Acylindrical Accessibility for Groups

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Abstract

We define the notion of acylindrical splitting (graph of groups) of a group. We bound the combinatorics of acylindrical splittings for f.g. freely indecomposable groups with no 2–torsion. Our arguments imply a theorem of J. Hass on finiteness of acylindrical surfaces in closed 3–manifolds [Ha], finiteness of isomorphism classes of small splittings for (torsion-free) freely indecomposable hyperbolic groups as well as finiteness results for small splittings of f.p. Kleinian and semisimple discrete groups acting on non-positively curved simply connected manifolds.

The notion of accessibility in groups was introduced by M. Dunwoody [Du1] who proved that f.p. groups can split over finite groups only in a bounded way, namely the combinatorics of the corresponding graphs of groups is bounded. Dunwoody's result, which was generalized by M. Bestvina and M. Feighn [Be-Fe1] to splittings over small groups, can be viewed as a generalization of the well known Grushko theorem for free products. In this paper we generalize Grushko theorem in a different direction and get a new kind of accessibility. We do not impose any conditions on edge or vertex groups of the corresponding graphs of groups by themselves, but we do impose a restriction on the relation between them. The accessibility we get, holds for (certain) finitely generated groups, a class for which Dunwoody's accessibility does not hold ([Be-Fe2]), [Du2]).

A splitting (graph of groups) of a group G is reduced if the label of every vertex of valence 2 properly contains the labels of both edges incident to it. Let T be the Bass-Serre tree for a given splitting of the group G. We say that the splitting (graph of groups) and T are k-acylindrical if they are reduced, T is minimal, and for all elements $g \in G$; $g \neq 1$ if g fixes (pointwise) a path in T, this path includes not more than k edges.

Since our definition might seem a bit technical, we list some natural examples:

(i) $G = A *_C B \qquad \forall a \in A \qquad a \notin C \qquad aCa^{-1} \cap C = id$. a 2–acylindrical splitting.

(ii) $G = A*_C = \langle A, t | tC_1t^{-1} = C_2 \rangle \qquad \forall a \in A \qquad aC_1a^{-1} \cap C_2 = id$. A 1-acylindrical splitting.

(iii) Let G be a (torsion-free) hyperbolic group. Every small splitting of G can be written as a 4–acylindrical splitting (see proposition 3.7).

(iv) An incompressible surface S in a compact 3–manifold M is called k-cylindrical if one of the components of $M \setminus S$ contains at most k non-homotopic cylinders. The splitting of $\pi_1(M)$ along $\pi_1(S)$ is $(2k+2)$ acylindrical.

A (Gromov) hyperbolic group has a natural action on a hyperbolic space, its own Cayley graph. This is definitely the "source" of many global results on hyperbolic groups. Our arguments suggest a limited optional substitution for f.g. groups, a k-acylindrical tree. From our point of view the exsitence of a k-acylindrical splitting joined with the inexistence of a splitting over a cyclic subgroup for given f.p. group should provide information on its global structure. In this paper our aim is to bound the combinatorics of all k-acylindrical splittings of a given f.g. freely indecomposable group. Freely indecomposable f.p. groups are shown to have "acylindrical super accessibility" (theorem 3.3), namely there exist finitely many splittings that cover all k-acylindrical splittings of such groups. This last result allow us to deduce the finiteness of isomorphism classes of small splittings for (torsion-free) freely indecomposable hyperbolic groups (theorem 3.3) as well as Kleinian groups with no rank 2 parabolics and semisimple groups acting freely and discretely on non-positively curved simply-connected manifolds (corollary 3.9). In addition we get a stronger version of J. Hass' theorem on the finiteness of acylindrical surfaces in 3-manifolds [Ha] (theorems 3.5, 3.6). To stress our point of view on the information carried by the existence of an acylindrical splitting for a group, we add an appendix in which we apply Barge-Ghys bounded cocycles [Ba-Gh] for groups acting freely on real trees, to show $H_b^2(G,R) \neq 0$ for non-cyclic groups G with a k-acylindrical splitting and no 2-torsion. In all our arguments we assume the groups in question have no 2-torsion. In fact some of what we do generalize in the presence of 2-torsion (at least for f.p. groups), but assumptions and assertions become more technical so it seems to us better to discuss 2-torsion elsewhere. The same remark holds for the assumption of torsion-free in the case of hyperbolic groups. In contrast with our theorem regarding finiteness of isomorphism classes of small splittings for freely indecomposable hyperbolic groups, free groups have lots of non-isomorphic ones:

$$\frac{\{x^m\}}{\{x\} \qquad \{x^m, y\}}$$

In the first section we use the Bestvina-Paulin method ([Be], [Pa]) to bound the image of generators in groups with no cyclic splittings. In section 2 we elaborate our argument to deduce a similar result for freely indecomposable groups up to isomorphsim. In section 3 we use these bounds to get our finiteness results. The appendix studies the bounded cohomology of groups with acylindrical splittings.

The main techniques we are using are based on a joint work of the author with E. Rips ([Ri-Se1], [Ri-Se2]). The whole concept of this paper would have never been occured to us without Joel Hass' question on the possibility to generalize his 3-manifold result. We are greatly indebted to him for that and for his continuing interest. We thank Etienne Ghys, Frederic Paulin and Leonid Potyagailo for fruitful discussions around questions discussed in this paper.

1. Groups with no cyclic splittings

To demonstrate our approach we first assume our finitely generated group with no 2-torsion $G = \langle g_1, \cdots, g_t \rangle$ has no splitting over a cyclic subgroup. Let $k \geq 1$ be given and let $T_1, T_2, \cdots$ be a sequence of Bass-Serre trees for non-conjugate k-acylindrical splittings of G. Clearly for each tree T_m we have a natural action $G \times T_m \to T_m$. Let $f_m(u) = \max_{1 \leq j \leq t} d_{T_m}(u, g_j u)$ be a function on the vertices of T_m . The functions $\{f_m\}_{m=1}^{\infty}$ are discrete so they achieve a minimum. Let $u_m \in T_m$ be a vertex on which the minimum is obtained and let $\mu_m = \max_{1 \leq j \leq t} d_{T_m}(u_m, g_j u_m)$. Our aim is to get a global bound on the μ_m 's under our assumptions on G.

From now on we assume $\mu_m \to \infty$ and use the Bestvina-Paulin method (see [Be], [Pa]) which is an elaborate application of the Gromov topology on metric spaces [Gr1]. Let X_m

be the pointed metric space $\left(\frac{1}{\mu_m}T_m, u_m\right)$ endowed with left isometric action of the group G. Clearly, the convex hull of the image of u_m under a finite set $P \subset G$ is compact, and since this convex hull is a finite tree of bounded combinatorics and length of edges (the bound is independent of m), these convex hulls form a totally bounded sequence of metric spaces. The last simple observations allow us to apply the following:

Theorem 1.1 ([Pa], 2.3) *Let $\{X_m\}_{m=1}^{\infty}$ be a sequence of δ_m-hyperbolic spaces with $\delta_\infty = \underline{\lim}\, \delta_m < \infty$. Let G be a countable group isometrically acting on X_m. Suppose there exists a base point u_m in X_m such that for every finite subset P of G, the closed convex hull of the images of u_m under P is compact and these convex hulls are totally bounded metric spaces. Then there is a subsequence converging in the Gromov topology to a $50\delta_\infty$-hyperbolic space X_∞ endowed with an isometric action of G.*

Our spaces $\{X_m\}_{m=1}^{\infty}$ are trees, so they are 0–hyperbolic and therefore any metric space which is the limit of a subsequence is a real tree. Let Y be the limit of a subsequence (still denoted) X_m.

Proposition 1.2:

(i) *The stabilizers of edges of Y are cyclic.*
(ii) *The stabilizers of tripods (convex hull of three points which are not on an interval in Y) are trivial.*

Proof: Let $h_1, h_2 \in G$ fix a segment $[x,y] \subseteq Y$. Let $D = d_Y(x,y)$ and let $x_m, y_m \in X_m$ be sequences converging to x and y correspondingly. By theorem 1.2 for all $m > m_0$ we have:

$$\max\left(d_{X_m}(h_1(x_m), x_m), d_{X_m}(h_1(y_m), y_m)\right) < \frac{D}{10}$$
$$\max\left(d_{X_m}(h_2(x_m), x_m), d_{X_m}(h_2(y_m), y_m)\right) < \frac{D}{10}$$
$$d_{X_m}(x_m, y_m) > \frac{9D}{10}$$

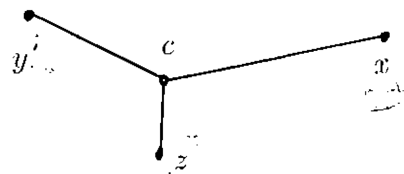
Since X_m is a discrete tree, h_1 and h_2 act as translations of a line on the segment $[\overline{x}_m, \overline{y}_m] \subset [x_m, y_m]$, $d_{X_m}(x_m, \overline{x}_m) = d_{X_m}(\overline{y}_m, y_m) = \frac{D}{10}$. Translations on a line do commute, so the commutator $[h_1, h_2]$ fixes a segment $[\tilde{x}_m, \tilde{y}_m]$, $d_{X_m}(\tilde{x}_m, \tilde{y}_m) \geq \frac{D}{10}$ located at the center of the interval $[x_m, y_m]$. Therefore $[h_1, h_2]$ fixes a segment of length $\frac{\mu_m \cdot D}{10}$ in all trees T_m for $m > m_0$. For $m > m_1$, $\frac{\mu_m \cdot D}{10} > k+1$, hence $[h_1, h_2]$ fixes a path of length at least $k+1$, but our trees are assumed k-acylindrical so $[h_1, h_2] = 1$ and stabilizers of edges are abelian. Moreover since h_1 and h_2 are commuting elements acting as translations on a line, they are hyperbolic elements with a common axis. Since our trees are k-acylindrical they therefore generate a cyclic group, so stabilizers of edges of Y are cyclic.

Under the notations above let h fix $[x,y] \subseteq Y$ and suppose $h^\ell = 1$. For all $m > m_2$ we have:

$$\max\left(d_{X_m}(h(x_m), x_m), d_{X_m}(h(y_m), y_m)\right) \leq \frac{D}{20\ell}$$
$$d_{X_m}(x_m, y_m) \geq \frac{9D}{10}$$

Then h acts as translation on $[\hat{x}_m, \hat{y}_m] \subset [x_m, y_m]$, $d_{X_m}(\hat{x}_m, \hat{y}_m) = \frac{D}{10}$ and $[\hat{x}_m, \hat{y}_m]$ located at the center of $[x_m, y_m]$. But if h acts as translation, so does h^ℓ and in particular $h^\ell \neq 1$, a contradiction. Therefore stabilizers of edges are torsion-free and we have (i).

Let $x, y, z \in Y$ be a tripod fixed by an element $g \in G$, and let $c \in Y$ be its three valence vertex. Let $x_m, y_m, z_m \in X_m$ be sequences converging to x, y, z correspondingly and let c_m



be three valence vertex of the tripod x_m, y_m, z_m. Let $D = \min(d_Y(x,c), d_Y(y,c), d_Y(z,c))$, and let m_3 be large enough so that for all $m > m_3$:

$$\max(d_{X_m}(g(x_m), x_m), d_{X_m}(g(y_m), y_m), d_{X_m}(g(z_m), z_m)) < \frac{D}{10}$$
$$\min(d_{X_m}(x_m, c_m), d_{X_m}(y_m, c_m), d_{X_m}(z_m, c_m)) > \frac{9D}{10}$$

Since X_m is a tree, the above inequality implies $g(c_m) = c_m$ for all $m > m_3$, and the same holds, therefore, for a $\frac{D}{10}$ neighborhood of c_m in the tripod x_m, y_m, z_m. In particular g fixes a segment of length $\frac{\mu_m D}{10}$ in T_m, so by picking m large enough and by our k-acylindrical assumption, g is trivial.

□

Proposition 1.3: *Let* $[x,y] \subseteq Y$ *be a segment and suppose* $H = stab([x,y]) \neq \{id\}$. *Then for all* $[x_0, y_0] \subseteq [x,y]$ $\quad stab([x_0,y_0]) = stab([x,y])$.

Proof: Assume the converse. Let h_1 fix $[x_0, y_0]$ but not $[x,y]$, let $h \in H$; $h \neq 1$ and suppose (w.l.o.g.) y is not fixed by h_1. By our assumptions $h_1 h(y) = h_1(y)$, on the other hand $h(h_1(y)) \neq h_1(y)$ since h does not fix a tripod by the previous proposition. But since $h, h_1 \in stab([x_0, y_0])$ h and h_1 do commute, a contradiction.

□

Proposition 1.3 shows the action of G on the R-tree Y satisfying the ascending chain condition, so it is small in Rips' sense [Ri]. Rips has completely classified small actions with inversions for finitely presented groups, so we do not really need the assumption of no 2-torsion in this case, but we prefer to restrict ourselves to actions with no inversions. According to Rips' theory the action is divided into axial, interval exchange transformation (IET), indiscrete minimal actions of the free group (e.g. Levitt type) and discrete components. In order to show G splits over a locally cyclic subgroup we need to analyze each of these components separately.

Indiscrete minimal actions of the free group. When the action is minimal and the ACC condition holds, the stabilizer of a given edge is the stabilizer of the whole component [Ri]. Since by proposition 1.2 the stabilizer of a tripod is trivial, the stabilizer of indiscrete actions of the free group components, minimal IET components and minimal axial components which are not isometric to a real line are trivial. Therefore, in the presence of a component with minimal indiscrete action of the free group, G has the form $A * F_n$ where F_n is a free group on n generators according to [Ri], so G is even freely decomposable.

IET components. Again, in our case the stabilizer of a minimal IET component is trivial, so we are in an identical situation to the one described in [Ri-Se1] for studying the automorphism group of a hyperbolic group. The discussion there is detailed and shows our group G is either a free product with a Fuchsian group or it splits over a finite or infinite cyclic group.

The axial components. First suppose an axial component is not isometric to the real line. Then the action is minimal, stabilizers of edges are trivial and we have one of the following presentations for G :

(i) $G = A *_Z F_2$

(ii) $G = A *_{Z_\ell} B_{1,k}$ where ℓ/k and:

$$B_{1,k} = \left\{a, b | [a, b]^k = 1\right\} .$$

(iii) $G = A * F_2$

(see [Ri] for details). Again our group splits over a cyclic group. For the real line case we have $G = A *_C B$ where C is the stabilizer of this real line component which is (torsion-free) cyclic by proposition 1.2.

The discrete case. We have in fact reduced our problem to the standard Bass-Serre theory, and since all edge stabilizers are cyclic (if G is not infinite cyclic) our group G admits a splitting with cyclic stabilizers.

Recall, that the whole construction of the tree Y is based on the existence of a subsequence of k-acylindrical splittings with $\mu_m = \max_{1 \leq j \leq t} d_{X_m}(g_j(u_m), u_m) \to \infty$. Therefore, we have:

Theorem 1.5 *Let $G = \langle g_1, \cdots, g_t \rangle$ be a f.g. group with no 2-torsion and no cyclic splitting. Let $\{T_m\}_{m=1}^{\infty}$ be the Bass-Serre trees for the k-acylindrical splittings of G. There exists a constant λ_k and vertices $u_m \in T_m$ so that $\max_{1 \leq j \leq t} d_{T_m}(u_m, g_j(u_m)) \leq \lambda_k$.*

2. Freely indecomposable groups

In this section we use a modification of the argument presented so far together with techniques introduced in [Ri-Se1] to get a bound on the image of the generators of a f.g. freely indecomposable group G with no 2-torsion in all isomorphism classes of k-acylindrical splittings. Let $T_1, T_2, \cdots$ be a sequence of Bass-Serre trees for non-isomorphic k-acylindrical splittings of G. Let u_m be a vertex of T_m and $\varphi_m \in \mathrm{Aut}(G)$ satisfy: $\max_{1 \leq j \leq t} d_{T_m}(u_m, \varphi_m(g_j)u_m) = \min_{\substack{\varphi \in \mathrm{Aut}(G) \\ u \in T}} \max_{1 \leq j \leq t} d_{T_m}(u, \varphi(g_j)u)$ and let $\mu_m = \max_{1 \leq j \leq t} d_{T_m}(u_m, \varphi_m(g_j)u_m)$.

We assume, as we did in section 1, that $\mu_m \to \infty$ and construct the pointed metric spaces $X_m = \left(\frac{1}{\mu_m} T_m, u_m\right)$ edowed with left isometric action of the group G via the isomorphism φ_m . The spaces X_m satisfy the assumptions of theorem 1.1 so there exists a subsequence (still denoted) X_m converges to Y, a real tree endowed with a G-action. Propositions 1.2 and 1.3 remains valid for the action of G on Y, so once again we have Rips' ACC condition and we can use his theory to analyze the action of G on Y.

Indiscrete actions of the free group. If Y contains such component (e.g. Levitt components), G can be represented as $G = A * F_n$ where F_n is a f.g. free group, in particular G is freely decomposable, a contradiction.

IET components. The stabilizer of an IET component is trivial since the stabilizers of tripods are trivial. In [Ri-Se1] we analyze the IET case in details and show one can find an automorphism $\psi \in \mathrm{Aut}(G)$ such that:

$$\max_{1 \leq j \leq t} d_Y(id, g_j(id)) > \max_{1 \leq j \leq t} d_Y(id, \psi(g_j)(id))$$

but if such inequality holds for Y it clearly holds for T_m for large enough m, a contradiction to the choice of φ_m .

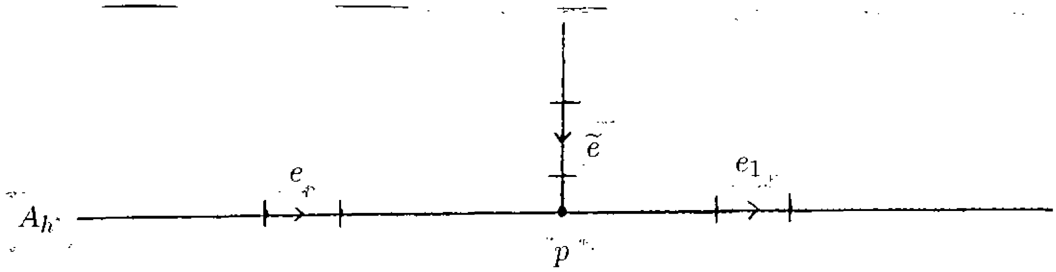
The axial components. Suppose first an axial component is not isometric to the real line. Then the action is minimal, stabilizers of edges are trivial and using the same argument we have used for the IET components [Ri-Se1], there exists an isomorphism $\psi \in \mathrm{Aut}(G)$ that maps the generators of G to elements with shorter length in Y, a contradiction to the choice of φ_m . For the real line components we need the following lemma:

Lemma 2.1 *Let T be a k-acylindrical tree for the group G. If $H < G$ and H is solvable then either H is contained in a stabilizer of a vertex of T or H is abelian.*

Proof: Suppose H is not contained in a stabilizer of a vertex of T. Let $T_H \subseteq T$ be a minimal subtree for the action of H. Since T_H is k-acylindrical (not necessarily reduced) for the action of H, if it is isometric to a real line H is abelian since it has no 2–torsion. At this stage one may apply our bounded cohomology appendix or simply the following:

Claim 2.2 Let T be a k-acylindrical tree for a group H which is not isometric to a real line. Then H contains a free subgroup.

Proof: By the k-acylindrical condition if all elements of H are elliptics, H has a fixed point (we do not need to assume H is f.g.) and the action is trivial. Therefore, there exists a hyperbolic element $h \in H$. Let A_h be the axis of h, let e be an oriented edge in A_h , and let $\tilde{e}$ be some image of e outside A_h (there is such because T is minimal and not isometric to a real line). Let $L \subseteq T$ be the convex hull of $\tilde{e}$ and A_h , let p be the three valence vertex in L, and let e_1 be the image of e under some power of h, such that e_1 and e belong to distinct components of $L \setminus \{p\}$:



Clearly, one of the elements $h_1 \in H$; $h_1(\tilde{e}) = e$ or $h_2 \in H$; $h_2(\tilde{e}) = e_1$ is hyperbolic (w.l.o.g. h_2) and its axis has an edge included in A_h . Therefore, the group generated by sufficiently high powers of h and h_2 is free (see [Be-Fe1]).

□

Having proved the lemma, we may conclude that A, the subgroup of G corresponding to an axial component which is isometric to a real line, is abelian. From our $k-$ acylindrical condition, it follows that for large enough m the elements of A are hyperbolic, so A is cyclic and we are actually in the discrete case.

The discrete case. Left with the standard Bass-Serre theory, we are guaranted stabilizers of edges are cyclic and non-trivial because G is freely indecomposable and propositions 1.2, 1.3. Our treatment is similar to the one given in [Ri-Se1] for the discrete case (which appears in an effective version in [Ri-Se2] as well), although our case is somewhat easier since we deal with trees and not with the Cayley graph of a hyperbolic group.

Our limiting (discrete) tree Y is a pointed metric space (Y, y_0) which is the limit of the pointed metric spaces (X_m, u_m) . To get a shortening argument for the discrete case, we divide our treatment into two cases, the first is when y_0 belongs to the interior of an edge of Y and the second occurs when y_0 is a vertex of Y.

Case 1 $y_0 \subset \text{int}(e)$, $e = [v_0, v_1]$ is an edge of Y. Let $z \in \text{stab}(e) = C$ and let $\overline{e}$ denote the edge corresponding to e in Y/G . Again we need to split our treatment:

Case 1A $\overline{e}$ is a separating edge in Y/G .

In this case we can clearly write G as $A*_C B$, where by our construction G is properly included in both A and B. We will define automorphisms of G (in fact Dehn twists) that will reduce the length of elements of A and B appear in the generators of G, in such a way that for large enough m the distance from u_m to $\varphi_m(g_j)(u_m)$ will decrease for those g_j which are not elements of C . This will clearly lead to a contradiction to the way we have chosen the automorphisms φ_m .

For each j let g_j be given in a reduced form with respect to the above splitting of G : $\quad g_j = a_1^j b_1^j \cdots a_{q_j}^j b_{q_j}^j \quad 1 \leq j \leq t \quad a_i^j \in A;\ b_i^j \in B \quad a_1^j$ or $b_{q_j}^j$ may be the identity element. Let $q = \max\limits_{1 \leq j \leq t} q_j$, $\varepsilon' = \min\left(d_Y(y_0, v_0), d_Y(y_0, v_1)\right)$ and ε be the minimum between ε' and the shortest length of an edge of Y (Recall, Y is discrete and G is f.g. so ε is positive). Since the pointed metric spaces (X_m, u_m) converge in the Gromov topology to Y (theorem 1.1), for large enough m we have:

$$
(2.1)\qquad
\begin{aligned}
&|d_{X_m}\left(\varphi_m\left(z^\delta a_i^j z^{-\delta}\right)(u_m), u_m\right) - d_Y\left(a_i^j(y_0), y_0\right)| < \varepsilon_1 \\
&|d_{X_m}\left(\varphi_m\left(z^\delta b_i^j z^{-\delta}\right)(u_m), u_m\right) - d_Y\left(b_i^j(y_0), y_0\right)| < \varepsilon_1 \\
&d_{X_m}\left(\varphi_m\left(z^\delta\right)(u_m), u_m\right) < \varepsilon_1 \\
&|d_{X_m}\left(\varphi_m\left(a_1^j b_1^j \cdots a_i^j b_i^j\right)(u_m), u_m\right) - d_Y\left(a_1^j b_1^j \cdots a_i^j b_i^j(y_0), y_0\right)| < \varepsilon_1 \\
&|d_{X_m}\left(\varphi_m\left(a_{i_1}^j\right)(u_m), \varphi_m\left(z^\delta a_{i_2}^j z^{-\delta}\right)(u_m)\right) - d_Y\left(a_{i_1}^j(y_0), z^\delta a_{i_2}^j z^{-\delta}(y_0)\right)| < \varepsilon_1 \\
&|d_{X_m}\left(\varphi_m\left(b_{i_1}^j\right)(u_m), \varphi_m\left(z^\delta b_{i_2}^j z^{-\delta}\right)(u_m)\right) - d_Y\left(b_{i_1}^j(y_0), z^\delta b_{i_2}^j z^{-\delta}(y_0)\right)| < \varepsilon_1
\end{aligned}
$$

where $1 \leq i, i_1, i_2 \leq q_j$, $\quad \varepsilon_1 = \frac{\varepsilon}{100q}$, $\quad \delta = 0;\ \pm 1$.

Proposition 2.3 *Let* $w_m \in \left[u_m, \varphi_m\left(a_i^j\right)(u_m)\right]$, $\quad w_m' \in \left[u_m, \varphi_m\left(b_i^j\right)(u_m)\right]$ *satisfy:*

$$d_{X_m}(u_m, w_m) = \frac{\varepsilon}{2} \qquad d_{X_m}\left(u_m, w_m'\right) = \frac{\varepsilon}{2} .$$

Then for either $\delta = +1$ *or* $\delta = -1$:

$$d_{X_m}\left(u_m, \varphi_m\left(z^{\delta}\right)(w_m)\right) < d_{X_m}(u_m, w_m) < d_{X_m}\left(u_m, \varphi_m\left(z^{-\delta}\right)(w_m)\right)$$
$$d_{X_m}\left(\varphi_m\left(z^{\delta}\right)(w_m), \varphi_m\left(z^{-\delta}\right)(w_m)\right) < 2\varepsilon_1$$
$$d_{X_m}\left(u_m, \varphi_m\left(z^{-\delta}\right)\left(w_m'\right)\right) < d_{X_m}\left(u_m, w_m'\right) < d_{X_m}\left(u_m, \varphi_m\left(z^{\delta}\right)\left(w_m'\right)\right)$$
$$d_{X_m}\left(\varphi_m\left(z^{\delta}\right)\left(w_m'\right), \varphi_m\left(z^{-\delta}\right)\left(w_m'\right)\right) < 2\varepsilon_1 .$$

Proof: $\varphi_m(z^{\delta})$ acts isometrically on X_m . By the k-acylindrical condition and our requirements on $\varphi_m\left(z^{\delta}a_i^j z^{-\delta}\right)(u_m)$, w_m cannot be fixed by $\varphi_m(z^{\delta})$. In addition we have by (2.1):

$$d_{X_m}\left(\varphi_m\left(z^{\delta}\right)(u_m), u_m\right) < \varepsilon_1$$
$$w_m, \varphi_m(z^{\pm 1})w_m \in \bigcap_{\substack{\delta=0,\pm 1\\ 1\leq j\leq t\\ 1\leq i\leq q_j}} \left[u_m, \varphi_m\left(z^{\delta}a_i^j\right)(u_m)\right]$$

So:

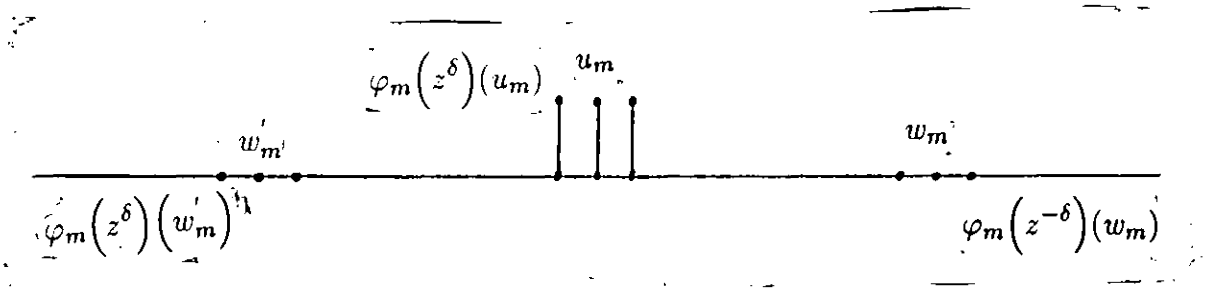
$$d_{X_m}\left(\varphi_m\left(z^{\delta}\right)(w_m), \varphi_m\left(z^{-\delta}\right)(w_m)\right) < 2\varepsilon_1$$

and either for $\delta = 1$ or $\delta = -1$:

$$d_{X_m}\left(u_m, \varphi_m\left(z^{\delta}\right)(w_m)\right) < d_{X_m}(u_m, w_m) < d_{X_m}\left(u_m, \varphi_m\left(z^{-\delta}\right)(w_m)\right) .$$

The inequalities for w_m' follow by an identical argument.

□



Proposition 2.4 *Assume (W.l.o.g.)* $\delta = 1$ *in proposition 2.3. Then for all* $1 \leq j \leq t$, $1 \leq i \leq q_j$:

$$d_{X_m}\left(\varphi_m\left(a_i^j\right)(u_m), u_m\right) > d_{X_m}\left(\varphi_m\left(za_i^j z^{-1}\right)(u_m), u_m\right)$$
$$d_{X_m}\left(\varphi_m\left(a_i^j\right)(u_m), u_m\right) - d_{X_m}\left(\varphi_m\left(za_i^j z^{-1}\right)(u_m), u_m\right) < 2\varepsilon_1$$
$$d_{X_m}\left(\varphi_m\left(b_i^j\right)(u_m), u_m\right) > d_{X_m}\left(\varphi_m\left(z^{-1}b_i^j z\right)(u_m), u_m\right)$$
$$d_{X_m}\left(\varphi_m\left(b_i^j\right)(u_m), u_m\right) - d_{X_m}\left(\varphi_m\left(z^{-1}b_i^j z\right)(u_m), u_m\right) < 2\varepsilon_1$$

Proof: By the inequalities (2.1) we get:

$$d_{X_m}\left(\varphi_m\left(a_i^j\right)(u_m), u_m\right) = 2d_{X_m}(u_m, w_m) + d_{X_m}\left(\varphi_m\left(a_i^j\right)(w_m), w_m\right)$$

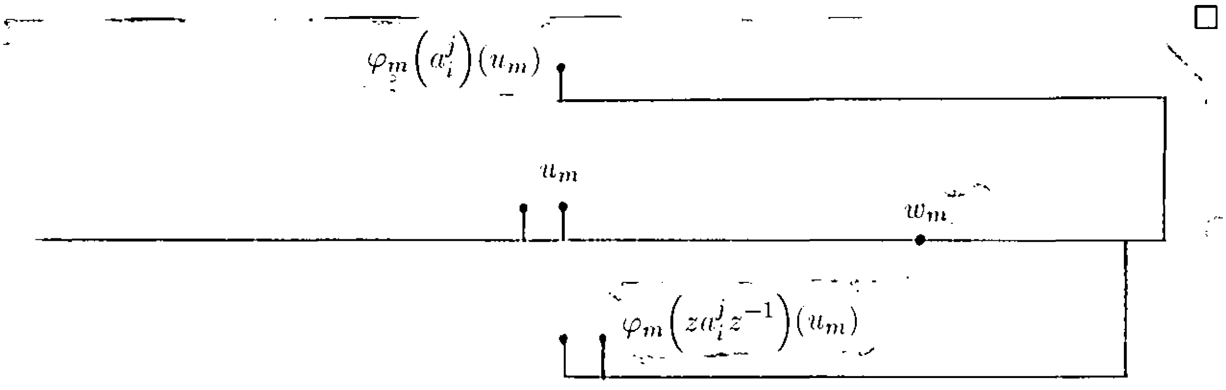
$$d_{X_m}\left(\varphi_m\left(za_i^j z^{-1}\right)(u_m), u_m\right) = d_{X_m}\left(\varphi_m\left(za_i^j\right)(w_m), \varphi_m(z)(w_m)\right)$$
$$+ d_{X_m}(\varphi_m(z)(w_m), u_m) + d_{X_m}\left(\varphi_m\left(za_i^j\right)(w_m), \varphi_m\left(za_i^j z^{-1}\right)(u_m)\right) =$$
$$= d_{X_m}\left(\varphi_m\left(a_i^j\right)(w_m), w_m\right) + d_{X_m}(\varphi_m(z)(w_m), u_m) + d_{X_m}\left(w_m, \varphi_m\left(z^{-1}\right)(u_m)\right)$$

But proposition 2.3 gives us:

$$d_{X_m}(u_m, w_m) > d_{X_m}(\varphi_m(z)(w_m), u_m)$$
$$d_{X_m}(u_m, w_m) - d_{X_m}(\varphi_m(z)(w_m), u_m) < \varepsilon_1$$

So we have proved the proposition for the a_i^j 's. A similar argument for the b_i^j 's completes our proof.

□



Theorem 2.5 *Under the notations above, let* Ψ *be an automorphism of* G *given by:*

$$\forall a \in A \qquad \Psi(a) = zaz^{-1}$$
$$\forall b \in B \qquad \Psi(b) = z^{-1}bz$$

Then for m *large enough (so that inequalities (2.1) hold):*

$$\max_{1\le j\le t} d_{X_m}(\varphi_m(g_j)(u_m), u_m) > \max_{1\le j\le t} d_{X_m}(\varphi_m \cdot \Psi(g_j)(u_m), u_m) \ .$$

Proof: For any j for which $g_j \notin C$ we have by the inequalities (2.1):

$$d_{X_m}(\varphi_m(g_j)(u_m), u_m) = \sum_{i=1}^{q_j} d_{X_m}(\varphi_m\left(a_i^j\right)(u_m), u_m) +$$
$$+ \sum_{i=1}^{q_j} d_{X_m}\left(\varphi_m\left(b_i^j\right)(u_m), u_m\right) - (q_j - 1)[d_{X_m}(u_m, w_m) +$$
$$+ d_{X_m}\left(u_m, w_m'\right) - d_{X_m}\left(w_m, w_m'\right)] > d_{X_m}(\varphi_m\left(za_i^j z^{-1}\right)(u_m), u_m) +$$
$$+ d_{X_m}\left(\varphi_m\left(z^{-1}b_i^j z\right)(u_m), u_m\right) - (q_j - 1)[d_{X_m}(u_m, \varphi_m(z)(w_m) +$$
$$+ d_{X_m}\left(u_m, \varphi_m\left(z^{-1}\right)(w_m)\right) - d_{X_m}\left(\varphi_m(z)(w_m), \varphi_m\left(z^{-1}\right)\left(w_m^1\right)\right)] =$$
$$= d_{X_m}(\varphi_m \cdot \Psi(g_j)(u_m), u_m)$$

Clearly, for m large enough the maximum length of the generators is obtained for some $g_{j_0} \notin C$ so our theorem follows.

$\square$

Theorem 2.5 demonstrates our general approach for the discrete case, and is clearly a contradiction to the way the couples (φ_m, u_m) were chosen, so case 1A can not occur.

Case 1B $\bar{e}$ is a non-separating edge in Y/G.

In this case we can write G as $A*_C$ where $C \neq A$, since otherwise G is solvable and by lemma 2.2 if G is not cyclic, it admits no k-acylindrical splittings at all. Our treatment for the non-separating case is very similar to the separating one.

For each j let g_j be given in a reduced form with respect to the above splitting $G = \langle A, f | fcf^{-1} = h(f) \rangle$: $g_j = a_1^j f^{n_1^j} a_2^j f^{n_2^j} \cdots a_{q_j}^j f^{n_{q_j}^j}$ $a_i^j \in A$, a_1^j or $f^{n_{q_j}^j}$ may be the identity element. As we did in case $1A$ let $q = \max_{1 \leq j \leq t} q_j$, $\varepsilon' = \min(d_Y(y_0, v_0), d_Y(y_0, v_1))$ and ε the minimum between ε' and the shortest length of an edge of Y. By the convergence in the Gromov topology of the pointed metric spaces (X_m, u_m) to Y (theorem 1.1) we have for large enough m the following inequalities:

$$
\begin{aligned}
& g_j \in C \Rightarrow d_{X_m}(\varphi_m(g_j)(u_m), u_m) < \varepsilon_1 \\
& |d_{X_m}\left(\varphi_m\left(a_i^j\right)(u_m), u_m\right) - d_Y\left(a_i^j(y_0), y_0\right)| < \varepsilon \\
& d_{X_m}\left(\varphi_m\left(z^\delta\right)(u_m), u_m\right) < \varepsilon_1 \\
(2.2) \qquad & |d_{X_m}\left(\varphi_m\left(\left(fz^\delta\right)^{n_i^j}\right)(u_m), u_m\right) - d_Y\left(f^{n_i^j}(y_0), y_0\right)| < \varepsilon_1 \\
& |d_{X_m}\left(\varphi_m\left(a_1^j\left(fz^\delta\right)^{n_1^j} \cdots a_i^j\left(fz^\delta\right)^{n_i^j}\right)(u_m), u_m\right) - \\
& d_Y\left(a_1^j f^{n_1^j} \cdots a_i^j f^{n_i^j}(y_0), y_0\right)| < \varepsilon_1 \\
& |d_{X_m}\left(\varphi_m\left(a_{i_1}^j\right)(u_m), \varphi_m\left(a_{i_2}^j\right)(u_m)\right) - d_Y\left(a_{i_1}^j(y_0), a_{i_2}^j(y_0)\right)| < \varepsilon_1
\end{aligned}
$$

where $1 \leq i_1, i_2, \ i \leq q_j$, $\varepsilon_1 = \frac{\varepsilon}{100q}$, $\delta = 0, \ \pm 1, \pm 2$.

By an identical argument we use to prove proposition 2.3, and from the inequalities (2.2), we obtain the following:

Proposition 2.6 *Let* $w_m \in [u_m, \varphi_m(f)(u_m)]$, $w'_m \in [u_m, \varphi_m(f^{-1})(u_m)]$ *satisfy:*

$$d_{X_m}(u_m, w_m) = \frac{\varepsilon}{2} \quad d_{X_m}\left(u_m, w'_m\right) = \frac{\varepsilon}{2}$$

Then for either $\delta = +1$ *or* $\delta = -1$:

$$
\begin{aligned}
& d_{X_m}\left(u_m, \varphi_m\left(z^\delta\right)(w_m)\right) < d_{X_m}(u_m, w_m) < d_{X_m}\left(u_m, \varphi_m\left(z^{-\delta}\right)(w_m)\right) \\
& d_{X_m}\left(\varphi_m\left(z^\delta\right)(w_m), \varphi_m\left(z^{-\delta}\right)(w_m)\right) < 2\varepsilon_1 \\
& d_{X_m}\left(u_m, \varphi_m\left(z^{-\delta}\right)\left(w'_m\right)\right) < d_{X_m}\left(u_m, w'_m\right) < d_{X_m}\left(u_m, \varphi_m\left(z^\delta\right)\left(w'_m\right)\right) \\
& d_{X_m}\left(\varphi_m\left(z^\delta\right)\left(w'_m\right), \varphi_m\left(z^{-\delta}\right)\left(w'_m\right)\right) < 2\varepsilon_1
\end{aligned}
$$

Proposition 2.7 *Assume (w.l.o.g.)* $\delta = 1$ *in the previous proposition. Then:*

$$d_{X_m}(\varphi_m(f)(u_m), w_m) > d_{X_m}(\varphi_m(fz)(u_m), w_m)$$

$$d_{X_m}(\varphi_m(f)(u_m), u_m) - d_{X_m}(\varphi_m(fz)(u_m), u_m) < \varepsilon_1$$

Proof: By the inequalities (2.2):

$$d_{X_m}(\varphi_m(f)(u_m), u_m) = d_{X_m}(\varphi_m(f)(u_m), w_m) + d_{X_m}(w_m, u_m) <$$
$$d_{X_m}(\varphi_m(fz)(u_m), \varphi_m(z)(w_m)) + d_{X_m}(\varphi_m(z)(w_m), u_m) =$$
$$d_{X_m}(\varphi_m(fz)(u_m), u_m) .$$

□

<u>Theorem 2.8</u> *Let* $u'_m = \varphi_m(z)(u_m)$ *and let* Ψ *be an automorphism of* G *given by:*

$$\Psi(a) = a \qquad \forall a \in A$$
$$\Psi(f) = fz^2$$

Then for m *large enough (so that inequalities (2.2) hold):*

$$\max_{1 \le j \le t} d_{X_m}(\varphi_m(g_j)(u_m), u_m) > \max_{1 \le j \le t} d_{X_m}\left(\varphi_m \circ \Psi(g_j)\left(u'_m\right), u'_m\right)$$

Proof: By the inequalities (2.2) the maximum above obtained for $g_j \notin C$, and clearly for $g_j \in C$:

$$d_{X_m}(\varphi_m(g_j)(u_m), u_m) = d_{X_m}\left(\varphi_m \circ \Psi(g_j)\left(u'_m\right), u'_m\right)$$

so for shortening purposes we may assume $g_j \notin C$.

By proposition 2.7 and (2.2):

$$d_{X_m}(\varphi_m(f)(u_m), u_m) = d_{X_m}\left(\varphi_m(f)\left(u'_m\right), u'_m\right) >$$
$$d_{X_m}\left(\varphi_m(f_z)\left(u'_m\right), u'_m\right) > d_{X_m}\left(\varphi_m\left(fz^2\right)\left(u'_m\right), u'_m\right) =$$
$$d_{X_m}\left(\varphi_m \circ \Psi(f)\left(u'_m\right), u'_m\right)$$
$$d_{X_m}\left(\varphi_m\left(f^{n_i^j}\right)(u_m), u_m\right) - d_{X_m}\left(\varphi_m \circ \Psi\left(f^{n_i^j}\right)\left(u'_m\right), u'_m\right) =$$
$$n_i^j d_{X_m}\left(w_m, \varphi_m\left(z^2\right)(w_m)\right)$$

For $a \in A$ we have:

$$d_{X_m}(\varphi_m(a)(u_m), u_m) = d_{X_m}\left(\varphi_m(a)\left(w'_m\right), w'_m\right) +$$
$$2d_{X_m}\left(w'_m, u_m\right) > d_{X_m}\left(\varphi_m(a)\left(w'_m\right), w'_m\right) + 2d_{X_m}\left(w'_m, u'_m\right) =$$
$$d_{X_m}\left(\varphi_m \circ \Psi(a)\left(u'_m\right), u'_m\right) .$$

(2.3) $$d_{X_m}(\varphi_m(a)(u_m), u_m) - d_{X_m}\left(\varphi_m \circ \Psi(a)\left(u'_m\right), u'_m\right) = 2d_{X_m}\left(w'_m, \varphi_m(z)\left(w'_m\right)\right)$$

The above inequalities conclude our proof for generators g_j of length 1 with respect to the splitting $G = A *_C$. We continue by scanning the subwords of g_j from left to right, essentially in a similar way we've dealt with amalgamated products. Let $\tau = \tau_1 f^{n_i^j}$ be a left subword of some reduced form of a generator g_j and suppose:

$$d_{X_m}(\varphi_m(\tau_1)(u_m), u_m) > d_{X_m}\left(\varphi_m \circ \Psi(\tau_1)\left(u_m'\right), u_m'\right)$$
$$d_{X_m}(\varphi_m(\tau)(u_m), u_m) > d_{X_m}\left(\varphi_m \circ \Psi(\tau)\left(u_m'\right), u_m'\right).$$

(The first assumption should hold only if $\tau_1 \neq 1$).

<u>Claim 2.9</u> With the notations above:

$$d_{X_m}\left(\varphi_m\left(\tau a_{i+1}^j\right)(u_m), u_m\right) > d_{X_m}\left(\varphi_m \circ \Psi\left(\tau a_{i+1}^j\right)\left(u_m'\right), u_m'\right)$$

Proof: First assume $n_i^j < 0$. Then:

$$d_{X_m}\left(\varphi_m \circ \Psi\left(\tau a_{i+1}^j\right)\left(u_m'\right), u_m'\right) = d_{X_m}\left(\varphi_m \circ \Psi(\tau)\left(u_m'\right), u_m'\right) +$$
$$d_{X_m}\left(\varphi_m \circ \Psi\left(a_{i+1}^j\right)\left(u_m'\right), u_m'\right) - \left[d_{X_m}\left(u_m', w_m\right) + d_{X_m}\left(u_m', w_m'\right) - d_{X_m}\left(w_m, w_m'\right)\right]$$
$$d_{X_m}\left(\varphi_m\left(\tau a_{i+1}^j\right)(u_m), u_m\right) = d_{X_m}(\varphi_m(\tau)(u_m), u_m) +$$
$$d_{X_m}\left(\varphi_m\left(a_{i+1}^j\right)(u_m), u_m\right) - \left[d_{X_m}(u_m, w_m) + d_{X_m}\left(u_m, w_m'\right) - d_{X_m}\left(w_m, w_m'\right)\right]$$

so the claim follows in this case from (2.3).

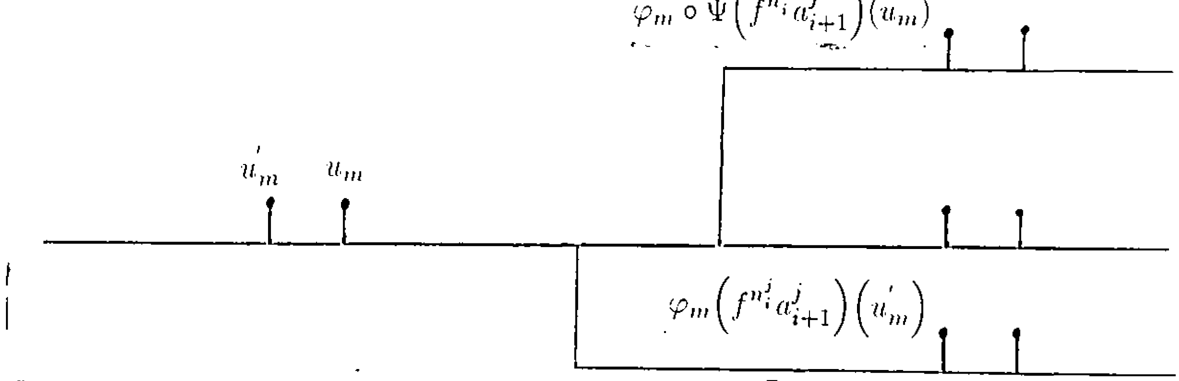
Second assume $n_i^j > 0$. Then:

$$d_{X_m}\left(\varphi_m \circ \Psi\left(\tau a_{i+1}^j\right)\left(u_m'\right), u_m'\right) = d_{X_m}\left(\varphi_m \circ \Psi(\tau_1)\left(u_m'\right), u_m'\right) +$$
$$d_{X_m}\left(\varphi_m \circ \Psi\left(f^{n_i^j} a_{i+1}^j\right)\left(u_m'\right), u_m'\right) - \left[d_{X_m}\left(u_m', w_m\right) + d_{X_m}\left(u_m', w_m'\right) - d_{X_m}\left(w_m, w_m'\right)\right]$$

In parallel with inequality (2.3) we obtain:

$$d_{X_m}\left(\varphi_m\left(f^{n_i^j} a_{i+1}^j\right)\left(u_m\right), u_m\right) - d_{X_m}\left(\varphi_m \circ \Psi\left(f^{n_i^j} a_{i+1}^j\right)(u_m'), u_m'\right)$$
$$\geq 2 d_{X_m}\left(\varphi_m\left(z^2\right)(w_m), w_m\right)$$

and the claim follows, since we have assumed the inequality for τ_1 . □



Now suppose τ , a left subword of one of the g_j has the reduced form $\tau = \tau_1 a_i^j$, and assume we have shortening inequalities for shorter subwords.

Claim 2.10 With the notations above:

$$d_{X_m}\left(\varphi_m\left(\tau f^{n_i^j}\right)(u_m), u_m\right) > d_{X_m}\left(\varphi_m \circ \Psi\left(\tau f^{n_i^j}\right)\left(u_m^{'}\right), u_m^{'}\right)$$
$$d_{X_m}\left(\varphi_m\left(\tau f^{n_i^j}\right)(u_m), u_m\right) - d_{X_m}\left(\varphi_m \circ \Psi\left(\tau f^{n_i^j}\right)\left(u_m^{'}\right), u_m^{'}\right)$$
$$- \left[d_{X_m}(\varphi_m(\tau)(u_m), u_m) - d_{X_m}\left(\varphi_m \circ \Psi(\tau)\left(u_m^{'}\right), u_m^{'}\right)\right]$$
$$\geq d_{X_m}(\varphi_m(z)(w_m), w_m) \ .$$

Proof: First assume $n_i^j > 0$. Then:

$$d_{X_m}\left(\varphi_m \circ \Psi\left(\tau f^{n_i^j}\right)\left(u_m^{'}\right), u_m^{'}\right) = d_{X_m}\left(\varphi_m \circ \Psi(\tau)\left(u_m^{'}\right), u_m^{'}\right) +$$
$$d_{X_m}\left(\varphi_m \circ \Psi\left(f^{n_i^j}\right)\left(u_m^{'}\right), u_m^{'}\right) - [d_{X_m}\left(u_m^{'}, w_m\right) +$$
$$d_{X_m}\left(u_m^{'}, w_m^{'}\right) - d_{X_m}\left(w_m, w_m^{'}\right)]$$

But by proposition 2.7:

$$d_{X_m}\left(\varphi_m\left(f^{n_i^j}\right)\left(u_m^{'}\right), u_m^{'}\right) - d_{X_m}\left(\varphi_m \circ \Psi\left(f^{n_i^j}\right)(u_m), u_m\right) =$$
$$n_i^j d_{X_m}\left(\varphi_m\left(z^2\right)(w_m), w_m\right) > d_{X_m}(\varphi_m(z)(w_m), w_m)$$

and our claim follows for a positive exponent.

Now assume $n_i^j < 0$:

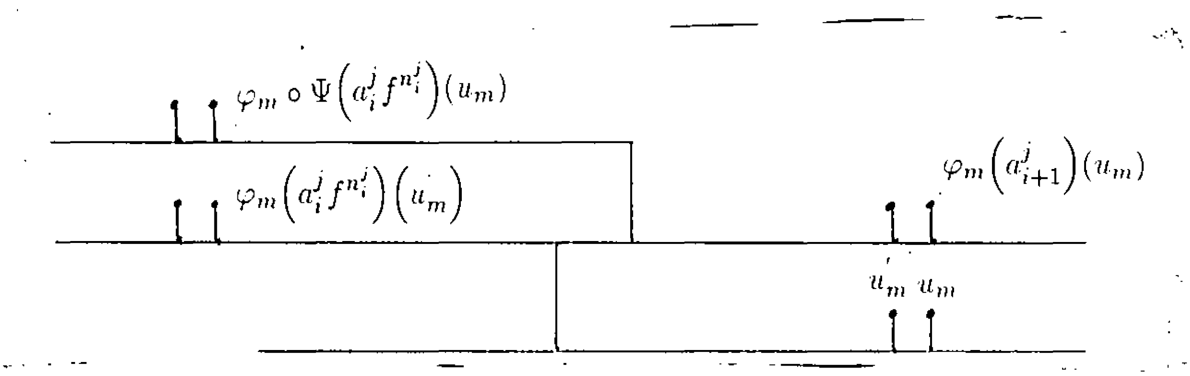
$$d_{X_m}\left(\varphi_m \circ \Psi\left(\tau f^{n_i^j}\right)\left(u_m^{'}\right), u_m^{'}\right) = d_{X_m}\left(\varphi_m \circ \Psi(\tau_1)\left(u_m^{'}\right), u_m^{'}\right) +$$
$$d_{X_m}\left(\varphi_m \circ \Psi\left(a_i^j f^{n_i^j}\right)\left(u_m^{'}\right), u_m^{'}\right) - [d_{X_m}\left(u_m^{'}, w_m\right) +$$
$$d_{X_m}\left(u_m^{'}, w_m^{'}\right) - d_{X_m}\left(w_m, w_m^{'}\right)]$$

In parallel with inequality (2.3), as we obtain in claim 2.9, we have:

$$d_{X_m}\left(\varphi_m\left(a_i^j f^{n_i^j}\right)\left(u_m\right), u_m\right) - d_{X_m}\left(\varphi_m \circ \Psi\left(a_i^j f^{n_i^j}\right)(u_m^{'}), u_m^{'}\right)$$
$$\geq d_{X_m}\left(\varphi_m\left(a_i^j\right)\left(u_m\right), u_m\right) - d_{X_m}\left(\varphi_m \circ \Psi\left(a_i^j\right)(u_m^{'}), u_m^{'}\right) +$$
$$d_{X_m}(\varphi_m(z)(w_m), w_m)$$

and the claim follows.

□



Claims 2.9 and 2.10 complete the proof of theorem 2.8.

□

Theorem 2.8 is clearly a contradiction to the way we've chosen the couples (φ_m, u_m) and therefore case 1B can not occur. We are left with the possibility of y_0 being a vertex of the limiting tree Y.

<u>Case 2</u> y_0 is a vertex of the limiting tree Y.

Our treatment in this case is very similar to the one we have already discussed, although we do not have a distinguished edge in this case, so we compose the given automorphisms φ_m with automorphisms that will make all edges attached to the vertex y_0 shorter, our system of generators become shorter and we get a contradiction to the way the couples (u_m, φ_m) were chosen. Let $\bar{e}_1, \cdots, \bar{e}_s$ be the edges attached to the vertex correponds to y_0 in Y/G, let $C_1, \cdots, C_s$ be their cyclic stabilizers and let $z_\ell \in C_\ell$. Again we need to consider two cases which are parallel to 1A and 1B.

<u>Case 2A</u> The edge $\bar{e}_\ell$ is a bridge in the graph Y/G.

Let L_1^ℓ and L_2^ℓ be the two components of $Y/G \setminus \bar{e}_\ell$, suppose $y_0 \in L_1^\ell$ and let A_ℓ be the fundamental group of the graph of groups L_1^ℓ and B_ℓ of L_2^ℓ . (By the construction of Y , C_ℓ is properly included in both A_ℓ and B_ℓ). For each j let g_j be given in a reduced form with respect to the above splitting:

$$g_j = a_1^j b_1^j \cdots a_{q_j}^j b_{q_j}^j$$

$a_i^j \in A_\ell$, $b_i^j \in B_\ell$ a_1^j or $b_{q_j}^j$ may be the identity. Let ε be the length of the shortest edge in Y. By the convergence of (X_m, u_m) to (Y, y_0) the inequalities (2.1) hold for large enough m for $z = z_\ell$.

<u>**Proposition 2.11**</u> *Let* $w_m \in \left[u_m, \varphi_m\left(b_i^j\right)(u_m)\right]$ *satisfy* $d_{X_m}(u_m, w_m) = \frac{\varepsilon}{2}$. *For either* $\delta = +1$ or $\delta = -1$:

$$d_{X_m}\left(u_m, \varphi_m\left(z_\ell^\delta\right)(w_m)\right) < d_{X_m}(u_m, w_m) < d_{X_m}\left(u_m, \varphi_m\left(z_\ell^{-\delta}\right)(w_m)\right)$$

$$d_{X_m}\left(\varphi_m\left(z_\ell^\delta\right)(w_m), \varphi_m\left(z_\ell^{-\delta}\right)(w_m)\right) < 2\varepsilon_1 .$$

Proof: identical with the proof of proposition 2.3.

<u>**Theorem 2.12**</u> *Assume proposition 2.11 holds for* $\delta = 1$. *Let* Ψ_ℓ *be an automorphism of* G *given by:*

$$\forall a \in A \qquad \Psi_\ell(a) = a$$

$$\forall b \in B \qquad \Psi_\ell(b) = z_\ell b z_\ell^{-1} .$$

Then for m *large enough we have:*

(i) $$d_{X_m}\left(\varphi_m\left(b_i^j\right)(u_m), u_m\right) > d_{X_m}\left(\varphi_m \circ \Psi_\ell\left(b_i^j\right)(u_m), u_m\right)$$

(ii) $$g_j \in A_\ell \Rightarrow d_{X_m}(\varphi_m(g_j)(u_m), u_m) = d_{X_m}(\varphi_m \circ \Psi_\ell(g_j)(u_m), u_m)$$

(iii) $$g_j \notin A_\ell \Rightarrow d_{X_m}(\varphi_m(g_j)(u_m), u_m) > d_{X_m}(\varphi_m \circ \Psi_\ell(g_j)(u_m), u_m) .$$

Proof: By the inequalities (2.1):

$$d_{X_m}\left(\varphi_m\left(z_\ell b_i^j z_\ell^{-1}\right)(u_m), u_m\right) = d_{X_m}\left(\varphi_m\left(z_\ell b_i^j\right)(w_m), \varphi_m(z_\ell)(w_m)\right) +$$
$$d_{X_m}(\varphi_m(z_\ell)(w_m), u_m) + d_{X_m}\left(\varphi_m\left(z_\ell b_i^j\right)(w_m), \varphi_m\left(z_\ell b_i^j z_\ell^{-1}\right)(u_m)\right) <$$
$$d_{X_m}\left(\varphi_m\left(b_i^j\right)(w_m), w_m\right) + 2d_{X_m}(w_m, u_m) =$$
$$d_{X_m}\left(\varphi_m\left(b_i^j\right)(u_m), u_m\right) .$$

(ii) is obvious since $\Psi_\ell(g_j) = g_j$ when $g_j \in A_\ell$.

(iii) follows from (i), since if $\tilde{q}_j$ is the number of appearances of b_i^j in g_j ($\tilde{q}_j$ is either q_j or $q_j - 1$ and since $g_j \notin A_\ell$ $\tilde{q}_j$ is positive) we obtain:

$$d_{X_m}(\varphi_m(g_j)(u_m), u_m) - d_{X_m}(\varphi_m \circ \Psi_\ell(g_j)(u_m), u_m) =$$
$$\tilde{q}_j d_{X_m}(\varphi_m(z_\ell)(w_m), (w_m))$$

□

<u>**Case 2B**</u> $Y/G \setminus \bar{e}_\ell$ is a connected graph.

Let A_ℓ be the fundamental group of the graph of groups $Y/G \setminus \bar{e}_\ell$ (as we have noted before $A_\ell \neq C_\ell$ by the construction of Y) $G = A_\ell *_{C_\ell}$. For each j let g_j be given in a reduced form with respect to the given splitting:

$$g_j = a_1^j f^{n_1^j} \cdots a_{q_j}^j f^{n_{q_j}^j}$$

$a_i^j \in A_l$, a_1^j or $f^{n_{q_j}^j}$ may be the identity. From the convergence of (X_m, u_m) to (Y, y_0) in the Gromov topology (theorem 1.1), for m large enough the inequalities (2.2) hold. In addition propositions 2.6 and 2.7 remain valid, so we have the following:

<u>**Theorem 2.13**</u> *Let Ψ_ℓ be an automorphism of G given by:*

$$\Psi_\ell(a) = a \qquad \forall a \in A$$
$$\Psi_\ell(f) = f z_\ell$$

Then for m large enough (such that inequalities 2.2 hold) we have:

(i) $$g_j \in A \Rightarrow d_{X_m}(\varphi_m(g_j)(u_m), u_m) = d_{X_m}(\varphi_m \circ \Psi_\ell(u_m), u_m)$$

(ii) $$g_j \notin A \Rightarrow d_{X_m}(\varphi_m(g_j)(u_m), u_m) > d_{X_m}(\varphi_m \circ \Psi_\ell(u_m), u_m)$$

Proof: (i) is trivial from the definition of Ψ_ℓ . (ii) follows from the inequalities (2.2) and proposition 2.7, since if $\tau = \tau_1 a_i^j$ then:

$$d_{X_m}(\varphi_m(\tau)(u_m), u_m) - d_{X_m}(\varphi_m \circ \Psi_\ell(\tau)(u_m), u_m) =$$
$$d_{X_m}(\varphi_m(\tau_1)(u_m), u_m) - d_{X_m}(\varphi_m \circ \Psi_\ell(\tau_1)(u_m), u_m)$$

and for $\tau = \tau_1 f^{n_i^j}$ we have:

$$d_{X_m}(\varphi_m(\tau)(u_m), u_m) - d_{X_m}(\varphi_m \circ \Psi_\ell(\tau)(u_m), u_m) =$$
$$d_{X_m}(\varphi_m(\tau_1)(u_m), u_m) - d_{X_m}(\varphi_m \circ \Psi_\ell(\tau_1)(u_m), u_m) +$$
$$|n_i^j| d_{X_m}(w_m, \varphi_m(z_\ell)(w_m)) .$$

□

Theorem 2.14 *Under the notations above let* $\Psi = \Psi_1 \circ \cdots \circ \Psi_s$ *. Then for* m *large enough (so that inequalities (2.2) hold for* $\ell = 1, \cdots, s$ *) we obtain:*

$$\max_{1 \leq j \leq t} d_{X_m}(\varphi_m(g_j)(u_m), u_m) > \max_{1 \leq j \leq t} d_{X_m}(\varphi_m \circ \Psi(u_m), u_m)$$

Proof: Clearly by inequalities (2.2) the above maximum obtained for $g_j \notin stab(y_0)$. Therefore, each g_j which obtain the maximum satisfies $g_j \notin A_\ell$ for some $\ell = \ell(j)$, so its translation of u_m is reduced by composing φ_m with Ψ_ℓ . Since the translation of u_m by g_j does not increase by composing with the other Ψ_ℓ 's the theorem follows.

□

The whole construction of the tree Y based on the existence of a subsequence of k-acylindrical splittings for G with $\mu_m = \max_{1 \leq j \leq t} d_{X_m}(\varphi_m(g_j)(u_m), u_m) \to \infty$. Since theorems 2.5, 2.8, 2.14 cancel all possibilities for the existence of a tree Y obtained by our construction, we have shown:

Theorem 2.15 *Let* $G = \langle g_1, \cdots, g_t \rangle$ *be a f.g. freely indecomposable group with no 2-torsion. Let* $\{T_m\}_{m=1}^{\infty}$ *be the Bass-Serre trees for the* k*-cylindrical splittings of G. There exists a constant* λ_k *, vertices* $u_m \in T_m$ *and automorphisms* $\varphi_m \in \mathrm{Aut}(G)$ *such that:*

$$\max_{1 \leq j \leq t} d_{T_m}(u_m, \varphi_m(g_j)(u_m)) \leq \lambda_k .$$

3. Acylindrical Accessibility and small splittings

Having bounded the image of given generators, we immediately obtain the following (which we call "acylindrical accessibility"):

Theorem 3.1 *Let* G *be a f.g. freely indecomposable group with no 2-torsion. For each* k *there exists an integer* $\nu(k, G)$ *so that the number of vertices and edges in all* k*-acylindrical splittings of* G *does not exceed* $\nu(k, G)$ *.*

Proof: This clearly follows from theorem 2.15, since the number of G-orbits of edges and vertices is bounded by the total length of a set of generators.

□

Remarks:

1. Acylindrical accessibility holds for (certain) f.g. groups, a class for which Dunwoody's accessibility fails to hold ([Du2], see also [Be-Fe2]).

2. Our bound $\nu(k, G)$ is not qualitative, i.e. we do not get a bound on $\nu(k, G)$ in terms of other algebraic invariants of the group. This again differs from both Dunwoody [Du1] and Bestvina-Feighn [Be-Fe1] accessibilities.

Now, we assume in addition that our group G is finitely presented. Let X be the Cayley complex for the group $G = \langle g_1, \cdots, g_t | r_1, \cdots, r_s \rangle$. By the Cayley complex we mean adding a 2-cell for every conjugate of each of the defining relations to the Cayley graph of G (cf. [Ly-Sc], ch. 3). Let $R = \max_{1 \le j \le s} |r_j|$ and let B_R be the ball of radius R in X.

Definition 3.2 *Let (T, t_0) be a minimal pointed G-tree. The convex hull of t_0 under the elements of B_R equipped with notations for the image of t_0 under each of the elements of B_R is called the R-state of T, $S_R(T)$.*

Since we have already bounded the length of the generators (theorem 2.15), there are only finitely many possibilities for R-states of k-acylindrical splittings (trees) of our group G. Given an R-state S, we know how each of the relations $r_1, \cdots, r_s$ collapses to a finite tree. Moreover if two conjugates of r_{j_1} and r_{j_2} have an edge in common, we know how to collapse both to a finite tree. Therefore, we can start with the Cayley complex X and collapse each of the faces to a given finite tree according to the R-state S. This collapsing procedure is equivariant by definition, so we get a minimal G-tree T. Let $\hat{T}$ be a minimal G-tree with the same R-state S. By our construction we have the following commutative diagram:

$$\begin{array}{ccc} G \times T & \rightarrow & T \\ \downarrow p & & \downarrow p \\ G \times \hat{T} & \rightarrow & \hat{T} \end{array}$$

$$\forall g \in G \qquad \forall t \in T \qquad g(p(t)) = p(g(t)) \; .$$

This leads us to the following, which we call "acylindrical super accessibility".

Theorem 3.3 *Let G be a f.p. freely indecomposable group with no 2-torsion. Then for each k there exist G-trees $T_1, \cdots, T_{n(G,k)}$ such that every k-acylindrical splitting is covered (in the above commutative diagram sense) up to isomorphism by one of the T_i 's.*

The finiteness of R-states and G-trees obtained from them gives several fairly direct corollaries. We start with k-acylindrical surfaces in 3–manifolds, for which we need the following theorem due to Stallings, Epstein and Waldhausen:

Theorem 3.4 ([Cu-Sh], 2.3.1) *Let M be a compact, orientable 3-manifold. For any non-trivial splitting of $\pi_1(M)$ there exists a non-empty system $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_m$ of incompressible surfaces in M, none of which is boundary parallel, such that $\mathrm{im}(\pi_1(\Sigma_i) \to \pi_1(M))$ is contained in an edge group for $i = 1, \cdots, m$ and $\mathrm{im}(\pi_1(N) \to \pi_1(M))$ is contained in a vertex group for each component of $M \setminus \Sigma$. Moreover, if $K \subseteq \partial M$ is a subcomplex such that $\mathrm{im}\left(\pi_1\left(\hat{K}\right) \to \pi_1(M)\right)$ is contained in a vertex group for each component $\hat{K}$ of K we may take Σ to be disjoint from K.*

Recall, a k-acylindrical surface in a hyperbolic manifold M is an incompressible surface S such that one of the pieces of $M \setminus S$ contains at most k non-homotopic cylinders.

Theorem 3.5 *A compact, boundary irreducible, acylindrical hyperbolic 3-manifold contains only finitely many closed k-acylindrical surfaces.*

Proof: Let M be a 3-manifold satisfying the conditions above. Each closed k-acylindrical surface defines a $(2k+2)$-acylindrical spliting of $\pi_1(M)$. $\pi_1(M)$ is clearly f.p. and freely indecomposable, so by theorem 3.3 there exist finitely many $\pi_1(M)$-trees $T_1, \cdots, T_n$ such that each $(2k+2)$-acylindrical splitting is covered by one of the T_i 's:

$$\begin{array}{ccc} \pi_1(M) \times T_i & \to & T_i \\ \downarrow p & & \downarrow p \\ \pi_1(M) \times T & \to & T \end{array}$$

Clearly, in a splitting of $\pi_1(M)$ along the fundamental group of a closed incompressible surface, the fundamental group of a boundary component stabilizes a vertex. Therefore, if this is not the case with T_i , we may look at the minimal folding of T_i obtained by identifying vertices, such that in this minimal folding each boundary component fixes a vertex (we still denote such a minimal folding by T_i). Now we are able to apply theorem 3.4 and conclude that each edge group of T_i contains the fundamental group of some closed incompressible surface Σ_i^ℓ . This implies that each T_i can cover finitely many splittings along a k-acylindrical surface and our theorem follows.

□

Theorem 3.5 lead us to J. Hass' theorem on acylindrical surfaces in closed 3-manifolds:

Theorem 3.6 [Ha] *There are only finitely many acylindrical surfaces in a closed 3-manifold.*

Proof: It is clear that a Seifert fibered and Solv manifolds do not contain acylindrical surfaces, so it is enough to discuss compact, acylindrical hyperbolic ones.

□

Remark To prove theorems 3.5 and 3.6 we do not need the whole strength of our machinery. It is enough to discuss f.p. groups with no cyclic splittings (theorem 1.5) and instead of Rips' theory [Ri] it suffices to use Morgan-Shalen results [Mo-Sh].

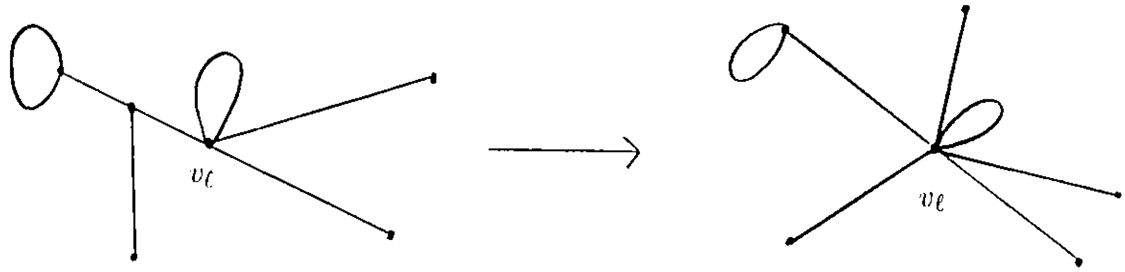
A natural application of acylindrical super accessibility is small splittings. M. Bestvina and M. Feighn have bounded the combinatorics of such splittings for f.p. groups [Be-Fe1]. Our theory implies a much stronger finiteness result of isomorphism classes of such splittings, under more restrictive algebraic conditions. Small splittings are closely related with the structure of the automorphism group for hyperbolic groups, and a more detailed and deeper discussion of both is given in [Ri-Se1] and [Se].

Proposition 3.7 *Let G be a f.g. torsion-free group such that the normalizer of every cyclic subgroup is cyclic. Then every splitting of G with cyclic edge stabilizers is obtained from an action on a 4-acylindrical tree.*

Proof: If $G = \langle A, t | t z_1 t^{-1} = z_2 \rangle$ for some subgroup A, then clearly both z_1 and z_2 are not non-trivial powers. If $G = A *_C B$ and $C = \langle c \rangle$, then c is not a power in either A or B.

Let Y be a minimal, reduced tree on which G acts with cyclic edge stabilizers and let Y/G be the corresponding graph of groups. We define an equivalence relation on the edges of Y/G, so that $\bar{e}_i \sim \bar{e}_j$, if $\text{stab}(\bar{e}_i)$ and $\text{stab}(\bar{e}_j)$ are contained in a common cyclic subgroup. For each equivalence class we pick a vertex v_ℓ of Y/G that contains the maximal cyclic group corresponding to the given equivalence class. Now we modify the graph Y/G such that all edges in the equivalence class corresponding to the vertex v_ℓ have it as one of their vertices (this can be done because G has the algebraic properties given above). The obtained grpah of groups gives an action of G on a tree $\tilde{Y}$ which is 4-acylindrical.

□



Theorem 3.8 *If G satisfies the conditions of proposition 3.7 and in addition finitely presented, freely indecomposable, then G admits only finitely many isomorphism classes of splittings with cyclic edge stabilizers.*

Proof: By the previous proposition every small splitting can be viewed as a 4-acylindrical one. By theorem 3.3 there are only finitely many G-trees $T_1, \cdots, T_n$ that covers all small splittings of G. Since G is freely indecomposable, the T_i define small splittings of G with cyclic edge stabilizers. Therefore, each small splitting is obtained from one of the T_i 's by a sequence of Stallings folds ([Be-Fe1]). Each of the last folds either reduce the combinatorics of the obtained graph of groups or enlarge edge stabilizers:

Type IA-reduce the combinatorics and perhaps enlarge edge stabilizers.
Type IIA-enlarge one of the edge stabilizers.
Type IIIA-reduce the combinatorics and perhaps enlarge edge stabilizers.

Since the combinatorics of each graph of groups correspond to one of the T_i 's is bounded and every edge stabilizer is of finite index in a maximal cyclic subgroup in G our theorem follows.

□

Corollary 3.9 Let G be a f.p. torsion-free freely indecomposable group that satisfies one of the following:

(i) G is (Gromov) hyperbolic.
(ii) G is discrete in $\mathrm{Isom}^+(H^n)$ and have no rank 2 parabolics.
(iii) G contains no rank 2 free abelian subgroups and admits a semisimple discrete action on a non-positively curved simply connected manifold.

Then G admits only finitely many isomorphism classes of small splittings.

Proof: We need to show the groups in question satisfy the assumptions of proposition 3.7, i.e. the normalizer of a cyclic group is cyclic. For hyperbolic groups this is proved in ([Gr], ch. 8). A torsion-free element that stabilizes a semi-simple cyclic Kleinian subgroup must have identical axis with the cyclic subgroup. Since we assume our group G is torsion-free, it must belong to the cyclic group that map this axis to itself. If the cyclic subgroup is parabolic, then every normalizing element in G fixes the same point at infinity, so the assumption of 3.7 follows since G has no rank 2 free abelian subgroups. If G is semisimple discrete group of isometries of a Hadamard manifold, if $t,\ z \in G$ and $tzt^{-1} \in \langle z \rangle$, then $tzt^{-1} = z^{\pm 1}$ and $[t^2, z] = 1$. Therefore, it remains to show that if $h_1^n = z,\ h_2^m = z$ then $[h_1, h_2] = 1$ since we have assumed G has no rank 2 abelian subgroups. Let $\Gamma = \langle h_1, h_2 \rangle$ and let A denote the center of Γ , $\langle z \rangle < A$. By ([BGS], 7.2) $\mathrm{MIN}(A)$ splits as $S_1 \times R^l$ where all elements of A act as (id, γ) and all elements of Γ act as (γ_1, γ_2) where γ_2 is a translation. Now, $[h_1, h_2]$ clearly fixes R^l in this splitting, it is not of finite order since G is torsion-free, so if not trivial it has to be of infinite order. But $[h_1, h_2]$ and z generate a rank 2 free abelian subgroup of G in that case, a contradiction.

□

Remarks:

(i) Proposition 3.7 can be modified for groups with higher rank abelian subgroups.
(ii) Proposition 3.7 implies acylindrical accessibility for f.g., torsion-free, freely indecomposable Kleinian groups and semisimple discrete groups of isometries of a Hadamard manifold (this case is not covered by the Bestvina-Feighn accessibility).

Appendix: Bounded cohomology of groups with acylindrical splittings

Let G be a f.g. group with no 2-torsion and with a k-acylindrical splitting. Let T be the Bass-Serre tree corresponds to the acylindrical splitting of G. We follow Barge-Ghys construction [Ba-Gh] to study the bounded cohomology of G.

Let $\underset{u}{\rightarrow}$ be an oriented path in T. For each oriented segment $[A,B] \subseteq T$ let $\Phi_{\underset{u}{\rightarrow}}(A,B)$ be the maximal integer n for which there exist $g_1, \cdots, g_n$ such that:

(i) $g_i \underset{u}{\rightarrow} \subseteq [A,B]$
(ii) the interiors of $g_i \underset{u}{\rightarrow}$ are disjoint for distinct elements.
(iii) the orientation of $g_i \underset{u}{\rightarrow}$ agrees with that of $[A,B]$.

Let the weight of $[A,B]$, $W_{\underset{u}{\rightarrow}}(A,B)$ be given by:

$$W_{\underset{u}{\rightarrow}}(A,B) = \Phi_{\underset{u}{\rightarrow}}(A,B) - \Phi_{\underset{u}{\rightarrow}}(B,A)$$

Let $*$ be (an arbitrary) base point in T. For each $g_1, g_2 \in G$ we define:

$$c_{\underset{u}{\rightarrow}}(g_1,g_2) = W_{\underset{u}{\rightarrow}}(*, g_1 g_2(*)) - W_{\underset{u}{\rightarrow}}(*, g_1(*)) - W_{\underset{u}{\rightarrow}}(*, g_2(*)) \, .$$

Proposition 4.1 [Ba-Gh]

(i) $c_{\underset{u}{\rightarrow},*}$ *defines a bounded 2-cocycle on* G .
(ii) *The cohomology class of* $c_{\underset{u}{\rightarrow},*}$ *is trivial.*
(iii) *The bounded cohomology class corresponds to* $c_{\underset{u}{\rightarrow},*}$ *does not depend on the base point* $*$.

Let $g_0 \in C$ satisfy:

$$d_T(t, g_0(t)) > k \text{ for all } t \in T \, .$$

Let A be the axis of g_0 , let $* \in A$, $\underset{u}{\rightarrow} = [*, g_0(*)]$ and $f_*^{g_0} : G \rightarrow Z$ be given by:

$$f_*^{g_0}(g) = W_{\underset{u}{\rightarrow}}(*, g(*))$$

Lemma 4.2 *Let* $c_{g_0} = \delta f_*^{g_0}$. *If* $[c_{g_0}]$ *is null bounded cohomology class then there exists a homomorphism* $h : G \rightarrow R$, *such that* $h(g_0) = 1$.

Proof: Clearly, we have $\Phi_{\rightarrow}(*, g_0^n(*)) = n$ and since T is k-acylindrical and G has no 2–torsion $\Phi_{\rightarrow}(g_0^n(*), *) = 0$, so we have $f_*^{g_0}(g_0^n) = n$. Now, if $[c_{g_0}]$ is a null bounded cohomology class, there exists a bounded functional $b : G \rightarrow R$ such that $c_{g_0} = \delta b = \delta f_*^{g_0}$ or $\delta(b - f_*^{g_0}) = 0$, which implies $b - f_*^{g_0}$ is a homomorphism $h : G \rightarrow R$.

But $|b(g_0^n)| \leq K$ for some global constant K and $f_*^{g_0}(g_0^n) = n$ which clearly give us $h(g_0) = 1$.

□

Theorem 4.3 *Let G be a non-cyclic group with no 2-torsion which posess a k-acylindrical splitting. Then $H_b^2(G, R) \neq 0$.*

Proof: If T is a minimal tree for a group G and G is abelian, then T is either a point or a real line. In both cases if G is not infinite cyclic, T is not a k-acylindrical tree for G. If G is not abelian and has no 2–torsion, then the commutator subgroup of G can not fix a vertex in a k-acylindrical tree. Since our tree T is k-acylindrical not all elements of the commutator subgroup are therefore elliptics. Let $\overline{g} \in [G, G]$ be a hyperbolic element and $g_1 = (\overline{g})^{k+1}$. Since $g_1 \in [G, G]$ there is no homomorphism from G to R that gives g_1 the value 1, so by lemma 4.2 $[C_{g_0}]$ is not a null bounded cohomology class.

□

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