hat{\pi}_-)_1^* \mathcal{O}_{N^-} \tag{6}$$

as follows. Consider the commutative diagram

$$\begin{tikzcd}
\hat{\mathcal{T}}_1 = \mathcal{T} \times_\Sigma \mathcal{T} \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathcal{T} \arrow[r, "\pi_+"] \arrow[d, "\pi_-"] & \Sigma \\
\mathcal{T} \arrow[r, "\pi_+"] \arrow[d, "\pi_-"'] & \Sigma & \\
\Sigma & &
\end{tikzcd}$$

where p_1 projects onto the first component and p_2 projects onto the second component of $\hat{\mathcal{T}}_1 = \mathcal{T} \times_\Sigma \mathcal{T}$. Notice that $(\hat{\pi}_+)_1 = \pi_+ \circ p_2$ and $(\hat{\pi}_-)_1 = \pi_- \circ p_1$. Clearly we have $p_2^* V_{\mathcal{T}} = V_{p_1}$ and thus $p_2^* \mathcal{O}_{V_{\mathcal{T}}} = \mathcal{O}_{V_{p_1}}$.

Moreover the short exact sequence $0 \to V_{p_1} \to V_{\hat{\mathcal{T}}_1} \to p_1^* V_{\mathcal{T}} \to 0$ provides $\mathcal{O}_{V_{p_1}} \otimes p_1^* \mathcal{O}_{V_{\mathcal{T}}} = \mathcal{O}_{V_{\hat{\mathcal{T}}_1}}$. Finally recall the isomorphism (4). Now define the isomorphism (6) as $(-)^{p_1^* \operatorname{rank}(V_{\mathcal{T}})}$ times the following composition of isomorphisms:

$$
\begin{aligned}
\mathcal{O}_{V_{\hat{\mathcal{T}}_1}} \otimes (\hat{\pi}_+)_1^* \mathcal{O}_{N^-} &= \mathcal{O}_{V_{p_1}} \otimes p_1^* \mathcal{O}_{V_{\mathcal{T}}} \otimes p_2^* \pi_+^* \mathcal{O}_{N^-} \\
&= p_2^* \mathcal{O}_{V_{\mathcal{T}}} \otimes p_1^* \mathcal{O}_{V_{\mathcal{T}}} \otimes p_2^* \pi_+^* \mathcal{O}_{N^-} \\
&= p_1^* \mathcal{O}_{V_{\mathcal{T}}} \otimes p_2^* (\mathcal{O}_{V_{\mathcal{T}}} \otimes \pi_+^* \mathcal{O}_{N^-}) \\
&= p_1^* \mathcal{O}_{V_{\mathcal{T}}} \otimes p_2^* \pi_-^* \mathcal{O}_{N^-} \\
&= p_1^* (\mathcal{O}_{V_{\mathcal{T}}} \otimes \pi_+^* \mathcal{O}_{N^-}) \\
&= p_1^* \pi_-^* \mathcal{O}_{N^-} \\
&= (\hat{\pi}_-)_1^* \mathcal{O}_{N^-}
\end{aligned}
$$

Suppose E is a flat vector bundle over M. In a similar manner, using (5), we obtain a canonic isomorphism of flat vector bundles

$$
(\hat{\pi}_-)_1^* E|_\Sigma = (\hat{\pi}_+)_1^* E|_\Sigma \tag{7}
$$

as the following composition of isomorphisms:

$$
(\hat{\pi}_-)_1^* E|_\Sigma = p_1^* \pi_-^* E|_\Sigma = p_1^* \pi_+^* E|_\Sigma = p_2^* \pi_-^* E|_\Sigma = p_2^* \pi_+^* E|_\Sigma = (\hat{\pi}_+)_1^* E|_\Sigma
$$

Set $\hat{\mathcal{T}} := \bigsqcup_{k \geq 0} \hat{\mathcal{T}}_k$. Define $\hat{\pi}_\pm : \hat{\mathcal{T}} \to \Sigma$ by $(\hat{\pi}_\pm)_k$ on $\hat{\mathcal{T}}_k$. For connected components $S, S' \subseteq \Sigma$ introduce a topology on $\hat{\mathcal{T}}(S, S') := \hat{\pi}_-^{-1}(S) \cap \hat{\pi}_+^{-1}(S')$ by parameterizing the possibly broken trajectories with the help of f and taking the topology induced from $C^0([a,b], M)$, where $a = f(S')$, $b = f(S)$. Topologize $\hat{\mathcal{T}} = \bigsqcup_{S, S' \subseteq \Sigma} \hat{\mathcal{T}}(S, S')$ as the disjoint union.

Proposition 2. *The space $\hat{\mathcal{T}}$ is compact.*

Proof. Clearly it suffices to show that $\hat{\mathcal{T}}(S, S')$ is compact, for Σ has only finitely many connected components. The compactness of $\hat{\mathcal{T}}(S, S')$ follows from the theorem of Arzela–Ascoli. Indeed, the subset $\hat{\mathcal{T}}(S, S') \subseteq C^0([a,b], M)$ is equicontinuous. For $t \in [a,b]$ which is non-critical this follows from the fact that these are flow lines of a locally non-vanishing vector field. For critical t one has to use the very explicit form of the singularities of X, cf. [BH01].

To see that $\hat{\mathcal{T}}(S, S') \subseteq C^0([a,b], M)$ is closed suppose $\sigma_k \in \hat{\mathcal{T}}(S, S')$ converges uniformly to $\sigma \in C^0([a,b], M)$. Then σ too is parametrized by f, can only pass through finitely many critical points, departs at S, ends at S' and away from the critical points σ too is a flow line of $\frac{X}{|X|}$. Hence $\sigma \in \hat{\mathcal{T}}(S, S')$. $\square$

Recall that a smooth manifold with corners P of dimension n is a paracompact Hausdorff space modeled on $\{(x_1,\dots,x_n)|x_i\geq 0\}\subseteq\mathbb{R}^n$. The subset of points which in one (and hence every) chart have exactly k coordinates zero is called the *k–corner of P* and denoted by $\partial_k P$. It inherits the structure of a smooth manifold of dimension $n-k$. The subset $\partial P:=\partial_1 P\cup\dots\cup\partial_n P$ is closed and inherits the structure of a topological manifold. The pair $(P,\partial P)$ is a topological manifold with boundary.

Theorem 3. *Let X be a Morse–Bott–Smale vector field on M. There is a canonic way to equip the space $\hat{\mathcal{T}}$ with the structure of a (compact) smooth manifold with corners such that the following hold:*

(i) *The inclusion $\hat{\mathcal{T}}_k\subseteq\hat{\mathcal{T}}$ is a diffeomorphism onto the k–corner.*
(ii) *The mapping $\hat\pi_+:\hat{\mathcal{T}}\to\Sigma$ is smooth.*
(iii) *The mapping $\hat\pi_-:\hat{\mathcal{T}}\to\Sigma$ is a smooth fiber bundle whose fibers are (compact) smooth manifolds with corners.*
(iv) $\operatorname{rank}(V_{\hat{\mathcal{T}}})=\hat\pi_-^*\operatorname{ind}-\hat\pi_+^*\operatorname{ind}-1:\hat{\mathcal{T}}\to\mathbb{N}_0$, *where $V_{\hat{\mathcal{T}}}$ denotes the vertical bundle of $\hat\pi_-:\hat{\mathcal{T}}\to\Sigma$.*
(v) *If $X=-\operatorname{grad}_g(f)$ and $\hat{\mathcal{T}}(S,S')\neq\emptyset$ then $f(S)>f(S')$.*
(vi) *The isomorphism* (4) *extends (uniquely) to a smooth isomorphism of orientation bundles $\mathcal{O}_{V_{\hat{\mathcal{T}}}}\otimes\hat\pi_+^*\mathcal{O}_{N^-}=\hat\pi_-^*\mathcal{O}_{N^-}$. Via $\mathcal{O}_{V_{\hat{\mathcal{T}}}}|_{\hat{\mathcal{T}}_1}=\mathcal{O}_{V_{\hat{\mathcal{T}}_1}}$ its restriction to $\hat{\mathcal{T}}_1$ identifies with* (6)*.*
(vii) *If E is a flat vector bundle over M then the isomorphism* (5) *extends (uniquely) to a smooth isomorphism of flat vector bundles $\hat\pi_-^*E|_\Sigma=\hat\pi_+^*E|_\Sigma$. Its restriction to $\hat{\mathcal{T}}_1$ identifies with* (7)*.*

The proof of Theorem 3 is contained in Appendix A.

2.6. **Compactification of W^-.** For $k\geq 0$ define a smooth manifold

$$(\hat W^-)_k:=\underbrace{\mathcal{T}\times_\Sigma\mathcal{T}\times_\Sigma\cdots\times_\Sigma\mathcal{T}}_{k\text{ factors}}\times_\Sigma W^-$$

Here $\mathcal{T}\times_\Sigma W^-=\{(x,y)\in\mathcal{T}\times W^-\mid\pi_+(x)=p_-(y)\}$ and similarly for more factors. Define a smooth mapping $\hat i_k:(\hat W^-)_k\to M$ by composing the projection to the last factor with $i_-:W^-\to M$. Define a smooth fiber bundle $(\hat p_-)_k:(\hat W^-)_k\to\Sigma$ by composing the projection to the first factor with the mapping $\pi_-:\mathcal{T}\to\Sigma$ in the case $k\geq 1$ and composing with $p_-:W^-\to\Sigma$ if $k=0$. Note that by definition we have $(\hat W^-)_0=W^-$, $\hat i_0=i_-$ and $(\hat p_-)_0=p_-$.

Let $V_{(\hat{W}^-)_k}$ denote the vertical bundle of $(\hat{p}_-)_k : (\hat{W}^-)_k \to \Sigma$. Define an isomorphism of orientation bundles

$$(-)^{p_1^* \operatorname{rank}(V_{\mathcal{T}})} : \mathcal{O}_{V_{(\hat{W}^-)_1}} = (\hat{p}_-)_1^* \mathcal{O}_{N^-} \tag{8}$$

as follows. Consider the commutative diagram

$$\begin{tikzcd}
(\hat{W}^-)_1 = \mathcal{T} \times_\Sigma W^- \arrow[r, "p_2"] \arrow[d, "p_1"'] & W^- \arrow[r, "i_-"] \arrow[d, "p_-"] & M \\
\mathcal{T} \arrow[r, "\pi_+"] \arrow[d, "\pi_-"'] & \Sigma & \\
\Sigma & &
\end{tikzcd} \tag{9}$$

where p_1 and p_2 denote the projection to the first and second factor, respectively. Notice that $\hat{i}_1 = i_- \circ p_2$ and $(\hat{p}_-)_1 = \pi_- \circ p_1$. Clearly we have $p_2^* V_{W^-} = V_{p_1}$ and thus $p_2^* \mathcal{O}_{V_{W^-}} = \mathcal{O}_{V_{p_1}}$. Moreover the short exact sequence

$$0 \to V_{p_1} \to V_{(\hat{W}^-)_1} \to p_1^* V_{\mathcal{T}} \to 0$$

provides $\mathcal{O}_{V_{p_1}} \otimes p_1^* \mathcal{O}_{V_{\mathcal{T}}} = \mathcal{O}_{V_{(\hat{W}^-)_1}}$. Finally recall (1) and (4). Now define the isomorphism (8) as $(-)^{p_1^* \operatorname{rank}(V_{\mathcal{T}})}$ times the following composition of isomorphisms:

$$\begin{aligned}
\mathcal{O}_{V_{(\hat{W}^-)_1}} &= \mathcal{O}_{V_{p_1}} \otimes p_1^* \mathcal{O}_{V_{\mathcal{T}}} \\
&= p_2^* \mathcal{O}_{V_{W^-}} \otimes p_1^* \mathcal{O}_{V_{\mathcal{T}}} \\
&= p_2^* p_-^* \mathcal{O}_{N^-} \otimes p_1^* \mathcal{O}_{V_{\mathcal{T}}} \\
&= p_1^* (\pi_+^* \mathcal{O}_{N^-} \otimes \mathcal{O}_{V_{\mathcal{T}}}) \\
&= p_1^* \pi_-^* \mathcal{O}_{N^-} \\
&= (\hat{p}_-)_1^* \mathcal{O}_{N^-}
\end{aligned}$$

Suppose E is a flat vector bundle over M. In a similar manner, using (2) and (5), define an isomorphism of flat vector bundles

$$\hat{i}_1^* E = (\hat{p}_-)_1^* E|_\Sigma \tag{10}$$

as the following composition of isomorphisms:

$$\hat{i}_1^* E = p_2^* i_-^* E = p_2^* p_-^* E|_\Sigma = p_1^* \pi_+^* E|_\Sigma = p_1^* \pi_-^* E|_\Sigma = (\hat{p}_-)_1^* E|_\Sigma$$

Set $\hat{W}^- := \bigsqcup_{k \geq 0} (\hat{W}^-)_k$. Define $\hat{i} : \hat{W}^- \to M$ using $\hat{i}_k$ on $(\hat{W}^-)_k$. Define $\hat{p}_- : \hat{W}^- \to \Sigma$ using $(\hat{p}_-)_k$ on $(\hat{W}^-)_k$. Let $S \subseteq \Sigma$ be a connected component. We are going to topologize $\hat{W}_S^- := \hat{p}_-^{-1}(S)$. Set $b := f(S)$ and $a := \min\{f(x) \mid x \in M\}$. Every element $x \in \hat{W}_S^-$ can be considered as an unparametrized path σ from S to $\hat{\imath}(x)$ which we parametrize with the help of f, i.e. $\sigma \in C^0([a,b], M)$

$$f(\sigma(t)) = \begin{cases} a+b-t & \text{for } a \le t \le a+b-f(\hat{\imath}(x)) \text{ and} \\ f(\hat{\imath}(x)) & \text{for } a+b-f(\hat{\imath}(x)) \le t \le b. \end{cases}$$

We equip $\hat{W}_S^-$ with the topology induced from $C^0([a,b], M)$ and topologize $\hat{W}^- = \bigsqcup_{S\subseteq\Sigma} \hat{W}_S^-$ as the disjoint union.

Proposition 3. *The space $\hat{W}^-$ is compact.*

The proof of Proposition 3 is exactly the same as the proof of Proposition 2.

Theorem 4. *Let X be a Morse–Bott–Smale vector field on M. There is a canonic way to equip the space $\hat{W}^-$ with the structure of a (compact) smooth manifold with corners such that the following hold:*

(i) *The inclusion $(\hat{W}^-)_k \subseteq \hat{W}^-$ is a diffeomorphism onto the k–corner of $\hat{W}^-$.*
(ii) *The mapping $\hat{\imath} : \hat{W}^- \to M$ is smooth.*
(iii) *The mapping $\hat{p}_- : \hat{W}^- \to \Sigma$ is a smooth fiber bundle whose fibers are (compact) smooth manifolds with corners.*
(iv) $\operatorname{rank}(V_{\hat{W}^-}) = \hat{p}_-^* \operatorname{ind} : \hat{W}^- \to \mathbb{N}_0$, *where $V_{\hat{W}^-}$ denotes the vertical bundle of $\hat{p}_- : \hat{W}^- \to \Sigma$.*
(v) *If $X = -\operatorname{grad}_g(f)$ then $\hat{f}(\hat{W}_S^-) \le f(S)$ for all connected components $S \subseteq \Sigma$, where $\hat{f} := f \circ \hat{\imath} : \hat{W}^- \to \mathbb{R}$.*
(vi) *The isomorphism* (1) *extends (uniquely) to a smooth isomorphism of orientation bundles $\mathcal{O}_{V_{\hat{W}^-}} = \hat{p}_-^* \mathcal{O}_{N^-}$. Via $\mathcal{O}_{V_{\hat{W}^-}}|_{(\hat{W}^-)_1} = \mathcal{O}_{V_{(\hat{W}^-)_1}}$ its restriction to $(\hat{W}^-)_1$ identifies with* (8).
(vii) *If E is a flat vector bundle over M then the isomorphism* (2) *extends (uniquely) to a smooth isomorphism of flat vector bundles $\hat{\imath}^* E = \hat{p}_-^* E|_\Sigma$. Its restriction to $(\hat{W}^-)_1$ identifies with* (10).

The proof of Theorem 4 is contained in Appendix A.

3. The geometric complex

3.1. **Fiber integration.** We will heavily use fiber integration. So let us fix a sign convention, which will be the same as in [BT82]. Suppose $\pi : F \to B$ is a smooth fiber bundle. Here F, B and the fibers $F_x = \pi^{-1}(x)$ may have corners and for simplicity we assume the fibers to be compact. However none of these manifolds is assumed to be connected or of constant dimension.[8] Let $V_F \to F$ denote the vertical bundle of π. Finally suppose we have a flat vector bundle $E \to B$. Given a differential form $\alpha \in \Omega(F; \pi^* E \otimes \mathcal{O}_{V_F})$ we are going to define a form $\pi_*\alpha \in \Omega(B; E)$ as follows. Suppose $x \in B$ and $X_1, \dots, X_k \in T_xB$. Then the degree k component of $\pi_*\alpha \in \Omega(B; E)$ is defined by:

$$(\pi_*\alpha)(X_1, \dots, X_k) := \int_{F_x} i_{\tilde{X}_k} \cdots i_{\tilde{X}_1} \alpha$$

Here $\tilde{X}_i$ are vector fields on the total space of F defined along the fiber F_x which project to X_i. The pull back of $i_{\tilde{X}_k} \cdots i_{\tilde{X}_1} \alpha$ to the fiber F_x is a form $\Omega(F_x; \pi^* E_x \otimes \mathcal{O}_{F_x})$ which can be integrated over F_x to yield an element of E_x. The outcome does not depend on the choice of $\tilde{X}_i$. For the fiber bundle $\pi : F \to B$ the fiber integration

$$\pi_* : \Omega(F; \pi^* E \otimes \mathcal{O}_{V_F}) \to \Omega(B; E)$$

has the following properties which follow easily from our definition.

Suppose $f : B' \to B$ is a smooth mapping. Then we have a pull back bundle $f^*\pi : f^*F \to B'$ and a smooth mapping $\pi^*f : f^*F \to F$ which restricts to a diffeomorphism $(f^*F)_x \to F_{f(x)}$ for every $x \in B'$.

$$\begin{tikzcd} f^*F \arrow[r, "\pi^*f"] \arrow[d, "f^*\pi"'] & F \arrow[d, "\pi"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

Hence its tangent mapping induces an isomorphism of vector bundles $V_{f^*F} = (\pi^*f)^*V_F$ over the total space of f^*F. Particularly we get a canonic isomorphism of orientation bundles $\mathcal{O}_{f^*V_F} = (\pi^*f)^* \mathcal{O}_{V_F}$ and thus a canonic isomorphism of flat vector bundles $(\pi^*f)^*(\pi^*E \otimes \mathcal{O}_{V_F}) = (f^*\pi)^* f^*E \otimes \mathcal{O}_{V_{f^*F}}$. Using this identification we have

$$(f^*\pi)_* \circ (\pi^*f)^* = f^* \circ \pi_* : \Omega(F; \pi^*E \otimes \mathcal{O}_{V_F}) \to \Omega(B'; f^*E)$$

the *naturality of fiber integration.*

Suppose $\tilde{\pi} : \tilde{F} \to F$ and $\pi : F \to B$ are two fiber bundles. Then $\pi \circ \tilde{\pi} : \tilde{F} \to B$ is a fiber bundle too and we have a canonic short exact sequence of vector bundles $0 \to V_{\tilde{\pi}} \to V_{\pi\circ\tilde{\pi}} \to \tilde{\pi}^* V_\pi \to 0$ over $\tilde{F}$. This provides us with a canonic isomorphism of orientation bundles $\mathcal{O}_{V_{\pi\circ\tilde{\pi}}} = \mathcal{O}_{V_{\tilde{\pi}}} \otimes \tilde{\pi}^* \mathcal{O}_{V_\pi}$ and thus with a canonic isomorphism $(\pi \circ \tilde{\pi})^* E \otimes \mathcal{O}_{V_{\pi\circ\tilde{\pi}}} = \tilde{\pi}^*(\pi^* E \otimes \mathcal{O}_{V_\pi}) \otimes \mathcal{O}_{V_{\tilde{\pi}}}$. Via this identification we have

$$\pi_* \circ \tilde{\pi}_* = (\pi \circ \tilde{\pi})_* : \Omega(\tilde{F}; (\pi \circ \tilde{\pi})^* E \otimes \mathcal{O}_{V_{\pi\circ\tilde{\pi}}}) \to \Omega(B; E)$$

[8] Thus even if we fiber integrate a form of homogeneous degree the outcome need not be of homogeneous degree.

the (covariant) *functoriality of fiber integration.*

Suppose $\pi : F \to B$ is a fiber bundle and suppose E' and E are two flat vector bundles over B. For $\alpha \in \Omega(F; \pi^* E \otimes \mathcal{O}_{V_F})$ and $\beta \in \Omega(B; E')$ we have

$$\pi_*(\pi^* \beta \wedge \alpha) = \beta \wedge \pi_* \alpha \in \Omega(B; E' \otimes E)$$

a *push–pull formula.* Here we only use the natural isomorphism $\pi^* E' \otimes \pi^* E = \pi^*(E' \otimes E)$.

Suppose $F \to B$ is a fiber bundle and suppose B has no boundary. Then the 1–corner $\pi|_{\partial_1 F} : \partial_1 F \to B$ is a smooth fiber bundle with possibly non-compact fibers. Using the *outward pointing normal first* convention we get a canonic isomorphism of orientation bundles $\mathcal{O}_{V_F}|_{\partial_1 F} = \mathcal{O}_{V_{\partial_1 F}}$ and thus a canonic isomorphism of flat vector bundles $(\pi^* E \otimes \mathcal{O}_{V_F})|_{\partial_1 F} = (\pi|_{\partial_1 F})^* E \otimes \mathcal{O}_{V_{\partial_1 F}}$. Note that this is the same isomorphisms as the one we get from the short exact sequence

$$0 \to V_{\partial_1 F} \to V_F|_{\partial_1 F} \to V_F|_{\partial_1 F}/V_{\partial_1 F} \to 0$$

with the right hand side quotient oriented by the outward pointing normal. Via this identification we find

$$\pi_* \circ d = d \circ \pi_* + A \circ (\pi|_{\partial_1 F})_* : \Omega(F; \pi^* E \otimes \mathcal{O}_{V_F}) \to \Omega(B; E)$$

where $A := (-)^k : \Omega^k(B; E) \to \Omega^k(B; E)$ is a grading homomorphism.

3.2. **Integration and the differential.** Let X be a Morse–Bott–Smale vector field on a closed manifold M, cf. Definition 3. Let E be a flat vector bundle over M. Define a vector space

$$C(X; E) := \Omega(\Sigma; E|_\Sigma \otimes \mathcal{O}_{N^-})$$

and an integration

$$\mathrm{Int} := (\hat{p}_-)_* \hat{i}^* = (p_-)_* i_-^* : \Omega(M; E) \to C(X; E).$$

Note that for $\alpha \in \Omega(M; E)$ we have $\hat{i}^* \alpha \in \Omega(\hat{W}^-; \hat{i}^* E)$ which via the isomorphisms in Theorem 4(vi) and (vii) can be interpreted as an element of $\Omega(\hat{W}^-; \hat{p}_-^* E|_\Sigma \otimes \hat{p}_-^* \mathcal{O}_{N^-} \otimes \mathcal{O}_{V_{\hat{W}^-}})$ and thus $(\hat{p}_-)_* \hat{i}^* \alpha$ is indeed an element of $\Omega(\Sigma; E|_\Sigma \otimes \mathcal{O}_{N^-}) = C(X; E)$. Similarly, using the isomorphisms in Theorem 3(vi) and (vii), we define

$$u := (\hat{\pi}_-)_* \hat{\pi}_+^* = (\pi_-)_* \pi_+^* : C(X; E) \to C(X; E)$$

and

$$\delta := d + uA : C(X; E) \to C(X; E)$$

where $A := (-)^k : \Omega^k(\Sigma; E|_\Sigma \otimes \mathcal{O}_{N^-}) \to \Omega^k(\Sigma; E|_\Sigma \otimes \mathcal{O}_{N^-})$ is a grading homomorphism, cf. section 3.1.

Proposition 4. $\mathrm{Int} \circ d = \delta \circ \mathrm{Int}$.

Proof. From section 3.1 we obtain $(\hat{p}_-)_*d = d(\hat{p}_-)_* + A((\hat{p}_-)|_{\partial_1\hat{W}^-})_*$ and thus:

$$\operatorname{Int}\circ d = d\circ\operatorname{Int} + A((\hat{p}_-)|_{\partial_1\hat{W}^-})_*\hat{i}^*$$

Using the description of the 1–boundary of $\hat{W}^-$ from Theorem 4 and the obvious relation $(\pi_-)_*(-)^{\operatorname{rank}(V_{\mathcal{T}})}\pi_+^* = AuA$ we get:

$$\begin{aligned}
((\hat{p}_-)|_{\partial_1\hat{W}^-})_*\hat{i}^* &= (\pi_-\circ p_1)_*(-)^{p_1^*\operatorname{rank}(V_{\mathcal{T}})}(i_-\circ p_2)^* \\
&= (\pi_-)_*(-)^{\operatorname{rank}(V_{\mathcal{T}})}(p_1)_*p_2^*i_-^* \\
&= (\pi_-)_*(-)^{\operatorname{rank}(V_{\mathcal{T}})}\pi_+^*(p_-)_*i_-^* \\
&= AuA\operatorname{Int}
\end{aligned}$$

Therefore $\operatorname{Int} d = d\operatorname{Int} + A^2uA\operatorname{Int} = (d+uA)\operatorname{Int} = \delta\operatorname{Int}$. □

Since the integration $\operatorname{Int} : \Omega(M;E) \to C(X;E)$ is onto we must have $\delta^2 = 0$ as a consequence of $d^2 = 0$. However, we will give a direct argument below.

Proposition 5. $\delta^2 = 0$.

Proof. Exactly as in the proof of Proposition 4 we derive $ud = du + uAu$ using the description of the 1–boundary of $\hat{\mathcal{T}}$ from Theorem 3. Then $\delta^2 = duA + uAd + uAuA = (du - ud + uAu)A = 0$, where we also made use of the obvious $dA = -Ad$. □

Definition 4 (Geometric complex)**.** The complex $(C(X;E),\delta)$ is called the *geometric complex with values in the flat vector bundle E*. It is a generalization of the deRham complex and the classical Morse complex at the same time. The homomorphism of chain complexes $\operatorname{Int} : \Omega(M;E) \to C(X;E)$ is called the *integration homomorphism.*

3.3. **Grading and filtration.** The geometric complex $C(X;E)$ can be graded and filtered in a compatible way. For a connected component $S \subseteq \Sigma$ let us write $E_S := E_\Sigma|_S = (E|_\Sigma \otimes \mathcal{O}_{N^-})|_S$. Define

$$C_p^q(X;E) := \bigoplus_{\operatorname{ind}(S)\geq p} \Omega^{q-\operatorname{ind}(S)}(S;E_S)$$

where the direct sum is over all connected components $S \subseteq \Sigma$ of index larger or equal to p. Let us also introduce the notation $C^q(X;E) := C_0^q(X;E)$, and note that $\bigoplus_q C^q(X;E) = C(X;E)$. Finally let us define $C_p(X;E) = \bigoplus_q C_p^q(X;E)$. Clearly $C_p^q(X;E) = C^q(X;E) \cap C_p(X;E)$. Of course the grading and the filtration stem from a bigrading on $C(X;E)$. However, since the bigrading will not be compatible with the differential we will not consider it in the sequel.

Proposition 6. *The differential of the geometric complex preserves the grading and the filtrations as indicated below:*

$$\delta : C^q_p(X;E) \to C^{q+1}_p(X;E)$$

Proof. This is obviously true for the deRham differential. So it suffices to consider $u = (\pi_-)_*(\pi_+)^*$. Let α' be a form of degree $|\alpha'|$ supported on a connected component $S' \subseteq \Sigma$. Let $S \subseteq \Sigma$ be another connected component and let α denote the restriction of $(\pi_-)_*(\pi_+)^*\alpha'$ to S. That is we integrate $(\pi_+)^*\alpha'$ only over $\mathcal{T}(S,S')$. We may assume $\mathcal{T}(S,S') \neq \emptyset$. Since $\operatorname{rank}(V_{\mathcal{T}(S,S')}) = \operatorname{ind}(S) - \operatorname{ind}(S') - 1$, see Theorem 3(iv), the degree of α will be $|\alpha'| - \operatorname{ind}(S) + \operatorname{ind}(S') + 1$. This shows $\delta : C^q \to C^{q+1}$. Since it is the rank of a vector bundle we must have $\operatorname{ind}(S) - \operatorname{ind}(S') - 1 \geq 0$ and thus $\delta : C_p \to C_p$. □

A Morse–Bott–Smale vector field on M also provides a filtration of $\Omega^*(M)$ as follows. For a connected component $S \subseteq \Sigma$ let $\hat{W}^-_S$ denote its compactified unstable manifold and let $\hat{i}_S : \hat{W}^-_S \to M$ denote the restriction of the smooth map $\hat{i} : \hat{W}^- \to M$, cf. Theorem 4. Let $\Omega^q_p(M;E)$ denote the space of forms $\alpha \in \Omega^q(M;E)$ which vanish on some open neighborhood of $M(p) := \bigcup_{\operatorname{ind}(S)<p} \hat{i}_S(\hat{W}^-_S)$. Clearly $\Omega^*_{p+1}(M;E) \subseteq \Omega^*_p(M;E)$ and this filtration is preserved by the deRham differential. Note that $(\Omega^*_p(M;E), d)$ computes the relative cohomology $H^*(M, M(p); E)$.

Proposition 7. *The integration preserves the grading and the filtration as indicated below.*

$$\operatorname{Int} : \Omega^q_p(M;E) \to C^q_p(X;E)$$

Proof. Let α' be a form on M of homogeneous degree $|\alpha'|$ and let $S \subseteq \Sigma$ be a connected component. Let α denote the restriction of $(p_-)_* i^*\alpha'$ to S. That is we only integrate over W^-_S. Since $\operatorname{rank}(V_{W^-_S}) = \operatorname{ind}(S)$, see Theorem 4(iv) we must have $|\alpha| = |\alpha'| - \operatorname{ind}(S)$. This implies $\operatorname{Int} : \Omega^q \to C^q$. The rest is obvious. □

3.4. **Cohomology of the geometric complex.** Let us consider the filtered graded complex $C^q_p(X;E)$, see Proposition 6. It gives rise to a spectral sequence $(E_k C^q_p(X;E), \delta_k)$. Since $C^*_0(X;E) = C^*(X;E)$ and $C^*_{\dim M+1}(X;E) = 0$ the spectral sequence will converge to $HC^*(X;E)$. Moreover, for the E_1–term we obviously have $E_1 C^q_p(X;E) = H^{q-p}(\Sigma_p; E_{\Sigma_p})$. Here $\Sigma_p = \bigsqcup_{\operatorname{ind}(S)=p} S$ denotes the critical manifold of index p and $E_{\Sigma_p} = E_\Sigma|_{\Sigma_p} = (E|_\Sigma \otimes \mathcal{O}_{N^-})|_{\Sigma_p}$. This proves the second part of Theorem 1.

Via this identification the differential on the E_1–term has a particularly simple form. All components but

$$\delta_1 : H^{q-p}(\Sigma_p; E_{\Sigma_p}) \to H^{q-p}(\Sigma_{p+1}; E_{\Sigma_{p+1}}) \tag{11}$$

vanish, and (11) is given by $\delta_1 = (-)^{q-p}(\hat\pi_-)_* \circ (\hat\pi_+)^*$. Here

$$(\hat\pi_+)^* : H^*(\Sigma_p; E_{\Sigma_p}) \to H^*\big(\hat{\mathcal{T}}(\Sigma_{p+1}, \Sigma_p); (\hat\pi_+)^*E\big)$$

is given by pull back along $\hat\pi_+ : \hat{\mathcal{T}}(\Sigma_{p+1}, \Sigma_p) \to \Sigma_p$, and

$$(\hat\pi_-)_* : H^*\big(\hat{\mathcal{T}}(\Sigma_{p+1}, \Sigma_p); (\hat\pi_+)^*E\big) \to H^*(\Sigma_{p+1}; E_{\Sigma_{p+1}})$$

is the map induced by fiber integration along $\hat\pi_- : \hat{\mathcal{T}}(\Sigma_{p+1}, \Sigma_p) \to \Sigma_{p+1}$. For this note that since the index difference is one, the fibers of the bundle $\hat\pi_- : \hat{\mathcal{T}}(\Sigma_{p+1}, \Sigma_p) \to \Sigma_{p+1}$ are zero dimensional, hence there is no boundary, and fiber integration does indeed induce a mapping in cohomology.

Remark 3. The differentials δ_k for $k \geq 1$ should be considered as generalization of the classical *counting of instantons*. These are differentials of finite dimensional complexes and can be considered the *combinatorics of the dynamic*. If X is a Morse–Smale vector field then (E_1C^*, δ_1) is the familiar Morse complex and $\delta_k = 0$ for all $k \geq 2$.

Recall the filtration $\Omega^q_p(M; E)$ introduced in section 3.3. Clearly we have $\Omega^*_0(M; E) = \Omega^*(M; E)$ and $\Omega^*_{\dim M+1}(M; E) = 0$. So the associated spectral sequence $(E_k\Omega^*(M; E), \delta_k)$ will converge to $H^*(M; E)$. Let $W^-_{\Sigma_p}$ denote the unstable manifold of the critical manifold of index p. Using the description of the boundary of $\hat W^-_{\Sigma_p}$ in Theorem 4 and the fact that $i : W^-_{\Sigma_p} \to M$ is an embedding we obtain a homomorphism

$$\Omega^*_p(M; E)/\Omega^*_{p+1}(M; E) \xrightarrow{i^*} \Omega^*_c(W^-_{\Sigma_p}; i^*E) \tag{12}$$

where the latter denotes forms with compact support. This is a quasi isomorphism, i.e. induces an isomorphism in cohomology, since $\Omega^*_p(M; E)$ computes $H^*(M, M(p); E)$. It follows from the Thom isomorphism theorem that $\mathrm{Int} : E_1\Omega^q_p(M; E) \to E_1C^q_p(X; E)$, cf. Proposition 7, is an isomorphism. The first part of Theorem 1 now follows from a standard spectral sequence argument.

Remark 4. Note that this argument actually shows that

$$\mathrm{Int} : (E_k\Omega^*_p(M; E), \delta_k) \to (E_kC^*_p(X; E), \delta_k)$$

is an isomorphism for all $k \geq 1$. That is, the spectral sequences induced by $\Omega^*_p(M; E)$ and $C^*_p(X; E)$ are isomorphic.

Corollary 1. *For the Euler characteristics we have:*

$$\chi(M) = \sum_q (-)^q \chi(\Sigma_q) \tag{13}$$

Proof. The Euler characteristics $\chi(M)$ equals the Euler characteristics of $H^*(M)$ which equals the Euler characteristics of the geometric complex $\chi(HC^*)$ by Theorem 1(i). Since the spectral sequence $E_k C^*$ converges to HC^* we get $\chi(HC^*) = \chi(E_1 C^*)$ which implies (13) in view of Theorem 1(ii). □

Remark 5. Suppose X is a vector field whose zero set Σ is a closed submanifold but not necessarily non-degenerate. For every connected component $S \subseteq \Sigma$ one can define an index $\mathrm{IND}(S) = \mathrm{IND}_X(S)$ generalizing the Hopf index, see [D95], and one has $\chi(M) = \sum_{S \subseteq \Sigma} \mathrm{IND}(S)$. If the vector field $-X$ is Morse–Bott then the indices are related by $\mathrm{IND}_X(S) = (-)^{\mathrm{ind}_{-X}(S)} \chi(S)$. This also implies Corollary 1 without using the Smale condition. Finally notice that if the zero set of X is degenerate $\mathrm{IND}(S)$ may be non-zero even if $\chi(S) = 0$.

The generalized Hopf index can easily be defined as follows. Let U be a connected open neighborhood of S. Then the vector field provides a mapping $X : (U, U \setminus S) \to (TU, TU \setminus U)$. The Hopf index then is the unique integer which makes the following diagram commutative:

$$\begin{tikzcd} H^n(TU, TU \setminus U; \pi^*_{TU}\, \mathcal{O}_U) \arrow[r, "X^*"] \arrow[d, equal, "\mathrm{Thom}"'] & H^n(U, U \setminus S; \mathcal{O}_U) \arrow[d, equal, "\mathrm{Thom}"] \\ H^0(U) = \mathbb{R} \arrow[r, "\mathrm{IND}(S)"] & \mathbb{R} = H^{\dim(S)}(S; \mathcal{O}_S) \end{tikzcd}$$

Corollary 2 (Morse inequalities)**.** *For every $q_0 \in \mathbb{Z}$ we have*

$$\sum_{q \geq q_0} (-)^{q-q_0} b^q(M) \leq \sum_{q \geq q_0} (-)^{q-q_0} \sum_p b^{q-p}(\Sigma_p)$$

where $b^q(M) := \dim H^q(M; E)$ and $b^q(\Sigma_p) = \dim H^q(\Sigma_p; E_{\Sigma_p})$.

Proof. Recall that for a finite dimensional complex C^* and any $q_0 \in \mathbb{Z}$ we have:

$$\sum_{q \geq q_0} (-)^{q-q_0} \dim HC^q \leq \sum_{q \geq q_0} (-)^{q-q_0} \dim C^q$$

Applying this to the complex $(E_k C^*(X; E), \delta_k)$ we obtain

$$\sum_{q \geq q_0} (-)^{q-q_0} \dim E_{k+1} C^q \leq \sum_{q \geq q_0} (-)^{q-q_0} \dim E_k C^q$$

which in turn implies:

$$\sum_{q\geq q_0}(-)^{q-q_0}\dim E_\infty C^q \leq \sum_{q\geq q_0}(-)^{q-q_0}\dim E_1 C^q$$

The statement now follows from Theorem 1. □

4. Torsion

4.1. **Euler structures.** The importance of Euler structures for torsion comes from the fact that they permit to remove the metric ambiguity of analytic and geometric torsion, see below. In the case $\chi(M)=0$ Euler structures where introduced by Turaev [T90] for exactly this purpose. A detailed discussion of the general case can be found in [BH]. In this paper we will only consider Euler structures over $\mathbb{R}$. Hence the space of Euler structures will be an affine version of $H_1(M;\mathbb{R})$, i.e. $H_1(M;\mathbb{R})$ acts free and transitively on the set of Euler structures.

In [BH] the set of Euler structures $\mathfrak{Eul}_{x_0}(M)$ based at $x_0 \in M$ was defined as the set of equivalence classes $[X, c]$, where X is a vector field with non-degenerate zeros and $c \in C_1(M;\mathbb{R})$ is a singular one chain with $\partial c = \sum_{x\in\mathcal{X}} \mathrm{IND}(x)x - \chi(M)x_0$. Here $\mathcal{X}$ denotes the (finite) set of zeros of X and $\mathrm{IND}(x) = \mathrm{IND}_X(x)$ its Hopf index at x. Notice that $\mathrm{IND}(x) = \pm 1$ for we assume X to be non-degenerate. Every such chain c is called *Euler chain* or *Turaev spider* for X based at x_0. Since $\sum_{x\in\mathcal{X}} \mathrm{IND}(x) = \chi(M)$ every such vector field admits Euler chains. By definition $[X_1, c_1] = [X_2, c_2]$ iff $c_2 - c_1 = c(X_1, X_2)$ modulo boundaries. Here $c(X_1, X_2)$ is the singular one chain obtained from the zero set of a non-degenerate homotopy from X_1 to X_2. Note that

$$\partial c(X_1, X_2) = \sum_{x\in\mathcal{X}_2} \mathrm{IND}_{X_2}(x)x - \sum_{x\in\mathcal{X}_1} \mathrm{IND}_{X_1}(x)x$$

and that changing the homotopy changes $c(X_1, X_2)$ by a boundary only. The action of $[\sigma] \in H_1(M;\mathbb{R})$ on $[X, c] \in \mathfrak{Eul}_{x_0}(M)$ is defined by $[X, c] + [\sigma] := [X, c + \sigma]$. This action clearly is free and transitive. If x_1 is another base point and τ a path from x_0 to x_1 then

$$[X, c] \mapsto [X, c] - \chi(M)\tau := [X, c - \chi(M)\tau] \tag{14}$$

defines an $H_1(M;\mathbb{R})$–equivariant (affine) bijection $\mathfrak{Eul}_{x_0}(M) \to \mathfrak{Eul}_{x_1}(M)$. Clearly this does only depend on the homology class of the path from x_0 to x_1. If $\chi(M) = 0$ it does not depend on the base point at all and the set of Euler structures is defined without reference to a base point, cf. [T90].

We will also use another way to represent Euler structures which is better suited to remove ambiguities from the analytic torsion. For a base point $x_0 \in M$ define $\mathfrak{Eul}^*_{x_0}(M)$ as the set of equivalence classes $[g, \alpha]$, where g is a Riemannian metric on M and $\alpha \in \Omega^{\dim M-1}(M \setminus \{x_0\}; \mathcal{O}_M)$ with $d\alpha = E_g$, where $E_g \in \Omega^{\dim M}(M; \mathcal{O}_M)$ denotes the Euler form of g. By definition $[g_1, \alpha_1] = [g_2, \alpha_2]$ iff $\alpha_2 - \alpha_1 = \mathrm{cs}(g_1, g_2)$ modulo $d\big(\Omega^{\dim M-1}(M; \mathcal{O}_M)\big)$, where $\mathrm{cs}(g_1, g_2)$ denotes the Chern–Simon invariant. There is a free and transitive action of $H^{\dim M-1}(M; \mathcal{O}_M)$ on $\mathfrak{Eul}^*_{x_0}$ defined by $[g, \alpha] + [\beta] := [g, \alpha - \beta]$. Recall that Poincaré duality provides an isomorphism $H_1(M; \mathbb{R}) = H^{\dim M-1}(M; \mathcal{O}_M)$, where $[\sigma] \in H_1(M; \mathbb{R})$ corresponds to $[\beta] \in H^{\dim M-1}(M; \mathcal{O}_M)$ iff $\int_\sigma \omega = \int_M \omega \wedge \beta$ for all closed one forms $\omega \in \Omega^1(M; \mathbb{R})$. There is a canonic affine isomorphism $\mathfrak{Eul}_{x_0}(M) = \mathfrak{Eul}^*_{x_0}(M)$ which can be regarded as affine version of Poincaré duality. To define it we need two preliminary propositions. For details see [BH].

For a closed one form $\omega \in \Omega^1(M; \mathbb{R})$, a Riemannian metric g, a form $\alpha \in \Omega^{\dim M-1}(M \setminus \{x_0\}; \mathcal{O}_M)$ with $d\alpha = E_g$ on $M \setminus \{x_0\}$ and a smooth function $f : M \to \mathbb{R}$ such that $\omega' := \omega - df$ vanishes in a neighborhood of x_0 define:

$$\mathcal{S}(\omega, g, \alpha; f) := \int_{M \setminus \{x_0\}} \omega' \wedge \alpha - \int_M f E_g + \chi(M) f(x_0)$$

Proposition 8. *The quantity $\mathcal{S}(\omega, g, \alpha; f)$ does not depend on the choice of f and will be denoted by $\mathcal{S}(\omega, g, \alpha)$. It should be considered a regularization of the possibly divergent integral $\int_{M \setminus \{x_0\}} \omega \wedge \alpha$. It has the following properties:*

(i) $\mathcal{S}(\omega, g, \alpha)$ *is linear in* ω.
(ii) $\mathcal{S}(dh, g, \alpha) = -\int_M h E_g + \chi(M) h(x_0)$ *for all smooth functions* $h : M \to \mathbb{R}$.
(iii) $\mathcal{S}(\omega, g, \alpha + \beta) - \mathcal{S}(\omega, g, \alpha) = \int_M \omega \wedge \beta$ *for all closed* $\beta \in \Omega^{\dim M-1}(M; \mathcal{O}_M)$.
(iv) *If* $[\alpha_1, g_1] = [\alpha_2, g_2] \in \mathfrak{Eul}^*_{x_0}(M)$ *then* $\mathcal{S}(\omega, g_2, \alpha_2) - \mathcal{S}(\omega, g_1, \alpha_1) = \int_M \omega \wedge \mathrm{cs}(g_1, g_2)$ *for all closed one forms* ω.

Proof. The independence of f follows from a straight forward calculation using $d\alpha = E_g$, $\int_M E_g = \chi(M)$ and Stokes' theorem. All of the properties are obvious, except (iv). For this one use $d\,\mathrm{cs}(g_1, g_2) = E_{g_2} - E_{g_1}$ and Stokes' theorem. For details see [BH, Lemma 2]. □

For a closed one form $\omega \in \Omega^1(M; \mathbb{R})$, a vector field X with non-degenerate zero set $\mathcal{X}$, a Riemannian metric g and a smooth function $f : M \to \mathbb{R}$ such that $\omega' := \omega - df$ vanishes in a neighborhood of $\mathcal{X}$

define:

$$\mathcal{R}(\omega, X, g; f) := \int_{M\setminus\mathcal{X}} \omega' \wedge X^*\Psi_g - \int_M fE_g + \sum_{x\in\mathcal{X}} \operatorname{IND}(x)f(x)$$

Here Ψ_g denotes the global angular form, cf. [BT82], also called Mathai–Quillen form in [BZ92]. Recall that Ψ_g is define as follows: Let $n = \dim(M)$ and let $\pi_{TM} : TM \to M$ be the tangent bundle equipped with the Levi–Civita connection. Let $\operatorname{vol}_g \in \Omega^n(TM; \pi_{TM}^* \mathcal{O}_M)$, the global volume form which vanishes when contracted with horizontal vectors on TM and which assigns to an n-tuple of vertical vectors 'their volume times their orientation'. Moreover let ξ denote the Euler vector field on TM which assigns to a point $v \in TM$ the vertical vector $-v \in T_v(TM)$. Then

$$\Psi_g := \frac{\Gamma(n/2)}{(2\pi)^{n/2}|\xi|^n} i_\xi \operatorname{vol}_g \in \Omega^{n-1}(TM \setminus M; \pi_{TM}^* \mathcal{O}_M).$$

Proposition 9. *The quantity $\mathcal{R}(\omega, X, g; f)$ does not depend on the choice of f and will be denoted by $\mathcal{R}(\omega, X, g)$.*[9] *It should be considered a regularization of the possibly divergent integral $\int_{M\setminus\mathcal{X}} \omega \wedge X^*\Psi_g$. It has the following properties:*

- (i) *$\mathcal{R}(\omega, X, g)$ is linear in ω.*
- (ii) *$\mathcal{R}(dh, X, g) = -\int_M hE_g + \sum_{x\in\mathcal{X}} \operatorname{IND}(x)h(x)$ for all smooth functions $h : M \to \mathbb{R}$.*
- (iii) *$\mathcal{R}(\omega, X, g_2) - \mathcal{R}(\omega, X, g_1) = \int_M \omega \wedge \operatorname{cs}(g_1, g_2)$ for two Riemannian metrics g_1 and g_2.*
- (iv) *$\mathcal{R}(\omega, X_2, g) - \mathcal{R}(\omega, X_1, g) = \int_{c(X_1,X_2)} \omega$ for two vector fields X_1 and X_2 with non-degenerate zeros.*

Proof. The independence of f follows from straight forward calculation using $d\Psi_g = E_g$, Stokes' theorem and $\lim_{\epsilon\to 0} \int_{\partial B_\epsilon(x)} X^*\Psi_g = -\operatorname{IND}(x)$ with $B_\epsilon(x)$ denoting the ϵ–ball around x. The first two properties are obvious. For (iii) use $\Psi_{g_2} - \Psi_{g_1} = \operatorname{cs}(g_1, g_2)$, $E_{g_2} - E_{g_1} = d\operatorname{cs}(g_1, g_2)$ and Stokes' theorem. Concerning (iv) it is easy to see that it suffices to consider the case where there is a homotopy $\mathbb{X}$ from X_1 to X_2 such that ω is exact on a neighborhood of the zero set of $\mathbb{X}$. Let $p : I \times M \to M$ denote the projection. The homotopy $\mathbb{X}$ provides a section of p^*TM which we may assume to be transversal to the zero section. Choose f such that $p^*\omega' = p^*\omega - p^*df$ vanishes on the zero set of $\mathbb{X}$. Then

$$\sum_{x\in\mathcal{X}_2} \operatorname{IND}_{X_2}(x)f(x) - \sum_{x\in\mathcal{X}_1} \operatorname{IND}_{X_1}(x)f(x) = \int_{c(X_1,X_2)} df = \int_{c(X_1,X_2)} \omega$$

[9] The quantity $\mathcal{R}(\omega, X, g)$ was considered by Bismut–Zhang [BZ92] for $X = -\operatorname{grad}_g(h)$ and h a Morse function.

and

$$\begin{aligned}\int_M \omega' \wedge X_2^*\Psi_g - \int_M \omega' \wedge X_1^*\Psi_g &= \int_{I\times M} d\big(p^*\omega' \wedge \mathbb{X}^*\tilde{p}^*\Psi_g\big)\\ &= -\int_{I\times M} p^*(\omega' \wedge E_g) = 0.\end{aligned}$$

Here $\tilde{p} : p^*TM \to TM$ denotes the canonic vector bundle homomorphism over $p : I \times M \to M$. More details can be found in [BH, Section 3]. $\square$

One can now define the affine isomorphism $\mathfrak{Eul}_{x_0}(M) = \mathfrak{Eul}^*_{x_0}(M)$ by saying that $[X, c]$ corresponds to $[g, \alpha]$ iff

$$\mathcal{R}(\omega, X, g) - \mathcal{S}(\omega, \alpha, g) - \int_c \omega = 0$$

for all closed one forms $\omega \in \Omega^1(M; \mathbb{R})$. In view of the last two Propositions this is indeed a well defined isomorphism affine over the Poincaré duality $H_1(M; \mathbb{R}) = H^{\dim M-1}(M; \mathcal{O}_M)$. We will from now on drop the notation $\mathfrak{Eul}^*_{x_0}(M)$ and feel free to represent an Euler structure $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M)$ as an equivalence class $[X, c]$ or as an equivalence class $[g, \alpha]$.

For the Morse–Bott case we introduce yet another way to represent Euler structures. This can be considered a localization of the Euler structure at the critical manifold of a Morse–Bott vector field. Fix a Morse–Bott vector field X with critical manifold Σ and define $\mathfrak{Eul}^X_{x_0}(M)$ as the set of equivalence classes $[X, c, \{\mathfrak{e}_S\}]$, where for every connected component $S \subseteq \Sigma$, $\mathfrak{e}_S \in \mathfrak{Eul}_{x_S}(S)$ with $x_S \in S$ and $c \in C_1(M; \mathbb{R})$ is a one chain with $\partial c = \sum_{S\subseteq\Sigma} \mathrm{IND}(S)x_S - \chi(M)x_0$. Since we have $\sum_{S\subseteq\Sigma} \mathrm{IND}(S) = \chi(M)$ such c indeed exist. Given two such data $(X, c^1, \{\mathfrak{e}^1_S\})$ and $(X, c^2, \{\mathfrak{e}^2_S\})$ with $\mathfrak{e}^i_S \in \mathfrak{Eul}_{x^i_S}(S)$ we choose paths τ_S in S from x^1_S to x^2_S. Then $\mathfrak{e}^1_S - \chi(S)\tau_S \in \mathfrak{Eul}_{x^2_S}(S)$ and hence $(\mathfrak{e}^1_S - \chi(S)\tau_S) - \mathfrak{e}^2_S \in H_1(S; \mathbb{R})$. We define $[X, c^1, \{e^1_S\}] = [X, c^2, \{e^2_S\}]$ if for one (and hence every) choice of paths τ_S as above we have

$$\sum_{S\subseteq\Sigma} (-)^{\mathrm{ind}_{-X}(S)}\big((\mathfrak{e}^1_S - \chi(S)\tau_S) - \mathfrak{e}^2_S\big) = c^2 - c^1 - \sum_{S\subseteq\Sigma} \mathrm{IND}(S)\tau_S$$

in $H_1(M; \mathbb{R})$. This is indeed independent of the paths $\{\tau_S\}$ for we have $\mathrm{IND}(S) = (-)^{\mathrm{ind}_{-X}(S)}\chi(S)$ for every $S \subseteq \Sigma$. For $[\sigma] \in H_1(M; \mathbb{R})$ the action is defined by $[X, c, \{\mathfrak{e}_S\}] + [\sigma] := [X, c + \sigma, \{\mathfrak{e}_S\}]$. Note that for $[\sigma_S] \in H_1(S; \mathbb{R})$ we have

$$[X, c, \{\mathfrak{e}_S + [\sigma_S]\}] = [X, c + \sum_{S\subseteq\Sigma} (-)^{\mathrm{ind}_{-X}(S)}\sigma_S, \{\mathfrak{e}_S\}] \tag{15}$$

Note how this specializes to the previous definition if X has non-degenerate zeros only, for in this case there is a unique base point and a unique Euler structure on every $S = \{\mathrm{pt}\}$. Also note how this localization degenerates for the Morse–Bott–Smale vector field $X \equiv 0$.

Suppose $[X, c, \{\mathfrak{e}_S\}]$ as above and suppose the Euler structures $\mathfrak{e}_S \in \mathfrak{Eul}_{x_S}(S)$ are represented as $\mathfrak{e}_S = [Y_S, c_S]$. With the help of a tubular neighborhood of Σ, a connection and a fiber metric on the normal bundle of Σ and a bump function we can construct a globally defined vector field Y which restricts to Y_S and is supported in a small neighborhood of Σ. Consider the homotopy $t \mapsto X + tY$. Using the Morse–Bott assumption for X we see that for sufficiently small $\epsilon > 0$ the zeros $\mathcal{X}^+$ of $X^+ := X + \epsilon Y$ will all be non-degenerate and coincide with the set of zeros of Y_S. For the index of such a zero x we have $\mathrm{ind}^M_{-X^+}(x) = \mathrm{ind}^S_{-Y_S}(x) + \mathrm{ind}^M_{-X}(S)$. Therefor $c^+ := c + \sum_{S\subseteq\Sigma}(-)^{\mathrm{ind}_{-X}(S)} c_S$ will be an Euler chain for X^+, and hence $[X^+, c^+] \in \mathfrak{Eul}_{x_0}(M)$. This Euler structure does not depend on the choices we made and the construction provides a natural affine isomorphism $\mathfrak{Eul}^X_{x_0}(M) = \mathfrak{Eul}_{x_0}(M)$. Obviously we have

Lemma 1. *Suppose X is a Morse–Bott vector field, $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M)$ and $\mathfrak{e}_S \in \mathfrak{Eul}_{x_S}(S)$ for every connected component $S \subseteq \Sigma$. Then there exists an Euler chain $c \in C_1(M;\mathbb{R})$ such that $\mathfrak{e} = [X, c, \{\mathfrak{e}_S\}]$. Moreover, every other such Euler chain differs from c by a boundary.*

For a Morse–Bott vector field X, a closed one form $\omega \in \Omega^1(M;\mathbb{R})$, a Riemannian metric g and a Morse vector field Y on Σ define

$$\mathcal{R}(\omega, X, g; Y) := \mathcal{R}(\omega, X^+, g) - \sum_{S\subseteq\Sigma}(-)^{\mathrm{ind}_{-X}(S)}\mathcal{R}(\omega, Y|_S, g|_S)$$

where $X^+ = X + \epsilon Y$ is a small perturbation of X as above.

Proposition 10. *The quantity $\mathcal{R}(\omega, X, g; Y)$ does not depend on Y nor on the other choices entering the definition of the perturbation X^+ and will be denoted by $\mathcal{R}(\omega, X, g)$. It should be considered a regularization of the possibly divergent integral $\int_{M\setminus\Sigma} \omega \wedge X^*\Psi_g$. It has the following properties:*

(i) *$\mathcal{R}(\omega, X, g)$ is linear in ω.*
(ii) *$\mathcal{R}(dh, X, g) = -\int_M hE_g + \sum_{S\subseteq\Sigma}(-)^{\mathrm{ind}_{-X}(S)}\int_S hE_{g|_S}$ for all smooth functions $h : M \to \mathbb{R}$.*
(iii) *$\mathcal{R}(\omega, X, g_2) - \mathcal{R}(\omega, X, g_1) = \int_M \omega\wedge\mathrm{cs}(g_1, g_2) - \sum_{S\subseteq\Sigma}(-)^{\mathrm{ind}_{-X}(S)}\int_S \omega\wedge \mathrm{cs}(g_1|_S, g_2|_S)$ for two Riemannian metrics g_1 and g_2.*

(iv) *If there exists a function f such that $\omega' = \omega - df$ vanishes in a neighborhood of Σ then:*

$$\mathcal{R}(\omega, X, g) = \int_{M\setminus\Sigma} \omega' \wedge X^*\Psi_g - \int_M fE_g + \sum_{S\subseteq\Sigma} (-)^{\mathrm{ind}_{-X}(S)} \int_S fE_{g|_S}$$

Proof. All this is an immediate consequences of Proposition 9. For the independence of Y note that a homotopy from Y_1 to Y_2 as vector fields on Σ provides a homotopy from X_1^+ to X_2^+ via a similar construction. The zero set of this homotopy will coincide with the zero set of the homotopy from Y_1 to Y_2 we started with. However, the orientations they inherit differ by $(-)^{\mathrm{ind}_{-X}(S)}$. Now apply Proposition 9(iv) twice. □

Proposition 11. *Suppose $\mathfrak{e}' = [g, \alpha]$ and $\mathfrak{e} = [X, c, \{\mathfrak{e}_S\}]$ are two Euler structures based at x_0 and suppose $\mathfrak{e}_S = [g|_S, \alpha_S]$. Then $\mathfrak{e} = \mathfrak{e}'$ iff*

$$0 = \mathcal{R}(\omega, X, g) - \mathcal{S}(\omega, \alpha, g) - \int_c \omega + \sum_{S\subseteq\Sigma} (-)^{\mathrm{ind}_{-X}(S)} \mathcal{S}(\omega, \alpha_S, g|_S)$$

for all closed one forms $\omega \in \Omega^1(M; \mathbb{R})$.

Proof. This is immediate from our definitions. □

4.2. **Analytic torsion.** Let M be a closed manifold, E a flat vector bundle over M and let $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M)$ be a an Euler structure based at $x_0 \in M$. Consider the deRham complex $\Omega^*(M; E)$, the determinant line of its cohomology $\det H^*(M; E)$ and the one dimensional vector space:

$$\mathrm{Det}_{x_0}(M; E) := \det H^*(M; E) \otimes (\det E_{x_0})^{-\chi(M)} \tag{16}$$

The analytic torsion will be a non-zero element in $\mathrm{Det}_{x_0}(M; E)$ defined up to multiplication by ± 1, equivalently a scalar product on $\mathrm{Det}_{x_0}(M; E)$, associated with the Euler structure $\mathfrak{e}$.

To define the analytic torsion choose a Riemannian metric g on M, a fiber metric μ on E and $\alpha \in \Omega^{\dim M - 1}(M \setminus \{x_0\}; \mathcal{O}_M)$ so that $d\alpha = E_g$ on $M \setminus \{x_0\}$ and so that $\mathfrak{e} = [g, \alpha] \in \mathfrak{Eul}_{x_0}(M)$, cf. section 4.1.

The deRham complex $\Omega(M; E)$ inherits a scalar product which permits to define the formal adjoint $d^\sharp : \Omega(M; E) \to \Omega(M; E)$ of the deRham differential d and the Laplacian $\Delta := d \circ d^\sharp + d^\sharp \circ d$. Clearly $\Delta^q : \Omega^q(M; E) \to \Omega^q(M; E)$ is a symmetric positive semi-definite elliptic second order differential operator. Therefor we have the *Ray–Singer torsion*, see [RS71]:

$$\log T_{\mathrm{an}}^{E,g,\mu} := \frac{1}{2} \sum_q (-)^{q+1} q \log \det{}' \Delta^q \in \mathbb{R}$$

The fiber metric μ on E determines a fiber metric on its determinant bundle $\det E$ and a canonic length one section e of $\det E\otimes\mathcal{O}_E$. There is a unique one form $\theta=\theta^E_\mu\in\Omega^1(M;\mathbb{R})$ such that $\nabla^{\det E\otimes\mathcal{O}_E}_Y e=\theta(Y)e$.[10] Since the connection on $\det E\otimes\mathcal{O}_E$ is flat θ is a closed one form. It measures to what extend the fiber metric μ on $\det E$ is not parallel. Indeed, one can easily check $\theta(Y)=-\frac{1}{2}\operatorname{tr}_\mu\nabla^E_Y\mu$, which could serve as alternative definition, cf. [BH], [BZ92]. The cohomology class $\Theta_E:=[\theta]\in H^1(M;\mathbb{R})$ does not depend on μ and is a complete obstruction to the existence of a fiber metric on E such that the induced fiber metric on $\det E$ is parallel.[11] For later use let us state

Lemma 2.

(i) *Suppose c is a closed path starting and ending at x_0. Then parallel transport along c provides an automorphism of* $\det E_{x_0}$ *which is given by multiplication with $\pm e^{-\langle\Theta_E,[c]\rangle}$ where $[c]\in H_1(M;\mathbb{R})$ denotes the homology class represented by c.*

(ii) *Let c be a path from x_0 to x_1, suppose μ is a fiber metric on E and let $||\cdot||^\mu_{\det E_{x_i}}$ denote the induced scalar product on* $\det E_{x_i}$. *Parallel transport along c will map $||\cdot||^\mu_{\det E_{x_0}}$ to $e^{\int_c\theta^\mu_E}\cdot||\cdot||^\mu_{\det E_{x_1}}$.*

Define a scalar product on $H^*(M;E)$ by restricting the scalar product on $\Omega^*(M;E)$ to harmonic forms. Let $||\cdot||^{E,g,\mu}_{\text{Hodge}}$ denote the induced scalar product on $\det H^*(M;E)$. Let $||\cdot||^\mu_{x_0}$ denote the scalar product on $(\det E_{x_0})^{\chi(M)}$ induced from $\mu|_{E_{x_0}}$. Define a linear isomorphism

$$\tau^{E,g,\alpha,\mu}_{\text{an}}:(\det E_{x_0})^{\chi(M)}=\det H^*(M;E)$$

so that it maps $||\cdot||^\mu_{x_0}$ to $T^{E,g,\mu}_{\text{an}}e^{-\mathcal{S}(\theta,g,\alpha)}||\cdot||^{E,g,\mu}_{\text{Hodge}}$. This is well defined up to sign and will be considered as element in $\operatorname{Det}_{x_0}(M;E)$, cf. (16).

Proposition 12. *$\tau^{E,g,\alpha,\mu}_{\text{an}}\in\operatorname{Det}_{x_0}(M;E)$ is independent of μ and does not depend on the representative of the Euler structure $\mathfrak{e}=[g,\alpha]$. Hence we will denote it by $\tau^{E,\mathfrak{e}}_{\text{an}}$.*

Proof. Let g_1 and g_2 be two Riemannian metrics on M, let μ_1 and μ_2 be two fiber metrics on E and suppose $[g_1,\alpha_1]=\mathfrak{e}=[g_2,\alpha_2]$, see section 4.1. Let h denote the unique smooth function so that $||\cdot||^{\mu_2}_{\det E}=$

[10]The orientation bundle $\mathcal{O}_E$ just comes in to ensure the existence of a canonic globally defined length one section e. The bundle $\det E$ has locally defined length one sections with a sign ambiguity. One could as well use these to first locally define θ. Then one has to observe that the sign ambiguity does not affect θ and that the locally defined one forms actually define a global one form. To circumvent these inconveniences we made use of $\mathcal{O}_E$.

[11]Note that the theta in [BZ92] is -2 times the θ here.

$e^h||\cdot||^{\mu_1}_{\det E}$ where $||\cdot||^{\mu_i}_{\det E}$ denotes the fiber metric on $\det E$ induced by μ_i. Then we clearly have $\log\big(||\cdot||^{\mu_2}_{x_0}/||\cdot||^{\mu_1}_{x_0}\big) = \chi(M)h(x_0)$. Moreover, $-dh = \theta_2 - \theta_1$ where $\theta_i := \theta^{\mu_i}_E$. With our notation the anomaly formula for the Ray–Singer torsion, see [BZ92, Theorem 0.1] reads:

$$\log \frac{T^{E,g_2,\mu_2}_{\mathrm{an}}||\cdot||^{E,g_2,\mu_2}_{\mathrm{Hodge}}}{T^{E,g_1,\mu_1}_{\mathrm{an}}||\cdot||^{E,g_1,\mu_1}_{\mathrm{Hodge}}} = \int_M hE_{g_1} + \int_M \theta_2 \wedge \mathrm{cs}(g_1, g_2)$$

Using Proposition 8 this implies

$$\log \frac{e^{-\mathcal{S}(\theta_2,g_2,\alpha_2)}T^{E,g_2,\mu_2}_{\mathrm{an}}||\cdot||^{E,g_2,\mu_2}_{\mathrm{Hodge}}}{e^{-\mathcal{S}(\theta_1,g_1,\alpha_1)}T^{E,g_1,\mu_1}_{\mathrm{an}}||\cdot||^{E,g_1,\mu_1}_{\mathrm{Hodge}}} = \chi(M)h(x_0)$$

and the proposition follows. □

Definition 5 (Analytic torsion)**.** The element $\tau^{E,\mathfrak{e}}_{\mathrm{an}} \in \mathrm{Det}_{x_0}(M;E)$ is well defined up to a sign and will be called the *analytic torsion*.[12]

Remark 6. For every $\sigma \in H_1(M;\mathbb{R})$ and every $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M)$ we have $\tau^{E,\mathfrak{e}+\sigma}_{\mathrm{an}} = \tau^{E,\mathfrak{e}}_{\mathrm{an}} \cdot e^{-\langle\Theta_E,\sigma\rangle}$. A path τ from x_0 to x_1 provides an isomorphism $\mathfrak{Eul}_{x_0}(M) = \mathfrak{Eul}_{x_1}(M)$ denoted by $\mathfrak{e} \mapsto \mathfrak{e} - \chi(M)\tau$, see (14) in section 4.1. Moreover parallel transport along τ provides an identification $\mathrm{Det}_{x_0}(M;E) = \mathrm{Det}_{x_1}(M;E)$. Via the latter we have $\tau^{E,\mathfrak{e}}_{\mathrm{an}} = \tau^{E,\mathfrak{e}-\chi(M)\tau}_{\mathrm{an}}$. This follows from Proposition 8 and Lemma 2(ii).

4.3. **Geometric torsion.** Let X be Morse–Bott–Smale vector field on a closed manifold M, let E be a flat vector bundle over M and let $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M)$ be an Euler structure based at $x_0 \in M$. Consider the geometric complex $C^*(M;E)$, cf. section 3.2, let $\det HC^*(X;E)$ denote the determinant line of its cohomology and consider the one dimensional vector space:

$$\mathrm{Det}_{x_0}(X;E) = \det HC^*(X;E) \otimes (\det E_{x_0})^{-\chi(M)}$$

The geometric torsion will be a non-zero element in $\mathrm{Det}_{x_0}(X;E)$ defined up to multiplication by ± 1, equivalently a scalar product on $\mathrm{Det}_{x_0}(X;E)$, associated to the Euler structure $\mathfrak{e}$.

In order to define the geometric torsion choose an Euler structure $\mathfrak{e}_S \in \mathfrak{Eul}_{x_S}(S)$ for every connected component $S \subseteq \Sigma$ and choose c such that $\mathfrak{e} = [-X, c, \{\mathfrak{e}_S\}]$, see Lemma 1. This gives rise to an isomorphism, well defined up to multiplication by ± 1,

$$(\det E_{x_0})^{\chi(M)} \stackrel{c}{=} \bigotimes_{S\subseteq\Sigma}\Big(\big(\det(E_S)_{x_S}\big)^{\chi(S)}\Big)^{(-)^{\mathrm{ind}(S)}} \tag{17}$$

[12] The sign ambiguity of the analytic torsion can be removed using a homology orientation of M, i.e. an orientation of $\det H^*(M;\mathbb{R})$, see [T02].

as follows. We choose a path c_S from x_0 to x_S and use parallel transport along c_S to get an isomorphism $\det E_{x_0} = \det E_{x_S}$. Taking tensor products (and duals) and using Corollary 1 we obtain an isomorphism

$$(\det E_{x_0})^{\chi(M)} \xrightarrow{\{c_S\}} \bigotimes_{S\subseteq\Sigma}\big((\det E_{x_S})^{\chi(S)}\big)^{(-)^{\mathrm{ind}(S)}} \tag{18}$$

Choose an orientation of $(\mathcal{O}_{N^-})_{x_S}$ to get an identification $(\mathcal{O}_{N^-})_{x_S} = \mathbb{R}$ and hence an identification $(E_S)_{x_S} = E_{x_S}$. Set

$$c_0 := c - \sum_{S\subseteq\Sigma}(-)^{\mathrm{ind}(S)}\chi(S)c_S \in C_1(M;\mathbb{R})$$

and note that $\partial c_0 = 0$. Now define (17) as $e^{-\langle\Theta_E,[c_0]\rangle}$ times the isomorphism (18). It follows immediately from Lemma 2(i) that the resulting isomorphism (17) does not depend on the choice of paths $\{c_S\}$ and actually only depends on $\mathfrak{e}$ and $\{\mathfrak{e}_S\}$ — up to sign.

As explained in section 3.3 the geometric complex is a filtered graded complex $C^q_p(X;E)$. In view of Theorem 1(ii) the associated spectral sequence converges to $HC^*(X;E)$. This provides us with a natural identification

$$\det HC^* = \det GHC^* = \det E_\infty C^* = \cdots = \det E_2C^* = \det E_1C^*$$

where $C^* = C^*(X;E)$ and GHC^* denotes the associated graded of HC^* which inherits a filtration as usual from the filtration on C^*. For more details on determinant lines see appendix B. Together with the description of the E_1–term in Theorem 1(ii) this yields an isomorphism:

$$\det HC^*(X;E) = \bigotimes_{S\subseteq\Sigma}\big(\det H^*(S;E_S)\big)^{(-)^{\mathrm{ind}(S)}} \tag{19}$$

The tensor product of the analytic torsions provides an isomorphism

$$\bigotimes_{S\subseteq\Sigma}\big((\det E_{x_S})^{\chi(S)}\big)^{(-)^{\mathrm{ind}(S)}} = \bigotimes_{S\subseteq\Sigma}\big(\det H^*(S;E_S)\big)^{(-)^{\mathrm{ind}(S)}} \tag{20}$$

Let $\tau^{E,X,c,\{\mathfrak{e}_S\}}_{\mathrm{geom}} \in \mathrm{Det}_{x_0}(X;E)$ denote the element corresponding to the composition of (17), (20) and (19). Using Remark 6 and (15) we immediately obtain

Proposition 13. *The element $\tau^{E,X,c,\{\mathfrak{e}_S\}} \in \mathrm{Det}_{x_0}(X;E)$ does not depend on the choice of $\mathfrak{e}_S$, c and hence will be denoted by $\tau^{E,\mathfrak{e},X}_{\mathrm{geom}}$.*

Definition 6 (Geometric torsion)**.** The element $\tau^{E,\mathfrak{e},X}_{\mathrm{geom}} \in \mathrm{Det}_{x_0}(X;E)$ is well defined up to a sign and called the *geometric torsion.*[13]

[13]The sign ambiguity of the geometric torsion can be removed using a homology orientation, that is an orientation of $\det H^*(M;\mathbb{R})$, see [T02].

Remark 7. For every $\sigma \in H_1(M;\mathbb{R})$ and every $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M)$ we have $\tau_{\text{geom}}^{E,\mathfrak{e}+\sigma,X} = \tau_{\text{geom}}^{E,\mathfrak{e},X} \cdot e^{-\langle\Theta_E,\sigma\rangle}$. A path τ from x_0 to x_1 provides an isomorphism $\mathfrak{Eul}_{x_0}(M) = \mathfrak{Eul}_{x_1}(M)$ denoted by $\mathfrak{e} \mapsto \mathfrak{e} - \chi(M)\tau$, see (14) in section 4.1. Moreover parallel transport along τ provides an identification $\mathrm{Det}_{x_0}(X;E) = \mathrm{Det}_{x_1}(X;E)$. Via the latter we have $\tau_{\text{geom}}^{E,\mathfrak{e},X} = \tau_{\text{geom}}^{E,\mathfrak{e}-\chi(M)\tau,X}$. Note that this is analogous to the properties of the analytic torsion, cf. Remark 6.

Remark 8. Note how the geometric torsion specializes to the analytic torsion for the Morse–Bott–Smale vector field $X \equiv 0$.

Also note how the geometric torsion specializes to the combinatorial torsion for a Morse–Smale vector field X. More precisely, if $\mathfrak{e} = [-X, c]$ then the geometric torsion coincides with the composition:

$$(\det E_{x_0})^{\chi(M)} \stackrel{c}{=} \det C^*(X;E) = \det HC^*(X;E)$$

This follows from Proposition 20 in appendix B, which is a very week statement in this special case.

Recall from Theorem 1 that the integration induces an isomorphism $H^*(M;E) = HC^*(X;E)$. So we get an induced isomorphism:

$$\mathrm{Det}_{x_0}(M;E) = \mathrm{Det}_{x_0}(X;E) \tag{21}$$

The following is the main result of this paper.

Theorem 5. *Let X be a Morse–Bott–Smale vector field on a closed manifold M, let E a flat vector bundle over M and suppose $\mathfrak{e}$ is an Euler structure based at $x_0 \in M$. Then the isomorphism* (21) *maps the analytic torsion $\tau_{\text{an}}^{E,\mathfrak{e}}$ to the geometric torsion $\tau_{\text{geom}}^{E,\mathfrak{e},X}$, up to sign.*

If X is a Morse–Smale pair then this is due to Bismut–Zhang, see [BZ92], which itself is an extension of a result of Cheeger [C77], [C79] and Müller [M78]. In this formulation it can be found in [BH, Theorem 5]. The proof of Theorem 5 is contained in section 5 below and will make use of the classical Bismut–Zhang theorem, that is we will assume Theorem 5 proved for Morse–Smale vector fields.

4.4. **Derivation of Theorem 2 from Theorem 5.** We continue to use the notation from section 4.3. So we have an Euler structure $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M)$ and Euler structures $\mathfrak{e}_S \in \mathfrak{Eul}_{x_S}(S)$ for every connected component $S \subseteq \Sigma$ and we have a one chain c such that $\mathfrak{e} = [-X, c, \{\mathfrak{e}_S\}]$.

Choose a fiber metric μ on E and let μ_S denote the induced fiber metric on E_S. Choose a Riemannian metric g on M and let g_S denote the restriction to $S \subseteq \Sigma$. Choose α so that $\mathfrak{e} = [g, \alpha]$ and for every connected component $S \subseteq \Sigma$ choose α_S with $\mathfrak{e}_S = [g_S, \alpha_S]$, cf. section 4.1. From Proposition 11 we have:

$$\mathcal{R}(\theta, -X, g) - \mathcal{S}(\theta, \alpha, g) - \int_c \theta + \sum_{S \subseteq \Sigma} (-)^{\operatorname{ind}(S)} \mathcal{S}(\theta, \alpha_S, g_S) = 0 \tag{22}$$

In view of Theorem 5 we have a commutative diagram:

$$\begin{tikzcd}
\bigotimes\big((\det(E_S)_{x_S})^{\chi(S)}\big)^{\pm} \arrow[d, "\bigotimes \tau_{\mathrm{an}}^{E_S, \mathfrak{e}_S}"'] & (\det E_{x_0})^{\chi(M)} \arrow[l, "c"'] \arrow[d, "\tau_{\mathrm{geom}}^{E, \mathfrak{e}, X}"] \arrow[dr, bend left, "\tau_{\mathrm{an}}^{E, \mathfrak{e}}"] & \\
\bigotimes\big(\det H^*(S; E_S)\big)^{\pm} & \det HC^*(X; E) \arrow[l, "(19)"'] & \det H^*(M; E) \arrow[l, "\mathrm{Int}"']
\end{tikzcd}$$

The tensor products are over all connected components $S \subseteq \Sigma$ and the powers $\pm$ are abbreviations for $(-)^{\operatorname{ind}(S)}$.

Note that Int maps $||\cdot||^{E,g,\mu}_{\mathrm{Hodge}}$ to $(T^M_{\mathrm{met}}(E, X, g, \mu))^{-1} ||\cdot||^{E,X,g,\mu}_{\mathrm{geom}}$ by the very definition of the metric torsion. Applying Proposition 21 in appendix B once for every E_k–term we see that (19) maps $||\cdot||^{E,X,g,\mu}_{\mathrm{geom}}$ to:

$$(T^M_{\mathrm{comb}}(E, X, g, \mu))^{-1} \cdot \bigotimes_{S \subseteq \Sigma} \big(||\cdot||^{E_S, g_S, \mu_S}_{\mathrm{Hodge}}\big)^{(-)^{\operatorname{ind}(S)}}$$

It follows from Lemma 2(ii) that c maps $||\cdot||^{\mu}_{x_0}$ to:

$$e^{\int_c \theta} \cdot \bigotimes \big(||\cdot||^{\mu_S}_{x_S}\big)^{(-)^{\operatorname{ind}(S)}}$$

Therefore the commutativity of the diagram implies:

$$\begin{aligned}
T^M_{\mathrm{an}}(E, g, \mu) e^{-\mathcal{S}(\theta, \alpha, g)} &(T^M_{\mathrm{met}}(E, X, g, \mu))^{-1} (T^M_{\mathrm{comb}}(E, X, g, \mu))^{-1} \\
&= e^{\int_c \theta} \cdot \prod_{S \subseteq \Sigma} \Big(T^M_{\mathrm{an}}(E_S, g_S, \mu_S) e^{-\mathcal{S}(\theta, \alpha_S, g_S)} \Big)^{(-)^{\operatorname{ind}(S)}}
\end{aligned}$$

Together with (22) Theorem 2 follows at once.

Remark 9. The proof actually shows that Theorem 5 is equivalent to Theorem 2. In view of the Bismut–Zhang theorem, see [BZ92, Theorem 0.2] we may assume Theorem 5 proved for Morse–Smale vector fields, see also [BH, Theorem 5].

5. Proof of main Theorem 5

5.1. **Homotopies.** A Morse–Bott homotopy $\mathbb{X} = \{X_s\}_{s \in \mathbb{R}}$ from a Morse–Bott–Smale vector field X^- to another Morse–Bott–Smale vector field X^+ is a smooth family of vector fields for which there exists $\epsilon > 0$ so that $X_s = X^-$ when $s \leq -1 + \epsilon$ and $X_s = X^+$ when $s \geq 1 - \epsilon$.

To such a homotopy one associate a Morse–Bott vector field Y on the smooth manifold $N := M \times [-1, 1]$ defined by

$$Y(x, s) := X_s(x) + 1/2(s^2 - 1)\frac{\partial}{\partial s}.$$

The homotopy is called Morse–Bott–Smale homotopy if Y is a Morse–Bott–Smale vector field. Be aware that it is in general not possible to find a Morse–Bott–Smale homotopy $\mathbb{X}$ connecting two given Morse–Bott–Smale vector fields X^- and X^+.

It is obvious from the definition of Y that the discussion in sections 2 and 3 as well as the compacity of $\hat{\mathcal{T}}$ and $\hat{W}^-$ continue to hold. We will decorate Σ, $\mathcal{T}$, u and Int by $X^\pm$ or Y to indicate the relevant vector field.

We will use several canonic identifications. First, Σ_Y identifies with the disjoint union of Σ_{X^-} and Σ_{X^+}. Note that $N_Y^-|_{\Sigma_{X^-}} = N_{X^-}^-$ hence $\mathrm{ind}_Y|_{\Sigma_{X^-}} = \mathrm{ind}_{X^-} : \Sigma_{X^-} \to \mathbb{N}_0$ and $\mathcal{O}_{N_Y^-}|_{\Sigma_{X^-}} = \mathcal{O}_{N_{X^-}^-}$. Moreover $N_{X^+}^- \subseteq N_Y^-|_{\Sigma_{X^+}}$ with a canonically oriented one dimensional quotient. Therefore $\mathrm{ind}_Y|_{\Sigma_{X^+}} = 1 + \mathrm{ind}_{X^+} : \Sigma_{X^+} \to \mathbb{N}_0$ and $\mathcal{O}_{N_Y^-}|_{\Sigma_{X^-}} = \mathcal{O}_{N_{X^+}^-}$. For the last one we use the *inward pointing normal last* convention.

Let $p : N \to M$ denote the canonical projection and suppose E is a flat vector bundle over M. From our identifications we obtain

$$C^*(Y; p^*E) = C^*(X^-; E) \oplus C^{*-1}(X^+; E)$$

as graded vector spaces and $\delta_Y = \left(\begin{smallmatrix} \delta_{X^-} & 0 \\ v_{\mathbb{X}} & \delta_{X^+} \end{smallmatrix}\right)$. Define a grading homomorphism $B = (-)^k : C^k(X^+; E) \to C^k(X^+; E)$ and let

$$\varphi_{\mathbb{X}} := B \circ v_{\mathbb{X}} : C^*(X^-; E) \to C^*(X^+; E).$$

The equation $\delta_Y^2 = 0$ implies $v_{\mathbb{X}} \circ \delta_{X^-} + \delta_{X^+} \circ v_{\mathbb{X}} = 0$ and hence

$$\varphi_{\mathbb{X}} \circ \delta_{X^-} = \delta_{X^+} \circ \varphi_{\mathbb{X}}. \tag{23}$$

Moreover the integration $\mathrm{Int}_Y : \Omega(N; p^*E) \to C(Y; p^*E)$ decomposes as $\mathrm{Int}_Y = \left(\begin{smallmatrix} \mathrm{Int}_{X^-} \circ \mathrm{inc}_-^* \\ h_{\mathbb{X}} \end{smallmatrix}\right)$ where $\mathrm{inc}_\pm : M \to M \times \{\pm 1\} \subseteq N$ denotes the canonical inclusion. Slightly generalizing Proposition 4 to the boundary case we have:

$$\mathrm{Int}_Y \circ d = \delta_Y \circ \mathrm{Int}_Y - B \circ \mathrm{Int}_{X^+} \circ \mathrm{inc}_+^* \tag{24}$$

The extra term stems from the additional 1–boundary component of Y's unstable manifold which coincides with the main stratum of the unstable manifold of X^+. The sign B stems partially from Stokes' theorem for fiber integration, see section 3.1, and partially from our identification $\mathcal{O}_{N_Y^-}|_{\Sigma_{X^+}} = \mathcal{O}_{N_{X^+}^-}$ using the inward pointing normal last convention. Define:

$$H_{\mathbb{X}} := B \circ h_{\mathbb{X}} \circ p^* : \Omega^*(M; E) \to C^{*-1}(X^+; E)$$

Equation (24) implies:

$$h_{\mathbb{X}} \circ d = v_{\mathbb{X}} \circ \mathrm{Int}_{X^-} \circ \mathrm{inc}_-^* + \delta_{X^+} \circ h_{\mathbb{X}} - B \circ \mathrm{Int}_{X^+} \circ \mathrm{inc}_+^*$$

and thus:

$$H_{\mathbb{X}} \circ d + \delta_{X^+} \circ H_{\mathbb{X}} = \varphi_{\mathbb{X}} \circ \mathrm{Int}_{X^-} - \mathrm{Int}_{X^+} \tag{25}$$

So we get an induced mapping in cohomology $\varphi_{\mathbb{X}} : HC^*(X^-; E) \to HC^*(X^+; E)$ which satisfies $\varphi_{\mathbb{X}} \circ \mathrm{Int}_{X^-} = \mathrm{Int}_{X^+}$ and in view of Theorem 1(i) is an isomorphism which does not depend on the particular homotopy from X^- to X^+. Notice also, that for a constant homotopy $\mathbb{X}$ we have $\varphi_{\mathbb{X}} = \mathrm{id}$ on chain level.

Also note that $\varphi_{\mathbb{X}} : C_p^*(X^-; E) \to C_p^*(X^+; E)$ preserves the filtration and hence it induces homomorphism intertwining the spectral sequences: $\varphi_{\mathbb{X}} : E_k C^*(X^-; E) \to E_k C^*(X^+; E)$.

5.2. **A small perturbation.** Let X be a Morse–Bott–Smale vector field on a closed manifold M with critical manifold $\Sigma := \Sigma_X$. Let Y be a Morse–Smale vector field on Σ. As in section 4.1 use a tubular neighborhood of Σ to extend Y to a globally defined vector field which is supported in a small neighborhood of Σ. Define a homotopy $\mathbb{X}$ from X to $X^+ := X + Y$ by $\mathbb{X}_t := X + tY$. By choosing the extension of Y to M carefully we may assume that the following hold true:

(i) X^+ is Morse–Smale and $\Sigma_{X^+} \subseteq \Sigma$.
(ii) We have $N_{X^+}^- = N_X^- \oplus N_{Y|_\Sigma}^-$, $\mathrm{ind}_{X^+}^M = \mathrm{ind}_X^M|_{\Sigma_{X^+}} + \mathrm{ind}_{Y|_\Sigma}^\Sigma$ and $\mathcal{O}_{N_{X^+}^-} = \mathcal{O}_{N_X^-} \otimes \mathcal{O}_{N_{Y|_\Sigma}^-}$.
(iii) X^+ is tangential to Σ. For x and x' in a connected component $S \subseteq \Sigma$ we have $\hat{\mathcal{T}}_{X^+}(x, x') = \hat{\mathcal{T}}_{Y|_S}(x, x')$.
(iv) Suppose $S \neq S' \subseteq \Sigma$ are two different connected components of Σ, $x \in \Sigma_{X^+} \cap S$, $x' \in \Sigma_{X^+} \cap S'$ and suppose $\hat{\mathcal{T}}_X(S, S') = \emptyset$. Then $\hat{\mathcal{T}}_{X^+}(x, x') = \emptyset$.
(v) The homotopy $\mathbb{X}$ is Morse–Bott–Smale.
(vi) The homotopy $\mathbb{X}$ is tangential to Σ, i.e. every $\mathbb{X}_t$ is tangential to Σ. For every connected component $S \subseteq \Sigma$ and $x \in S$ we have $\hat{\mathcal{T}}_{\mathbb{X}}(x, S) = \hat{W}_{Y|_S}^-(x)$.
(vii) Suppose $S \neq S' \subseteq \Sigma$ are two different connected components of Σ, $x \in \Sigma_{X^+} \cap S$ and $\hat{\mathcal{T}}_X(S, S') = \emptyset$. Then $\hat{\mathcal{T}}_{\mathbb{X}}(x, S') = \emptyset$.

With the help of (i) we can define an interesting filtration on the geometric complex $C(X^+;E)$ by:

$$C(X^+;E)_p = \bigoplus_{p'\geq p} \Omega(\Sigma_{X^+}\cap\Sigma_{p'};E\otimes\mathcal{O}_{N^-_{X^+}})$$

Because of (iv) the differential on $C(X^+;E)$ preserves this filtration. We thus get a spectral sequence $E_kC(X^+;E)$ converging to $HC(X^+;E)$. In view of (ii) and (iii) we can compute the E_1–term as:

$$E_1C^q(X^+;E)_p = HC^{q-p}(Y|_{\Sigma_p};E_{\Sigma_p})$$

Here $E_{\Sigma_p} = E|_{\Sigma_p}\otimes\mathcal{O}_{N^-_X}|_{\Sigma_p}$. As explained in section 5.1 the Morse–Bott–Smale homotopy, cf. (v), provides a homomorphism of graded complexes which in view of (vii) preserves the filtrations:

$$\varphi_{\mathbb{X}} : C^q_p(X;E)\to C^q(X^+;E)_p \tag{26}$$

From (vi) we see that the induced mapping on E_1–terms, cf. Theorem 1(ii), is the isomorphism induced by integration:

$$\begin{tikzcd} E_1C^q_p(X;E) \arrow[r, "\varphi_{\mathbb{X}}"] \arrow[d, equal] & E_1C^q(X^+;E)_p \arrow[d, equal] \\ H^{q-p}(\Sigma_p;E_{\Sigma_p}) \arrow[r, "\mathrm{Int}"] & HC^{q-p}(Y|_{\Sigma_p};E_{\Sigma_p}) \end{tikzcd}$$

5.3. **Proof of Theorem 5.** The main ingredient for the proof will be the Bismut–Zhang theorem for Morse–Smale vector fields. We will apply it once for every connected component $S\subseteq\Sigma$ and once for M. Another major input will be a theorem of Maumary [M69]. This piece of homological algebra describes the behavior of torsion with respect to spectral sequences and generalizes the well known behavior of torsion with respect to short exact sequences. The geometric ingredient is the very special Morse–Bott–Smale homotopy constructed in section 5.2.

We continue to use the notation from section 5.2. Let $\mathfrak{e}\in\mathfrak{Eul}_{x_0}(M)$ be the given Euler structure and choose Euler structures $\mathfrak{e}_S\in\mathfrak{Eul}_{x_S}(S)$. Choose a one chain c such that $\mathfrak{e}=[-X,c,\{\mathfrak{e}_S\}]$ and choose one chains c_S such that $\mathfrak{e}_S=[-Y_S,c_S]$, cf. Lemma 1. Set

$$c^+ := c+\sum_{S\subseteq\Sigma}(-)^{\mathrm{ind}(S)}\chi(S)c_S. \tag{27}$$

This is an Euler chain for X^+ and in view of section 4.1 we know $\mathfrak{e}=[-X^+,c^+]$.

$$
\begin{tikzcd}
 & \det H^*(M;E) \arrow[dl, bend right, "\mathrm{Int}_X"'] \arrow[d, "\mathrm{Int}_{X^+}"] & (\det E_{x_0})^{\chi(M)} \arrow[l, "\tau_{\mathrm{an}}^{E,\mathfrak{e}}"'] \arrow[dl, "\tau_{\mathrm{geom}}^{E,\mathfrak{e},X^+}"] \arrow[d, "c^+"] \\
\det HC^*(X;E) \arrow[r, "\varphi_{\mathbb{X}}"] \arrow[d, equal] & \det HC^*(X^+;E) \arrow[r, equal] \arrow[d, equal] & \det C^*(X^+;E) \arrow[dd, equal] \\
\det E_\infty C^*(X;E) \arrow[r, "\varphi_{\mathbb{X}}"] \arrow[d, equal] & \det E_\infty C^*(X^+;E) \arrow[d, equal] & \\
\det E_1 C^*(X;E) \arrow[r, "\varphi_{\mathbb{X}}"] \arrow[d, equal] & \det E_1 C^*(X^+;E) \arrow[r, equal] \arrow[d, equal] & \det GC^*(X^+;E) \arrow[d, equal] \\
\bigotimes\big(\det H^*(S;E_S)\big)^{\pm} \arrow[r, "\mathrm{Int}_{Y_S}"] & \bigotimes\big(\det HC^*(Y_S;E_S)\big)^{\pm} \arrow[r, equal] & \bigotimes\big(\det C^*(Y_S;E_S)\big)^{\pm} \\
 & \bigotimes\big((\det(E_S)_{x_S})^{\chi(S)}\big)^{\pm} \arrow[ul, bend left, "\bigotimes\left(\tau_{\mathrm{an}}^{E_S,\mathfrak{e}_S}\right)^{\pm}"] \arrow[u, "\bigotimes\left(\tau_{\mathrm{geom}}^{E_S,\mathfrak{e}_S}\right)^{\pm}"'] \arrow[ur, bend right, "\bigotimes(c_S)^{\pm}"'] & \\
 & (\det E_{x_0})^{\chi(M)} \arrow[u, "c"'] &
\end{tikzcd}
$$

Figure 1. A commutative diagram for the proof of Theorem 5. The tensor products are over all connected components $S \subseteq \Sigma$, and the powers $\pm$ are abbreviations for $(-)^{\mathrm{ind}(S)}$.

Consider the diagram in figure 1. The vertical identifications in the left column stem from the spectral sequence of the filtered complex $C^q_p(X;E)$. The corresponding vertical identifications in the middle column stem from the spectral sequence of the filtered complex $C^q(X^+;E)_p$. The isomorphisms labeled c, c^+ and c_S are the ones constructed using parallel transport along these one chains, see section 4.3. We claim that this diagram commutes — up to sign.

Indeed, the upper right part commutes because of the Bismut–Zhang theorem applied to the Morse–Smale vector field X^+, but see also Remark 8. The same argument applied to the Morse–Smale vector field Y_S on S (and taking tensor products) shows that the lower half circle commutes. The upper left part commutes in view of section 5.1. The squares involving the mapping $\varphi_{\mathbb{X}}$ commute because these mappings are induced from a filtration preserving map $\varphi_{\mathbb{X}} : C^q_p(X;E) \to C^q(X^+;E)_p$ and because of the description of $\varphi_{\mathbb{X}}$ on the E_1–term in section 5.2. The big square in the right central part of the diagram commutes in view of a theorem due to Maumary, see Proposition 20 in appendix B. The remaining lower right square obviously commutes.

Note that $\tau_{\mathrm{geom}}^{E,\mathfrak{e},X} \in \mathrm{Det}_{x_0}(X;E)$ can be seen in the diagram as the composition of $\otimes\tau_{\mathrm{an}}^{E_S,\mathfrak{e}_S}\circ c$ with the left vertical isomorphisms. In order to prove Theorem 5 we have to show that this isomorphism coincides with the composition $\mathrm{Int}_X\circ\tau_{\mathrm{an}}^{E,\mathfrak{e}}$ at the very top. In view of the commutativity of the diagram it remains to show that the isomorphism obtained by starting at the very bottom and going to the upper right corner along the right edge coincides with the identity on $(\det E_{x_0})^{\chi(M)}$. But this follows immediately from (27) and the proof is complete.

6. Torsion of smooth bundles

6.1. **Bundles.** Suppose $\pi : M \to N$ is a smooth bundle with M and N closed manifolds, and suppose (f', g') is a Morse–Smale pair on N. Denote by $Y := -\operatorname{grad}_{g'}(f')$ the corresponding Morse–Smale vector field, and let Σ' denote its critical manifold which is a discrete set. Define a Morse–Bott function f on M by $f = f'\circ\pi$. Let us write Σ for the critical manifold of f. Clearly $\Sigma = \pi^{-1}(\Sigma')$, and $\mathrm{ind}_\Sigma = \pi^*\,\mathrm{ind}_{\Sigma'} : \Sigma \to \mathbb{N}_0$. Let us also introduce the notation $M_z := \pi^{-1}(z)$ for the fiber over $z \in N$.

Proposition 14.

(i) *There exists a Riemannian metric g on M so that (f, g) is a Morse–Bott–Smale pair and so that $X := -\operatorname{grad}_g(f)$ satisfies $d_x\pi(X(x)) = Y(\pi(x))$ for all $x \in M$.*

(ii) *One can choose g so that $d_x\pi$ restricted to $(T_x(M_{\pi(x)}))^\perp$ is an isometry onto $T_{\pi(x)}N$, and so that the fibers M_z for $z \in \Sigma'$ are isometric.*

Proof. One chooses a connection in the bundle π (i.e. for any $x \in M$ a subspace $H_x \subset T_xM$ depending smoothly on x, so that $d_x\pi : H_x \to T_{\pi(x)}N$ is an isomorphism), and a smooth family of Riemannian metrics g_z, $z \in N$, on the fibers M_z. Clearly T_xM decomposes as $T_xM = H_x \oplus V_x$ with $V_x = T_x(M_{\pi(x)})$. Declare H_x and V_x orthogonal and define g_x so that $d_x\pi$ is an isometry when restricted H_x and equal to $(g_z)_x$ on V_x.

To insure that the metric g is compatible with f one chooses a sufficiently small contractible neighborhood $U \subset N$ of Σ' and an identification of the bundle $\pi : \pi^{-1}(U) \to U$ with the trivial bundle $\mathrm{pr}_1 : U \times F \to U$. We chose our connection so that its restriction to $\pi^{-1}(U)$ corresponds to the trivial connection in the bundle pr_1, and the family of metrics g_z, for $z \in U$, corresponds to a constant family of Riemannian metrics on F. Clearly if g' is compatible with f' then, with the above choices, g is compatible with f and satisfies the desired properties. $\square$

Suppose E is a flat vector bundle over M. For $z \in N$ let $E_z := E|_{M_z}$. Denote by H^r the flat bundle over N whose fiber above z is $H^r(M_z; E_z)$. Note that H^r is equipped with an obvious flat connection; any path from z_1 to z_2 induces an isomorphism from $H^r(M_{z_1}; E_{z_1})$ to $H^r(M_{z_2}; E_{z_2})$ which depends only on the homotopy class of the path, and therefore defines a flat connection. Note that g_z and μ_z induce, via Hodge theory, a scalar product on $H^r(M_z; E_z)$ and therefore a fiber metric μ^r on the bundle H^r.

Proposition 15. *Suppose X and Y are as in Proposition 14. Then:*

(i) *The spectral sequence of the geometric complex $C^*(X; E)$ is isomorphic to the Leray–Serre spectral sequence of the bundle $\pi : M \to N$ associated with the cell structure given by Y.*

(ii) $\big(E_1C^*(X; E), \delta_1\big)$ *identifies to* $\bigoplus_r S^r\big(C^*(Y; H^r), \delta\big)$.[14]

Proof. Observe that the filtration on $\Omega^*(M; E)$ induced by X, see section 3.3, induces the Leray–Serre spectral sequence of the bundle. But the spectral sequence of $\Omega^*(M; E)$ is isomorphic to the spectral sequence associated to $C^*(X; E)$, see Remark 4 in section 3.4. This proves (i).

Using $\pi^*(N_z^-) = N_{M_z}^-$ for $z \in \Sigma'$, we get $\pi^*(\mathcal{O}_{N_z^-}) = \mathcal{O}_{N_{M_z}^-}$ and hence $H^*(M_z; E_z \otimes \mathcal{O}_{N_{M_z}^-}) = H^*(M_z; E_z) \otimes \mathcal{O}_{N_z^-}$. Now (ii) follows easily from the description of the differential δ_1 on $E_1C^*(X; E)$ in section 3.4. $\square$

Suppose that X resp. Y are as in Proposition 14. The isomorphisms between the deRham cohomology and the cohomology of the geometric complex provided by 'integration theory' induces the isomorphism

$$\det \mathrm{Int}_E^X : \det H^*(M; E) \to \det H^*(X; E)$$

and for any $r \geq 0$ the isomorphisms

$$\det \mathrm{Int}_{H^r}^Y : \det H^*(N; H^r) \to \det H^*(Y; H^r).$$

The last isomorphisms induce the isomorphism

$$\det \mathrm{Int}_E^Y : \bigotimes_r \big(\det H^*(N; H^r)\big)^{\epsilon(r)} \to \bigotimes_r \big(\det H^*(Y; H^r)\big)^{\epsilon(r)}$$

[14]Here $S^r(C^*, \delta^*)$ denotes the r–suspension of the cochain complex (C^*, δ^*), i.e. the degree q component of of $S^r(C^*, \delta^*)$ is exactly (C^{q-r}, δ^{q-r}).

defined by $\bigotimes_r(\det \mathrm{Int}^Y_{H^r})^{\epsilon(r)}$. Here $V^{\epsilon(r)}$ resp. $\alpha^{\epsilon(r)}$ denote V resp. α if r even and the dual V^* resp. $(\alpha^*)^{-1}$ if r is odd.

The identification in Proposition 15(ii) induces an isomorphism

$$E_2C^*(X;E) \to \bigoplus_r H^{*-r}(Y;H^r)$$

and then an isomorphism:

$$\det(\pi, X, Y) : \det H^*(X;E) \to \bigotimes_r \big(\det H^*(Y;H^r)\big)^{\epsilon(r)}$$

Proposition 16. *The bundle* π *induces a canonical isomorphism* $\det(\pi)$: $\det H^*(M;E) \to \bigotimes_r\big(\det H^*(B;H^r)\big)^{\epsilon(r)}$ *so that for* X *and* Y *as in Proposition 14 the following diagram is commutative:*

$$\begin{CD}
\det H^*(M;E) @>{\det(\pi)}>> \bigotimes_r\big(\det H^*(N;H^r)\big)^{\epsilon(r)}\\
@V{\det \mathrm{Int}^E_X}VV @VV{\det \mathrm{Int}^Y_E}V\\
\det H^*(X;E) @>{\det(\pi,X,Y)}>> \bigotimes_r\big(\det H^*(Y;H^r)\big)^{\epsilon(r)}
\end{CD}$$

Proof. Choose a pair (X,Y) as in Proposition 14 and define

$$\det(\pi) := (\det \mathrm{Int}^E_Y)^{-1} \circ \det(\pi, X, Y) \circ \mathrm{Int}^E_X\,.$$

We claim that this does not depend on the choice of (X,Y). Since given two such pairs (X_-,Y_-) and (X_+,Y_+) one can, by Proposition 14, construct a Morse–Bott–Smale homotopy which satisfies $d_x\pi(\mathbb{X}(x,t)) = \mathbb{Y}(\pi(x),t)$ the arguments in section 5.1 imply the independence of the choice of (X,Y). ☐

Suppose now that g resp. g' are Riemannian metrics on M resp. N. Recall that g and μ induce a scalar product on $\det H^*(M;E)$. Similarly g' and μ^r induce scalar products on $\det H^*(N;H^r)$. Denote by

$$\log \mathrm{vol}_{\mathrm{an}}(\pi) := \log \mathrm{vol} \det(\pi)$$

and

$$\log \mathrm{vol}_{\mathrm{comb}}(\pi) := \log \mathrm{vol} \det(\pi, X, Y).$$

Proposition 16 implies

$$\begin{aligned}\log \mathrm{vol}_{\mathrm{an}}(\pi) - \log \mathrm{vol}_{\mathrm{comb}}(\pi) = {}& \log T^M_{\mathrm{met}}(E,X,g,\mu)\\ &- \sum_r (-)^r \log T^N_{\mathrm{met}}(H^r, Y, g', \mu^r) \quad (28)\end{aligned}$$

6.2. **Torsion for bundles.** As in section 6.1, let $\pi : M \to N$ be a smooth bundle with M and N closed smooth manifolds, E a flat vector bundle over M equipped with a fiber metric μ, and let g resp. g' be a Riemannian metric on M resp. N. In the previous section we have introduced flat bundles H^r with fiber metrics μ^r. Denote by θ resp. θ^r the closed one forms associated to the fiber metric μ resp. μ^r. Let X and Y be vector fields as in Proposition 14.

Consider the spectral sequence associated with the geometric complex $C(X;E)$. For any $k \geq 1$ denote

$$\rho_k^{E,X,g,\mu} := \frac{1}{2}\sum_q (-)^{q+1} q \log \det{}' \Delta_k^q$$

where Δ_k^q denote the Laplacians in the cochain complexes of Euclidean vector spaces $(E_k C^*(X;E), \delta_k)$ and $\det{}'$ denotes the product of non-zero eigen values. Note that for $k \geq 1$ $E_k C^*(X;E)$ is a finite dimensional vector space, and for k large $\log \det{}' \Delta_k^q = 0$. Then the calculation of ρ_k for $k \geq 1$ is, in principle, a matter of finite dimensional linear algebra.

Theorem 2 applied to X combined with Proposition 15 gives

$$\begin{aligned}\log T_{\text{an}}^M(E,g,\mu) = \sum_{z\in\Sigma'} (-)^{\text{ind}(z)} \log T_{\text{an}}^{M_z}(E_z, g_z, \mu_z) \\ + \sum_r (-)^r \log T_{\text{comb}}^N(H^r, Y, \mu^r) + \sum_{k\geq 2} \rho_k^{E,X,g,\mu} \\ + \log T_{\text{met}}^M(E, X, g, \mu) + \mathcal{R}(\theta, -X, g)\end{aligned} \tag{29}$$

Since Σ' is zero dimensional, the definition of $T_{\text{comb}}^N(\cdot, Y, \cdot)$ does not involve any Riemannian metric on N. For the metric g' on N the Bismut–Zhang theorem gives

$$\begin{aligned}\log T_{\text{comb}}^N(H^r, Y, \mu^r) = \log T_{\text{an}}^N(H^r, g', \mu^r) \\ - \log T_{\text{met}}^N(H^r, Y, g', \mu^r) - \mathcal{R}(\theta^r, -Y, g').\end{aligned} \tag{30}$$

Combining (28), (29) and (30) one obtains the following statement

Theorem 6. *Let $\pi : M \to N$ be a smooth fiber bundle with M and N closed manifolds. Suppose g resp. g' is a Riemannian metric on M resp. N, X and Y are vector fields as in Proposition 14, and suppose*

E is a flat vector bundle over M equipped with a fiber metric μ. Then:

$$\begin{aligned}\log T_{\mathrm{an}}^M(E,g,\mu) &= \sum_r (-)^r \log T_{\mathrm{an}}^N(H^r,g',\mu^r) \\ &\quad + \mathcal{R}(\theta,-X,g) - \sum_r (-)^r \mathcal{R}(\theta^r,-Y,g') + \log \mathrm{vol}_{\mathrm{an}}(\pi) \\ &\quad - \log \mathrm{vol}_{\mathrm{comb}}(\pi) + \sum_{k\geq 2} \rho_k^{E,X,g,\mu} \\ &\quad + \sum_{z\in\Sigma'} (-)^{\mathrm{ind}(z)} \log T_{\mathrm{an}}^{M_z}(E_z,g_z,\mu_z)\end{aligned}$$

Combining with the metric anomaly for torsion,[15] cf. [BZ92], Theorem 6 implies the main result of [LST98, Theorems 0.2 and 0.4]. As in [LST98], two cases are of particular interest.

Bundles with base a sphere. Suppose that N is homotopy equivalent to S^n. Then for $k \geq 1$ all $d_k^{p,r} : E_k^{p,r} \to E_k^{p+k,r-k+1}$ are zero but $d_n^{0,r} : E_n^{0,r} \to E_n^{n,r-n+1}$. Here we use the notation $E_k^{p,r} := E_k C_p^{p+r}(X;E)$ and $d_k^{p,r}$ for the restriction $\delta_k : E_k^{p,r} \to E_k^{p+k,r-k+1}$ in order to stay with the traditional notation. Then one obtains the following long exact sequence

$$\cdots \to E_n^{0,r-1} \xrightarrow{d_n^{0,r-1}} E_n^{n,r-n} \xrightarrow{i^{r-n}} H^r(X;E) \xrightarrow{p^r} E_n^{0,r} \xrightarrow{d_n^{0,r}} E_n^{n,r-n+1} \to \cdots \tag{31}$$

When $H^r(X;E)$ is identified to $H^r(M;E)$ via the integration isomorphism we refer to the sequence

$$\cdots \to E_n^{0,r-1} \xrightarrow{d_n^{0,r-1}} E_n^{n,r-n} \xrightarrow{i^{r-n}} H^r(M;E) \xrightarrow{p^r} E_n^{0,r} \xrightarrow{d_n^{0,r}} E_n^{n,r-n+1} \to \cdots \tag{32}$$

as the *Wang exact sequence.*

The Wang exact sequence can be regarded as an acyclic cochain complex of finite dimensional Euclidean spaces where each term is equipped with the scalar product described above. In particular $H(M;E)$ is equipped with the scalar product induced from μ and g. Denote by T_W the torsion of the complex (32). Recall that for an acyclic cochain complex of Euclidean vector spaces (C^*,d^*) the logarithm of the torsion is defined by $1/2\sum_q(-)^{q+1}q\log\det\Delta^q = \sum_q(-)^q\log\mathrm{vol}(\overline{d^q})$ where Δ^q is the Laplacian acting on C^q. Recall that given a linear map

[15] which describes how the analytic torsion T_{an} changes when one changes the Riemannian metric g

$\alpha : V \to W$ between two finite dimensional Euclidean spaces one denotes by $\log \mathrm{vol}(\overline{\alpha})$ the quantity, cf. appendix B.2,

$$\log \mathrm{vol}(\overline{\alpha}) := \log \mathrm{vol}\big(\alpha : (\ker \alpha)^{\perp} \to \alpha(V)\big).$$

Proposition 17. *One has:*

$$\log T_W = \log T^M_{\mathrm{met}}(E, X, g, \mu) + \rho_n^{E,X,g,\mu}$$

Proof. When $H^*(X; E)$ is equipped with a scalar product compatible to the scalar product on E^*_∞ the torsion of the exact sequence (31) is

$$\rho_n^{E,X,\mu,g} = \sum_r (-)^r \log \mathrm{vol}(\overline{d_n^{0,r}})$$

simply because $\log \mathrm{vol}(\overline{i^{n-r}}) = 0 = \log \mathrm{vol}(\overline{p^r})$.

To see this observe that in the commutative diagram of Euclidean vector spaces below the left side arrow is surjective, the right side arrow injective and the lower horizontal row is short exact. Observe that by construction the scalar product on E_∞ is induced from the scalar product on E_n and the scalar product on $H(X; E)$ is chosen so that it induces the scalar products on E_∞ in the lower horizontal row sequence.

$$\begin{tikzcd} E_n^{n,r-n} \arrow[r, "i^{r-n}"] \arrow[d] & H^r(X;E) \arrow[r, "p^r"] \arrow[d, equal] & E_n^{n,r-n} \\ E_\infty^{n,r-n} \arrow[r, "i_\infty^{n,r-n}"] & H^r(X;E) \arrow[r, "p_\infty^{0,r}"] & E_\infty^{0,r}. \arrow[u] \end{tikzcd}$$

Give all these we conclude that

$$\log \mathrm{vol}(\overline{i^{r-n}}) = \log \mathrm{vol}(\overline{i_\infty^{n,r-n}})$$
$$\log \mathrm{vol}(\overline{p^r}) = \log \mathrm{vol}(\overline{p_\infty^{0,r}})$$
$$\log \mathrm{vol}(\overline{i_\infty^{n,r-n}}) = 0 = \log \mathrm{vol}(\overline{p_\infty^{0,r}})$$

which imply the statement.

Since in the Wang exact sequence $H^r(X; E)$ is replaced by $H^r(M; E)$ with the scalar product induced by Hodge theory the torsion T_W of the Wang sequence is corrected by $T^M_{\mathrm{met}}(E, X, g, \mu)$. □

Consequently, for $n \geq 2$, one obtains from (29):

$$\log T^M_{\mathrm{an}}(E, \mu, g) = \sum_{z \in \Sigma'} (-)^{\mathrm{ind}(z)} \log T^{M_z}_{\mathrm{an}}(E_z, g_z, \mu_z) + \sum_r (-)^r \log T^{S^n}_{\mathrm{comb}}(H^r, Y, \mu^r) + \log T_W + \mathcal{R}(\theta, -X, g).$$

For $n = 1$ one obtains from Theorem 2:

$$\log T_{\mathrm{an}}^{M}(E,\mu,g) = \sum_{z\in\Sigma'} (-)^{\mathrm{ind}(z)} \log T_{\mathrm{an}}^{M_z}(E_z, g_z, \mu_z) + \log T_W + \mathcal{R}(\theta, -X, g).$$

Sphere bundles. Suppose that the fiber M_z is homotopy equivalent to S^n. Then for $k \geq 1$ all $d_k^{p,r} : E_k^{p,r} \to E_k^{p+k,r-k+1}$ are zero but $d_{n+1}^{p,n} : E_{n+1}^{p,n} \to E_{n+1}^{p+n+1,0}$. From this we derive the following long exact sequence

$$\cdots \to E_{n+1}^{p-n-1,n} \xrightarrow{d_{n+1}^{p-n-1,n}} E_{n+1}^{p,0} \xrightarrow{i^p} H^p(X;E) \xrightarrow{p^p} E_{n+1}^{p-n,n} \xrightarrow{d_{n+1}^{p-n,n}} E_{n+1}^{p+1,0} \to \cdots$$

When $H^q(X;E)$ is identified to $H^q(M;E)$ via 'integration theory' the sequence

$$\cdots \to E_{n+1}^{p-n-1,n} \xrightarrow{d_{n+1}^{p-n-1,n}} E_{n+1}^{p,0} \xrightarrow{i^p} H^p(M;E) \xrightarrow{p^p} E_{n+1}^{p-n,n} \xrightarrow{d_{n+1}^{p-n,n}} E_{n+1}^{p+1,0} \to \cdots \tag{33}$$

will be referred to as the *Gysin long exact sequence.*

The Gysin long exact sequence can be regarded as an acyclic cochain complex of finite dimensional Euclidean spaces where each term is regarded with the scalar product described above. Denote by T_G the torsion of the complex (33).

Proposition 18. *One has:*

$$\log T_G = \log T_{\mathrm{met}}^{M}(E, X, g, \mu) + \rho_{n+1}^{E,X,g,\mu}$$

The proof is identical to the proof of Proposition 17. Consequently we obtain from (29):

$$\log T_{\mathrm{an}}^{M}(E,\mu,g) = \sum_{z\in\Sigma'} (-)^{\mathrm{ind}(z)} \log T_{\mathrm{an}}^{M_z}(E_z, g_z, \mu_z) + \log T_{\mathrm{comb}}^{N}(H^0, Y, \mu^0) + (-)^n \log T_{\mathrm{comb}}^{N}(H^n, Y, \mu^n) + \log T_G + \mathcal{R}(\theta, -X, g)$$

Note that if $E = \pi^* E'$ is the pull back of a flat bundle E' over N then, as flat bundles over N, we have $H^0 = H^n = E'$. Also note that for $n \geq 2$ every flat bundle is E over M is the pull back of a flat bundle E' over N.

Appendix A. Proofs of Theorems 3 and 4

The following is a sometimes a simple way to recognize a smooth manifold with corners.

Lemma 3. *If $\mathcal{P}$ is a smooth manifold with corners, $\mathcal{O}$ and $\mathcal{S}$ smooth manifolds, $p : \mathcal{P} \to \mathcal{O}$ and $s : \mathcal{S} \to \mathcal{O}$ smooth maps so that p and s are transversal then $p^{-1}(s(\mathcal{S}))$ is a smooth submanifold with corners of $\mathcal{P}$. Here p is called transversal to s if its restriction to each k–boundary $\partial_k\mathcal{P}$ is transversal to s.*

A.1. **Some notations.** Let $\cdots < c_i < c_{i+1} < \cdots$ denote the set of all critical values of f. Choose $\epsilon > 0$ sufficiently small, so that $c_i^+ < c_{i+1}^-$, for all $i \in \mathbb{Z}$, where $c_i^\pm := c_i \pm \epsilon$. Define:

$$\begin{aligned}
M_i &:= f^{-1}(c_i)\\
M_i^\pm &:= f^{-1}(c_i^\pm)\\
M(i) &:= f^{-1}(c_{i-1}, c_{i+1})\\
\Sigma_i &:= \Sigma \cap M_i\\
W^\pm_{\Sigma_i} &:= p_\pm^{-1}(\Sigma_i) = \bigsqcup_{S\subseteq\Sigma_i} W^\pm_S\\
S_i^\pm &:= W^\pm_{\Sigma_i} \cap M_i^\pm\\
\mathbf{S}_i &:= S_i^+ \times_{\Sigma_i} S_i^-\\
W^\pm(i) &:= W^\pm_{\Sigma_i} \cap M(i)\\
\mathbf{S}W(i) &:= S_i^+ \times_{\Sigma_i} W^-(i)
\end{aligned}$$

These definitions are to some extent illustrated in Figure 2. We have:

(i) $\mathbf{S}_i \subseteq M_i^+ \times M_i^-$.
(ii) $\mathbf{S}W(i) \subseteq S_i^+ \times W^-(i) \subseteq M_i^+ \times M(i)$.
(iii) $M_i^\pm \subseteq M$ are smooth submanifolds of codimension 1.
(iv) $M(i)$ is an open submanifold of M.
(v) M_i is not a manifold, however $\dot{M}_i := M_i \setminus \Sigma_i$ and $\dot{M}_i^\pm := M_i^\pm \setminus S_i^\pm$ are smooth codimension 1 submanifolds of M.

Let Φ_t be the flow associated to the vector field $\frac{X}{|X|^2}$ on $M \setminus \Sigma$ and consider the diffeomorphisms, see Figure 3,

$$\psi_i : M_i^- \to M_{i-1}^+, \quad \psi_i(x) := \Phi_{c_i-\epsilon-(c_{i-1}-\epsilon)}(x)$$

and

$$\varphi_i^\pm : \dot{M}_i^\pm \to \dot{M}_i, \quad \varphi_i^\pm(x) := \Phi_{\pm\epsilon}(x),$$

as well as the submersion

$$\varphi(i) : M(i) \setminus \big(W^+(i) \cup W^-(i)\big) \to \dot{M}_i, \quad \varphi(i)(x) := \Phi_{h(x)-c_i}(x).$$

The maps $\varphi_i^\pm$ and $\varphi(i)$ extend to continuous maps

$$\varphi_i^\pm : M_i^\pm \to M_i \quad \text{and} \quad \varphi(i) : M(i) \to M_i.$$

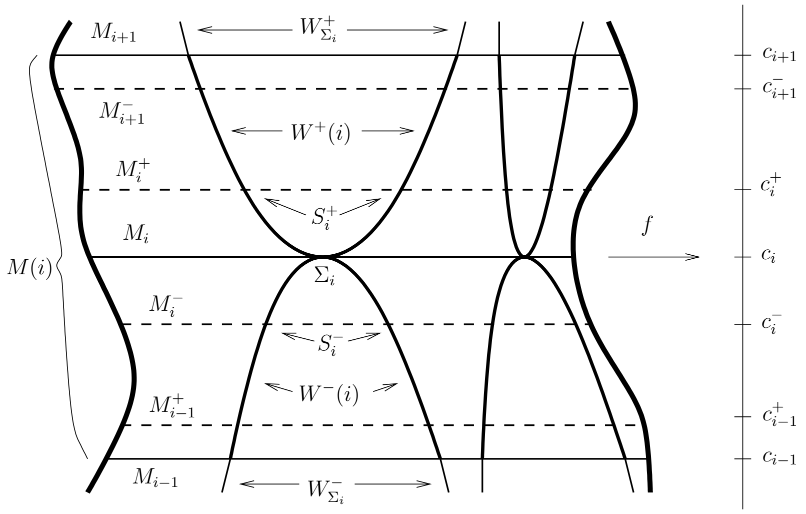

FIGURE 2. Notations.

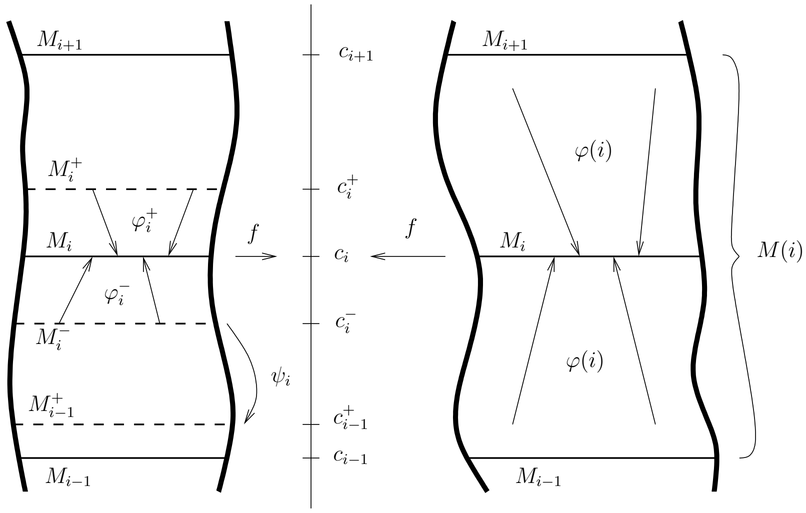

FIGURE 3. Retractions.

This can be seen from explicit formulae for $\varphi_i^\pm$ and $\varphi(i)$ in the neighborhood of $S_i^\pm$ resp. $\big(W^+(i) \cup W^-(i)\big)$, cf. [BFK]. Define

$$P_i := \big\{(x,y) \in M_i^+ \times M_i^- \;\big|\; \varphi_i^+(x) = \varphi_i^-(y)\big\},$$

and denote by $p_i^\pm : P_i \to M_i^\pm$ the canonical projections. One can verify the following

Lemma 4. *$P_i \subseteq M_i^+ \times M_i^-$ is a smooth submanifold with boundary ∂P_i diffeomorphic to $\mathbf{S}_i \subset M_i^+ \times M_i^-$. Precisely we have:*

(P1) $p_i^\pm : P_i \setminus \partial P_i \to \dot M_i^\pm$ *are diffeomorphisms.*

(P2) *The restriction of* $p_i^+ \times p_i^-$ *to* ∂P_i *is a diffeomorphism onto* $\mathbf{S}_i$*, each* $p_i^\pm$ *restricted to* ∂P_i *identifies with the projection onto* $S_i^\pm$.

Geometrically P_i *represents the space of all trajectories, possibly broken, from* M_i^+ *to* M_i^-.

Next we define

$$Q(i) := \big\{(x,y) \in M_i^+ \times M(i) \bigm| \varphi_i^+(x) = \varphi(i)(y)\big\}.$$

Equivalently, $Q(i)$ consists of pairs of points (x,y), $x \in M_i^+$, $y \in M(i)$, which lie on the same possibly broken trajectory. Moreover let $l_i : Q(i) \to M_i^+$ and $r_i : Q(i) \to M(i)$ denote the canonical projections. One can verify the following

Lemma 5. $Q(i) \subseteq M_i^+ \times M(i)$ *is a smooth submanifold with boundary* $\partial Q(i)$ *diffeomorphic to* $\mathbf{S}W(i) \subset M_i^+ \times M(i)$*. Precisely we have:*

(Q1) $l_i : Q(i) \setminus \partial Q(i) \to \dot M_i^+$ *is a smooth bundle with fiber an open segment.* $r_i : Q(i)\setminus\partial Q(i) \to M(i)\setminus W^-(i)$ *is a diffeomorphism.*

(Q2) *The restriction of* $l_i \times r_i$ *to* $\partial Q(i)$ *is a diffeomorphism onto* $\mathbf{S}W(i)$*, i.e.* l_i *resp.* r_i *restricted to* $\partial Q(i)$ *identifies with the projection onto* S_i^+ *resp.* $W^-(i)$.

Since P_i and $Q(i)$ are smooth manifolds with boundaries

$$\mathcal{P}_{r,r-k} := P_r \times P_{r-1} \times \cdots \times P_{r-k} \tag{34}$$

and

$$\mathcal{P}_r(r-k) := P_r \times \cdots \times P_{r-k+1} \times Q(r-k) \tag{35}$$

are smooth manifolds with corners.

A.2. **Proof of Theorem 3.** We will equip

$$\hat{\mathcal{T}}(\Sigma_{r+1}, \Sigma_{r-k-1}) := \hat\pi_-^{-1}(\Sigma_{r+1}) \cap \hat\pi_+^{-1}(\Sigma_{r-k-1}) = \bigsqcup_{\substack{S \subseteq \Sigma_{r+1} \\ S' \subseteq \Sigma_{r-k-1}}} \hat{\mathcal{T}}(S, S')$$

with the structure of a smooth manifold with corners. If $k \leq -2$ the statement is empty. If $k = -1$ there is nothing to check. So we suppose $k \geq 0$. We are going to apply Lemma 3.

Consider the smooth manifold with corners $\mathcal{P} = \mathcal{P}_{r,r-k}$ from (34) above, the smooth manifold $\mathcal{O} := \prod_{i=r}^{r-k}(M_i^+ \times M_i^-)$ and the smooth manifold $\mathcal{S} := S_{r+1}^- \times M_r^- \times \cdots \times M_{r-k+1}^- \times S_{r-k-1}^+$. In order to define the maps p and s we consider

$$\omega_i : M_i^- \to M_i^- \times M_{i+1}^+, \quad \omega_i(x) := (x, \psi_i(x))$$

and

$$\tilde{p}_i : P_i \to M_i^+ \times M_i^-, \quad \tilde{p}_i(y) := \big(p_i^+(y), p_i^-(y)\big).$$

We also denote by $\alpha : S_{r+1}^- \to M_r^+$ resp. $\beta : S_{r-k-1}^+ \to M_{r-k}^-$ the restriction of ψ_{r+1} resp. ψ_{r-k}^{-1} to S_{r+1}^- resp. S_{r-k-1}^+. Finally we set,

$$s := \alpha \times \omega_r \times \cdots \times \omega_{r-k+1} \times \beta : \mathcal{S} \to \mathcal{O}$$

and

$$p := \tilde{p}_r \times \cdots \times \tilde{p}_{r-k} : \mathcal{P} \to \mathcal{O}.$$

The verification of the transversally of p and s follows easily from (P1), (P2) and the Morse–Smale condition and is done exactly as in [BH01]. It is easy to see that $p^{-1}(s(\mathcal{S}))$ identifies to $\hat{\mathcal{T}}(\Sigma_{r+1}, \Sigma_{r-k-1})$ as topological spaces and we leave this verification to the reader. The rest is straight forward.

A.3. **Proof of Theorem 4.** We are going equip the compact topological space $X := \hat{W}^-_{\Sigma_{r+1}}$ with the structure of a smooth manifold with corners. Let us write $\hat{i} := \hat{i}^-_{\Sigma_{r+1}} : X \to M$. For any integer k denote by $X(r-k) := \hat{i}^{-1}(M(r-k))$. First we will put the structure of smooth manifold with corners on $X(r-k)$ so that the restriction of $\hat{i}$ is smooth. Second we check that $X(r-k)$ and $X(r-k')$ induce on the intersection $X(r-k) \cap X(r-k')$ the same smooth structure. These facts imply that X has a canonical structure of smooth manifold with corners and that $\hat{i}$ is a smooth map.

To accomplish first step we proceed in exactly the same way as in the proof of Theorem 3. Consider the smooth manifold with corners $\mathcal{P} := \mathcal{P}_r(r-k)$ from (35), the smooth manifold $\mathcal{O} := \prod_{i=r}^{r-k+1}(M_i^+ \times M_i^-) \times M_{r-k}^+$ and the smooth manifold $\mathcal{S} := S_{r+1}^- \times M_r^- \times \cdots \times M_{r-k+1}^-$. Define

$$p := \tilde{p}_r \times \cdots \times \tilde{p}_{r-k+1} \times l_{r-k} : \mathcal{P} \to \mathcal{O}$$

and

$$s := \alpha \times \omega_r \times \cdots \times \omega_{r-k+1} : \mathcal{S} \to \mathcal{O}.$$

Again we will apply Lemma 3. The verification of the transversality follows from (P1), (P2), (Q1), (Q2) and the Morse–Smale condition, exactly as explained in [BH01]. It is easy to see that $p^{-1}(s(\mathcal{S}))$ identifies to $X(r-k)$. The rest is straight forward.

Appendix B. Some homological algebra

B.1. **Finite dimensional complexes.** All vector spaces in this section are supposed to be finite dimensional and over a field $\mathbb{K}$ of characteristics zero. For such a vector space V we define its *determinant* by $\det V := \Lambda^{\dim(V)}V$. If we have a short exact sequence of vector spaces $0 \to U \to V \to W \to 0$ we obtain a canonic isomorphism $\det V = \det U \otimes \det W$. This is constructed using a section $W \to V$ to get an isomorphism $V = U \oplus W$. This induces an isomorphism $\det V = \det U \otimes \det W$ which does not depend on the section.

If $V = V^{\mathrm{even}} \oplus V^{\mathrm{odd}}$ is a $\mathbb{Z}_2$–graded vector space we define its determinant by $\det V := \det V^{\mathrm{even}} \otimes \det(V^{\mathrm{odd}})^*$. Here $(V^{\mathrm{odd}})^*$ denotes the dual of V^{odd}. Note that we have a natural identification $\det((V^{\mathrm{odd}})^*) = (\det V^{\mathrm{odd}})^*$. Again, a short exact sequence $0 \to U \to V \to W \to 0$ of $\mathbb{Z}_2$–graded vector spaces gives rise to a natural isomorphism $\det V = \det U \otimes \det W$ by combining the isomorphisms for the even and the odd part.

Suppose $C = C^{\mathrm{even}} \oplus C^{\mathrm{odd}}$ is a $\mathbb{Z}_2$–graded complex with a degree differential of odd degree $d : C^{\mathrm{even}} \to C^{\mathrm{odd}}$ and $d : C^{\mathrm{odd}} \to C^{\mathrm{even}}$. Then we get three $\mathbb{Z}_2$–graded vector spaces; $Z := \ker d$, $B := \operatorname{img} d$ and $HC := Z/B$. Moreover we get two short exact sequences of $\mathbb{Z}_2$–graded vector spaces $0 \to B \to Z \to HC \to 0$ and $0 \to Z \to C \xrightarrow{d} B^{\mathrm{op}} \to 0$. Here B^{op} denotes the $\mathbb{Z}_2$–graded vector space B equipped with the opposite grading, $(B^{\mathrm{op}})^{\mathrm{even}} := B^{\mathrm{odd}}$ and $(B^{\mathrm{op}})^{\mathrm{odd}} := B^{\mathrm{even}}$. Note that we have a natural isomorphism $\det B^{\mathrm{op}} = (\det B)^*$. The two short exact sequences provide natural isomorphisms

$$\det Z = \det B \otimes \det HC \quad \text{and} \quad \det C = \det Z \otimes \det B^{\mathrm{op}}.$$

Together with $\det B \otimes \det B^{\mathrm{op}} = \mathbb{K}$ they give rise to a natural isomorphism

$$\det C = \det HC. \tag{36}$$

Suppose we have a short exact sequence of $\mathbb{Z}_2$–graded complexes $0 \to C_0 \to C_1 \to C_2 \to 0$. Its long exact sequence in cohomology

$$\begin{tikzcd} H^{\mathrm{even}}C_0 \arrow[r] & H^{\mathrm{even}}C_1 \arrow[r] & H^{\mathrm{even}}C_2 \arrow[d] \\ H^{\mathrm{odd}}C_2 \arrow[u] & H^{\mathrm{odd}}C_1 \arrow[l] & H^{\mathrm{odd}}C_0 \arrow[l] \end{tikzcd}$$

can be used to make the $\mathbb{Z}_2$–graded vector space $HC_0 \oplus (HC_1)^{\mathrm{op}} \oplus HC_2$ a $\mathbb{Z}_2$–graded complex with vanishing cohomology. From (36) we thus obtain a natural isomorphism:

$$\det HC_0 \otimes \det HC_2 = \det HC_1 \tag{37}$$

The following is a reformulation of a well known fact, which for instance can be found in [M66, Theorem 3.2].

Proposition 19. *Suppose* $0 \to C_0 \to C_1 \to C_2 \to 0$ *is a short exact sequence of* $\mathbb{Z}_2$*–graded complexes. Then the following diagram of canonic isomorphisms commutes up to sign:*

$$\begin{array}{ccc} \det C_0 \otimes \det C_2 & = & \det C_1 \\ \| & & \| \\ \det HC_0 \otimes \det HC_2 & = & \det HC_1 \end{array}$$

Suppose C is a $\mathbb{Z}_2$–graded complex equipped with a filtration

$$\cdots \supseteq C_p \supseteq C_{p+1} \supseteq \cdots$$

which is preserved by the differential. We assume $C_p = 0$ for p sufficiently large and $C_p = C$ for p sufficiently small. Moreover we assume that the filtration is compatible with the grading, that is $C_p = C_p^{\text{even}} \oplus C_p^{\text{odd}}$ where $C_p^{\text{even}} := C_p \cap C^{\text{even}}$ and $C_p^{\text{odd}} := C_p \cap C^{\text{odd}}$. In this situation we have a spectral sequence (E_kC, δ_k) converging to HC. Every E_kC is a $\mathbb{Z}_2$–graded complex with differential δ_k. For the E_1–term we have $E_1C = HGC$ as $\mathbb{Z}_2$–graded vector spaces. Here $GC := \bigoplus_p C_p/C_{p+1}$ denotes the associated graded of C considered as $\mathbb{Z}_2$–graded complex, and HGC its cohomology, a $\mathbb{Z}_2$–graded vector space. We will not consider the $\mathbb{Z}$–grading GC and HGC inherit from the filtration. Applying (36) we obtain a canonic isomorphism $\det C = \det GC = \det HGC = \det E_1C$. The cohomology HC inherits the structure of a $\mathbb{Z}_2$–graded vector space with filtration from C. Let GHC denote the associated graded considered as $\mathbb{Z}_2$–graded vector space. Again, we will not consider the $\mathbb{Z}$–grading GHC inherits. Since the spectral sequence converges we have a canonic isomorphism $GHC = E_\infty C$ which provides us with a canonic isomorphism $\det C = \det HC = \det GHC = \det E_\infty C$. Since $HE_kC = E_{k+1}C$ we also have a canonic isomorphism $\det E_k = \det E_{k+1}C$ for every $k \geq 1$. Also note that $E_k = E_\infty$ for k sufficiently large.

The following can be found in [F92, Theorem 1.3], but see also [M69, Théorèm 4.4] for a more general result.

Proposition 20. *Let* C *be a filtered* $\mathbb{Z}_2$*–graded complex. Then the following diagram of canonic isomorphism commutes up to sign:*

$$\begin{array}{ccccccccc} \det HGC & = & \det GC & = & \det C & = & \det HC & = & \det GHC \\ \| & & & & & & & & \| \\ \det E_1C & = & \det E_2C & = & \det E_3C & = & \cdots & = & \det E_\infty C \end{array}$$

B.2. **Finite dimensional complexes with scalar product.** All vector spaces (complexes) in this section are suppose to be finite dimensional and over $\mathbb{R}$. We will work with $\mathbb{Z}_2$–graded complexes and all determinant lines are understood in the graded sense. If a $\mathbb{Z}_2$–graded complex $C = C^{\text{even}} \oplus C^{\text{odd}}$ comes with a scalar product g we will assume $g = g^{\text{even}} \oplus g^{\text{odd}}$, i.e. the even and the odd part are orthogonal. In such a situation $B = \operatorname{img} d$, $Z = \ker d$ and $HC = Z/B$ are $\mathbb{Z}_2$–graded vector spaces with scalar products. We let $||\cdot||_{\det C}$, $||\cdot||_{\det B}$, $||\cdot||_{\det Z}$ and $||\cdot||_{\det HC}$ denote the induced norms on $\det C$, $\det B$, $\det Z$ and $\det HC$, respectively. Moreover let $d^\sharp$ denote the adjoint of the differential d and $\Delta = d^\sharp d + dd^\sharp$ the Laplacian.

For a linear isomorphism between finite dimensional Euclidean spaces $\alpha : V \to W$ we set $\operatorname{vol}\alpha := \sqrt{\det(\alpha^\sharp\alpha)}$. For an arbitrary linear mapping $\alpha : V \to W$ between Euclidean spaces we let $\overline{\alpha} : (\ker\alpha)^\perp \to \operatorname{img}\alpha$ denote its restriction and thus have a volume $\operatorname{vol}(\overline{\alpha})$. Moreover, if $\beta : U \to U$ is a linear mapping between finite dimensional vector spaces we write $\det{}'\beta$ for the product of non-zero eigen values. Clearly $\operatorname{vol}(\overline{\alpha}) = \sqrt{\det{}'(\alpha^\sharp\alpha)}$. For a $\mathbb{Z}_2$–graded complex C with scalar product define:

$$T_C := \frac{\operatorname{vol}\big(\overline{d : C^{\text{even}} \to 01zC^{\text{odd}}}\big)}{\operatorname{vol}\big(\overline{d : C^{\text{odd}} \to C^{\text{even}}}\big)} = \frac{\sqrt{\det{}'(d^\sharp d : C^{\text{even}} \to C^{\text{even}})}}{\sqrt{\det{}'(d^\sharp d : C^{\text{odd}} \to C^{\text{odd}})}}$$

Remark 10. Suppose we have a $\mathbb{Z}$–graded complex $C = \bigoplus_q C^q$ with scalar product. It gives rise to a $\mathbb{Z}_2$–graded complex with scalar product in the usual way. Let Δ^q denote the Laplacian acting on C^q. A two line computation using

$$\det{}'(\Delta^q) = \det{}'(d^\sharp d : C^q \to C^q)\det{}'(dd^\sharp : C^q \to C^q)$$

and

$$\det{}'(dd^\sharp : C^q \to C^q) = \det{}'(d^\sharp d : C^{q-1} \to C^{q-1})$$

shows

$$\log T_C = \frac{1}{2}\sum_q (-)^{q+1} q \log\det{}'\Delta^q$$

However, for this to make sense it is essential to have a $\mathbb{Z}$–grading at hand.

Proposition 21. *Let C be a $\mathbb{Z}_2$–graded complex with scalar product. Then the canonic isomorphism $\det C = \det HC$ maps $||\cdot||_C$ to $T_C||\cdot||_{HC}$.*

Proof. The short exact sequence $0 \to B \to Z \to HC \to 0$ of $\mathbb{Z}_2$–graded vector spaces induces an isomorphism $\det Z = \det B \otimes \det HC$

which clearly maps $||\cdot||_{\det Z}$ to $||\cdot||_{\det B}\otimes||\cdot||_{\det HC}$. The short exact sequence $0\to Z\to C\to B^{\mathrm{op}}\to 0$ of $\mathbb{Z}_2$–graded vector spaces induces an isomorphism $\det C=\det Z\otimes\det B^{\mathrm{op}}$ which maps $||\cdot||_{\det C}$ to $T_C||\cdot||_{\det Z}\otimes||\cdot||_{\det B^{\mathrm{op}}}$. Putting this together we find $||\cdot||_{\det C}=T_C||\cdot||_{\det HC}$ via $\det C=\det HC$. ☐

Proposition 22. *Suppose* $0\to C_0\to C_1\to C_2\to 0$ *is a short exact sequence of* $\mathbb{Z}_2$*–graded complexes with compatible scalar products, i.e.* $C_0\to C_1$ *is an isometry onto its image and* C_2 *carries the quotient scalar product. Then the canonic isomorphism*

$$\det HC_1=\det HC_0\otimes\det HC_2$$

maps $T_{C_1}||\cdot||_{HC_1}$ *to* $T_{C_0}||\cdot||_{HC_0}\otimes T_{C_2}||\cdot||_{HC_2}$.

Proof. The compatibility condition for the scalar products guarantees that the isomorphism $\det C_0\otimes\det C_2=\det C_1$ maps $||\cdot||_{\det C_0}\otimes||\cdot||_{\det C_2}$ to $||\cdot||_{\det C_1}$. Now the statement follows from Proposition 19 and Proposition 21. ☐

Let C be a filtered $\mathbb{Z}_2$–graded complex as considered in section B.1 and suppose C is equipped with a scalar product. Then HC, GC and HGC inherit scalar products in an obvious way.

Proposition 23. *Suppose* C *is a filtered* $\mathbb{Z}_2$*–graded complex with scalar product. Then the canonic isomorphism*

$$\det HC=\det GHC=\det E_\infty C=\cdots=\det E_1C=\det HGC$$

maps $T_C||\cdot||_{HC}$ *to* $T_{GC}||\cdot||_{HGC}$.

Proof. This follows immediately from Proposition 20, Proposition 21 applied to C and GC as well as the fact that the natural isomorphism $\det C=\det GC$ maps $||\cdot||_{\det C}$ to $||\cdot||_{\det GC}$. ☐

References

[AB95] D.M. Austin and P.J. Braam, *Morse–Bott theory and equivariant cohomology.* The Floer memorial volume, 123–183. Progr. Math. **133**, Birkhäuser, Basel, 1995.

[BZ92] J.M. Bismut and W. Zhang, *An extension of a theorem by Cheeger and Müller.* With an appendix by F. Laudenbach. Astrisque **205**(1992).

[BZ94] J.M. Bismut and W. Zhang, *Milnor and Ray–Singer metrics on the equivariant determinant of a flat vector bundle.* Geom. Funct. Anal. **4**(1994), 136–212.

[BT82] R. Bott and L.W. Tu, *Differential forms in algebraic topology.* Graduate Texts in Mathematics **82**. Springer–Verlag, New York–Berlin, 1982.

[B99] D. Burghelea, *Removing metric anomalies from Ray–Singer torsion*, Lett. Math. Phys. **47**(1999), 149–158.

[BFK99] D. Burghelea, L. Friedlander and T. Kappeler, *Torsions for manifolds with boundary and glueing formulas*. Math. Nachr. **208**(1999), 31–91.

[BFK01] D. Burghelea, L. Friedlander and T. Kappeler, *Relative torsion*. Commun. Contemp. Math. **3**(2001), 15–85.

[BFK] D. Burghelea, L. Friedlander and T. Kappeler, *Elementary Morse Theory*, Lectures Notes in preparation.

[BH01] D. Burghelea and S. Haller, *On the topology and analysis of a closed one form. I (Novikov's theory revisited)*. Essays on geometry and related topics, 133–175. Monogr. Enseign. Math. **38**, Enseignement Math., Geneva, 2001.

[BH] D. Burghelea and S. Haller, *A Riemannian invariant, Euler structures and some topological applications*, preprint `math.DG/0310154`.

[BH04] . . .

[C77] J. Cheeger, *Analytic torsion and Reidemeister torsion*, Proc. Nat. Acad. Sci. U.S.A. **74**(1977), 2651–2654.

[C79] J. Cheeger, *Analytic torsion and the heat equation*, Ann. of Math. **109**(1979), 259–322.

[D95] A. Dold, *Lectures on algebraic topology*. Reprint of the 1972 edition. Classics in Mathematics. Springer–Verlag, Berlin, 1995.

[F92] D.S. Freed, *Reidemeister torsion, spectral sequences, and Brieskorn spheres*, J. Reine Angew. Math. **429**(1992), 75–89.

[LST98] W. Lück, T. Schick and T. Thielmann, *Torsion and fibrations*, J. Reine Angew. Math. **498**(1998), 1–33.

[M69] S. Maumary, *Contributions à la théorie du type simple d'homotopie*, Comment. Math. Helv. **44**(1969), 410–437.

[M66] J. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc. **72**(1966), 358–426.

[M78] W. Müller, *Analytic torsion and R-torsion of Riemannian manifolds*, Adv. in Math. **28**(1978), 233–305.

[RS71] D.B. Ray and I.M. Singer, *R–torsion and the Laplacian on Riemannian manifolds*, Adv. in Math. **7**(1971), 145–210.

[T90] V. Turaev, *Euler structures, non-singular vector fields and Reidemeister-type torsions*, Math. USSR–Izv. **34**(1990), 627–662.

[T02] V. Turaev, *Torsions of* 3*–dimensional manifolds*. Progress in Mathematics **208**. Birkhäuser Verlag, Basel, 2002.

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