Background independent geometry and Hopf cyclic cohomology

Alain Connes and Henri Moscovici

Abstract

This is primarily a survey of the way in which Hopf cyclic cohomology has emerged and evolved, in close relationship with the application of the noncommutative local index formula to transverse index theory on foliations. The Diff-invariant geometric framework that allowed us to treat the 'space of leaves' of a general foliation provides a "background independent" set-up for geometry that could be of relevance to the handling of the the background independence problem in quantum gravity. With this potential association in mind, we have added some new material, which complements the original paper and is also meant to facilitate its understanding. Section 2 gives a detailed description of the Hopf algebra that controls the 'affine' transverse geometry of codimension n foliations, and Section 5 treats the relative version of Hopf cyclic cohomology in full generality, including the case of Hopf pairs with noncompact isotropy.

Introduction

Coincidentally, or perhaps as a reflection of "ontogeny recapitulates phylogeny" in the world of mathematical ideas, the brand of cyclic cohomology related to Hopf algebras came about in a strikingly similar fashion to cyclic cohomology itself, both being motivated by the index theory of abstract elliptic operators, at successive stages.

The original impetus for the development of cyclic cohomology (as enunciated in [3], see also [22], and presented in [4]), was to construct invariants for K-theory classes that perform the function of the classical Chern-Weil theory in the general framework of operator algebras. Starting from the index pairing between the K-homology class of a p-summable Fredholm module $(\mathfrak{H}, \gamma, F)$ and the K-theory class $[e] \in K_0(\mathcal{A})$ of an idempotent in an involutive algebra $\mathcal{A} \subset \mathcal{L}(\mathfrak{H})$, in the graded case to fix the ideas,

$$\text{Index}(e\, F^+\, e) \;=\; (-1)^n\, Tr(\gamma\, e[F,e]^{2n})\,, \quad \forall\, 2n \geq p \geq 1\,, \tag{0.1}$$

one arrived, by regarding $de = i[F,e]$ as a 'quantized' differential, to the multilinearized form of the right hand side,

$$\tau_F(a^0, a^1, \ldots, a^{2n}) = Tr\left(\gamma a^0 [F, a^1] \ldots [F, a^{2n}]\right), \qquad a^i \in \mathcal{A}, \tag{0.2}$$

that turned out to encode the quintessential features of the cyclic cohomology theory for algebras.

The non-additive category of algebras and algebra homomorphisms was replaced in [5] by the additive category of modules over the cyclic category Λ, allowing the realization of cyclic cohomology as an Ext functor. This enlargement of the scope of the theory played an essential role years later, when the authors were faced with the formidable looking task of concretely computing in the geometrically interesting case of foliations the 'theoretical' answer provided by the universal local index formula [8]. The gist of that formula is that, in the unbounded version of the index pairing (0.1), it replaces the 'global' cocycle (0.2) by a universal finite linear combination of 'local' cocycles of the form

$$\phi(a^0, \ldots, a^m) = \int\!\!\!\!\!- a^0 [D, a^1]^{(k_1)} \ldots [D, a^m]^{(k_m)}\, |D|^{-(m+2|\mathbf{k}|)}\,, \tag{0.3}$$

where $T^{(k)}$ stands for the kth iterated commutator of the operator T with D^2 and $\int\!\!\!\!\!-$ is an extension of the Dixmier trace given by residues of spectral zeta-functions.

In the case of transversely hypoelliptic operators on foliations, algebraic manipulations with the commutators appearing in (0.3) led to the emergence of the Hopf algebra $\mathcal{H}_n$, that plays for the transverse frame bundle to a foliation the role of the affine group of the frame bundle to a manifold.

Recognizing the cyclic module structure associated to the Hopf algebra $\mathcal{H}_n$, and intrinsically related to the 'characteristic' cochains (0.3), provided precisely the missing principle to organize the computation. We settled the index problem in [9], as briefly sketched in §1, by proving that the cyclic cohomology of the above cyclic module is in fact isomorphic to the Gelfand-Fuks cohomology, in both the 'absolute' and the 'relative' case. This isomorphism is concretely illustrated in the codimension 1 case in §6, for the Godbillon-Vey class and also for the transverse fundamental class.

The emergent Hopf cyclic structure applies to arbitrary Hopf algebras, in particular to quantum groups, and gives rise to characteristic classes associated to Hopf actions, cf. [9, 11], also §3 and §4 below. The algebraic machinery developed in the process has been extended by Hajac-Kahalkhali-Rangipour-Sommerhäuser to a theory with coefficients [19, 20]. The characteristic map associated to a Hopf module algebra with invariant trace has been generalized to the case of higher traces by Crainic [15] and by Gorokhovsky [18]. It was further extended by Khalkhali and Rangipour [24], who upgraded it to cup products in Hopf-cyclic cohomology. For these developments we refer the reader to the cited papers.

The geometric framework that allowed us to treat the 'space of leaves' of a general foliation is Diff-invariant and therefore provides a 'background independent' set-up for geometry that could be of relevance in dealing with the background independence problem in quantum gravity. With this potential association in mind, we have added some new material. In §2 we give a detailed description of the Hopf algebra $\mathcal{H}_n$ and of its 'standard' module-algebra representation, while §5 treats the relative version of Hopf cyclic cohomology in full generality; thus, besides the relative Hopf cyclic cohomology of the pair $(\mathcal{H}_n, \mathfrak{o}_n)$, that has played a crucial role in understanding the Chern character of the hypoelliptic signature operator, one can now handle pairs with noncompact isotropy, such as $(\mathcal{H}_{n+1}, \mathfrak{o}_{n,1})$.

Contents

1 Background independent geometry and the local index formula

In the noncommutative approach a geometric structure on a 'space' is specified by means of a spectral triple $(\mathcal{A}, \mathfrak{H}, D)$. $\mathcal{A}$ is an involutive algebra of bounded operators in a Hilbert space $\mathfrak{H}$, and represents the 'local coordinates' of the space. D is an unbounded selfadjoint operator on $\mathfrak{H}$, which has bounded commutators with the 'coordinates', and whose inverse D^{-1} corresponds to the infinitesimal line element ds in Riemannian geometry. In addition to its metric significance, D carries an important topological meaning, that of a K-homology cycle which represents the fundamental class of the 'space' which is the spectrum of $\mathcal{A}$.
When this space is an ordinary spin manifold M, *i.e.* when the algebra $\mathcal{A} = C^\infty(M)$, one obtains a natural spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ by fixing a Riemannian metric on M and taking for D the Dirac operator in the Hilbert space $\mathfrak{H}$ of square integrable spinors.

At first sight, *i.e.* when viewed from the classical viewpoint of Riemannian geometry, the transition from the local differential geometric set-up to the operator theoretic framework might appear as a mere translation. That this is far from being the case, even in the classical framework, requires an explanation which we now give below.
In order to define the transverse geometry, *i.e.* the geometry of the 'space' of leaves, for a general foliation, one is confronted with the problem of finding a geometric structure that is *invariant* under all diffeomorphisms of a given manifold M. Indeed, the action of the holonomy on a complete transversal M to a foliation is as wild (in general) as that of an arbitrary (countable) subgroup of $\mathrm{Diff}(M)$, and invariance under holonomy is a necessary constraint when passing to the space of leaves.
The standard geometric notions are of course *equivariant* with respect to $\mathrm{Diff}(M)$, but they are not *invariant* . In fact, it is well known that the group

of isometries of a Riemannian manifold N is a finite dimensional Lie group and is thus incomparably smaller than the group $\mathrm{Diff}(M)$ of any manifold.

The first virtue of the operator theoretic framework of noncommutative geometry is that it only requires invariance to hold at the level of the *principal symbol* (in classical pseudodifferential terms) of the operator D. When D is an elliptic operator the gain is non-existent since in that case the symbol specifies the metric. But the first main point is that the theory applies with no change when D is only *hypoelliptic*, and this allows to treat 'para-Riemannian' spaces, which admit groups of isometries as large as diffeomorphism groups. This allows to handle n-dimensional geometry in the following "background independent" way [8]. One first replaces a given manifold M^n (with no extra structure except an orientation) by the total space of the bundle $PM = F^+M/SO(n)$, where F^+M is the $GL^+(n,\mathbb{R})$–principal bundle of oriented frames on M^n. The sections of $\pi : PM \to M$ are precisely the Riemannian metrics on M but unlike the space of such metrics the space P is still a finite dimensional manifold. The total space PM itself admits a canonical, and thus $\mathrm{Diff}^+(M)$-invariant, 'para-Riemannian' structure, which can be described as follows. The vertical subbundle $\mathcal{V} \subset T(PM)$, $\mathcal{V} = \mathrm{Ker}\,\pi_*$, carries natural Euclidean structures on each of its fibers, determined solely by fixing once and for all a choice of a $GL^+(n,\mathbb{R})$-invariant Riemannian metric on the symmetric space $GL^+(n,\mathbb{R})/SO(n)$. On the other hand, the quotient bundle $\mathcal{N} = T(PM)/\mathcal{V}$ comes equipped with a tautologically defined Riemannian structure: every point $p \in PM$ is an Euclidean structure on $T_{\pi(p)}(M)$ which is identified to $\mathcal{N}_p$ via π_*.
Since no non-canonical choice were involved so far, the obtained structure on PM is *invariant* under the canonical lift of the action of $\mathrm{Diff}^+(M)$. In particular any hypoelliptic operator whose *principal symbol* only depends upon the above 'para-Riemannian' structure will have the required invariance to yield a spectral triple governing the geometry in a "background independent" manner. Since the object of our interest is the K-homology class of the spectral triple (and we can freely use the Thom isomorphism to pass from the base M to the total space PM in an invariant manner), we shall take for D the hypoelliptic signature operator. The precise construction of D, to be recalled below, involves the choice of a connection on the frame bundle but this choice does not affect the *principal symbol* of D and thus plays an innocent role which does not alter the fundamental $\mathrm{Diff}^+(M)$-invariance of the spectral triple. More precisely, we have shown in [8] that it does define in

full generality a spectral triple on the crossed product of PM by $\mathrm{Diff}^+(M)$.

It is worth mentioning at this point that this construction, besides allowing to handle arbitrary foliations, could be of relevance in handling the basic problem of *background independence*, which is inherent to any attempt at a quantization of the theory of gravitation.

The *hypoelliptic signature operator* D is uniquely determined by the equation $Q = D|D|$, where Q is the operator

$$Q = (d_V^* \, d_V - d_V \, d_V^*) \oplus \gamma_V \, (d_H + d_H^*) \,,$$

acting on the Hilbert space of L^2-sections

$$\mathfrak{H}_{PM} = L^2(\wedge^{\cdot} \mathcal{V}^* \otimes \wedge^{\cdot} \mathcal{N}^*, \, \varpi_P) \,;$$

here d_V denotes the vertical exterior derivative, γ_V is the usual grading for the vertical signature operator, d_H stands for the horizontal exterior differentiation with respect to a fixed connection on the frame bundle, and ϖ_P is the $\mathrm{Diff}^+(M)$-invariant volume form on PM associated to the connection. When $n \equiv 1 \,\mathrm{or}\, 2 \,(\mathrm{mod}\, 4)$, for the vertical component to make sense, one has to replace PM with $PM \times S^1$ so that the dimension of the vertical fiber be even.

The above construction allows to associate to any transversely oriented foliation $\mathcal{F}$ of a manifold V a spectral triple encoding the geometry of $V/\mathcal{F}$ in the following sense. If M is a complete transversal and Γ is the corresponding holonomy pseudogroup, then the pair $(\mathfrak{H}_{PM}, D)$ described above can be completed to a spectral triple $(\mathcal{A}_\Gamma, \mathfrak{H}_{PM}, D)$, by taking as $\mathcal{A}_\Gamma$ the convolution algebra of the smooth ètale groupoid associated to Γ. The spectral triple $(\mathcal{A}_\Gamma, \mathfrak{H}_{PM}, D)$ represents the desired geometric structure for $V/\mathcal{F}$.

Using hypoelliptic calculus, which in particular provides a noncommutative residue functional $\displaystyle{-\!\!\!\!\!\!\int}$ extending the Dixmier trace, we proved in [8, Part I] that such a spectral triple $(\mathcal{A}_\Gamma, \mathfrak{H}_{PM}, D)$ fulfills the hypotheses of the operator theoretic local index theorem of [8, Part II]. Therefore, its character-index $ch_*(D) \in HC_*(\mathcal{A}_\Gamma)$ can be expressed in terms of residues of spectrally defined zeta-functions, and is given by a cocycle $\{\phi_q\}$ in the (b, B) bi-complex of $\mathcal{A}_\Gamma$ whose components are of the following form

$$\phi_q(a^0, \dots, a^q) = \sum_{\mathbf{k}} c_{q,\mathbf{k}} \,{-\!\!\!\!\!\!\int} a^0 [Q, a^1]^{(k_1)} \dots [Q, a^q]^{(k_q)} \, |Q|^{-q-2|k|} \,; \qquad (1.1)$$

we have used here the abbreviations $T^{(k)} = \nabla^k(T)$ and $\nabla(T) = D^2T - TD^2$,

$$\mathbf{k} = (k_1, \ldots, k_q), \qquad |\mathbf{k}| = k_1 + \ldots + k_q, \quad \text{and}$$

$$c_{q,\mathbf{k}} = \frac{(-1)^{|\mathbf{k}|}\sqrt{2i}}{k_1! \ldots k_q! \, (k_1+1) \ldots (k_1 + \ldots + k_q + q)} \, \Gamma\left(|k| + \frac{q}{2}\right).$$

The summation necessarily involves finitely many nonzero terms for each ϕ_q, and q cannot exceed $\dfrac{n(n+1)}{2} + 2n$.

In practice, the actual computation of the expression (1.1) is exceedingly difficult to perform. Even in the case of codimension $n = 1$, when there are only two components $\{\phi_1, \phi_3\}$, the order of magnitude of the number of terms one needs to handle is 10^3. Thus, a direct evaluation of (1.1) for an arbitrary codimension n is impractical.

There are two reduction steps which help alleviate, to some extent, the complexity of the problem. First, by enlarging if necessary the pseudogroup Γ, one may assume that M is a flat affine manifold. There is no loss of generality in making this assumption as long as the affine structure is not required to be preserved by Γ. Thus, one can equip M with a flat connection, and since the horizontal component of the operator Q is built out of the connection, its expression gets simplified to the fullest extent possible. It is also important to note that Q is affiliated with the universal enveloping algebra of the group of affine motions of $\mathbb{R}^n$, in the sense that it is of the form

$$Q = R(Q_{\mathrm{alg}}), \quad \text{with} \quad Q_{\mathrm{alg}} \in \left(\mathfrak{A}(\mathbb{R}^n \rtimes \mathfrak{gl}(n,\mathbb{R})) \otimes \mathrm{End}(E)\right)^{SO(n)},$$

where R is the right regular representation of $\mathbb{R}^n \rtimes GL(n,\mathbb{R})$ and E is a unitary $SO(n)$–module.

Secondly, one can afford to work at the level of the principal bundle F^+M, since the descent to the quotient bundle PM only involves the simple operation of taking $SO(n)$-invariants.

The strategy that led to the unwinding of the formula (1.1) essentially evolved from the following observation. The built-in affine invariance of the operator Q, allows to reduce the noncommutative residue functional involved in the cochains

$$\phi(a^0, \ldots, a^q) = ⨍ a^0[Q, a^1]^{(k_1)} \ldots [Q, a^q]^{(k_q)} \, |Q|^{-(q+2|\mathbf{k}|)}, \qquad (1.2)$$

to a genuine integration, and thus replace them by cochains of the form

$$\psi(a^0, \ldots, a^q) = \tau_\Gamma(a^0\, h^1(a^1) \ldots h^q(a^q))\,; \tag{1.3}$$

here τ_Γ is the canonical trace on $\mathcal{A}_\Gamma$ and $h^1, \ldots, h^q$ are 'transverse' differential operators acting on the algebra $\mathcal{A}_\Gamma$. Under closer scrutiny, which will be discussed in great detail in the next section, these transverse differential operators turn out to arise from the action of a canonical Hopf algebra $\mathcal{H}_n$, depending only on the codimension n. Furthermore, the cochains (1.3) will be recognized to belong to the range of a certain cohomological characteristic map.

2 The Hopf algebra $\mathcal{H}_n$ and its standard action

Let $F\mathbb{R}^n$ be the frame bundle on $\mathbb{R}^n$, identified to $\mathbb{R}^n \times GL(n, \mathbb{R})$ in the usual way: the 1-jet at $0 \in \mathbb{R}^n$ of the map $\phi : \mathbb{R}^n \to \mathbb{R}^n$,

$$\phi(t) = x + \mathbf{y} t\,, \qquad x,\, t \in \mathbb{R}^n\,, \quad \mathbf{y} \in GL(n, \mathbb{R})$$

is identified to the pair $(x, \mathbf{y}) \in GL(n, \mathbb{R})$. We endow it with the trivial connection, given by the matrix-valued 1-form $\omega = (\omega^i_j)$ where, with the usual summation convention,

$$\omega^i_j \;:=\; (\mathbf{y}^{-1})^i_\mu\, d\mathbf{y}^\mu_j \;=\; (\mathbf{y}^{-1}\, d\mathbf{y})^i_j\,, \qquad i, j = 1, \ldots, n\,. \tag{2.1}$$

The corresponding basic horizontal vector fields on $F\mathbb{R}^n$ are

$$X_k \;=\; y^\mu_k\, \partial_\mu\,, \quad k = 1, \ldots, n\,, \quad \text{where} \quad \partial_\mu = \frac{\partial}{\partial\, x^\mu}\,. \tag{2.2}$$

We denote by $\theta = (\theta^k)$ the canonical form of the frame bundle

$$\theta^k \;:=\; (\mathbf{y}^{-1})^k_\mu\, dx^\mu \;=\; (\mathbf{y}^{-1}\, dx)^k\,, \qquad k = 1, \ldots, n \tag{2.3}$$

and then let

$$Y^j_i \;=\; y^\mu_i\, \partial^j_\mu\,, \quad i, j = 1, \ldots, n\,, \quad \text{where} \quad \partial^j_\mu := \frac{\partial}{\partial\, y^\mu_j}\,, \tag{2.4}$$

be the fundamental vertical vector fields associated to the standard basis of $\mathfrak{gl}(n,\mathbb{R})$ and generating the canonical right action of $GL(n,\mathbb{R})$ on $F\mathbb{R}^n$. At each point of $F\mathbb{R}^n$, $\{X_k, Y_i^j\}$, resp. $\{\theta^k, \omega_j^i\}$, form bases of the tangent, resp. cotagent space, dual to each other:

$$\begin{aligned} \langle \omega_j^i, Y_k^\ell \rangle &= \delta_k^i \delta_j^\ell\,, \quad \langle \omega_j^i, X_k \rangle \,=\, 0\,, \\ \langle \theta^i, Y_k^\ell \rangle &= 0\,, \qquad \langle \theta^i, X_j \rangle \,=\, \delta_j^i\,. \end{aligned} \tag{2.5}$$

The group of diffeomorphisms $\mathcal{G}_n := \mathrm{Diff}\,\mathbb{R}^n$ acts on $F\mathbb{R}^n$, by the natural lift of the tautological action to the frame level:

$$\widetilde{\varphi}(x,\mathbf{y}) := (\varphi(x), \varphi'(x)\cdot \mathbf{y})\ , \quad \text{where} \quad \varphi'(x)^i_j = \partial_j\, \varphi^i(x)\,. \tag{2.6}$$

Viewing here $\mathcal{G}_n$ as a discrete group, we form the crossed product algebra

$$\mathcal{A}_n \,:=\, C_c^\infty(F\mathbb{R}^n) \rtimes \mathcal{G}_n\,.$$

As a vector space, it is spanned by monomials of the form $f\,U_\varphi^*$, where $f \in C_c^\infty(F\mathbb{R}^n)$ and U_φ^* stands for $\widetilde{\varphi}^{-1}$, while the product is given by the multiplication rule

$$f_1\,U_{\varphi_1}^* \cdot f_2\,U_{\varphi_2}^* = f_1(f_2 \circ \widetilde{\varphi}_1)\,U_{\varphi_2\varphi_1}^*\,. \tag{2.7}$$

Alternatively, $\mathcal{A}_n$ can be regarded as the subalgebra of the endomorphism algebra $\mathcal{L}\,(C_c^\infty(F\mathbb{R}^n))$ of the vector space $C_c^\infty(F\mathbb{R}^n)$, generated by the multiplication and the translation operators

$$M_f(\xi) \;=\; f\,\xi\,, \quad f \in C_c^\infty(F\mathbb{R}^n)\,,\, \xi \in C_c^\infty(F\mathbb{R}^n) \tag{2.8}$$

$$U_\varphi^*(\xi) \;=\; \xi \circ \widetilde{\varphi}\,, \qquad \varphi \in \mathcal{G}_n\,,\, \xi \in C_c^\infty(F\mathbb{R}^n)\,. \tag{2.9}$$

Since the right action of $GL(n,\mathbb{R})$ on $F\mathbb{R}^n$ commutes with the action of $\mathcal{G}_n$, at the Lie algebra level one has

$$U_\varphi\, Y_i^j\, U_\varphi^* \,=\, Y_i^j\,, \qquad \varphi \in \mathcal{G}_n\,. \tag{2.10}$$

This allows to promote the vertical vector fields to derivations of $\mathcal{A}_n$. Indeed, setting

$$Y_i^j(f\,U_\varphi^*) \,=\, Y_i^j(f)\,U_\varphi^*\,, \quad f\,U_\varphi^* \in \mathcal{A}_n\,, \tag{2.11}$$

the extended operators satisfy the derivation rule

$$Y_i^j(a\,b) \;=\; Y_i^j(a)\,b \,+\, a\,Y_i^j(b)\,, \quad a,b\in\mathcal{A}_n\,, \tag{2.12}$$

We shall also prolong the horizontal vector fields to linear transformations $X_k\in\mathcal{L}\,(\mathcal{A}_n)$, in a similar fashion:

$$X_k(f\,U_\varphi^*) = X_k(f)\,U_\varphi^*\,, \quad f\,U_\varphi^*\in\mathcal{A}_n\,. \tag{2.13}$$

The resulting operators are no longer $\mathcal{G}_n$-invariant. Instead of (2.10), they satisfy

$$U_\varphi\,X_k\,U_\varphi^* \;=\; X_k \,-\, \gamma_{jk}^i(\varphi^{-1})\,Y_i^j\,, \tag{2.14}$$

where $\varphi\mapsto\gamma_{jk}^i(\varphi)$ is a group 1-cocycle on $\mathcal{G}_n$ with values in $C^\infty(F\mathbb{R}^n)$; specifically,

$$\gamma_{jk}^i(\varphi)(x,\mathbf{y}) \;=\; \big(\mathbf{y}^{-1}\cdot\varphi'(x)^{-1}\cdot\partial_\mu\varphi'(x)\cdot\mathbf{y}\big)_j^i\;\mathbf{y}_k^\mu\,. \tag{2.15}$$

The above expression comes out readily from the pull-back formula for the connection,

$$\widetilde{\varphi}^*(\omega_j^i) \;=\; \omega_j^i \,+\, \gamma_{jk}^i(\varphi)\,\theta^k\,; \tag{2.16}$$

indeed, if $(\widetilde{x},\widetilde{\mathbf{y}}) := \widetilde{\varphi}(x,\mathbf{y}) = (\varphi(x),\varphi'(x)\cdot\mathbf{y})$, then

$$\begin{aligned}
\widetilde{\mathbf{y}}^{-1}\,d\widetilde{\mathbf{y}} \;&=\; \mathbf{y}^{-1}\,\varphi'(x)^{-1}\,(d\varphi'(x)\,\mathbf{y} \,+\, \varphi'(x)\,d\mathbf{y})\\
&=\; \mathbf{y}^{-1}\,d\mathbf{y} \,+\, \big(\mathbf{y}^{-1}\,\varphi'(x)^{-1}\,\partial_\mu\varphi'(x)\,\mathbf{y}\big)\;dx^\mu\\
&=\; \mathbf{y}^{-1}\,d\mathbf{y} \,+\, \big(\mathbf{y}^{-1}\,\varphi'(x)^{-1}\,\partial_\mu\varphi'(x)\,\mathbf{y}\big)\;\mathbf{y}_k^\mu\,\theta^k\,.
\end{aligned}$$

In view of the $\mathcal{G}_n$-invariance of θ, (2.16) makes the cocycle property of γ obvious. To obtain (2.14), one just has to use that $\{\theta^k,\,(\widetilde{\varphi}^{-1})^*(\omega_i^j)\}$ is the dual basis to $\{U_\varphi\,X_k\,U_\varphi^*,\,Y_i^j\}$, cf. (2.5).

As a consequence of (2.14), the operators $X_k\in\mathcal{L}\,(\mathcal{A}_n)$ are no longer derivations of $\mathcal{A}_n$, but satisfy instead a non-symmetric Leibniz rule:

$$X_k(a\,b) \;=\; X_k(a)\,b \,+\, a\,X_k(b) \,+\, \delta_{jk}^i(a)\,Y_i^j(b)\,, \quad a,b\in\mathcal{A}_n\,, \tag{2.17}$$

where the linear operators $\delta_{jk}^i\in\mathcal{L}\,(\mathcal{A}_n)$ are defined by

$$\delta_{jk}^i(f\,U_\varphi^*) \;=\; \gamma_{jk}^i(\varphi)\,f\,U_\varphi^*\,. \tag{2.18}$$

Indeed, on taking $a = f_1\, U^*_{\varphi_1}$, $b = f_2\, U^*_{\varphi_2}$, one has

$$\begin{aligned} X_k(a\cdot b) &= X_k(f_1\, U^*_{\varphi_1}\cdot f_2\, U^*_{\varphi_2}) \,=\, X_k(f_1\cdot U^*_{\varphi_1}\, f_2\, U_{\varphi_1})\, U^*_{\varphi_2\varphi_1} \\ &= X_k(f_1)\, U^*_{\varphi_1}\cdot f_2\, U^*_{\varphi_2} \,+\, f_1\, U^*_{\varphi_1}\cdot X_k(f_2\, U^*_{\varphi_2}) \\ &+\, f_1\, U^*_{\varphi_1}\cdot (U_{\varphi_1}\, X_k\, U^*_{\varphi_1} \,-\, X_k)(f_2\, U^*_{\varphi_2})\,, \end{aligned}$$

which together with (2.14) and the cocycle property of γ^i_{jk} imply (2.17). The same cocycle property shows that the operators δ^i_{jk} are derivations:

$$\delta^i_{jk}(a\, b) \,=\, \delta^i_{jk}(a)\, b \,+\, a\, \delta^i_{jk}(b)\,, \quad a, b\in \mathcal{A}_n\,, \tag{2.19}$$

The operators $\{X_k,\, Y^i_j\}$ satisfy the commutation relations of the group of affine transformations of $\mathbb{R}^n$:

$$\begin{aligned} [Y^j_i, Y^\ell_k] &= \delta^j_k Y^\ell_i - \delta^\ell_i Y^j_k\,, \\ [Y^j_i, X_k] &= \delta^j_k X_i\,, \qquad\qquad [X_k, X_\ell] \,=\, 0\,. \end{aligned} \tag{2.20}$$

The successive commutators of the operators δ^i_{jk}'s with the X_ℓ's yield new generations of

$$\delta^i_{jk|\ell_1\ldots\ell_r} \,:=\, [X_{\ell_r},\ldots [X_{\ell_1}, \delta^i_{jk}]\ldots]\,, \tag{2.21}$$

which involve multiplication by higher order jets of diffeomorphisms

$$\begin{aligned} \delta^i_{jk|\ell_1\ldots\ell_r}\, (f\, U^*_\varphi) &= \gamma^i_{jk|\ell_1\ldots\ell_r}\, f\, U^*_\varphi\,, \qquad \text{where} \\ \gamma^i_{jk|\ell_1\ldots\ell_r} &= X_{\ell_r}\cdots X_{\ell_1}(\gamma^i_{jk})\,. \end{aligned} \tag{2.22}$$

Evidently, they commute among themselves:

$$[\delta^i_{jk|\ell_1\ldots\ell_r},\, \delta^{i'}_{j'k'|\ell'_1\ldots\ell_r}] \,=\, 0\,. \tag{2.23}$$

The operators $\delta^i_{jk|\ell_1\ldots\ell_r}$ are not all distinct; the order of the first two lower indices or of the last r indices is immaterial. Indeed, performing the matrix multiplication in (2.15) gives the expression

$$\gamma^i_{jk}(\varphi)(x, \mathbf{y}) \,=\, (\mathbf{y}^{-1})^i_\lambda\, (\varphi'(x)^{-1})^\lambda_\rho\, \partial_\mu \partial_\nu \varphi^\rho(x)\, \mathbf{y}^\nu_j\, \mathbf{y}^\mu_k\,, \tag{2.24}$$

which is clearly symmetric in the indices j and k. The symmetry in the last r indices follows from the definition (2.21) and the fact that, the connection being flat, the horizontal vector fields commute. It can also be directly seen from the explicit formula

$$\begin{aligned} &\gamma^i_{jk|\ell_1\dots\ell_r}(\varphi)(x,\mathbf{y}) \,= \\ &= \,(\mathbf{y}^{-1})^i_\lambda \partial_{\beta_r}\dots\partial_{\beta_1}\left((\varphi'(x)^{-1})^\lambda_\rho \partial_\mu\partial_\nu\varphi^\rho(x)\right)\mathbf{y}^\nu_j\mathbf{y}^\mu_k\mathbf{y}^{\beta_1}_{\ell_1}\dots\mathbf{y}^{\beta_r}_{\ell_r}\,. \end{aligned} \tag{2.25}$$

The commutators between the Y^λ_ν's and δ^i_{jk}'s can be obtained from the explicit expression (2.15) of the cocycle γ , by computing its derivatives in the direction of the vertical vector fields. Denoting by E^λ_ν the $n\times n$ matrix whose entry at the λ-th row and the μ-th column is 1 and all others are 0, one has:

$$\begin{aligned} &Y^\lambda_\nu(\gamma^i_{jk}(\varphi))(x,\mathbf{y}) \,= \\ &= \,\frac{d}{dt}|_{t=0}\left(\left(\exp(-tE^\lambda_\nu)\,\mathbf{y}^{-1}\,\varphi'(x)^{-1}\,\partial_\mu\varphi'(x)\,\mathbf{y}\exp(tE^\lambda_\nu)\right)^i_j\,(\mathbf{y}\exp(tE^\lambda_\nu))^\mu_k\right) \\ &= \,-[E^\lambda_\nu,\,\mathbf{y}^{-1}\,\varphi'(x)^{-1}\,\partial_\mu\varphi'(x)\,\mathbf{y}]^i_j\,\mathbf{y}^\mu_k \,+\, \left(\mathbf{y}^{-1}\,\varphi'(x)^{-1}\,\partial_\mu\varphi'(x)\,\mathbf{y}\right)^i_j\,(\mathbf{y}E^\lambda_\nu)^\mu_k \\ &= \,\left(\mathbf{y}^{-1}\,\varphi'(x)^{-1}\,\partial_\mu\varphi'(x)\,\mathbf{y}\right)^i_\nu\,\delta^\lambda_j\,\mathbf{y}^\mu_k \,-\, \delta^i_\nu\,\left(\mathbf{y}^{-1}\,\varphi'(x)^{-1}\,\partial_\mu\varphi'(x)\,\mathbf{y}\right)^\lambda_j\,\mathbf{y}^\mu_k \\ &+ \,\left(\mathbf{y}^{-1}\,\varphi'(x)^{-1}\,\partial_\mu\varphi'(x)\mathbf{y}\right)^i_j\,\mathbf{y}^\mu_\nu\,\delta^\lambda_k\,. \end{aligned}$$

It follows that

$$[Y^\lambda_\nu,\delta^i_{jk}] \,=\, \delta^\lambda_j\,\delta^i_{\nu k} \,+\, \delta^\lambda_k\,\delta^i_{j\nu} \,-\, \delta^i_\nu\,\delta^\lambda_{jk}\,.$$

More generally, using (2.20) and the very definition (2.21), one obtains from this, by induction,

$$[Y^\lambda_\nu,\delta^i_{j_1j_2|j_3\dots j_r}] \,=\, \sum_{s=1}^r \delta^\lambda_{j_s}\,\delta^i_{j_1j_2|j_3\dots j_{s-1}\nu j_{s+1}\dots j_r} \,-\, \delta^i_\nu\,\delta^\lambda_{j_1j_2|j_3\dots j_r}\,. \tag{2.26}$$

By a transient abuse of notation, we now regard X_k, Y^i_j and $\delta^i_{jk|\ell_1\dots\ell_r}$ as abstract symbols, preserving the convention that in the designation of the latter the order of the first two lower indices or of the last r indices is unimportant, and make the following definition.

Definition 1. *Let $\mathcal{H}_n$ be the universal enveloping algebra of the Lie algebra $\mathfrak{h}_n$ with basis*

$$\{X_\lambda,\, Y_\nu^\mu,\, \delta^i_{jk|\ell_1\dots\ell_r} | 1 \le \lambda, \mu, \nu, i \le n,\, 1 \le j \le k \le n,\, 1 \le \ell_1 \le \dots \le \ell_r \le n\}$$

and the following presentation:

$$\begin{aligned}
[X_k, X_\ell] &= 0, && (2.27)\\
[Y_i^j, Y_k^\ell] &= \delta_k^j Y_i^\ell - \delta_i^\ell Y_k^j, && (2.28)\\
[Y_i^j, X_k] &= \delta_k^j X_i, && (2.29)\\
[X_{\ell_r}, \delta^i_{jk|\ell_1\dots\ell_{r-1}}] &= \delta^i_{jk|\ell_1\dots\ell_r}, && (2.30)\\
[Y_\nu^\lambda, \delta^i_{j_1j_2|j_3\dots j_r}] &= \sum_{s=1}^{r} \delta_{j_s}^\lambda\, \delta^i_{j_1j_2|j_3\dots j_{s-1}\nu j_{s+1}\dots j_r} - \delta_\nu^i\, \delta^\lambda_{j_1j_2|j_3\dots j_r}, && (2.31)\\
[\delta^i_{jk|\ell_1\dots\ell_r}, \delta^{i'}_{j'k'|\ell'_1\dots\ell'_r}] &= 0\,. && (2.32)
\end{aligned}$$

We shall endow $\mathcal{H}_n = \mathfrak{A}(\mathfrak{h}_n)$ with a canonical Hopf structure, which is non-cocommutative and therefore different from the standard structure of a universal enveloping algebra.

Proposition 2. 1^0. *The formulae*

$$\begin{aligned}
\Delta X_k &= X_k \otimes 1 + 1 \otimes X_k + \delta^i_{jk} \otimes Y_i^j\,, && (2.33)\\
\Delta Y_i^j &= Y_i^j \otimes 1 + 1 \otimes Y_i^j\,, && (2.34)\\
\Delta \delta^i_{jk} &= \delta^i_{jk} \otimes 1 + 1 \otimes \delta^i_{jk}\,, && (2.35)
\end{aligned}$$

uniquely determine a coproduct $\Delta : \mathcal{H}_n \to \mathcal{H}_n \otimes \mathcal{H}_n$, which makes $\mathcal{H}_n$ a bialgebra with respect to the product $m : \mathcal{H}_n \otimes \mathcal{H}_n \to \mathcal{H}_n$ and the counit $\varepsilon : \mathcal{H}_n \to \mathbb{C}$ inherited from $\mathfrak{A}(\mathfrak{h}_n)$.

2^0. *The formulae*

$$\begin{aligned}
S(X_k) &= -X_k + \delta^i_{jk}\, Y_i^j\,, && (2.36)\\
S(Y_i^j) &= -Y_i^j\,, && (2.37)\\
S(\delta^i_{jk}) &= -\delta^i_{jk}\,, && (2.38)
\end{aligned}$$

uniquely determine an anti-homomorphism $S : \mathcal{H}_n \to \mathcal{H}_n$, which provides the antipode that turns $\mathcal{H}_n$ into a Hopf algebra.

Proof. 1^0. Once its existence is established, the uniqueness of the coproduct Δ satisfying (2.33)–(2.35) is obvious. Indeed, these formulae prescribe the values of the algebra homomorphism $\Delta : \mathfrak{A}(\mathfrak{h}_n) \to \mathfrak{A}(\mathfrak{h}_n) \otimes \mathfrak{A}(\mathfrak{h}_n)$ on a set of generators. (Note however that, in view of (2.33), they *do not* define a Lie algebra homomorphism $\mathfrak{h}_n \to \mathfrak{h}_n \otimes \mathfrak{h}_n$).
To prove the existence of the coproduct, one checks that the presentation (2.27)–(2.32) is preserved by Δ; this ensures that Δ extends to an *algebra homorphism* $\mathfrak{A}(\mathfrak{h}_n) \to \mathfrak{A}(\mathfrak{h}_n) \otimes \mathfrak{A}(\mathfrak{h}_n)$.

One then has to verify *coassociativity* and *counitality.* We skip the straightforward details. An alternate argument will emerge later (see Remark 5).

2^0. Similarly, one shows that S defines an an anti-homomorphism $\mathcal{H}_n \to \mathcal{H}_n$ by checking that the presentation (2.27)–(2.32) is anti-preserved. Since the *antipode* axioms are multiplicative, it suffices to verify them on a set of generators. This is precisely how the formulae (2.36)–(2.38) were obtained, and is easy to verify. $\square$

The abuse of notation made in the definition 1 will be rendered completely innocuous by the next result.

Proposition 3. 1^0. *The subalgebra of $\mathcal{L}(\mathcal{A}_n)$ generated by the linear operators $\{X_k,\, Y^i_j,\, \delta^i_{jk} \mid i,j,k = 1,\ldots,n\}$ is isomorphic to the algebra $\mathcal{H}_n$.*

2^0. *The ensuing action of $\mathcal{H}_n$ turns $\mathcal{A}_n$ into a left $\mathcal{H}_n$-module algebra.*

Proof. 1^0. The notation needed to specify a Poincaré-Birkhoff-Witt basis for $\mathfrak{A}(\mathfrak{h}_n)$ involves two kinds of multi-indices. The first kind are of the form

$$I = \left\{ i_1 \leq \ldots \leq i_p \,;\, \binom{j_1}{k_1} \leq \ldots \leq \binom{j_q}{k_q} \right\} ,$$

while the second kind are of the form $K = \{\kappa_1 \leq \ldots \leq \kappa_r\}$, where

$$\kappa_s = \begin{pmatrix} i_s & & \\ j_s & k_s & \mid \ell^s_1 \leq \ldots \leq \ell^s_{p_s} \end{pmatrix} , \qquad s = 1, \ldots, r \,;$$

in both cases the inner multi-indices are ordered lexicographically.

The PBW basis of $\mathfrak{A}(\mathfrak{h}_n)$ will consist of elements of the form $\delta_K\, Z_I$, ordered lexicographically, where

$$Z_I = X_{i_1} \dots X_{i_p}\, Y_{k_1}^{j_1} \dots Y_{k_q}^{j_q} \quad \text{and} \quad \delta_K = \delta_{j_1\, k_1 | \ell_1^1 \dots \ell_{p_1}^1}^{i_1} \dots \delta_{j_r\, k_r | \ell_1^r \dots \ell_{p_r}^r}^{i_r} .$$

We need to prove that if $c_{I,\kappa} \in \mathbb{C}$ are such that

$$\sum_{I,K} c_{I,K}\, \delta_K\, Z_I\,(a) \;=\; 0\,, \quad \forall\, a \in \mathcal{A}_n\,, \tag{2.39}$$

then $c_{I,K} = 0$, for any (I, K).
To this end, we evaluate (2.39) on all monomials $a \;=\; f\, U_\varphi^*$ at the point

$$u_0 \;=\; (x = 0, \mathbf{y} = \mathbf{I}) \in F\mathbb{R}^n = \mathbb{R}^n \times GL(n, \mathbb{R}^n)\,.$$

In particular, for any fixed but arbitrary $\varphi \in \mathcal{G}_n$, one obtains

$$\sum_I \left(\sum_K c_{I,K}\, \gamma_\kappa(\varphi)(u_0) \right) (Z_I\, f)(u_0) = 0\,, \quad \forall\, f \in C_c^\infty(F\mathbb{R}^n)\,. \tag{2.40}$$

Since the Z_I's form a PBW basis of $\mathfrak{A}(\mathbb{R}^n \rtimes \mathfrak{gl}(n, \mathbb{R}))$, which can be viewed as the algebra of left-invariant differential operators on $F\mathbb{R}^n$, the validity of (2.40) for any $f \in C_c^\infty(F\mathbb{R}^n)$ implies the vanishing for each I of the corresponding coefficient. One therefore obtains, for any fixed I,

$$\sum_K c_{I,K}\, \gamma_K(\varphi)(u_0) \;= 0\,, \quad \forall\, \varphi \in \mathcal{G}_n\,. \tag{2.41}$$

To prove the vanishing of all the coefficients, we shall use induction on the *height* of $K \;=\; \{\kappa_1 \leq \ldots \leq \kappa_r\}$; the latter is defined by counting the total number of horizontal derivatives of its largest components:

$$|K| \;=\; \ell_1^r + \cdots + \ell_{p_r}^r\,.$$

We start with the case of height 0, when the identity (2.41) reads

$$\sum_K c_{I,K}\, \gamma_{j_1 k_1}^{i_1}(\varphi)(u_0) \cdots \gamma_{j_r k_r}^{i_r}(\varphi)(u_0) \;= 0\,, \quad \forall\, \varphi \in \mathcal{G}_n\,.$$

Choosing φ in the subgroup $\mathcal{G}_n^{(2)}(0) \subset \mathcal{G}_n$ consisting of the diffeomorphisms whose 2-jet at 0 is of the form

$$J_0^2(\varphi)^i(x) \;=\; x^i \;+\; \frac{1}{2}\sum_{j,k=1}^{n} \xi^i_{jk} x^j x^k, \quad \xi \in \mathbb{R}^{n^3}, \quad \xi^i_{jk} = \xi^i_{kj}\,,$$

and using (2.24), one obtains:

$$\sum_K c_{I,K}\, \xi^{i_1}_{j_1 k_1} \cdots \xi^{i_r}_{j_r k_r} \;=0, \quad \xi^i_{jk} \in \mathbb{R}^{n^3}, \quad \xi^i_{jk} = \xi^i_{kj}\,.$$

It follows that all coefficients $c_{I,K} \;=\; 0$.

Let now $N \in \mathbb{N}$ be the largest height of occurring in (2.41). By varying φ in the subgroup $\mathcal{G}_n^{(N+2)}(0) \subset \mathcal{G}_n$ of all diffeomorphisms whose $(N+2)$-jet at 0 has the form

$$\begin{aligned} J_0^{N+2}(\varphi)^i(x) \quad &= \; x^i + \tfrac{1}{(N+2)!} \textstyle\sum_{j,k,\alpha_1,\ldots,\alpha_{N+2}} \xi^i_{jk\alpha_1\ldots\alpha_N} x^j x^k x^{\alpha_1}\cdots x^{\alpha_N}, \\ \xi^i_{jk\alpha_1\ldots\alpha_N} \quad &\in \mathbb{C}^{n^{N+3}}, \; \xi^i_{jk\alpha_1\ldots\alpha_N} = \xi^i_{kj\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}, \; \forall \text{ permutation } \sigma\,, \end{aligned}$$

and using (2.25) instead of (2.24), one derives as above the vanishing of all coefficients $c_{I,\kappa}$ with $|\kappa| \;=\; N$. This lowers the height in (2.41) and thus completes the induction.

2^0. Due to its multiplicative nature, it suffices to check the compatibility property

$$h(ab) \;=\; \sum_{(h)} h_{(1)}(a)\, h_{(2)}(b)\,, \qquad h \in \mathcal{H}_n\,, \quad a, b \in \mathcal{A}_n \tag{2.42}$$

only on generators. This is precisely what the formulae (2.12), (2.17) and (2.19) verify. $\square$

As a matter of fact, the action of the algebra $\mathcal{H}_n$ on $\mathcal{A}_n$ is not only faithful but even *multi-faithful*, in the sense made clear by the result that follows. In order to state it, we associate to an element $h^1 \otimes \ldots \otimes h^p \in \mathcal{H}_n^{\otimes p}$ a *multi-differential* operator acting on $\mathcal{A}_n$, as follows:

$$\begin{aligned} T(h^1 \otimes \ldots \otimes h^p)\,(a^1 \otimes \ldots \otimes a^p) \;&=\; h^1(a^1)\cdots h^p(a^p)\,, \\ \text{where} \quad h^1, \ldots, h^p \in \mathcal{H}_n \quad &\text{and} \quad a^1, \ldots, a^p \in \mathcal{A}_n\,. \end{aligned} \tag{2.43}$$

Proposition 4. *The linearization* $T : \mathcal{H}_n^{\otimes p} \longrightarrow \mathcal{L}(\mathcal{A}_n^{\otimes p}, \mathcal{A}_n)$ *of the above assignment is injective for each* $p \in \mathbb{N}$.

Proof. For $p = 1$, T gives the standard action of $\mathcal{H}_n$ on $\mathcal{A}_n$, which was just shown to be faithful. To prove that $\operatorname{Ker} T = 0$ for an arbitrary $p \in \mathbb{N}$, assume that

$$H = \sum_\rho h_\rho^1 \otimes \cdots \otimes h_\rho^p \in \operatorname{Ker} T .$$

After fixing a Poincaré-Birkhoff-Witt basis as above, we may uniquely express each h_ρ^j in the form

$$h_\rho^j = \sum_{I_j, K_j} C_{\rho, I_j, K_j} \, \delta_{K_j} \, Z_{I_j} , \quad \text{with} \quad C_{\rho, I_j, K_j} \in \mathbb{C} .$$

Evaluating $T(H)$ on elementary tensors of the form $f_1 U_{\varphi_1}^* \otimes \cdots \otimes f_p U_{\varphi_p}^*$, one obtains

$$\sum_{\rho, I, K} C_{\rho, I_1, K_1} \cdots C_{\rho, I_p, K_p} \, \delta_{K_1} \left(Z_{I_1}(f_1) U_{\varphi_1}^* \right) \cdots \, \delta_{K_p} \left(Z_{I_p}(f_p) U_{\varphi_p}^* \right) = 0 .$$

Evaluating further at a point $u_1 = (x_1, \mathbf{y}_1) \in F\mathbb{R}^n$, and denoting

$$u_2 = \widetilde{\varphi}_1(u_1) , \ldots , u_p = \widetilde{\varphi}_{p-1}(u_{p-1}) ,$$

the above identity gives

$$\begin{aligned} \sum_{\rho, I, K} C_{\rho, I_1, K_1} \cdots C_{\rho, I_p, K_p} \quad &\cdot \quad \gamma_{K_1}(\varphi_1)(u_1) \cdots \gamma_{K_p}(\varphi_p)(u_p) \\ &\cdot \quad Z_{I_1}(f_1)(u_1) \cdots Z_{I_p}(f_p)(u_p) = 0 . \end{aligned}$$

Let us fix points $u_1, \ldots, u_p \in F\mathbb{R}^n$ and then diffeomorphisms $\psi_0, \psi_1, \ldots, \psi_p$, such that

$$u_2 = \widetilde{\psi}_1(u_1) , \ldots , u_p = \widetilde{\psi}_{p-1}(u_{p-1}) .$$

Following a line of reasoning similar to that of the preceeding proof, and iterated with respect to the points $u_1, \ldots, u_p$, we can infer that for each p-tuple of indices of the first kind $(I_1, \ldots, I_p)$ one has

$$\sum_{\rho, K} C_{\rho, I_1, K_1} \cdots C_{\rho, I_p, K_p} \cdot \gamma_{K_1}(\varphi_1)(u_1) \cdots \gamma_{K_p}(\varphi_p)(u_p) = 0 .$$

Similarly, making repeated use of diffeomorphisms of the form

$$\psi_k \circ \varphi \quad \text{with} \quad \varphi \in \mathcal{G}_n^{(N)}(u_k)\,, \quad k = 1, \ldots, p\,,$$

for sufficiently many values of N, we can eventually conclude that for any $(K_1, \ldots, K_p)$

$$\sum_\rho C_{\rho,\, I_1, K_1} \cdots C_{\rho,\, I_p, K_p} \;=\; 0\,.$$

This proves that $H \;=\; 0$. ◻

Remark 5. The coproduct $\Delta : \mathcal{H}_n \to \mathcal{H}_n \otimes \mathcal{H}_n$ and its fundamental properties are completely determined by the action of $\mathcal{H}_n$ on its *standard module algebra* $\mathcal{A}_n$.

Indeed, the compatibility rule (2.42) can be rewritten as

$$T(\Delta h)(a \otimes b) \;=\; h(ab)\,, \quad \forall\, h \in \mathcal{H}_n\,,\; a, b \in \mathcal{A}_n\,. \tag{2.44}$$

By Proposition 4 this completely determines Δ. Furthermore, the coassociativity of Δ becomes a consequence of the associativity of $\mathcal{A}_n$, because after applying T it amounts to the identity

$$h((ab)c) \;=\; h(a(bc))\,, \quad \forall\, h \in \mathcal{H}_n\,,\; a, b \in \mathcal{A}_n\,.$$

Similarly, the counitality is a byproduct of the unitality of $\mathcal{A}$; transported via T it becomes tantamount to

$$h((a\,1) \;=\; h(1\,a) \;=\; h(a)\,, \quad \forall\, h \in \mathcal{H}_n\,,\; a \in \mathcal{A}_n\,.$$

3 Invariant trace and characteristic cochains

An important feature of the standard module algebra is that it carries an *invariant trace*, uniquely determined up to a scaling factor. It is defined as the linear functional $\tau : \mathcal{A}_n \to \mathbb{C}$,

$$\tau\,(f\, U_\varphi^*) \;=\; \begin{cases} \int_{F\mathbb{R}^n} f\, \varpi\,, & \text{if} \quad \varphi = Id\,, \\ \\ 0\,, & \qquad\quad \text{otherwise}\,, \end{cases} \tag{3.1}$$

where ϖ is the volume form on the frame bundle

$$\varpi \;=\; \bigwedge_{k=1}^{n} \theta^k \wedge \bigwedge_{(i,j)} \omega^i_j \qquad \text{(lexicographically ordered)}\,.$$

The invariance property is relative to the *modular character* $\delta : \mathcal{H}_n \to \mathbb{C}$, that extends the *trace* character of $\mathfrak{gl}(n,\mathbb{R})$, and is defined on generators as follows:

$$\delta(Y_i^j) = \delta_i^j, \quad \delta(X_k) = 0, \quad \delta(\delta^i_{jk}) = 0, \qquad i,j,k = 1,\ldots,n\,. \tag{3.2}$$

Proposition 6. 1^0. *For any $a, b \in \mathcal{A}_n$ and $h \in \mathcal{H}_n$ one has*

$$\tau(a\,b) \;=\; \tau(b\,a)\,, \qquad \tau(h(a)) \;=\; \delta(h)\,\tau(a)\,. \tag{3.3}$$

2^0. *For any $h \in \mathcal{H}_n$ and $a, b \in \mathcal{A}_n$ one has*

$$\tau(h(a)\,b) \;=\; \tau\,(a\,\widetilde{S}(h)(b))\,, \tag{3.4}$$

with

$$\widetilde{S}(h) \;=\; \sum_{(h)} \delta(h_{(1)})\,S(h_{(2)})\,, \tag{3.5}$$

satisfying the anti-involutive property

$$\widetilde{S}^2 \;=\; Id\,. \tag{3.6}$$

Proof. 1^0. The trace property is a consequence of the $\mathcal{G}_n$-invariance of the volume form ϖ. In turn, the latter follows from the fact that

$$\widetilde{\varphi}^*(\theta) = \theta \quad \text{and} \quad \widetilde{\varphi}^*(\omega) \;=\; \omega \;+\; \gamma \cdot \theta\,, \quad \text{(cf. (2.16)};$$

therefore,

$$\widetilde{\varphi}^*(\varpi) \;=\; \bigwedge_{k=1}^{n} \theta^k \wedge \bigwedge_{(i,j)} \left(\omega^i_j + \gamma^i_{j\ell}(\varphi)\theta^\ell\right) \;=\; \bigwedge_{k=1}^{n} \theta_k \wedge \bigwedge_{(i,j)} \omega^i_j\,.$$

Passing to the $\mathcal{H}_n$-invariance property in (3.3), it suffices to verify it on generators. Evidently, both sides vanish if $h = \delta^i_{jk}$. On the other hand, its restriction to $\mathbb{R}^n \ltimes \mathfrak{gl}(n,\mathbb{R})$ is just the restatement, at the level of the Lie algebra, of the right semi-invariance property for the left Haar measure on $\mathbb{R}^n \ltimes GL(n,\mathbb{R})$.

2^0. Using the 'product rule' (2.12) for vertical vector fields, in conjunction with the invariance property (3.3) applied to the product of two elements $a, b \in \mathcal{A}_n$, one obtains

$$\tau\,(Y_i^j(a)\,b) \;=\; -\,\tau\,(a\,Y_i^j(b)) \;+\; \delta_i^j\,\tau\,(a\,b)\,, \qquad \forall\, a, b \in \mathcal{A}_n\,. \tag{3.7}$$

On the other hand, for the basic horizontal vector fields, (2.17) and (3.4) give

$$\begin{aligned} \tau\,(X_k(a)\,b) \;&=\; -\,\tau\,(a\,X_k(b)) \;-\; \tau\,(\delta_{jk}^i(a)\,Y_i^j(b)) \\ &=\; -\,\tau\,(a\,X_k(b)) \;+\; \tau\,(a\,\delta_{jk}^i(Y_i^j(b))\,; \end{aligned} \tag{3.8}$$

the second equality uses the 1-cocycle nature of γ_{jk}^i. The same property implies

$$\tau\,(\delta_{jk}^i(a)\,b) \;=\; -\,\tau\,(a\,\delta_{jk}^i(b))\,, \qquad \forall\, a, b \in \mathcal{A}_n\,. \tag{3.9}$$

Thus, the generators of $\mathcal{H}_n$ satisfy an *integration by parts* identity of the form (3.4). Being multiplicative, this rule extends to all elements $h \in \mathcal{H}_n$. Furthermore, since the pairing $(a,b) \mapsto \tau(a\,b)$ is obviously non-degenerate, the integration by parts formula uniquely determines an anti-involutive algebra homomorphism $\widetilde{S} : \mathcal{H}_n \to \mathcal{H}_n$. The equations (3.7)–(3.9) show that $\widetilde{S}$ fulfills (3.5) on generators, and therefore for all $h \in \mathcal{H}_n$. $\square$

We now define an *elementary characteristic cochain* as a cochain $\phi \in C^p(\mathcal{A}_n)$ of the form

$$\phi(a^0, \ldots, a^q) = \tau\left(h^0(a^0) \cdots h^p(a^p))\right),\;\; h^0, ..., h^p \in \mathcal{H}_n,\;\; a^0, ..., a^p \in \mathcal{A}_n.$$

The subspace of $C^p(\mathcal{A}_n)$ spanned by such cochains will be denoted $C^p_\tau(\mathcal{A}_n)$. The collection of all characteristic cochains

$$\mathcal{A}_n^\natural = \bigoplus_{p \geq 0} C^p_\tau(\mathcal{A}_n) \tag{3.10}$$

forms a module over the cyclic category Λ, more precisely a Λ-submodule of the cyclic module $\mathcal{A}_n^\natural$. Indeed, it is very easy to check that the canonical

Λ-operators

$$
\begin{aligned}
(\delta_i\varphi)(a^0,\dots,a^p) &= \varphi(a^0,\dots,a^i\,a^{i+1},\dots,a^p)\,, \quad i=0,1,\dots,p-1\\
(\delta_p\varphi)(a^0,\dots,a^p) &= \varphi(a^p\,a^0,a^1,\dots,a^{p-1})\\
(\sigma_j\varphi)(a^0,\dots,a^p) &= \varphi(a^0,\dots,a^j,1,a^{j+1},\dots,a^p)\,, \quad j=0,1,\dots,p\\
(\tau_p\,\varphi)(a^0,\dots,a^p) &= \varphi(a^p,a^0,\dots,a^{p-1})\,.
\end{aligned}
\tag{3.11}
$$

preserve the property of a cochain of being characteristic.

Note that, because of the $\mathcal{H}_n$-invariance of the trace (3.3),

$$C^0_\tau(\mathcal{A}_n) \,=\, \mathbb{C}\,\tau\,.$$

More generally, using the integration by parts property (3.4), any characteristic cochain can be put in the *standard* form

$$\phi(a^0,\dots,a^p) \,=\, \sum_{i=1}^r \tau\,(a^0\,h_i^1(a^1)\cdots h_i^p(a^p))\,, \quad h_i^j\,\in\mathcal{H}_n\,;$$

moreover, because of the non-degeneracy of the bilinear form $(a,b)\mapsto\tau(a\,b)$, the p-differential operator

$$P(a^1,\dots,a^p) \,=\, \sum_{i=1}^r h_i^1(a^1)\cdots h_i^p(a^p)$$

is uniquely determined. In conjunction with Proposition 4, this means that the linear map $T^\natural : \mathcal{H}_n^{\otimes p} \longrightarrow C^p_\tau(\mathcal{A}_n)$,

$$T^\natural(h^1\otimes\ldots\otimes h^p) \,:=\, T(1\otimes h^1\otimes\ldots\otimes h^p) \tag{3.12}$$

is an isomorphism. We can thus transfer via this isomorphism the cyclic structure of (3.10).

Proposition 7. *The operators*

$$
\begin{aligned}
\delta_0(h^1\otimes\ldots\otimes h^{p-1}) &= 1\otimes h^1\otimes\ldots\otimes h^{p-1}\\
\delta_j(h^1\otimes\ldots\otimes h^{p-1}) &= h^1\otimes\ldots\otimes\Delta h^j\otimes\ldots\otimes h^{p-1}, \quad 1\le j\le p-1\\
\delta_p(h^1\otimes\ldots\otimes h^{p-1}) &= h^1\otimes\ldots\otimes h^{p-1}\otimes 1\\
\sigma_i(h^1\otimes\ldots\otimes h^{p+1}) &= h^1\otimes\ldots\otimes\varepsilon(h^{i+1})\otimes\ldots\otimes h^{p+1}, \quad 0\le i\le p\\
\tau_p(h^1\otimes\ldots\otimes h^p) &= (\Delta^{p-1}\widetilde{S}(h^1))\cdot h^2\otimes\ldots\otimes h^p\otimes 1\,,
\end{aligned}
$$

define a Λ-module structure on $\mathcal{H}_n^\natural = \mathbb{C}\oplus\bigoplus_{p\ge1}\mathcal{H}_n^{\otimes p}$.

Proof. All we need to check is that the isomorphism $T^\natural$ intertwines the above Λ-operators with those of (3.11). This is easy, and not surprising, for the face operators $\{\delta_i\,,\,0\le i\le p-1\}$ and the degeneracy operators $\{\sigma_j\,,\,0\le j\le p\}$, because it amounts to the natural equivalence between the cosimplicial structure of the *coalgebra* $\mathcal{H}_n$ and that of the algebra $\mathcal{A}_n$. By contrast, the full Hopf algebraic structure of $\mathcal{H}_n$ is reflected in the expression of the cyclic operator, which is obtained as follows:

$$\begin{aligned}
\tau_p(T^\natural(h^1\otimes\ldots\otimes h^p))(a^0,\ldots,a^p) &= T^\natural(h^1\otimes\ldots\otimes h^p)(a^p,a^0,\ldots,a^{p-1}) \\
=\tau(a^p\,h^1(a^0)\cdots h^p(a^{p-1})) &= \tau(h^1(a^0)\cdots h^p(a^{p-1})\,a^p) \\
&= \tau(a^0\,\widetilde{S}(h^1)\left((h^1)(a^0)\cdots h^p(a^{p-1})\,a^p\right)).
\end{aligned}$$

In the last two lines we have used the trace property and the integration by parts formula (3.5). To arrive at the expression in the statement, it remains to employ the compatibility of $\mathcal{H}_n$-action with the algebra product. ☐

4 Cyclic cohomology for Hopf algebras

We now consider an arbitrary Hopf algebra $\mathcal{H}$ over a field k containing $\mathbb{Q}$, with unit $\eta: k\to\mathcal{H}$, counit $\varepsilon:\mathcal{H}\to k$ and antipode $S:\mathcal{H}\to\mathcal{H}$. As the discussion of $\mathcal{H}_n$ clearly indicates, the initial datum should also include a 'modular' component, playing the role of the modular function of a Lie group. To be consistent with the intrinsic duality of the Hopf algebra framework, this modular datum should also be self-dual. One is thus led to postulate the existence of a *modular pair* (δ,σ), consisting of a character $\delta\in\mathcal{H}^*$,

$$\delta(ab)=\delta(a)\delta(b)\,,\quad\forall\,a,b\in\mathcal{H},\qquad\delta(1)=1\,,$$

and a group-like element $\sigma\in\mathcal{H}$,

$$\Delta(\sigma)=\sigma\otimes\sigma\,,\qquad\varepsilon(\sigma)=1,$$

related by the condition

$$\delta(\sigma)=1. \tag{4.1}$$

The convolution of the antipode by the character δ gives rise to the *twisted antipode* $\widetilde{S} = S_\delta : \mathcal{H} \to \mathcal{H}$,

$$\widetilde{S}(h) = \sum_{(h)} \delta(h_{(1)})\ S(h_{(2)}) \quad , \quad h \in \mathcal{H}.$$

The basic properties of the ordinary antipode are inherited by the twisted antipode. In particular, we shall freely use the following facts that can be easily checked:

$$\begin{aligned} \widetilde{S}(h^1h^2) &= \widetilde{S}(h^2)\widetilde{S}(h^1)\,, \qquad \forall h^1, h^2 \in \mathcal{H}\,, \qquad \widetilde{S}(1) = 1\,; \\ \Delta\widetilde{S}(h) &= \sum_{(h)} S(h_{(2)}) \otimes \widetilde{S}(h_{(1)})\,, \quad \forall h \in \mathcal{H}\,; \\ \varepsilon \circ \widetilde{S} &= \delta, \qquad \delta \circ \widetilde{S} = \varepsilon. \end{aligned}$$

Since the concrete Hopf algebras often arise as generalized symmetries acting on algebras, it is reasonable to take into account this dynamical aspect in the process of defining the Hopf-cyclic cohomology.

Definition 8. *Let $\mathcal{A}$ be an $\mathcal{H}$-module algebra. A linear form $\tau : \mathcal{A} \to k$ will be called a σ-*trace *if*

$$\tau(ab) = \tau(b\sigma(a))\,, \qquad \forall a, b \in A. \tag{4.2}$$

*A σ-trace τ will be called $(\mathcal{H}, \delta)$-*invariant *if*

$$\tau(h(a)) \;=\; \delta(h)\,\tau(a)\,, \qquad \forall a \in A,\, h \in \mathcal{H}\,. \tag{4.3}$$

We remark that the invariance condition (4.3) is equivalent to the apparently stronger 'integration by parts' property

$$\tau(h(a)\,b) \;=\; \tau(a\,\widetilde{S}(h)(b))\,, \qquad \forall a \in A,\, h \in \mathcal{H}\,. \tag{4.4}$$

Indeed, the former is obtained from the latter by specializing $b = 1$. Conversely, one has (omitting the summation sign in the Sweedler notation)

$$\begin{aligned} \tau(a\,\widetilde{S}(h)(b)) &= \delta(h_{(1)})\,\tau(a\,S(h_{(2)}(b)) \;=\; \tau\left(h_{(1)}(a\,S(h_{(2)})(b))\right) \\ &= \tau\left(h_{(1)}(a)\,h_{(2)}(S(h_{(3)})(b))\right) = \tau(h_{(1)}(a)\,\varepsilon(h_{(2)})\,b) = \tau(h(a)\,b)\,. \end{aligned}$$

Motivated by the discussion in the previous section, we shall enforce the existence of a natural characteristic homomorphism, as a guiding principle for the definition of Hopf-cyclic cohomology. Specifically, the definition should satisfy the following:

Ansatz. *Let $(\mathcal{A}, \tau)$ be an an $\mathcal{H}$-module algebra with $(\mathcal{H}, \delta)$-invariant σ-trace. Then the assignment*

$$h^1 \otimes \ldots \otimes h^n \longmapsto \chi_\tau(h^1 \otimes \ldots \otimes h^n) \in C^n(\mathcal{A}), \quad \forall\, h^1, \ldots, h^n \in \mathcal{H},$$
$$\chi_\tau(h^1 \otimes \ldots \otimes h^n)(a^0, \ldots, a^n) = \tau(a^0 h^1(a^1) \ldots h^n(a^n)), \quad \forall\, a^0, \ldots, a^n \in \mathcal{A},$$

defines a canonical map $\chi^*_\tau : HC^*(\mathcal{H}) \to HC^*(\mathcal{A})$.

If $\mathcal{H}$ happens to admit a multi-faithful action on a module algebra, then the Ansatz would dictate, as in Proposition 7, the following cyclic structure on

$$\begin{aligned}
\mathcal{H}^\natural_{(\delta,\sigma)} \;&=\; \mathbb{C} \oplus \bigoplus_{n \geq 1} \mathcal{H}^{\otimes n} \;: \qquad\qquad (4.5)\\
\delta_0(h^1 \otimes \ldots \otimes h^{n-1}) \;&=\; 1 \otimes h^1 \otimes \ldots \otimes h^{n-1},\\
\delta_j(h^1 \otimes \ldots \otimes h^{n-1}) \;&=\; h^1 \otimes \ldots \otimes \Delta h^j \otimes \ldots \otimes h^{n-1}, \qquad 1 \leq j \leq n-1\\
\delta_n(h^1 \otimes \ldots \otimes h^{n-1}) \;&=\; h^1 \otimes \ldots \otimes h^{n-1} \otimes \sigma,\\
\sigma_i(h^1 \otimes \ldots \otimes h^{n+1}) \;&=\; h^1 \otimes \ldots \otimes \varepsilon(h^{i+1}) \otimes \ldots \otimes h^{n+1}, \qquad 0 \leq i \leq n\,,\\
\tau_n(h^1 \otimes \ldots \otimes h^n) \;&=\; (\Delta^{p-1} \widetilde{S}(h^1)) \cdot h^2 \otimes \ldots \otimes h^n \otimes \sigma.
\end{aligned}$$

The insertion of σ in δ_n and τ_n accounts for the passage from an ordinary trace to a twisted trace. Note that

$$\tau_1{}^2(h) \;=\; \widetilde{S}(\widetilde{S}(h)\,\sigma)\sigma \;=\; \sigma^{-1}\, \widetilde{S}^2(h)\,\sigma, \quad \forall\, h \in \mathcal{H}\,,$$

therefore the fact that τ_1 is cyclic is equivalent to the identity

$$\widetilde{S}^2 \;=\; \mathrm{Ad}\,\sigma\,. \qquad (4.6)$$

Definition 9. *A modular pair (δ, σ) is said to be* in involution *if the δ-twisted antipode $\widetilde{S} = S_\delta$ satisfies the condition (4.6).*

The pleasant surprise is that, besides (4.6), there are no more conditions needed for the existence of a cyclic structure based on the above operators. The key calculation to prove this makes the object of the following lemma.

Lemma 10. *Given a Hopf algebra $\mathcal{H}$ endowed with a modular pair (δ,σ) one has, for any $h^1,\ldots,h^n\in\mathcal{H}$,*

$$\tau_n^{n+1}(h^1\otimes\ldots\otimes h^n) \,=\, \sigma^{-1}\,\widetilde{S}^2(h^1)\,\sigma\otimes\ldots\otimes\sigma^{-1}\,\widetilde{S}^2(h^n)\,\sigma.$$

Proof. Omitting the summation sign, we write τ_n in the form

$$\tau_n(h^1\otimes h^2\otimes\ldots\otimes h^n) = S(h^1_{(n)})h^2\otimes S(h^1_{(n-1)})h^3\otimes\ldots\otimes\widetilde{S}(h^1_{(1)})\sigma.$$

Upon iterating once, one obtains $\quad \tau_n^2(h^1\otimes\ldots\otimes h^n) \,=$

$$\begin{aligned}
&= S(S(h^1_{(n)})_{(n)}h^2_{(n)}))S(h^1_{(n-1)})h^3\otimes\cdots\otimes\widetilde{S}(S(h^1_{(n)})_{(1)}h^2_{(1)})\sigma\\
&= S(h^2_{(n)})S(S(h^1_{(n)(1)}))S(h^1_{(n-1)})h^3\otimes\ldots\otimes\ \widetilde{S}(h^2_{(1)})\widetilde{S}(S(h^1_{(n)(n)}))\sigma\\
&= S(h^2_{(n)})S(h^1_{(n-1)}S(h^1_{(n)}))h^3\otimes\ldots\otimes\widetilde{S}(h^2_{(1)})\widetilde{S}(S(h^1_{(2n-1)}))\sigma;
\end{aligned}$$

successive use of the basic identities for the counit and the antipode leads to

$$\begin{aligned}
&= S(h^2_{(n)})h^3\ldots\otimes S(h^2_{(2)})\ S(S(h^1_{(2)}))\widetilde{S}(h^1_{(1)})\sigma\otimes\widetilde{S}(h^2_{(1)})\widetilde{S}(S(h^1_{(3)}))\sigma\\
&= \Delta^{(n-1)}\widetilde{S}(h^2)\cdot h^3\otimes\ldots\otimes\sigma\otimes\widetilde{S}^2(h^1)\,\sigma\,;
\end{aligned}$$

for the last equality we have used, with $k=h^1_{(1)}$,

$$\begin{aligned}
S(S(k_{(2)}))\widetilde{S}(k_{(1)}) &= S(S(k_{(3)}))\delta(k_{(1)})S(k_{(2)}) = \delta(k_{(1)})S\left(k_{(2)}S(k_{(3)})\right)\\
&= \delta(k_{(1)})S(\varepsilon(k_{(2)})1) = \delta(k_{(1)})\varepsilon(k_{(2)})\\
&= \delta\left(k_{(1)}\varepsilon(k_{(2)})\right) = \delta(k)\,.
\end{aligned}$$

By induction, one obtains for any $j=1,\ldots,n$,

$$\tau_n^j(h^1\otimes\ldots\otimes h^n) = \Delta^{n-1}\widetilde{S}(h^j)\cdot h^{j+1}\otimes\ldots\otimes h^n\otimes\sigma\otimes\widetilde{S}^2(h^1)\sigma\otimes\ldots\otimes\widetilde{S}^2(h^{j-1})\sigma.$$

A last iteration finally gives

$$\tau_n^{n+1}(h^1\otimes\ldots\otimes h^n) = \Delta^{n-1}\widetilde{S}(\sigma)\cdot\widetilde{S}^2(h^1)\sigma\otimes\ldots\otimes\widetilde{S}^2(h^n)\sigma.$$

$\square$

Theorem 11. *Let $\mathcal{H}$ be a Hopf algebra endowed with a modular pair (δ, σ). Then $\mathcal{H}^{\natural}_{(\delta,\sigma)} = \mathbb{C} \oplus \bigoplus_{n \geq 1} \mathcal{H}^{\otimes^n}$ equipped with the operators (4.5) is a Λ-module if and only if the modular pair (δ, σ) is in involution.*

Proof. One needs to check that the operators defined in (4.5) satisfy the following relations, that give the standard presentation of the cyclic category $\Lambda = \Delta C$ ([5], [25]):

$$\delta_j \delta_i = \delta_i \delta_{j-1}, \; i < j, \qquad \sigma_j \sigma_i = \sigma_i \sigma_{j+1}, \; i \leq j \tag{4.7}$$

$$\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & i < j \\ 1_n & \text{if } i = j \text{ or } i = j+1 \\ \delta_{i-1} \sigma_j & i > j+1; \end{cases} \tag{4.8}$$

$$\begin{aligned}
\tau_n \delta_i = \delta_{i-1} \tau_{n-1}, \qquad & 1 \leq i \leq n, \quad \tau_n \delta_0 = \delta_n && (4.9)\\
\tau_n \sigma_i = \sigma_{i-1} \tau_{n+1}, \qquad & 1 \leq i \leq n, \quad \tau_n \sigma_0 = \sigma_n \tau^2_{n+1} && (4.10)\\
\tau_n^{n+1} \;\; = \;\; & 1_n \, . && (4.11)
\end{aligned}$$

The cyclicity condition (4.11) holds by virtue of Lemma 10 and the hypothesis that the modular pair (δ, σ) is in involution.
The verification of the remaining relations is straightforward. As typical illustrations, we check below some of the compatibility relations. For $i = 1$ in (4.9), one has

$$\begin{aligned}
& \tau_n \delta_1 (h^1 \otimes \ldots \otimes h^{n-1}) \;=\; \tau_n (h^1_{(1)} \otimes h^1_{(2)} \otimes h^2 \otimes \ldots \otimes h^{n-1}) \\
= \;& \Delta^{n-1} \widetilde{S}(h^1_{(1)}) \cdot h^1_{(2)} \otimes h^2 \otimes \ldots \otimes h^{n-1} \otimes \sigma \\
= \;& S(h^1_{(1)(n)}) h^1_{(2)} \otimes S(h^1_{(1)(n-1)}) h^2 \otimes \ldots \otimes S(h^1_{(1)(2)}) h^{n-1} \otimes \widetilde{S}(h^1_{(1)(1)}) \sigma \\
= \;& \varepsilon(h^1_{(n)}) 1 \otimes S(h^1_{(n-1)}) h^2 \otimes \ldots \otimes S(h^1_{(1)}) h^{n-1} \otimes \widetilde{S}(h^1_{(1)}) \sigma \\
= \;& 1 \otimes S(h^1_{(n-1)}) h^2 \otimes \ldots \otimes S(h^1_{(1)}) h^{n-1} \otimes \widetilde{S}(h^1_{(1)}) \sigma \\
= \;& \delta_0 \tau_{n-1} (h^1 \otimes \ldots \otimes h^{n-1}),
\end{aligned}$$

while for $i = 0$ the verification is even easier:

$$\begin{aligned}
& \tau_n \delta_0 (h^1 \otimes \ldots \otimes h^{n-1}) \;=\; \tau_n (1 \otimes h^1 \otimes \ldots \otimes h^{n-1}) = \\
= \;& \Delta^{n-1} \widetilde{S}(1) \cdot h^1 \otimes \ldots \otimes h^{n-1} \otimes \sigma = h^1 \otimes \ldots \otimes h^{n-1} \otimes \sigma \\
= \;& \delta_n (h^1 \otimes \ldots \otimes h^{n-1}).
\end{aligned}$$

Passing now to the relations (4.10), when $i = 0$ the left hand side gives

$$\begin{aligned}
&\tau_n\sigma_0(h^1\otimes\ldots\otimes h^{n+1}) = \varepsilon(h^1)\tau_n(h^2\otimes\ldots\otimes h^{n+1}) = \\
&= \varepsilon(h^1)\sum S(h^2_{(n)})h^3\otimes\ldots\otimes S(h^2_{(2)})h^{n+1}\otimes\widetilde{S}(h^2_{(1)})\sigma,
\end{aligned}$$

which is reproduced by the right hand side as follows:

$$\begin{aligned}
&\sigma_n\tau^2_{n+1}(h^1\otimes\ldots\otimes h^{n+1}) = \\
= \quad &\sigma_n\left(\sum S(h^2_{(n+1)})h^3\otimes\ldots\otimes S(h^2_{(2)})\sigma\otimes\widetilde{S}(h^2_{(1)})\widetilde{S}^2(h^1)\sigma\right) \\
= \quad &\sum\varepsilon(\widetilde{S}(h^2_{(1)})\widetilde{S}^2(h^1)\sigma)S(h^2_{(n+1)})h^3\otimes\ldots\otimes S(h^2_{(2)})\sigma \\
= \quad &\varepsilon(\sigma^{-1}\widetilde{S}^2(h^1)\sigma)\sum\delta(h^2_{(1)})S(h^2_{(n+1)})h^3\otimes\ldots\otimes S(h^2_{(2)})\sigma \\
= \quad &\varepsilon(h^1)S(h^2_{(n)})h^3\otimes\ldots\otimes S(h^2_{(2)})h^{n+1}\otimes\widetilde{S}(h^2_{(1)})\sigma.
\end{aligned}$$

For $i = 1$ the left hand side is

$$\begin{aligned}
&\tau_n\sigma_1(h^1\otimes\ldots\otimes h^{n+1}) = \varepsilon(h^2)\tau_n(h^1\otimes h^3\otimes\ldots\otimes h^{n+1}) \\
= \quad &\varepsilon(h^2)\cdot\Delta^{n-1}\widetilde{S}(h^1)\cdot h^3\otimes\ldots\otimes h^{n+1}\otimes\sigma,
\end{aligned}$$

and the right hand side amounts to the same:

$$\begin{aligned}
&\sigma_0\tau_{n+1}(h^1\otimes\ldots\otimes h^{n+1}) = \\
&\sum\sigma_0(S(h^1_{(n+1)})h^2\otimes\ldots\otimes S(h^1_{(2)})h^{n+1}\otimes\widetilde{S}(h^1_{(1)}))\sigma \\
= \quad &\sum\varepsilon(h^2)\cdot\varepsilon(h^1_{(n+1)})\cdot S(h^1_{(n)})h^3\otimes\ldots\otimes S(h^1_{(2)})h^{n+1}\otimes\widetilde{S}(h^1_{(1)})\sigma \\
= \quad &\sum\varepsilon(h^2)\cdot S(h^1_{(n-1)})h^3\otimes\ldots\otimes S(h^1_{(2)})h^{n+1}\otimes\widetilde{S}(h^1_{(1)})\sigma.
\end{aligned}$$

Similar calculations hold for $i = 2,\ldots n$. $\square$

For completeness, we record below the normalized bi-complex

$$(CC^{*,*}(\mathcal{H};\delta,\sigma),\, b,\, B)$$

that computes the Hopf-cyclic cohomology of $\mathcal{H}$ with respect to a modular pair in involution (δ,σ):

$$CC^{p,q}(\mathcal{H};\delta,\sigma) \;=\; \begin{cases}\bar{C}^{q-p}(\mathcal{H};\delta,\sigma)\,, & q\geq p\,,\\ 0\,, & q<p\,,\end{cases}$$

where

$$\bar{C}^n(\mathcal{H};\delta,\sigma) \;=\; \begin{cases} \bigcap_{i=0}^{n-1} \mathrm{Ker}\ \sigma_i\,, & n \geq 1\,, \\ \mathbb{C}\,, & n = 0\,; \end{cases}$$

the operator

$$b : \bar{C}^{n-1}(\mathcal{H};\delta,\sigma) \to \bar{C}^n(\mathcal{H};\delta,\sigma), \qquad b = \sum_{i=0}^{n} (-1)^i \delta_i$$

has the form $\qquad b(\mathbb{C}) \;=\; 0 \;$ for $\; n = 0\,,$

$$\begin{aligned} b\,(h^1 \otimes \ldots \otimes h^{n-1}) \;&=\; 1 \otimes h^1 \otimes \ldots \otimes h^{n-1} \\ &+\; \sum_{j=1}^{n-1} (-1)^j \sum_{(h_j)} h^1 \otimes \ldots \otimes h^j_{(1)} \otimes h^j_{(2)} \otimes \ldots \otimes h^{n-1} \\ &+\; (-1)^n h^1 \otimes \ldots \otimes h^{n-1} \otimes 1\,, \end{aligned}$$

while the B-operator $B \,:\, \bar{C}^{n+1}(\mathcal{H};\delta,\sigma) \;\to\; \bar{C}^n(\mathcal{H};\delta,\sigma)$ is defined by the formula

$$B = A \circ B_0\,, \quad n \geq 0\,,$$

where

$$B_0(h^1 \otimes \ldots \otimes h^{n+1}) \;=\; \begin{cases} (\Delta^{n-1}\widetilde{S}(h^1)) \cdot h^2 \otimes \ldots \otimes h^{n+1}\,, & n \geq 1\,, \\ \delta(h^1)\,, & \text{for} \;\; n = 0 \end{cases}$$

and

$$A = 1 + \lambda_n + \cdots + \lambda_n^n\,, \qquad \text{with} \qquad \lambda_n = (-1)^n \tau_n\,.$$

The groups $\{HC^n(\mathcal{H};\delta,\sigma)\}_{n\in\mathbb{N}}$ are computed from the first quadrant total complex $(TC^*(\mathcal{H};\delta,\sigma),\, b + B)$,

$$TC^n(\mathcal{H};\delta,\sigma) \;=\; \sum_{k=0}^{n} CC^{k,n-k}(\mathcal{H};\delta,\sigma)\,,$$

and the periodic groups $\{HP^i(\mathcal{H};\delta,\sigma)\}_{i\in\mathbb{Z}/2}$ are computed from the full total complex $(TP^*(\mathcal{H};\delta,\sigma),\, b + B)$,

$$TP^i(\mathcal{H};\delta,\sigma) \;=\; \sum_{k\in\mathbb{Z}} CC^{k,i-k}(\mathcal{H};\delta,\sigma)\,.$$

A class of useful examples is that of universal enveloping algebras of Lie algebras. If $\mathcal{H} = \mathfrak{A}(\mathfrak{g})$, with $\mathfrak{g}$ a Lie algebra over k, together with a character $\delta : \mathfrak{g} \to k$, then the periodic Hopf-cyclic cohomology of the $(\mathfrak{A}(\mathfrak{g}); \delta, 1)$ reduces to the Lie algebra homology with coefficients in the 1-dimensional $\mathfrak{g}$-module k_δ associated to the character (cf. [9, §7, Prop. 7], see also Thm. 15 below):

$$HP^* \left(\mathfrak{A}(\mathfrak{g}); \delta, 1\right) \simeq \sum_{i \equiv * (2)}^{\oplus} H_i \left(\mathfrak{g}, k_\delta\right).$$

Dually, if $\mathcal{H} = \mathcal{H}(G)$ is the Hopf algebra of regular (i.e. polynomial) functions on a unipotent affine algebraic group G with Lie algebra $\mathfrak{g}$, then its periodic cyclic cohomology with respect to the trivial modular pair $(\varepsilon, 1)$ is isomorphic to the Lie algebra cohomology:

$$HP^* \left(\mathcal{H}(G); \varepsilon, 1\right) \simeq \sum_{i \equiv * (2)}^{\oplus} H^i \left(\mathfrak{g}, k\right).$$

In these instances $\sigma = 1$. On the other hand, the class of quasitriangular Hopf algebras called *ribbon algebras*, and in particular the q-deformations $\mathfrak{A}_q(\mathfrak{g})$ of the classical Lie algebras, provide examples of dual pairs with $\sigma \neq 1$ and $\delta = \varepsilon$ (see [11, §4]).

Our prototype of a Hopf algebra with modular pair in involution was of course $(\mathcal{H}_n; \delta, 1)$. In [9, §7, Theorem 11] we established a canonical isomorphism

$$\kappa_n^* : \sum_{i \equiv * (2)}^{\oplus} H^i(\mathfrak{a}_n; \mathbb{C}) \xrightarrow{\simeq} HP^* \left(\mathcal{H}_n; \delta, 1\right), \tag{4.12}$$

between the Gelfand-Fuks cohomology of the Lie algebra $\mathfrak{a}_n$ of formal vector fields on $\mathbb{R}^n$ and the periodic Hopf-cyclic cohomology of $(\mathcal{H}_n; \delta, 1)$. Moreover, the isomorphism κ_n^* is implemented at the cochain level by a homomorphism constructed out of a (fixed but arbitrary) torsion-free connection.

A similar statement takes place at the *relative* level, and it actually plays a key role in understanding the Chern character of the hypoelliptic signature operator, namely the existence of a canonical isomorphim

$$\kappa_{n,\mathrm{SO}(n)}^* : \sum_{i \equiv * (2)}^{\oplus} H^i(\mathfrak{a}_n, \mathrm{SO}(n); \mathbb{C}) \xrightarrow{\simeq} HP^* \left(\mathcal{H}_n, \mathrm{SO}(n); \delta, 1\right) \tag{4.13}$$

While the meaning of the relative Hopf cyclic group $HP^*(\mathcal{H}_n, \mathrm{SO}(n); \delta, 1)$ happened to be quite clear in that particular context, it was not so in the case of non-compact isotropy, for instance for the Lorentz group $\mathrm{SO}(n-1, 1)$. The treatment of the general case makes the object of the next section.

5 Relative Hopf cyclic cohomology

Let $\mathcal{H}$ be an arbitrary Hopf algebra over a field F containing $\mathbb{Q}$, with unit $\eta : F \to \mathcal{H}$, counit $\varepsilon : \mathcal{H} \to F$ and antipode $S : \mathcal{H} \to \mathcal{H}$, and let $\mathcal{K}$ be a Hopf subalgebra of $\mathcal{H}$. We consider the tensor product over $\mathcal{K}$

$$\mathcal{C} \,=\, \mathcal{C}(\mathcal{H}, \mathcal{K}) \,:=\, \mathcal{H} \otimes_{\mathcal{K}} F \,, \tag{5.1}$$

where $\mathcal{K}$ acts on $\mathcal{H}$ by right multiplication and on F by the counit. It is a left $\mathcal{H}$-module, which can be identified with the quotient module $\mathcal{H}/\mathcal{H}\mathcal{K}^+$, $\mathcal{K}^+ = \mathrm{Ker}\, \varepsilon|\mathcal{K}$, via the isomorphism induced by the map

$$h \in \mathcal{H} \longmapsto \dot{h} = h \otimes_{\mathcal{K}} 1 \in \mathcal{H} \otimes_{\mathcal{K}} F \,. \tag{5.2}$$

Moreover, $\mathcal{C} = \mathcal{C}(\mathcal{H}, \mathcal{K})$ is an $\mathcal{H}$-module coalgebra. Indeed, its coalgebra structure is given by the coproduct

$$\Delta_{\mathcal{C}} \left(h \otimes_{\mathcal{K}} 1\right) \,=\, \left(h_{(1)} \otimes_{\mathcal{K}} 1\right) \otimes \left(h_{(2)} \otimes_{\mathcal{K}} 1\right) , \tag{5.3}$$

inherited from that on $\mathcal{H}$, $\Delta h = h_{(1)} \otimes h_{(2)}$ and is compatible with the action of $\mathcal{H}$ on $\mathcal{C}$ by left multiplication:

$$\Delta_{\mathcal{C}} \left(h' \cdot h \otimes_{\mathcal{K}} 1\right) \,=\, \Delta h' \cdot \Delta_{\mathcal{C}} \left(h \otimes_{\mathcal{K}} 1\right) ;$$

similarly, there is an inherited counit

$$\varepsilon_{\mathcal{C}} \left(h \otimes_{\mathcal{K}} 1\right) \,=\, \varepsilon(h) \,, \quad \forall\, h \in \mathcal{H} \,, \tag{5.4}$$

that satisfies

$$\varepsilon_{\mathcal{C}} \left(h' \cdot h \otimes_{\mathcal{K}} 1\right) \,=\, \varepsilon(h)\, \varepsilon_{\mathcal{C}} \left(h \otimes_{\mathcal{K}} 1\right) .$$

Let us now fix a right $\mathcal{H}$-module M that is also a left $\mathcal{H}$-comodule,

$${}_M\Delta(m) \,=\, m_{(-1)} \otimes m_{(0)} \,\in \mathcal{H} \otimes M \,,$$

and assume that it is a *stable anti-Yetter-Drinfeld module*, cf. [19], that is

$$\begin{aligned} m_{(0)}\, m_{(-1)} &= m\,; & (5.5)\\ {}_M\Delta(m\,h) &= S(h_{(3)})\, m_{(-1)}\, h_{(1)} \otimes m_{(0)}\, h_{(2)}\,. & (5.6)\end{aligned}$$

The following result provides, in the extended framework of [20], a direct generalization to the relative case with coefficients of the definition (4.5) suggested by the Ansatz of §4.

Theorem 12. *Let $C^*(\mathcal{H},\mathcal{K};M) = \{C^n(\mathcal{H},\mathcal{K};M) := M \otimes_{\mathcal{K}} \mathcal{C}^{\otimes n}\}_{n\geq 0}$, where $\mathcal{C} = \mathcal{H} \otimes_{\mathcal{K}} F$ and the tensor product over $\mathcal{K}$ is with respect to the diagonal action on $\mathcal{C}^{\otimes n}$. The following operators are well-defined and endow $C^*(\mathcal{H},\mathcal{K};M)$ with a cyclic structure:*

$$\begin{aligned}
\delta_0(m \otimes_{\mathcal{K}} c^1 \otimes \ldots \otimes c^{n-1}) &= m \otimes_{\mathcal{K}} \dot{1} \otimes c^1 \otimes \ldots \otimes \ldots \otimes c^{n-1}, \qquad (5.7)\\
\delta_i(m \otimes_{\mathcal{K}} c^1 \otimes \ldots \otimes c^{n-1}) &= m \otimes_{\mathcal{K}} c^1 \otimes \ldots \otimes c^i_{(1)} \otimes c^i_{(2)} \otimes \ldots \otimes c^{n-1},\\
&\qquad \forall \quad 1 \leq i \leq n-1\,;\\
\delta_n(m \otimes_{\mathcal{K}} c^1 \otimes \ldots \otimes c^{n-1}) &= m_{(0)} \otimes_{\mathcal{K}} c^1 \otimes \ldots \otimes c^{n-1} \otimes \dot{m}_{(-1)}\,;\\
\sigma_i(m \otimes_{\mathcal{K}} c^1 \otimes \ldots \otimes c^{n+1}) &= m \otimes_{\mathcal{K}} c^1 \otimes \ldots \otimes \varepsilon(c^{i+1}) \otimes \ldots \otimes c^{n+1},\\
&\qquad \forall \quad 0 \leq i \leq n\,;\\
\tau_n(m \otimes_{\mathcal{K}} \dot{h}^1 \otimes c^2 \otimes \ldots \otimes c^n) &= m_{(0)} h^1_{(1)} \otimes_{\mathcal{K}} S(h^1_{(2)}) \cdot (c^2 \otimes \ldots \otimes c^n \otimes \dot{m}_{(-1)}).
\end{aligned}$$

Proof. One can directly check that the above operators are defined in a consistent fashion and satisfy the defining relations of the cyclic category. This is however redundant, because they can simply be obtained by transport of structure. Indeed, since $\mathcal{C}$ is an $\mathcal{H}$-module coalgebra, one can apply [20, Theorem 2.1] that provides a cyclic structure on the collection of spaces

$$\tilde{C}^*(\mathcal{H},\mathcal{K};M) = \{\tilde{C}^n(\mathcal{H},\mathcal{K};M) := M \otimes_{\mathcal{H}} \mathcal{C}^{\otimes n+1}\}_{n\geq 0}\,, \qquad (5.8)$$

where $\mathcal{H}$ acts diagonally on $\mathcal{C}^{\otimes n+1}$, as follows:

$$\begin{aligned}
\tilde{\delta}_i(m \otimes_{\mathcal{H}} c^0 \otimes \ldots \otimes c^{n-1}) &= m \otimes_{\mathcal{H}} c^0 \otimes \ldots \otimes c^i_{(1)} \otimes c^i_{(2)} \otimes \ldots \otimes c^{n-1},\\
&\qquad \forall \quad 0 \leq i \leq n-1\,;\\
\tilde{\delta}_n(m \otimes_{\mathcal{H}} c^0 \otimes \ldots \otimes c^{n-1}) &= m_{(0)} \otimes_{\mathcal{H}} c^0_{(2)} \otimes c^1 \otimes \ldots \otimes c^{n-1} \otimes m_{(-1)} c^0_{(1)};\\
\tilde{\sigma}_i(m \otimes_{\mathcal{H}} c^0 \otimes \ldots \otimes c^{n+1}) &= m \otimes_{\mathcal{H}} c^0 \otimes \ldots \otimes \varepsilon_{\mathcal{C}}(c^{i+1}) \otimes \ldots \otimes c^{n+1},\\
&\qquad \forall \quad 0 \leq i \leq n\,;\\
\tilde{\tau}_n(m \otimes_{\mathcal{H}} c^0 \otimes \ldots \otimes c^n) &= m_{(0)} \otimes_{\mathcal{H}} c^1 \otimes \ldots \otimes c^n \otimes m_{(-1)} c^0.
\end{aligned}$$

To recast the cyclic module (5.8) in the form that appears in the statement we shall make use of the 'transfer' isomorphism

$$\Psi^* : C^*(\mathcal{H}, \mathcal{K}; M) \longrightarrow \tilde{C}^*(\mathcal{H}, \mathcal{K}; M)$$

whose n-th component is defined by the simple formula

$$\Psi^n(m \otimes_{\mathcal{K}} c^1 \otimes \ldots \otimes c^n) \, = \, m \otimes_{\mathcal{H}} \dot{1} \otimes c^1 \otimes \ldots \otimes c^n \,, \tag{5.9}$$

which is obviously consistent.

We claim that the expression

$$\Phi^n(m \otimes_{\mathcal{H}} \dot{h}^0 \otimes c^1 \otimes \ldots \otimes c^n) \, = \, m \, h^0_{(1)} \otimes_{\mathcal{K}} S(h^0_{(2)}) \cdot (c^1 \otimes \ldots \otimes c^n), \tag{5.10}$$

is also well-defined and gives the inverse operator to Ψ^n. Note first that the right hand side only depends on the class $c^0 = \dot{h}^0 \in \mathcal{C}$; indeed, for any $g \in \mathcal{H}$ and $k \in \mathcal{K}^+$,

$$\begin{aligned} m \, g^0_{(1)} k_{(1)} \otimes_{\mathcal{K}} S(k_{(2)}) S(g^0_{(2)}) \cdot (c^1 \otimes \ldots \otimes c^n) \;&=\\ m \, g^0_{(1)} k_{(1)} S(k_{(2)}) \otimes_{\mathcal{K}} S(g^0_{(2)}) \cdot (c^1 \otimes \ldots \otimes c^n) \;&=\\ m \, g^0_{(1)} \varepsilon(k) \otimes_{\mathcal{K}} S(g^0_{(2)}) \cdot (c^1 \otimes \ldots \otimes c^n) \;&=\; 0\,. \end{aligned}$$

Let us now check the consistency of the definition (5.10). On replacing the elementary tensor $\dot{h}^0 \otimes c^1 \otimes \ldots \otimes c^n \, \in \mathcal{C}^{\otimes n+1}$ in the left hand side by

$$h \cdot (\dot{h}^0 \otimes c^1 \otimes \ldots \otimes c^n) \, = \, \dot{\widehat{h_{(1)} h^0}} \otimes h_{(2)} \cdot (c^1 \otimes \ldots \otimes c^n) \,, \qquad h \in \mathcal{H} \,,$$

one obtains in the right hand side

$$\begin{aligned} m \, h_{(1)} h^0_{(1)} \otimes_{\mathcal{K}} S(h^0_{(2)}) S(h_{(2)}) \, h_{(3)} \cdot (c^1 \otimes \ldots \otimes c^n) \;&=\\ m \, h_{(1)} h^0_{(1)} \otimes_{\mathcal{K}} S(h^0_{(2)}) \varepsilon(h_{(2)}) \cdot (c^1 \otimes \ldots \otimes c^n) \;&=\\ m \, h_{(1)} \varepsilon(h_{(2)}) h^0_{(1)} \otimes_{\mathcal{K}} S(h^0_{(2)}) \cdot (c^1 \otimes \ldots \otimes c^n) \;&=\\ m \, h h^0_{(1)} \otimes_{\mathcal{K}} S(h^0_{(2)}) \cdot (c^1 \otimes \ldots \otimes c^n) \quad&, \end{aligned}$$

which correctly corresponds to $\Phi^n(mh \otimes_{\mathcal{H}} \dot{h}^0 \otimes c^1 \otimes \ldots \otimes c^n)$.

Next, Φ^n is obviously a left inverse to Ψ^n. Conversely, one has

$$\begin{aligned} (\Psi^n \circ \Phi^n)(m \otimes_{\mathcal{H}} \dot{h}^0 \otimes \ldots \otimes \dot{h}^n) \;&=\; m \, h^0_{(1)} \otimes_{\mathcal{H}} \dot{1} \otimes S(h^0_{(2)}) \cdot (\dot{h}^1 \otimes \ldots \otimes \dot{h}^n) \\ &=\; m \otimes_{\mathcal{H}} \dot{\widehat{h^0_{(1)}}} \otimes h^0_{(2)} S(h^0_{(3)}) \cdot (\dot{h}^1 \otimes \ldots \otimes \dot{h}^n) \\ &=\; m \otimes_{\mathcal{H}} \dot{\widehat{h^0_{(1)}}} \varepsilon(h^0_{(2)}) \otimes \dot{h}^1 \otimes \ldots \otimes \dot{h}^n \\ &=\; m \otimes_{\mathcal{H}} \dot{h}^0 \otimes \dot{h}^1 \otimes \ldots \otimes \dot{h}^n \,. \end{aligned}$$

To achieve the proof, it remains to notice that defining the face, degeneracy and cyclic operators by transport of structure,

$$\begin{aligned}
\delta_i &= \Phi^n \circ \tilde{\delta}_i \circ \Psi^n\,, && 0 \leq i \leq n-1\,;\\
\sigma_i &= \Phi^n \circ \tilde{\sigma}_i \circ \Psi^n\,, && 0 \leq i \leq n\,;\\
\tau_n &= \Phi^n \circ \tilde{\tau}_n \circ \Psi^n\,,
\end{aligned}$$

their expressions are precisely as stated. □

Definition 13. *The relative Hopf cyclic cohomology $HC^*(\mathcal{H}, \mathcal{K}; M)$ of the pair $\mathcal{K} \subset \mathcal{H}$ with coefficients in the stable anti-Yetter-Drinfeld module-comodule M is the cyclic cohomology of the cyclic module (5.7).*

Dually, one can define the *relative Hopf cyclic homology* $HC_*(\mathcal{H}, \mathcal{K}; M')$ of the pair $(\mathcal{H}, \mathcal{K})$ *with coefficients* in the dual module-comodule $M' = \mathrm{Hom}(M, F)$ as the cyclic homology of the cyclic module

$$C_*(\mathcal{H}, \mathcal{K}; M') = \{C_n(\mathcal{H}, \mathcal{K}; M') := \mathrm{Hom}_{\mathcal{K}}(\mathcal{C}^{\otimes n}, M')\}_{n\geq 0}\,, \tag{5.11}$$

obtained by dualizing the cyclic structure of Theorem 12:

$$\begin{aligned}
(d_0\varphi)(c^1 \otimes \ldots \otimes c^{n-1}) &= \varphi(\dot{1} \otimes c^1 \otimes \ldots \otimes \ldots \otimes c^{n-1}),\\
(d_i\varphi)(c^1 \otimes \ldots \otimes c^{n-1}) &= \varphi(c^1 \otimes \ldots \otimes c^i_{(1)} \otimes c^i_{(2)} \otimes \ldots \otimes c^{n-1}),\\
&\qquad\qquad \forall \quad 1 \leq i \leq n-1\,;\\
(d_n\varphi)(c^1 \otimes \ldots \otimes c^{n-1})(m) &= \varphi(c^1 \otimes \ldots \otimes c^{n-1} \otimes \dot{m}_{(-1)})(m_{(0)})\,;\\[1em]
(s_i\varphi)(c^1 \otimes \ldots \otimes c^{n+1}) &= \varphi(c^1 \otimes \ldots \otimes \varepsilon(c^{i+1}) \otimes \ldots \otimes c^{n+1}),\\
&\qquad\qquad \forall \quad 0 \leq i \leq n\,;\\
(t_n\varphi)(\dot{h}^1 \otimes c^2 \otimes \ldots \otimes c^n)(m) &= \varphi\left(S(h^1_{(2)}) \cdot (c^2 \otimes \ldots \otimes c^n \otimes \dot{m}_{(-1)})\right)(m_{(0)} h^1_{(1)}).
\end{aligned}$$

Remark 14. *The restriction to $\mathcal{K}$ of the left action of $\mathcal{H}$ on $\mathcal{C} = \mathcal{H} \otimes_{\mathcal{K}} F$ can also be regarded as 'adjoint action', induced by conjugation.*

Indeed, for $k \in \mathcal{K}$,

$$k_{(1)}\, h\, S(k_{(2)}) \otimes_{\mathcal{K}} 1 \;=\; k_{(1)}\, h \otimes_{\mathcal{K}} \varepsilon(k_{(2)})\, 1 \;=\; k\, h \otimes_{\mathcal{K}} 1\,.$$

We now specialize the above notions to the Lie algebra case, in order to verify that they coincide with the usual definitions of relative Lie algebra homology and cohomology.

Let $\mathfrak{g}$ be a Lie algebra over the field F, let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra, and let M be a $\mathfrak{g}$-module. We equip M with the *trivial* $\mathfrak{g}$-comodule structure

$$_M\Delta(m) \;=\; 1 \otimes m \;\in \mathcal{H} \otimes M\,, \tag{5.12}$$

and note the condition (5.5) is then trivially satisfied, while (5.6) follows from (5.12) and the cocommutativity of the universal enveloping algebra $\mathfrak{A}(\mathfrak{g})$. The relative Lie algebra homology and cohomology of the pair $\mathfrak{h} \subset \mathfrak{g}$ with coefficients in M, resp. M', are computed from the Chevalley-Eilenberg (cf. [2]) complexes

$$\begin{aligned} &\{C_*(\mathfrak{g}, \mathfrak{h}; M), {}_M\delta\}, \quad C_n(\mathfrak{g}, \mathfrak{h}; M) \;:=\; M \otimes_{\mathfrak{h}} \bigwedge^{n} (\mathfrak{g}/\mathfrak{h})\,, \\ \text{resp.} \quad &\{C^*(\mathfrak{g}, \mathfrak{h}; M'), d_{M'}\}, \quad C^n(\mathfrak{g}, \mathfrak{h}; M') \;:=\; \mathrm{Hom}_{\mathfrak{h}}\left(\bigwedge^{n}(\mathfrak{g}/\mathfrak{h}), M'\right)\;; \end{aligned}$$

the action of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$ is induced by the adjoint representation and the differentials are given by the formulae

$$\begin{aligned} {}_M\delta(m \otimes_{\mathfrak{h}} \quad & \dot X_1 \wedge \ldots \wedge \dot X_{n+1}) \;=\; \sum_{i=1}^{n+1} (-1)^{i+1} m X_i \otimes_{\mathfrak{h}} \dot X_1 \wedge \ldots \wedge \check{\dot X_i} \ldots \wedge \dot X_{n+1} \\ + \quad & \sum_{i<j} (-1)^{i+j} m \otimes_{\mathfrak{h}} \widehat{[X_i, X_j]}^{\,\cdot} \wedge \dot X_1 \wedge \ldots \wedge \check{\dot X_i} \ldots \check{\dot X_j} \ldots \wedge \dot X_{n+1} \end{aligned} \tag{5.13}$$

$$\begin{aligned} (d_{M'}\varphi) \quad & (\dot X_1 \wedge \ldots \wedge \dot X_{n+1}) \;=\; \sum_{i=1}^{n+1} (-1)^{i+1} X_i \cdot \varphi(\dot X_1 \wedge \ldots \wedge \check{\dot X_i} \ldots \wedge \dot X_{n+1}) \\ + \quad & \sum_{i<j} (-1)^{i+j} \varphi(\widehat{[X_i, X_j]}^{\,\cdot} \wedge \dot X_1 \wedge \ldots \wedge \check{\dot X_i} \ldots \check{\dot X_j} \ldots \wedge \dot X_{n+1}) \end{aligned} \tag{5.14}$$

where $\dot X \in \mathfrak{g}/\mathfrak{h}$ stands for the class modulo $\mathfrak{h}$ of $X \in \mathfrak{g}$ and the superscript $\check{}$ signifies the omission of the indicated variable.

Theorem 15. *There are canonical isomorphisms between the periodic relative Hopf cyclic cohomology (resp. homology) of the pair $\mathfrak{A}(\mathfrak{h}) \subset \mathfrak{A}(\mathfrak{g})$, with coefficients in any $\mathfrak{g}$-module M (resp. its dual M'), and the relative Lie algebra homology (resp. cohomology) with coefficients of the pair $\mathfrak{h} \subset \mathfrak{g}$:*

$$\begin{aligned} HP^*(\mathfrak{A}(\mathfrak{g}), \mathfrak{A}(\mathfrak{h}); M) &\cong \bigoplus_{n \equiv * \bmod 2} H_n(\mathfrak{g}, \mathfrak{h}; M) \\ \textit{resp.} \quad HP_*(\mathfrak{A}(\mathfrak{g}), \mathfrak{A}(\mathfrak{h}); M') &\cong \bigoplus_{n \equiv * \bmod 2} H^n(\mathfrak{g}, \mathfrak{h}; M'). \end{aligned}$$

Proof. The pattern of the proof is exactly the same as in the "absolute" case, cf. [9, §7, Prop.7]. For the convenience of the reader, we shall repeat the main steps in order to make sure that they apply in the relative case as well.

First of all, the Hopf cyclic cohomology $HC^*(\mathfrak{A}(\mathfrak{g}), \mathfrak{A}(\mathfrak{h}); M)$ can be computed from the normalized bi-complex $\big(\bar{C}^*(\mathfrak{A}(\mathfrak{g}), \mathfrak{A}(\mathfrak{h}); M), {}_M b, {}_M B\big)$ where

$$\bar{C}^n(\mathfrak{A}(\mathfrak{g}), \mathfrak{A}(\mathfrak{h}); M) = \bigcap_{i=0}^{n-1} \operatorname{Ker} \sigma_i\,, \quad \forall n \geq 1, \qquad \bar{C}^0(\mathfrak{A}(\mathfrak{g}), \mathfrak{A}(\mathfrak{h}); M) = F;$$

the operator ${}_M b = \sum_{i=0}^{n} (-1)^i \delta_i : \bar{C}^{n-1} \to \bar{C}^n$ has the expression

$$\begin{aligned} {}_M b(m \otimes_{\mathcal{K}} c^1 \otimes \ldots \otimes c^{n-1}) &= m \otimes_{\mathcal{K}} \dot{1} \otimes c^1 \otimes \ldots \otimes \ldots \otimes c^{n-1} \\ &+ \sum_{i=1}^{n-1} (-1)^i m \otimes_{\mathcal{K}} c^1 \otimes \ldots \otimes c^i_{(1)} \otimes c^i_{(2)} \otimes \ldots \otimes c^{n-1}, \\ &+ (-1)^n m \otimes_{\mathcal{K}} c^1 \otimes \ldots \otimes c^{n-1} \otimes \dot{1} \end{aligned} \tag{5.15}$$

involving only the coalgebra structure, while ${}_M B : \bar{C}^{n+1} \to \bar{C}^n$ is given by the formula

$${}_M B = \sum_{i=0}^{n} (-1)^{ni} \, {\tau_n}^i \circ \sigma_{-1}\,, \qquad \text{where} \qquad \sigma_{-1} = \sigma_n \circ \tau_{n+1}$$

is the extra degeneracy operator

$$\sigma_{-1}(m \otimes_{\mathcal{K}} \dot{h}^1 \otimes c^2 \otimes \ldots \otimes c^{n+1}) = m h^1_{(1)} \otimes_{\mathcal{K}} S(h^1_{(2)}) \cdot (c^2 \otimes \ldots \otimes c^{n+1}).$$

We recall that the antisymmetrization map $\alpha^n : \bigwedge^n(\mathfrak{g}/\mathfrak{h}) \to \mathcal{C}(\mathfrak{A}(\mathfrak{g}), \mathfrak{A}(\mathfrak{h}))^{\otimes n}$,

$$\alpha^n(\dot{X}_1 \wedge \ldots \wedge \dot{X}_n) := \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \dot{X}_{\sigma(1)} \otimes \ldots \otimes \dot{X}_{\sigma(n)},$$

has a left inverse

$$\mu^n : \mathcal{C}(\mathfrak{A}(\mathfrak{g}), \mathfrak{A}(\mathfrak{h}))^{\otimes n} \to \bigwedge^n(\mathfrak{g}/\mathfrak{h}), \qquad \mu^n \circ \alpha^n = \mathrm{Id},$$

which is the unique DGA map whose first-degree component μ^1 is the canonical projection of $\mathcal{C} = \mathcal{C}(\mathfrak{A}(\mathfrak{g}), \mathfrak{A}(\mathfrak{h})) \simeq S(\mathfrak{g})/S(\mathfrak{g})S(\mathfrak{h})^+$ onto $\mathfrak{g}/\mathfrak{h}$.
In view of Remark 14, both maps commute with the adjoint representation, and therefore induce well-defined maps

$$\begin{aligned} {}_M\alpha^n : C_n(\mathfrak{g}, \mathfrak{h}; M) = M \otimes_{\mathfrak{h}} \bigwedge^n(\mathfrak{g}/\mathfrak{h}) \quad &\to \quad \bar{C}^n(\mathfrak{A}(\mathfrak{g}), \mathfrak{A}(\mathfrak{h}); M), \\ \text{resp.} \quad {}_M\mu^n : M \otimes_{\mathfrak{A}(\mathfrak{h})} \mathcal{C}^{\otimes n} \quad &\to \quad M \otimes_{\mathfrak{h}} \bigwedge^n(\mathfrak{g}/\mathfrak{h}). \end{aligned}$$

By transposition one also obtains dual maps

$$\begin{aligned} \mu^n_{M'} : \mathrm{Hom}_{\mathfrak{h}}\left(\bigwedge^n(\mathfrak{g}/\mathfrak{h}), M'\right) \quad &\to \quad \mathrm{Hom}_{\mathfrak{A}(\mathfrak{h})}(\mathcal{C}^{\otimes n}, M'), \\ \text{resp.} \quad \alpha^n_{M'} : \mathrm{Hom}_{\mathfrak{A}(\mathfrak{h})}(\mathcal{C}^{\otimes n}, M') \quad &\to \quad \mathrm{Hom}_{\mathfrak{h}}\left(\bigwedge^n(\mathfrak{g}/\mathfrak{h}), M'\right). \end{aligned}$$

From (5.15), one sees that ${}_Mb = \mathrm{Id} \otimes b$, where b corresponds to trivial coefficients. In particular,

$${}_Mb \circ {}_M\alpha \;=\; \mathrm{Id} \otimes (b \circ \alpha) \;=\; 0,$$

and likewise,

$${}_M\mu \circ {}_Mb \;=\; \mathrm{Id} \otimes (\mu \circ b) \;=\; 0.$$

Therefore, both

$${}_M\alpha : (C_*(\mathfrak{g}, \mathfrak{h}; M), 0) \to \big(\bar{C}^*(\mathfrak{A}(\mathfrak{g}), \mathfrak{A}(\mathfrak{h}); M), b\big)$$

and

$${}_M\mu : \big(\bar{C}^*(\mathfrak{A}(\mathfrak{g}), \mathfrak{A}(\mathfrak{h}); M), b\big) \to (C_*(\mathfrak{g}, \mathfrak{h}; M), 0)$$

are chain maps with ${}_M\mu \circ {}_M\alpha = \mathrm{Id}$. Moreover, as in the case of trivial coefficients (cf. [1], see [23, XVIII.7] for full details) they induce isomorphism between the corresponding cohomology groups.

On the other hand, for any $\sigma \in S_{n+1}$ one has

$$\sigma_{-1}(m \otimes_{\mathcal{K}} \dot{X}^{\sigma(1)} \otimes \ldots \otimes \dot{X}^{\sigma(n+1)}) = m\dot{X}^{\sigma(1)} \otimes_{\mathcal{K}} \dot{X}^{\sigma(2)} \otimes \ldots \otimes \dot{X}^{\sigma(n+1)}$$
$$- \sum_{i=2}^{n+1} m \otimes_{\mathcal{K}} \dot{X}^{\sigma(2)} \otimes \ldots \otimes \widehat{[X^{\sigma(1)}, \dot{X}^{\sigma(i)}]} \otimes \ldots \otimes \dot{X}^{\sigma(n+1)}.$$

By a routine calculation one then sees that, up to a normalizing factor $c_n \neq 0$,

$${}_M B \circ {}_M\alpha \simeq {}_M\alpha \circ {}_M\delta\,.$$

Thus, when regarding $(C_*(\mathfrak{g}, \mathfrak{h}; M), 0, {}_M\delta)$ as a bi-complex with degree $+1$ differential 0 and degree -1 differential ${}_M\delta$, ${}_M\alpha$ is a bi-chain map, which is a quasi-isomorphism with respect to the degree $+1$ differential. From the long exact sequence for (b, B) bi-complexes [5] it follows that ${}_M\alpha$ is also quasi-isomorphism for the corresponding total complexes. $\square$

Finally, the characteristic map in cyclic cohomology (cf. §4, Ansatz) associated an $\mathcal{H}$-module algebra $\mathcal{A}$ with $\mathcal{H}$-invariant trace has a relative version, which we describe below, in the framework of [20]. The higher cup products defined in [24] should also admit relative counterparts.

In addition to the datum of §1, we consider an $\mathcal{H}$-module algebra $\mathcal{A}$ together with an *$\mathcal{H}$-invariant twisted M'-trace* ϕ on $\mathcal{A}$. By definition, this means that

$$\phi \in \mathrm{Hom}_{\mathcal{H}}(\mathcal{A}, M') \tag{5.16}$$

and

$$\phi(a\, m_{(-1)}(b))(m_{(0)}) = \phi(b\, a)(m), \qquad a, b \in \mathcal{A}, \quad m \in M. \tag{5.17}$$

Examples of such data are provided by actions of Hopf algebras with modular pairs (δ, σ) on algebras admitting δ-invariant σ-traces and they abound in 'nature' (cf. [10, 11]). Furthermore, the construction of the characteristic map associated to such a trace extends to the present situation in a straightforward fashion. Indeed, the assignment

$$\chi_\phi(m \otimes_{\mathcal{H}} h^0 \otimes \ldots \otimes h^n)(a^0, \ldots, a^n) = \phi(h^0(a^0)h^1(a^1) \ldots h^n(a^n))(m),$$
$$h^0, \ldots, h^n \in \mathcal{H}, \quad a^0, \ldots, a^n \in \mathcal{A}, \quad m \in M$$

preserves the cyclic structures and therefore defines a map from the 'absolute' Hopf cyclic cohomology of $\mathcal{H}$ with coefficients in M to the cyclic cohomology of $\mathcal{A}$:

$$\chi_\phi^* : HC^*(\mathcal{H}; M) \longrightarrow HC^*(\mathcal{A}). \tag{5.18}$$

The 'relative' version of (5.18) arises from the inherited action of the coalgebra $\mathcal{C} = \mathcal{C}(\mathcal{H}, \mathcal{K})$ on the subalgebra of $\mathcal{K}$-invariant elements

$$\mathcal{A}_{\mathcal{K}} \,=\, \{a \in \mathcal{A} \mid \quad k(a) \,=\, \varepsilon(k)\, a\,, \quad \forall\, k \in \mathcal{K}\}\,.$$

It is defined by a similar formula,

$$\begin{aligned}\chi_{\phi,\mathcal{K}}(m \otimes_{\mathcal{K}} c^1 \otimes c^2 \otimes \ldots \otimes c^n) \,&=\, \phi(a^0 c^1(a^1) \ldots c^n(a^n))(m)\,, \\ a^0, \ldots, a^n \in \mathcal{A}_{\mathcal{K}}, \quad & c^1, \ldots, c^n \in \mathcal{C}, \quad m \in M,\end{aligned}$$

that again induces at the level of cyclic cohomology the *relative characteristic map*:

$$\chi_{\phi,\mathcal{K}}^* : HC^*(\mathcal{H}, \mathcal{K}; M) \longrightarrow HC^*(\mathcal{A}_{\mathcal{K}}). \tag{5.19}$$

6 The Hopf algebra $\mathcal{H}_1$ and its Hopf-cyclic classes

In this last section we describe in detail the basic Hopf-cyclic cocycles of the Hopf algebra $\mathcal{H}_1$, corresponding to the Godbillon-Vey class, to the Schwarzian derivative and to the transverse fundamental class, and illustrate the isomorphism (4.12).

We begin by specializing the presentation of the Hopf algebra $\mathcal{H}_n$ (cf. §2, Proposition 2) to the codimension 1 case. As an algebra, $\mathcal{H}_1$ it coincides with the universal enveloping algebra of the Lie algebra with basis $\{X, Y, \delta_n\,; n \geq 1\}$ and brackets

$$[Y, X] = X\,,\; [Y, \delta_n] = n\,\delta_n\,,\; [X, \delta_n] = \delta_{n+1}\,,\; [\delta_k, \delta_\ell] = 0\,, \quad n, k, \ell \geq 1\,,$$

where we have used the abbreviated notation

$$X = X_1\,,\; Y = Y_{11}^1\,,\; \delta_1 = \delta_{11}^1\,,\; \delta_2 = \delta_{11;1}^1\,,\; \delta_3 = \delta_{11;11}^1\,,\; \delta_4 = \delta_{11;111}^1\,, \ldots\,.$$

As a Hopf algebra, the coproduct $\Delta : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_1$ is determined by

$$\Delta Y = Y \otimes 1 + 1 \otimes Y \,, \quad \begin{aligned} \Delta X &= X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y \\ \Delta \delta_1 &= \delta_1 \otimes 1 + 1 \otimes \delta_1 \end{aligned}$$

and the multiplicativity property

$$\Delta(h^1 h^2) = \Delta h^1 \cdot \Delta h^2 \,, \quad h^1, h^2 \in \mathcal{H}_1 \;;$$

the antipode is determined by

$$S(Y) = -Y \,,\, S(X) = -X + \delta_1 Y \,,\, S(\delta_1) = -\delta_1$$

and the anti-isomorphism property

$$S(h^1 h^2) = S(h^2)\, S(h^1) \,, \quad h^1, h^2 \in \mathcal{H}_1 \;;$$

finally, the counit is

$$\varepsilon(h) = \text{constant term of} \quad h \in \mathcal{H}_1 \,.$$

We next recall the 'standard' actions $\mathcal{H}_1$ (cf. §2, Proposition 3). Given a one-dimensional manifold M^1 and a discrete subgroup $\Gamma \subset \mathrm{Diff}^+(M^1)$, $\mathcal{H}_1$ acts on the crossed product algebra

$$\mathcal{A}_\Gamma = C_c^\infty(J_+^1(M^1)) \rtimes \Gamma \,,$$

by a Hopf action, where $J_+^1(M^1) = F^+ M^1$ is the oriented 1-jet bundle over M^1. We use the coordinates in $J_+^1(M^1)$ given by the Taylor expansion,

$$j(s) = x + s\, y + \cdots \,, \qquad y > 0 \,,$$

and let diffeomorphisms act in the obvious functorial manner on the 1-jets,

$$\widetilde{\varphi}(x, y) = (\varphi(x),\, \varphi'(x) \cdot y) \,.$$

The canonical action of $\mathcal{H}_1$ is then given as follows:

$$Y(f U_\varphi^*) = y\, \frac{\partial f}{\partial y}\, U_\varphi^* \quad , \qquad X(f U_\varphi^*) = y\, \frac{\partial f}{\partial x}\, U_\varphi^* \,,$$

$$\delta_n(f U_\varphi^*) \;=\; y^n\, \frac{d^n}{dx^n}\, (\log \varphi'(x))\; f U_\varphi^* \,,$$

where we have identified $F^+M^1 \simeq M^1 \times \mathbb{R}^+$ and denoted by (x,y) the coordinates on the latter.

The volume form $\dfrac{dx \wedge dy}{y^2}$ on $F^+M^1)$ is invariant under $\mathrm{Diff}^+(M^1)$ and gives rise to the following trace $\tau : \mathcal{A}_\Gamma \to \mathbb{C}$,

$$\tau(fU_\varphi^*) \;=\; \begin{cases} \displaystyle\int_{F^+M^1)} f(y,y_1)\,\frac{dy \wedge dy_1}{y_1^2} & \text{if } \varphi = 1\,, \\ 0 & \text{if } \varphi \neq 1\,. \end{cases}$$

This trace is δ-invariant with respect to the action $\mathcal{H}_1 \otimes \mathcal{A}_\Gamma \to \mathcal{A}_\Gamma$ and with the modular character $\delta \in \mathcal{H}_1^*$, determined by

$$\delta(Y) = 1, \quad \delta(X) = 0, \quad \delta(\delta_n) = 0\,;$$

the invariance property is given by the identity

$$\tau(h(a)) = \delta(h)\,\tau(a)\,, \qquad \forall \quad h \in \mathcal{H}_1\,.$$

The fact that

$$S^2 \neq Id\,,$$

is automatically corrected by twisting with δ. Indeed, $\widetilde{S} = \delta * S$ satisfies the involutive property

$$\widetilde{S}^2 = Id\,. \tag{6.1}$$

One has

$$\widetilde{S}(\delta_1) \;=\; -\delta_1\,, \quad \widetilde{S}(Y) = -Y + 1\,, \quad \widetilde{S}(X) = -X + \delta_1 Y\,.$$

Equation (6.1) shows that the pair $(\delta, 1)$ given by the character δ of $\mathcal{H}_1$ and the group-like element $1 \in \mathcal{H}_1$ is a modular pair in involution. Thus the Hopf-cyclic cohomology $HC^*(\mathcal{H}_1; \delta, 1)$ is well-defined and, for each pair (M^1, Γ) as above, the assignment

$$\chi_\tau(h^1 \otimes \ldots \otimes h^n)(a^0, \ldots, a^n) = \tau(a^0\, h^1(a^1) \ldots h^n(a^n))\,, \quad h^i \in \mathcal{H}_1,\ a^j \in \mathcal{A}\,,$$

induces a characteristic homomorphism

$$\chi_\tau^* \,:\, HC^*\,(\mathcal{H}_1; \delta, 1) \;\to\; HC^*\,(\mathcal{A}_\Gamma)\,. \tag{6.2}$$

Proposition 16. *The element* $\delta_1 \in \mathcal{H}_1$ *is a Hopf cyclic cocycle, which gives a nontrivial class*

$$[\delta_1] \in HP^1(\mathcal{H}_1; \delta, 1).$$

Proof. Indeed, the fact that δ_1 is a 1-cocycle is easy to check:

$$b(\delta_1) = 1 \otimes \delta_1 - \Delta\delta_1 + \delta_1 \otimes 1 = 0,$$

while

$$\tau_1(\delta_1) = \widetilde{S}(\delta_1) = S(\delta_1) = -\delta_1 .$$

The non-triviality of to the periodic class $[\delta_1]$ is a consequence of Proposition 19 below. Alternatively, one can remark that its image under the above characteristic map (6.2), $\chi_\tau^*([\delta_1]) \in HC^1(\mathcal{A}_\Gamma)$, is precisely the anabelian 1-trace of [6] (cf. also [7, III. 6. γ]), and the latter is known to give a nontrivial class on the transverse frame bundle to codimension 1 foliations. $\square$

We shall now describe another Hopf cyclic 1-cocycle, intimately related to the classical Schwarzian

$$\{y\,;x\} := \frac{d^2}{dx^2}\left(\log\frac{dy}{dx}\right) - \frac{1}{2}\left(\frac{d}{dx}\left(\log\frac{dy}{dx}\right)\right)^2,$$

which plays a prominent role in the transverse geometry of modular Hecke algebras (cf. [13, 14]).

Proposition 17. *The element* $\delta_2' := \delta_2 - \frac{1}{2}\delta_1^2 \in \mathcal{H}_1$ *is a Hopf cyclic cocycle, whose action on the crossed product algebra* $\mathcal{A}_\Gamma = C_c^\infty(J_+^1(M^1)) \rtimes \Gamma$ *is given by the Schwarzian derivative*

$$\delta_2'(fU_\varphi^*) = y_1^2\,\{\varphi(y)\,;y\}\,fU_\varphi^*.$$

Its class

$$[\delta_2'] \in HC^1(\mathcal{H}_1; \delta, 1)$$

is equal to $B[c]$, *where* c *is the following Hochschild* 2*-cocycle:*

$$c := \delta_1 \otimes X + \frac{1}{2}\,\delta_1^2 \otimes Y .$$

Proof. We shall give the detailed computation, in order to illustrate the (b, B) bi-complex for Hopf cyclic cohomology. Let us compute $b(c)$. One has

$$b(\delta_1 \otimes X) = 1 \otimes \delta_1 \otimes X - (\delta_1 \otimes 1 + 1 \otimes \delta_1) \otimes X +$$
$$\delta_1 \otimes (X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y) - \delta_1 \otimes X \otimes 1 = \delta_1 \otimes \delta_1 \otimes Y$$

Also

$$b(\delta_1^2 \otimes Y) = 1 \otimes \delta_1^2 \otimes Y - (\delta_1^2 \otimes 1 + 2\delta_1 \otimes \delta_1 + 1 \otimes \delta_1^2) \otimes Y +$$
$$\delta_1^2 \otimes (Y \otimes 1 + 1 \otimes Y) - \delta_1^2 \otimes Y \otimes 1 = -2\delta_1 \otimes \delta_1 \otimes Y$$

This shows that

$$b(c) \;=\; 0\,,$$

so that c is a Hochschild cocycle.

Let us now compute $B(c)$. First, we recall that

$$B_0(h^1 \otimes h^2) = \widetilde{S}(h^1)h^2\,.$$

Since $\widetilde{S}(\delta_1) \;=\; -\delta_1$, one then has

$$B_0(c) = -\delta_1\, X + \frac{1}{2}\delta_1^2\, Y\,.$$

Since $\widetilde{S}(Y) = -Y + 1$ and $\widetilde{S}(X) = -X + \delta_1 Y$, it follows that

$$\begin{aligned}
\widetilde{S}(B_0 c) &= \widetilde{S}(X)\delta_1 + \frac{1}{2}\widetilde{S}(Y)\delta_1^2 \\
&= (-X + \delta_1 Y)\delta_1 + \frac{1}{2}(-Y + 1)\delta_1^2 \\
&= -X\delta_1 + \delta_1^2 Y + \delta_1^2 - \frac{1}{2}(\delta_1^2 Y + \delta_1^2) \\
&= -X\delta_1 + \frac{1}{2}\delta_1^2 Y + \frac{1}{2}\delta_1^2\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
B(c) \;=\; B_0 c - \widetilde{S}(B_0 c) \quad &= \\
-\delta_1\, X + \frac{1}{2}\delta_1^2\, Y - (-X\delta_1 + \frac{1}{2}\delta_1^2 Y + \frac{1}{2}\delta_1^2) \quad &= \quad \delta_2'\,.
\end{aligned}$$

which shows that the class of δ_2' is trivial in the periodic cyclic cohomology $HP^*(\mathcal{H}_1; \delta, 1)$. ◻

Let us illustrate the isomorphism (3.12) by determining the 2-cocycle that represents the 'universal' transversal fundamental class in codimension 1.

Proposition 18. *The* 2*-cochain*

$$\Pi := X \otimes Y - Y \otimes X - \delta_1 \, Y \otimes Y \, \in \, \mathcal{H}_1 \otimes \mathcal{H}_1$$

is a cyclic 2*-cocycle, whose class* $[\Pi] \in HC^2(\mathcal{H}_1; \delta, 1)$ *corresponds by* $T^\natural$ *to the transverse fundamental class.*

Proof. The transverse fundamental class in $HC^2(\mathcal{A}_1)$ is given by the 2-cocycle

$$F(f^0 U^*_{\varphi_0}, f^1 U^*_{\varphi_1}, f^2 U^*_{\varphi_2}) = \begin{cases} \displaystyle\int_{F^+\mathbb{R}} f^0 \, \widetilde{\varphi}_0^*(df^1) \, \widetilde{\varphi}_0^* \widetilde{\varphi}_1^*(df^2) \, , & \text{if} \quad \varphi_2 \varphi_1 \varphi_0 = Id \, , \\ 0 \, , & \text{otherwise} \, . \end{cases}$$

In terms of the standard basis of $T^*(F^+\mathbb{R})$, one has

$$df^1 \; = \; X(f^1) \, y^{-1} dx \; + \; Y(f^1) \, y^{-1} dy \, ,$$

therefore

$$\begin{aligned} \widetilde{\varphi}_0^*(df^1) \quad &= \quad \widetilde{\varphi}_0^* \left(X(f^1) \right) \cdot \widetilde{\varphi}_0^*(y^{-1}dx) \; + \; \widetilde{\varphi}_0^* \left(Y(f^1) \right) \cdot \widetilde{\varphi}_0^*(y^{-1}dy) \\ &= \quad \left(\widetilde{\varphi}_0^*(X(f^1)) + \gamma_1(\varphi_0) \, \widetilde{\varphi}_0^*(Y(f^1)) \right) \, y^{-1}dx + \widetilde{\varphi}_0^* \left(Y(f^1) \right) \, y^{-1}dy \, , \end{aligned}$$

where, consistently with the abbreviation δ_1, $\gamma_1 = \gamma_{11}^1$.
On substituting the above expression of $\widetilde{\varphi}_0^*(df^1)$ together with the similar one for $\widetilde{\varphi}_0^* \widetilde{\varphi}_1^*(df^2)$ in the formula defining F, after using the cocycle identity

$$\gamma_1(\varphi_1 \varphi_0) \; = \; \widetilde{\varphi}_0^*(\gamma_1(\varphi_1)) + \gamma_1(\varphi_0) \, ,$$

one observes that the result is identical to $T^\natural(\Pi)(f^0 U^*_{\varphi_0}, f^1 U^*_{\varphi_1}, f^2 U^*_{\varphi_2})$. ☐

We now proceed to illustrate the isomorphism

$$\kappa_1^* \, : \, \sum_{i \equiv * \, (2)}^{\oplus} \quad H^i(\mathfrak{a}_1, \mathbb{C}) \xrightarrow{\simeq} HP^* \, (\mathcal{H}_1; \delta, 1) \tag{6.3}$$

between the Gelfand-Fuks cohomology of the Lie algebra of formal vector fields on $\mathbb{R}^1$ and the periodic Hopf-cyclic cohomology of $(\mathcal{H}_1; \delta, 1)$.
We recall (cf. [16, 17]) that the cohomology $H^*(\mathfrak{a}_1, \mathbb{R})$ is finite dimensional and the only nontrivial groups are:

$$H^0(\mathfrak{a}_1, \mathbb{R}) = \mathbb{R} \cdot 1 \qquad \text{and} \qquad H^3(\mathfrak{a}_1, \mathbb{R}) = \mathbb{R} \cdot gv\,,$$

where

$$gv(p_1\,\partial_x, p_2\,\partial_x, p_3\,\partial_x) = \begin{vmatrix} p_1(0) & p_2(0) & p_3(0) \\ p_1'(0) & p_2'(0) & p_3'(0) \\ p_1''(0) & p_2''(0) & p_3''(0) \end{vmatrix}\,.$$

The class of the unit constant $[1] \in HC^0(\mathcal{H}_1; \delta, 1)$ is trivial in $HP^0(\mathcal{H}_1; \delta, 1)$, since $B(Y) = 1$. On the other hand,

$$\kappa_1^*(1) \;=\; F\,, \tag{6.4}$$

which thus gives the generator of $HP^0(\mathcal{H}_1; \delta, 1)$. Formula (6.4) is easy to check using Proposition 18 and applying the definition of κ_1^*, cf. [9] or [12], in the case of 0-dimensional Lie algebra cochains.

On the other hand, the evaluation of κ_1^* on the the Godbillon-Vey class requires some calculations, whose details will be given below, after we recall the original definition of this class.

Let V be a closed, smooth manifold, foliated by a transversely oriented codimension 1 foliation $\mathcal{F}$. Then $T\mathcal{F} = \operatorname{Ker}\omega \subset TV$, for some $\omega \in \Omega^1(V)$ such that $\omega \wedge d\omega = 0$. Equivalently, $d\omega = \omega \wedge \alpha$ for some $\alpha \in \Omega^1(V)$, which implies $d\alpha \wedge \omega = 0$. In turn, the latter ensures that $d\alpha = \omega \wedge \beta$, $\beta \in \Omega^1(V)$. Thus, $\alpha \wedge d\alpha \in \Omega^3(V)$ is closed. Its de Rham cohomology class,

$$\mathrm{GV}(V, \mathcal{F}) = [\alpha \wedge d\alpha] \in H^3(V, \mathbb{R})\,,$$

is independent of the choices of ω and α and represents the original definition of the Godbillon-Vey class.

The Godbillon-Vey class acquires a universal status when viewed as a characteristic class (cf. [21]) associated to the Gelfand-Fuks cohomology of the Lie algebra $\mathfrak{a}_1 = \mathbb{R}\,[[x]]\,\partial_x$ of formal vector fields on $\mathbb{R}$.

Given any oriented 1-dimensional manifold M^1, the Lie algebra cocycle gv can be converted into a 3-form on the jet bundle (of orientation preserving jets)

$$J_+^\infty(M^1) = \varprojlim_n J_+^n(M^1)\,,$$

invariant under the pseudogroup $\mathcal{G}^+(M^1)$ of all orientation preserving local diffeomorphisms of M^1. Indeed, sending the formal vector field

$$p = j_0^\infty \left(\left. \frac{dh_t}{dt} \right|_{t=0} \right) \in \mathfrak{a}_1\,,$$

where $\{h_t\}$ is a 1-parameter family of local diffeomorphisms of $\mathbb{R}$ preserving the origin, to the $\mathcal{G}^+(M^1)$-invariant vector field

$$j_0^\infty \left(\left. \frac{d(f\circ h_t)}{dt} \right|_{t=0} \right) \in T_{j_0^\infty(f)}\, J_+^\infty(M^1)$$

gives a natural identification of the Lie algebra complex of $\mathfrak{a}_1$ with the invariant forms on the jet bundle,

$$\theta \in C^\bullet(\mathfrak{a}_1) \mapsto \theta \in \Omega^\bullet(J_+^\infty(M^1))^{\mathcal{G}^+(M^1)}\,.$$

In local coordinates on $J_+^\infty(M^1)$, given by the coefficients of the Taylor expansion at 0,

$$f(s) = x + s\,y + s^2 y_1 + \cdots\,, \qquad y > 0\,,$$

one has

$$\begin{array}{rcl} dx & = & y\,\theta^0 \\ dy & = & y\,\theta^1 + 2\,y_1\,\theta^0 \\ dy_1 & = & y\,\theta^2 + 2\,y_1\,\theta^1 + 3\,y_2\,\theta^0\,, \end{array}$$

therefore

$$gv \equiv \theta^0 \wedge \theta^1 \wedge \theta^2 \;=\; \frac{1}{y^3}\, dx \wedge dy \wedge dy_1 \in \Omega^3(J_+^\infty(M^1))^{\mathcal{G}^+(M^1)}\,.$$

Given a codimension 1 foliation $(V, \mathcal{F})$ as above, one can find an open covering $\{U_i\}$ of V and submersions $f_i : U_i \to T_i \subset \mathbb{R}$, whose fibers are plaques of $\mathcal{F}$, such that $f_i = g_{ij} \circ f_j$ on $U_i \cap U_j$, with $g_{ij} \in \mathcal{G}^+(M^1)$ a 1-cocycle. Then $M^1 = \bigcup_i T_i \times \{i\}$ is a complete transversal. Let $J^\infty(\mathcal{F})$ denote the bundle over V whose fiber at $x \in U_i$ consists of the ∞-jets of local submersions of the form $\varphi \circ f_i$ with $\varphi \in \mathcal{G}^+(M^1)$. Using the $\mathcal{G}^+(M^1)$-invariance of $gv \in \Omega^3(J^\infty(M^1))$ one can pull it back to a closed form $gv(\mathcal{F}) \in \Omega^3(J^2(\mathcal{F}))$. Its de Rham class $[gv(\mathcal{F})] \in H^3(J^2(\mathcal{F}), \mathbb{R})$, when viewed as a class in $H^3(V, \mathbb{R})$ (the fibers of $J^2\mathcal{F}$ being contractible), is precisely the Godbillon-Vey class $\mathrm{GV}(V, \mathcal{F})$.

Proposition 19. *The canonical cochain map associated to the trivial connection on $F^+\mathbb{R}$ sends the universal Godbillon-Vey cocycle gv to the Hopf cyclic cocycle δ_1, implementing the identity*

$$\kappa_1^*([gv]) \;=\; [\delta_1]\,.$$

Proof. The definition of κ_1^* involves two steps (see [9], [12]). The first turns the Lie algebra cocycle $gv \in C^2(\mathfrak{a}_1, \mathbb{R})$, or equivalently, the form on the jet bundle $J^2_+(\mathbb{R})$

$$gv \;=\; \frac{1}{y^3}\, dx \wedge dy \wedge dy_1\,,$$

into a group 1-cocycle $C_{1,0}(gv)$ on $\mathcal{G}_1 = \mathrm{Diff}^+(\mathbb{R})$ with values in currents on $J^1_+(\mathbb{R})$. The second step takes its image in the cyclic bi-complex under the canonical map Φ. One thus obtains a cyclic cocycle

$$(\Phi\,(C_{1,0}\,(gv)\,))\,(f^0\,U^*_{\varphi_0}, f^1\,U^*_{\varphi_1})$$

which is automatically supported at the identity, i.e. it is nonzero only when $\varphi_1\varphi_0 = 1$. Moreover, it is of the form

$$(\Phi\,(C_{1,0}\,(gv)\,))\,(f^0\,U^*_{\varphi}, f^1\,U^*_{\varphi^{-1}}) \;=\; -\langle\, C_{1,0}(gv)(1,\varphi),\; f^0 \cdot \widetilde{\varphi}^*(f^1)\,\rangle\,.$$

Applying the definition given in [12], one gets

$$\langle C_{1,0}(gv)(1,\varphi), f\rangle \;=\; \int_{\Delta^1 \times F^+\mathbb{R}} f\,\widetilde{\sigma}(1,\varphi)^*\,(gv)$$

where Δ^1 is the 1-simplex and

$$\widetilde{\sigma}(1,\varphi) : \Delta^1 \times J^1_+(\mathbb{R}) \to J^\infty_+(\mathbb{R})$$

has the expression

$$\widetilde{\sigma}(1,\varphi)(t,x,y) \;=\; \sigma_{(1-t)\nabla_0+t\nabla_0^{\varphi}}(x,y)\,,$$

where the meaning of the notation used is as follows.
First, ∇_0 stands for the trivial linear connection on $\mathbb{R}$, given by the connection form on $F^+\mathbb{R}$

$$\omega_0 = y^{-1}\,dy\,,$$

while ∇_0^{φ} denotes its transform under the prolongation

$$\widetilde{\varphi}(x,y) = (\varphi(x),\, \varphi'(x)\cdot y)\,,$$

of the diffeomorphism $\varphi \in \mathcal{G}_1$; the latter corresponds to the connection form

$$\widetilde{\varphi}^*(\omega_0) = \frac{1}{\varphi'(x)\,y}\,\left(\varphi'(x)\,dy + \varphi''(x)\cdot y\,dx\right) = y^{-1}\,dy + \frac{d}{dx}(\log\varphi'(x))\,dx\,.$$

Furthermore, for any linear connection ∇ on $\mathbb{R}$, σ_∇ denotes the jet

$$\sigma_\nabla(x,y) = j_0^\infty\,(Y(s))$$

of the local diffeomorphism

$$s \;\mapsto\; Y(s) := \exp_y^\nabla\left(s\,y\,\frac{d}{dx}\right)\,, \qquad s\in\mathbb{R}\,.$$

Now $Y(s)$ satisfies the geodesics ODE

$$\begin{cases} \ddot{Y}(s) + \Gamma^1_{1\,1}(Y(s))\cdot\dot{Y}(s)^2 = 0\,, \\ Y(0) = x\,, \\ \dot{Y}(0) = y\,. \end{cases}$$

Since we only need the 2-jet of the exponential map, suffices to retain that

$$Y(0) = x\,,\quad \dot{Y}(0) = y \quad\text{and}\quad \ddot{Y}(0) = -\Gamma^1_{1\,1}(x)\,y^2\,.$$

Thus,

$$\sigma_\nabla(x,y) \;=\; x + y\,s \;-\; \Gamma^1_{1\,1}(x)\,y^2\,s^2 \;+\; \text{higher order terms}\,.$$

In our case $\nabla \;=\; (1-t)\nabla_0 \;+\; t\nabla_0^{\varphi}$, which gives

$$\Gamma^1_{1\,1}(t,x) \;=\; t\,\frac{d}{dx}(\log\varphi'(x))\,,$$

and therefore

$$\widetilde{\sigma}(1,\varphi)(t,x,y) \;=\; x + y\,s \;-\; t\,\frac{d}{dx}(\log\varphi'(x))\,y^2\,s^2 \;+\; \text{higher order terms}\,.$$

It follows that

$$\widetilde{\sigma}(1,\varphi)^*(dx) \;=\; dx\,, \qquad \widetilde{\sigma}(1,\varphi)^*(dy) \;=\; dy$$

and

$$\begin{aligned}\widetilde{\sigma}(1,\varphi)^*(dy_1) \;=\; -\left(\frac{d}{dx}(\log\varphi'(x))\,dt \;+\; t\frac{d}{dx}\left(\frac{d}{dx}(\log\varphi'(x))\right)\,dx\right)y^2 \\ -\,2t\,\frac{d}{dx}(\log\varphi'(x))\,y\,dy\,.\end{aligned}$$

Hence on $\Delta^1 \times F^+\mathbb{R}$,

$$\widetilde{\sigma}(1,\varphi)^*(gv) \;=\; -\frac{1}{y}\,\frac{d}{dx}(\log\varphi'(x))\,dt\wedge dx\wedge dy\,.$$

Going back to the definition of the group cochain, one gets

$$\begin{aligned}\langle C_{1,0}(gv)(1,\varphi), f\rangle \;&=\; -\int_{F^+\mathbb{R}} f(x,y)\cdot\int_0^1 dt\cdot\frac{1}{y}\,\frac{d}{dx}(\log\varphi'(x))dx\wedge dy \\ &=\; -\int_{F^+\mathbb{R}} f(x,y)\left(y\;\frac{d}{dx}(\log\varphi'(x))\right)\,\frac{dx\wedge dy}{y^2}\,,\end{aligned}$$

which finally gives

$$\begin{aligned}&(\,\Phi\,(C_{1,0}\,(gv)))(f^0\,U_\varphi^*, f^1\,U_{\varphi^{-1}}^*) \;= \\ =\;& \int_{F^+\mathbb{R}} f^0\cdot\widetilde{\varphi}^*(f^1)\cdot\left(y\,\frac{d}{dx}(\log\varphi'(x))\right)\,\frac{dx\wedge dy}{y^2} \\ =\;& \tau(f^0U_\varphi^*\cdot\delta_1(f^1U_{\varphi^{-1}}^*)) \;=\; \chi_\tau(\delta_1)(f^0\,U_\varphi^*, f^1\,U_{\varphi^{-1}}^*)\,.\end{aligned}$$

$\square$

References

[1] Cartier, P., Cohomologie de coalgèbres, Exposés 4, 5 in *Séminaire Sophus Lie, 1955-56*, Faculté des Sciences de Paris, 1957.

[2] Chevalley, C. and Eilenberg, S. , Cohomology theory for Lie groups and Lie algebras, *Trans. Amer. Math. Soc.* **63** (1948), 66–76.

[3] Connes, A., Spectral sequence and homology of currents for operator algebras, *Math. Forschungsinst. Oberwolfach Tagungsber.* 41/81; *Funktionalanalysis und C^*-Algebren*, 27-9/3-10, 1981.

[4] Connes, A. , Noncommutative differential geometry, *Inst. Hautes Etudes Sci. Publ. Math.* **62** (1985), 257-360.

[5] Connes, A., Cohomologie cyclique et foncteur Ext^n, *C.R. Acad. Sci. Paris*, Ser. I Math., **296** (1983), 953-958.

[6] Connes, A. , Cyclic cohomology and the transverse fundamental class of a foliation. In *Geometric methods in operator algebras*, (Kyoto, 1983), pp. 52-144, *Pitman Res. Notes in Math.* **123** Longman, Harlow, 1986.

[7] Connes, A. , **Noncommutative geometry**, Academic Press, 1994.

[8] Connes, A. and Moscovici, H. , The local index formula in noncommutative geometry, *GAFA* **5** (1995), 174-243.

[9] Connes, A. and Moscovici, H. , Hopf algebras, cyclic Cohomology and the transverse index theorem, *Commun. Math. Phys.* **198** (1998), 199-246 .

[10] Connes, A. and Moscovici, H. , Cyclic cohomology and Hopf algebras, *Letters Math. Phys.* **48** (1999), 97-108.

[11] Connes, A. and Moscovici, H. , Cyclic Cohomology and Hopf algebra symmetry, *Letters Math. Phys.* **52** (2000), 1-28.

[12] Connes, A. and Moscovici, H., Differentiable cyclic cohomology and Hopf algebraic structures in transverse geometry, In **Essays on Geometry and Related Topics**, pp. 217-256, *Monographie No. 38 de L'Enseignement Mathématique*, Genève, 2001.

[13] Connes, A. and Moscovici, H., Modular Hecke algebras and their Hopf symmetry, *Moscow Math. J.* **4** (2004), 67–109 (arXiv:math.QA/0301089).

[14] Connes, A. and Moscovici, H., Rankin-Cohen brackets and the Hopf algebra of transverse geometry, *Moscow Math. J.* **4** (2004), 111–130 (arXiv:math.QA/0304316).

[15] Crainic, M ., Cyclic cohomology of Hopf algebras, *J. Pure and Appl. Algebra* **166** (2002), 29–66.

[16] Gelfand, I. M. and Fuks, D. B. , Cohomology of the Lie algebra of formal vector fields, *Izv. Akad. Nauk SSSR* **34** (1970), 322-337.

[17] Godbillon, G. , Cohomologies d'algèbres de Lie de champs de vecteurs formels, *Seminaire Bourbaki (1971/1972), Exposé No.421,* Lecture Notes in Math. Vol. **383**, Springer, Berlin (1974), 69-87.

[18] Gorokhovsky, A., Secondary classes and cyclic cohomology of Hopf algebras, *Topology*, **41** (2002), 993 – 1016.

[19] Hajac, P. M., Khalkhali, M., Rangipour, B., Sommerhäuser, Y., Stable anti-Yetter-Drinfeld modules, *C. R. Math. Acad. Sci. Paris*, **338** (2004), 587–590.

[20] Hajac, P. M., Khalkhali, M., Rangipour, B., Sommerhäuser, Y., Hopf-cyclic homology and cohomology with coefficients, *C. R. Math. Acad. Sci. Paris*, **338** (2004), 667–672.

[21] Haefliger, A. , Sur les classes caractéristiques des feuilletages, *Seminaire Bourbaki (1971/1972), Exposé No.412,*

[22] Khalkhali, M., Very Basic Noncommmutative Geometry, arXiv:math.KT/0408416.

[23] Kassel, C. , **Quantum Groups**, Springer-Verlag, Berlin, 1995.

[24] Khalkhali, M., Rangipour, B., Cup products in Hopf-cyclic cohomology, *C. R. Math. Acad. Sci. Paris*, **340** (2005), 9–14.

[25] Loday, J.L., **Cyclic Homology**, Springer-Verlag, 1992, 1998.