On the integer points of some toric varieties

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1. Let T be an algebraic torus defined over $\mathbb{Q}$. We shall describe a class of affine varieties over $\mathbb{Z}$, say $\{X_{\mathfrak{a}}\}$, each of these varieties $X_{\mathfrak{a}}$ contains a T- orbit $Y_{\mathfrak{a}}$ as a (Zariski) open dense subset. Moreover, any two of these varieties are $\mathbb{Q}$ - (but not, in general, $\mathbb{Z}$-) isomorphic. Our primary interest lies in studying the distribution of the integer points $Y_{\mathfrak{a}}(\mathbb{Z})$ in the real locus $Y_{\mathfrak{a}}(\mathbb{R})$ of $Y_{\mathfrak{a}}$. To this end, we develop a theory of ideals on T and, for a grossencharacter χ of T, define a *Draxl* L - function $L(T;s,\chi)$ known, [2], to be meromorphic in the half-plane $\{s| \in \mathbb{C}, Res > 0\}$. The standard analytic argument gives now an asymptotic formula for the number of integer points of bounded height; moreover,under certain restrictions on T one can prove that the integer points are equidistributed with respect to a properly chosen measure on $Y_{\mathfrak{a}}(\mathbb{R})$, and in this case one obtains an asymptotic formula for the number of integer points in a "smooth" domain on $Y_{\mathfrak{a}}(\mathbb{R})$. As an application of these results, one can generalise and strengthen my results, [9], on the integer points of norm - form varieties. Actually the theory developed here is applicable to a wider class of toric varieties associated with T, however for a general toric variety the Diophantine problem is not as clear-cut as in the special case to be considered here.

As usual, $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ stand for the fields of rational, real, and complex numbers respectively, and $\mathbb{Z}$ denotes the ring of integers in $\mathbb{Q}$; we write B^* for the group of units of a ring B. In what follows a number field is, by definition, a finite extension of $\mathbb{Q}$, and $Gal(K|k)$ denotes the Galois group of a normal extension $K|k$ of number fields. Sometimes $|S|$ denotes the cardinality of a (finite) set S.

2. Let $K|k$ be a finite normal extension of number fields, and let $Cat(K|k)$ denote the category of algebraic tori defined over k which split over K; let $T \in Cat(K|k)$. The algebraic torus T is uniquely defined by an integral representation $\rho : G \to GL(d, \mathbb{Z}), G := Gal(K|k)$; here d is equal to the dimension of T (as an algebraic variety). This representation is afforded by the module of characters

$$\hat{T} = \{x | x \in \mathbb{Z}^d, \sigma x = \rho(\sigma) x\};$$

let

$$\hat{T}^* = \{x | x' \in \mathbb{Z}^d, \sigma x = x\rho(\sigma^{-1})\}$$

be the dual module (given a set S, we write S^d for the set of columns of the length d with entries in S, the upper affix $'$ denotes matrix transposition). For a commutative group S and u in S^d,we let $u^\sigma = w$ with $w_i = \prod_{j=1}^{d} w_j^{r_{ji}(\sigma)}, \sigma \in G, \rho(\sigma) = (r_{ij}(\sigma))_{1 \leq i,j \leq d}$. One may view T as a (covariant) functor from the category of commutative k-algebras to the category of abelian groups. Let A be a commutative k - algebra, and let $B = A \otimes_k K$. One sets, by definition,

$$T(A) = \left\{\alpha | \alpha \in (B^*)^d, \sigma\alpha = \alpha^\sigma \quad \text{for} \quad \sigma \in G\right\},$$

where $\sigma(u \otimes x) := u \otimes \sigma x$ for $u \in A, x \in K, \sigma \in G$. Let $I(K)$ and $I_o(K)$ denote the group of fractional ideals of K and the monoid of integral ideals of K respectively. By definition,

$$I(T) = \left\{\mathfrak{a} | \mathfrak{a} \in I(K)^d, \sigma\mathfrak{a} = \mathfrak{a}^\sigma \quad \text{for} \quad \sigma \in G\right\},$$

and $I_o(T) = I(T) \cap I_o(K)^d$. To describe the structure of the group $I(T)$ and of the moniod $I_o(T)$ we introduce the following notation. For $u \in S, y' \in \mathbb{Z}^d$, let

$u^y = v, v \in S^d, v_i = u^{y_i}, 1 \leq i \leq d$ (assuming S is a commutative group); if $y_i \geq 0$ for $1 \leq i \leq d$ we write $y \geq 0$. Let p be a prime divisor in k, choose a prime $\mathfrak{p}$ in $I_o(K)$ with $\mathfrak{p}|p$ and write $p = \prod\limits_{\tau \in G/G_{\mathfrak{p}}} (\tau\mathfrak{p})^{e(p)}$, where $G_{\mathfrak{p}} = \{\sigma | \sigma\mathfrak{p} = \mathfrak{p} \text{ for } \sigma \in G\}$ is the decomposition group at $\mathfrak{p}$ and $e(p)$ is the ramification index of p in K. Let

$$C^* = \{a | a \in \hat{T}^*, \sigma a \geq 0 \quad \text{for} \quad \sigma \in G\}, C^*_{\mathfrak{p}} := C^* \cap (\hat{T}^*)^{G_{\mathfrak{p}}}$$

We introduce the group

$$I^{(p)}(T) = \{\mathfrak{a}_a | \mathfrak{a}_a = \prod_{\tau \in G/G_{\mathfrak{p}}} (\tau\mathfrak{p})^{\tau a}, a \in (\hat{T}^*)^{G_{\mathfrak{p}}}\}$$

of p-primary ideals, and the monoid

$$I_p(T) = \{\mathfrak{a}_a | \mathfrak{a}_a = \prod_{\tau \in G/G_{\mathfrak{p}}} (\tau\mathfrak{p})^{\tau a}, a \in C^*_{\mathfrak{p}}\}$$

of p - primary intergral ideals.

If $C^* \neq \{0\}$, then on choosing a in $C^* \backslash \{0\}$ and letting $b = \sum\limits_{\sigma \in G} \sigma a$ we see that $\hat{T}^G \neq \{0\}$ (since $b \in \hat{T}^{*G} \backslash \{0\}$). Conversely, if $\hat{T}^G \neq \{0\}$ one may choose a in $\hat{T}^{*G} \backslash \{0\}$ as a basis vector in $\hat{T}^*$, then $a \in C^*$ and therefore $C^* \neq \{0\}$. If $\hat{T}^G \neq \{0\}$, the torus is to be called *isotropic*; if $\hat{T}^G = \{0\}$, then T is *anisotropic*. In what follows it is assumed that *T is isotropic and $C^* \neq \{0\}$*, unless an explicit assumption to the contrary has been made.

Proposition 1. *We have:*

$$I(T) = \prod_p I^{(p)}(T), \qquad \text{and} \qquad I_o(T) = \prod_p I_p(T); \tag{1}$$

furthermore, after a possible change of basis in $\hat{T}$, it may be assumed that $I_p(T)$ generates the group $I^{(p)}(T)$. Here p ranges over all the prime divisors of of k.

Proof. Let $\mathfrak{b} \in I^{(p)}(T)$, say $\mathfrak{b} = \mathfrak{a}_a$. We have $\mathfrak{b}_i^\sigma = \prod_{j=1}^{d} \mathfrak{b}_j^{r_{ji}(\sigma)}$ for $\sigma \in G$, or $\mathfrak{b}^\sigma = \prod_{\tau \in G/G_{\mathfrak{p}}} (\tau \mathfrak{p})^{\sigma^{-1}\tau a} = \sigma \mathfrak{a}_a = \sigma \mathfrak{b}$, so that $\mathfrak{b} \in I(T)$. Moreover, if $a \in C^*$ then $\tau a \geq 0$ for each τ, and therefore $\mathfrak{b} \in I_o(T)$. Thus $I^{(p)}(T) \subseteq I(T), I_p(T) \subseteq I_o(T)$ for every p. Let $\mathfrak{a} \in I(T)$; we write $\mathfrak{a}_i = \prod_{\tau \in G/G_{\mathfrak{p}}} (\tau\mathfrak{p})^{a_i(\tau)}\mathfrak{b}_i$ with $(p, \mathfrak{b}_i) = 1$. Since $\sigma\mathfrak{a} = \mathfrak{a}^\sigma$ for $\sigma \in G$, it follows that $\sigma a(\tau) = a(\sigma\tau)$ for $\sigma \in G$; therefore we may let $a(\tau) = \tau b$ with $b \in (\hat{T}^*)^{G_{\mathfrak{p}}}$. This proves (1). Let now $a \in C^* \backslash \{0\} (C^* \neq \{0\}$ by assumption !), and let $b = \sum_{\sigma \in G} \sigma a$. On changing basis in $\hat{T}$, if necessary, we may assume that $a > 0$; then $b > 0$. Clearly, $b \in C^* \cap (\hat{T}^*)^G$. Let $c \in (\hat{T}^*)^{G_{\mathfrak{p}}}$, then $Nb + c \in C^*_{\mathfrak{p}}$ for a sufficiently large positive N; thus $c = c_1 - c_2, \{c_1, c_2\} \subseteq C^*_{\mathfrak{p}}$. Therefore $C^*_{\mathfrak{p}}$ generates $(\hat{T}^*)^{G_{\mathfrak{p}}}$, as asserted.

For $\mathfrak{a} \in I(T)$, let $N\mathfrak{a} = \prod_{j=1}^{d} N\mathfrak{a}_j$ and $N_{K/k}\mathfrak{a} = \prod_{j=1}^{d} N_{K/k}\mathfrak{a}_j$. Clearly, $N_{K/k}\mathfrak{a} = p^{\|a\|/e(p)}$ for $\mathfrak{a}_a \in I^{(p)}(T)$, where $\| a \| := \sum_{\sigma \in G} |\sigma a|$, and $|a| := \sum_{j=1}^{d} a_j$ for $a \in \hat{T}^*$. Let

$$C^*(m) = \{a | a \in C^* \ , \ \| a \| = m\},$$

and let

$$\kappa = \min \{m | m \in \mathbb{Z}, m > 0, C^*(m) \neq \phi\} \tag{2}$$

Let $\chi : I_o(T) \to \mathbb{C}_1 \cup \{0\}$ satisfy two conditions : (i) it is multiplicative, that is $\chi(\mathfrak{a}_1\mathfrak{a}_2) = \chi(\mathfrak{a}_1)\chi(\mathfrak{a}_2)$ for $\mathfrak{a}_j \in I_o(T), j = 1, 2$, and (ii) $\chi^{-1}(\{0\}) = \prod_{p \in S} I_p(T)$ with $|S| < \infty$ (here $|S|$ stands for the cardinality of the set S, and $\mathbb{C}_1 := \{z | z \in \mathbb{C}^*, |z| = 1\}$).

The L-function of T associated to χ is defined by a Dirichlet series

$$L(T;s,\chi) = \sum_{\mathfrak{a}\in I_o(T)} \chi(\mathfrak{a})N\mathfrak{a}^{-s/\kappa} \tag{3}$$

The series (3) converges absolutely for $Res > 1$ since

$$L(T;s,\chi) = \prod_p L_p(T;s,\chi) \tag{4}$$

with

$$L_p(T;s,\chi) = \sum_{\mathfrak{a}\in I_p(T)} \chi(\mathfrak{a})N\mathfrak{a}^{-s/\kappa}$$

by Proposition 1, or

$$L_p(T;s,\chi) = \sum_{a\in C^*_{\mathfrak{p}}} \chi(\mathfrak{a}_a)Np^{-\|a\|s/e(p)\kappa}, \tag{5}$$

so that the Euler product (4) may be majorised by the product

$$\prod_p (1 + Np^{-Res/e(p)}(1 + A.Np^{-Res/\kappa e(p)}))$$

with some positive A. Let $\mathfrak{P} \in I_o(T)$, we say that $\mathfrak{P}$ is a **prime ideal** if $\mathfrak{a}|\mathfrak{P} \Rightarrow \mathfrak{a} = \mathfrak{P}$ for $\mathfrak{a} \in I_o(T)\backslash\{1\}$; a prime ideal $\mathfrak{P}$ is called a **strict prime** if $\mathfrak{a}|\mathfrak{P}^n \Rightarrow \exists m(\mathfrak{a} = \mathfrak{P}^m)$ with n, m ranging over non-negative integers. Let $\mathcal{P}(T)$ denote the set of prime ideals, and let $\mathcal{P}_{st}(T)$ stand for the subset of strict primes.

Proposition 2 (i) *The set $\mathcal{P}(T)$ generates the group $I(T)$*; (ii) *there is a sequence of polynomials $Q_p(t)$ in $\mathbb{Q}[t]$ such that $Q_p(t) = 1(mod\ t^\kappa)$ for each p, and*

$$L(T;s,\chi) = \prod_{\mathfrak{P}\in\mathcal{P}_{st}(T)} (1 - \chi(\mathfrak{P})N\mathfrak{P}^{-s/\kappa})^{-1} \prod_p Q_p(Np^{-s/\kappa e(p)}) \tag{6}$$

for $Res > 1$.

Proof. Since $\mathcal{P}(T) \subseteq \bigcup_p I_p(T)$ and $I_p(T)$ is generated by the finite set $\mathcal{P}(T) \cap I_p(T)$, assertion (i) follows from Proposition 1. Let $\mathcal{P}_p = \mathcal{P}_{st}(T) \cap I_p(T)$; it follows from a theorem on the solutions of linear homogeneous Diophantine equations, [14, theorem 2.5], that

$$L_p(T; s, \chi) = \prod_{\mathfrak{P} \in \mathcal{P}_p} (1 - \chi(\mathfrak{P}) N\mathfrak{P}^{-s/\kappa})^{-1} Q_p(Np^{-s/\kappa e(p)}) \tag{7}$$

Identity (6) is a consequence of (4) and (7).

Remark 1. An element of $I_o(T)$ is called an *integral* ideal, the elements of $\bigcup_p I_p(T)$ are *primary* ideals. By Proposition 1, an integral ideal can be uniquely factorised into primary ideals; however, factorisation of primary ideals into primes is not, in general, unique (cf. [14]).

Needless to say, one does not expect the function

$$s \mapsto L(T; s, \chi) \tag{8}$$

to possess an analytic continuation beyond the half-plane $\{s | Re s > 1\}$ of absolute convergence of the series (2) for a generic multiplicative character χ. In the next sections we introduce grossencharacters, and following [2] prove that the function (8) allows for analytic continuation to a meromorphic function in the half-plane $\mathbb{C} = \{s | Re s > \kappa/\kappa + 1\}$ when χ is a grossencharacter.

3. Let L be a number field, we denote by $S_1(L), S_2(L), S_0(L)$ the sets of all the real places, all the complex places, and all the finite places of L respectively, and write $S_\infty(L) := S_1(L) \cup S_2(L), S(L) := S_\infty(L) \cup S_0(L)$. As usual, L_p stands for the completion of $L, p \in S(L)$; let $\mathfrak{o}$ (and $\mathfrak{O}$) denote the ring of integers of k (and K), then $\mathfrak{o}_p$ (and $\mathfrak{O}_\mathfrak{p}$)

stands for the ring integers in k_p (and in $K_{\mathfrak{p}}$), $p \in S_0(K), \mathfrak{p} \in S_0(K)$. Given a commutative $\mathfrak{o} - algebra\ A$, let $B := A \otimes_{\mathfrak{o}} \mathfrak{O}$, and let, by definition

$$T(A) = \left\{ \alpha | \alpha \in B^{*d}, \sigma\alpha = \alpha^{\sigma} \quad \text{for} \quad \sigma \in G \right\}.$$

Let $G_{\mathfrak{p}} = Gal(K_{\mathfrak{p}} | k_p)$ be the ramification group at $\mathfrak{p}$, and let

$$T(\Pi_p) = \left\{ \alpha | \alpha = (\ldots, \tau\pi^{\tau a}, \ldots), \tau \in G/G_{\mathfrak{p}} \ , \ a \in (\hat{T}^*)^{G_{\mathfrak{p}}} \right\},$$

where π is a fixed prime in $K^*_{\mathfrak{p}}$. Clearly $T(\Pi_p) \cong I^{(p)}(T)$.

<u>**Lemma 1.**</u> *Let $p \epsilon S_0(k)$, then $T(k_p) = T(\mathfrak{o}_p).\ T(\Pi_p)$.*

<u>**Proof.**</u> By definition,

$$T(k_p) = \left\{ \alpha | \alpha \epsilon (B^*_p)^d \ , \ \sigma\alpha = \alpha^{\sigma} \quad \text{for} \quad \sigma \epsilon G \right\},$$

and

$$T(\mathfrak{o}_p) = \left\{ \alpha | \alpha \epsilon (A^*_p)^d \ , \ \sigma\alpha = \alpha^{\sigma} \quad \text{for} \quad \sigma \epsilon G \right\},$$

where $B_p = k_p \otimes_k K$, $A_p = \mathfrak{o}_p \otimes_{\mathfrak{o}} \mathfrak{O}$. Let $\alpha \epsilon T(k_p)$. Since $B^*_p = \prod_{\tau \in G/G_{\mathfrak{p}}} K^*_{\tau\mathfrak{p}}$, one may write $\alpha = (\ldots, \alpha_{\tau}, \ldots)$ with $\alpha_{\tau} \epsilon (K^*_{\tau\mathfrak{p}})^d$; moreover, it follows from the equation $\sigma\alpha = \alpha^{\sigma}$ that $\alpha_{\tau} = \tau\alpha_1^{\tau^{-1}}$. On writing $\alpha_1 = \epsilon\pi^a$ with $\epsilon \in (\mathfrak{O}^*_{\mathfrak{p}})^d, a \in \hat{T}^*$ one deduces from the same equation that $\sigma\epsilon = \epsilon^{\sigma}, \sigma a = a$ for $\sigma \in G_{\mathfrak{p}}$. On the other hand, if $\alpha_{\tau} = (\tau\epsilon^{\tau-1})(\tau\pi^{\tau a})$ with $a \in (\hat{T}^*)^{G_{\mathfrak{p}}}$ and ϵ satisfying $\sigma\epsilon = \epsilon^{\sigma}$ for $\sigma \in G_{\mathfrak{p}}$, $\epsilon \in (\mathfrak{O}^*_{\mathfrak{p}})^d$ then $\alpha \in T(k_p)$. Since $A^*_p = \prod_{\tau \in G/G_{\mathfrak{p}}} \mathfrak{O}^*_{\tau\mathfrak{p}}$, the same argument shows that

$$T(\mathfrak{o}_p) = \left\{ \alpha | \alpha = (\ldots, \tau\epsilon^{\tau^{-1}}, \ldots), \epsilon \in (\mathfrak{O}^*_{\mathfrak{p}})^d, \sigma\epsilon = \epsilon^{\sigma} \quad \text{for} \quad \sigma \in G_{\mathfrak{p}} \right\}$$

This proves the lemma.

Let μ be the rank of the $\mathbb{Z}$-module $\hat{T}^G$.

Lemma 2. *Let $p \in S_\infty(k)$, then*

$$T(k_p) \cong \mathbb{R}_+^{*r_p} \times \left(\mathbb{Z}/2\mathbb{Z}\right)^{\nu_p} \times \left(S^1\right)^{d_p}, \tag{9}$$

where r_p is the rank of the $\mathbb{Z}$- module $\hat{T}^{G_\mathfrak{p}}$ for a place $\mathfrak{p}$ in *$S_\infty(K)$ above p.*

Proof. If $k_p = K_\mathfrak{p}$ then $T(k_p) = (k_p^*)^d$, and (9) is immediate. Otherwise, $k_p = \mathbb{R}, K_\mathfrak{p} = \mathbb{C}$, and we have:

$$T(k_p) = \left\{\alpha|\alpha_i = (..., \alpha_{\tau i}, ...), \tau \in G_\mathfrak{p} \quad , \quad \alpha_{\tau i} \in \mathbb{C}^* \quad , \quad \sigma\alpha = \alpha^\sigma \quad \text{for} \quad \sigma \in G\right\},$$

where $G_\mathfrak{p} = \{1, \sigma_\mathfrak{p}\}$, and $\sigma_\mathfrak{p}\alpha_{1i} = \bar{\alpha}_{1i}$ for $1 \leq i \leq d$. As in the proof of Lemma 1, it follows that $\alpha_\tau = \tau\alpha_1^{\tau^{-1}}$, and $\sigma_\mathfrak{p}\alpha_1 = \alpha_1^{\sigma_\mathfrak{p}}$. Let $l = \{e_1, ...e_d\}$ be a new basis in $\hat{T}$ such that $\sigma e_i = e_i$ for $i \leq \mu$, $\sigma \in G$, $\sigma_\mathfrak{p} e_i = e_i$ for $i \leq r_p$, and $\sigma_\mathfrak{p} e_i = -e_i$ for $i > r_p$. This basis induces an isomorphism $l : T(k_p) \to (\mathbb{R}^*)^{r_p} \times (S^1)^{d_p}$ with $d_p = d - r_p$. This proves the lemma.

Let A_k be the adele ring of k, and let Φ be a finite subset of $S(k)$ with $\Phi \supseteq S_\infty(k)$; one introduces a group

$$T_\Phi(A_k) = \prod_{p\in\Phi} T(k_p) \times \prod_{p\notin\Phi} T(\mathfrak{o}_p),$$

by definition, $T(A_k) = \bigcup_{\Phi\supseteq S_\infty(k)} T_\Phi(A_k)$. The group $T(k)$ may be regarded as a discrete subgroup of $T(A_k)$. Let $T_\infty(A_k) = \prod_{p\in S_\infty(k)} T(k_p)$; by lemma 2, there is an isomorphism

$$l : T_\infty(A_k) \underset{\cong}{\rightarrow} (\mathbb{R}_+^*)^{\mu+r} \times (\mathbb{Z}/2\mathbb{Z})^\nu \times (S^1)^{d_\infty}$$

with $\mu + r = \sum_{p \in S_\infty(k)} r_p$, $\nu \leq \mu + r$. For $x \in \hat{T}$, $\alpha \in T(A)$ we let $x(\alpha) = \prod_{j=1}^{d} \alpha_j^{x_j}$, where A is an $\mathfrak{o}-$(*or a* $k-$) algebra; it follows from definitions that $x(\alpha) \in A^*$ if $x \in \hat{T}^G, \alpha \in T(A)$. Let

$$T^1(A_k) = \left\{ \alpha | \alpha \in T(A_k), |x(\alpha)| = 1 \quad \text{for} \quad x \in \hat{T}^G \right\},$$

where, by definition, $|\beta| = \prod_{p \in S(k)} |\beta|_p$ for $\beta \in A_k^*, |.|_p$ being the normalised absolute value on k_p. Clearly, $T(A_k)/T^1(A_k) \cong (\mathbb{R}_+^*)^\mu$ and we may write $T(A_k) = (\mathbb{R}_+^*)^\mu \times T^1(A_k)$ if we embed $(\mathbb{R}_+^*)^\mu$ in one of the groups $T(k_p), p \in S_\infty(k)$; by the product formula,
$T(k) \subseteq T^1(A_k)$. Obviously, $T_\infty(A_k) = (\mathbb{R}_+^*)^\mu \times T_\infty^1(A_k)$ with
$T_\infty^1(A_k) \cong (\mathbb{R}_+^*)^r \times (\mathbb{Z}/2\mathbb{Z})^\nu \times (S^1)^{d_\infty}$, and on writing
$T_\Phi^1(A_k) = T_\infty^1(A_k) \times \prod_{p \in \Phi \setminus S_\infty(k)} T(k_p) \times \prod_{p \notin \Phi} T(\mathfrak{o}_p)$ one obtains
$T^1(A_k) = \bigcup_{\phi \supseteq S_\infty(k)} T_\Phi^1(A_k)$. The main theme of the rest of this section is an interplay between "idèle groups", that is different objects defined in terms of $T(A_k)$, and "ideal groups", that is some objects defined in terms of $I(T)$. Let $\phi_p : I^{(p)}(T) \to T(\Pi_p)$ be the isomorpishm defined by $\phi_p : \mathfrak{a}_a \mapsto (..., \tau\pi^{\tau a}, ...)$, $a \in (\hat{T}^*)^{G_\mathfrak{p}}$. One defines a monomorphism

$\phi(\Phi) = \prod_{p \notin \Phi} \phi_p \quad : \quad I_\Phi(T) \hookrightarrow T^1(A_k)$, where we let $I_\Phi(T) = \prod_{p \notin \Phi} I^{(p)}(T)$. It follows from Lemma 1 that

$$T^1(A_k) = T_\Phi^1(A_k) \ . \ \phi(\Phi)(I_\Phi(T)). \tag{10}$$

Let $\mathfrak{f}_o = \prod_{p \in S_o(k)} p^{m_p}$, $m_p \in \mathbb{Z}$, $m_p \geq 0$, and $m_p = 0$ for $p \notin \Phi$; let $\mathfrak{f}_\infty$ be a subgroup of $(\mathbb{Z}/2\mathbb{Z})^\nu$, and write $\mathfrak{f} = (\mathfrak{f}_o, \mathfrak{f}_\infty)$. For $\alpha \in T(k)$ relation $\alpha = 1(\mathfrak{f})$ means that
$\alpha_i = 1(\mathfrak{p}^{m_p}), 1 \leq i \leq d, p \in S_o(k)$ and $\iota(\alpha) \in \mathfrak{f}_\infty$, where $\iota : T^1(A_k) \to (\mathbb{Z}/2\mathbb{Z})^\nu$ is the natural epimorphism and $T(k)$ is regarded as a subgroup of $T^1(A_k)$.

Let $I_\mathfrak{f}^{pr}(T) = \{(\alpha) | \alpha \in T(k), \alpha = 1(\mathfrak{f})\}$, and let $Cl_\mathfrak{f}(T) = I_\Phi(T)/I_\mathfrak{f}^{pr}(T)$ be the corresponding ray class group modulo $\mathfrak{f}$. The group $Cl_\mathfrak{f}(T)$ is known to be finite, [11] (cf. also [15, p148], and [2, p453]). By (10),

$$Cl_{\mathfrak{f}}(T) \cong T^1(A_k) \;/\; T^1_{\Phi}(A_k)T_{\mathfrak{f}}(k), \tag{11}$$

where $T_{\mathfrak{f}}(k) := \{\alpha | \alpha \in T(k) \;,\; \alpha = 1(\mathfrak{f})\}$. Let $T_{\mathfrak{f}}(\mathfrak{o}) = T(\mathfrak{o}) \cap T_{\mathfrak{f}}(k)$; by a generalisation of the Dirichlet unit theorem, [13],

$$T_{\mathfrak{f}}(\mathfrak{o}) \cong \mathbb{Z}^r \times \mathcal{A} \qquad \text{with} \qquad |\mathcal{A}| < \infty.$$

Therefore

$$T^1_{\infty}(A_k) \;/\; T_{\mathfrak{f}}(\mathfrak{o}) \cong \mathcal{T}_{\mathfrak{f}}, \tag{12}$$

where $\mathcal{T}_{\mathfrak{f}} := (\mathbb{Z}/2\mathbb{Z})^{\nu_o} \times (S^1)^{d_\infty + r}$, $\nu_o \leq \nu$. Let $T_{\mathfrak{f}}(A_k) = \mathfrak{f}_\infty \times \prod_{p \in S_o(k)} T_{\mathfrak{f}}(\mathfrak{o}_p)$ with $T_{\mathfrak{f}}(\mathfrak{o}_p) := \{\alpha | \alpha \in T(\mathfrak{o}_p), \alpha = 1(\mathfrak{p}^{m_p})\}$, and let $\mathfrak{G}(\mathfrak{f}) = T^1(A_k)/T(k)T_{\mathfrak{f}}(A_k)$; we write, for brevity, $\mathfrak{G} := \mathfrak{G}(\mathfrak{f})$.

Proposition 3. *We have*

$$\mathfrak{G} \cong \mathcal{T}_{\mathfrak{f}} \times Cl_{\mathfrak{f}} \times \mathcal{B}_{\mathfrak{f}} \qquad \text{with} \qquad |\mathcal{B}_{\mathfrak{f}}| < \infty. \tag{13}$$

Proof. It follows from Lemma 1 and equation (10) that

$$\mathfrak{G} \cong T^1_{\infty}(A_k) \times \prod_{p \in \Phi_o} T(k_p)/T_{\mathfrak{f}}(\mathfrak{o}_p) \times \phi_{\Phi}(I_{\Phi}(T)/\mathfrak{f}_\infty . T(k)$$

where we write, for brevity, $\Phi_o := \Phi \backslash S_\infty(k)$. Since

$$I^{pr}_{\mathfrak{f}}(T) \cong T_{\mathfrak{f}}(k)/T_{\mathfrak{f}}(\mathfrak{o}),$$

relation (12) and the definition of $Cl_{\mathfrak{f}}(T)$ give (13) with

$$\mathcal{B}_{\mathfrak{f}} = \Big(\prod_{p \in \Phi_o} T(k_p)/T_{\mathfrak{f}}(\mathfrak{o}_p)\Big) / \Big(T(k)/T_{\mathfrak{f}}(k)\Big). \tag{14}$$

Finiteness of the group $\mathcal{B}_{\mathfrak{f}}$ follows from the relation $|Cl(T)| < \infty$, where $Cl(T) := I(T)/I^{pr}(T)$ is the class group of T; here $I^{pr}(T) := \{(\alpha)|\alpha \in T(k)\}$.

Remark 2. As $\mathfrak{f}$ varies, the groups $\mathcal{B}_{\mathfrak{f}}$ show to what extent the weak approximation principle for the set Φ_o fails; in particular, it follows from the theorem on weak approximation in an algebraic torus, [12, §7.3], [15, §VI.6], that $\mathcal{B}_{\mathfrak{f}} = \{1\}$ if $(\mathfrak{f}_o, D(K|k)) = 1$, where $D(K|k)$ denotes the discriminant of the extension $K|k$.

Composing the map $\phi(\Phi)$ with the natural epimorphism $T^1(A_k) \to \mathfrak{G}$, one obtians a map $g(\mathfrak{f}) : I_\Phi(T) \to \mathfrak{G}$; we are interested in the distribution of integral ideals with respect to the (normalised) Haar measure on $\mathfrak{G}$.

4. Let $\chi : T^1(A_k) \to \mathbb{C}_1$ be a continuous homomorphism, then $\{p|p \in S_o(k), \chi(T(\mathfrak{o}_p)) \neq 1\}$ is finite and, moreover, for each p in$S_o(k)$ there is an integer m such that

$$\{\alpha|\alpha \in T(\mathfrak{o}_p) \ , \ \alpha = 1(\mathfrak{p}^m)\} \subseteq Ker\chi.$$

Therefore, on writing $\mathfrak{f}' \leq \mathfrak{f}$ when $\mathfrak{f}'_\infty \subseteq \mathfrak{f}_\infty$ and $\mathfrak{f}'_o|\mathfrak{f}_o$, one may let $\mathfrak{f}(\chi) = min\{\mathfrak{f}|T_{\mathfrak{f}}(A_k) \subseteq Ker\chi\}$. Composing χ with $\phi(\Phi)$ for $\Phi = S_\infty(k) \cup \Phi_o$, $\Phi_o = \{p|p \in S_o(k) \ , \ p|\mathfrak{f}_o\}$ we define a multiplicative function $\chi : I_\Phi(T) \to \mathbb{C}_1$ that can be extended to a character $\chi : I_o(T) \to \mathbb{C}^1 \cup \{0\}$ with $\chi^{-1}(\{0\}) = \prod_{p \in \Phi_o} I_p(T)$. This character is said to be a (normalised) grossencharacter if $T(k) \subseteq Ker\chi$; let $Gr_{\mathfrak{f}}(T)$ be the group of all the grossencharacters χ with $\mathfrak{f}(\chi) \leq \mathfrak{f}$, and let $Gr(T) := \bigcup_{\mathfrak{f}} Gr_{\mathfrak{f}}(T)$ be the group of all the grossencharacters. It follows from Proposition 3 that $Gr_{\mathfrak{f}}(T) = \mathfrak{G}(\mathfrak{f})^\perp$; on the other hand, $Gr(T) = (T^1(A_k)/(T(k))^\perp$ by definition (here $G^\perp$ denotes the group of (continuous) characters of a (locally compact) group G).

Example 1. Let $T = G_{m,k}$, then the group $Gr(T)$ coincides with the group $Gr(k)$ of all the normalised grossencharacters of k, and $L(T; s, \chi)$, for $\chi \in Gr(T)$, is just a Hecke L-function $L(\chi, s)$ known to be a meromorphic function of s with the only pole at $s = 1$ when $\chi = 1$, and having no poles if $\chi \neq 1$.

Example 2. Let $k_o|k$ be a finite extension of number fields, and let $T = Res_{k_o/k}T_o$, where T_o is a torus defined over k_o. Then $T(k) = T_o(k_o)$, and $T(A_k) = T_o(A_{k_o})$, [16]; therefore $Gr(T) = Gr(T_o)$, $Gr_{\mathfrak{f}}(T) = Gr_{\mathfrak{f}}(T_o)$ if $\mathfrak{f}$ is defined over k, and $L(T; s, \chi)$ is equal to $L(T_o; s, \chi)$ up to a finite number of Euler factors (however, $\mathfrak{f}(\chi)$ is not necessarily equal to the conductor, say $\mathfrak{f}'(\chi)$, of χ in T_o, although $\mathfrak{f}'(\chi) \leq \mathfrak{f}(\chi)$ over k_o).

Lemma 3 *Let $\{T, T_o\} \subseteq Cat(K|k)$, and let $f : \hat{T}_o^* \to \hat{T}^*$ be a G-homomorphism. Then there is a natural homomorphism $f_A : T_o(A) \to T(A)$ for each k-algebra A.*

Proof Let $B = A \otimes_k K$. On writing $f_A(\alpha) = \beta$ with $\beta_j = \prod_i \alpha_i^{a_{ij}}$ for $\alpha \in B^{*d_o}$, $\beta \in B^{*d}$, where d and d_o denote the dimensions of T and T_o respectively and where $(a_{ij})_{\substack{1 \leq i \leq d_o \\ 1 \leq j \leq d}}$ is the matrix of the transformation f in an appropriate basis, one notes that $f_A(\sigma\alpha) = \sigma f(\alpha)$ and $f_A(\alpha^\sigma) = f_A(\alpha)^\sigma$, and the assertion follows.

We are now ready to prove the following important result due to P Draxl, [2].

Proposition 4. *For $T \in Cat(K|k)$,* let $\chi \in Gr(T)$. *There are number fields k_j and characters χ_j such that $\chi_j \in Gr(k_j)$, $k \subseteq k_j \subseteq K$, $1 \leq j \leq B$, and*

$$L(T; s, \chi) = \prod_{j=1}^{B} L(\chi_j, s) L_1(T; s, \chi), \tag{15}$$

where the function $s \mapsto L_1(T; s, \chi)$ is holomorphic and has no zeros in the half-plane

$\mathbb{C}_\kappa := \{s | s \in \mathbb{C}, Res > \frac{\kappa}{\kappa+1}\}$ *(with κ defined by* (2)*).*

Proof The finite set $C^*(m)$ being G-invariant, one can write $C^*(m) = \bigcup_{i=1}^{B(m)} D_i$, where $D_i = G.a_i$ is a G- orbit. Thus

$$\sum_{\substack{\mathfrak{a}_a \in I_p(T) \\ a \in C^*(m)}} \chi(\mathfrak{a}_a) = \sum_{i=1}^{B(m)} S_i(\chi) \quad , \quad S_i(\chi) := \sum_{\substack{\mathfrak{a}_a \in I_p(T) \\ a \in D_i}} \chi(\mathfrak{a}_a). \tag{16}$$

Let $H_i = \{\sigma | \sigma \in G \ , \ \sigma.a_i = a_i\}$ be the stabilizer of a_i, and let k_i be the H_i-invariant subfield of K, so that $H_i = Gal(K|k_i)$, $1 \leq i \leq B(m)$. Let $T_i = Res_{k_i/k} G_{m,k_i}$, we have

$$\hat{T}_i^* = \{ \sum_{\tau \in \frac{G}{H_i}} \alpha(\tau) e_\tau \Big| \alpha(\tau) \in \mathbb{Z} \ , \ \sigma e_\tau = e_{\sigma\tau} \quad \text{for} \quad \sigma \in G \ , \ \tau \in \tfrac{G}{H_i} \}$$

One defines a G-homomorphism $f_i : \hat{T}_i^* \to \hat{T}^*$ by letting $f_i(e_\tau) = \tau.a_i$. By lemma 3, there is a homomorphism $\bar{f}_i : T_i(A_k) \to T(A_k)$; let $\chi_i = \chi \circ \bar{f}_i$, clearly $\chi_i \in Gr(T_i)$. Moreover, $\chi(\mathfrak{a}_a) = \chi_i(\mathfrak{a}_{e_\tau})$ for $a = \tau.a_i$, $\mathfrak{a}_a \in I_p(T)$, $\mathfrak{a}_{e_\tau} \in I_p(T_i)$, and therefore

$$S_i(\chi) = \sum_{\substack{\tau \in \frac{G}{H_i} \\ \mathfrak{a}_{e_\tau} \in I_p(T_i)}} \chi_i(\mathfrak{a}_{e_\tau}). \tag{17}$$

Suppose $e(p) = 1$ (that is, $p \nmid D(K|k)$), then p splits completely in k_i, and the sum in (17) is equal to

$$S_i(\chi) = \sum_{\mathfrak{p}|p} \chi_i(\mathfrak{p}), \tag{18}$$

where $\mathfrak{p}$ ranges over all the prime divisors of p in k_i (we have identified $Gr(T_i)$ with $Gr(k_i)$, cf. Examples 1 and 2). On the other hand, if we take $m = \kappa$ it follows from (5) that

$$L_p(T; s, \chi) = 1 + \sum_{i=1}^{B} S_i(\chi) Np^{-s} + O\Big(Np^{-(Res)(\kappa+1)/\kappa}\Big), \tag{19}$$

where $B := B(\kappa)$. Equation (15) and the assertion of Proposition 4 follow from (16), (18), (19), and (4).

Remark 3. Equation (15) defines the function (8) in the half-plane $\mathbb{C}_\kappa$; by Proposition 4, it is holomorphic in $\mathbb{C}_\kappa \setminus \{1\}$ and has a pole of order

$$b(\chi) := card\ \{j | 1 \leq j \leq B \quad , \quad \chi_j = 1\} \tag{20}$$

at $s = 1$. Proceeding by induction on m, one can continue this function to a meromorphic function in $\mathbb{C}_+ := \{s | s \in \mathbb{C}\ ,\ Res > 0\}$, [2]. It is an interesting open problem to describe the class $\mathcal{K}$ of tori for which L-functions (8) are meromorphic in $\mathbb{C}$. This class is closed with respect to the restriction of scalars (Example 2). By example 1, $G_{m,k} \in \mathcal{K}$. More generally, given extensions $k_j | k\ ,\ 1 \leq j \leq \nu$, of number fields one defines a *norm-form torus* T by letting

$$T(A) = \left\{ \alpha \middle| \alpha \in \prod_{i=1}^{\nu} B_i^* \quad , \quad N_{B_i/A}\alpha_i = N_{B_1/A}\alpha_1 \quad , \quad B_i := A \bigotimes_k k_i \quad , \quad 1 \leq i \leq \nu \right\};$$

the L-functions of this torus T are known to have the line $\{s | s \in \mathbb{C}\ ,\ Res = 0\}$ as their natural boundary for analytic continuation, unless either $|I| \leq 1$, or $|I| = 2$ and $[k_i : k] = 2$ for $i \in I\ ,\ I := \{i | 1 \leq i \leq \nu\ ,\ k_i \neq k\}$, in which cases $T \in \mathcal{K}$, [4], [8].

Remark 4 By construction, if $\chi = 1$ then $b(\chi) = B$. The converse assertion is not, in general, true (cf. for instance, [5]); however, if $C^*(\kappa)$ generates $\hat{T}^*$ then a weaker implication

$$b(\chi) = B \Longrightarrow \chi \quad \text{is of finite order} \tag{21}$$

has been proved, [2], to hold true. We do not know whether (21) holds for any torus T.

A well-known argument (cf, for instance, [7]) rooted in the classical analytic number theory allows us to deduce the following estimates for character sums over integral ideals and over prime ideals from Proposition 4.

Theorem 1 *Let $\chi \in Gr(T)$. Then*

$$\sum_{\substack{\mathfrak{a} \in I_o(T) \\ N\mathfrak{a} < y^\kappa}} \chi(\mathfrak{a}) = yP_\chi(\log y) + O(y^{1-\gamma}), \gamma > 0 \tag{22}$$

where $P_\chi(t)$ is a polynomial of degree $b(\chi) - 1$ if $b(\chi) \geq 1$, and $P_\chi(log x) = 0$ if $b(\chi) = 0$;

$$\sum_{\substack{\mathfrak{p} \in \mathcal{P}(T) \\ N\mathfrak{p} < y^\kappa}} \chi(\mathfrak{p}) = b(\chi) \int_2^y \frac{du}{log\ u} + O(ye^{-\gamma_1 \sqrt{log\ y}}), \gamma_1 > 0. \tag{23}$$

We omit the proof of this theorem.

Let, in notation of (13),

$$\mathfrak{G}_o := \left(\mathbb{Z}/_{2\mathbb{Z}}\right)^{\nu_o} \times Cl_\mathfrak{f} \times \mathcal{B}_\mathfrak{f}, \tag{24}$$

so that

$$\mathfrak{G} \cong \mathcal{T}_\mathfrak{f}^{(o)} \times \mathfrak{G}_o, \tag{25}$$

where $\mathcal{T}_\mathfrak{f}^{(o)} = (S^1)^{d_\infty + r}$ is a flat torus. Let $gr_\mathfrak{f}(T)$ be the subgroup of $Gr_\mathfrak{f}(T)$ consisting of all the characters of finite order; clearly,

$$gr_\mathfrak{f}(T) \cong \mathfrak{G}_o^\perp. \tag{26}$$

Corollary 1. *Let $A \in \mathfrak{G}_o$; let $\mathcal{N}(A, y) = |\mathfrak{N}(a, y)|$ with*

$$\mathfrak{N}(A, y) := \{\mathfrak{a} | \mathfrak{a} \in I_o(T)\ ,\ (\mathfrak{a}, \mathfrak{f}_o) = 1\ ,\ g(\mathfrak{f})(\mathfrak{a}) \in A, N\mathfrak{a} < y^\kappa\},$$

and $\pi(A, y) = |\mathfrak{M}(A, y)|$ with $\mathfrak{M}(A, y) := \mathfrak{N}(A, y) \cap \mathcal{P}(T)$. Then

$$N(A, y) = y \sum\nolimits_\chi^* \overline{\chi(A)} P_\chi(log\ y) + O(y^{1-\gamma})\ ,\ \gamma > 0, \tag{27}$$

and

$$\pi(A, y) = \left(\sum\nolimits_\chi^* \overline{\chi(A)} b(\chi)\right) \int_2^y \frac{du}{log\ u} + O\left(ye^{-\gamma_1 \sqrt{log\ y}}\right), \gamma_1 > 0, \tag{28}$$

where

$$\sum_{\chi}{}^{*} := \frac{1}{|\mathfrak{G}_o|} \sum_{\chi \in gr_{\mathfrak{f}}(T)} .$$

Proof Relation (27) (resp. (28)) follows from (26) and (22) (resp. (23)).

Corollary 2. *If $I_o(T) \cap A \neq \phi$ then*

$$\mathcal{N}(A, y) = yP_A(log\ y) + O(y^{1-\gamma}), \gamma > 0, \tag{29}$$

where $P_A(t)$ is a non-vanishing polynomial of degree lower than B.

Proof By (28),

$$\pi(I_{\mathfrak{f}}^{pr}(T), Y) = \sum_{\chi}{}^{*} b(\chi) \frac{y}{log\ y}(1 + O(1)) \gg \frac{y}{log\ y};$$

therefore it follows from (27) that $\mathcal{N}(I_{\mathfrak{f}}^{pr}(T), y) \gg y$. Let $\mathfrak{a}_o \in I_o(T) \cap A$. Clearly

$$\left\{\mathfrak{a}_o(\alpha) \middle| (\alpha) \in I_{\mathfrak{f}}^{pr}(T) \cap I_o(T)\ ,\ N(\alpha) < y^{\kappa} N\mathfrak{a}_o^{-1}\right\} \subseteq \mathfrak{N}(A, y),$$

so that $\mathcal{N}(A, y) \geq \mathcal{N}(I_{\mathfrak{f}}^{pr}(T), yN\mathfrak{a}_o^{-1}) \gg y$, and (29) follows from (27).

Remark 5. Let τ be a smooth subset of $\mathcal{T}_{\mathfrak{f}}^{(o)}$ (in the sense of [6], [7]), let $\mathfrak{N}(\tau, A, y) = \mathfrak{N}(A, y) \cap g(\mathfrak{f})^{-1}(\tau)$, and let $\mathfrak{M}(\tau, A, y) = \mathfrak{N}(\tau, A, y) \cap \mathcal{P}(T)$. If the torus T satisfies condition (21) and $B = 1$, then the considerations of [6], [7] may be easily adopted to the present situation and one can prove asymptotic formulae for the number of integral ideals $|\mathfrak{N}(\tau, A, y)|$ and for the number of prime ideals $|\mathfrak{M}(\tau, A, y)|$. Thus in this case both integral and prime ideals from a given class A in $\mathfrak{G}_o$ are spatially equidistributed in the sense of E. Hecke. If condition (21) is satisfied but $B > 1$, we can still obtain an asymptotic formula for $|\mathfrak{N}(\tau, A, y)|$ gaining, however, only a power of $log\ y$ in the error term.

Corollary 3. *Suppose that $B = 1$, and let $\mathcal{H} = \{\chi | \chi \in gr_{\mathfrak{f}}(T)\ ,\ b(\chi) = 1\}$; write $H = \mathcal{H}^{\perp}$.* Then

$$\pi(A, y) = \frac{1}{|H|} \int_2^y \frac{du}{\log y} + O\left(y e^{-\gamma_1 \sqrt{\log y}}\right) \quad \text{for} \quad A \in H, \tag{30}$$

and $\pi(A, y) = O(y e^{-\gamma_1 \sqrt{\log y}})$ for *$A \notin H$. Moreover,* if *$A \in H$ then* (29) *holds with $P_A(t) = c_A, c_A > 0,\ \gamma > 0$.*

Proof. The estimates for $\pi(A, y)$ follow from (28). The last assertion follows from (27) and (30).

Remark 6. Already in the case of norm-form tori relation (29) (with $P_A(t) \neq 0$) does not imply that the class A contains infinitely many prime ideals, [5]. One should note that if $B = 1$, then a better estimate $\pi(A, y) = O(y^{1-1/\kappa+1})$ for $A \neq H$ follows from the definition of the characters $\chi_j\ ,\ 1 \leq j \leq B$, in (15).

5. Let $T \in Cat(K|\mathbb{Q})$ with $[K : \mathbb{Q}] = n$. For $\mathfrak{a} \in I(T)$, we define an affine toric variety $X_{\mathfrak{a}}$ as follows. Let $\mathfrak{a}_i = \sum_{j=1}^{n} \omega_j^{(i)} \mathbb{Z},\ 1 \leq i \leq d$; let us introduce a set of independent variables $\{x_{ij} | 1 \leq i \leq d\ ,\ 1 \leq j \leq n\}$ and write

$$t_i = \sum_{j=1}^{n} x_{ij} \omega_j^{(i)}\ ,\ \sigma t_i := \sum_{j=1}^{n} x_{ij} \sigma \omega_j^{(i)} \text{ for } \sigma \in G\ ,\ 1 \leq i \leq d.$$

The variety $X_{\mathfrak{a}}$ is defined by the set of equations:

$$\sigma t = t^{\sigma} \quad , \quad \sigma \in G; \tag{31}$$

this is a system of polynomial equations with integral rational coefficients, so that $X_{\mathfrak{a}}$ is a variety of dimension d defined over $\mathbb{Z}$. The torus T may be embedded in $X_{\mathfrak{a}}$ as an open subset $Y_{\mathfrak{a}}$ defined by the condition $\prod_{i=1}^{d} t_i \neq 0$. We shall study the distribution of integer points $Y_{\mathfrak{a}}(\mathbb{Z})$ in the real locus $Y_{\mathfrak{a}}(\mathbb{R})$. By definition, $T_\infty(A_{\mathbb{Q}}) = T(\mathbb{R})$ and therefore, by lemma 2, $T_\infty(A_{\mathbb{Q}}) \cong \mathbb{R}_+^{*r+\mu} \times (\mathbb{Z}/2\mathbb{Z})^\nu \times (S^1)^{d_\infty}$ with $\nu = r+\mu$, $\nu + d_\infty = d$; by construction, $Y_{\mathfrak{a}}(\mathbb{R})$ is homeomorphic to $T_\infty(A_{\mathbb{Q}})$. Thus there is a homeomorphism

$$l_u : Y_{\mathfrak{a}}(\mathbb{R}) \to \mathbb{R}_+^{*r+\mu} \times (\mathbb{Z}/2\mathbb{Z})^\nu \times (S^1)^{d_\infty}; \tag{32}$$

to describe this homeomorphism explicitly, let us choose a basis $\{e_i | 1 \le i \le d\}$ in $\hat{T}$ satisfying the same conditions as in the proof of lemma 2: $e_i \in \hat{T}^G$ for $i \le \mu$, $e_i \in \hat{T}^{G_{\mathfrak{p}}}$ for $i \le r+\mu$, and $\sigma_{\mathfrak{p}} e_i = -e_i$ for $i > r+\mu$; here $\mathfrak{p}$ is a fixed prime in $S_\infty(K)$ and $\sigma_{\mathfrak{p}}$ denotes the generator of $G_{\mathfrak{p}}$ if $S_\infty(K) = S_2(K)$, $G_{\mathfrak{p}} = \{1\}$ with $d_\infty = 0, r+\mu = d$ if K is a totally real field. We introduce new coordinates, say,

$$u_i = \prod_{j=1}^{d} t_j^{m_{ji}} \quad \text{with} \quad (m_{ij})_{1 \le i,j \le d} \in SL(d,\mathbb{Z}) \tag{33}$$

corresponding to the chosen basis $\{e_i | 1 \le i \le d\}$. By (31),

$$u_i(a) \in \mathbb{R}^* \text{ for } i \le r+\mu, \text{ and } u_i(a) \in \mathbb{C}_1 \text{ for } i > r+\mu \ , \ a \in Y_{\mathfrak{a}}(\mathbb{R}),$$

and the homeomorphism (32) may be described as follows: let $a \in Y_{\mathfrak{a}}(\mathbb{R})$, then, by definition, $l_u(a) = b$ with $b_i = |u_i(a)|$ for $i \le r+\mu$, $b_{i+j} = \frac{u_i(a)}{|u_i(a)|}$ for $i \le \nu$, $1 \le j \le \nu$, $b_{i+\nu} = u_i(a)$ for $i > \nu$ (here we let $t_i(a) = \sum\limits_{j=1}^{n} a_{ij}\lambda_{\mathfrak{p}}(\omega_j^{(i)})$, where $\lambda_{\mathfrak{p}} : K \to \mathbb{C}$ denotes the isomorphism corresponding to $\mathfrak{p}$, and define u by (33)). Let

$$U(y) = \left\{ a | a \in Y_{\mathfrak{a}}(\mathbb{R}) \ , \ |Nt(a)| \le y^\kappa \ , \ \frac{1}{y} \le |u_j(a)| \le y \text{ for } \mu < j \le \nu \right\},$$

where $Nt := \prod\limits_{i=1}^{d} \prod\limits_{\sigma \in G} \sigma t_i$.

<u>**Proposition 5.**</u> *If T is an anisotropic torus, then*

$$\text{card } \{U(y) \cap Y_{\mathfrak{a}_o}(\mathbb{Z})\} = c_1 (log\ y)^r \left(1 + O(\frac{1}{log y})\right), c_1 > 0, \tag{34}$$

where $\mathfrak{a}_{oi} = \mathfrak{O}$ for $1 \leq i \leq d$, $\mathfrak{a} = \mathfrak{a}_o$.

Proof If T is anisotrpic, then $I_o(T) = \{1\}$. Therefore the map $a \mapsto t(a)$ establishes an one-to-one correspondence between $Y_{\mathfrak{a}_o}(\mathbb{Z})$ and $T(\mathbb{Z})$. Relation (34) follows from the unit theorem, [13].

Returning to our study of isotropic tori we are now ready to prove the following estimate for the number of integer points in the "cube-like" domain $U(y)$.

Theorem 2. *Let $Z_{\mathfrak{f}} = \{a | a \in Y_{\mathfrak{a}}(\mathbb{Z}), t(a) = 1(\mathfrak{f})\}$. We have:*

$$\text{card } (Z_{\mathfrak{f}} \cap U(y)) = c_2 y(log\ y)^b \left(1 + O(\frac{1}{log\ y})\right), \tag{35}$$

with $c_2 > 0$ if $\mathbb{Z}_{\mathfrak{f}} \neq \phi$, $b \geq 0$, $b \in \mathbb{Z}$.

Proof. The map $t : a \mapsto t(a)$ is an one-to-one map from $Y_{\mathfrak{a}}(\mathbb{Z})$ to $\mathfrak{a} \cap T(\mathbb{Q})$, and $\{(t(a)) | a \in Z_{\mathfrak{f}}\} \subseteq I_{\mathfrak{f}}^{pr}(T)$. Thus $\{(t(a)) | a \in Z_{\mathfrak{f}}\} = \{\mathfrak{b}.\mathfrak{a} | \mathfrak{b} \in A^{-1} \cap I_o(T)\}$, where $A \in Cl_{\mathfrak{f}}(T), \mathfrak{a} \in A$. Therefore (35) follows from (27) and the unit theorem [13]. Moreover, if $Z_{\mathfrak{f}} \neq \phi$ then $c_2 > 0$ by Corollary 2.

Remark 7. Introducing a height function $h : Y_{\mathfrak{a}}(\mathbb{R}) \to \mathbb{R}_+$ one can rewrite the asymptotic formula (35) as follows:

$$\text{card } \left\{a | a \in Z_{\mathfrak{f}}\ ,\ h(a) \leq y\right\} = c_2 y(log\ y)^b \left(1 + O(\frac{1}{log\ y})\right),$$

where $h(a) := \max_{\mu < j \leq \nu} \{|Nt(a)|^{1/\kappa}\ ,\ |u_j(a)|\ ,\ |u_j(a)|^{-1}\}$. One should note that $X_{\mathfrak{a}}$ may be regarded as an affine toric variety corresponding to the cone C^* (cf. for instance, [3, §1.2]). Let us recall now that a Draxl L-function $L_k(T, C, z, \chi, S; s)$, [2], depends on two parameters: z in $\hat{T}^G$ and a z-admissible cone C. In our case $z \in \hat{T}^G$, $z_i = \sum_{\sigma \in G} \sum_{j=1}^{d} r_{ij}(\sigma)$ for $1 \leq i \leq d$, and C is a cone in $\hat{T}$ dual to C^*. It is clear from the definitions that the choice of z determines the height h. On the other hand, a different choice of the basic cone C^* would lead to a different theory of ideals; a Diophantine problem relating to such a generalisation is yet to be discovered.

Remark 8. If $B = 1$ and condition (21) is satisfied, one can prove an equidistribution formula in the spirit of my work on norm-form varieties (see [7], [9] and references therein). Namely, the number of integer points in a smooth subset of $Y_{\mathfrak{a}}(\mathbb{R})$ is seen to be asymptotically proportional to the (properly defined) measure of this set.

Remark 9. Given a torus T defined over a number field k, one can pass to the torus $Res_{k/\mathbb{Q}}T$ and then apply the theory developed in this section.

Remark 10. In my work on norm- form tori (loc. cit.) a few questions have been left open; as an application of the theory developed here, one can now answer these questions. Let T be the norm-form torus defined in Remark 3. First of all, we note that condition (21) is satisfied for the torus $Res_{k/\mathbb{Q}}T$ since in this case $C^*(\kappa)$ generates $\hat{T}^*$. Thus one can treat general norm-form tori without making additional assumptions on the fields $k, k_1, ..., k_\nu$. Moreover, if $k_1, ..., k_\nu$ are linearly disjoint over k, then $B = 1$, and we obtain a generalisation of results in [9] to arbitrary ground fields. Finally let us recall that in the previous work, [7], [9], I restricted myself to the variety given by the equations

$$|f_1(x_1)| = ... = |f_\nu(x_\mu)|,$$

rather than treating the original variety

$$V : f_1(x_1) = ... = f_\nu(x_\nu)$$

(here f_j is a full norm-form associated to a module in k_j , $1 \leq j \leq \nu$). This restriction may also be removed now. In view of the detailed considerations in [9], we may omit the details here.

Remark 11 We have not tried to obtain sharp error terms in the asymptotic formulae; in (23) and (28) one can, of course, improve the error term slightly, even if we require the estimates to be uniform in all parameters (cf [1]). To determine the exact value of γ in (22) and (29) may be of some interest also (cf. [7], where this question has been touched upon in the case of norm-form tori). A preliminary report on this work, [10], may appear elsewhere.

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Appendix

Exercises in analytic arithmetic on an algebraic torus, the corrected version of the Max-Planck-Institut für Mathematik Preprint 93-82 (1993)

Exercises in Analytic Arithmetic on an Algebraic Torus

B.Z. Moroz

Dedicated to Professor F. Hirzebruch
with deep respect and gratitude

1. The multidimensional arithmetic of E. Hecke, [4], [5], [7], may be regarded as a study in analytic number theory on the torus $Res_{k/\mathbb{Q}}G_{m,k}$ for a number field k of finite degree over the field $\mathbb{Q}$ of rational numbers. Here we shall try to generalise these considerations to an arbitrary algebraic torus defined over a number field. After applying Weil's restriction of scalars, if necessary, we may suppose that our torus T is defined over $\mathbb{Q}$; it splits over a finite normal extension $K|\mathbb{Q}$. Let $G = Gal(K|\mathbb{Q})$ be the Galois group of K, let $[K : \mathbb{Q}] = n$ be its degree, and let $d = \dim T$ denote the dimension of T. Such a torus is uniquely defined by an integral representation

$$\rho : G \longrightarrow GL(d, \mathbb{Z}) \ ,$$

where $\mathbb{Z}$ is the ring of rational integers, [12] (cf. also [15]). Consider a G-module $K[x]$, $x := \{x_{ij} | \ 1 \leq i \leq d, \ 1 \leq j \leq n\}$, choose an integral basis $\{\omega_i | \ 1 \leq i \leq n\}$ of $K|\mathbb{Q}$, and let

$$t_i = \sum_{j=1}^{n} x_{ij}\omega_j \quad , \quad 1 \leq i \leq d \ .$$

Equations

$$\sigma t_i = t_i^{\sigma} \quad , \quad \sigma \in G \quad , \quad 1 \leq i \leq d \ ,$$

where

$$\sigma t_i := \sum_{j=1}^{n} x_{ij}\, \sigma\omega_j \ , \ t_i^\sigma := \prod_{j=1}^{d} t_j^{r_{ji}(\sigma)} \ , \ \rho(\sigma) = (r_{ij}(\sigma)) \ , \ 1 \le i,j \le d \ ,$$

define an algebraic variety, say

$$X = Spec\ \mathbb{Q}[x]/J \ ,$$

J being the defining ideal of X; the torus T may be regarded as a Zariski open subset of X given by the condition $\prod_{1\le i\le d} t_i \ne 0$. We view $X(\mathbb{Z})$ as a generalisation of the ring of integers of an algebraic number field (if $T = Res_{k/\mathbb{Q}} G_{m,k}$ one may identify $X(\mathbb{Z})$ with the ring of integers of k), and intend to play the usual game of analytic number theory on this set.

2. On choosing a fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ we shall regard the field $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$, as a subfield of the field $\mathbb{C}$ of complex numbers. For a commutative k–algebra A, $k \subseteq \mathbb{C}$, let $A_K = A \otimes_{k_0} K$, where $k_0 = K \cap k$ (the fields k and K are linearly disjoint over k_0 since $K|\mathbb{Q}$ is normal). If one defines an embedding

$$t : \ T(A) \longrightarrow A_K^{*^d}$$

in a natural way, $T(A)$ may be viewed as a subset of G_0–invariants, where $G_0 := Gal(K|k_0)$, that is to say

$$T(A) = \{t(a) |\ a \in A^{nd},\ \sigma t(a) = t^\sigma(a) \quad \text{for} \quad \sigma \in G_0\}$$

(a word about notation, $t(a) := (t_1(a), \ldots, t_d(a))$, $t_i(a) = \sum_{j=1}^{n} a_{ij}\omega_j$, $a = \{a_{ij}|\ 1 \le i \le d,\ 1 \le j \le n\}$, $t^\sigma := (t_1^\sigma, \ldots, t_d^\sigma)$, etc.). Since

$$X(\mathbb{Q})\backslash T(\mathbb{Q}) \subseteq \bigcup_{i=1}^{d} \ell_i \ , \ \ell_i := \{x |\ x \in \mathbb{Q}^{nd},\ x_{ij} = 0 \text{ for } 1 \le j \le d\} \ ,$$

we may often replace $X(A)$ by $T(A)$ causing no damage to the type of problems discussed here.

Before proceeding any further let us introduce the G–module of characters

$$\hat{T} = \{x |\ x \in \mathbf{Z}^d,\ \sigma x = \rho(\sigma)x \text{ for } \sigma \in G\} \quad ,$$

and its dual

$$\hat{T}^* = \{y |\ y^t \in \mathbf{Z}^d,\ \sigma y = y\rho(\sigma^{-1}) \text{ for } \sigma \in G\} \quad ,$$

where the upper affix ^{t} denotes matrix transposition. The G–module

$$M = \{t^x |\ x \in \hat{T},\ \sigma t^x = t^{\sigma x} \text{ for } \sigma \in G\} \quad ,$$

and its submonoid

$$M_0 = \{t^x |\ x \in \hat{T},\ x \geq 0\}$$

furnish us with a convenient parametrization of $T(A)$. Here $t^x := \prod_{i=1}^{d} t_i^{x_i}$, and $x \geq 0$ means $x_i \geq 0$ for $1 \leq i \leq d$.

3. Let $I(K)$ and $I_0(K)$ denote the group of fractional ideals of K and the monoid of integral ideals of K respectively, and let

$$\begin{aligned} I(T) &= \{\mathfrak{A} |\ \mathfrak{A} \in I(K)^d,\ \sigma\mathfrak{A}_j = \prod_{i=1}^{d} \mathfrak{A}_i^{r_{ij}(\sigma)} \quad \text{for} \quad \sigma \in G,\ 1 \leq j \leq d\} \quad , \\ I_0(T) &= I(T) \cap I_0(K)^d \ . \end{aligned}$$

One defines the norm homomorphism $N:\ I(T) \to \mathbf{Q}_+^*$ by letting $N\mathfrak{A} = \prod_{1 \leq j \leq d} N\mathfrak{A}_j$ for $\mathfrak{A} \in I(T)$. We say that $\mathfrak{A}$ is a *primary* ideal if $\mathfrak{A} \in I_0(T)$ and $N\mathfrak{A}$ is a prime power in $\mathbf{Q}$. For a rational prime p, let

$$I_p(T)(:= I_p) = \{\mathfrak{A} |\ \mathfrak{A} \in I_0(T),\ N\mathfrak{A} = p^n \text{ for some } n\}$$

be the submonoid of p–primary ideals. To analyze the structure of I_p let us introduce the G–module of one–parameter subgroups

$$M_u = \{u^y |\ y \in \hat{T}^*,\ \sigma u^y = u^{\sigma y}\} \quad ,$$

where $u^y := (u^{y_1}, \ldots, u^{y_d})$. Clearly, $(\sigma x)(\sigma u^y) = x(u^y)$ if we let $x(u^y) := (u^y)^x = u^{y \cdot x}$ for $x \in \hat{T}$, $u^y \in M_u$.

Let us choose a prime $\mathfrak{p}$ in $I(K)$ dividing p, and let

$$G_{\mathfrak{p}} = \{\sigma |\ \sigma\mathfrak{p} = \mathfrak{p},\ \sigma \in G\}$$

be the decomposition group of $\mathfrak{p}$, so that

$$p = \prod_{\tau \bmod G_{\mathfrak{p}}} (\tau\mathfrak{p})^{e(p)} \qquad \text{in} \qquad I(K) \ ,$$

where τ ranges over G. Let $\mathfrak{A} \in I_p$, then

$$\mathfrak{A}_j = \prod_{\tau \bmod G_{\mathfrak{p}}} (\tau\mathfrak{p})^{a_j(\tau)} \qquad \text{with} \qquad a_j(\tau) \in \mathbf{Z},\ \ a_j(\tau) \geq 0 \ ,$$

and

(1)
$$\sigma\mathfrak{A}_j = \prod_{i=1}^{d} \mathfrak{A}_i^{r_{ij}(\sigma)} = \prod_{\tau \bmod G_{\mathfrak{p}}} (\tau\mathfrak{p})^{(a(\tau)\cdot\rho(\sigma))_j} \ .$$

On the other hand,

$$\sigma\mathfrak{A}_j = \prod_{\tau \bmod G_{\mathfrak{p}}} (\sigma\tau\mathfrak{p})^{a_j(\tau)} \ ,$$

and in particular

$$\tau^{-1}\mathfrak{A}_j = \mathfrak{p}^{a_j(\tau)}\mathfrak{A}'_j \qquad \text{with} \qquad \mathfrak{p} \not| \mathfrak{A}'_j \ .$$

But

$$\tau^{-1}\mathfrak{A}_j = \mathfrak{p}^{(a(e)\rho(\tau^{-1}))_j}\, \mathfrak{A}'_j \qquad \text{with} \qquad \mathfrak{p} \not| \mathfrak{A}'_j \ ,$$

in view of (1). Therefore

$$a(\tau) = a \cdot \rho(\tau^{-1}) \ ,$$

and, moreover,

$$a \cdot \rho(\sigma) = a \qquad \text{for} \qquad \sigma \in G_{\mathfrak{p}} \ ,$$

where we write $a(e) = a$ and denote by e the unit element of G. Thus (cf. [1])

(2)
$$I_p = I_p(T) = \{\mathfrak{A}_a |\ \mathfrak{A}_a = \prod_{\tau \bmod G_{\mathfrak{p}}} (\tau\mathfrak{p})^{\tau \cdot a},\ a \in C_{\mathfrak{p}}^{*}\} \ ,$$

where

$$C^* = \{a|\ a \in \hat{T}^*,\ \sigma \cdot a \geq 0 \text{ for } \sigma \in G\} \ ,$$

and

$$C^*_{\mathfrak{p}} = C^* \cap (\hat{T}^*)^{G_{\mathfrak{p}}} \ .$$

If $C^* \neq \{0\}$ let $a \in C^*\backslash\{0\}$; clearly

$$\sum_{\sigma \in G} \sigma a \in (\hat{T}^*)^G \backslash \{0\} \ ,$$

so that $\hat{T}^G \neq \{0\}$, and T is not anisotropic. Therefore $I_0(T) = \{1\}$, and consequently $T(\mathbb{Z}) = X(\mathbb{Z})$ for an anisotropic torus T. Suppose now that T is not anisotropic (that is $\hat{T}^G \neq \{0\}$), then after a possible change of basis in T it may be assumed that $C^* \cap (\hat{T}^*)^G \neq \{0\}$, and in particular $C^*_{\mathfrak{p}} \neq \{0\}$.

Let

$$\chi :\ I_0(T) \longrightarrow \mathbb{C}_1 \cup \{0\}$$

be such a homomorphism that

$$\chi^{-1}(\{0\}) = \prod_{p \in S} I_p \qquad \text{with} \qquad \#S < \infty \ ;$$

here $\mathbb{C}_1 := \{z|\ z \in \mathbb{C},\ |z| = 1\}$. Let

(3) $$L(\chi, s) = \sum_{\mathfrak{A} \in I_0(T)} \chi(\mathfrak{A}) N\mathfrak{A}^{-s} \ ;$$

clearly

(4) $$L(\chi, s) = \prod_p L_p(\chi, s) \ ,$$

where p ranges over all the rational primes, and

$$L_p(\chi, s) = \sum_{\mathfrak{A} \in I_p} \chi(\mathfrak{A}) N\mathfrak{A}^{-s} \ .$$

Both the Dirichlet series (3) and the Euler product (4) converge absolutely for $Re\, s > 1$. By a well-known theorem (going back to D. Hilbert), the cone C^* and therefore the monoid I_p are finitely generated. The generators of I_p are the *prime ideals* of T; it can be shown that the theorem on the uniqueness of factorization of the primary ideals into primes does **not** hold in this generality. Let $\mathcal{P}(T)$ be the set of all the prime ideals in $I_0(T)$, and let $\mathfrak{P} \in \mathcal{P}(T)$; we say that $\mathfrak{P}$ is a *strict prime* if

$$\mathfrak{A} | \mathfrak{P}^n \Longrightarrow (\mathfrak{A} = \mathfrak{P}^m \quad \text{for some} \quad m) \ .$$

Let $\mathcal{P}_s(T)$ be the subset of the strict primes. From a theorem in combinatorics, [14, theorem 2.5], one concludes that

$$L(\chi, s) = \prod_{\mathfrak{P} \in \mathcal{P}_s(T)} (1 - \chi(\mathfrak{P}) N\mathfrak{P}^{-s})^{-1} \prod_p Q_p(p^{-s}) \ , \tag{5}$$

with $Q_p(x) \in \mathbb{C}[x]$, $Q_p(0) = 1$.

Lemma 1. For $\mathfrak{A}_a \in I_p$ one has

$$N\mathfrak{A}_a = p^{b(a)} \quad , \quad b(a)e(p) = a \cdot z \ , \tag{6}$$

with $z_i = \sum_{\sigma \in G, 1 \le j \le d} r_{ij}(\sigma)$; moreover, $z \in \hat{T}^G$.

Proof. Let $N\mathfrak{p} = p^{f(p)}$. It follows from (2) that

$$N\mathfrak{A}_a = p^{f(p)b_1} \qquad \text{with} \qquad b_1 = \sum_{\tau \bmod G_\mathfrak{p}} |\tau a| \ ,$$

where $|a| := \sum_{j=1}^{d} a_j$ for $a \in \hat{T}^*$. Since $C_\mathfrak{p}^* \subseteq (\hat{T}^*)^{G_\mathfrak{p}}$ we have

$$b_1 = \frac{1}{|G_\mathfrak{p}|} \sum_{\sigma \in G} |\sigma a| = \frac{1}{|G_\mathfrak{p}|} \sum_{\sigma \in G} \sum_{1 \le i,j \le d} a_i r_{ij}(\sigma^{-1}) \ .$$

Relation (6) follows now from the equation $|G_\mathfrak{p}| = e(p)f(p)$; the last assertion is obvious.

Write now

(7) $$L_p(\chi, s) = \sum_{n=0}^{\infty} p^{-ns} \sum_{\substack{a\cdot z = e(p)n \\ \mathfrak{A}_a \in I_p}} \chi(\mathfrak{A}_a) \ .$$

For $H \subseteq G$, let $C_H^* = C^* \cap (\hat{T}^*)^H$, and let

$$\beta(H) := \min \{a \cdot z | \ a \neq 0, \ a \in C_H^*\} \ .$$

By construction,

$$\beta(H) = \left(\min \left\{ \sum_{\tau \bmod H} |\tau\alpha| \middle| a \neq 0, \ a \in C_H^* \right\}\right) \cdot |H| \ ,$$

and therefore

(8) $$|H| \leq \beta(H) < \infty \ .$$

Clearly $\beta(H_1) \leq \beta(H_2)$ if $H_1 \subseteq H_2$, so that

(9) $$\min_{H \subseteq G} \beta(H) = \beta_0 \quad , \quad \beta_0 = \beta(\{e\}) \ .$$

By (7)–(9),

(10) $$L_p(\chi, s) = 1 + \sum_{e(p)n \geq \beta_0} p^{-ns} \sum_{\substack{a\cdot z = e(p)n \\ \mathfrak{A}_a \in I_p}} \chi(\mathfrak{A}_a) \ .$$

Lemma 2. Both the Dirichlet series (3) and the Euler product (4) converge absolutely for $Re\, s > \frac{1}{\beta_0}$.

Proof. It follows from (10) and the definitions (3), (4).

Clearly

$$\mathfrak{A}_a \in I_p \quad , \quad a \cdot z = \beta(G_{\mathfrak{p}}) \Longrightarrow \mathfrak{A}_a \quad \text{is prime} \ .$$

Let

$$\mathcal{P}_m(T) = \{\mathfrak{A}_a |\ a \in C^*,\ a \cdot z = \beta_0\}$$

be the set of the *minimal primes*. It follows from (5) that

$$L(\chi, s) = \prod_{\mathfrak{P} \in \mathcal{P}_m(T)} (1 - \chi(\mathfrak{P}) N\mathfrak{P}^{-s})^{-1} L^{(1)}(\chi, s) \quad , \tag{11}$$

where

$$L^{(1)}(\chi, s) = \prod_{\mathfrak{P} \in \mathcal{P}_s(T) \setminus \mathcal{P}_m(T)} (1 - \chi(\mathfrak{P}) N\mathfrak{P}^{-s})^{-1} \prod_p Q_p^{(1)}(p^{-s}) \tag{12}$$

with $Q_p^{(1)}(x) \in \mathbb{C}[x]$, $Q_p^{(1)}(0) = 1$, and the Euler product (12) converges absolutely for $Re\, s > \frac{1}{\beta_0 + 1}$.

Corollary 1. The set

$$D(\beta) = \{a |\ a \in C^*,\ a \cdot z = \beta\} \quad , \quad \beta > 0 \ ,$$

is a finite G–invariant set.

Proof. It follows from Lemma 1 that $D(\beta)$ is G–invariant since $z \in \hat{T}^G$; moreover, $a \cdot z = \sum_{\sigma \in G} |\sigma a| \geq |a|$ for $a \in C^*$, and therefore

$$|D(\beta)| \leq card\{a |\ a \in \mathbb{Z}^d,\ a \geq 0,\ |a| \leq \beta\} < \infty \ .$$

Let

$$D(\beta_0) = \bigcup_{i=1}^{B} D_i$$

be the decomposition of the set $D(\beta_0)$ into G–orbits

$$D_i = G \cdot a^{(i)} \quad , \quad 1 \leq i \leq B \ ,$$

and let

$$\overline{D}_i(p) = \{\mathfrak{A}_a |\ \mathfrak{A}_a \in I_p(T),\ a \in D_i\} \ .$$

We have

$$(13) \qquad \prod_{\mathfrak{P}\in\mathcal{P}_m(T)} (1-\chi(\mathfrak{P})N\mathfrak{P}^{-s})^{-1} = f(s) \prod_{1\le i\le B}^{p} \ell_p^{(i)}(s)$$

with

$$(14) \qquad \ell_p^{(i)}(s) = \prod_{\mathfrak{P}\in\overline{D}_i(p)} (1+\chi(\mathfrak{P})p^{-\beta_0 s/e(p)}) \ ,$$

where $f(s)$ is equal to an Euler product absolutely convergent for $Re\, s > \frac{1}{2\beta_0} \ge \frac{1}{\beta_0+1}$.

Let

$$H_i = \{\sigma |\ \sigma \in G,\ \sigma a^{(i)} = a^{(i)}\}$$

be the stabiliser of $a^{(i)}$, and let

$$k_i = \{x |\ x \in K,\ \sigma x = x \text{ for } \sigma \in H_i\}$$

be the subfield of K corresponding to H_i; let

$$T_i = Res_{k_i/\mathbb{Q}} G_{m,k_i} \quad , \quad 1 \le i \le B \ ,$$

so that

$$\hat{T}_i^* = \left\{ \sum_{\sigma \bmod H_i} \alpha(\sigma)\sigma |\ \sigma \in G,\ \alpha(\sigma) \in \mathbb{Z} \right\} \ .$$

There is an injective homomorphism $f_i :\ \hat{T}_i^* \to \hat{T}^*$, uniquely defined by the condition $f_i(\sigma) = \sigma \cdot a^{(i)}$; clearly $f_i(\hat{T}_i^*)$ coincides with the submodule $[D_i]$ generated in $\hat{T}^*$ by D_i. By construction,

$$I(T_i) = \{\mathfrak{A} |\ \mathfrak{A}_1 \in I(k_i),\ \mathfrak{A}_j = \mathfrak{A}_1^{\sigma_j},\ 1 \le j \le d_i\} \ ,$$

where $G = \bigcup_{1\le j\le d_i} H_i\sigma_j$, $d_i = |D_i| = [k_i : \mathbb{Q}]$. Therefore we can define a homomorphism

$$\chi_i :\ I_0(k_i) \longrightarrow \mathbb{C}_1 \cup \{0\}$$

as follows: let $\mathfrak{B}_1 \in I_0(k_i)$ with $N\mathfrak{B}_1 = p^{\ell}$ for a rational prime p, and let $\mathfrak{B}_j = \mathfrak{B}_1^{\sigma_j}$, $1 \le j \le d_i$; then $\mathfrak{B} \in I_p(T_i)$, say $\mathfrak{B} = \mathfrak{A}_a$ with $a \in \hat{T}_i^*$, and we may set $\chi_i(\mathfrak{B}_1) = \chi(\mathfrak{A}_{f_i(a)})$ for the uniquely defined ideal $\mathfrak{A}_{f_i(a)}$ in $I_p(T)$. Let

$$(15) \qquad L(\chi_i, s) = \prod_{\mathfrak{p}\in I(k_i)} (1-\chi_i(\mathfrak{p})N\mathfrak{p}^{-s})^{-1} \ .$$

Proposition 1. We have

$$L(\chi, s) = \prod_{i=1}^{B} L(\chi_i, \beta_0 s) L^{(2)}(\chi, s) \quad , \tag{16}$$

where $L^{(2)}(\chi, s)$ is represented by an Euler product absolutely convergent for $Re\ s > \frac{1}{\beta_0+1}$; moreover,

$$\chi_i = 1 \quad \text{for} \quad 1 \leq i \leq B \Longleftrightarrow \chi|_{\mathcal{P}_m(T)} = 1 \quad . \tag{17}$$

Proof. In view of (11) – (15), it suffices to note that

$$\overline{D}_i(p) = \{\mathfrak{A}_{f_i(a)}|\ \mathfrak{A}_a = \mathfrak{B} \text{ with } N\mathfrak{B}_1 = p,\ \mathfrak{B} \in I_p(T_i)\} \quad .$$

Proposition 1 may be regarded as a formal counterpart of a theorem of Draxl's (cf. [1], equation (2.1)).

4. Now we are ready to proceed to the main part of this investigation and to comment on the structure of $X(\mathbf{Z})$ as a discrete subset of $X(\mathbb{R})$. To begin with let

$$G_2 = Gal(K|\ K \cap \mathbb{R}) \quad ,$$

so that

$$|G_2| = \begin{cases} 1 & \text{if } K \subseteq \mathbb{R} \\ 2 & \text{otherwise} \end{cases} \quad .$$

Since both $\hat{T}/\hat{T}^{G_2}$ and $\hat{T}^{G_2}/\hat{T}^{G}$ are torsion–free there is a $\mathbf{Z}$–basis $\{u_j|1 \leq j \leq d\}$ of $\hat{T}$ such that $\{u_j|\ 1 \leq j \leq \mu\}$ is a basis of $\hat{T}^G$, while $\{u_j|\ 1 \leq j \leq \mu + r\}$ is a basis of $\hat{T}^{G_2}$. Clearly

$$T(\mathbb{R}) = \{a|\ a \in \mathbb{R}^{nd},\ u^{\tau}(a) = \tau u(a) \text{ for } \tau \in G_2\} \quad ,$$

and we can define a surjective map

$$\begin{aligned} f:\ T(\mathbb{R}) &\longrightarrow \mathbb{R}^{*\mu+r} \times (S^1)^{d_1} \quad , \\ a &\longmapsto \left(u_1(a), \ldots, u_{\mu+r}(a), \ldots, \frac{u_i(a)}{|u_i(a)|}, \ldots\right) \quad , \end{aligned}$$

where $\mu + r + d_1 = d$, $d_1 \geq 0$, $i > \mu + r$. By a generalisation of the Dirichlet unit theorem, [12], [13],

$$T(\mathbf{Z}) \cong \mathbf{Z}^r \times \mathfrak{A} \quad \text{with} \quad |\mathfrak{A}| < \infty \ ;$$

therefore $T(\mathbb{R})/T(\mathbf{Z}) \cong \mathbf{R}_+^{*\mu} \times \mathcal{T}$, where

$$\mathcal{T} = (S^1)^{d-\mu} \times (\mathbf{Z}/2\mathbf{Z})^{r_0}$$

and $r_0 \leq \mu + r$.

Given a set

$$S = \{\infty\} \cup S_0 \quad , \quad S_0 \subseteq \{p |\ p \text{ is a rational prime}\} \ ,$$

let

$$T_A(S) = \prod_{p \in S} T(\mathbf{Q}_p) \times \prod_{p \notin S} T(\mathbf{Z}_p) \ ,$$

and let

$$T_A = \bigcup_{|S| < \infty} T_A(S) \ .$$

Clearly $T_A = T(A_{\mathbf{Q}})$, where $A_{\mathbf{Q}}$ is the adèle–algebra over $\mathbf{Q}$. Let

$$T_A^1 = \{a |\ a \in T_A,\ |x(a)| = 1 \text{ for } x \in \hat{T}^G\} \ ;$$

clearly $T(\mathbf{Q}) \subseteq T_A^1$ (if one identifies $T(\mathbf{Q})$ with its image under the diagonal embedding into T_A). By a well-known theorem, [12], [15], $T_A^1/T(\mathbf{Q})$ is a compact group. We have

$$T(\mathbf{Q}_p) = \{\alpha | \alpha \in K_{\mathfrak{p}}^{*d}, \sigma\alpha = \alpha^\sigma \qquad \text{for} \qquad \sigma \in G_{\mathfrak{p}}\} \ ,$$

where $\mathfrak{p}$ is a fixed prime in $I(K)$ with $\mathfrak{p}|p$. Therefore there is a natural embedding $g : I_p \hookrightarrow T_A(S)$ with $S = \{\infty, p\}$ such that $g(I_p)_q = 1$ for $q \notin S$, and $g(\mathfrak{A}_a)_p = \alpha$ for $\mathfrak{A}_a \in I_p$, with $\alpha \in T(\mathbf{Q}_p)$, $\alpha = \pi^a \varepsilon$, where $\varepsilon \in \mathfrak{o}_{\mathfrak{p}}^{*d}$, $\mathfrak{p} = (\pi)$, $\mathfrak{o}_{\mathfrak{p}}$ being the ring of integers of $K_{\mathfrak{p}}$; moreover, it may be assumed that $g(I_p) \subseteq T_A^1$ if one adjusts $g(I_p)_\infty$ properly. One extends g to an embedding

$$g : \ I_0(T) \hookrightarrow T_A^1 \ .$$

Given a character

$$\tilde{\chi}: T_A^1/T(k) \longrightarrow \mathbb{C}_1 \ ,$$

the set $S_0 = \{p|\ \tilde{\chi}_p(T(\mathbf{Z}_p)) \neq 1\}$ is finite; for $p \notin S_0$ we let $\chi_p = \tilde{\chi}_p \circ g$, if $p \in S_0$ let $\chi_p(I_p) = 0$. This procedure gives rise to the group $Gr(T)$ of *Hecke characters*

$$\chi: I_o(T) \longrightarrow \mathbb{C}_1 \cup \{0\} \ .$$

If $\chi \in Gr(T)$ then $\chi_i \in Gr(k_i)$, $1 \leq i \leq B$, where $Gr(k)$ denotes the group of all the Grössencharakteren of a number field k. The following result may be regarded as a corollary of Satz 1 in [1].

Corollary 2. Suppose that $\chi \in Gr(T)$. Then equation (16) defines $L(\chi, s)$ as a meromorphic function of s in the halfplane $\left\{s|\ s \in \mathbb{C},\ Re\, s > \frac{1}{\beta_0+1}\right\}$, with the only possible pole at $s = 1/\beta_0$.

Proof. It is an immediate consequence of Proposition 1 since $L(\chi_i, s)$, $1 \leq i \leq B$, is a Hecke L–function of k_i in this case.

The usual machinery of analytic number theory (see, for instance, [9] and references therein) yields now the following results:

$$card\left\{\mathfrak{p}|\ \mathfrak{p} \in \mathcal{P}(T),\ N\mathfrak{p} < y^{\beta_0}\right\} = B\int_2^y \frac{du}{\log u} + O\left(ye^{-c\sqrt{\log y}}\right) \ , \tag{18}$$

with $c > 0$,

and

$$card\left\{\mathfrak{A}|\ \mathfrak{A} \in I_0(T),\ N\mathfrak{A} < y^{\beta_0}\right\} = yp(\log y) + O(y^{1-c_1}) \ , \tag{19}$$

with $c_1 > 0$,

where $p(x) \in \mathbb{C}[x]$, $\deg p = B - 1$.

The infinite component $\tilde{\chi}_\infty$ in the decomposition $\tilde{\chi} = \tilde{\chi}_\infty \cdot \prod_p \chi_p$ may be regarded as a character of $T(\mathbb{R})/T(\mathbf{Z})$, say

$$\tilde{\chi}_\infty : \mathbf{R}_+^{*\mu} \times \mathcal{T} \longrightarrow \mathbf{C}_1 \ .$$

The grossencharacter χ obtained from $\tilde{\chi}$ is said to be *normalised* if $\tilde{\chi}_\infty|_{\mathbf{R}_+^{*\mu}} = 1$. Write

$$\mathfrak{f}_\infty(\chi) = \{\alpha |\ \alpha \in (\mathbf{Z}/2\mathbf{Z})^{r_0},\ \tilde{\chi}_\infty(\alpha) \neq 1\} \ ,$$

and let $\mathfrak{f}_0(\chi) = \prod_p p^{m_p}$, where

$$m_p = \min \{m |\ \alpha \in T(\mathbf{Z}_p),\ \alpha = 1(\mathrm{mod}\ p^m) \Longrightarrow \tilde{\chi}_p(\alpha) = 1\} \ .$$

The pair $\mathfrak{f}(\chi) = (\mathfrak{f}_\infty(\chi), \mathfrak{f}_0(\chi))$ is said to be the conductor of χ. The group $Gr_0(T, \mathfrak{f})$ of all the normalised grossencharacters having a given conductor $\mathfrak{f}$ is isomorphic to $\mathbf{Z}^{d-\mu} \times \mathfrak{B}(\mathfrak{f})$, where $\mathfrak{B}(\mathfrak{f})$ is a finite Abelian group. Moreover, $\mathfrak{B}(\mathfrak{f})$ may be chosen to coincide with the subgroup of all the characters of finite order in $Gr_0(T, \mathfrak{f})$. Let

$$\mathfrak{B}(\mathfrak{f})^\perp = \{\mathfrak{A} |\ \mathfrak{A} \in I_0(T),\ \chi(\mathfrak{A}) = 1 \text{ for } \chi \in \mathfrak{B}(\mathfrak{f})\} \ ,$$

and let

$$I_0^{\mathfrak{f}}(T) = \{\mathfrak{A} |\ \chi(\mathfrak{A}) \neq 0 \text{ for } \chi \in Gr_0(T, \mathfrak{f})\} \ .$$

The ray class group $H(\mathfrak{f}) := I_0^{\mathfrak{f}}(T)/\mathfrak{B}(\mathfrak{f})^\perp$ is finite, [12] (cf. also [15]), and $\mathfrak{B}(\mathfrak{f})$ may be regarded as the group of characters of $H(\mathfrak{f})$. In a usual way one obtains the following asymptotic formulae for the number of integral ideals and for the number of the prime ideals in a given ideal class. Let $A \in H(\mathfrak{f})$, we have

(20)
$$card\left\{\mathfrak{p} |\ \mathfrak{p} \in \mathcal{P}(T) \cap A,\ N\mathfrak{p} < y^{\beta_0}\right\}$$
$$= \left(\sum_{\chi \in \mathfrak{B}(\mathfrak{f})}{}^{*}\ \overline{\chi(A)} g(\chi)\right) \int_2^y \frac{du}{\log u} + O\left(y e^{-c_2\sqrt{\log y}}\right) \ ,$$

and

(21)
$$card\left\{\mathfrak{A} |\ \mathfrak{A} \in A,\ N\mathfrak{A} < y^{\beta_0}\right\} = y \sum_{\chi \in \mathfrak{B}(\mathfrak{f})}{}^{*}\ \overline{\chi(A)} p_\chi(\log y) + O(y^{1-c_3}) \ ,$$

where $c_2 > 0$, $c_3 > 0$, $\sum^* := \frac{1}{|H(\mathfrak{f})|}\sum$, p_χ is a polynomial of degree $g(\chi) - 1$ whose coefficients may depend on χ, $g(\chi) := card\{i |\ 1 \leq i \leq B,\ \chi_i = 1\}$ (if $g(\chi) = 0$ we let $p_\chi = 0$).

Although our ultimate purpose is to investigate the distribution of integer points on X in the real locus $X(\mathbb{R})$, the methods of this paper fall short of such a goal, and we should be content with somewhat weaker results on the integer points of the variety Y defined as follows. For $a \in K^{*d}$ let $\epsilon(a,\sigma) = (\sigma a)(a^\sigma)^{-1}$, and write $\epsilon(a) : \sigma \mapsto \epsilon(a,\sigma)$, $\sigma \in G$; define an equivalence relation $\sim$:

$$\epsilon(a) \sim \epsilon(a') \iff \epsilon(a) = \epsilon(a')\epsilon(b) \text{ for some } b \text{ in } E_K^d \ ,$$

where E_K denotes the group of units of K, and let

$$A = \{\epsilon(a) |\ a \in K^{*d},\ \epsilon(a,\sigma) \in E_K^d \text{ for } \sigma \in G\} \ .$$

Let B be a set of representatives for $A/\sim$ containing the identity $\epsilon^{(0)}$ (here $\epsilon^{(0)} := \epsilon(1)$, $\epsilon_i(1,\sigma) = 1$ for $1 \leq i \leq d$). We set

$$Y = \bigcup_{\epsilon \in B} V_\epsilon \ ,$$

the variety V_ϵ being defined by the equations

$$\sigma t = \epsilon(\sigma) t^\sigma \quad , \quad \sigma \in G \ ;$$

clearly $V_{\epsilon^{(0)}} = X$, so that $X \subseteq Y$. The open subset $\tilde{V}_\epsilon$ of V_ϵ defined by the condition $\prod_{i=1}^{d} t_i \neq 0$ is a T–homogeneous space, and we identify $\tilde{V}_\epsilon(\mathbb{R})$ with $T(\mathbb{R})$. Moreover,

$$(t(a)) \in I_0(T) \iff a \in \tilde{Y}(\mathbb{Z}) \ ,$$

with $\tilde{Y} := \bigcup_{\epsilon \in B} \tilde{V}_\epsilon$. Making use of the theory developed here we obtain now an estimate for the number of integer points on Y in the "rectangular" compact domain $U(y)$ in $T(\mathbb{R})$ given as follows:

$$U(y) = \left\{a |\ a \in T(\mathbb{R}),\ |Nt(a)| < y^{\beta_0},\ \ y^{-1} \leq |u_j(a)| \leq y \text{ for } \mu + 1 \leq j \leq \mu + r\right\} \ ,$$

where $Nt(a) := \prod_{i=1}^{d} \prod_{\sigma \in G} (\sigma t_i)(a)$.

Corollary 3. Let $\mathfrak{A}_0 \in I_0^{\mathfrak{f}}(T)$, and let

$$M(\mathfrak{A}_0) = \{a | \; a \in Y(\mathbb{Z}), \; (t(a)) \subseteq \mathfrak{A}_0, \; (t(a)) \in \mathfrak{B}_0(\mathfrak{f})^{\perp}\} \; .$$

We have

$$card(U(y) \cap M(\mathfrak{A}_0)) = c_1(\mathfrak{A}_0) y (\log y)^{b+r} \left(1 + O\Big(\frac{1}{\log y}\Big)\right) \; , \tag{22}$$

with $0 \leq b \leq B - 1$.

Proof. Clearly

$$a \in M(\mathfrak{A}_0) \iff (t(a)) = \mathfrak{A}\mathfrak{A}_0 \;\; \text{with} \;\; \mathfrak{A} \in A \;\; ,$$

where $A \in H(\mathfrak{f})$, $\mathfrak{A}_0 \in A^{-1}$. By the unit theorem,

$$card\{a | \; (t(a)) = (t(a_0)), \; a \in M(\mathfrak{A}_0) \cap U(y)\} = c_2 (\log y)^r \left(1 + O\Big(\frac{1}{\log y}\Big)\right) \; .$$

Relation (22) follows from this estimate when combined with (21).

Proposition 2. If T is anisotropic then

$$card(X(\mathbb{Z}) \cap U(y)) = c_3 (\log y)^r \left(1 + O\Big(\frac{1}{\log y}\Big)\right) \; . \tag{23}$$

Proof. In this case $I_0(T) = \{1\}$, so that $X(\mathbb{Z}) \cap U(y)$ coincides with $T(\mathbb{Z}) \cap U(y)$. Therefore (23) follows from the unit theorem.

Remark 1. The constants $c_1(\mathfrak{A}_0)$ and c_3 can be explicitly evaluated; if $M(\mathfrak{A}_0) \neq \{0\}$ (resp. $X(\mathbb{Z}) \neq \{0\}$) then $c_1(\mathfrak{A}_0) > 0$ (resp. $c_3 > 0$).

5. Proposition 2 provides a complete solution of the problem of counting integer points on an anisotropic torus, although further refinements in the spirit of [3] may be probably obtained. Thus henceforth we assume again that the torus T under consideration is not anisotropic. The deeper results on the spatial ("multidimensional") distribution of the integer points as well as of the integral (or of the prime) ideals depend on the following condition

$$\chi_i = 1 \;\text{ for }\; 1 \leq i \leq B \Longrightarrow \chi \in \mathfrak{B}(\mathfrak{f}) \;\text{ for some } \mathfrak{f} \tag{24}$$

to be satisfied. If (24) holds and $B = 1$ then a complete analysis in the spirit of [8], [9], [11] is possible. If (24) holds but $B \neq 1$ we can still prove a spatial equidistribution theorem for integral ideals gaining, however, only a power of logarithm of the main term in the error term (this being insufficient for finer applications to an equidistribution theorem for integer points, as exhibited in [11]).

In view of (17), condition (24) holds true (with an even stronger conclusion) if the set $\mathcal{P}_m(T)$ of minimal primes generates the monoid $I_0(T)$ of integral ideals. The following observation [1, Satz 1] lies deeper, and it is more useful.

Lemma 3. If $C^*(\beta_0)$ generates the group $\hat{T}^*$ then (24) holds true. Here$C^*(m) := \{a | a \in C^*, a \cdot z = m\}$, $m \in \mathbf{Z}$, $m \geq 1$.

Proof. It is an immediate consequence of the last assertion in [1, Satz 1].

Example 1. The norm-form (or Vinogradov) torus T can be defined as follows. Let k be a field of algebraic numbers of finite degree over $\mathbb{Q}$; let $k_i | k$, $1 \leq i \leq \nu$, be a finite extension. The torus T_k is defined by the following condition (cf. [1]):

$$T_k(B) = \left\{ b | \; b \in \prod_{i=1}^{\nu} (B \otimes_k k_i)^*, \; N_{B\otimes_k k_1/B} b_1 = N_{B\otimes k_i/B} b_i, \;\; 1 \leq i \leq \nu \right\}$$

for any k-algebra B; we let $T = Res_{k/\mathbb{Q}} T_k$. It follows from Lemma 3 that the torus T satisfies (24), and therefore one can prove a theorem on the equidistribution of integral ideals having equal norms (cf. [8], where $k = \mathbf{Q}$ and the fields k_i are assumed to be linearly disjoint over k). Moreover, if the fields k_i, $1 \leq i \leq \nu$, are linearly disjoint over k then $B = 1$; therefore a complete theory in the spirit of [8], [9], [11] (where we have assumed $k = \mathbf{Q}$) can be developed in this case.

An open question. A Draxl L–function $L(s, \chi)$ of an algebraic torus is known to be meromorphic in the half–plane $\{s|\ s \in \mathbf{C},\ Re\, s > 0\}$, [1]. Moreover, if T is a norm–form torus considered in Example 1, then $L(s, \chi)$ has the line $\{s|\ s \in \mathbf{C},\ Re\, s = 0\}$ as its natural boundary for analytic continuation, unless either $\#\{i|\ k_i \neq k\} \leq 1$, or $\#\{i|\ k_i \neq k\} = 2$ and $[k_i : k] \leq 2$ for each i in which cases $L(s, \chi)$ is meromorphic on the whole complex plane, [6], [10]. Therefore we may ask under what conditions on T the function $s \mapsto L(s, \chi)$ can be analytically continued to a meromorphic function on $\mathbf{C}$.

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