SEMI LOCAL LEVI-FLAT EXTENSIONS

NIKOLAY SHCHERBINA* AND GIUSEPPE TOMASSINI**

Contents

1. Introduction.

Let G be a domain in $\mathbb{C}_z \times \mathbb{R}_u \subset \mathbb{C}^2_{z,w}$ $(w = u + iv)$. Let $\varphi : b\,G \to \mathbb{R}_v$ be a continuous function and $\Gamma(\varphi)$ its graph. The extendability of φ to a continuous function $\Phi : \overline{G} \to \mathbb{R}_v$ with the Levi-flat (i.e. foliated by holomorphic curves) graph $\Gamma(\Phi)$ has been studied by several authors ([BG], [BK], [E], [Kr], [Sh], [CS] in the case G is bounded, and [ST] in the unbounded case) under the assumption that bG is smooth and strictly pseudoconvex (i.e. $G \times \mathbb{R}_v$ is a strictly pseudoconvex domain of $\mathbb{C}^2_{z,w}$). In this paper we study a *semi-local* version of the extension problem, namely, when the function φ is prescribed on an open subset U of bG, where bG is smooth and strictly pseudoconvex. Namely, we define a notion of the hull $\mathcal{E}(U) \subset \overline{G}$ of U and prove that every continuous function $\varphi : U \to \mathbb{R}_v$ has a Levi-flat extension to $\mathcal{E}(U)$.

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Given an open subset $\mathbb{D}$ of $\mathbb{C}^n$ we denote $\mathcal{A}(\mathbb{D})$ the Fréchet algebra $\mathcal{O}(D) \cap C^0(\overline{\mathbb{D}})$. If K is a compact subset of $\overline{\mathbb{D}}$ let us denote by $\widehat{K}_{\mathcal{A}(\mathbb{D})}$ (or simply by $\widehat{K}$) the $\mathcal{A}(\mathbb{D})$-hull of K.

If $\mathbb{D}$ is a strictly pseudoconvex domain in $\mathbb{C}^n$ and V is an open subset of $b\mathbb{D}$, we define the set $\mathcal{E}(V)$ to be equal to the union of the hulls $\widehat{K}_{\mathcal{A}(\mathbb{D})}$ for all compact subsets K of V. It is easy to see that if $K_1 \subset K_2 \subset K_3 \subset \cdots$ is an exhaustion of V by compact subsets, then $\mathcal{E}(V) = \bigcup\limits_n \widehat{K}_n$. Denote also by $E(V)$ the CR-hull of V. Since the domain $\mathbb{D}$ is assumed to be strictly pseudoconvex, CR-functions defined on V admit holomorphic extension to a one-sided neighbourhood $\mathbf{V}$ of V in the closed domain $\overline{\mathbb{D}}$ and then $E(V)$ is just the union of V with the hull of holomorphy of the interior part of $\mathbf{V}$.

The first result of this paper describes some properties of the set $\mathcal{E}(V)$.

Theorem 1. *Let $\mathbb{D}$ be a strictly pseudoconvex domain in $\mathbb{C}^{n+1}_{z,w}$, where $z = (z_1, z_2, ..., z_n), w = u + iv$, and let V be an open subset of $b\mathbb{D}$. Then*

1) *$\mathcal{E}(V)$ is an open subset of $\overline{\mathbb{D}}$ such that $\mathcal{E}(V) \cap b\mathbb{D} = V$ and $\mathcal{E}(V) \cap \mathbb{D}$ is pseudoconvex.*

If $\mathbb{D}$ is of the form $\mathbb{D} = G \times \mathbb{R}_v$, where G is a domain in $\mathbb{C}^n_z \times \mathbb{R}_u$, and V is of the form $V = U \times \mathbb{R}_v$, where $U \subset bG$, then

2) *$\mathcal{E}(V)$ is invariant under translation in v-direction. In particular, there is an open subset $\mathcal{E}(U)$ of $\overline{G}$ such that $\mathcal{E}(V) = \mathcal{E}(U) \times \mathbb{R}_v$.*

If, moreover, $n = 1$ and U is the union of simply-connected subdomains of bG, then

3) *$\mathcal{E}(U \times \mathbb{R}_v) = E(U \times \mathbb{R}_v)$.*

In the case when G is a topological 3-ball the set $\mathcal{E}(U)$ enjoies the following property.

Theorem 2. *Let $G \subset \mathbb{C}_z \times \mathbb{R}_u$ be a domain diffeomorphic to a 3-ball such that $G \times \mathbb{R}_v \subset \mathbb{C}^2_{z,w}$ is strictly pseudoconvex. Let U be an open subset of bG and let $\{U_\alpha\}$ be the connected components of U. Then the sets $\mathcal{E}(U_\alpha)$ are connected and disjoint and, moreover, $\mathcal{E}(U) = \bigcup\limits_\alpha \mathcal{E}(U_\alpha)$.*

Note that the statement of Theorem 2 is in general false if G is not diffeomorphic to a 3-ball, as it is shown by Example 1 in Section 7.

In view of part 3 in Theorem 1 one can raise the following question: *How to describe the CR-hull of the surface $U \times \mathbb{R}_v$?* A general construction of the hull of holomorphy of a rigid domain was given in the classical paper [Br]. This result was generalised in [CS], [Ki] and [TT]. Since in our case the surface $U \times \mathbb{R}_v$ is already contained in the boundary of strictly pseudoconvex domain $G \times \mathbb{R}_v$, it is possible to provide a concrete geometric description of its CR-hull. Such description is given in Section 4 of this paper and is used later in Example 2 presented in Section 7.

Let, as above, $G \subset \mathbb{C}_z \times \mathbb{R}_u$ be a domain such that $G \times \mathbb{R}_v \subset \mathbb{C}^2_{z,w}$ is strictly pseudoconvex, U an open subset of bG and $\varphi : U \to \mathbb{R}_v$ a continuous function. Consider the sets

$$V_\varphi^+ = \{(z,w) \in U \times \mathbb{R}_v : \varphi(z,u) < v\}$$

and

$$V_\varphi^- = \{(z,w) \in U \times \mathbb{R}_v : v < \varphi(z,u)\}.$$

Then the following theorem holds.

Theorem 3. *The sets $\mathcal{E}(V_\varphi^\pm)$ can be represented in the following form:*

$$\mathcal{E}(V_\varphi^+) = \{(z,w) \in \overline{G} \times \mathbb{R}_v : v > \Phi^+(z,u), (z,u) \in \mathcal{E}(U)\}, \tag{1}$$

$$\mathcal{E}(V_\varphi^-) = \{(z,w) \in \overline{G} \times \mathbb{R}_v : v < \Phi^-(z,u), (z,u) \in \mathcal{E}(U)\}, \tag{2}$$

for some functions $\Phi^\pm$ defined on the set $\mathcal{E}(U)$.

Having defined the functions $\Phi^\pm$, we can formulate the main result of this paper.

Theorem 4. *Let G be a domain in $\mathbb{C}_z \times \mathbb{R}_u$ such that $G \times \mathbb{R}_v \subset \mathbb{C}^2_{z,w}$ is strictly pseudoconvex. Let U be an open subset of bG and $\mathcal{E}(U)$ the defined above open subset of $\overline{G}$. Then for every $\varphi \in C(U)$ there exist two functions $\Phi^\pm \in C(\mathcal{E}(U))$ such that $\Phi^\pm|_U = \varphi$ and their graphs $\Gamma(\Phi^\pm)$ are Levi-flat over $\mathcal{E}(U) \cap G$.*

Moreover, for any function $\Phi \in C(\mathcal{E}(U))$ such that $\Phi|_U = \varphi$ and $\Gamma(\Phi)$ is Levi-flat over $\mathcal{E}(U) \cap G$ one has

$$\Phi^-(z,u) \le \Phi(z,u) \le \Phi^+(z,u)$$

for each point $(z,u) \in \mathcal{E}(U)$.

For bounded domains $G \subset \mathbb{C}_z \times \mathbb{R}_u$ such that $G \times \mathbb{R}_v$ is strictly pseudoconvex the following characterization was given in Theorem 1 of [CS]: there is a domain $\mathcal{G}$ over $\mathbb{C}_z$ and functions h^- and $-h^+$ continuous in $\overline{\mathcal{G}}$, smooth and strictly subharmonic in $\mathcal{G}$, such that

$$G = \{(z, u) : z = \pi(\zeta),\, h^-(\zeta) < u < h^+(\zeta), \zeta \in \mathcal{G}\},$$

where $\pi : \mathcal{G} \to \mathbb{C}_z$ is the projection. It is easy to see that the same characterization holds true for unbounded domains G with the only difference that one of the functions h^- of $-h^+$ can be identically equal to $-\infty$. Following the notations in [CS] we denote by

$$bG^{\pm} = \{(z, u) : z = \pi(\zeta),\, u = h^{\pm}(\zeta),\, \zeta \in \mathcal{G}\}$$

respectively the "upper" and the "lower" part of bG (if $h^- \equiv -\infty$ or $h^+ \equiv +\infty$, then the respective set bG^- or bG^+ is empty).

Now we can formulate our result on maximality of $\mathcal{E}(U)$.

Theorem 5. *Let $G \subset \mathbb{C}_z \times \mathbb{R}_u$ be a domain such that $G \times \mathbb{R}_v$ is strictly pseudoconvex. Let U be an open subset of bG such that each connected component U_α of U can be represented as $U_\alpha = U_\alpha{}' \setminus K_\alpha$, where $U_\alpha{}'$ is a simply-connected subdomain of either bG^- or bG^+ and K_α is a compact subset of $U_\alpha{}'$. Then there is a function $\varphi \in C(U)$ which does not admit a continuous Levi-flat extension to any domain $\Omega \subset \overline{G}$ such that $\mathcal{E}(U) \subsetneq \Omega$.*

Remark. It is easy to see that in the case when the domain G is diffeomorphic to a 3-ball it is enough to assume that each connected component U_α of U is contained either in bG^- or bG^+.

Remark. The main part of our results holds true for domains G with much weaker regularity assumptions as in [CS] and [ST]. We present them for strictly pseudoconvex case to avoid additional technical difficulties.

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2. Preliminaries.

Lemma 1. *Let $\mathbb{D}$ be an unbounded strictly pseudoconvex domain in $\mathbb{C}^n$. Let K be a compact subset of $b\mathbb{D}$. Then there exists a bounded strictly pseudoconvex subdomain $\mathbb{D}'$ of $\mathbb{D}$ such that $K \subset b\mathbb{D}'$ and the $\mathcal{A}(\mathbb{D})$-hull $\widehat{K}_{\mathcal{A}(\mathbb{D})}$ is equal to the $\mathcal{A}(\mathbb{D}')$-hull $\widehat{K}_{\mathcal{A}(\mathbb{D}')}$ of K. In particular, in the case $n = 2$, this implies that the set $\mathbb{D} \setminus \widehat{K}_{\mathcal{A}(\mathbb{D})}$ is pseudoconvex.*

Proof. Let $\mathbb{B}_R = \{z \in \mathbb{C}^n : |z| < R\}$. Since $\widehat{K}_{\mathcal{A}(\mathbb{D})}$ is a compact subset of $\overline{\mathbb{D}}$, we can choose $R \in \mathbb{R}$ sufficiently large so that $\widehat{K}_{\mathcal{A}(\mathbb{D})} \subset \mathbb{B}_R \cap \overline{\mathbb{D}}$ and $b\mathbb{B}_R$ intersects $b\mathbb{D}$ transversally. Let us set $\mathbb{D}_1 = \mathbb{B}_R \cap \mathbb{D}$. Take $r < R$ such that the inclusion $\widehat{K}_{\mathcal{A}(\mathbb{D})} \subset \mathbb{B}_r \cap \overline{\mathbb{D}}$ still holds true and consider a neighbourhood $\mathbf{V}$ of $b\mathbb{B}_R \cap b\mathbb{D}$ such that $\mathbf{V} \cap \overline{\mathbb{B}}_r = \emptyset$. $b\mathbb{D}_1$ is piece-wise smooth so, given an open neighbourhood $\mathbf{V}$ of $b\mathbb{B}_R \cap b\mathbb{D}$, there exists a strictly pseudoconvex domain $\mathbb{D}'$ with the properties $\mathbb{D}' \subset \mathbb{D}_1$ and $\mathbb{D}_1 \setminus \mathbb{D}' \subset \mathbf{V}$ (see e.g. [T]). We claim that $\mathbb{D}'$ satisfies the properties of Lemma 1. Indeed, since $\mathcal{A}(\mathbb{D})$ is a subalgebra of $\mathcal{A}(\mathbb{D}')$, one has $\widehat{K}_{\mathcal{A}(\mathbb{D}')} \subset \widehat{K}_{\mathcal{A}(\mathbb{D})}$. Assume, by a contradiction, that $\widehat{K}_{\mathcal{A}(\mathbb{D})} \setminus \widehat{K}_{\mathcal{A}(\mathbb{D}')} \neq \varnothing$ and let $P \in \widehat{K}_{\mathcal{A}(\mathbb{D})} \setminus \widehat{K}_{\mathcal{A}(\mathbb{D}')}$. Then, by the definition of hull, there is a function $f \in \mathcal{A}(\mathbb{D}')$ such that $|f(P)| > 1$ and $||f||_K < 1$. We are going to approximate f by functions of $\mathcal{A}(\mathbb{D})$ what will provide the desired contradiction. In view of a theorem of Kerzman [K], for every $\varepsilon > 0$ there exists a function g holomorphic in a neighbourhood $\mathbf{W}$ of $\overline{\mathbb{D}}'$ such that $||g - f||_{\overline{\mathbb{D}}'} < \varepsilon$. Now we use the fact that $\overline{\mathbb{D}}$ has a basis of Stein neighbourhoods [T] to choose a Stein domain $\boldsymbol{\mathcal{U}} \supset \overline{\mathbb{D}}$ such that $\boldsymbol{\mathcal{U}} \cap \mathbb{B}_r \Subset \mathbf{W}$. Since $\boldsymbol{\mathcal{U}} \cap \mathbb{B}_r$ is a sublevel set in $\boldsymbol{\mathcal{U}}$ of a plurisubharmonic function $\rho(z) = |z|^2$, we know by a theorem of Narasimhan [N] that $\boldsymbol{\mathcal{U}} \cap \mathbb{B}_r$ is Runge in $\boldsymbol{\mathcal{U}}$. Hence, by the choice of r, there exists a function $\tilde{g}$ holomorphic in $\boldsymbol{\mathcal{U}}$ such that $||\tilde{g} - g||_{K \cup \{P\}} < \varepsilon$. It follows now that $g_1 = \tilde{g}|_{\overline{\mathbb{D}}}$ satisfies $||g_1 - f||_{K \cup \{P\}} < 2\varepsilon$. Thus, if ε is chosen sufficiently small, g_1 satisfies the same condition as f, $|g_1(P)| > 1$ and $||g_1||_K < 1$ which gives us a contradiction.

The last statement of lemma on pseudoconvexity of $\mathbb{D} \setminus \widehat{K}_{\mathcal{A}(\mathbb{D})}$ follows directly from strict pseudoconvexity of $\mathbb{D}$ and from the fact, which can be found for example in [St], that in the case $n = 2$ the set $\widehat{K}_{\mathcal{A}(\mathbb{D}')}$ is pseudoconcave in $\mathbb{D}'$. $\square$

3. Some properties of $\mathcal{E}(U)$.

Proof of Theorem 1. 1) Let P be a point of the set $\mathcal{E}(V)$. Consider an exhaustion $K_1 \subset K_2 \subset K_3 \subset \cdots$ of V by compact subsets such that for each $n = 1, 2, \ldots$ the set K_n is a subset of the interior part of K_{n+1}. Then, by definition of $\mathcal{E}(V)$, $P \in \widehat{K}_n$ for some $n \in \mathbb{N}$. Since $\mathbb{D}$ is strictly pseudoconvex, it follows that there is a one-sided neighbourhood $\mathbf{V}_{n+1}$ of the interior part of K_{n+1} (relatively to $b\mathbb{D}$) in $\overline{\mathbb{D}}$ filled by holomorphic discs with boundary on K_{n+1}. Consider $\delta_0 > 0$ such that for any vector $\boldsymbol{e}$ of length 1 one has $(K_n + \delta \boldsymbol{e}) \cap \overline{\mathbb{D}} \subset \mathbf{V}_{n+1}$ and $(\widehat{K}_n + \delta \boldsymbol{e}) \cap b\mathbb{D} \subset K_{n+1}$ for all δ, $0 \leq \delta < \delta_0$. For any such δ consider the set $E_{n,\delta} = (\widehat{K}_n + \delta \boldsymbol{e}) \cap \overline{\mathbb{D}}$. By Lemma 1, the set $\mathbb{D} \setminus \widehat{K}_n$ is pseudoconvex, consequently the set $E_{n,\delta}$ is pseudoconcave in $\mathbb{D} \setminus \mathbf{V}_{n+1}$ and, therefore, it is contained in the $\mathcal{A}(\mathbb{D})$-hull $\widehat{\overline{\mathbf{V}}}_{n+1}$ of $\overline{\mathbf{V}}_{n+1}$. (This is clear because if not, there exist $f \in \mathcal{A}(\mathbb{D})$ and $Q \in E_{n,\delta}$ such that $|f| < 1$ on $E_{n,\delta} \cap \overline{\mathbf{V}}_{n+1}$ and $|f(Q)| = 1$. Consider the function $\varrho = \log|f| + \varepsilon(|z|^2 + |w|^2)$; ϱ is strictly plurisubharmonic in $\mathbb{D} \setminus \{f = 0\}$ and, if ε is small enough, then for some sublevel set $L_a = \{\varrho < a\}$ we have $E_{n,\delta} \cap \overline{V}_{n+1} \Subset L_a$, $E_{n,\delta} \subset \overline{L}_a$ and $E_{n,\delta} \cap bL_a \neq \varnothing$ which contradicts the pseudoconcavity of $E_{n,\delta}$). Therefore,

$$E_{n,\delta} \subset \widehat{\overline{\mathbf{V}}}_{n+1} \subset \widehat{K}_{n+1}.$$

(The inclusion $\widehat{\overline{\mathbf{V}}}_{n+1} \subset \widehat{K}_{n+1}$ is a consequence of the maximum principle owing the fact that $\mathbf{V}_{n+1}$ is filled by holomorphic discs with boundary on K_{n+1}.) Hence, if $P + \delta \mathbf{e} \in \overline{\mathbb{D}}$, one has $P + \delta \mathbf{e} \in \widehat{K}_{n+1} \subset \mathcal{E}(V)$ for all δ such that $0 \leq \delta < \delta_0$. This proves that $\mathcal{E}(V)$ is an open subset of $\overline{\mathbb{D}}$.

The property $\mathcal{E}(V) \cap b\mathbb{D} = V$ follows directly from strict pseudoconvexity of $\mathbb{D}$ and the definition of $\mathcal{E}(V)$. Pseudoconvexity of the set $\mathcal{E}(V) \cap \mathbb{D}$ is the consequence of an obvious fact that the interior $\mathrm{int}\widehat{K}_n$ of each set $\widehat{K}_n$ is pseudoconvex and the equality $\mathcal{E}(V) \cap \mathbb{D} = \bigcup_{n=1}^{\infty} \mathrm{int}\widehat{K}_n$, which can be easily obtained from the property $\widehat{K}_n \cap \mathbb{D} \subset \mathrm{int}\widehat{K}_{n+1}$ proved above.

2) Let t be any real number and let $\mathbf{e}_v$ be the unit vector in the v-direction. Then for each $n \in \mathbb{N}$ one has $K_n + t\mathbf{e}_v \subset K_{n+k}$ for all k big enough, therefore $\widehat{K}_n + t\mathbf{e}_v \subset \widehat{K}_{n+k}$ and consequently

$$\mathcal{E}(V) + t\mathbf{e}_v = \bigcup_n (\widehat{K}_n + t\mathbf{e}_v) \subset \bigcup_n \widehat{K}_{n+k} = \mathcal{E}(V).$$

3) It follows from strict pseudoconvexity of $G\times\mathbb{R}_v\subset\mathbb{C}^2_{z,w}$ that the CR-hull $E(U\times\mathbb{R}_v)$ of the set $U\times\mathbb{R}_v$ is single-sheeted, i.e. it is a subdomain of $\overline{G}\times\mathbb{R}_v$. Since, by part 1, $(\mathcal{E}(U)\cap G)\times\mathbb{R}_v$ is pseudoconvex, we conclude that $E(U\times\mathbb{R}_v)\subset\mathcal{E}(U\times\mathbb{R}_v)$. Hence, we only need to prove the other inclusion $\mathcal{E}(U\times\mathbb{R}_v)\subset E(U\times\mathbb{R}_v)$.

Let $E_1\subset E_2\subset E_3\subset\cdots$ be an exhaustion of U by compact subsets such that each set E_n is contained in the interior part of E_{n+1}. Consider also an exhaustion of $U\times\mathbb{R}_v$ by compact subsets $K_n=E_n\times[-n,n]$. It follows from our assumption on simply-connectedness of the components of U, that for each $n\in\mathbb{N}$ there exist finitely many disjoint smooth simply-connected domains $D^p_n\Subset U$ such that $E_n\subset\cup_p D^p_n$. Consider piece-wise smooth 2-spheres

$$S^p_n=((bD^p_n)\times[-n-1,n+1])\cup(D^p_n\times\{-n-1\})\cup(D^p_n\times\{n+1\})$$

contained in bG, their smooth generic perturbations $\tilde{S}^p_n\subset bG\times\mathbb{R}_v$ and given by Kruzhilin [Kr] 3-balls B^p_n with the boundary $\tilde{S}^p_n$ foliated by holomorphic discs. Since $bB^p_n=\tilde{S}^p_n\subset bG\times\mathbb{R}_v\setminus K_n$, we conclude for each p from pseudoconvexity of the domain $G\times\mathbb{R}_v\setminus\widehat{K}_n$ and from "Kontinuitätsatz" that $B^p_n\cap\widehat{K}_n=\varnothing$ and that there is a neighbourhood $\mathbf{V}^p_n$ of B^p_n in $G\times\mathbb{R}_v$ such that $\mathbf{V}^p_n\subset E(U\times\mathbb{R}_v)$. Denote by $\mathcal{W}^p_n$ the bounded component of the set $G\times\mathbb{R}_v\setminus B^p_n$. Then, by strict pseudoconvexity of $G\times\mathbb{R}_v$, there is a neighbourhood $\mathbf{W}^p_n$ of $(bG\times\mathbb{R}_v)\cap\overline{\mathcal{W}}^p_n$ in $\overline{G}\times\mathbb{R}_v$ which is contained in $E(U\times\mathbb{R}_v)$. Since $\mathbf{V}^p_n\subset E(U\times\mathbb{R}_v)$ and $\mathbf{W}^p_n\subset E(U\times\mathbb{R}_v)$, we conclude from Hartogs theorem on removability of compact singularities that $\mathcal{W}^p_n\subset E(U\times\mathbb{R}_v)$ for all p. Then the inclusion $\widehat{K}_n\subset\bigcup_p\mathcal{W}^p_n$ implies that $\widehat{K}_n\subset E(U\times\mathbb{R}_v)$ and hence $\mathcal{E}(U\times\mathbb{R}_v)=\bigcup_n\widehat{K}_n\subset E(U\times\mathbb{R}_v)$. The proof of Theorem 1 is now completed. □

Proof of Theorem 2. We consider first the special case when the domain G is bounded. Then the set $\mathcal{E}(U)$ cannot have connected components with part of the boundary in two different components U_{α_1} and U_{α_2}. Indeed, let F be the connected component of bU_{α_1} such that U_{α_1} and U_{α_2} are contained in different connected components of the set $bG\setminus F$. Since F is connected, and since bG is diffeomorphic to a 2-sphere, the set $bG\setminus F$ is the union of simply-connected domains $\{V_\beta\}$. We can assume that the domains V_β are enumerated in such a way that $U_{\alpha_1}\subset V_1$. Let $D^1\subset D^2\subset\cdots$ be an exhaustion of the set $bG\setminus F$ by open subsets such that for each $k=1,2,...$ one has $D^k\Subset D^{k+1}$, the set D^k is the union of finitely many simply-connected domains with smooth boundary and for each β the set $D^k_\beta = D^k \cap V_\beta$ consists of at most one connected component. Observe that according to our enumeration for the domains $\{D^k_\beta\}$ one has $U_{\alpha_1} \subset \bigcup\limits_{k=1}^{+\infty} D^k_1$.

Let $E_1 \subset E_2 \subset \cdots$ be an exhaustion of U by compact subsets such that $E_n \Subset \overset{\circ}{E}_{n+1}$ and $E_n \cap U_\alpha$ is connected for every $n \in \mathbb{N}$ and every α. Let $\widehat{K}_n$ be the $\mathcal{A}(G \times \mathbb{R}_v)$-hull of $K_n = E_n \times [-n, n]$. Since $\bigcup\limits_k D^k = bG \setminus F \supset U$, it follows that for each n there is a number $k(n)$ such that $E_n \subset D^{k(n)}$. For each β such that $D^{k(n)} \cap V_\beta \neq \emptyset$ consider a piece-wise smooth 2-sphere

$$S^{k(n)}_\beta = ((bD^{k(n)}_\beta) \times [-n-1, n+1]) \cup (D^{k(n)}_\beta \times \{-n-1\}) \cup (D^{k(n)}_\beta \times \{n+1\})$$

contained in bG. Smoothing the edges of $S^{k(n)}_\beta$ and perturbing it a little bit we get a smooth generic 2-sphere $\widetilde{S}^{k(n)}_\beta \subset bG$. It follows then from Kružilin [Kr] (see also Bedford-Klingenberg [BK]), that there is a 3-ball $B^{k(n)}_\beta$ with the boundary $\widetilde{S}^{k(n)}_\beta$ foliated by holomorphic discs. Since, by our construction, the domains $D^{k(n)}_\beta$ are disjoint, the 3-balls $B^{k(n)}_\beta$ will also be disjoint. Since each ball $B^{k(n)}_\beta$ is foliated by holomorphic discs with the boundary $bS^{k(n)}_\beta \subset bG \setminus K_n$, we conclude from Lemma 1 and "Kontinuitätsatz" that $B^{k(n)}_\beta \cap \widehat{K}_n = \varnothing$.

For each β denote by $\mathcal{W}^{k(n)}_\beta$ the bounded component of the set $\overline{G} \times \mathbb{R}_v \setminus B^{k(n)}_\beta$. Then we have $\widehat{K}_n \subset \bigcup\limits_\beta \mathcal{W}^{k(n)}_\beta$. Consider the sets $\mathcal{W}_1 = \bigcup\limits_{n=1}^{+\infty} \mathcal{W}^{k(n)}_1$ and $\mathcal{W}_2 = \bigcup\limits_{n=1}^{+\infty} \bigcup\limits_{\beta \neq 1} \mathcal{W}^{k(n)}_\beta$. Since for each n the sets $\mathcal{W}^{k(n)}_\beta$ are relatively open in $\overline{G} \times \mathbb{R}_v$ and disjoint, and since for each β the family of sets $\mathcal{W}^{k(n)}_\beta$ is increasing with respect to n, it follows that the sets $\mathcal{W}_1$ and $\mathcal{W}_2$ are disjoint and relatively open in $\overline{G} \times \mathbb{R}_v$. The inclusion $\widehat{K}_n \subset \bigcup\limits_\beta \mathcal{W}^{k(n)}_\beta$ for each n implies that $\mathcal{E}(U \times \mathbb{R}_v) = \bigcup\limits_n \widehat{K}_n \subset \mathcal{W}_1 \cup \mathcal{W}_2$. Since $U_{\alpha_1} \times \mathbb{R}_v \subset \mathcal{W}_1$ and $U_{\alpha_2} \times \mathbb{R}_v \subset (U \setminus U_{\alpha_1}) \times \mathbb{R}_v \subset \mathcal{W}_2$, we conclude that the set $\mathcal{E}(U)$ cannot have connected components with parts of the boundary in two different connected components U_{α_1} and U_{α_2}.

In order to prove that $\mathcal{E}(U) = \bigcup\limits_\alpha \mathcal{E}(U_\alpha)$ it is enough to show that for each connected component U_{α_1} of U one has $\mathcal{E}(U) \cap \mathcal{W}_1 = \mathcal{E}(U_{\alpha_1})$ (see the definition of $\mathcal{W}_1$ above). Since $\mathcal{E}(U) \times \mathbb{R}_v = \bigcup\limits_n \widehat{K_n}$ and $\mathcal{E}(U_{\alpha_1}) \times \mathbb{R}_v =$

$\bigcup\limits_n \widehat{K}_n^{U_{\alpha_1}}$ (here $K_n^{U_{\alpha_1}} = K_n \cap (U_{\alpha_1} \times \mathbb{R}_v)$), it is sufficient to prove that for each n we have that $\widehat{K}_n \cap \mathcal{W}_1^{k(n)} = \widehat{K}_n^{U_{\alpha_1}}$. It follows from Lemma 1 that for each n the set $\mathcal{W}_1^{k(n)} \setminus (\widehat{K}_n \cap \mathcal{W}_1^{k(n)})$ is pseudoconvex and, therefore, $\widehat{K}_n \cap \mathcal{W}_1^{k(n)} \subset \widehat{K}_n^{U_{\alpha_1}}$. The opposite inclusion $\widehat{K}_n^{U_{\alpha_1}} \subset \widehat{K}_n \cap \mathcal{W}_1^{k(n)}$ is obvious, since $\widehat{K}_n^{U_{\alpha_1}} \subset \widehat{K}_n$. Hence $\widehat{K}_n \cap \mathcal{W}_1^{k(n)} = \widehat{K}_n^{U_{\alpha_1}}$ for each n and then

$$\mathcal{E}(U) \cap \mathcal{W}_1 = \bigcup_n (\widehat{K}_n \cap \mathcal{W}_1^{k(n)}) = \bigcup_n \widehat{K}_n^{U_{\alpha_1}} = \mathcal{E}(U_{\alpha_1}).$$

The connectedness of the sets $\mathcal{E}(U_\alpha)$ follows from the fact that $\mathcal{E}(U_\alpha) \times \mathbb{R}_v = \bigcup\limits_n \widehat{K}_n^{U_\alpha}$ and the connectedness of each set $\widehat{K}_n^{U_\alpha}$ which is due to Shilov Idempotent Theorem (see, for example [G], p. 88). This completes the proof in the case of bounded domain G.

Consider now the general case when the domain G is unbounded. It follows from the argument of Proposition 3.4 in [ST] that there is an exhaustion of G by bounded domains G_n such that $G_n \times \mathbb{R}_v$ are piece-wise strictly pseudoconvex and $G \cap \mathbb{B}_n \subset G_n$, where $\mathbb{B}_n$ is the ball in $\mathbb{C}_z \times \mathbb{R}_u$ of radius n with centre at the origin. Moreover, since the domain G is assumed to be diffeomorphic to a 3-ball, the covering model $\mathcal{G}$ of G is simply-connected, hence, we can use in the mentioned above construction of G_n an exhaustion of $\mathcal{G}$ by simply-connected subdomains to insure that each domain G_n is also diffeomorphic to a 3-ball. Smoothing the edges of G_n (see e.g. [T]), we get an exhaustion of G by bounded domains $\tilde{G}_n$ diffeomorphic to a 3-ball such that $\tilde{G}_n \times \mathbb{R}_v$ are strictly pseudoconvex and $G \cap \mathbb{B}_n \subset \tilde{G}_n$. Denote by U_n the interior part of the set $U \cap b\tilde{G}_n$. In view of the argument of Lemma 1, the set $\mathcal{E}(U_n)$ defined with respect to the domain G is the same as being defined with respect to $\tilde{G}_n$. It is also easy to see that $\mathcal{E}(U) = \bigcup\limits_{n=1}^{+\infty} \mathcal{E}(U_n)$. Therefore, applying the argument of the first part of the proof to $\tilde{G}_n$ and U_n and passing to the limit as $n \to \infty$ we obtain the required properties of $\mathcal{E}(U)$ in the general case. The proof of Theorem 2 is now completed. $\square$

Remark. In the proof of Theorem 2 the use of Bedford-Klingenberg Theorem is similar to the arguments of Eroskin [Er].

4. Characterization of the CR-hull of rigid surfaces.

Let, as above, U be an open subset of bG, where G is a domain in $\mathbb{C}_z \times \mathbb{R}_u$ such that $G \times \mathbb{R}_v$ is strictly pseudoconvex. Since the CR-hull $E(U \times \mathbb{R}_v)$ of $U \times \mathbb{R}_v$ is equal to the union of CR-hulls of its connected components, and since for any exhaustion $U_1 \Subset U_2 \Subset U_3 \Subset \cdots$ of U one has $E(U \times \mathbb{R}_v) = \cup_{n=1}^{\infty} E(U_n \times \mathbb{R}_v)$, it is enough to give a description of the CR-hull of $U \times \mathbb{R}_v$ for U being a smooth bounded domain in bG which is in general position with respect to the canonical projection $p : \overline{G} \to \overline{\mathcal{G}}$. The last assumption means that all the selfintersections and intersections with $b\mathcal{G}$ of the curve $p(bU)$ are transversal. In the rest of this section we will always assume that $U \subset bG$ is a smooth bounded domain in general position. In the case of an arbitrary domain U a characterization similar to the given below in Proposition 1 can be easily obtained by passing to the limit.

Let $U^- = U \cap bG^-$ and $U^+ = U \cap bG^+$, where bG^- and bG^+ are the "lower" and the "upper" part of bG, respectively, defined in the Introduction. Denote by Σ the union of all connected components of the set $p(U^-) \cap p(U^+)$ whose closure has a nonempty intersection with bG. Further, denote $\mathcal{U}^- = p(U^-) \setminus \Sigma$ and $\mathcal{U}^+ = p(U^+) \setminus \Sigma$. Consider the function f^- defined on $b\mathcal{U}^-$ which equals h^- on the set $b\mathcal{U}^- \setminus b\Sigma$ and h^+ on the set $b\mathcal{U}^- \cap b\Sigma$. Let F^- be the harmonic extension of f^- to $\mathcal{U}^-$. Similarly, consider the function f^+ defined on $b\mathcal{U}^+$ which equals h^+ on the set $b\mathcal{U}^+ \setminus b\Sigma$ and h^- on the set $b\mathcal{U}^+ \cap b\Sigma$. Let F^+ be the harmonic extension of f^+ to $\mathcal{U}^+$. Denote

$$\boldsymbol{\Sigma} = \{(z,u) : z = \pi(\zeta),\, h^-(\zeta) \le u \le h^+(\zeta),\, \zeta \in \Sigma\},$$

$$\boldsymbol{\mathcal{U}}^- = \{(z,u) : z = \pi(\zeta),\, h^-(\zeta) \le u < F^-(\zeta),\, \zeta \in \mathcal{U}^-\},$$

$$\boldsymbol{\mathcal{U}}^+ = \{(z,u) : z = \pi(\zeta),\, F^+(\zeta) < u \le h^+(\zeta),\, \zeta \in \mathcal{U}^+\},$$

and

$$\mathbf{U} = \boldsymbol{\Sigma} \cup \boldsymbol{\mathcal{U}}^- \cup \boldsymbol{\mathcal{U}}^+.$$

Now we observe that the sets $\boldsymbol{\Sigma}$, $\boldsymbol{\mathcal{U}}^-$ and $\boldsymbol{\mathcal{U}}^+$ are disjoint and, therefore, $\mathbf{U}$ is a one-sheeted subdomain of $\overline{G}$. Indeed, the set $\boldsymbol{\Sigma}$ is disjoint from the sets $\boldsymbol{\mathcal{U}}^-$ and $\boldsymbol{\mathcal{U}}^+$, since the corresponding set Σ is disjoint from the sets $\mathcal{U}^-$ and $\mathcal{U}^+$. To show that the set $\boldsymbol{\mathcal{U}}^-$ is disjoint from the set $\boldsymbol{\mathcal{U}}^+$ we consider an arbitrary connected component $\mathcal{V}$ of the set $\mathcal{U}^- \cap \mathcal{U}^+$. Note that, by definition of the sets $\mathcal{U}^-$ and $\mathcal{U}^+$, the set $\overline{\mathcal{V}}$ is relatively compact in $\mathcal{G}$. Denote by γ^- the set $b\mathcal{U}^- \cap \overline{\mathcal{V}}$ and by γ^+ the set $b\mathcal{U}^+ \cap \overline{\mathcal{V}}$. Since the functions h^- and $-h^+$ are strictly subharmonic, and since the functions F^- and F^+ are harmonic, one has

$h^-(\zeta) < F^-(\zeta) < h^+(\zeta)$ for all $\zeta \in \mathcal{U}^-$ and $h^+(\zeta) < F^+(\zeta) < h^+(\zeta)$ for all $\zeta \in \mathcal{U}^+$. Hence, $F^-(\zeta) < h^+(\zeta) = F^+(\zeta)$ for $\zeta \in \gamma^+$ and $F^-(\zeta) = h^-(\zeta) < F^+(\zeta)$ for $\zeta \in \gamma^-$. Since $b\mathcal{V}{=}\gamma^- \cup \gamma^+$, it follows that $F^-(\zeta) < F^+(\zeta)$ for all $\zeta \in \overline{\mathcal{V}}$ which implies that the sets $\boldsymbol{\mathcal{U}}^-$ and $\boldsymbol{\mathcal{U}}^+$ are disjoint. Now we can formulate the main statement of this section.

Proposition 1. *Let G be a domain in $\mathbb{C}_z \times \mathbb{R}_u$ such that $G \times \mathbb{R}_v$ is strictly pseudoconvex. Let U be an open bounded subdomain of bG such that bU is smooth and in general position with respect to the canonical projection $p : \overline{G} \to \overline{\mathcal{G}}$. Then for the CR-hull $E(U \times \mathbb{R}_v)$ of the set $U \times \mathbb{R}_v$ one has $E(U \times \mathbb{R}_v) = \mathbf{U} \times \mathbb{R}_v$.*

Proof. We show first that the CR-hull $E(U \times \mathbb{R}_v)$ of $U \times \mathbb{R}_v$ is single-sheeted, i.e. it is a subdomain of $\overline{G} \times \mathbb{R}_v$. Let $W_1 \Subset W_2 \Subset W_3 \Subset \cdots$ be an exhaustion of $U \times \mathbb{R}_v$ by open subsets. For each $n = 1, 2, \ldots$ consider a bounded strictly pseudoconvex subdomain G_n of $G \times \mathbb{R}_v$ such that $W_n \subset bG_n$. It follows then from [St] that $E(W_n)$ is equal to the complement in $\overline{G}_n$ of the $\mathcal{A}(G_n)$-hull of $bG_n \smallsetminus W_n$. In particular, $E(W_n)$ is single-sheeted. Since $E(U \times \mathbb{R}_v)$ is equal to the union of the increasing sequence of hulls $E(W_n)$, we conclude that $E(U \times \mathbb{R}_v)$ is also single-sheeted.

Now we observe that the domain $\mathbf{U} \times \mathbb{R}_v$ is pseudoconvex. Indeed, pseudoconvexity on the part $(\mathbf{U} \cap bG) \times \mathbb{R}_v$ of its boundary follows from strict pseudoconvexity of $G \times \mathbb{R}_v$. For proving pseudoconvexity of $\mathbf{U} \times \mathbb{R}_v$ inside $G \times \mathbb{R}_v$ we denote by I the set of those points of selfintersection of $p(bU)$ which are contained in $\overline{\Sigma}$. Further, we denote by $\mathbf{I}$ the set

$$\{(z,u) \in G : z = p(\zeta), h^-(\zeta) < u < h^+(\zeta), \zeta \in I\}.$$

It is easy to see that the set $\mathbf{I}$ is a finite union of the vertical (i.e. parallel to the u-axis) open segments. The "interior" part $(b\mathbf{U} \cap G) \times \mathbb{R}_v$ of the boundary of $\mathbf{U} \times \mathbb{R}_v$ is a Levi-flat surface constituted by pieces

$$\{(z,w) \in G \times \mathbb{R}_v : z = \pi(\zeta), u = F^-(\zeta), \zeta \in \mathcal{U}^-\},$$

$$\{(z,w) \in G \times \mathbb{R}_v : z = \pi(\zeta), u = F^+(\zeta), \zeta \in \mathcal{U}^+\},$$

and $\mathbf{I} \times \mathbb{R}_v$ each of which is foliated by holomorphic curves. Levi-flatness of the first two sets follows directly from harmonicity of the functions F^- and F^+. The third set is just the finite union of holomorphic strips each of which is a segment in u-direction multiplied by $\mathbb{R}_v$. Since the surface $(b\mathbf{U} \cap G) \times \mathbb{R}_v$ is Levi-flat, we conclude that the domain $\mathbf{U} \times \mathbb{R}_v$

is pseudoconvex on this part of its boundary and, therefore, the domain $\mathbf{U} \times \mathbb{R}_v$ itself is pseudoconvex.

In the remaining part of the proof we use "Kontinuitätssatz" to show that $\mathbf{U} \times \mathbb{R}_v \subset E(U \times \mathbb{R}_v)$. Since $\mathbf{U} \times \mathbb{R}_v$ is proved to be pseudoconvex, it will imply that $E(U \times \mathbb{R}_v) = \mathbf{U} \times \mathbb{R}_v$.

Note first that strict pseudoconvexity of the domain $G \times \mathbb{R}_v$ implies that there is a one-sided neighbourhood $\mathbf{V} \times \mathbb{R}_v$ of $U \times \mathbb{R}_v$ filled by holomorphic discs with the boundary on $U \times \mathbb{R}_v$. $\mathbf{V} \times \mathbb{R}_v$ is obviously contained in $E(U \times \mathbb{R}_v)$. We proceed further in two steps. In the first step we "fill" the set $\mathbf{\Sigma} \times \mathbb{R}_v$ by a continuous family of holomorphic discs. More precisely, for each point $Q \in \mathbf{\Sigma}$ we construct a family of holomorphic discs $\{D_t\}$, $0 \le t \le 1$, such that $D_0 \subset \mathbf{V} \times \mathbb{R}_v$, $bD_t \subset \mathbf{V} \times \mathbb{R}_v$ for all $t \in [0,1]$ and $D_t \cap (\{Q\} \times \mathbb{R}_v) \neq \varnothing$ for some t. Since the surface $U \times \mathbb{R}_v$ is rigid, and since Q is an arbitrary point of $\mathbf{\Sigma}$, this will imply by "Kontinuitätssatz" that the whole set $\mathbf{\Sigma} \times \mathbb{R}_v$ is contained in $E(U \times \mathbb{R}_v)$. In the second step we "fill" the rest of $\mathbf{U} \times \mathbb{R}_v$ by a continuous family of holomorphic discs. Namely, for each point $Q \in \boldsymbol{\mathcal{U}}^- \cup \boldsymbol{\mathcal{U}}^+$ we construct holomorphic discs $\{D_t{}'\}$, $0 \le t \le 2$, such that $D_0' \subset \mathbf{V} \times \mathbb{R}_v$, $bD_t' \subset (\mathbf{\Sigma} \cup \mathbf{V}) \times \mathbb{R}_v$ for all $t \in [0,2]$ and $D_t{}' \cap (\{Q\} \times \mathbb{R}_v) \neq \varnothing$ for some t. Then, as above, by "Kontinuitätssatz" we conclude that $E(U \times \mathbb{R}_v) = \mathbf{U} \times \mathbb{R}_v$ as it is stated in proposition.

Step 1. Let Q be a point of the set $\mathbf{\Sigma}$. To simplify the proof we assume that the set $\mathbf{\Sigma}$ is connected. In the general case one can apply the argument below to each of the connected components of $\mathbf{\Sigma}$. Consider a point ζ_0 contained in the interior of one of the arcs constituting the set $b\,\mathcal{G} \cap \overline{\Sigma}$. Since the set $\Sigma = p(\mathbf{\Sigma})$ is connected, there is a smooth curve $\gamma(t) \subset \Sigma$, $0 \le t \le 1$, connecting the points ζ_0 and $p(Q) \in \Sigma$ and transversal to $b\mathcal{G}$ at the point ζ_0. For each $T \le 1$ denote by γ_{T} the part of γ corresponding to $t \in [0,T]$. Let $\mathcal{W}$ be a neighbourhood of ζ_0 in $\overline{\mathcal{G}}$ so small that $p^{-1}(\mathcal{W}) \subset \mathbf{V}$. Choose $\delta > 0$ small enough so that for the δ-neighbourhood $\mathcal{V}_\delta(\gamma)$ of γ in $\overline{\mathcal{G}}$ one has $\mathcal{V}_\delta(\gamma) \cap b\Sigma \subset \mathcal{W}$. Consider $T_0 < 1$ such that $\mathcal{V}_\delta(\gamma_{T}) \subset \mathcal{W}$ for all $T \in [0, T_0]$. For $\delta_1 \in (0,\delta)$ and for each $T \in [T_0, 1]$ consider the domain $\mathcal{A}_T = \mathcal{V}_\delta(\gamma_{T}) \setminus \overline{\mathcal{V}}_{\delta_1}(\gamma_{T}) \subset \overline{\mathcal{G}}$. The boundaries of the domains $\mathcal{A}_T$ are smooth everywhere except for some points of the set $b\Sigma \cap \mathcal{W}$ where they have transversal intersection with $b\mathcal{G}$. It is easy to see that the domains $\mathcal{A}_T$ depend continuously on T. For each $T \in [T_0, 1]$ define the function $\phi_T(\zeta)$ on $b\mathcal{A}_T$ to be equal to $h^-(\zeta)$ on $b\mathcal{V}_\delta(\gamma_{T}) \cap \mathcal{G}$, to be equal to $h^+(\zeta)$ on $b\mathcal{V}_{\delta_1}(\gamma_{T}) \cap \mathcal{G}$ and finally to be equal to $h^-(\zeta) = h^+(\zeta)$ on $b\mathcal{A}_T \cap b\mathcal{G}$. Then $\{\phi_T\}$ is a family of functions of Hölder class $C^{1/2}(b\mathcal{A}_T)$ depending continuously on T (each of these functions will behave like $\pm\sqrt{\operatorname{dist}(\zeta, b\mathcal{G})}$ as $\zeta \to b\mathcal{G}$).

Let F_T be the harmonic extension of ϕ_T to $\mathcal{A}_T$. Since each domain $\mathcal{A}_T$ is constructed to be simply-connected, for each $T \in [T_0, 1]$ there exists a holomorphic function G_T defined on $\mathcal{A}_T$ and continuous $\overline{\mathcal{A}}_T$ such that $\operatorname{Re} G_T = F_T$ (these functions will actually be also in some Hölder class on $\bar{\mathcal{A}}_T$). One can obviously choose the functions G_T to be continuously dependent of T. Consider the family of holomorphic discs

$$D_T = \{(z, w) : z = \pi(\zeta), w = G_T(\zeta), \zeta \in \bar{\mathcal{A}}_T\}, \, T_0 \leq T \leq 1.$$

Then, by our construction, D_T depend continuously on T, for each T one has $bD_T \subset U \times \mathbb{R}_v$ and, since $\mathcal{A}_{T_0} \subset \mathcal{W}$, we conclude that $D_{T_0} \subset \mathbf{V} \times \mathbb{R}_v$. Moreover, the family of domains $\mathcal{A}_T$ and the functions F_T were constructed in such a way that the graph of some function F_T in our family have to contain the point Q and, hence, the corresponding holomorphic disc D_T will contain a point of $\{Q\} \times \mathbb{R}_v$. It follows then from "Kontinuitätssatz" that this point of $\{Q\} \times \mathbb{R}_v$ is contained in $E(U \times \mathbb{R}_v)$. Since the surface $U \times \mathbb{R}_v$ is rigid, the whole line $\{Q\} \times \mathbb{R}_v$ is contained in $E(U \times \mathbb{R}_v)$, and since Q is an arbitrary point of $\boldsymbol{\Sigma}$, we finally conclude that $\boldsymbol{\Sigma} \times \mathbb{R}_v \subset E(U \times \mathbb{R}_v)$.

Step 2. Here we need to prove that $\boldsymbol{\mathcal{U}}^- \cup \boldsymbol{\mathcal{U}}^+ \subset E(U \times \mathbb{R}_v)$. Since the argument for both sets $\boldsymbol{\mathcal{U}}^-$ and $\boldsymbol{\mathcal{U}}^+$ is the same, we will show that $\boldsymbol{\mathcal{U}}^- \subset E(U \times \mathbb{R}_v)$. We can assume in what follows that the set $\boldsymbol{\mathcal{U}}^-$ is connected. If not, we use the same argument for each connected component of $\boldsymbol{\mathcal{U}}^-$ separately. In particular, our assumption implies that $\mathcal{U}^-$ is a (connected) domain in $\overline{\mathcal{G}}$.

Let Q be a point of $\boldsymbol{\mathcal{U}}^-$. Then $Q = (\pi(\zeta_Q), u_Q)$ for some $\zeta_Q \in \mathcal{U}^-$. It follows from the definition of $\boldsymbol{\mathcal{U}}^-$ that $\varepsilon \stackrel{\text{def}}{=} F^-(\zeta_Q) - u_Q > 0$, where F^- is the function defining the "upper" (with respect to the u-direction) part of $b\boldsymbol{\mathcal{U}}^-$. Remind, that by definition of the function $f^- = F^-|_{b\mathcal{U}^-}$ it is continuous everywhere except maybe for finitely many points contained in $b\mathcal{U}^- \cap b\mathcal{U}^+$.

Now we consider a new domain $\widetilde{\mathcal{U}}^-$ obtained by shrinking the domain $\mathcal{U}^-$ on the part $b\mathcal{U}^- \setminus b\Sigma$ of its boundary and enlarging it on the resting part $b\mathcal{U}^- \cap b\Sigma$ of the boundary so that the domain $\widetilde{\mathcal{U}}^-$ will have a smooth boundary. Further, we approximate the function f^- by a smooth function $\tilde{f}^-$ defined on $b\widetilde{\mathcal{U}}^-$. More precisely, let $\widetilde{\mathcal{U}}^-$ be a smoothly bounded domain over $\mathcal{G}$ with projection $\tilde{\pi} : \widetilde{\mathcal{U}}^- \to \mathcal{G}$ and let $\tilde{f}^-$ be a smooth function on $b\widetilde{\mathcal{U}}^-$ with harmonic extension $\widetilde{F}^-$ to $\widetilde{\mathcal{U}}^-$ such that $\tilde{\pi}(\widetilde{\mathcal{U}}^-) \subset \mathcal{U}^- \cup \Sigma$,

$$\Gamma(\tilde{f}^-) \stackrel{\text{def}}{=} \{(z, u) : z = \pi(\tilde{\pi}(\xi)), u = \tilde{f}^-(\xi), \xi \in b\widetilde{\mathcal{U}}^-\} \subset U^- \cup \boldsymbol{\Sigma}$$

and $\widetilde{F}^-(\xi_Q) > F^-(\zeta_Q) - \varepsilon/2$, where $\xi_Q = \tilde{\pi}^{-1}(\zeta_Q)$. The last inequality implies that $u_Q < \widetilde{F}^-(\xi_Q) - \varepsilon/2$. The problem now is that the domain $\widetilde{\mathcal{U}}^-$ might be multi-connected, while for being able to use "Kontinuitätssatz" we need to have a simply-connected domain. That is why we need to modify the domain $\widetilde{\mathcal{U}}^-$ and the function $\tilde{f}^-$ further. To do this we use the argument of Lemma 5.1 in [ST]. For the convenience of reading we remind the construction used there.

Let $\Gamma_1, \Gamma_2, \dots, \Gamma_m$ be the connected components of $b\widetilde{\mathcal{U}}^-$. Consider the universal covering $\tilde{\tilde{\pi}} : \Delta \to \widetilde{\mathcal{U}}^-$ of $\widetilde{\mathcal{U}}^-$ by the unit disc Δ. Let $\gamma_1, \gamma_2, \dots, \gamma_{m-1}$ be a disjoint family of smooth curves in $\widetilde{\mathcal{U}}^-$ such that each curve γ_i, $i = 1, \dots, m-1$, connects the corresponding curves Γ_i and Γ_{i+1} and non of the curves γ_i contains the point ξ_Q. Since, by the definition of $\mathcal{U}^-$ and Σ, non of the connected components of $b\mathcal{U}^-$ is contained in Σ, we can choose the curves γ_i to be contained in $\tilde{\pi}^{-1}(\mathcal{U}^-)$. Then the domain $\widetilde{\mathcal{U}}^- \setminus \bigcup_{i=1}^{m-1} \gamma_i$ is simply-connected. Hence, the set $\tilde{\tilde{\pi}}^{-1}(\tilde{\mathcal{U}}^- \setminus \bigcup_{i=1}^{m-1} \gamma_i)$ is the disjoint union of a countable family of simply-connected subdomains $\{\mathcal{F}_\sigma\}$ of Δ. For each σ denote by $\mathcal{F}_\sigma^*$ the union of the set $\overline{\mathcal{F}}_\sigma \cap \Delta$ with all the domains $\mathcal{F}_{\sigma'}$ such that $\overline{\mathcal{F}}_\sigma \cap \overline{\mathcal{F}}_{\sigma'} \neq \varnothing$.

Choose now one of the domains $\{\mathcal{F}_\sigma\}$ and denote it by $\mathcal{F}_0$. Denote by $\mathcal{F}_1$ the corresponding domain $\mathcal{F}_0^*$. Define inductively for each $n = 2, 3, \dots$ a domain $\mathcal{F}_n$ as the union of all the domains $\mathcal{F}_\sigma^*$ such that $\mathcal{F}_\sigma \subset \mathcal{F}_{n-1}$. It follows directly from the definition that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ and that $\bigcup_{n=0}^{+\infty} \mathcal{F}_n = \Delta$.

The following lemma was proved in [ST] (Lemma 5.1).

Lemma. *There exists a positive number $\theta < 1$ such that the inequality for harmonic measure*

$$\omega(\eta, b\mathcal{F}_n \cap \Delta, \mathcal{F}_n) \leq \theta^n$$

holds true for all $\eta \in \overline{\mathcal{F}}_0$ and all $n = 1, 2, \dots$.

Denote by η_Q the point $\tilde{\tilde{\pi}}^{-1}(\xi_Q) \cap \mathcal{F}_0$. Let $\tilde{\tilde{f}}_n^-$ be the function on $b\mathcal{F}_n$ such that $\tilde{\tilde{f}}_n^-(\eta) = \tilde{f}^-(\tilde{\tilde{\pi}}(\eta))$ for $\eta \in b\mathcal{F}_n \cap b\Delta$ and $\tilde{\tilde{f}}_n^-(\eta) = h^-(\tilde{\pi}(\tilde{\tilde{\pi}}(\eta)))$ for $\eta \in b\mathcal{F}_n \cap \Delta$. Consider the harmonic extension $\tilde{\tilde{F}}_n^-$ of the function $\tilde{\tilde{f}}_n^-$ to the domain $\mathcal{F}_n$. It follows then from Lemma 5.1 in [ST] recalled above that $\tilde{\tilde{F}}_n^- > \tilde{F}^-(\xi_Q) - \varepsilon/2$ if n is big enough. In particular, $u_Q < \tilde{\tilde{F}}_n^-(\eta_Q)$. Note, that by the construction of $\mathcal{F}_n$ and $\tilde{\tilde{F}}_n^-$ one has

$\tilde{\pi}(\tilde{\tilde{\pi}}(\mathcal{F}_n)) \subset \mathcal{U}^- \cup \Sigma$ and

$$\Gamma(\tilde{\tilde{F}}_n^-) \stackrel{\text{def}}{=} \{(z,u) : z = \pi(\tilde{\pi}(\tilde{\tilde{\pi}}(\eta))), u = \tilde{\tilde{f}}_n^-(\eta), \eta \in b\mathcal{F}_n\} \subset U^- \cup \mathbf{\Sigma}.$$

Now we are in position to construct the family of holomorphic discs required by "Kontinuitätssatz". Define for $t \in [1,2]$ the function $\phi_t{}'$ on $b\mathcal{F}_n$ as $\phi_t{}'(\eta) = (1-t)\tilde{\tilde{f}}_n^- + (2-t)h^-(\tilde{\pi}(\tilde{\tilde{\pi}}(\eta)))$ and for each t consider the harmonic extension $\tilde{\tilde{F}}_t$ of the function $\phi_t{}'$ to the domain $\mathcal{F}_n$. Since $\mathcal{F}_n$ is a simply-connected domain with a piece-wise smooth boundary, and since the functions $\phi_t{}'$ are smooth on $b\mathcal{F}_n$, it follows that there is a one-parameter family of holomorphic functions $G_t{}'$ defined $\mathcal{F}_n$ and continuous on $\overline{\mathcal{F}}_n$ which depend continuously on the parameter t and such that $\operatorname{Re} G_t{}' = \tilde{\tilde{F}}_t$. Consider the family of holomorphic discs

$$D_t{}' = \{(z,w) : z = \pi(\tilde{\pi}(\tilde{\tilde{\pi}}(\eta))), w = G_t{}'(\eta), \eta \in \overline{\mathcal{F}}_n\},\ 1 \le t \le 2.$$

Then, by our construction of $\mathcal{F}_n$ and $\phi_t{}'$, for each $t \in [1,2]$ one has $bD_t{}' \subset (U^- \cup \mathbf{\Sigma}) \times \mathbb{R}_v$ and, since $\tilde{\tilde{F}}_1(\eta) = h^-(\tilde{\pi}(\tilde{\tilde{\pi}}(\eta)))$ for all $\eta \in b\mathcal{F}_n$, one also has $bD_1{}' \subset U \times \mathbb{R}_v$.

To finish the proof we use the fact that the domain $\mathcal{F}_n$ is simply-connected. Namely, consider a one-parameter family of closed smoothly bounded simply-connected subdomains $\mathcal{A}_t{}'$, $0 \le t \le 1$, of $\mathcal{F}_n$ which depends continuously on t and such that $\mathcal{A}_1{}' = \mathcal{F}_n$ and $\mathcal{A}_0{}'$ is a point in $\mathcal{F}_n$. For each $t \in [0,1]$ let $\tilde{\tilde{F}}_t$ be the harmonic extension of the function $\phi_t{}'(\eta) = h^-(\tilde{\pi}(\tilde{\tilde{\pi}}(\eta)))$ from $b\mathcal{A}_t{}'$ to $\mathcal{A}_t{}'$. Consider a one-parameter family of holomorphic functions $G_t{}'$ defined on the respective domains $\mathcal{A}_t{}'$ which depend continuosly on t and such that $\operatorname{Re} G_t{}' = \tilde{\tilde{F}}_t$. Then for the family of holomorphic discs

$$D_t{}' = \{(z,w) : z = \pi(\tilde{\pi}(\tilde{\tilde{\pi}}(\eta))), w = G'_t(\eta), \eta \in \bar{\mathcal{A}}_n\},$$

$0 \le t \le 1$, one obviously has that $D_0{}'$ is just a point in $U \times \mathbb{R}_v$ and, for each $t \in [0,1]$, $bD_t{}' \subset U \times \mathbb{R}_v$. If we consider the whole family of discs $D_t{}'$, $0 \le t \le 2$, then it will depend continuously on t, D_0 will be a point in $U \times \mathbb{R}_v$ and for each t one will have $bD_t{}' \subset (U^- \cup \mathbf{\Sigma}) \times \mathbb{R}_v$. Observe now that by our construction one has

$$\bigcup_{0 \le t \le 2} \pi_{z,u}(D_t{}') = \{(z,u) : \pi(\tilde{\pi}(\tilde{\tilde{\pi}}(\eta))), h^-(\tilde{\pi}(\tilde{\tilde{\pi}}(\eta))) \le u \le \tilde{\tilde{F}}_n^-(\eta), \eta \in \mathcal{F}_n\},$$

where $\pi_{z,u} : \mathbb{C}^2_{z,w} \to \mathbb{C}_z \times \mathbb{R}_u$ is the projection. Since $u_Q < \tilde{\tilde{F}}_n^-(\eta_Q)$, it follows that there is a disc $D_{t_0}{}'$ in our family such that $Q \in \pi_{z,u}(D_{t_0}{}')$. This means that there is a point of $\{Q\} \times \mathbb{R}_v$ contained in $D_{t_0}{}'$. It follows then from "Kontinuitätssatz" that this point is contained in the CR-hull of the set $(U^- \cup \mathbf{\Sigma}) \times \mathbb{R}_v$, which is, in view of Step 1, contained in $E(U \times \mathbb{R}_v)$. Rigidity of the domain $E(U \times \mathbb{R}_v)$ imply now that the line $\{Q\} \times \mathbb{R}_v$ is contained in $E(U \times \mathbb{R}_v)$ and then, by the choice of Q, the whole set $\boldsymbol{\mathcal{U}}^- \cup \boldsymbol{\mathcal{U}}^+$ is contained in $E(U \times \mathbb{R}_v)$. This complete the argument of Step 2 and the proof of Proposition 1. $\square$

5. Proof of Theorems 3 and 4.

Proof of Theorem 3. The fact that the sets $\mathcal{E}(V_\varphi^\pm)$ are open in $\overline{G} \times \mathbb{R}_v$ follows from the part 1 of Theorem 1 and the fact that these sets can be represented in the form (1) and (2) follows from the same argument as in the part 2 of Theorem 1. The property $\Phi^\pm|_U = \varphi$ follows directly from the definition of $\mathcal{E}(V_\varphi^\pm)$. $\square$

Proof of Theorem 4. We start with the proof of continuity of the functions $\Phi^\pm$ at the points of the set U. Since the domain $G \times \mathbb{R}_v$ is strictly pseudoconvex, there are one-sided neighbourhoods $\mathbf{V}^+$ and $\mathbf{V}^-$ of the sets V_φ^+ and V_φ^- in $\overline{G} \times \mathbb{R}_v$, respectively, filled by holomorphic discs with the boundary on the respective set V_φ^+ and V_φ^-. Since $\mathbf{V}^+ \subset \mathcal{E}(V_\varphi^+)$ and $\mathbf{V}^- \subset \mathcal{E}(V_\varphi^-)$, and since directly from the definition of $\Phi^\pm$ one has $\Phi^-(z,u) \leq \Phi^+(z,u)$ for $(z,u) \in \mathcal{E}(U)$, we conclude that $\Phi^\pm$ are continuous at the points of U.

To prove continuity of the functions $\Phi^\pm$ in the interior points of $\mathcal{E}(U)$ we need the following

Lemma 2. *There is an increasing sequence of open relatively compact subsets $\{\mathcal{E}_n\}$ of $\mathcal{E}(U)$ such that*

1) *$\mathcal{E}_n \times \mathbb{R}_v$ is piece-wise strictly pseudoconvex for every $n \in \mathbb{N}$*

and

2) *For each compact subset K of $\mathcal{E}(U)$ one has $K \subset \overline{\mathcal{E}}_n$ for all n big enough.*

Proof. Let, as above, $E_1 \subset E_2 \subset E_3 \subset \cdots$ be an exhaustion of U by compact subsets and let $K_n = E \times [-n, n]$. Since $\mathcal{E}(U) = \bigcup_n \pi_{z,u}(\widehat{K}_n)$, where $\pi_{z,u} : \mathbb{C}^2_{z,w} \to \mathbb{C}_z \times \mathbb{R}_u$ is the projection, it is enough to construct for each $n \in \mathbb{N}$ an open subset $\mathcal{E}_n$ of $\mathcal{E}(U)$ satisfying property 1 and such that $\pi_{z,u}(\widehat{K}_n) \subset \overline{\mathcal{E}}_n$. For n being fixed, we can assume in the construction of $\mathcal{E}_n$ that the domain G is bounded. If not, we use the same argument as in Lemma 1. Namely, for r large enough consider the connected component G_r of $G \cap \mathbb{B}_r$ containing E_n on the boundary. Smoothing the edges of G_r (see e.g. [T]), we obtain a bounded domain $\tilde{G}_r$ such that $\tilde{G}_r \times \mathbb{R}_v$ is strictly pseudoconvex. Finally, we can replace the domain G by $\tilde{G}_r$ and the set U by the interior part of the set $U \cap b\tilde{G}_r$.

Let ϱ be the defining function of the domain G, i.e. $G = \{(z,u) : \varrho(z,u) < 0\}$ and $\operatorname{grad} \varrho \neq 0$ on bG. For each $\delta > 0$ consider the domain $G^\delta = \{(z,u) : \varrho(z,u) < \delta\}$ and the set

$$U_n^\delta = \{(z,u) : \varrho(z,u) = \delta,\ \operatorname{dist}((z,u), E_n) < \varepsilon_n/2\},$$

where $\varepsilon_n = \operatorname{dist}(E_n, bU)$ and E_n is the corresponding compact from the exhaustion of U. Let $\mathcal{E}(U_n^\delta)$ be the set defined in the same way as the set $\mathcal{E}(U)$ with G and U replaced by G^δ and U_n^δ, respectively. Since for all δ small enough the domains $G^\delta \times \mathbb{R}_v$ are strictly pseudoconvex, there are uniformly big one-sided neighbourhoods $\mathbf{V}_n^\delta$ of the sets $U_n^\delta \times \mathbb{R}_v$ in the respective domains $\overline{G}^\delta \times \mathbb{R}_v$ filled by holomorphic discs with the boundary on the corresponding set $U_n^\delta \times \mathbb{R}_v$. Then, for some $\delta_0 > 0$ one has $K_n \subset \mathbf{V}_n^{\delta_0} \subset \mathcal{E}(U_n^{\delta_0}) \times \mathbb{R}_v$. Therefore, by the definition of $\mathcal{E}(U_n^{\delta_0})$, one also has that $\widehat{K}_n \subset \mathcal{E}(U_n^{\delta_0}) \times \mathbb{R}_v$, i.e. $\pi_{z,u}(\widehat{K}_n) \subset \mathcal{E}(U_n^{\delta_0})$.

It also follows from strict pseudoconvexity of the domains $G^\delta \times \mathbb{R}_v$ that there are uniformly big one-sided neighbourhoods $\tilde{\boldsymbol{V}}_n^\delta$ of the sets $(bG^\delta \smallsetminus \overline{U}_n^\delta) \times \mathbb{R}_v$ in the respective domains $\overline{G}^\delta \times \mathbb{R}_v$ filled by holomorphic discs with the boundary on the corresponding set $(bG^\delta \smallsetminus \overline{U}_n^\delta) \times \mathbb{R}_v$. Since, by definition, the set $\mathcal{E}(U_n^\delta) \times \mathbb{R}_v$ is the union of the hulls of compacts exhausting the set $U_n^\delta \times \mathbb{R}_v$, and since, by Lemma 1, the discs filling $\tilde{\boldsymbol{V}}_n^\delta$ cannot intersect these hulls, it follows that $bG \cap \mathcal{E}(U_n^\delta) \subset U$ and, therefore, $\mathcal{E}(U_n^\delta) \cap \overline{G} \subset \mathcal{E}(U)$ for all δ small enough. We can assume that this inclusion holds for δ_0 chosen above (otherwise we choose the smallest of these two δ's). In view of Theorem 1, the interior part of the set $\mathcal{E}(U_n^{\delta_0}) \times \mathbb{R}_v$ is pseudoconvex, hence the function

$$h(z,u) = -\log \operatorname{dist}((z,u), b\mathcal{E}(U_n^{\delta_0}) \times \mathbb{R}_v)$$

is plurisubharmonic in the interior part of the set $\mathcal{E}(U_n^{\delta_0}) \times \mathbb{R}_v$. Smoothing this function and adding the function $\theta(|z|^2 + u^2)$, $\theta > 0$ being small enough, we get a strictly plurisubharmonic function $\tilde{h}(z,u)$. Choose $\eta > 0$ sufficiently big such that the domain $\mathcal{E}_\eta(U_n^{\delta_0}) = \{(z,u) : \tilde{h}(z,u) < \eta\}$ is close enough to $\mathcal{E}(U_n^{\delta_0})$ and $b\mathcal{E}_\eta(U_n^{\delta_0})$ is transversal to bG. Define

$\mathcal{E}_n = \mathcal{E}_\eta(U_n^{\delta_0}) \cap G$. It follows then from the construction of $\mathcal{E}_\eta(U_n^{\delta_0})$ that the domain $\mathcal{E}_n \times \mathbb{R}_v$ is piece-wise strictly pseudoconvex and that $\pi_{z,u}(\widehat{K}_n) \subset \overline{\mathcal{E}}_n$. This completes the proof of the lemma. $\square$

Now we can proceed the same way as in [ST]. Namely, for each $n \in \mathbb{N}$ consider the subset $A_n = bG \cap \overline{\mathcal{E}}_n$ of $b\mathcal{E}_n$. Further, for each $s > 0$ define the subset $B_{n,s}$ of $b\mathcal{E}_n$:

$$B_{n,s} = \{P \in b\mathcal{E}_n : \text{dist } (P, bG) \geq 1/s\}.$$

Consider a one-parameter family of continuous functions $\varphi_{n,t}$, $t \in \mathbb{R}$, defined on $b\mathcal{E}_n$, continuously depending on the parameter t such that: $\varphi_{n,t} = \varphi$ on A_n, where φ is the given continuous function on U, $\varphi_{n,t} = t$ on $B_{n,|t|}$ and $\varphi_{n,t_1} \leq \varphi_{n,t_2}$ if $t_1 \leq t_2$. Then, in view of the construction of $\mathcal{E}_n$, we can apply Theorem 2 of [CS] to the functions $\varphi_{n,t}$ and obtain for each t a function $\Phi_{n,t} \in C(\overline{\mathcal{E}}_n)$ such that $\Phi_{n,t} = \varphi_{n,t}$ on $b\mathcal{E}_n$ and the graph $\Gamma(\Phi_{n,t}) \smallsetminus \Gamma(\varphi_{n,t})$ is (locally) foliated by 1-dimensional complex submanifolds. Define functions $\widetilde{\Phi}^{\pm}$ on $\mathcal{E}(U)$ as

$$\widetilde{\Phi}^{+}(P) = \lim_{n\to+\infty} \lim_{t\to+\infty} \Phi_{n,t}(P) \quad \text{and} \quad \widetilde{\Phi}^{-}(P) = \lim_{n\to+\infty} \lim_{t\to-\infty} \Phi_{n,t}(P).$$

Remark. We will prove later (see Lemma 5) that the functions $\widetilde{\Phi}^{\pm}$ coincide with $\Phi^{\pm}$, respectively, and then we only use the notation $\Phi^{\pm}$.

In the next lemma we prove the uniform boundedness of the functions $\Phi_{n,t}$ on compact subsets of $\mathcal{E}(U)$.

Lemma 3. *For each compact subset K of $\mathcal{E}(U)$ there exist a constant $C > 0$ and a number N such that $|\Phi_{n,t}(P)| \leq C$ for all $P \in K$, $t \in \mathbb{R}$ and $n \geq N$.*

Proof. Let K be a compact subset of $\mathcal{E}(U)$ and let $\mathcal{K}_1^{\pm} \subset \mathcal{K}_2^{\pm} \subset \mathcal{K}_3^{\pm} \subset \cdots$ be exhaustions by compact subsets of the sets $V_\varphi^{\pm}$, respectively, such that for every $n \in \mathbb{N}$ each set $\mathcal{K}_n^{\pm}$ is a subset of the interior part of the respective set $\mathcal{K}_{n+1}^{\pm}$. Since, by Theorem 3, $\pi_{z,u}(\mathcal{E}(V_\varphi^{\pm})) = \mathcal{E}(U)$, it follows that there is a number L such that $K \subset \pi_{z,u}(\widehat{\mathcal{K}}_L^{\pm})$. Consider a number N such that $\pi_{z,u}(\widehat{\mathcal{K}}_L^{\pm}) \subset \mathcal{E}_N$. Then for each $n \geq N$ and $t \in \mathbb{R}$ the graph $\Gamma(\Phi_{n,t})$ of the function $\Phi_{n,t}$ does not intersect the sets $\widehat{\mathcal{K}}_L^{\pm}$.

Since the argument is the same for both sets $\widehat{\mathcal{K}}_L^{\pm}$, we prove this property for the set $\widehat{\mathcal{K}}_L^{+}$.

Indeed, for $n \geq N$ and $t \in \mathbb{R}$ being fixed, it follows from continuity of $\Phi_{n,t}$ on $\mathcal{E}_n$ and compactness of $\widehat{\mathcal{K}}_L^{+}$ that $\Gamma(\Phi_{n,t} - C) \cap \widehat{\mathcal{K}}_L^{+} = \varnothing$ for $C > 0$ big enough. Since, by definition of $\varphi_{n,t}$ and the choice of $\mathcal{E}_N$, $\Gamma(\varphi_{n,t} - s) \cap \widehat{\mathcal{K}}_L^{+} = \varnothing$ for $s \geq 0$, and since, by Lemma 1, the domain $G \times \mathbb{R}_v \setminus \widehat{\mathcal{K}}_L^{+}$ is pseudoconvex, we conclude from Levi-flatness of $\Gamma(\Phi_{n,t} - s)$, $0 \leq s \leq C$, that $\Gamma(\Phi_{n,t}) \cap \widehat{\mathcal{K}}_L^{+} = \varnothing$.

To complete the proof of Lemma 3 we observe that the properties $\Gamma(\Phi_{n,t}) \cap \widehat{\mathcal{K}}_L^{\pm} = \varnothing$ and $K \subset \pi_{z,u}(\widehat{\mathcal{K}}_L^{\pm})$ imply that $|\Phi_{n,t}(P)| \leq C$ for all $P \in K$, $t \in \mathbb{R}$ and $n \geq N$, where $C = \max\{|v| : (z, u + iv) \in \mathcal{K}_L^{\pm}\}$. $\square$

To show that the graphs $\Gamma(\widetilde{\Phi}^{\pm}) \setminus \Gamma(\varphi)$ are foliated by holomorphic curves we follow the argument of section 6 in [ST]. Namely, from Lemma 3 we conclude, using Lemmas 3.2-3.5 in [Sh], that for each point $P = (z_0, u_0) \in \mathcal{E}(U) \cap G$ there exists $R^* > 0$ (which depends only on dist $(P, b\mathcal{E}(U))$ and the constant C given by Lemma 3 for compact K being a fixed neighbourhood of P) such that for each $t \in \mathbb{R}$ and all n big enough there is a holomorphic function

$$f_{n,t}^P(z) = u_{n,t}^P(z) + iv_{n,t}^P(z) : \Delta_{R^*}(z_0) \to \mathbb{C}_w$$

with the properties $u_{n,t}^P(z_0) = u_0$ and $\Gamma(f_{n,t}^P) \subset \Gamma(\Phi_{n,t})$, where $\Delta_{R^*}(z_0) = \{|z - z_0| < R^*\}$. Then, in view of Montel's theorem, we can pass to the limit as $t \to +\infty$ and $n \to +\infty$ (for details see the proof of Lemma 6.2 in [ST]) and obtain a holomorphic function

$$f^P(z) = u^P(z) + iv^P(z) : \Delta_{R^*}(z_0) \to \mathbb{C}_w$$

such that $f^P(z_0) = u_0$ and $\Gamma(f^P) \subset \Gamma(\widetilde{\Phi}^{+})$. Similarly, as $t \to -\infty$ and $n \to +\infty$ we get the corresponding holomorphic disc in $\Gamma(\widetilde{\Phi}^{-})$. Finally, using the argument of Proposition 6.3 in [ST], we obtain foliations of the whole graphs $\Gamma(\widetilde{\Phi}^{\pm}) \setminus \Gamma(\varphi)$ by holomorphic curves constituted by mentioned above holomorphic discs.

For the leaves of these foliations we have the following

Lemma 4. *Each holomorphic leaf contained in one of the sets* $\Gamma(\tilde{\Phi}^{\pm}) \setminus \Gamma(\varphi)$ *has an accumulation point in* $\Gamma(\varphi)$.

Proof. We argue by contradiction assuming that there is a holomorphic leaf $\mathcal{L}$ contained in one of the sets $\Gamma(\tilde{\Phi}^{\pm}) \setminus \Gamma(\varphi)$ such that $\mathcal{L}$ has no accumulation point in $\Gamma(\varphi)$. Since both cases can be treated in the same way, we can assume that $\mathcal{L} \subset \Gamma(\tilde{\Phi}^-) \setminus \Gamma(\varphi)$. The inclusion $\pi_{z,u}(\mathcal{L}) \subset \mathcal{E}(U)$ implies that $\pi_{z,u}(\mathcal{L}) \cap \pi_{z,u}(\widehat{K}_n) \neq \varnothing$ for some $n \in \mathbb{N}$. Let V be a relatively compact neighbourhood of the set $\pi_{z,u}(\widehat{K}_n)$ in $\mathcal{E}(U)$ such that $\operatorname{dist}(b\mathcal{E}(U) \cap G, V) > 0$. It follows then from our assumption and from continuity of $\tilde{\Phi}^{\pm}$ near U, that $\pi_{z,u}(\mathcal{L}) \cap V \Subset G$. Since, by Lemma 3, $\tilde{\Phi}^-$ is bounded over compact subsets of $\mathcal{E}(U)$, we have from the construction of holomorphic leaves in $\Gamma(\tilde{\Phi}^-)$ that the closure $\overline{\mathcal{L}}$ of the leaf $\mathcal{L}$ in $\overline{V} \times \mathbb{R}_v$ is compact and that the set $\overline{\mathcal{L}} \cap (V \times \mathbb{R}_v)$ is foliated by holomorphic curves. Therefore, the domain $V \times \mathbb{R}_v \setminus \overline{\mathcal{L}}$ is pseudoconvex near $\overline{\mathcal{L}}$. Then, considering the function $\varrho(P) = -\log \operatorname{dist}(P, \overline{\mathcal{L}}) + \varepsilon(|z|^2 + |w|^2)$ with small enough $\varepsilon > 0$ and smoothing it, if necessary, we construct (as a superlevel set of the constructed function) a strictly pseudoconcave neighbourhood $\mathcal{V}$ of $\overline{\mathcal{L}}$ in $V' \times \mathbb{R}_v$ (we mean here the strict pseudoconcavity of $\mathcal{V}$ near the part $b\mathcal{V} \cap (V' \times \mathbb{R}_v)$ of its boundary) such that $\mathcal{V} \Subset G \times \mathbb{R}_v$, where V' is a slightly smaller than V neighbourhood of $\pi_{z,u}(\widehat{K}_n)$ in $\mathcal{E}(U)$. Since the set $\overline{\mathcal{V}}$ is compact, there is a big enough constant C such that $(\overline{\mathcal{V}} + C\mathbf{e}_v) \cap \widehat{K}_n = \varnothing$. Then the property $\pi_{z,u}(\mathcal{L}) \cap \pi_{z,u}(\widehat{K}_n) \neq \varnothing$ implies that for some $t \geq 0$ the set $\overline{\mathcal{V}} + (C - t)\mathbf{e}_v$ will touch for the first moment the set $\widehat{K}_n$. This contradicts pseudoconvexity of $G \times \mathbb{R}_v \setminus \widehat{K}_n$ given by Lemma 1. $\square$

Continuity of the functions $\widetilde{\Phi}^{\pm}$ in the interior points of $\mathcal{E}(U)$ follows now directly from the argument of Lemma 7.2 in [ST] with Lemma 7.1 replaced by just proved Lemma 4. Hence, we have proved that the functions $\widetilde{\Phi}^{\pm}$ define continuous Levi-flat extensions of the given function $\varphi \in C(U)$ to the domain $\mathcal{E}(U)$.

Lemma 5. *The functions $\Phi^{\pm}$ defining the sets $W^{\pm}$ coincide with the respective functions $\widetilde{\Phi}^{\pm}$.*

Proof. Since the argument is the same for both functions, we prove the statement for the function Φ^+. First, we will show that $\Phi^+ \leq \widetilde{\Phi}^+$ in $\mathcal{E}(U)$. Consider for each $\varepsilon > 0$ and $m \in \mathbb{N}$ the graph $\Gamma(\varphi_{m,t} + \varepsilon)$ of the function $\varphi_{m,t} + \varepsilon$. It follows then from the definitions of V_{φ}^+, $\varphi_{m,t}$ and Theorem 3 that for m, ε and t (big enough) being fixed one has $\Gamma(\varphi_{m,t} + \varepsilon) \subset \widehat{\mathcal{K}}_n^+$ if n is sufficiently big. Therefore, by the maximum principle, we conclude from the Levi-flatness of $\Gamma(\Phi_{m,t}+\varepsilon)$ that $\Gamma(\Phi_{m,t}+\varepsilon)\subset\widehat{\mathcal{K}}_n^+$. This implies, by the definition of $\widetilde{\Phi}^+$ and $\mathcal{E}(V_\varphi^+)$, that $\Gamma(\widetilde{\Phi}^+ + \varepsilon)\subset\overline{\mathcal{E}(V_\varphi^+)}$. Hence, $\Phi^+\le\widetilde{\Phi}^+ + \varepsilon$ for each $\varepsilon>0$ and, therefore, $\Phi^+\le\widetilde{\Phi}^+$.

To prove the other inequality $\widetilde{\Phi}^+\le\Phi^+$ we argue by contradiction and assume that $\Phi^+(P_0)<\widetilde{\Phi}^+(P_0)$ for some point $P_0\in\mathcal{E}(U)$. It follows then from the definition of Φ^+ and $\mathcal{E}(V_\varphi^+)$ that there exist a number $n_0\in\mathbb{N}$ and a point $(P_0,v_0)\in\widehat{\mathcal{K}}_{n_0}^+$ such that $v_0<\widetilde{\Phi}^+(P_0)$. Therefore, by the definition of $\widetilde{\Phi}^+$, there are $m_0\in\mathbb{N}$ and $t_0\in\mathbb{R}$ such that $v_0<\Phi_{m_0,t_0}(P_0)$. We can also assume that m_0 is chosen so big that the domain $\mathcal{E}_{m_0}$ (where the function Φ_{m_0,t_0} is defined) has the property $\operatorname{dist}\left(b\mathcal{E}_{m_0}\cap G,\pi_{z,u}(\widehat{\mathcal{K}}_{n_0}^+)\right)>0$. Since $\Gamma(\Phi_{m_0,t_0}-C)$ does not intersect $\widehat{\mathcal{K}}_{n_0}^+$ for C big enough, and since, by the choice of m_0, $\Gamma(\varphi_{m_0,t_0}-s)$ does not intersect $\widehat{\mathcal{K}}_{n_0}^+$ for all $s\ge0$, we conclude from pseudoconvexity of the domain $G\times\mathbb{R}_v\smallsetminus\widehat{\mathcal{K}}_{n_0}^+$ (see Lemma 1) and from Levi-flatness of $\Gamma(\Phi_{m_0,t_0}-s)$, $0\le s\le C$, that $\Gamma(\Phi_{m_0,t_0}-s)\cap\widehat{\mathcal{K}}_{n_0}^+=\varnothing$ for all $0\le s\le C$. This contradicts the inequality $v_0<\Phi_{m_0,t_0}(P_0)$ and proves Lemma 5. $\square$

The last statement of Theorem 4, saying that for each function $\Phi\in C(\mathcal{E}(U))$ such that $\Phi|_U=\varphi$ and $\Gamma(\Phi)\smallsetminus\Gamma(\varphi)$ is Levi-flat one has $\Phi^-(z,u)\le\Phi(z,u)\le\Phi^+(z,u)$ for all $(z,u)\in\mathcal{E}(U)$, follows from pseudoconvexity of the domains $G\times\mathbb{R}_v\smallsetminus\left(\widehat{\mathcal{K}}_n^-\cup\widehat{\mathcal{K}}_n^+\right)$ by the same argument as in the proof of Lemma 3. The proof of Theorem 4 is now completed. $\square$

Remark. Denote $U_c=bG\setminus\overline{U}$ and let $\mathcal{E}^*(U)$ be the union of all connected components of the set $\mathcal{E}(U\cup U_c)$ which have some points of U on the boundary. Consider a function $\psi\in C(U_c)$. Applying Theorem 4 to the set $U\cup U_c$ instead of U and to the function on $U\cup U_c$ which is equal to φ on U and to ψ on U_c, we obtain Levi-flat extensions of φ to the set $\mathcal{E}^*(U)$ which is in general bigger than $\mathcal{E}(U)$ (see Example 1 in Section 7). If the domain G is diffeomorphic to a 3-ball, then, in view of Theorem 2, $\mathcal{E}^*(U)=\mathcal{E}(U)$ and these Levi-flat extensions of φ do not depend on the choice of ψ, being equal to the functions $\Phi^\pm$ given by Theorem 4 for the set U and the function φ.

6. On maximality of $\mathcal{E}(U)$.

Proof of Theorem 5. We can assume, without loss of generality, that the set U is connected. If not, we can apply the argument below to all the connected components of U. Since both cases $U \subset bG^-$ and $U \subset bG^+$ can be treated in the same way, in what follows we assume that $U \subset bG^-$. Consider an exhaustion $U_1 \Subset U_2 \Subset U_3 \Subset \cdots$ of U by domains with smooth boundary and for each $n = 1, 2, \dots$ let $\mathcal{U}_n = p(U_n)$, where $p : \overline{G} \to \overline{\mathcal{G}}$ is the canonical projection. Further, consider $\mathcal{U}' = p(U')$, where $U' \subset bG^-$ is the simply-connected domain given by assumption of the theorem. Denote by $\mathcal{A}_n$ the set consisting of all compact connected components of $\mathcal{U}' \setminus \mathcal{U}_n$. Then the sets ${\mathcal{U}_n}' = \mathcal{U}_n \cup \mathcal{A}_n$ are simply-connected subdomains of $\mathcal{U}'$ such that ${\mathcal{U}_1}' \Subset {\mathcal{U}_2}' \Subset {\mathcal{U}_3}' \Subset \cdots \subset \mathcal{U}'$ and $\mathcal{U}' = \cup_{n=1}^{+\infty} {\mathcal{U}_n}'$. For each $n = 1, 2, \dots$ define the function ${H_n}'$ on the domain $\mathcal{U}'$ to be equal to h^- on the set $\mathcal{U}' \setminus \mathcal{A}_n$ and to be the harmonic extension of h^- from $b\mathcal{A}_n$ to $\mathcal{A}_n$ on the set $\mathcal{A}_n$. Further, define the function ${H_n}''$ on the domain $\mathcal{U}'$ to be equal to h^- on the set $\mathcal{U}' \setminus {\mathcal{U}_n}'$ and to be the harmonic extension of h^- from $b{\mathcal{U}_n}'$ to ${\mathcal{U}_n}'$ on the set ${\mathcal{U}_n}'$. Then all the defined functions are subharmonic on $\mathcal{U}'$ and $\cdots \le {H_3}' \le {H_2}' \le {H_1}' \le {H_1}'' \le {H_2}'' \le {H_3}'' \le \cdots$. Let $H'(\zeta) = \lim_{n \to +\infty} {H_n}'(\zeta)$ and $H''(\zeta) = \lim_{n \to +\infty} {H_n}''(\zeta)$, $\zeta \in \mathcal{U}'$. Define the subdomain $\mathbf{U}$ of $\overline{G}$ by

$$\mathbf{U} = \{(z, u) : \, z = \pi(\zeta), H'(\zeta) < u < H''(\zeta), \zeta \in \mathcal{U}'\} \cup U.$$

Now we observe that $\mathbf{U} \subset \mathcal{E}(U)$. Indeed, let n be fixed and for each connected component Γ_n^α of the set $b\mathcal{U}_n$ consider the harmonic extension F_n^α of the function h^- from Γ_n^α to the domain $\mathcal{B}_n^\alpha$ bounded by Γ_n^α in $\mathcal{G}$. Note, that one of these domains is ${\mathcal{U}_n}'$ (it corresponds to $\Gamma_n^\alpha = b{\mathcal{U}_n}'$) and the others are just the connected components of the set $\mathcal{A}_n$. Since each of the domains $\mathcal{B}_n^\alpha$ is simply-connected, and since the function h^- is smooth in $\mathcal{G}$, for each α there is a function G_n^α holomorphic in $\mathcal{B}_n^\alpha$ and smooth on $\overline{\mathcal{B}}_n^\alpha$ such that $\operatorname{Re} G_n^\alpha = F_n^\alpha$. For each α and each $t \in \mathbb{R}$ consider the disc

$$D_{n,t}^\alpha = \{(z, w) : \, z = \pi(\zeta), w = G_n^\alpha(\zeta) + it, \zeta \in \mathcal{B}_n^\alpha\}$$

and then define the set $\mathcal{M}_n$ to be the union of the set $U_n \times \{0\}$ with the boundaries of all discs $D_{n,t}^\alpha$ such that $\overline{D}_{n,t}^\alpha \cap \{v = 0\} \neq \varnothing$. Since $\mathcal{M}_n$ is a compact subset of $U \times \mathbb{R}_v$, it follows that $\widehat{\mathcal{M}_n} \subset \mathcal{E}(U \times \mathbb{R}_v)$. Consider the set

$$\mathbf{U}_n = \{(z, u) : \, z = \pi(\zeta), H_n'(\zeta) < u < H_n''(\zeta), \zeta \in \mathcal{U}_n'\}.$$

Since the set $\mathbf{U}_n \times \{0\} \subset \mathbb{C}^2_{z,w}$ is obviously Levi-flat, and since, by definition of $\mathcal{M}_n$, its boundary is contained in $\mathcal{M}_n$, we conclude that $\mathbf{U}_n \times \{0\} \subset \widehat{\mathcal{M}}_n \subset \mathcal{E}(U \times \mathbb{R}_v)$. This implies that $\mathbf{U}_n \subset \mathcal{E}(U)$. The last inclusion holds true for all n, therefore, $\mathbf{U} = \bigcup_{n=1}^{+\infty} \mathbf{U}_n \subset \mathcal{E}(U)$ as it was claimed above.

To complete the proof of Theorem 5 we construct a function $\varphi \in C(U)$ such that $\mathbf{U}$ is the maximal domain where the Levi-flat extension of φ can be defined. Since, in view of Theorem 4, we always have a Levi-flat extension to $\mathcal{E}(U)$, it will imply that $\mathcal{E}(U) \subset \mathbf{U}$ and, hence, by the inclusion proved above, one has $\mathcal{E}(U) = \mathbf{U}$ and this domain is maximal for the Levi-flat extension of φ.

For each $n = 1, 2, \dots$ choose $t_n \in \mathbb{R}$ in such a way that all discs D^α_{n,t_n} are contained in $\{v \leq -n\}$. Further, for $(z, u) \in bU_n$ define $\varphi(z, u)$ to be equal to $\mathrm{Im} G^\alpha_n(\zeta) + t_n$, where ζ is the point of $b\mathcal{U}_n$ such that $(z, u) = (\pi(\zeta), h^-(\zeta))$. Since the sequence of sets $\{bU_n\}$ has no accumulation point in U, we can extend this function to a continuous function φ defined on the whole of U. We claim that $\mathbf{U}$ is the maximal domain for the Levi-flat extension of the defined function φ.

Assume, to get a contradiction, that there is a domain $\Omega \subset \overline{G}$, $\mathbf{U} \subsetneq \Omega$, such that our function φ has a continuous Levi-flat extension Φ to Ω. Consider a point $Q \in (b\mathbf{U} \setminus bG) \cap \Omega$. Then, by definition of $\mathbf{U}$, for each $n = 1, 2, \dots$ we can choose α_n in such a way that Q will be contained in the set of accumulation points for the sequence of graphs

$$\Gamma(F^{\alpha_n}_n) = \{(z, u) :\ z = \pi(\zeta), u = F^{\alpha_n}_n(\zeta), \zeta \in \overline{\mathcal{B}}^{\alpha_n}_n\}$$

of functions $F^{\alpha_n}_n$. By our assumption, the functions Φ is continuous on the set $\Gamma(F^{\alpha_n}_n) \subset \Omega$. Hence, for $c > 0$ big enough the disc $D^{\alpha_n}_{n,c}$ will not intersect the graph $\Gamma(\Phi)$ of Φ. Since, by our assumption, $\Gamma(\Phi)$ is Levi-flat, the complement to $\Gamma(\Phi)$ in $\mathbb{C}^2_{z,w}$ is locally pseudoconvex at the interior points of $\Gamma(\Phi)$. Applying "Kontinuitätssatz" to the family of discs $D^{\alpha_n}_{n,t}$, $t_n \leq t \leq c$, we conclude that $D^{\alpha_n}_{n,t} \cap \Gamma(\Phi) = \varnothing$ for all $t > t_n$. This means that the disc $D^{\alpha_n}_{n,t_n}$ is situated above $\Gamma(\Phi)$ in the v-direction. Then the definition of φ on bU_n and the choice of t_n in this definition imply that $\Phi \leq -n$ on $\Gamma(F^{\alpha_n}_n)$. Now the fact that Q is contained in the accumulation set of the sequence $\{\Gamma(F^{\alpha_n}_n)\}$ contradicts to our assumptions that $Q \in \Omega$ and $\Phi \in C(\Omega)$. $\square$

In view of Theorems 4 and 5 and the argument of Theorem 2 applied to the discs $D^\alpha_{n,t}$ instead of the Bedford-Klingenberg's discs, we have the following

Corollary. *Under the assumptions of Theorem 5 the sets $\mathcal{E}(U_\alpha)$ are disjoint and $\mathcal{E}(U) = \bigcup\limits_\alpha \mathcal{E}(U_\alpha)$.*

Remark. Surprisingly, the domain $\mathcal{E}(U)$ will not be maximal for Levi-flat extensions from U if we relax our assumption that every connected component of U is contained either in bG^- or in bG^+. Namely, there is a smoothly bounded simply-connected domain U in the boundary of the unit ball $\mathbb{B}$ of $\mathbb{C}_z \times \mathbb{R}_u$ such that every function $\varphi \in C(U)$ admits a continuous Levi-flat extension to a subdomain of $\mathbb{B}$ which is bigger than $\mathcal{E}(U)$. The authors are planning to present an example of such set U and to study this phenomena in more details in the next paper.

7. Examples.

Example 1. Let G be the solid torus in $\mathbb{C}_z \times \mathbb{R}_u$ defined by the inequality $(|z| - 2)^2 + u^2 < 1$. By a direct computation of the Levi-form one can easily see that the domain $G \times \mathbb{R}_v \subset \mathbb{C}^2_{z,w}$ is strictly pseudoconvex (see also Theorem 1 in [CS]). Consider the open subset $U = \{(z,u) \in bG : |z| < 2\}$ of bG. Then $U_c = bG \setminus \overline{U} = \{(z,u) \in bG : |z| > 2\}$. The sets U and U_c are obviously disjoint, but the set $\mathcal{E}(U \cup U_c) = \{(z,u) \in G : |u| < 1\}$ is connected (the set $\mathcal{E}(U \cup U_c) \times \mathbb{R}_v$ is foliated by annuli $A_C = \{(z,w) \in G \times \mathbb{R}_v : u = \operatorname{Re} C,\ v = \operatorname{Im} C\}$ where C satisfies the inequality $|\operatorname{Re} C| < 1\}$. $\square$

The following open problem is the motivation to our second example.

Let $\mathbb{D}$ be a strictly pseudoconvex domain in $\mathbb{C}^2$ with smooth boundary. Let S be a smooth 2-dimensional sphere embedded into $b\mathbb{D}$. Does it follow that there is a uniquely determined 3-dimensional ball B^3 embedded into $\mathbb{D}$ and foliated by holomorphic discs such that $bB^3 = S$?

Note, that by [BK] and [Kr] there are 2-spheres $S' \subset b\mathbb{D}$ arbitrarily close to S which satisfy all the requirements of the problem. A partial answer to the raised question is given by the following

Proposition 2. *If $\hat{S}$ is the $\mathcal{A}(\mathbb{D})$-hull of S, then the set $b\hat{S} \cap \mathbb{D}$ is the disjoint union of holomorphic discs with the boundary in S.*

Remark. We do not know if $\hat{S} \setminus S$ is a 3-ball or, even, if the set $\hat{S} \setminus S$ has an empty interior.

Proof. Since S locally divides $b\mathbb{D}$ in two connected components $b\mathbb{D}'$ and $b\mathbb{D}''$, we can consider two sequences of disjoint 2-spheres $S'_n \subset b\mathbb{D}'$ and $S''_n \subset b\mathbb{D}''$ converging to S. It is possible to choose all the spheres S'_n and S''_n in such a way that they satisfy the conditions of [Kr] and, hence, they bound 3-balls B'_n and B''_n, respectively, foliated by holomorphic discs. By "Kontinuitätssatz" we can easily see that, since the spheres S'_n and S''_n were chosen to be disjoint, all the balls B'_n and B''_n will also be disjoint. For each $n = 1, 2, \ldots$ denote by $\mathcal{M}_n$ the closed connected component of $\mathbb{D} \setminus (B'_n \cup B''_n)$ containing S on its boundary and by $\mathcal{N}_n$ the closed connected component of $b\mathbb{D} \setminus (S'_n \cup S''_n)$ containing S. Then $\{\mathcal{M}_n\}$ is a decreasing sequence of compact subsets of $\overline{\mathbb{D}}$, therefore $\mathcal{M} = \bigcap\limits_{n=1}^{\infty} \mathcal{M}_n$ is also a compact subset of $\overline{\mathbb{D}}$. We claim that $\mathcal{M} = \hat{S}$ and $b\mathcal{M} \cap \mathbb{D}$ is the disjoint union of holomorphic discs with boundary in S.

To show that $\hat{S} \subset \mathcal{M}$ we observe that, by the definition of $\mathcal{M}$, for each $\varepsilon > 0$ there are balls B'_n and B''_n contained in the ε-neighbourhood $U_\varepsilon(\mathcal{M})$ of $\mathcal{M}$. By Lemma 1, we know that the set $\mathbb{D} \setminus \hat{S}$ is pseudoconvex. Hence, applying "Kontinuitätssatz" to the discs foliating B'_n and B''_n, we conclude that $\hat{S} \cap B'_n = \varnothing$ and $\hat{S} \cap B''_n = \varnothing$. Since, due to Shilov Idempotent Theorem (see [G], p. 88), the set $\hat{S}$ is connected, one has $\hat{S} \subset \mathcal{M}_n$. This argument holds true for all $\varepsilon > 0$, therefore $\hat{S} \subset \bigcap\limits_{n=1}^{\infty} \mathcal{M}_n = \mathcal{M}$.

Now we prove that $b\mathcal{M} \cap \mathbb{D}$ is the disjoint union of holomorphic discs with boundary in S. In paricular, by virtue of the maximum modulus principle, this will imply the other inclusion $\mathcal{M} \subset \hat{S}$ and, hence, that $\mathcal{M} = \hat{S}$. Let Q be a point of $b\mathcal{M} \cap \mathbb{D}$. It follows then from the definition of $\mathcal{M}$ that either there is a sequence of points $Q_n \in B'_n$ or a sequence of points $Q_n \in B''_n$ converging to Q. Since the argument in both cases is the same, we assume that $Q_n \in B'_n$. Let D_n be the holomorphic disc given by [Kr] which is contained in B'_n and such that $Q_n \in D_n$. Since the area of D_n is less than the area of $S'_n = bB'_n$, and since the spheres S'_n converging to S can be chosen in such a way that the areas of S'_n are uniformly bounded, we know by [Bi] that there is a subsequence $\{D_{n_k}\}$ converging to a holomorphic disc D_Q. It follows then from the inclusion $D_{n_k} \subset \mathcal{M}_{n_k}$ and the definition of $\mathcal{M} = \bigcap_{n=1}^{\infty} \mathcal{M}_n$ that $D_Q \subset \mathcal{M}$. Since $D_{n_k} \subset B'_{n_k}$, and since, by the argument above, $B'_{n_k} \cap \mathcal{M} = \varnothing$, we conclude that D_Q does not intersect the interior part of $\mathcal{M}$ and, hence, $D_Q \subset b\mathcal{M} \cap \mathbb{D}$. To show the unicity of the holomorphic disc contained in $b\mathcal{M} \cap \mathbb{D}$ and passing through Q we argue by contradiction assuming that there is another disc $D \neq D_Q$. Since $Q \in D \cap D_Q$, and since the property of two holomorphic discs in $\mathbb{C}^2$ to have a nonempty intersection is stable under small perturbation, we conclude that $D \cap D_{n_k} \neq \varnothing$ for k big enough. Then the fact that $\mathcal{M} \cap B'_{n_k} = \varnothing$, $D_{n_k} \subset B'_{n_k}$ and $D \subset \mathcal{M}$ gives the desired contradiction thereby proving the unicity of D_Q. It is easy to see that the unicity of the holomorphic disc contained in $b\mathcal{M}$ and passing through a given point of $b\mathcal{M} \cap \mathbb{D}$ implies that the whole set $b\mathcal{M} \cap \mathbb{D}$ is the disjoint union of holomorphic discs with boundary lying in S. This completes the proof of Proposition 2. $\square$

Note, that if the domain $\mathbb{D}$ is bounded and diffeomorphic to a 4-ball, then S divides $b\mathbb{D}$ into two connected components $b\mathbb{D}'$ and $b\mathbb{D}''$. Hence, by an argument similar to the one in part 2 of Theorem 1, we have $\mathcal{E}(b\mathbb{D}') = E(b\mathbb{D}')$ and $\mathcal{E}(b\mathbb{D}'') = E(b\mathbb{D}'')$, the sets $\mathcal{E}(b\mathbb{D}')$ and $\mathcal{E}(b\mathbb{D}'')$ are disjoint and $\hat{S} = \overline{\mathbb{D}} \setminus (\mathcal{E}(b\mathbb{D}') \cup \mathcal{E}(b\mathbb{D}''))$. As we have already mentioned above, we do not know if the interior part of the set $\hat{S} = \overline{\mathbb{D}} \setminus (\mathcal{E}(b\mathbb{D}') \cup \mathcal{E}(b\mathbb{D}''))$ is empty. Surprisingly, for $\mathbb{D}$ being a rigid domain in $\mathbb{C}^2$ this can occur. More precisely, there is an open smoothly bounded domain U, diffeomorphic to a 2-disc on the boundary $b\mathbb{B}$ of the unit ball $\mathbb{B} \subset \mathbb{C}_z \times \mathbb{R}_u$ such that the interior part of the set $\mathbb{B} \times \mathbb{R}_v \setminus \big(\mathcal{E}(U \times \mathbb{R}_v) \cup \mathcal{E}((b\mathbb{B} \setminus \overline{U}) \times \mathbb{R}_v)\big)$ is nonempty. An example of such set U is presented below.

Example 2. Consider the unit ball $\mathbb{B}$ in $\mathbb{C}_z \times \mathbb{R}_u$. Then, according to our notations in Section 4, the covering model $\mathcal{G}$ of $\mathbb{B}$ is the unit disc $\Delta = \{|z| < 1\} \subset \mathbb{C}_z$ and for the functions $h^{\pm}$ defining $b\mathbb{B}$ we have $h^{\pm}(z) = \pm\sqrt{1 - |z|^2}$.

Consider the smooth 2-disc in $b\mathbb{B}$ with the projections $p(U^{\pm})$ of the sets $U^{\pm} = U \cap b\mathbb{B}^{\pm}$ on Δ as in fig. 1.

It follows then from our definition of the sets Σ and $\mathcal{U}^{\pm}$ (see Section 4) and from the description of $\mathcal{E}(U)$ given by Proposition 1 that the set

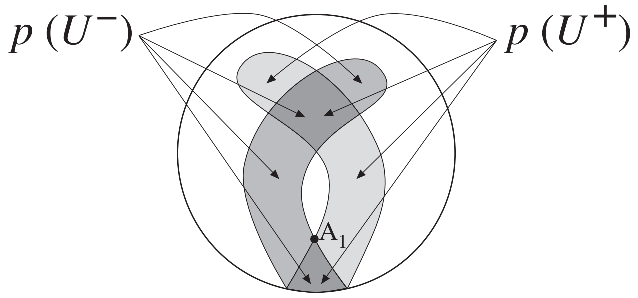


Figure 1

$b\mathcal{E}(U) \cap \mathbb{B}$ is the union of the graphs $\Gamma(F^{\pm})$ of harmonic functions $F^{\pm}$ defined on the corresponding domains $\mathcal{U}^{\pm}$ (see figg. 2 and 3, where the boundary data of $F^{\pm}$ are also indicated) and the segment $\mathbb{B} \cap \{z = A_1\}$.

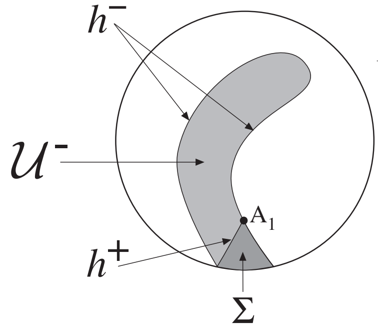


Figure 2

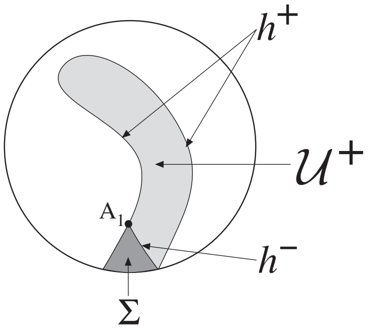


Figure 3

Similarly, the set $U_c = b\mathbb{B} \smallsetminus \overline{U}$ is also a 2-disc with the projections $p({U_c}^{\pm})$ of the sets ${U_c}^{\pm} = U_c \cap b\mathbb{B}^{\pm}$ on Δ as in fig. 4. The corresponding sets Σ_c and ${\mathcal{U}_c}^{\pm}$ are shown in figg. 5 and 6 as well as the boundary data of the functions ${F_c}^{\pm}$, whose graphs together with the segments $\mathbb{B} \cap \{z = A_i\}$, $i = 2, 3, 4$, constitute the set $b\mathcal{E}(U_c) \cap \mathbb{B}$. It is clear now that, since the graphs $\Gamma(F^{\pm})$ do not coincide with the graphs $\Gamma({F_c}^{\pm})$, and since the sets $\mathcal{E}(U)$ and $\mathcal{E}(U_c)$ are disjoint, there is an open subdomain Ω of $\mathbb{B}$, bounded by the graphs $\Gamma(F^{\pm})$, $\Gamma({F_c}^{\pm})$ and the segments $\mathbb{B} \cap \{z = A_i\}$, $i = 1, 2, 3, 4$, such that

$$\overline{\mathbb{B}} \times \mathbb{R}_v \smallsetminus (\mathcal{E}(U \times \mathbb{R}_v) \cup \mathcal{E}(U_c \times \mathbb{R}_v)) = \Omega \times \mathbb{R}_v.$$

Note also, that the boundary of the domain $\Omega \times \mathbb{R}_v$ consists of two Levi-flat surfaces: L_1, which is constituted by the hypersurfaces $\Gamma(F^{\pm}) \times \mathbb{R}_v$

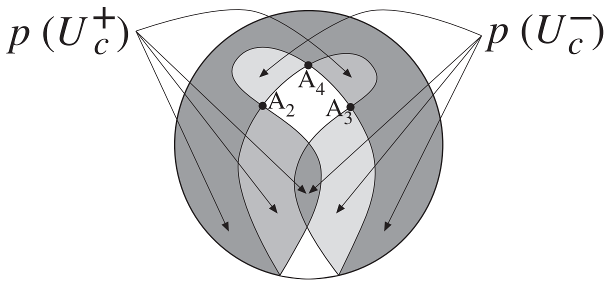
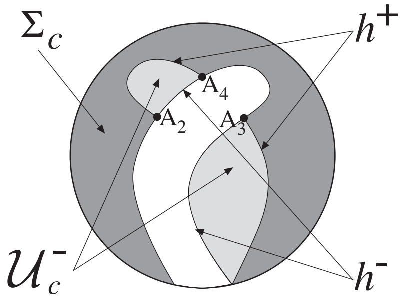


Figure 4



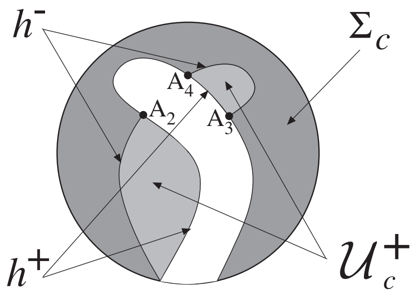


Figure 5

Figure 6

foliated by holomorphic discs and the strip $(\mathbb{B} \cap \{z = A_1\}) \times \mathbb{R}_v$, and L_2, which is constituted by the hypersurfaces $\Gamma(F_c{}^{\pm}) \times \mathbb{R}_v$ foliated by holomorphic discs and the strips $(\mathbb{B} \cap \{z = A_i\}) \times \mathbb{R}_v$, $i = 2, 3, 4,$. Both Levi-flat hypersurfaces L_1 and L_2 have the same boundary $bU \times \mathbb{R}_v$. $\square$

Finally, we formulate some questions which remains open.

Question 1. *Describe the set $\mathcal{E}(U)$ in the intrinsic terms of the space $\mathbb{C}_z \times \mathbb{R}_u$ (i.e. without using the domain $G \times \mathbb{R}_v \subset \mathbb{C}^2_{z,w}$ and the hulls $\widehat{K}_n$ of compacts K_n exhausting $U \times \mathbb{R}_v$).*

Question 2. *For each open set $U \subset bG$ find the maximal open subset $\mathcal{E}$ of $\overline{G}$ such that every function $\varphi \in C(U)$ has a continuous Levi-flat extension to $\mathcal{E}$.*

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Nikolay Shcherbina
Department of Mathematics
University of Göteborg

412 96 Göteborg
SWEDEN

Giuseppe Tomassini
Scuola Normale Superiore
Piazza dei Cavalieri 7
56126 Pisa
ITALY