THE MILNOR LATTICES OF THE ELLIPTIC HYPERSURFACE SINGULARITIES

by

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Introduction

To each isolated hypersurface singularity $f : (\mathbb{C}^k, \underset{\sim}{0}) \to (\mathbb{C}, 0)$, $k \equiv 3(4)$, is associated an even lattice, which is the homology group of the Milnor fibre in dimension $k-1$ provided with the symmetric intersection form. This lattice will be called Milnor lattice. One of the possible questions related to this lattice is the following: Which lattices occur as Milnor lattices of hypersurface singularities? One of the coarse invariants of the lattice is the signature (μ_o, μ_+, μ_-) , where μ_o, μ_+ resp. μ_- is the number of zeros, positive terms resp. negative terms on the diagonal after a diagonalization of the quadratic form over the real numbers. Then for certain values of (μ_o, μ_+) the answer to the above question is wellknown: For $\mu_+ = 0$ these are the root lattices of type A_n, D_n, E_6, E_7 and E_8 $(\mu_o = 0)$, and the orthogonal direct sum of E_6, E_7 resp. E_8 with a two-dimensional radical $(\mu_o = 2)$. For $\mu_+ = 1$ these are the lattices defined by the graph $T(p,q,r)$, $1/p + 1/q + 1/r < 1$ (cf. Fig. 2) plus a one-dimensional radical (here $\mu_o = 1$) . So the "first" open case is $\mu_+ = 2$, $\mu_o = 0$. If we restrict to the case $k = 3$, then the hypersurface singularities with $\mu_o + \mu_+ = 2$ are precisely the (minimally) elliptic hypersurface singularities studied and classified by Laufer [16] (cf. § 4). In this article we compute the Milnor lattices of these singularities and we calculate Dynkin diagrams of these singularities with respect to geometric bases. We use these results to give an arithmetic-combinatorial characterization of the occuring lattices.

The elliptic hypersurface singularities, which will be abbreviated EHS in the sequel, contain in particular all uniand bimodal singularites. These are precisely the hypersurface singularities in the class of Kodaira singularities. The Kodaira singularities are normal surface singularites of arbitrary embedding dimension, characterized by the resolution being a Kodaira elliptic curve fitting into Kodaira's classification, disregarding the embedding. These were studied in a joint paper with C.T.C. Wall [11]. The present paper can be considered as a sequel to this article. We computed in that paper the Milnor lattices of all Kodaira singularities of embedding dimension ebd ≤ 5 and gave a characterization of the occuring lattices. One of the purposes of this paper is to extend as far as possible the results on Milnor lattices of that paper to the larger and very natural class of elliptic Gorenstein singularities. The elliptic complete intersections were classified and equations for them given by C.T.C. Wall. This will be published elsewhere. Since the tables needed for the classification of the occuring lattices are already lengthy for the hypersurface case, and the enumeration of elliptic complete intersections (not to think of the case ebd = 5) involves a lot of cases, an analogous study of the Milnor lattices would be laborious. We also believe that an extension of our results to the case ebd ≤ 5 would be analogous. Therefore we restrict ourselves here to the hypersurface case. We adopt the notation of the above paper and refer to it for all information on the uni- and bimodal singularities, and we concentrate ourselves here on the remaining EHS .

The paper is organized as follows. In the first section we introduce the EHS and enumerate their dual graphs. In the second section we consider equations of the EHS . In the third section we compute Dynkin diagrams with respect to distinguished bases. These are used to determine the lattice structure of the Milnor lattices in § 4 . It turns out that there are many strange connections between the Milnor lattices of Kodaira singularities with ebd ≤ 5 and EHS . In § 5 we present an algorithmic characterization of the occuring lattices, which is an extension of the corresponding characterization in [11] . We conclude in § 6 by giving examples that the restriction to the case $k = 3$ is a real restriction concerning our original question: There exist singularities of corank 4 with $(\mu_o,\mu_+) = (0,2)$.

Acknowledgements:

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§ 1 The Elliptic Hypersurface Singularities

Let $\underset{\sim}{x}$ be a normal singularity of the two-dimensional complex space X and let $\pi : M \to X$ be the minimal resolution. The geometric genus h of X is the dimension of the complex vector space $H^1(M, \mathcal{O}_M)$, where $\mathcal{O}_M$ is the sheaf of holomorphic functions on M . A singularity $(X, \underset{\sim}{x})$ is called Gorenstein, if there exists a neighbourhood U of $\underset{\sim}{x}$ in X , and a nonvanishing holomorphic 2-form on $U - \{\underset{\sim}{x}\}$. Examples of Gorenstein singularities are isolated hypersurface singularities, i.e. $X = f^{-1}(0)$ for a function $f : \mathbb{C}^3 \to \mathbb{C}$, or more generally isolated complete intersection singularities, i.e. $X = F^{-1}(\underset{\sim}{0})$ for a mapping $F : \mathbb{C}^{2+1} \to \mathbb{C}^1$.

If $h = 0$, the singularity is called rational. These singularites were studied by many people, e.g. by Du Val [7]. The Gorenstein rational singularities are precisely the rational double points, i.e. the rational hypersurface singularities of corank 2 . They can be characterized in many different ways, see Durfee [6].

If $h = 1$, the singularity is called elliptic. The elliptic Gorenstein singularities were studied by Laufer [16] and Reid [21]. They are precisely the minimally elliptic singularities of Laufer [16]. They have arbitary large embedding dimension, and the embedding dimension can be computed as follows. Let $E = \pi^{-1}(\underset{\sim}{x})$ be the exceptional set of M and $E = \bigcup_i C_i$ its decomposition into irreducible components. The fundamental cycle $Z = \sum_i z_i C_i$ is the unique minimal positive cycle Z with $Z \cdot C_i \leq 0$ for all i . We call the number $D := -Z \cdot Z$ the grade of the singularity. It is shown in [16] and [21] that

(a) the multiplicity of the singularity is $\max(D,2)$

(b) its embedding dimension is $\max(D,3)$.

Thus the elliptic hypersurface singularities (EHS) are precisely the elliptic Gorenstein singularities with $D \leq 3$. For $D \leq 2$ they have corank 2 and for $D = 3$ they have corank 3 . The resolutions of the EHS have been classified by Laufer [16] and Reid [21]. We enumerate the possible types.

We distinguish between 4 types of EHS . The fundamental cycle Z is called <u>almost reduced</u> , if $z_i = 1$ except for nonsingular rational C_i, $C_i \cdot C_i = -2$. The <u>EHS of type 1</u> are those with reduced fundamental cycle, i.e. $z_i = 1$ for all i . The <u>EHS of type 2</u> are those with an almost reduced but not reduced fundamental cycle. The exceptional set E is in both cases a Kodaira elliptic curve, i.e. an exceptional fibre of a pencil of elliptic curves, imbedded in a certain way. For the EHS of type 1 it is of type $I_n (n \geq 0)$, II, III or IV in Kodaira's classification, for type 2 it is of type $I_n^* (n \geq 0)$, II^*, III^* or IV^*. See [11] for more details.

The <u>EHS of type 3</u> are those with a nonsingular rational component C_o with $z_o = 2$ and $C_o \cdot C_o = -3$. Here the grade D takes the values 2 and 3 . The exceptional sets look as follows. All components are rational and nonsingular, E has only normal crossings, and no two components meet more than once. So the resolutions can be described by the corresponding weighted dual graphs. We list the weighted dual graphs according to Laufer's classification. We use the following notation: To a component with normal degree $-b$ we associate a vertex

$$\ast b \ .$$

We abbreviate

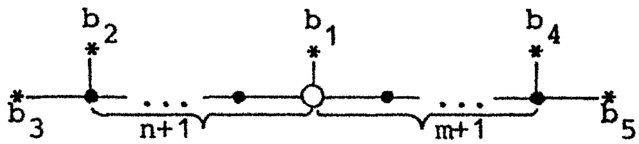
$$\bullet = *2 \; ,$$
$$\bigcirc = *3 \; .$$

We give Laufer's notation for the dual graph, where we extend the definition of $A_{n,**,o}$ to the case $n = 0$ for simplicity, setting

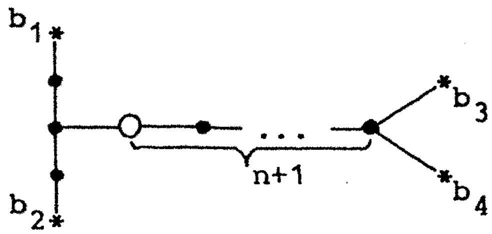
$$A_{0,**,o} = A_{*,o} + A_{*,o}$$

in this case. So $n,m \geq 0$ always. The condition $D \leq 3$ yields the following restrictions for the possible numbers $b_i \geq 2$:

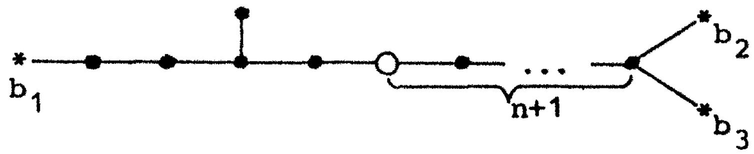
$$\sum_i (b_i - 2) \leq 1 \quad .$$



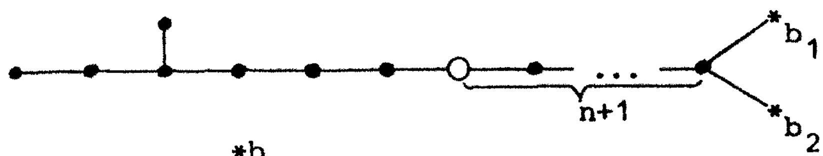
$A_{*,o} + A_{n,**,o} + A_{m,**,o}$



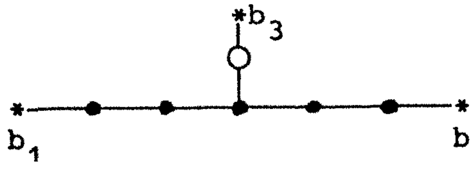
$A'_{3,**,o} + A_{n,**,o}$



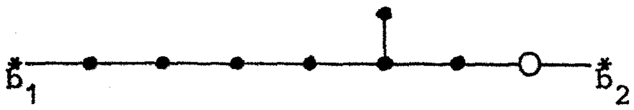
$D_{5,*,o} + A_{n,**,o}$



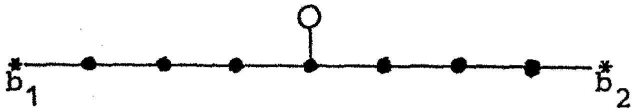
$E_{7,o} + A_{n,**,o}$



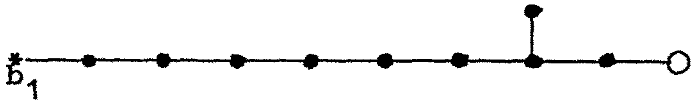
$A'_{5,**,o} + A_{*,o}$



$D_{7,*,o} + A_{*,o}$

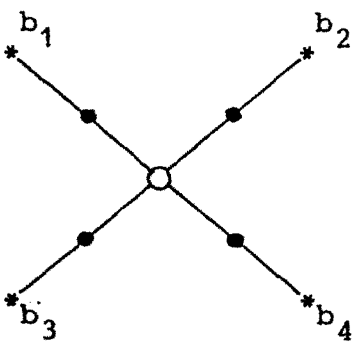


$A'_{7,**,o}$

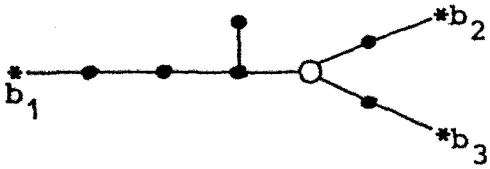


$D_{9,*,o}$

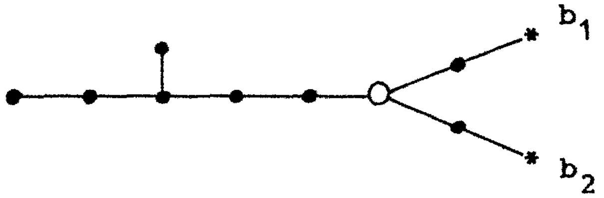
Finally the <u>EHS of type 4</u> are those with a nonsingular rational C_o with $z_o = 3$ and $C_o \cdot C_o = -3$. They all have grade $D = 3$. What is stated about the exceptional sets in the previous case, is also true in this case. So the resolutions can again be described by the weighted dual graphs, and we again list these graphs with the above conventions. Here all numbers b_i have to be 2 .



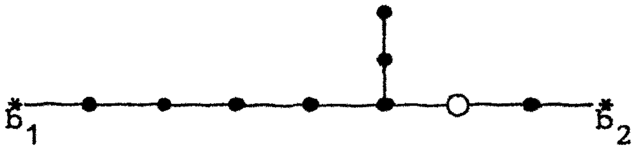
$A_{1,*,o} + A_{1,*,o} + A_{1,*,o} + A_{1,*,o}$



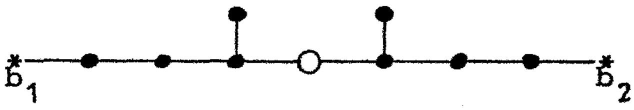
$A_{4,*,o} + A_{1,*,o} + A_{1,*,o}$



$E_{6,o} + A_{1,*,o} + A_{1,*,o}$

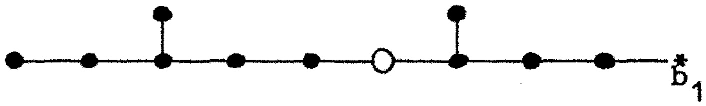


$A_{7,*,o} + A_{1,*,o}$

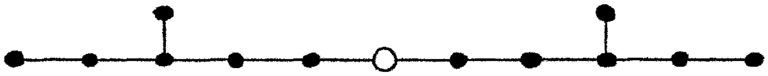


$A_{4,*,o} + A_{4,*,o}$

$A_{10,*,o}$



$E_{6,o} + A_{4,*,o}$



$E_{6,o} + E_{6,o}$

In general the above weighted dual graphs do not completely determine the associated singularity. We refer to [16] for the numbers of moduli of the deformations of the resolutions.

§ 2 Equations

Equations for the EHS have already been given by Laufer [16]. In Table 1 we have listed representatives of the functions which do not in any case agree with Laufer's, but are right equivalent to Laufer's functions. The functions are chosen to be appropriate for the computation of distinguished bases, cf. § 3 .

We compare the classification of EHS with Arnold's classification ([4]).

The EHS of type 1 are precisely the unimodal singularities, the EHS of type 2 the bimodal. For corank 2 these singularities have a nonzero 4-jet and for corank 3 the 3-jet defines a reduced plane cubic curve.

For $D = 2$, the EHS of type 3 have corank 2 , zero 4-jet and nonzero 5-jet and thus belong to the class $\underset{\sim}{N}$ in Arnold's notation. For $D = 3$ they have corank 3 and their 3-jet is of the form x^2y . Therefore they belong to the series $\underset{\sim}{V}$. Here the repeated line $x = 0$ gives the component C_o of the exceptional set.

The EHS of type 4 have corank 3 and 3-jet x^3 . Therefore they belong to the series $\underset{\sim}{V}'$. Again the repeated line x gives the component C_o .

Denote by m the number of moduli of the function f with respect to right equivalence. By Arnold [4] one has: For an EHS of type 1 , $m \geq 1$. We conjecture that we have in fact an equality in all cases. For all EHS of corank 2 it can be checked by the method of Arnold-Kushnirenko (cf.[4, p.20]). For the quasihomogeneous EHS the type number equals the inner modality (cf. also [25]). The inner modality is also equal to the number of moduli of the deformations of the resolutions given by Laufer [16] in the quasihomogeneous case.

We look at the EHS of type 3 and 4 more closely. To each EHS belonging to the $\underset{\sim}{V}$ - series is associated an EHS belonging to the $\underset{\sim}{N}$ - series by the process of Δ_x - reduction defined by C.T.C. Wall (cf. [23] for the following). Let $f(x,y,z) = 0$ be the equation of an EHS belonging to the $\underset{\sim}{V}$ - series. One can write f in the form

$$f(x,y,z) = x^2y + 2xb(z) + c(y,z)$$

with $\operatorname{ord} b \geq 3$, $\operatorname{ord} c \geq 4$. Then the discriminant

$$\Delta_x f(y,z) = b(z)^2 - yc(y,z)$$

defines an EHS belonging to the $\underset{\sim}{N}$ - series. Moreover the resolution graphs differ only by the selfintersection number of one component. The Milnor numbers satisfy $\mu(\Delta_x f) = 1 + \mu(f)$. In general up to two different singularities of the $\underset{\sim}{V}$ - series can give the same of the series $\underset{\sim}{N}$. The singularities of the series $\underset{\sim}{N}$ with a nonzero 5-jet were classified by Wall. The equivalence of names of Laufer and Wall is given in Table 1 . For a notation for the singularities of the series $\underset{\sim}{V}$ we follow Wall by prefixing V (resp. $V^{\#}$) to the name of $\Delta_x f$ to obtain a name for f . For the equivalence of names we again refer to Table 1 . We shall use these names in the sequel, where we abbreviate

$$NA_{n,m} = NA^{1}_{n,m} \quad , \quad NB^{n}_{(-1)} = NB^{1,n}_{0(-1)} \quad \text{etc.}$$

There is no similar notation for the eight EHS of type 4 . We therefore denote these singularities by $V'_{(i)}$, $1 \leq i \leq 8$, where the equivalence of names is again given in Table 1 .

There are 15 quasihomogeneous singularities among the EHS of type 3 and 4 . We give the weights w_i and degrees d of these singularities in Table 2 .

Table 1

Dual Graph	Normal degrees b_i	Notation	Equation	Numbers M_j
$A_{*,o}+A_{*,o}+A_{*,o}+A_{*,o}+A_{*,o}$	2,2,2,2,2	$NA^1_{0,0}$	x^5+z^5	4,4,4,4
$A_{*,o}+A_{n,**,o}+A_{*,o}+A_{*,o}$	2,2,2,2,2	$NA^1_{n,0}$	$(x^3-z^3)(x^2+z^{n+2})$	4+n,4,4,4
$A_{*,o}+A_{n,**,o}+A_{m,**,o}$	2,2,2,2,2	$NA^1_{n,m}$	$(2x-z)((x-z)^2+x^{n+2})(x^2+(x-z)^{m+2})$	4+m,4,4,4+n
$A'_{3,**,o}+A_{*,o}+A_{*,o}$	2,2,2,2	$NB^1_{0(-1)}$	$(x^2-z^2)(x^3-z^4)$	4,5,5,4
$A'_{3,**,o}+A_{n,**,o}$	2,2,2,2	$NB^{1,n}_{0(-1)}$	$((x-z)^2+x^{n+2})(x^3+(x-z)^4)$	4+n,5,4,5
$D_{5,*,o}+A_{*,o}+A_{*,o}$	2,2,2	$NB^1_{0(0)}$	$(x^2-z^2)(x^3-xz^3)$	4,6,5,4
$D_{5,*,o}+A_{n,**,o}$	2,2,2	$NB^{1,n}_{0(0)}$	$((x-z)^2+x^{n+2})(x^3+x(x-z)^3)$	4+n,6,4,5
$E_{7,o}+A_{*,o}+A_{*,o}$	2,2	$NB^1_{0(1)}$	$(x^2-z^2)(x^3-z^5)$	4,6,6,4
$E_{7,o}+A_{n,**,o}$	2,2	$NB^{1,n}_{0(1)}$	$((x-z)^2+x^{n+2})(x^3+(x-z)^5)$	4+n,6,4,6
$A'_{5,**,o}+A_{*,o}$	2,2,2	$NC^1_{(0)}$	$(x-z)(x^4+(x-z)^5)$	4,5,5,5
$D_{7,*,o}+A_{*,o}$	2,2	$NC^1_{(1)}$	$(x-z)(x^4+x(x-z)^4)$	4,5,5,6
$A'_{7,**,o}$	2,2	$NF^1_{(0)}$	x^5+z^6	5,5,5,5
$D_{9,*,o}$	2	$NF^1_{(1)}$	x^5+xz^5	5,5,5,6
$A_{*,o}+A_{*,o}+A_{*,o}+A_{*,o}+A_{*,o}$	2,2,2,2,3	$VNA^1_{0,0}$	$yx^2+y^4+z^4$	3,3,3,3,3
$A_{*,o}+A_{n,**,o}+A_{*,o}+A_{*,o}$	2,2,2,2,3	$VNA^1_{n,0}$	$yx^2+y^4-\frac{5}{2}y^2z^2-yz^3+x^bz^a$ $2a+3b=n+8$	3,3,3,3+i,3+j $i+j=n$

Table 1 (continued)

Dual Graph	Normal degrees b_i	Notation	Equation	Numbers M_j
$A_{*,o}+A_{n,**,o}+A_{m,**,o}$	2,2,2,2,3	$VNA^1_{n,m}$	$yx^2+(y-2z)^3y+(y-2z)^2y^2+y^{m+4}+x^b(y-2z)^a$ $2a+3b=n+8$	3+m,3,3,3+i,3+j i+j=n
$A_{*,o}+A_{*,o}+A_{*,o}+A_{m,**,o}$	2,2,2,2,3	$VNA^1_{O,m}$	$yx^2+\frac{1}{4}(y-z)^4-\frac{1}{2}(y-z)^2y^2+y^{m+4}$	3+m,3,3,3,3
$A_{*,o}+A_{n,**,o}+A_{m,**,o}$	3,2,2,2,2	$V^{\#}NA^1_{n,m}$	$(2y-z)x^2+(y-z)^2y^2+(y-z)^{n+4}+y^{m+4}$	3+n,3+m,3,3,3
$A'_{3,**,o}+A_{*,o}+A_{*,o}$	2,2,2,3	$VNB^1_{O(-1)}$	$yx^2+(y-z)^4+(y-z)^3y-y^5$	4,3,4,3,3
$A'_{3,**,o}+A_{n,**,o}$	2,2,2,3	$VNB^{1,n}_{O(-1)}$	$yx^2+(y-z)^3y+(y-z)^ax^b-y^5$ $2a+3b=n+8$	4,3,4,3+i,3+j i+j=n
$A'_{3,**,o}+A_{*,o}+A_{*,o}$	3,2,2,2	$V^{\#}NB^1_{O(-1)}$	$yx^2+y^4-y^2z^2-xz^3$	3,4,3,3,4
$A'_{3,**,o}+A_{n,**,o}$	3,2,2,2	$V^{\#}NB^{1,n}_{O(-1)}$	$yx^2+(y+z)^2y^2+y^{n+4}+x(y+z)^3$	3+n,4,3,3,4
$D_{5,*,o}+A_{*,o}+A_{*,o}$	2,2,3	$VNB^1_{O(O)}$	$yx^2+(y-z)^4+(y-z)^3y-(y-z)y^4$	5,3,4,3,3
$D_{5,*,o}+A_{n,**,o}$	2,2,3	$VNB^{1,n}_{O(O)}$	$yx^2+(y-z)^3y-(y-z)y^4+(y-z)^ax^b$ $2a+3b=n+8$	5,3,4,3+i,3+j i+j=n
$D_{5,*,o}+A_{*,o}+A_{*,o}$	3,2,2	$V^{\#}NB^1_{O(O)}$	$yx^2+\frac{1}{4}y^4+(y-z)^2y^2-(y-z)^5$	3,4,3,4,4
$D_{5,*,o}+A_{n,**,o}$	3,2,2	$V^{\#}NB^{1,n}_{O(O)}$	$yx^2+(y-z)^2y^2+y^{n+4}-(y-z)^5$	3+n,4,3,4,4
$E_{7,o}+A_{*,o}+A_{*,o}$	2,3	$VNB^1_{O(1)}$	$yx^2+(y-z)^4+(y-z)^3y-y^6$	5,3,5,3,3
$E_{7,o}+A_{n,**,o}$	2,3	$VNB^{1,n}_{O(1)}$	$yx^2+(y-z)^3y+(y-z)^ax^b-y^6$ $2a+3b=n+8$	5,3,5,3+i,3+j i+j=n

Table 1 (continued)

Dual Graph	Normal degree b_i	Notation	Equation	Numbers M_j
$A'_{5,**,o}+A_{*,o}$	2,2,3	$VNC^1_{(0)}$	$yx^2+(y-z)^4+y^5$	4,4,4,3,3
$A'_{5,**,o}+A_{*,o}$	3,2,2	$V^{\#}NC^1_{(0)}$	$yx^2+(y-z)y^3+(y-z)^3x$	3,4,4,3,4
$D_{7,*,o}+A_{*,o}$	2,3	$VNC^1_{(1)}$	$yx^2+(y-z)^4+(y-z)y^4$	5,4,4,3,3
$D_{7,*,o}+A_{*,o}$	3,2	$V^{\#}NC^1_{(1)}$	$yx^2+(y-z)y^3+(y-z)^5$	3,4,4,4,4
$A'_{7,**,o}$	2,3	$VNF^1_{(0)}$	$yx^2+y^4+xz^3$	4,4,4,3,4
$D_{9,*,o}$	3	$VNF^1_{(1)}$	$yx^2+y^4+z^5$	4,4,4,4,4
$A_{1,*,o}+A_{1,*,o}+A_{1,*,o}+A_{1,*,o}$	2,2,2,2	$V'_{(1)}$	$x^3+y^4+z^4$	3,3,3,3,3,3
$A_{4,*,o}+A_{1,*,o}+A_{1,*,o}$	2,2,2	$V'_{(2)}$	$x^3+y^4-z^2y^2-z^3x$	3,3,3,3,4,3
$E_{6,o}+A_{1,*,o}+A_{1,*,o}$	2,2	$V'_{(3)}$	$x^3+y^4-z^2y^2+z^5$	3,3,3,3,4,4
$A_{7,*,o}+A_{1,*,o}$	2,2	$V'_{(4)}$	$x^3+y(y+z)^3+xy^3$	3,4,3,3,4,3
$A_{4,*,o}+A_{4,*,o}$	2,2	$V'_{(5)}$	$x^3+(y-z)^2y^2-(y-z)^3x+xy^3$	4,4,3,3,3,3
$A_{10,*,o}$	2	$V'_{(6)}$	$x^3+y^4-xz^3$	4,4,3,3,4,3
$E_{6,o}+A_{4,*,o}$	2	$V'_{(7)}$	$x^3+(y+z)^2y^2+xy^3+(y+z)^5$	4,4,3,4,3,3
$E_{6,o}+E_{6,o}$		$V'_{(8)}$	$x^3+(y-z)^2y^2-(y-z)^5+y^5$	4,4,4,4,3,3

Table 2

a) Type 3

Sing.	w_1	w_2	w_3	d
$NA_{0,0}$	2	2	5	10
$NC_{(0)}$	4	5	12	24
$NC_{(1)}$	6	8	19	38
$NF_{(0)}$	5	6	15	30
$NF_{(1)}$	8	10	25	50

Sing.	w_1	w_2	w_3	d
$VNA_{0,0}$	2	2	3	8
$VNC_{(0)}$	4	5	8	20
$V^{\#}NC_{(0)}$	4	5	7	19
$VNC_{(1)}$	6	8	13	32
$V^{\#}NC_{(1)}$	6	8	11	30
$VNF_{(0)}$	5	6	9	24
$VNF_{(1)}$	8	10	15	40

b) Type 4

Sing.	w_1	w_2	w_3	d
$V'_{(1)}$	3	3	4	12
$V'_{(4)}$	6	7	9	27
$V'_{(6)}$	8	9	12	36

§ 3 Geometric Bases

Let $f : (\mathbb{C}^3, \underset{\sim}{0}) \to (\mathbb{C}, 0)$ be the germ of an analytic function with an isolated singularity at $\underset{\sim}{0}$. Let B_ε denote an open ball of radius ε in $\mathbb{C}^3$ around $\underset{\sim}{0}$. Then for sufficiently small $\varepsilon >> \delta >> 0$

$$X_\delta = f^{-1}(\delta) \cap B_\varepsilon$$

is the Milnor fibre of f [18]. The homology group

$$H = H_2(X_\delta, \mathbb{Z})$$

provided with the symmetric intersection form $\langle\, ,\, \rangle$ is called the Milnor lattice of f. It is an even lattice of rank μ. In a geometric way one can define certain special classes of bases of H : the distinguished and weakly distinguished bases of vanishing cycles (cf. e.g. [10]). A method for computing the intersection matrix corresponding to a distinguished basis of vanishing cycles is given by Gabrielov [13]. Intersection matrices for the uni- and bimodal singularities were already calculated by Gabrielov. We apply his method to compute intersection matrices for the other EHS. For that purpose one has to look at generic hyperplane sections $z = 0$ for a linear function $z : \mathbb{C}^3 \to \mathbb{C}$. For the uni- and bimodal singularities one can choose z such that $f|_{z=0}$ has a singularity of type A_2, A_3 or D_4. For the other EHS one can choose z as follows:

Type/Series	Singularity of $f\|_{z=0}$
Type 3: $\underset{\sim}{N}$	A_4
$\underset{\sim}{V}$	D_5
Type 4: $\underset{\sim}{V}'$	E_6

We have the following result:

Theorem 3.1

For each EHS there exists a distinguished basis of vanishing cycles $\{\underset{\sim}{e}_{mj} \mid 1 \leq j \leq \mu' , 1 \leq m \leq M_j\}$, where the numbers M_j are exhibited in Table 1 , with the following intersection matrix. Here $\{\underset{\sim}{e}_{mj}\}$ is ordered by the lexicographic order of the pairs (m,j) .

(i) The Dynkin diagram corresponding to the intersection matrix of $\{\underset{\sim}{e}_{1j} \mid 1 \leq j \leq \mu'\}$ is shown in Fig. 1 . It is a Dynkin diagram with respect to a distinguished basis of $f|_{z=0}$ for suitable z .

(ii) We have the following relations:

$$\langle \underset{\sim}{e}_{mj}, \underset{\sim}{e}_{mj'} \rangle = \langle \underset{\sim}{e}_{1j}, \underset{\sim}{e}_{1j'} \rangle ,$$

$$\langle \underset{\sim}{e}_{mj}, \underset{\sim}{e}_{m'j} \rangle = 1 \quad \text{for} \quad |m'-m| = 1 ,$$

$$\langle \underset{\sim}{e}_{mj}, \underset{\sim}{e}_{m'j'} \rangle = -\langle \underset{\sim}{e}_{1j}, \underset{\sim}{e}_{1j'} \rangle \quad \text{for} \quad |m'-m| = 1 , \ (m'-m)(j'-j) < 0 ,$$

$$\langle \underset{\sim}{e}_{mj}, \underset{\sim}{e}_{m'j'} \rangle = 0 \quad \text{for} \quad |m'-m| > 1 \quad \text{or} \quad (m'-m)(j'-j) > 0 .$$

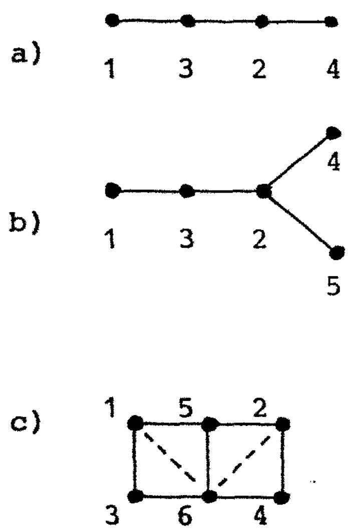


Fig.1: Dynkin diagrams of $f|_{z=0}$ for elliptic f belonging to the series a) $\underset{\sim}{N}$, b) $\underset{\sim}{V}$ c) $\underset{\sim}{V}'$.

Remarks on the proof of Theorem 3.1

Let $\Gamma_z(f)$ be the polar curve of f with respect to z, i.e. the set of critical points of the mapping $(z,f):\mathbb{C}^3 \to \mathbb{C}^2$, and let $\Gamma_z(f) = \bigcup_i \Gamma_i$ be the decomposition of $\Gamma_z(f)$ into irreducible components. Let α_i be the exponent of the first term in the Puiseux expansion of the plane curve Σ_i, which is the image of Γ_i under the mapping $(z,f):\mathbb{C}^3 \to \mathbb{C}^2$, and where we take (z,f) as coordinates. In order to compute an intersection matrix of f, one has to compute the numbers α_i, the distribution of the critical points of $f|_{z=\varepsilon}$ on the components Γ_i of the polar curve and an intersection matrix of a distinguished basis of vanishing cycles of $f|_{z=0}$ satisfying certain conditions, in particular the condition that the cycles must vanish in the critical points of $f|_{z=\varepsilon}$.

A convenient way of calculating the decomposition of Γ into irreducible components and the numbers α_i is by computing the beginning terms of the generalized Puiseux expansions following Maurer [17]. Let us assume that no component of $\Gamma_z(f)$ lies in a coordinate hyperplane. Otherwise one can do the same with fewer variables. In almost all cases the first step is sufficient for the further calculations. For simplicity we treat here this case, the general case is analogous. The first step gives a "first approximation" of a parametrization of an irreducible component Γ_i as follows

$$x = w^{a_1^{(i)}}\,(\alpha_1^{(i)} + u)$$

$$y = w^{a_2^{(i)}}\,(\alpha_2^{(i)} + v)$$

$$z = w^{a_3^{(i)}}$$

with $a_j^{(i)} \in \mathbb{N}_+$, $\alpha_j^{(i)} \in \mathbb{C} \setminus \{0\}$. The critical points of $f|_{z=\varepsilon}$ are given by replacing $z=\varepsilon$ in these equations. In most of the cases one can choose z and an equation f in the $\mu=$ constant - stratum of the singularity, such that all $\alpha_j^{(i)} \in \mathbb{R} \setminus \{0\}$ and $f(x,y,z)$ has real coefficients. Then the critical points and critical values of $f|_{z=\varepsilon}$ are approximately real and one can use the method of A'Campo [1] resp. Gusein-Zade [14] to compute the required intersection matrix of $f|_{z=0}$. The equations we used for calculation are given in Table 1 . In some cases the intersection matrix of $f|_{z=0}$ is not that given in Fig. 1 . In these cases one still has to do transformations to get the intersection matrix indicated in the table.

In some few cases we did not succeed in finding such a z and f as above and also easier methods were not applicable. In these cases we proceeded as follows. Using if possible symmetry properties (complex conjugated critical points and critical values)one can single out a set of possible intersection matrices for $\{\underset{\sim}{e}_{1j} \mid 1 \leq j \leq \mu'\}$ and determine for which of these possiblities the intersection matrix of the complete basis $\{\underset{\sim}{e}_{mj}\}$ gives the right quadratic form. It turns out in the considered cases that the remaining bases are all equivalent to the corresponding basis of Theorem 2.1 under the action of the braid group Z_μ (cf. [10]) . $\square$

Using transformations which transform weakly distinguished bases to weakly distinguished bases (like in [9]) one can derive from this theorem the following result, which was partly announced in [10]. Let S be a graph. For a vertex $v \in S$, the valence of v, val v, is the number of edges incident with v. Let $\zeta(S)$ be the number of cycles of S of the form $v_o, v_1, \ldots, v_r = v_o$ with val $v_o = 3$, val $v_i = 2$ for $i \neq 0$. Define

$$\sigma(S) = \sum_{\substack{v \in S \\ \text{val } v \geq 3}} (\text{val } v - 2) + \zeta(S) .$$

Theorem 3.2

Let f be an EHS which is not simply elliptic or of type $T_{p,q,r}$.
Then there exists a weakly distinguished basis $B = \{\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_\mu\}$ of f satisfying the following properties

a) $\langle \underset{\sim}{e}_{\mu-1}, \underset{\sim}{e}_\mu \rangle = 1$

$\langle \underset{\sim}{e}_\mu, \underset{\sim}{e}_i \rangle = 0, \langle \underset{\sim}{e}_{\mu-1}, \underset{\sim}{e}_i \rangle = \langle \underset{\sim}{e}_{\mu-2}, \underset{\sim}{e}_i \rangle$ for $1 \leq i \leq \mu-2$

b) For $i, j \in \{1, \ldots, \mu-2\}$, $i \neq j$, $\langle \underset{\sim}{e}_i, \underset{\sim}{e}_j \rangle \in \{0,1\}$ (The matrix $(-\langle \underset{\sim}{e}_i, \underset{\sim}{e}_j \rangle)_{1 \leq i,j \leq \mu-2}$ is therefore an indecomposable symmetric Cartanmatrix of negative type in the sense of [15]).

c) The subgraph S of the Dynkin diagram corresponding to $\{\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_{\mu-2}\}$ has no vertices of valence > 3.

c) For f of type l, $\sigma(S) = l$.

Remark

One can change the numbering of S to get other bases which are also weakly distinguished bases of the same singularity (cf. [8, Abb. 2 b]).

Some of the possible graphs S for each EHS are considered in § 5 . The types of the graphs considered are shown in Fig. 2 .

§ 4 Milnor Lattices

There are two ways to determine the Milnor lattices of our singularities.

First we can compute these lattices from the resolutions as in [11, Chap.4]. In particular one has the following formulas: Let

$$L = X \cap \partial \overline{B}_{\varepsilon}$$

denote the neighbourhood boundary of X. Then

$$\mu_o = \mathrm{rk}\ H_1(L, \mathbb{Z})$$

$$\mu_o + \mu_+ = 2h \quad \text{(Durfee [5])} .$$

This implies that the EHS are precisely the hypersurface singularities defined by a function germ $f : (\mathbb{C}^3, \underset{\sim}{0}) \to (\mathbb{C}, 0)$ with $\mu_o + \mu_+ = 2$. In addition $\mu_o \neq 0$ only for the (unimodal) simply elliptic singularities $(\mu_o = 2)$ and the (unimodal) singularities $T_{p,q,r}$ $(\mu_o = 1)$. An analogous consideration as in [11, Lemma 4.4.1] yields the following estimation for the minimal number $\lambda(G)$ of generators of $G = TH_1(L)$ (where the prefix T means the torsion subgroup):

$$\lambda(G) \leq 4 .$$

Alternatively we can use a basis of Theorem 3.2. We proceed here this way. The basis of Theorem 3.2 is a special basis in the sense of [9, Def. (1.2)]. This implies in particular that

$$H = K \perp U ,$$

i.e. H is the orthogonal direct sum of the lattice

$$K = \sum_{i=1}^{\mu-2} \mathbb{Z}\, \underset{\sim}{e}_i \quad ,$$

hence the sublattice determined by the graph S , and a unimodular hyperbolic plane U . We call K the <u>hyperbolic sublattice</u> of L , since it has signature $(t_-, t_o, t_+) = (\mu-3, 0, 1)$. By the arguments of [11, (4.5)] it is uniquely determined by H .

By the theorems of Nikulin [20] (cf. also [11, (4.5)]), the lattice K is determined by the corresponding discriminant quadratic form $q : G_H \to \mathbb{Q}/2\mathbb{Z}$ or the corresponding discriminant bilinear form $b : G_H \times G_H \to \mathbb{Q}/\mathbb{Z}$. The finite quadratic or bilinear forms are computed as follows (cf. also [11, (4.7)] .

Let S be one of the graphs of Fig. 2. First we get rid of some of the cycles by transformations as in [8 , § 1,2] , possibly going over to a graph with less vertices defining a stably equivalent lattice. Recall that two even lattices H_1 and H_2 are called stably equivalent, if there exist even unimodular lattices M_1 and M_2 such that $H_1 \perp M_1 \cong H_2 \perp M_2$. Two lattices are stably equivalent if and only if they have the same discriminant quadratic form. Then we extend the resulting graph by connecting new vertices of length -1 or 0 to the free ends, such that the new extended graph defines a unimodular lattice. We have thus constructed an imbedding of the original lattice or a stably equivalent one into a unimodular lattice. The orthogonal complement $H^{\perp}$ of the original lattice H or the corresponding stably equivalent one is then described by a matrix A of low rank. But

$$b_H = -b_{H^\perp}$$

and in particular

$$\operatorname{disc} H = |\det A|$$

where disc H denotes the discriminant of H, which is the order of G_H.

Thus the calculation of the discriminant bilinear form of H (which together with sign H(mod 8) determines the discriminant quadratic form of H [20, Theorem 1.11.3]) is reduced to the determination of the discriminant bilinear form of the lattice $H^\perp$ of low rank, which amounts to the inversion of the matrix A. The matrices A for the different types of graphs S are listed in Table 5. The resulting finite quadratic forms for the EHS of type 3 and 4 are listed in Table 3 resp. 4.

Here we use the following notation, which is also used in [11]: Each finite quadratic form splits as the orthogonal direct sum of the following forms:

(1) $w^{\varepsilon}_{p,k} : \mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Q}/2\mathbb{Z}$, where a generator of $\mathbb{Z}/p^k\mathbb{Z}$ is mapped to θp^{-k} and where p is a prime number, θ a p-adic unit and

$$\varepsilon = \left(\frac{\theta}{p}\right) \text{ (Legendresymbol) } \in \{+1,-1\} \text{ for } p \neq 2$$

and $$\varepsilon \equiv \theta \pmod 8, \quad \varepsilon \in \{+1,-1,+5,-5\} \quad \text{for } p=2.$$

(2a) $v_k : \mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z} \to \mathbb{Q}/2\mathbb{Z}$ given by

$$\frac{1}{2^k}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

(2b) $u_k : \mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z} \to \mathbb{Q}/2\mathbb{Z}$ given by

$$\frac{1}{2^k}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

For the systematic description of series we augment this notation by the discriminant quadratic forms of simple singularities:

$$q_{A_r} = \mathbb{Z}/(r+1)\mathbb{Z} \text{ , generator } x \text{ , } q(x) = (r+2)/(r+1)$$

$$q_{D_r} = \begin{cases} u_1 \text{ , } r \equiv 0 \pmod 8 \\ v_1 \text{ , } r \equiv 4 \pmod 8 \\ w^{\varepsilon}_{2,1} + w^{\varepsilon}_{2,1} \text{ , } \varepsilon = \pm 1 \text{ , } r \equiv -2\varepsilon \pmod 8 \\ w^{\varepsilon}_{2,2} \text{ , } \varepsilon = \pm 1 \text{ or } \pm 5 \text{ , } r \equiv -\varepsilon \pmod 8) \text{ .} \end{cases}$$

In the entry of $V^{\#}NA_{n,m}$,$\varepsilon \equiv 1-2m \pmod 8$ and

$$\eta(\varepsilon) \equiv \frac{\varepsilon-1}{2} \pmod 2 \text{ , } \eta(\varepsilon) \in \{0,1\}.$$

It turns out that there are many relations among the Milnor lattices of the EHS of type 3 and 4 and the singularities studied in [11]. There are many singularities with isomorphic Milnor lattices. If X and Y are singularities with isomorphic Milnor lattices, we write

$$X \cong Y$$

There are also singularities with stably equivalent Milnor lattices. We write

$$X \sim Y$$

if X and Y have stably equivalent Milnor lattices. Some of these relations are indicated in Tables 3 and 4 . In the case of a series indexed by a number n , the singularities of this series are only defined for $n \geq 0$, but there are also lattices defined for $n_o \leq n < 0$, cf. [11, Chapt. 5] . So a relation involving a singularity name with index $n < 0$ refers to the corresponding lattice. In all cases of pairs of singularities with isomorphic Milnor lattices, also the monodromy groups are isomorphic by [9] , but the characteristic polynomials of the classical monodromy operators are different.

We point out two remarkable strange correspondances: The quadratic forms of the left hand side of the rows of Table 3 correspond to the quadratic forms of the columns for I_n^* for $g=1,2$ of the tables in [11] in a certain way. We get in particular other examples of hypersurface and complete intersection singularities, which are not hypersurface singularities having the same Milnor lattices. Five of the eight EHS of type 4 have quadratic forms which are stably equivalent to the quadratic forms of the top entries of the columns for I_n^* for $g=1,2,3,4$ setting $n=-3$ in the tables in [11] .

Table 3

	$A_{*,o}$ $+A_{n,**,o}$ $+A_{m,**,o}$	$A'_{3,**,o}$ $+A_{n,**,o}$	$D_{5,*,o}$ $+A_{n,**,o}$	$E_{7,o}$ $+A_{n,**,o}$	$A'_{5,**,o}$ $+A_{*,o}$	$D_{7,*,o}$ $+A_{*,o}$	$A'_{7,**,o}$	$D_{9,*,o}$
D=2 $b_i=2$, all i	$NA_{m,n}$ $16+m+n$ 16 $q_{D_{4+m}}+q_{D_{8+n}}$ $m=1$: $\cong J'_{2,4+n}$	$NB^n_{(-1)}$ $18+n$ 12 $w^{-1}_{3,1}+q_{D_{8+n}}$ $\cong Q_{2,4+n}$	$NB^n_{(o)}$ $19+n$ 8 $w^1_{2,1}+q_{D_{8+n}}$ $\cong Z_{1,4+n}$	$NB^n_{(1)}$ $20+n$ 4 $q_{D_{8+n}}$ $\cong E_{3,4+n}$	$NC_{(o)}$ 19 8 $w^5_{2,3}$ $\cong VNC_{(1)}$ $\sim S_{11}$	$NC_{(1)}$ 20 4 $w^1_{2,1}+w^{-1}_{2,1}$ $\sim Z_{12}$	$NF_{(o)}$ 20 5 $w^{-1}_{5,1}$ $\cong VNF_{(1)}$ $\sim W_{12}$	$NF_{(1)}$ 21 2 $w^{-1}_{2,1}$ $\sim E_{13}$
$b_r=3$, r = max i =3	$VNA_{m,n}$ $15+m+n$ $32+4n$ $q_{D_{4+m}}+q_{A_{7+n}}$ $m=6$: $\sim K^\flat_{1,n}$	$VNB^n_{(-1)}$ $17+n$ $24+3n$ $w^{-1}_{3,1}+q_{A_{7+n}}$	$VNB^n_{(o)}$ $18+n$ $16+2n$ $w^1_{2,1}+q_{A_{7+n}}$	$VNB^n_{(1)}$ $19+n$ $8+n$ $q_{A_{7+n}}$ $\sim W^\#_{1,n-4}$	$VNC_{(o)}$ 18 16 $w^{-1}_{2,2}+w^{-5}_{2,2}$ $\cong NA_{1,1}$	$VNC_{(1)}$ 19 8 $w^5_{2,3}$ $\cong NC_{(o)}$ $\sim S_{11}$	$VNF_{(o)}$ 19 12 $w^{-1}_{2,2}+w^{-1}_{3,1}$ $\cong NB^1_{(-1)}$	$VNF_{(1)}$ 20 5 $w^{-1}_{5,1}$ $\cong NF_{(o)}$ $\sim W_{12}$
$b_1=3$, $b_i=2$, i>1	$V^\# NA_{m,n}$ $15+m+n$ 32 $w^\varepsilon_{2,3}+$ $+q_{D_{m+n+4-2\eta(\varepsilon)}}$ $m=2$: $\cong K'_{1,4+n}$	$V^\# NB^n_{(-1)}$ $17+n$ 28 $w^{-1}_{7,1}+q_{D_{3+n}}$ $\cong L_{1,4+n}$	$V^\# NB^n_{(o)}$ $18+n$ 20 $w^{-1}_{5,1}+q_{D_{6+n}}$ $\cong S_{1,4+n}$		$V^\# NC_{(o)}$ 18 19 $w^{-1}_{19,1}$ $\sim L^\#_{1,-3}$	$V^\# NC_{(1)}$ 19 10 $w^1_{2,1}+w^{-1}_{5,1}$		

Table 4

	$A_{1,*,o}+A_{1,*,o}+A_{1,*,o}+A_{1,*,o}$	$A_{4,*,o}+A_{1,*,o}+A_{1,*,o}$	$E_{6,o}+A_{1,*,o}+A_{1,*,o}$	$A_{7,*,o}+A_{1,*,o}$	$A_{4,*,o}+A_{4,*,o}$	$A_{10,*,o}$	$E_{6,o}+A_{4,*,o}$	$E_{6,o}+E_{6,o}$
	$V'_{(1)}$	$V'_{(2)}$	$V'_{(3)}$	$V'_{(4)}$	$V'_{(5)}$	$V'_{(6)}$	$V'_{(7)}$	$V'_{(8)}$
=2,	18	19	20	20	20	21	21	22
	27	18	9	9	12	4	6	3
ll i	$w^{-1}_{3,1}+w^{-1}_{3,1}+w^{-1}_{3,1}$	$w^{1}_{2,1}+w^{-1}_{3,1}+w^{1}_{3,1}$	$w^{-1}_{3,1}+w^{1}_{3,1}$	$w^{1}_{3,2}$	$w^{1}_{2,1}+w^{1}_{2,1}+w^{1}_{3,1}$	$w^{-1}_{2,2}$	$w^{1}_{2,1}+w^{1}_{3,1}$	$w^{1}_{3,1}$
				$\cong VNB^{1}_{(1)}$	$\cong W_{1,5}$	$\cong NB^{1}_{(1)} \cong E_{3,5}$		
	$\sim I_{1,-3}$	$\sim U_{1,-3}$	$\sim Q_{12}$	$\sim W^{\#}_{1,-3}$	$\sim W_{1,-3}$	$\sim E_{3,-3}$	$\sim Z_{13}$	$\sim E_{14}$

Legend for Tables 3 and 4:

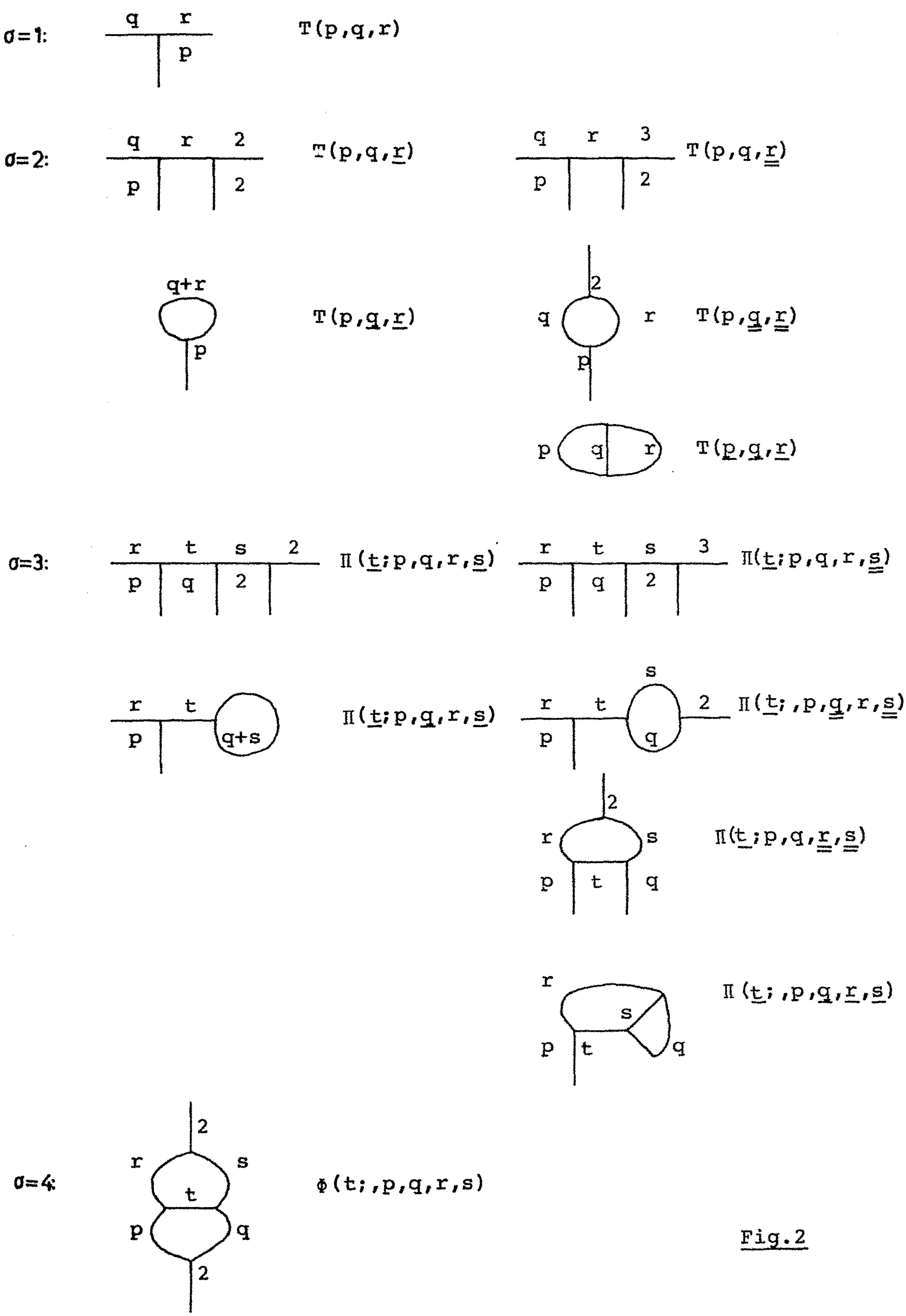
We have indicated in an entry: Name of the singularity

Milnor number μ

discriminant

discriminant quadratic form

singularities with isomorphic ($\cong$)
or stably equivalent ($\sim$) Milnor lattices



$\sigma=1$:
$T(p,q,r)$
$\sigma=2$:
$T(p,q,\underline{r})$
$T(p,q,\underline{\underline{r}})$
$T(p,\underline{q},\underline{r})$
$T(p,\underline{\underline{q}},\underline{\underline{r}})$
$T(\underline{p},\underline{q},\underline{r})$
$\sigma=3$:
$\Pi(\underline{t};p,q,r,\underline{s})$
$\Pi(\underline{t};p,q,r,\underline{\underline{s}})$
$\Pi(\underline{t};p,\underline{q},r,\underline{s})$
$\Pi(\underline{t};,p,\underline{\underline{q}},r,\underline{\underline{s}})$
$\Pi(\underline{t};p,q,\underline{\underline{r}},\underline{\underline{s}})$
$\Pi(\underline{t};,p,\underline{q},\underline{r},\underline{s})$
$\sigma=4$:
$\Phi(t;,p,q,r,s)$

Fig.2

Table 5:

$T(p,q,r)$:

$$\begin{pmatrix} 1-p & 1 & 1 \\ 1 & 1-q & 1 \\ 1 & 1 & 1-r \end{pmatrix}$$

$T(p,q,\underline{r})$:

$$\begin{pmatrix} 1-p & 1 & 1 & 0 \\ 1 & 1-q & 1 & 0 \\ 1 & 1 & 4-r & -2 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

$T(p,q,\underline{\underline{r}})$:

$$\begin{pmatrix} 1-p & 1 & 1 \\ 1 & 1-q & 1 \\ 1 & 1 & 8-r \end{pmatrix}$$

$T(p,\underline{\underline{q}},\underline{\underline{r}})$:

$$\begin{pmatrix} 1-p & 1 & 1 \\ 1 & 4-q & 3 \\ 1 & 3 & 4-r \end{pmatrix}$$

$T(\underline{p},\underline{q},\underline{r})$:

$$\begin{pmatrix} 3-p & 2 & 2 \\ 2 & 3-q & 2 \\ 2 & 2 & 3-r \end{pmatrix}$$

$\Pi(\underline{t};2,q,r,\underline{s})$:

$$\begin{pmatrix} 2-r & -2 & 0 & 0 & 0 \\ -2 & 4-t & 1 & 1 & 0 \\ 0 & 1 & 1-q & 1 & 0 \\ 0 & 1 & 1 & 4-s & -2 \\ 0 & 0 & 0 & -2 & 0 \end{pmatrix}$$

$\Pi(\underline{t};2,q,r,\underline{\underline{s}})$:

$$\begin{pmatrix} 2-r & -2 & 0 & 0 \\ -2 & 4-t & 1 & 1 \\ 0 & 1 & 1-q & 1 \\ 0 & 1 & 1 & 8-s \end{pmatrix}$$

$\Pi(\underline{t};,2,\underline{\underline{q}},r,\underline{\underline{s}})$:

$$\begin{pmatrix} 2-r & -2 & 0 & 0 \\ -2 & 4-t & 1 & 1 \\ 0 & 1 & 4-q & 3 \\ 0 & 1 & 3 & 4-s \end{pmatrix}$$

$\Pi(\underline{t};p,q,\underline{\underline{r}},\underline{\underline{s}})$:

$$\begin{pmatrix} 1-p & 0 & 1 & 0 & -1 \\ 0 & 1-q & 0 & 1 & 1 \\ 1 & 0 & 4-r & 2 & -1 \\ 0 & 1 & 2 & 4-s & 1 \\ -1 & 1 & -1 & 1 & 3-t \end{pmatrix}$$

<u>Table 5</u> (continued)

$\Pi(\underline{t};,2,\underline{q},\underline{r},\underline{s})$:

$$\begin{pmatrix} 3-q & 2 & 1 & 1 \\ 2 & 3-s & 1 & 1 \\ 1 & 1 & 4-r & -2 \\ 1 & 1 & -2 & 4-t \end{pmatrix}$$

$\Phi(t;,p,q,r,s)$:

$$\begin{pmatrix} 4-p & -2 & 1 & 0 & 1 \\ -2 & 4-q & 0 & 1 & 1 \\ 1 & 0 & 4-r & -2 & 1 \\ 0 & 1 & -2 & 4-s & 1 \\ 1 & 1 & 1 & 1 & 3-t \end{pmatrix}$$

$$T(p,\underline{q},\underline{r}) = A_{q+r-2} \perp A_{p-2}$$

$$\Pi(\underline{t};,p,\underline{q},r,\underline{s}) = A_{q+s-2} \perp T(p,r,t-2)$$

§ 5 Dynkin Diagrams

In this section we present an algorithm on graphs which produces a set of graphs, such that the lattices defined by these graphs are exactly the hyperbolic sublattices of the EHS with $\mu_+ = 2$. Moreover all these graphs occur as subgraphs of Dynkin diagrams with respect to weakly distinguished bases as in Theorem 3.2.

Let S be the Dynkin diagram corresponding to an indecomposable symmetric Cartanmatrix multiplied by -1 with only 0 or 1 outside the diagonal. That means that S is a connected unweighted graph with no multiple edges. We assume in addition that S has no vertices of valence >3 . We recall the definition of the transformation $\tau_{D,1}$ (cf. [11]) . Let D be a subgraph of S ,which is an extended classical Dynkin diagram, that is of type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7$ or $\tilde{E}_8$. Let e_1 be a vertex of S , which is neither contained in D nor connected by an edge to a vertex of D . Let $\underset{\sim}{w}$ be the distinguished isotropic vector of D , that is the sum of the longest root of the corresponding finite root system and the additional vector. Let

$$\underset{\sim}{e}_1^{(1)} = \underset{\sim}{w} - \underset{\sim}{e}_1 \ , \ \underset{\sim}{e}_i^{(1)} = \underset{\sim}{e}_i \quad \text{for} \quad i \neq 1 \ .$$

Then the new basis $\{\underset{\sim}{e}_i^{(1)}\}$ satisfies again $\langle \underset{\sim}{e}_i^{(1)}, \underset{\sim}{e}_i^{(1)} \rangle = -2$, since $\langle \underset{\sim}{w}, \underset{\sim}{e}_1 \rangle = \langle \underset{\sim}{w}, \underset{\sim}{w} \rangle = 0$. If $\langle \underset{\sim}{e}_1^{(1)}, \underset{\sim}{e}_i^{(1)} \rangle \in \{0,1\}$ for $i \neq 1$, and the new graph S corresponding to $\{\underset{\sim}{e}_i^{(1)}\}$ is connected, then we define

$$\tau_{D,1}(S) = S^{(1)}$$

If $S^{(1)}$ is not connected or if there exists an $i \neq 1$ with

$|<\underset{\sim}{e}_1^{(1)}, \underset{\sim}{e}_i^{(1)}>| > 1$, then $\tau_{D,1}$ is not defined. If finally for some i $<\underset{\sim}{e}_1^{(1)},\underset{\sim}{e}_i^{(1)}> = -1$, we shall continue transforming as follows. Define $\alpha_i(j)$ to be the transformation

$$\{\underset{\sim}{f}_1,\ldots,\underset{\sim}{f}_r\} \to \{\underset{\sim}{f}_1,\ldots,\underset{\sim}{f}_{j-1},s_{\underset{\sim}{f}_i}(\underset{\sim}{f}_j),\underset{\sim}{f}_{j+1},\ldots,\underset{\sim}{f}_r\}$$

where $s_{\underset{\sim}{f}_i}$ denotes the reflection corresponding to $\underset{\sim}{f}_i$, i.e.

$$s_{\underset{\sim}{f}_i}(\underset{\sim}{x}) = \underset{\sim}{x} + <\underset{\sim}{x},\underset{\sim}{f}_i>\underset{\sim}{f}_i \ .$$

Choose j_1 with $<\underset{\sim}{e}_1^{(1)},\underset{\sim}{e}_{j_1}^{(1)}> = -1$ and let

$$S^{(2)} = \alpha_{j_1}(1) \quad (S^{(1)}) \ .$$

Now there are again several possibilites for the new basis $\underset{\sim}{e}_i^{(2)}$:

a) $|<\underset{\sim}{e}_1^{(2)},\underset{\sim}{e}_i^{(2)}>| > 1$ for some $i \neq 1$.
b) $<\underset{\sim}{e}_1^{(2)},\underset{\sim}{e}_i^{(2)}> = -1$ for some $i \neq 1$ with $\underset{\sim}{e}_i^{(2)} \in D$.
c) $<\underset{\sim}{e}_1^{(2)},\underset{\sim}{e}_i^{(2)}> \in \{0,1\}$ for all $i \neq 1$.
d) $<\underset{\sim}{e}_1^{(2)},\underset{\sim}{e}_i^{(2)}> = -1$ for some $i \neq 1$ with $\underset{\sim}{e}_i^{(2)} \notin D$.

In cases a) and b) $\tau_{D,1}$ is not defined. In case c) the matrix corresponding to $\{\underset{\sim}{e}_i^{(2)}\}$ is again a Cartan matrix and the corresponding graph $S^{(2)}$ has no multiple edges. We define in this case

$$\tau_{D,1}(S) = S^{(2)} \ .$$

In case d) we continue as above until one of the other cases is reached, choosing j_2 and so on. One easily checks that cases a),b) or c) are reached after finitely many, say N , steps, and that $\tau_{D,1}$ does not depend on the choice of the sequence $j_1,j_2,\ldots,j_N$, if it is defined. The graphs different from S

obtained by these transformations or sequences of these transformations with the same D but different $\underset{\sim}{e}_l$ are called the <u>proper transforms</u> of S .

To a graph S is associated a weighted graph S' , called the <u>scheme</u> of S , by replacing subgraphs

$$\underset{e_1}{\bullet}\!-\!\!-\!\underset{e_2}{\bullet}\!-\!\!-\quad\cdots\quad-\!\!-\!\underset{e_l}{\bullet}$$

with $\operatorname{val} e_i = 2$ for $i \neq 1,l$, $\operatorname{val} e_i \neq 2$ for $i=1,l$ by an edge weighted by l between e_1 and e_l . The weights of S' are called the <u>weights</u> of S , the underlying unweighted graph the <u>shape</u> of S . If $\operatorname{val} e_i = 1$ for $i=1$ or $i=l$, then l is called an <u>outer weight</u>, otherwise it is called an <u>inner weight</u>. Let S be a graph with weights $w_1,\dots,w_r$. We define φ^+ to be the operation which associates to S the graph of the same shape with weights $w_1,w_2,\dots,w_i+1,\dots,w_r$, where w_i satisfies the following conditions:

(i) w_i is not an outer weight which is adjacent to two inner weights.

(ii) $w_i \neq 2,3$;

or $w_i = 2$ and there is an adjacent weight $w_j = 2$;

or $w_i = 3$ and there is <u>no</u> adjacent weight $w_j = 2$.

(iii) If w_i is an inner weight, then w_i is greater than or equal all the inner weights which belong to edges incident with the same two vertices.

Let S_1 and S_2 be graphs of the same shape with weights $w_1^{(1)},\dots,w_r^{(1)}$ resp. $w_1^{(2)},\dots,w_r^{(2)}$. We define $S_1 \leq S_2$, if the outer weights are the same and the inner weights satisfy $w_i^{(1)} \leq w_i^{(2)}$. The difference in the number of vertices is called the distance of S_1 and S_2 and denoted by $\delta(S_1,S_2)$.

Let $\mathcal{S}$ be a set of graphs as above. Define the set $(\hat{\mathcal{S}})^{\circ}$ as follows. This is the set of all graphs R such that for each chain

$$R_1 \leq R_2 \leq \ldots \leq R_k = R$$

with R_1 minimal and $\delta(R_{i+1},R_i)=1$, there is an $1 \leq i < k$ with $R_i \in \mathcal{S}$. Define $\check{\mathcal{S}}$ to be the set of graphs R with $R \leq S$ for $S \in \mathcal{S}$ and $\delta(R,S) \leq 1$.

Let $\mathcal{H}_i$ be the set of all graphs S with $\sigma(S) = i$, which define a hyperbolic lattice. Using the above definitions we shall define inductively subsets $\mathcal{S}_i$, $\mathcal{B}_i$, $\mathcal{F}_i \subset \mathcal{H}_i$ as follows. Assume $\mathcal{S}_i$ is already defined. Then let

$$\mathcal{B}'_i := (\varphi^+(\mathcal{S}_i) \cup \varphi^+(\varphi^+(\mathcal{S}_i)) \cap \mathcal{H}_i ,$$

$$\mathcal{B}_i := \check{\mathcal{B}}'_i \setminus (\check{\mathcal{B}}'_i \cap (\mathcal{F}_i \cup \mathcal{S}_i))$$

Define $\mathcal{F}_{i+1}$ to be the set of all transforms of the set $\mathcal{B}_i$, which lie in $\mathcal{H}_{i+1}$, under operations $\tau_{D,1}$. For $i=1$ the subdiagram D can only be of type $\tilde{E}_6,\tilde{E}_7$ or $\tilde{E}_8$, for $i > 1$ D has to be of type $\tilde{A}_r$ or $\tilde{D}_r$. Define

$$\mathcal{S}_{i+1} := (\widehat{\mathcal{F}_{i+1}})^{\circ} \cap \mathcal{H}_{i+1}$$

Finally define $\mathcal{F}_1 = \emptyset$ and $\mathcal{S}_1 \subset \mathcal{H}_1$ to be the set of all graphs of $\mathcal{H}_1$, which have no proper transforms.

With these definitions we have the following theorem

<u>Theorem 5.1</u>

(i) $\mathcal{B}_4 = \emptyset$ <u>and thus</u> $\mathcal{F}_i = \mathcal{S}_i = \mathcal{B}_i = \emptyset$ <u>for</u> $i \geq 5$.

(ii) <u>The hyperbolic sublattices of the EHS with</u> $\mu_+ = 2$ <u>of type</u> i <u>are exactly those given by the graphs of</u> $\mathfrak{Y}_i$.

The possible schemes of graphs of $\mathfrak{Y}_i$, $1 \leq i \leq 4$, are given in Fig. 2. We adopt the notation of [11] , where the case i=1,2 is already treated. We conclude with tables of the above defined sets which are nonempty. We start with $\mathfrak{Y}_1$, which is the set of graphs corresponding to the 14 unimodal exceptional singularities. Then we list the sets $\mathfrak{B}_{i-1}$, $\mathfrak{F}_i$, $\mathfrak{Y}_i$ for $i \geq 2$ in the following way. We list the possible types of graphs occuring in $\mathfrak{Y}_i$. For each type we give a table of those weights such that the corresponding graph is in $\mathfrak{Y}_i$ or in the set $\mathfrak{F}_i$, which "bounds" the set $\mathfrak{Y}_i$ from below. In the case that the weights define an element of $\mathfrak{Y}_i$, we indicate the name of the corresponding singularity in the associated entry. In the case that they define an element of $\mathfrak{F}_i$, we indicate the element of $\mathfrak{B}_{i-1}$, of which this element is a transform. Thus the set $\mathfrak{Y}_i$ is the set of those graphs corresponding to the entries with singularity names, the set $\mathfrak{F}_i$ is the set of those graphs corresponding to entries with notations of graphs of $\mathfrak{X}_{i-1}$, and finally the set $\mathfrak{B}_{i-1}$ is the set of graphs indicated in the entries of $\mathfrak{F}_i$. This is, however, not quite true for the case i=4 , where the sets $\mathfrak{F}_i$ and $\mathfrak{B}_{i-1}$ are bigger than the above sets. That means that they contain other graphs of other types, but which are not relevant for the set $\mathfrak{Y}_4$, and which we have not indicated.

In the case that there is a symmetry in the type of graph considered, we give only a part of the table, so it has to be completed according to this symmetry. By * in an entry we indicate that the corresponding graph does not define a hyperbolic lattice. Since the table for the graphs $\Phi(t;,p,q,r,s)$ (Table 9) gets rather involved, we make here other conventions. We indicate the region of graphs defining hyperbolic lattices by drawing the line which separates this region from that for which this is not true. Although many of the graphs of $\mathfrak{F}_4$ have more than one preimage in the set $\mathfrak{B}_3$, we only indicate one.

The tables now give the result of Theorem 5.1. $\square$

<u>Remarks</u>

1)Each of the graphs of $\mathcal{G}_i$, $1 \leq i \leq 4$, corresponds to the subgraph S of a weakly distinguished basis as in Theorem 3.2 of the corresponding singularity. Extend also the graphs of $\mathfrak{B}_i$, $\mathfrak{F}_i$ in an analogous way by two new vertices. Let $S \in \mathfrak{B}_i$ and $R \in \mathfrak{F}_{i+1}$ be a transform of S and denote the corresponding extended graphs by S' and R'. Then it seems that S' and R' are equivalent under the group Z^o generated by the braid group Z_μ and the symmetric group Σ_μ. This is the group of transformations of weakly distinguished bases (cf. [10]). Moreover the operation induced on the graphs S and R is just a transformation $\tau_{D,1}$ (resp. a sequence of these transformations). Unfortunately we have no general proof for this fact, but checked it in many cases.

2) Usually there corresponds to a relation $S_1 \leq S_2$ between two graphs $S_1, S_2 \in \mathcal{G}_i$ an adjacency relation between the corresponding singularities. So the tables also show some adjacency relations between the EHS.

3) The tables show that there are many graphs defining the same hyperbolic lattice. There are many other graphs not contained in one of the sets $\mathcal{S}_i$ defining hyperbolic sublattices of the EHS with $\mu_+ = 2$. In [11] we defined the notion of a small fundamental valuation for a graph corresponding to an indecomposable symmetric Cartanmatrix. Not all the graphs of the sets $\mathcal{S}_i$ possess a small fundamental valuation, but one can find for each EHS with $\mu_+ = 2$ a diagram (possibly not in any of the sets $\mathcal{S}_i$) with a small fundamental valuation.

<u>Table 6</u> $\mathcal{S}_1$

T(p,q,r)

p	q	r		p	q	r		p	q	r	
2	3	7	E_{12}	2	4	7	Z_{13}	2	5	6	W_{13}
2	3	8	E_{13}	3	3	4	Q_{10}	3	4	4	S_{11}
2	3	9	E_{14}	3	3	5	Q_{11}	3	4	5	S_{12}
2	4	5	Z_{11}	3	3	6	Q_{12}	4	4	4	U_{12}
2	4	6	Z_{12}	2	5	5	W_{12}				

<u>Table 7</u> $\mathfrak{F}_2$, $\mathfrak{S}_2$

T(p,q,$\underline{r}$)

p	q	r_o	$r=r_o$	$r=r_o+1+n$, $n \geq 0$
2	3	8	T(2,3,10)	$E_{3,n}$
2	4	6	T(2,4,8)	$Z_{1,n}$
3	3	5	T(3,3,7)	$Q_{2,n}$
2	5	5	T(2,6,6)	$W_{1,n}$
3	4	4	T(3,5,5)	$S_{1,n}$

T(p,$\underline{q}$,$\underline{r}$)

p	s_o	$q+r=s_o$	$q+r=s_o+1+n$, $n \geq 0$
2	12	T(2,5,7)	$W^{\#}_{1,n}$
3	10	T(3,4,6)	$S^{\#}_{1,n}$
4	9	T(4,4,5)	$U_{1,n}$

T(p,q,$\underline{\underline{r}}$)

p	q	r_o	$r=r_o$	$r=r_o+1$	$r=r_o+2$	$r=r_o+3$	$r=r_o+4$	$r=r_o+5$	$r=r_o+6$
2	3	8	T(2,3,11)	$E_{3,1}$	E_{18}	E_{19}	E_{20}	*	*
2	4	6	T(2,4,9)	$Z_{1,1}$	Z_{17}	Z_{18}	Z_{19}	*	*
3	3	5	T(3,3,8)	$Q_{2,1}$	Q_{16}	Q_{17}	Q_{18}	*	*
2	5	5	T(2,6,7)	$W^{\#}_{1,1}$	W_{17}	W_{18}	$E_{3,3}$	E_{20}	*
3	4	4	T(3,5,6)	$S^{\#}_{1,1}$	S_{16}	S_{17}	W_{18}	Z_{19}	*

Table 7 (continued)

$T(p,\underline{\underline{q}},\underline{\underline{r}})$

p	q	r=4	r=5	r=6	r=7	r=8
2	8		T(2,6,7)	$W^{\#}_{1,1}$	W_{17}	W_{18}
	7			T(2,5,8)	$W_{1,1}$	W_{17}
3	7	T(3,5,6)	$S^{\#}_{1,1}$	S_{16}	S_{17}	
	6		T(3,4,7)	$S_{1,1}$	S_{16}	
4	6	T(4,5,5)	$U_{1,1}$	U_{16}		
	5		T(4,4,6)	$U_{1,1}$		

$T(\underline{p},\underline{q},\underline{r})$

p	q	r=4	r=5	r=6
6	6	T(4,5,5)	$U_{1,1}$	U_{16}
	5		T(4,4,6)	$U_{1,1}$

Table 8 $\mathfrak{F}_3$, $\mathfrak{S}_3$

$\Pi(\underline{t}; p,q,r,\underline{s})$

p	q	r	s	t=3	t=4	t=5+n
2	2	2	$5+m, m\geq 0$		$T(2,6,\underline{5+m})$	$NA_{n,m}$
			4			$T(2,6,\underline{5+n})$
2	3	2	$5+m, m\geq 0$	$T(4,4,\underline{5+m})$	$VNA_{m+1,o}$	$V^{\#}NA_{m+1,n+1}$
			4	$T(4,4,\underline{4})$	$VNA_{o,o}$	$VNA_{n+1,o}$
			3		$T(4,4,\underline{4})$	$T(4,4,\underline{5+n})$

$\Pi(\underline{t}; ,p,\underline{q},r,\underline{s})$

p	r	q+s	t=3	t=4	t=5	t=6	t=7	t=8	t=9
2	2	$9+m, m\geq 0$	$T(5,\underline{5},\underline{4+m})$	$VNA_{o,m}$	$VNA_{1,m}$	$VNA_{2,m}$	$VNA_{3,m}$	$VNA_{4,m}$	etc.
		8		$T(3,5,\underline{4})$	$T(3,5,\underline{5})$	$T(3,5,\underline{6})$	$T(3,5,\underline{7})$	$T(3,5,\underline{8})$	etc.
2	3	$9+n, n\geq 0$	$T(6,\underline{5},\underline{4+n})$	$VNA_{1,n}$	$VNB^{n}_{(-1)}$	$VNB^{n}_{(o)}$	$VNB^{n}_{(1)}$	*	*
		8		$T(3,\underline{\underline{4}},\underline{\underline{8}})$	$T(3,\underline{\underline{5}},\underline{\underline{8}})$	$T(3,\underline{\underline{6}},\underline{\underline{8}})$	$T(3,\underline{\underline{7}},\underline{\underline{8}})$	*	*

$\Pi(\underline{t}; p,q,r,\underline{s})$

p	q	r	t	s=3	s=4	s=5	s=6	s=7	s=8	s=9
2	2	2	$5+n, n\geq 0$		$T(2,7,\underline{5+n})$	$NA_{1,n}$	$NB^{n}_{(-1)}$	$NB^{n}_{(o)}$	$NB^{n}_{(1)}$	*
			4			$T(2,\underline{\underline{5}},\underline{\underline{9}})$	$T(2,\underline{\underline{6}},\underline{\underline{9}})$	$T(2,\underline{\underline{7}},\underline{\underline{9}})$	$T(2,\underline{\underline{8}},\underline{\underline{9}})$	*
2	3	2	$4+n, n\geq 0$	$T(4,5,\underline{4+n})$	$VNA_{n,1}$	$V^{\#}NB^{n}_{(-1)}$	$V^{\#}NB^{n}_{(o)}$			*
			3		$T(4,\underline{\underline{4}},\underline{\underline{7}})$	$T(4,\underline{\underline{5}},\underline{\underline{7}})$	$T(4,\underline{\underline{6}},\underline{\underline{7}})$			*
2	2	3	7		$T(2,\underline{\underline{7}},\underline{\underline{10}})$	$NC_{(1)}$	$NF_{(1)}$	*	*	*
			6		$T(2,\underline{\underline{6}},\underline{\underline{10}})$	$NC_{(o)}$	$NF_{(o)}$	$NF_{(1)}$	*	*
			5		$T(2,\underline{\underline{5}},\underline{\underline{10}})$	$A_{1,o}$	$NC_{(o)}$	$NC_{(1)}$	*	*
			4			[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Table 8 (continued)

$\Pi(\underline{t};,p,\underline{\underline{q}},r,\underline{\underline{s}})$

p	r	t	q	s=3	s=4	s=5	s=6
2	2	4+n, $n \geq 0$	6	$T(4,5,\underline{4+n})$	$VNA_{n,1}$	$V^{\#}NB^{n}_{(-1)}$	$V^{\#}NB^{n}_{(o)}$
			5		$T(3,6,\underline{4+n})$	$V^{\#}NA_{n,1}$	$V^{\#}NB^{n}_{(-1)}$
		3	6		$T(\underline{4},\underline{6},\underline{7})$	$T(\underline{5},\underline{6},\underline{7})$	$T(\underline{6},\underline{6},\underline{7})$
			5			$T(\underline{5},\underline{5},\underline{7})$	$T(\underline{5},\underline{6},\underline{7})$
2	3	7	7		*	*	*
			6			*	*
			5		$T(3,\underline{\underline{7}},\underline{\underline{9}})$	*	*
		6	6	$T(4,\underline{\underline{6}},\underline{\underline{8}})$	$V^{\#}NC_{(1)}$	$VNF_{(1)}$	*
			5		$T(3,\underline{\underline{6}},\underline{\underline{9}})$	$VNC_{(1)}$	$VNF_{(1)}$
		5	6	$T(4,\underline{\underline{5}},\underline{\underline{8}})$	$V^{\#}NC_{(o)}$	$VNF_{(o)}$	$VNF_{(1)}$
			5		$T(3,\underline{\underline{5}},\underline{\underline{9}})$	$VNC_{(o)}$	$VNF_{(o)}$
		4	6	$T(4,\underline{\underline{4}},\underline{\underline{8}})$	$V^{\#}NB^{1}_{(-1)}$	$V^{\#}NC_{(o)}$	$V^{\#}NC_{(1)}$
			5		$T(3,\underline{\underline{4}},\underline{\underline{9}})$	$VNB^{1}_{(-1)}$	$V^{\#}NC_{(o)}$
		3	6		$T(\underline{4},\underline{6},\underline{8})$	$T(\underline{5},\underline{6},\underline{8})$	$T(\underline{6},\underline{6},\underline{8})$
			5			$T(\underline{5},\underline{5},\underline{8})$	$T(\underline{5},\underline{6},\underline{8})$

Table 8 (continued)

$\Pi(\underline{t}; ,p,q,\underline{\underline{r}},\underline{\underline{s}})$

p	q	r	s	t=3	t=4	t=5	t=6	t=7	t=8	t=9	t=10
2	2	6	7		$T(2,\underline{\underline{7}},\underline{\underline{10}})$	$NC_{(1)}$	$NF_{(1)}$	*	*	*	*
			6		$T(2,\underline{\underline{6}},\underline{\underline{10}})$	$NC_{(o)}$	$NF_{(o)}$	$NF_{(1)}$	*	*	*
			5		$T(2,\underline{\underline{5}},\underline{\underline{10}})$	$NB^{1}_{(-1)}$	$NC_{(o)}$	$NC_{(1)}$	*	*	*
			4			$T(2,\underline{\underline{5}},\underline{\underline{10}})$	$T(2,\underline{\underline{6}},\underline{\underline{10}})$	$T(2,\underline{\underline{7}},\underline{\underline{10}})$	$T(2,\underline{\underline{8}},\underline{\underline{10}})$	*	*
		5	7		$T(2,6,\underline{\underline{7}})$	$NB^{1}_{(o)}$	$NC_{(1)}$	*	*	*	*
			6		$T(2,6,\underline{\underline{6}})$	$NB^{1}_{(-1)}$	$NC_{(o)}$	$NC_{(1)}$	*	*	*
			5		$T(2,6,\underline{\underline{5}})$	$NA_{1,o}$	$NB^{1}_{(-1)}$	$NB^{1}_{(o)}$	$NB^{1}_{(1)}$	*	*
			4			$T(2,6,\underline{\underline{5}})$	$T(2,6,\underline{\underline{6}})$	$T(2,6,\underline{\underline{7}})$	$T(2,6,\underline{\underline{8}})$	$T(2,6,\underline{\underline{9}})$	*
		4	7			$T(2,6,\underline{\underline{7}})$	$T(2,\underline{\underline{7}},\underline{\underline{10}})$	*	*	*	*
			6			$T(2,6,\underline{\underline{6}})$	$T(2,\underline{\underline{6}},\underline{\underline{10}})$	$T(2,\underline{\underline{7}},\underline{\underline{10}})$	$T(2,\underline{\underline{8}},\underline{\underline{10}})$	*	*
			5			$T(2,6,\underline{\underline{5}})$	$T(2,6,\underline{\underline{6}})$	$T(2,6,\underline{\underline{7}})$	$T(2,6,\underline{\underline{8}})$	$T(2,6,\underline{\underline{9}})$	*
			4								

<u>Table 8</u> (continued)

$\Pi(\underline{t};p,q,\underline{\underline{r}},\underline{\underline{s}})$ (continued)

p	q	r	s	t=3	t=4	t=5	t=6	t=7	t=8	t=9	t=10
2	3	6	6	$T(4,\underline{\underline{6}},\underline{\underline{8}})$	$V^{\#}NC_{(1)}$	$VNF_{(1)}$	*	*	*	*	*
			5	$T(4,\underline{\underline{5}},\underline{\underline{8}})$	$V^{\#}NC_{(o)}$	$VNF_{(o)}$	$VNF_{(1)}$	*	*	*	*
			4	$T(4,\underline{\underline{4}},\underline{\underline{8}})$	$V^{\#}NB^{1}_{(1)}$	$V^{\#}NC_{(o)}$	$V^{\#}NC_{(1)}$		*	*	*
			3		$T(4,\underline{\underline{4}},\underline{\underline{8}})$	$T(4,\underline{\underline{5}},\underline{\underline{8}})$	$T(4,\underline{\underline{6}},\underline{\underline{8}})$		*	*	*
		5	6	$T(4,4,\underline{\underline{6}})$	$VNB^{1}_{(o)}$	$VNC_{(1)}$	*	*	*	*	*
			5	$T(4,4,\underline{\underline{5}})$	$VNB^{1}_{(-1)}$	$VNC_{(o)}$	$VNC_{(1)}$	*	*	*	*
			4	$T(4,4,\underline{\underline{4}})$	$VNA_{1,o}$	$VNB^{1}_{(-1)}$	$VNB^{1}_{(o)}$	$VNB^{1}_{(1)}$	*	*	*
			3		$T(4,4,\underline{\underline{4}})$	$T(4,4,\underline{\underline{5}})$	$T(4,4,\underline{\underline{6}})$	$T(4,4,\underline{\underline{7}})$	$T(4,4,\underline{\underline{8}})$	*	*
		4	6		$T(3,5,\underline{\underline{6}})$	$T(3,\underline{\underline{6}},\underline{\underline{9}})$	*	*	*	*	*
			5		$T(3,5,\underline{\underline{5}})$	$T(3,\underline{\underline{5}},\underline{\underline{9}})$	$T(3,\underline{\underline{6}},\underline{\underline{9}})$	$T(3,\underline{\underline{7}},\underline{\underline{9}})$	*	*	*
			4		$T(3,5,\underline{\underline{4}})$	$T(3,5,\underline{\underline{5}})$	$T(3,5,\underline{\underline{6}})$	$T(3,5,\underline{\underline{7}})$	$T(3,5,\underline{\underline{8}})$	$T(3,5,\underline{\underline{9}})$	*
			3								

Table 8 (continued)

$\Pi(\underline{t};p,\underline{\underline{q}},\underline{r},\underline{s})$

p	q	s	r	t=3	t=4	t=5	t=6	t=7	t=8	t=9	t=10
2	6	6	5	$T(\underline{6},\underline{6},\underline{8})$			*	*	*	*	*
		5	3		$T(4,\underline{\underline{4}},\underline{\underline{8}})$	$T(4,\underline{\underline{5}},\underline{\underline{8}})$	$T(4,\underline{\underline{6}},\underline{\underline{8}})$		*	*	*
		4	5	$T(\underline{4},\underline{6},\underline{8})$					*	*	*
		3	5		$T(4,\underline{\underline{4}},\underline{\underline{8}})$	$T(4,\underline{\underline{5}},\underline{\underline{8}})$	$T(4,\underline{\underline{6}},\underline{\underline{8}})$		*	*	*
	5	5	6	$T(4,4,\underline{\underline{6}})$	$VNB^{1}_{(o)}$	$VNC_{(1)}$	*	*	*	*	*
			5	$T(4,4,\underline{\underline{5}})$	$VNB^{1}_{(-1)}$	$VNC_{(o)}$	$VNC_{(1)}$	*	*	*	*
			4	$T(4,4,\underline{\underline{4}})$	$VNA_{1,o}$	$VNB^{1}_{(-1)}$	$VNB^{1}_{(o)}$	$VNB^{1}_{(1)}$	*	*	*
			3		$T(4,4,\underline{\underline{4}})$	$T(4,4,\underline{\underline{5}})$	$T(4,4,\underline{\underline{6}})$	$T(4,4,\underline{\underline{7}})$	$T(4,4,\underline{\underline{8}})$	*	*
		4	6		$T(3,5,\underline{\underline{6}})$	$T(3,\underline{\underline{6}},\underline{\underline{9}})$	*	*	*	*	*
			5		$T(3,5,\underline{\underline{5}})$	$T(3,\underline{\underline{5}},\underline{\underline{9}})$	$T(3,\underline{\underline{6}},\underline{\underline{9}})$	$T(3,\underline{\underline{7}},\underline{\underline{9}})$	*	*	*
			4		$T(3,5,\underline{\underline{4}})$	$T(3,5,\underline{\underline{5}})$	$T(3,5,\underline{\underline{6}})$	$T(3,5,\underline{\underline{7}})$	$T(3,5,\underline{\underline{8}})$	$T(3,5,\underline{\underline{9}})$	*
			3								

Table 9 $\mathcal{S}_4$: $\Phi(t;,p,q,r,s)$

$(p,r,t) = (3,4,7)$

s/q	4
4	A

$(p,r,t) = (4,4,6)$

s/q	3	4	5
5	E		
4	C	$V'_{(3)}$	
3		C	E

$(p,r,t) = (4,5,5)$

s/q	3	4
5	G	$V'_{(6)}$
4	K	$V'_{(4)}$
3		L

$(p,r,t) = (3,4,6)$

s/q	3	4	5
5		B	
4		C	
3			D

$(p,r,t) = (3,5,5)$

s/q	3	4
6		F
5		G
4	H	I
3		J

$(p,r,t) = (4,4,5)$

s/q	3	4	5	6
6				
5	I	$V'_{(4)}$	$V'_{(6)}$	
4	M	$V'_{(2)}$	$V'_{(4)}$	
3		M	I	

$(p,r,t) = (4,5,4)$

s/q	3	4	5	6
6	N			
5	I	$V'_{(4)}$	$V'_{(6)}$	
4	M	$V'_{(2)}$	$V'_{(4)}$	
3		O	P	

$(p,r,t) = (3,4,5),(5,4,3)$

s/q	3	4	5	6
7				
6				
5				
4	$\Pi(\underline{s};2,\underline{\underline{7}},3,\underline{\underline{q}})$			
3				

$(p,r,t) = (3,5,4)$

s/q	3	4	5	6
6				
5				
4	$\Pi(\underline{s};,2,3,\underline{\underline{7}},\underline{\underline{q}})$			
3				

$(p,r,t) = (4,4,4)$

s/q	3	4	5	6
6	Q	$V'_{(3)}$	$V'_{(7)}$	$V'_{(8)}$
5	M	$V'_{(2)}$	$V'_{(5)}$	$V'_{(7)}$
4	R	$V'_{(1)}$	$V'_{(2)}$	$V'_{(3)}$
3		R	M	Q

$(p,r,t) = (3,4,4),(4,4,3)$

s/q	3	4	5	6	7	8
8						
7						
6		$\Pi(\underline{s};,2,3,3,\underline{\underline{q}})$				
5						
4						
3						

Legend for Table 9

$A = \Pi(\underline{7};2,\underline{4},\underline{4},\underline{7})$

$B = \Pi(\underline{6};2,\underline{4},\underline{5},\underline{7})$

$C = \Pi(\underline{6};2,\underline{4},\underline{4},\underline{7})$

$D = \Pi(\underline{6};2,\underline{5},\underline{3},\underline{7})$

$E = \Pi(\underline{6},2,\underline{5},\underline{4},\underline{7})$

$F = \Pi(\underline{5};2,\underline{5},\underline{6},\underline{7})$

$G = \Pi(\underline{5};2,\underline{5},\underline{5},\underline{7})$

$H = \Pi(\underline{5};2,\underline{\underline{7}},3,\underline{\underline{3}})$

$I = \Pi(\underline{5};2,\underline{5},\underline{4},\underline{7})$

$J = \Pi(\underline{3},2,\underline{\underline{7}},3,\underline{\underline{5}})$

$K = \Pi(\underline{5};2,\underline{4},\underline{5},\underline{7})$

$L = \Pi(\underline{4},2,\underline{\underline{7}},3,\underline{\underline{5}})$

$M = \Pi(\underline{5};2,\underline{4},\underline{4},\underline{7})$

$N = \Pi(\underline{3};2,3,3,\underline{\underline{8}})$

$O = \Pi(\underline{4};2,3,3,\underline{\underline{5}})$

$P = \Pi(\underline{4};2,3,\underline{\underline{7}},\underline{\underline{5}})$

$Q = \Pi(\underline{3};2,3,3,\underline{\underline{7}})$

$R = \Pi(\underline{4};2,3,3,\underline{\underline{4}})$

The following graphs of $\mathfrak{F}_4 \cap \{\Phi(t;p,q,r,s)\}$ are not indicated:

$\Phi(4;2,6,4,s) = \Pi(\underline{s};2,3,3,\underline{\underline{5}})$, $3 \leq s \leq 7$

$\Phi(3;3,4,5,6) = \Pi(\underline{3};2,\underline{5},\underline{6},\underline{7})$

§ 6 Hypersurface Singularities of corank ≥ 4 with $(\mu_0, \mu_+) = (0,2)$

In this section we consider the hypersurface singularities of corank ≥ 4 with $(\mu_0, \mu_+) = (0,2)$ for a stably equivalent function germ $f : (\mathbb{C}^k, \underset{\sim}{0}) \to (\mathbb{C}, 0)$ with $k \equiv 3 \pmod 4$. We first look for quasihomogeneous singularities with these conditions. One can classify the possible weights and the degree of a quasihomogeneous function defining such a singularity using the procedure of [11 , (4.3)].

Let $f : \mathbb{C}^7 \to \mathbb{C}$ be a quasihomogeneous function with weights $w_1 \leq w_2 \leq \ldots \leq w_7$, $w_i \in \mathbb{N}$, and degree d, and assume that f is nondegenerate, i.e. $\underset{\sim}{0}$ is an isolated critical point of f. Then one can derive the following condition for the corresponding singularity to have $(\mu_0, \mu_+) = (0,2)$:

(1) $$\sum_{i=1}^{7} w_i < 3d < 2w_1 + \sum_{i=2}^{7} w_i$$

This condition shows that a function stably equivalent to a cubic form in 5 variables has $\mu_0 + \mu_+ > 2$. Therefore all singularities with $(\mu_0, \mu_+) = (0,2)$ have corank ≤ 4 and we may assume

(2) $$w_1 \leq w_2 \leq w_3 \leq w_4 < w_5 = w_6 = w_7 = \frac{d}{2}$$

For the quasihomogeneous function f to be nondegenerate, the following two necessary conditions have to be satisfied (cf. [3]):

(3) For $i=1,2,3,4$ there exists a natural number n_i and and index $j \in \{1,2,3,4\}$ such that

$$d = n_i w_i + w_j \ .$$

(4) Let $i,j \in \{1,2,3,4\}$, $i \neq j$, be given. Then a) d or b) $d-w_l$ for all $l \in \{1,2,3,4\} \setminus \{i,j\}$ can be expressed as a linear combination of w_i and w_j with non-negative integers as coefficients.

Now one can determine in a combinatorial way the possible 7-tuples of weights and associated degrees satisfying the conditions (1)-(4), and one gets the following four cases :

Weights w_i	d	Equation	μ	disc	Notation
2,2,2,2,3,3,3	6	$x_1^3+x_2^3+x_3^3+x_4^3+x_5^2+x_6^2+x_7^2$	16	64	O_{16}
6,8,8,9,12,12,12	24	$x_1^4+x_1x_4^2+x_2^3+x_3^3+x_5^2+x_6^2+x_7^2$	20	16	O_{20}
8,10,11,12,16,16,16	32	$x_1^4+x_1x_4^2+x_2x_3^2+x_2^2x_4+x_5^2+x_6^2+x_7^2$	21	8	O_{21}
12,15,16,18,24,24,24	48	$x_1^4+x_1x_4^2+x_3^3+x_2^2x_4+x_5^2+x_6^2+x_7^2$	22	3	O_{22}

These are all quasihomogeneous singularities of corank ≥ 4 with $(\mu_o,\mu_+) = (0,2)$.

The first singularity is O_{16} in Arnold's notation (cf. [2]). This is a 5-modal singularity, so the numbers of moduli of all these singularities are ≥ 5 . A Dynkin diagram with respect to a distinguished basis for O_{16} and O_{20} can be computed by Gabrielov's method [12]. Such a diagram is shown in [9, Fig. 6b] where one has to delete row 6 for O_{20} , and row 4 and 6 for O_{16} . The transformations in [9 , (4.3)] for O_{16} yield the Dynkin diagram with respect to a weakly distinguished basis shown in Fig. 3.

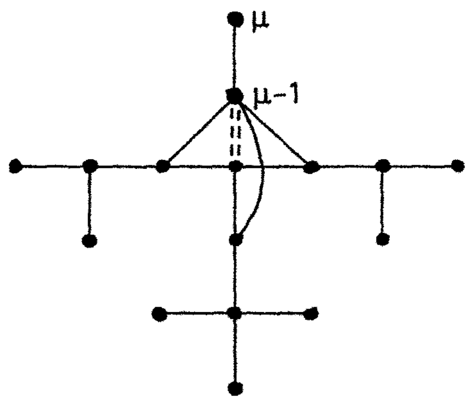


Fig. 3: A Dynkin diagram of O_{16}

This diagram satisfies the conditions a) and b) of Theorem 3.2, but the corresponding subgraph S has also a vertex of valence 4 . The number $\sigma(S)$ is again equal to 5 , the number of moduli. The Milnor lattice turns out to be

$$H = D_4 \perp D_4 \perp D_4 \perp U \perp U ,$$

so in particular

$$\lambda(G_H) = 6 \quad .$$

The singularity O_{16} is the first member of a whole family of singularities of corank 4 with $(\mu_o, \mu_+) = (0,2)$, which are all but O_{16} not quasihomogeneous. These are the following singularities

Notation	$O_{p,q,r,s}$
Equation	$x_1^p+x_2^q+x_3^r+x_4^s+x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4+x_5^2+x_6^2+x_7^2$
μ	$4+p+q+r+s$
δ	$(-1)^{\mu}64$

Here δ denotes the determinant of the intersection matrix, which is equal to

$$(-1)^{\mu}P(1) = (-1)^{\mu-} \operatorname{disc}(H) ,$$

where $P(t) = \det(t\cdot \mathrm{Id}-h_*)$ is the characteristic polynomial of the monodromy operator. This polynomial can be computed by the Ehlers-Varchenko method [22]. (In the above quasihomogeneous cases one can use the method of Milnor-Orlik [19]). The value of δ shows that all these singularities also have $(\mu_o,\mu_+) = (0,2)$.

We do not know whether these singularities are all singularities with these values of the invariants. So

Problem (cf. [24, Problem D])
Classify all function germs $f : (\mathbb{C}^7,\underset{\sim}{0}) \to (\mathbb{C},0)$ with an isolated critical point at the origin, which have corank 4 and $(\mu_o,\mu_+) = (0,2)$.

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