Extensions of simple modules for $G_2(p)$

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MPI / 93-31

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Abstract

Extensions of simple modules for the finite Chevalley group $G_2(p)$ with $p \geq 13$ are completely determined in this note.

1. Preliminaries

1.1 Computing all the extensions of simple modules for finite Chevalley groups is an interesting task in the modular representation theory. Alperin [1], Andersen, Jørgensen and Landrock [4], Sin [18-21], and Andersen [3] have computed extensions of simple modules for $SL(2,2^n)$, $SL(2,p^n)$, $SL(3,2^n)$ and $SU(3,2^n)$, $Sp(4,2^n)$ and $Suz(2^m)$, $G_2(2^n)$,$G_2(3^n)$ and ${}^2G_2(3^m)$, $SL(3,p)$ and $Sp(4,p)$, separately. In particular, Andersen [3] provides a way of computing extensions of simple modules for finite Chevalley groups from those for the ambient algebraic groups. In the present note, we shall compute completely extensions between simple modules for the finite Chevalley group of type G_2 over the prime field $\mathbb{F}_p$ of p elements with $p \geq 13$.

1.2 Let K be an algebraic closure of $\mathbb{F}_p$, and let G be a connected simply-connected simple algebraic group of type G_2 over K. We assume that G is defined and split over the prime field $\mathbb{F}_p$. For $n \geq 1$ we denote by G_n the kernel of the Frobenius morphism F^n on G, and by $G(n)$ the finite group consisting of the points of G over the field of p^n elements, i.e., the subgroup of G consisting of fixed points of the Frobenius morphism F^n on G. In particular, we write $G_2(p)$ in stead of $G(1)$.

We denote by T a split maximal torus in G and by B a Borel subgroup containing T. Let R denote the root system associated with $(G,\ T)$. We choose a set of positive roots R_+ in R to be

$$R_+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$$

such that B corresponds to $-R_+ = R_-$, where α and β are simple roots with α short. We let $X(T)$ be the character group of T, i.e., the weight lattice. Then the Weyl group W of G acts on it, and the dot action of W on $X(T)$ is given by

$$w\cdot\lambda = w(\lambda+\rho), \qquad \text{for all } \lambda \in X(T),\ \ w \in W$$

where ρ is half the sum of all the positive roots. We define the following subsets of the group:

$$X(T)_+ = \{\lambda \in X(T) \mid \langle \lambda, \gamma^\vee \rangle \geq 0,\ \ \gamma \in R_+\},$$

the set of dominant weights, and

$$X_n(T) = \{\lambda \in X(T)_+ \mid \langle \lambda, \gamma^\vee \rangle < p^n,\ \ \gamma \text{ simple root}\},$$

the set of p^n-restricted weights. Here $\gamma^\vee$ is the dual root of γ. Let ω_α and ω_β be the fundamental weights corresponding to α and β, respectively. Then we write a weight λ in $X(T)$ in terms of $\lambda = r\omega_\alpha + s\omega_\beta = (r,s)$. In particular, we have

$$X(T)_+ = \{\lambda = (r,s) \in X(T) \mid r, s \geq 0\},$$

and

$$X_1(T) = \{\lambda = (r,s) \in X(T) \mid 0 \leq r, s < p\}.$$

Finally we let C_0 be the lowest alcove in $X(T)_+$, i.e.,

$$C_0 = \{\lambda \in X(T)_+ \mid \langle \lambda+\rho, \alpha_0^\vee \rangle < p\},$$

where α_0 is the highest short root in R_+. Then the 'closure' $\bar{C}_0$ of C_0 is a fundamental domain of the action of the affine group $W_p = \langle \{s_{\gamma,n} \mid \gamma \in R_+, n \in \mathbb{Z}\}\rangle$ on $X(T)$, where $s_{\gamma,n}$ denotes the affine reflection defined by

$$s_{\gamma,n}\cdot\lambda = s_\gamma\cdot\lambda + np\gamma = \lambda + (np - \langle \lambda+\rho, \gamma^\vee \rangle)\,\gamma, \quad \lambda \in X(T).$$

Then we see that there are 12 alcoves in $X_1(T)$. The fact that there are 6 positive roots in R and 12 alcoves in $X_1(T)$ makes the computations here much more complicated. We have to do these very carefully.

1.3 Recall that $X(T)_+$ parametrizes the simple G-modules via highest weights. For $\lambda \in X(T)_+$ we denote by $L(\lambda)$ (respectively $V(\lambda)$) the simple G-module (respectively the Weyl module) with highest weight λ. When $\lambda \in X_n(T)$, the $L(\lambda)$'s stay simple upon restriction to $G(n)$ and any simple $G(n)$-module is isomorphic to exactly one of these. For any G-module M we denote by $M^{[n]}$ the Frobenius twist of M, that is, the new G-module structure on M defined by the composite

$$G \xrightarrow{F^n} G \longrightarrow GL(M).$$

It is well known that $M^{[n]}$ is trivial considered as G_n-module and that any G-module that becomes trivial upon restriction to G_n is of this form. The simple modules for G_n are the 'same' as those of for $G(n)$, i.e., the restrictions to G_n of the $L(\lambda)$'s with $\lambda \in X_n(T)$.

Let $\lambda \in X(T)_+$. It has the unique decomposition,

$$\lambda = \lambda^0 + p^n\lambda^1, \quad \lambda^0 \in X_n(T) \text{ and } \lambda^1 \in X(T)_+. \tag{1}$$

Then the Steinberg tensor product theorem asserts that

$$L(\lambda) \cong L(\lambda^0) \otimes L(\lambda^1)^{[n]}.$$

Since $G(n)$ consists of fixed points of the Frobenius morphism, we get

$$L(\lambda) \cong L(\lambda^0) \otimes L(\lambda^1)$$

as a $G(n)$-module.

1.4 For $\lambda \in X_n(T)$ we denote by $Q_n(\lambda)$ (respectively $U_n(\lambda)$) the injective G_n (respectively $G(n)$)-module, whose socle is $L(\lambda)$. In particular, the Steinberg module $St_n = L((p^n - 1)\rho)$ has the property that it is injective both as a G_n and as a $G(n)$-module. We have therefore

$$St_n = Q_n((p^n - 1)\rho) = U_n((p^n - 1)\rho).$$

Note that $Q_n(\lambda)$ is a G-summand of $St_n \otimes L((p^n - 1)\rho - \lambda^*)$. Here $\lambda^* = -w_0(\lambda)(= \lambda$, because $-w_o = 1$ for type G_2) is the highest weight of the dual module $L(\lambda)^*(\cong L(\lambda)$, because each simple G-module is self-dual), and w_0 is the longest element in the Weyl group W of G. Therefore $Q_n(\lambda)$ is also injective upon restriction to $G(n)$. Following Chastkofsky [5, Theorem 2] and Jantzen [15, Corollary 2], we decompose $Q_n(\lambda)$ into a direct sum of $U_n(\mu)$'s for all $\lambda \in X_n(T)$,

$$[Q_n(\lambda) : U_n(\mu)]_{G(n)} = \sum_{\nu \in X(T)+} [L(\mu) \otimes L(\nu) : L(\lambda + p^n \nu)]_G. \qquad (2)$$

Here $[Q_n(\lambda) : U_n(\mu)]_{G(n)}$ denotes the number of times of $U_n(\mu)$ appearing as $G(n)$-summand of $Q_n(\lambda)$ and $[M : L(\eta)]_G$ is the composition factor multiplicity of the simple G-module $L(\eta)$ in M. Moreover, we have $[Q_n(\lambda) : U_n(\lambda)]_{G(n)} = 1$, and if $\lambda \neq \mu$, then $[Q_n(\lambda) : U_n(\mu)]_{G(n)} \neq 0$ implies that $\langle \mu - \lambda, \alpha_0^\vee \rangle \geq p^n - 1$, and $\langle \nu, \alpha_0^\vee \rangle \leq h - 1$.

We list all the decompositions of $Q_1(\lambda)$'s for $n = 1$ in Table 2.

1.5 Following Jantzen [14], we call a G-module p^n-bounded if all its weights μ satisfy $\langle \mu, \gamma^\vee \rangle < 2p^n(h-1)$ (h being the Coxeter number, and in our case $h = 6$) for all $\gamma \in R_+$. Also, we call a weight μ p^n-bounded if the same inequality holds. It is well-known that when $p \geq 2h - 2$, $Q_n(\lambda)$ has a G-module structure for all $\lambda \in X_n(T)$. Moreover, $Q_n(\lambda)$ can be considered as the injective hull of $L(\lambda)$ in the category of p^n-bounded G-modules.

1.6 Andersen [3] provides a way of computing all $G(n)$-extensions from a knowledge of G-extensions. Andersen [3, Proposition 2.7] shows that (no restriction on p is needed here) for all λ, $\mu \in X_n(T)$ the restriction map

$$\mathrm{Ext}^1_G(L(\mu), L(\lambda)) \longrightarrow \mathrm{Ext}^1_{G(n)}(L(\mu), L(\lambda)) \qquad (3)$$

is injective. For $p \geq 3h - 3$, Andersen [3, Theorem 3.2] shows the following theorem.

Theorem Let λ and $\mu \in X_n(T)$ be such that $\langle \mu, \alpha_0^\vee \rangle \leq \langle \lambda, \alpha_0^\vee \rangle$. Suppose that the distances between λ (respectively μ) and any facet that lies in the 'closure' of the alcove to which λ (respectively μ) belongs are at least $2(h-1)$ (respectively $h-1$) (i.e., if p divides $\langle \lambda+\rho, \alpha_0^\vee \rangle + c$ (respectively $\langle \mu+\rho, \alpha_0^\vee \rangle + c$) for some $c \in \mathbb{Z}$, then $|c| \geq 2(h-1)$ (respectively $h-1$)). Then

$$\bigoplus_{\nu} \mathrm{Hom}_G\left(L(\mu), L(\nu^0) \otimes L(\nu^1)\right) \otimes \mathrm{Ext}^1_G(L(\nu), L(\lambda))$$
$$\cong \mathrm{Ext}^1_{G(n)}(L(\mu), L(\lambda)) \qquad (4)$$

where the sum is taken over all p^n-bounded weights $\nu = \nu^0 + p^n\nu^1$ with $\nu^0 \in X_n(T)$ and $\nu^1 \in X(T)_+$.

When (4) holds, if all $\mathrm{Soc}_G\left(L(\nu^0) \otimes L(\nu^1)\right)$ and all $\mathrm{Ext}^1_G(L(\nu), L(\lambda))$ are known, then $\mathrm{Ext}^1_{G(n)}(L(\mu), L(\lambda))$ can be easily determined. Liu and the author [17] determined all the extensions of simple G-modules for $p \geq 13$. As a by-product we also computed $\mathrm{Soc}_G(L(\lambda) \otimes L(\nu))$ for all $\lambda \in X_1(T)$ and $\nu \in \{\omega_\alpha, \ \ \omega_\beta\}$. In fact, we believe that our results are also valid when $p = 11$, however, we have to do more, because the Jantzen condition $\langle \lambda + \rho, \alpha_0^\vee \rangle \leq p(p - h - 2)$ is no longer satisfied for some p-bounded weights. For convenience we shall still restrict ourselves here to the case $p \geq 13$ only.

1.7 Let $\lambda \in X_n(T)$ and define $R_n(\lambda)$ to be the G-module for which the sequence

$$0 \longrightarrow L(\lambda) \longrightarrow Q_n(\lambda) \longrightarrow R_n(\lambda) \longrightarrow 0$$

is exact. Then, we have for all p^n-bounded weights ν

$$\mathrm{Ext}^1_G(L(\nu), L(\lambda)) = \mathrm{Hom}_G(L(\nu), R_n(\lambda)).$$

Restricting to $G(n)$, we get for all $\mu (\neq \lambda) \in X_n(T)$ an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{G(n)}(L(\mu), Q_n(\lambda)) &\longrightarrow \mathrm{Hom}_{G(n)}(L(\mu), R_n(\lambda)) \\ &\longrightarrow \mathrm{Ext}^1_{G(n)}(L(\mu), L(\lambda)) \longrightarrow 0. \end{aligned}$$

If $Q_n(\lambda) = U_1(\lambda)$ or $[Q_n(\lambda) : U_n(\mu)]_{G(n)} = 0$, then

$$\mathrm{Ext}^1_{G(n)}(L(\mu), L(\lambda)) = \mathrm{Hom}_{G(n)}(L(\mu), R_n(\lambda)).$$

Moreover, Andersen [3, Lemma 2.2] shows that $R_n(\lambda)$ is a G-submodule of $\sum_\nu Q_n(\nu^0) \otimes L(\nu^1)^{[n]}$, where ν runs over all p^n-bounded weights, each with multiplicity $\dim \mathrm{Ext}^1_G(L(\nu), L(\lambda))$. Following Andersen [3], in order to prove (4) is an isomorphism we need only to check that

$$\mathrm{Hom}_G\left(L(\mu), L(\nu^0) \otimes L(\nu^1)\right) = \mathrm{Hom}_{G(n)}\left(L(\mu), Q_n(\nu^0) \otimes L(\nu^1)\right), \tag{5}$$

because (5) implies that $L(\mu)$ is a $G(n)$-submodule of $R_n(\lambda) \subset \sum_\nu Q_n(\nu^0) \otimes L(\nu^1)^{[n]}$ if and only if $L(\mu)$ is a G-submodule of the G-socle of $R_n(\lambda)$ restricted to $G(n)$.

1.8 In the case $n = 1$, we have known $\mathrm{Ext}^1_G(L(\nu), L(\lambda))$ for all p-bounded weights ν from [17, Table 1] (note that $\nu^1 \in \{0, \ \ \omega_\alpha, \ \ \omega_\beta\}$ for each ν

with $\mathrm{Ext}^1_G(L(\nu), L(\lambda)) \neq 0$), and $\mathrm{Soc}_G\left(L(\nu^0)\otimes L(\nu^1)\right)$ from [17, Table 3 and Table 4] for all ν as above. Then $\bigoplus_\nu \mathrm{Hom}_G\left(L(\mu), L(\nu^0)\otimes L(\nu^1)\right)\otimes \mathrm{Ext}^1_G(L(\nu), L(\lambda))$ can be easily determined. On the other hand, we have

$$\begin{aligned}\dim \mathrm{Hom}_{\mathrm{G}_2(p)}\left(L(\mu), U_1(\nu^0)\otimes L(\nu^1)\right) &= [L(\mu)\otimes L(\nu^1) : L(\nu^0)]_{\mathrm{G}_2(p)},\\ \dim \mathrm{Hom}_G\left(L(\mu), Q_1(\nu^0)\otimes L(\nu^1)\right) &= [L(\mu)\otimes L(\nu^1) : L(\nu^0)]_G.\end{aligned} \tag{6}$$

If $Q_1(\nu^0) = U_1(\nu^0)$, and if the highest weights of all G-composition factors of $L(\mu)\otimes L(\nu^1)$ belong to the set $X_1(T)$, then

$$[L(\mu)\otimes L(\nu^1) : L(\nu^0)]_G = [L(\mu)\otimes L(\nu^1) : L(\nu^0)]_{\mathrm{G}_2(p)},$$

and then

$$\mathrm{Hom}_G\left(L(\mu), Q_1(\nu^0)\otimes L(\nu^1)\right) = \mathrm{Hom}_{\mathrm{G}_2(p)}\left(L(\mu), U_1(\nu^0)\otimes L(\nu^1)\right).$$

For most pairs $\{\lambda, \mu\}$ we have $Q_1(\nu^0) = U_1(\nu^0)$, and the composition factor multiplicity of simple G-module $L(\nu^0)$ in $L(\mu)\otimes L(\nu^1)$ is one, and the number of times of $L(\mu)$ appearing as G-summand of $\mathrm{Soc}_G\left(L(\nu^0)\otimes L(\nu^1)\right)$ is also one. So (5) holds, and then (4) holds for these pairs, even though we do not assume that $p \geq 3h-3$.

2. Computations

2.1 In this section we show how to compute $\mathrm{Ext}^1_{\mathrm{G}_2(p)}(L(\mu), L(\lambda))$ for all λ, $\mu \in X_1(T)$. Write $\lambda = (r, s)$ and $\mu = (u, v)$. Since

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{G}_2(p)}(L(\mu), L(\lambda)) &\cong \mathrm{Ext}^1_{\mathrm{G}_2(p)}(L(\lambda)^\star, L(\mu)^\star)\\ &\cong \mathrm{Ext}^1_{\mathrm{G}_2(p)}(L(\lambda), L(\mu)),\end{aligned} \tag{7}$$

we may assume that $2u+3v \leq 2r+3s$. We shall compute all $\mathrm{G}_2(p)$-extensions case by case, and in Table 1 we list all $\mu \in X_1(T)$ with $2u = 3v \leq 2r+3s$ for which $L(\mu)$ extend $L(\lambda)$ non-trivially(for $\mathrm{G}_2(p)$).

In Table 1, terms with a negative coordinate should as usual be dealt with in the following way: If $\mu = (u, v)$ with $u = -1$ or $v = -1$, then we cancel μ only. If $\mu = (u, v)$ with $u < -1$ or $v < -1$, then there are a simple reflection s in W and a weight $\mu' = (u', v')$ in the same column as μ such that $s\cdot\mu = \mu'$, and then we cancel μ and μ' simultaneously. The dimension of $\mathrm{Ext}^1_{\mathrm{G}_2(p)}(L(\mu), L(\lambda))$ can also be read from Table 1, namely as the number of times μ appearing in the row corresponding to λ. Note that this number is between 1 and 3.

2.2 Let $\mu \in X_1(T)$ be such that $\operatorname{Hom}_G(L(\mu), L(\nu^0) \otimes L(\nu^1)) \neq 0$. Then we have $\mu \prec \nu^0 + \nu^1$, where the partial ordering '$\prec$' of $X(T)$ is defined in the usual way, $\lambda \prec \mu$ if and only if $\mu - \lambda$ can be written as a linear combination of positive roots with non-negative integer coefficients. It is obvious that if $L(\eta)$ is a G-composition factor in $L(\mu) \otimes L(\nu^1)$, then $\eta \prec \mu + \nu^1 \prec \nu^0 + 2\nu^1$, and if $L(\eta)$ is also a $\mathrm{G}_2(p)$-composition factor in $L(\mu) \otimes L(\nu^1)$, then we still have $\eta \prec \mu + \nu^1 \prec \nu^0 + 2\nu^1$. On the other hand, if $\operatorname{Hom}_{\mathrm{G}_2(p)}(L(\mu), Q_1(\nu^0) \otimes L(\nu^1)) \neq \operatorname{Hom}_{\mathrm{G}_2(p)}(L(\mu), U_1(\nu^0) \otimes L(\nu^1))$, then there must be a weight $\eta (\neq \nu^0) \in X_1(T)$ such that $[Q_1(\nu^0) : U_1(\eta)]_{\mathrm{G}_2(p)} \neq 0$, and $\operatorname{Hom}_{\mathrm{G}_2(p)}(L(\mu), U_1(\eta) \otimes L(\nu^1)) \neq 0$, and hence $L(\eta)$ is a $\mathrm{G}_2(p)$-composition factor in $L(\mu) \otimes L(\nu^1)$. So we obtain from (1.4)

$$\langle \nu^0, \alpha_0^\vee \rangle + p - 1 \leq \langle \eta, \alpha_0^\vee \rangle \leq \langle \nu^0 + 2\nu^1, \alpha_0^\vee \rangle, \tag{8}$$

that is, $p \leq \langle 2\nu^1, \alpha_0^\vee \rangle + 1$. In our case, $2 \leq \langle \nu^1, \alpha_0^\vee \rangle \leq 3$, hence $p \leq 7$, which contradicts our assumption. Furthermore, if

$$[L(\mu) \otimes L(\nu^1) : L(\nu^0)]_G \neq [L(\mu) \otimes L(\nu^1) : L(\nu^0)]_{\mathrm{G}_2(p)},$$

then $L(\nu^0)$ must be a $\mathrm{G}_2(p)$-composition factor in a certain simple G-module $L(\eta) = L(\eta^0) \otimes L(\eta^1)^{[1]}$ with $\eta^1 \neq 0$ being restricted to $\mathrm{G}_2(p)$, and $L(\eta)$ is a G-composition factor in $L(\mu) \otimes L(\nu^1)$. Hence we get $\nu^0 \prec \eta^0 + \eta^1$ and $\nu^0 + (p-1)\eta^1 \prec \eta^0 + p\eta^1 \prec \mu + \nu^1 \prec \nu^0 + 2\nu^1$. Then we get (8) for such an η, this is still a contradiction. Now we can conclude that if $\mu \in X_1(T)$ satisfies $\operatorname{Hom}_G(L(\mu), L(\nu^0) \otimes L(\nu^1)) \neq 0$, then

$$\begin{aligned}\operatorname{Hom}_{\mathrm{G}_2(p)}(L(\mu), Q_1(\nu^0) \otimes L(\nu^1)) &= \operatorname{Hom}_{\mathrm{G}_2(p)}(L(\mu), U_1(\nu^0) \otimes L(\nu^1)) \\ &= \operatorname{Hom}_G(L(\mu), Q_1(\nu^0) \otimes L(\nu^1)).\end{aligned}$$

Hence we need only to check

$$\operatorname{Hom}_G(L(\mu), L(\nu^0) \otimes L(\nu^1)) = \operatorname{Hom}_G(L(\mu), Q_1(\nu^0) \otimes L(\nu^1)) \tag{9}$$

in stead of (5) which is more complicated.

When (9) fails, in order to compute $\operatorname{Ext}^1_{\mathrm{G}_2(p)}(L(\mu), L(\lambda))$, we have to determine $\operatorname{Soc}_{\mathrm{G}_2(p)} R_1(\lambda)$, and to decompose $Q_1(\lambda)$ (as an injective $\mathrm{G}_2(p)$-module) into the direct sum of $U_1(\eta)$'s (see (1.7)).

Remark In general, we have $p^n\langle\nu^1,\alpha_0^\vee\rangle \le \langle\nu^0+p^n\nu^1,\alpha_0^\vee\rangle \le 2p^n(h-1)$, then $\langle\nu^1,\alpha_0^\vee\rangle \le 2(h-1)$, and hence $p^n \le 4h-3$. So our conclusion is always true for p or n large enough.

2.3 We shall compute all non-trivial $\mathrm{Ext}^1_{G_2(p)}(L(\mu),L(\lambda))$ in three cases, and give examples to illustrate our method.

Case 1. When λ is far from the facets in an alcove C, we have

$$\mathrm{Hom}_G\left(L(\mu),L(\nu^0)\otimes L(\nu^1)\right)=\mathrm{Hom}_G\left(L(\mu),Q_1(\nu^0)\otimes L(\nu^1)\right),$$

for all p-restricted weight μ. Hence (5) holds for all μ as above. We therefore get all μ for which $L(\mu)$ extend $L(\lambda)$ non-trivially(for $G_2(p)$).

For example, take $\lambda=(r,s)$ in the top alcove with $2r+3s>4p-2, r<p-4, s<p-2$, then the p-bounded weight ν for which $L(\nu)$ extends $L(\lambda)$ (for G) has the following values: $(4p-r-3s-5,s), (-3p+r+3s+3,p-s-2), (-3p+r+3s+3,2p-s-2), (-3p+2r+3s+4,3p-r-2s-4), (2p-r-2,-p+r+s+1)$. Therefore, we have by [17, Table 3 and Table 4] that

$$\begin{aligned}
&\mathrm{Soc}_G\left(L(-3p+r+3s+3,p-s-2)\otimes L(0,1)\right)\\
&=2L(-3p+r+3s+3,p-s-2)\oplus L(-3p+r+3s+3,p-s-1)\\
&\oplus L(-3p+r+3s+3,p-s-3)\oplus L(-3p+r+3s+4,p-s-2)\\
&\oplus L(-3p+r+3s+2,p-s-2)\oplus L(-3p+r+3s+4,p-s-3)\\
&\oplus L(-3p+r+3s+2,p-s-1)\oplus L(-3p+r+3s+5,p-s-3)\\
&\oplus L(-3p+r+3s+1,p-s-1)\oplus L(-3p+r+3s+6,p-s-3)\\
&\oplus L(-3p+r+3s,p-s-1)\oplus L(-3p+r+3s+6,p-s-4)\\
&\oplus L(-3p+r+3s,p-s),
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Soc}_G\left(L(-4p+2r+3s+4,3p-r-2s-4)\otimes L(1,0)\right)\\
&=L(-4p+2r+3s+4,3p-r-2s-4)\\
&\oplus L(-4p+2r+3s+5,3p-r-2s-4)\\
&\oplus L(-4p+2r+3s+3,3p-r-2s-4)\\
&\oplus L(-4p+2r+3s+5,3p-r-2s-5)\\
&\oplus L(-4p+2r+3s+3,3p-r-2s-3)\\
&\oplus L(-4p+2r+3s+6,3p-r-2s-5)\\
&\oplus L(-4p+2r+3s+2,3p-r-2s-3),
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Soc}_G\,(L(p-r-2,-p+r+s+1)\otimes L(1,0))\\
&= L(p-r-2,-p+r+s+1)\oplus L(p-r-1,-p+r+s+1)\\
&= L(p-r-3,-p+r+s+1)\oplus L(p-r-1,-p+r+s)\\
&= L(p-r-3,-p+r+s+2)\oplus L(p-r,-p+r+s)\\
&= L(p-r-4,-p+r+s+2).
\end{aligned}$$

We can easily check for μ which is one of the highest weights among the above summands that $\dim \mathrm{Hom}_G\,(L(\mu), L(\nu^0)\otimes L(\nu^1)) = [L(\mu)\otimes L(\nu^1) : L(\nu^0)]_G = \mathrm{Hom}_G\,(L(\mu), Q_1(\nu^0)\otimes L(\nu^1))$. So Case 1 can be done in this way.

Case 2. When λ is close to a facet in an alcove C, (9) no longer holds for some p-bounded weights. In this case μ is in a facet which lies in the upper 'closure' of an alcove. However, if we interchange the roles of λ and μ, then (9) holds, and hence Case 2 will be done in virtue of (7).

For example, take $\lambda = (p-4, s)$ in the top alcove with $\frac{1}{3}(2p+7)\le s\le p-3$, $\mu = (-2p+3s-1, p-s-1)$ and $\mu = (-2p+3s-4, p-s)$. Note that $\nu = (-2p+3s-1, 2p-s-2)$ in this case, i.e., $\nu^0 = (-2p+3s-1, p-s-2)$ and $\nu^1 = (0,1)$. We have

$$\mathrm{Hom}_G\,(L(\mu), L(\nu^0)\otimes L(\nu^1)) = K,$$

whereas

$$\mathrm{Hom}_G\,(L(\mu), Q_1(\nu^0)\otimes L(\nu^1)) = K\oplus K.$$

However, if we interchange the roles of λ and μ, then we have

$$\begin{aligned}
&\mathrm{Hom}_G\,(L(p-4,s), L(p-1,s-1)\otimes L(0,1))\\
&= \mathrm{Hom}_G\,(L(p-4,s), Q_1(p-1,s-1)\otimes L(0,1)) = K,
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Hom}_G\,(L(p-4,s), L(p-1,s-2)\otimes L(0,1))\\
&= \mathrm{Hom}_G\,(L(p-4,s), Q_1(p-1,s-2)\otimes L(0,1)) = K,
\end{aligned}$$

respectively. So (9) still holds in this case.

Case 3. When λ is too close to a facet in an alcove C, (9) no longer holds for some p-bounded weights, even though we interchange the roles of λ and μ. In this case μ is close to a facet which lies in the upper 'closure' of an alcove. We have to pay more attention to this case in the next subsection.

2.4 Let us give an example to show how Case 3 can be done successfully. Take $\lambda = (p-3, s)$ in the top alcove with $\frac{1}{3}(2p+5) \le s \le p-3$, $\mu = (-2p+3s+1, p-s-2)$ and $(-2p+3s-2, p-s-1)$. Note that $\nu = (-2p+3s, 2p-s-2)$ in this case, i.e., $\nu^0 = (-2p+3s, p-s-2)$ and $\nu^1 = (0,1)$. We have

$$\mathrm{Hom}_G\left(L(\mu), L(\nu^0) \otimes L(\nu^1)\right) = K,$$

whereas

$$\mathrm{Hom}_G\left(L(\mu), Q_1(\nu^0) \otimes L(\nu^1)\right) = K \oplus K.$$

Even though we interchange the roles of λ and μ, we still have

$$\mathrm{Hom}_G\left(L(\lambda), L({}^{\star}\nu^0) \otimes L(\nu^1)\right) = K,$$

whereas

$$\mathrm{Hom}_G\left(L(\lambda), Q_1({}^{\star}\nu^0) \otimes L(\nu^1)\right) = K \oplus K,$$

where ${}^{\star}\nu^0 = (p-2, s)$ and $(p-2, s-1)$, respectively. So (9) fails. In order to compute

$$\mathrm{Ext}_{G_2(p)}\left(L(-2p+3s+1, p-s-2), L(p-3, s)\right)$$

and

$$\mathrm{Ext}_{G_2(p)}\left(L(-2p+3s-2, p-s-1), L(p-3, s)\right),$$

we need to determine $\mathrm{Soc}_{G_2(p)} R_1(p-3, s)$ because $Q_1(p-3, s) = U_1(p-3, s)$ (see (1.7)). It is easy to see that

$$\mathrm{Soc}_{G_2(p)}\left(\mathrm{Soc}_G R_1(p-3, s)\right) = \mathrm{Soc}_{G_2(p)} R_1(p-3, s),$$

and

$$\begin{aligned}\mathrm{Soc}_G R_1(p-3, s) = {} & L(-2p+3s, 2p-s-2) \oplus L(-p+3s-2, 2p-2s-1) \\ & L(p+1, s-2) \oplus L(-2p+3s, p-s-2) \oplus L(3p-3s-2, s).\end{aligned}$$

By decomposing the tensor product of two simple G-modules we have

$$\begin{aligned} & L(-2p+3s, p-s-2) \otimes L(0,1) \overset{G}{\longleftrightarrow} 2L(-2p+3s, p-s-2) \\ & \oplus L(-2p+3s, p-s-1) \oplus L(-2p+3s, p-s-3) \\ & \oplus 2L(-2p+3s+1, p-s-2) \oplus L(-2p+3s-1, p-s-2) \\ & \oplus L(-2p+3s+1, p-s-3) \oplus L(-2p+3s-1, p-s-1) \\ & \oplus L(-2p+3s+2, p-s-3) \oplus 2L(-2p+3s-2, p-s-1) \\ & \oplus L(-2p+3s+3, p-s-3) \oplus L(-2p+3s-3, p-s-1) \\ & \oplus L(-2p+3s+3, p-s-4) \oplus L(-2p+3s-3, p-s) \\ & \oplus L(0, p-s-2) \oplus L(0, p-s-1),\end{aligned}$$

where $\overset{G}{\longleftrightarrow}$ means that the two sides have the same G-composition factors. Therefore, we get

$$\begin{aligned}
&\mathrm{Soc}_G\,(L(-2p+3s,p-s-2)\otimes L(0,1))\\
&=\mathrm{Soc}_{G_2(p)}\,(L(-2p+3s,p-s-2)\otimes L(0,1))\\
&=2L(-2p+3s,p-s-2)\oplus L(-2p+3s,p-s-3)\\
&\oplus L(-2p+3s+1,p-s-2)\oplus L(-2p+3s-1,p-s-2)\\
&\oplus L(-2p+3s+1,p-s-3)\oplus L(-2p+3s-1,p-s-1)\\
&\oplus L(-2p+3s+2,p-s-3)\oplus L(-2p+3s-2,p-s-1)\\
&\oplus L(-2p+3s+3,p-s-3)\oplus L(-2p+3s-3,p-s-1)\\
&\oplus L(-2p+3s+3,p-s-4).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\mathrm{Soc}_G\,(L(-2p+3s-2,2p-2s-1)\otimes L(1,0))\\
&=\mathrm{Soc}_{G_2(p)}\,(L(-2p+3s-2,2p-2s-1)\otimes L(1,0))\\
&=L(-2p+3s-2,2p-2s-1)\oplus L(-2p+3s-1,2p-2s-1)\\
&\oplus L(-2p+3s-3,2p-2s-1)\oplus L(-2p+3s-1,2p-2s-2)\\
&\oplus L(-2p+3s-3,2p-2s)\oplus L(-2p+3s,2p-2s-2)\\
&\oplus L(-2p+3s-4,2p-2s),
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Soc}_G\,(L(1,s-2)\otimes L(1,0))=\mathrm{Soc}_{G_2(p)}\,(L(1,s-2)\otimes L(1,0))\\
&=L(1,s-2)\oplus L(2,s-2)\oplus L(0,s-2)\oplus L(2,s-3)\oplus L(0,s-1)\\
&\oplus L(3,s-3).
\end{aligned}$$

So we have

$$\begin{aligned}
&\mathrm{Ext}_{G_2(p)}\,(L(-2p+3s+1,p-s-2),L(p-3,s))=K\\
&\mathrm{Ext}_{G_2(p)}\,(L(-2p+3s-2,p-s-1),L(p-3,s))=K,
\end{aligned}$$

and hence (4) is still true. The same argument applies to all other pairs $\{\lambda,\mu\}$ in this case.

Acknowledgement

I am grateful to Max-Planck-Institut für Mathematik for financial support. Also, I would like to thank Prof. Dr. F. Hirzebruch for his hospitality during my stay. The author is also supported by the National Natural Science Foundation of China.

Table 1

$\lambda = (r, s)$	$\mu = (u, v)$ with $2u + 3v \leq 2r + 3s$, and $Ext^1_{G_2(p)}(L(\lambda), L(\mu)) \neq 0$	
$2r + 3s \leq p - 5$, $r + 3s = p - 4$, $r + 2s = p - 3$, $2r + 3s = 2p - 5$, $r + s = p - 2$ $s \leq \frac{1}{2}(p - 3)$, $r = p - 1$ $s \leq \frac{1}{3}(p - 4)$, $r + 3s = 2p - 4$ $r \leq \frac{1}{2}(p - 5)$ $r = s = p - 1$	none	
$2r + 3s > p - 5$, $r + 3s < p - 4$	$(p - r - 3s - 5, s)$	
$r + 3s > p - 4$, $r + 2s < p - 3$	$(-p + 2r + 3s + 4, p - r - 2s - 4)$	
$r + 2s > p - 3$, $2r + 3s < 2p - 5$	$(r, p - r - s - 3)$	
$2r + 3s > 2p - 5$, $r + s < p - 2$, $r + 3s < 2p - 4$	$(2p - r - 3s - 5, s)$	
$r + 3s > 2p - 4$, $r + s < p - 2$, $r > 1$	$(-2p + r + 3s + 3, p - s - 2)$ $(-2p + r + 3s + 4, p - s - 2)$ $(-2p + r + 3s + 2, p - s - 2)$ $(-2p + r + 3s + 4, p - s - 3)$ $(-2p + r + 3s + 2, p - s - 1)$ $(-2p + r + 3s + 5, p - s - 3)$ $(-2p + r + 3s + 1, p - s - 1)$	$(-2p + 2r + 3s + 4, 2p - r - 2s - 4)$
$(1, s)$, $\frac{1}{3}(2p - 4) \leq s$, $s \leq p - 4$	$(-2p + 3s + 4, p - s - 2)$ $(-2p + 3s + 5, p - s - 2)$ $(-2p + 3s + 3, p - s - 2)$ $(-2p + 3s + 3, p - s - 1)$ $(-2p + 3s + 6, p - s - 3)$ $(-2p + 3s + 2, p - s - 1)$	$(-2p + 3s + 6, 2p - 2s - 5)$

$\lambda = (r, s)$	$\mu = (u, v)$ with $2u + 3v \leq 2r + 3s$, and $Ext^1_{G_2(p)}(L(\lambda), L(\mu)) \neq 0$	
$(0, s)$, $\frac{1}{3}(2p-2) \leq s$, $s \leq p-3$	$(-2p+3s+4, 2p-2s-4)$ $(-2p+3s+4, p-s-2)$ $(-2p+3s+2, p-s-1)$ $(-2p+3s+1, p-s-1)$	
$(1, p-3)$	$(p-5, 1)$ $(p-4, 1)$ $(p-6, 1)$ $(p-4, 0)$ $(p-3, 0)$ $(p-7, 2)$	$(p-3, 1)$
$(0, p-2)$	$(p-3, 0)$ $(p-2, 0)$ $(p-4, 0)$ $(p-4, 1)$ $(p-5, 1)$	$(p-2, 0)$
$r+s = p-2$, $2 \leq r \leq \frac{1}{2}(p-3)$	$(-p+2s+1, r)$ $(-p+2s, r)$ $(-p+2s+2, r-1)$ $(-p+2s+3, r-1)$ $(-p+2s-1, r+1)$	(s, r)
$r+s > p-2$, $r+3s < 2p-5$, $r < p-1$	$(3p-2r-3s-6, -p+r+2s+2)$ $(p-r-2, -p+r+s+1)$ $(p-r-1, -p+r+s+1)$ $(p-r-3, -p+r+s+1)$ $(p-r-1, -p+r+s)$ $(p-r-3, -p+r+s+2)$ $(p-r, -p+r+s)$ $(p-r-4, -p+r+s+2)$	
$r+3s = 2p-5$, $\frac{1}{2}(p+5) \leq r$, $r \leq p-2$	$(p-r-2, -p+r+s+1)$ $(p-r-3, -p+r+s+1)$ $(p-r-1, -p+r+s)$ $(p-r-3, -p+r+s+2)$ $(p-r, -p+r+s)$ $(p-r-4, -p+r+s+2)$	$(p-r-1, p-s-3)$
$r+3s = 2p-4$, $\frac{1}{2}(p+1) \leq r$, $r \leq p-2$	$(-p+3s+2, p-2s-3)$ $(-p+3s+1, p-2s-3)$ $(-p+3s+3, p-2s-4)$ $(-p+3s, p-2s-2)$	$(-p+3s+2, p-s-2)$

$\lambda = (r, s)$	$\mu = (u, v)$ with $2u + 3v \leq 2r + 3s$, and $Ext^1_{G_2(p)}(L(\lambda), L(\mu)) \neq 0$	
$2r + 3s < 3p - 6$, $r + 3s > 2p - 2$, $r + s > p - 1$	$(-2p + 2r + 3s + 4, 2p - r - 2s - 4)$ $(3p - 2r - 3s - 6, -p + r + 2s + 2)$ $(-2p + r + 3s + 3, p - s - 2)$ $(-2p + r + 3s + 4, p - s - 2)$ $(-2p + r + 3s + 2, p - s - 2)$ $(-2p + r + 3s + 4, p - s - 3)$ $(-2p + r + 3s + 2, p - s - 1)$ $(-2p + r + 3s + 5, p - s - 3)$ $(-2p + r + 3s + 1, p - s - 1)$ $(-2p + r + 3s + 3, p - s - 2)$	$(p - r - 2, -p + r + s + 1)$ $(p - r - 1, -p + r + s + 1)$ $(p - r - 3, -p + r + s + 1)$ $(p - r - 1, -p + r + s)$ $(p - r - 3, -p + r + s + 2)$ $(p - r, -p + r + s)$ $(p - r - 4, -p + r + s + 2)$
$(3, p - 4)$	$(p - 2, 1)$ $(0, p - 3)$ $(p - 5, 0)$ $(p - 4, 0)$ $(p - 6, 0)$ $(p - 7, 1)$	$(p - 6, 2)$ $(p - 6, 2)$ $(p - 7, 3)$ $(p - 4, 1)$ $(p - 8, 3)$
$(p - 4, \frac{1}{3}(p + 2))$	$(0, \frac{1}{3}(2p - 2))$ $(p - 2, \frac{1}{3}(p - 4))$ $(2, \frac{1}{3}(p - 7))$ $(3, \frac{1}{3}(p - 7))$ $(3, \frac{1}{3}(p - 10))$ $(4, \frac{1}{3}(p - 10))$ $(0, \frac{1}{3}(p - 4))$	$(1, \frac{1}{3}(2p - 8))$ $(1, \frac{1}{3}(2p - 8))$ $(0, \frac{1}{3}(2p - 8))$ $(2, \frac{1}{3}(2p - 11))$ $(0, \frac{1}{3}(2p - 5))$ $(3, \frac{1}{3}(2p - 11))$
$2r + 3s = 3p - 6$, $6 \leq r \leq p - 5$	$(p - 2, -p + r + s + 2)$ $(p - r - 3, p - s - 2)$ $(p - r - 3, p - s - 2)$ $(p - r - 4, p - s - 1)$ $(p - r - 4, p - s - 2)$ $(p - r - 2, p - s - 3)$ $(p - r - 1, p - s - 3)$ $(p - r - 5, p - s - 1)$	$(0, -p + r + 2s + 2)$ $(p - r - 2, -p + r + s + 1)$ $(p - r - 1, -p + r + s + 1)$ $(p - r - 3, -p + r + s + 1)$ $(p - r - 1, -p + r + s)$ $(p - r, -p + r + s)$ $(p - r - 4, -p + r + s + 2)$
$r + 3s = 2p - 2$, $\frac{1}{2}(p + 1) \leq r$, $r \leq p - 6$	$(p - r - 4, p - s)$ $(1, p - s - 2)$ $(1, p - s - 2)$ $(2, p - s - 2)$ $(0, p - s - 2)$ $(2, p - s - 3)$ $(0, p - s - 1)$ $(3, p - s - 3)$	$(r + 2, s - 2)$ $(p - r - 2, -p + r + s + 1)$ $(p - r - 1, -p + r + s + 1)$ $(p - r - 3, -p + r + s + 2)$ $(p - r - 1, -p + r + s)$ $(p - r, -p + r + s)$ $(p - r - 4, -p + r + s + 2)$

$\lambda=(r,s)$	$\mu=(u,v)$ with $2u+3v\le 2r+3s$, and $Ext^1_{G_2(p)}(L(\lambda),L(\mu))\neq 0$	
$r+3s=2p-3$,	$(p-r-3,p-s-1)$	$(r+1,s-1)$
$\frac{1}{2}(p+3)\le r$,	$(0,p-s-2)$	$(p-r-1,-p+r+s+1)$
$r\le p-4$	$(1,p-s-2)$	$(p-r-3,-p+r+s+2)$
	$(1,p-s-3)$	$(p-r,-p+r+s)$
	$(2,p-s-3)$	
$r+s=p-1$,	$(p-s-4,s+1)$	$(s+2,p-s-3)$
$4\le r\le\frac{1}{2}(p-1)$	$(-p+2s+2,p-s-2)$	$(p-r-2,0)$
	$(-p+2s+2,p-s-2)$	$(p-r-1,0)$
	$(-p+2s+3,p-s-2)$	$(p-r-3,0)$
	$(-p+2s+1,p-s-1)$	$(p-r-3,1)$
	$(-p+2s+4,p-s-3)$	$(p-r-4,1)$
	$(-p+2s,p-s-1)$	
$2r+3s=3p-5$,	$(p-1,-p+r+s+1)$	$(p-r-2,p-s-2)$
$5\le r\le p-5$	$(p-r-2,-p+r+s+1)$	$(p-r-2,p-s-2)$
	$(p-r-3,-p+r+s+1)$	$(p-r-3,p-s-2)$
	$(p-r-1,-p+r+s)$	$(p-r-1,p-s-3)$
	$(p-r,-p+r+s)$	$(p-r-4,p-s-1)$
$(p-3,\frac{1}{3}(p+1))$	$(p-1,\frac{1}{3}(p-5))$	$(1,\frac{1}{3}(2p-7))$
	$(1,\frac{1}{3}(p-5))$	$(1,\frac{1}{3}(2p-7))$
	$(0,\frac{1}{3}(p-5))$	$(0,\frac{1}{3}(2p-4))$
	$(2,\frac{1}{3}(p-8))$	$(2,\frac{1}{3}(2p-10))$
	$(3,\frac{1}{3}(p-8))$	$(0,\frac{1}{3}(2p-7))$
$(2,p-3)$	$(p-1,0)$ $(p-4,0)$	$(p-4,1)$ $(p-4,1)$
	$(p-3,0)$ $(p-5,0)$	$(p-3,1)$ $(p-5,1)$
	$(p-5,1)$	$(p-3,0)$ $(p-2,0)$
		$(p-6,2)$
$(p-2,\frac{1}{3}(p-1))$	$(p-1,\frac{1}{3}(p-4))$	$(0,\frac{1}{3}(2p-5))$
	$(1,\frac{1}{3}(p-7))$	$(1,\frac{1}{3}(2p-5))$
	$(2,\frac{1}{3}(p-7))$	$(1,\frac{1}{3}(2p-8))$
$2r+3s>3p-3$,	$(-3p+2r+3s+4,2p-r-2s-4)$	$(3p-r-3s-5,s)$
$r+2s<2p-3$,	$(-3p+2r+3s+5,2p-r-2s-4)$	$(-2p+r+3s+3,p-s-2)$
$r<p-2$	$(-3p+2r+3s+3,2p-r-2s-4)$	$(-2p+r+3s+3,p-s-2)$
	$(-3p+2r+3s+5,2p-r-2s-5)$	$(-2p+r+3s+4,p-s-2)$
	$(-3p+2r+3s+3,2p-r-2s-3)$	$(-2p+r+3s+2,p-s-2)$

$\lambda = (r,s)$	$\mu = (u,v)$ with $2u+3v \le 2r+3s$, and $Ext^1_{G_2(p)}(L(\lambda), L(\mu)) \neq 0$	
$2r+3s > 3p-3$,	$(-3p+2r+3s+6, 2p-r-2s-5)$	$(-2p+r+3s+4, p-s-3)$
$r+2s < 2p-3$,	$(-3p+2r+3s+2, 2p-r-2s-3)$	$(-2p+r+3s+2, p-s-1)$
$r < p-2$	$(p-r-2, -p+r+s+1)$	$(-2p+r+3s+5, p-s-3)$
	$(p-r-2, -p+r+s+1)$	$(-2p+r+3s+1, p-s-1)$
	$(p-r-1, -p+r+s+1)$	
	$(p-r-3, -p+r+s+1)$	
	$(p-r-1, -p+r+s)$	
	$(p-r-3, -p+r+s+2)$	
	$(p-r, -p+r+s)$	
	$(p-r-4, -p+r+s+2)$	
$2r+3s = 3p-3$,	$(1, -p+r+s-1)$	$(r-2, s)$
$6 \le r \le p-3$	$(2, -p+r+s-1)$	$(p-r, p-s-2)$
	$(0, -p+r+s-1)$	$(p-r, p-s-2)$
	$(2, -p+r+s-2)$	$(p-r+1, p-s-2)$
	$(0, -p+r+s)$	$(p-r+1, p-s-3)$
	$(3, -p+r+s-2)$	$(p-r-1, p-s-1)$
	$(p-r-2, -p+r+s+1)$	$(p-r+2, p-s-3)$
	$(p-r-2, -p+r+s+1)$	$(p-r-2, p-s-1)$
	$(p-r-1, -p+r+s+1)$	
	$(p-r-3, -p+r+s+1)$	
	$(p-r-3, -p+r+s+2)$	
	$(p-r, -p+r+s)$	
	$(p-r-4, -p+r+s+2)$	
$2r+3s = 3p-4$,	$(1, -p+r+s)$	$(r-1, s)$
$4 \le r \le p-3$	$(1, -p+r+s-1)$	$(p-r-1, p-s-2)$
	$(2, -p+r+s-1)$	$(p-r, p-s-2)$
	$(p-r-2, -p+r+s+1)$	$(p-r-2, p-s-1)$
	$(p-r-1, -p+r+s+1)$	$(p-r+1, p-s-3)$
	$(p-r-3, -p+r+s+2)$	
	$(p-r-4, -p+r+s+2)$	
$r+2s = 2p-3$,	$(s, p-s-2)$	$(-p+2s+1, p-s-2)$
$5 \le r \le p-4$	$(s, p-s-2)$	$(-p+2s+1, p-s-2)$
	$(s-1, p-s-2)$	$(-p+2s, p-s-2)$
	$(s+1, p-s-3)$	$(-p+2s+2, p-s-3)$
	$(s-1, p-s-1)$	$(-p+2s+3, p-s-3)$
	$(s-2, p-s-1)$	$(-p+2s-1, p-s-1)$

$\lambda = (r,s)$	$\mu = (u,v)$ with $2u+3v \le 2r+3s$, and $Ext^1_{G_2(p)}(L(\lambda), L(\mu)) \neq 0$			
$(p-2, \frac{1}{2}(p-1))$	$(\frac{1}{2}(p-1), \frac{1}{2}(p-3))$		$(0, \frac{1}{2}(p-3))$	
	$(\frac{1}{2}(p-1), \frac{1}{2}(p-3))$		$(1, \frac{1}{2}(p-5))$	
	$(\frac{1}{2}(p-3), \frac{1}{2}(p-3))$		$(2, \frac{1}{2}(p-5))$	
	$(\frac{1}{2}(p+1), \frac{1}{2}(p-5))$			
	$(\frac{1}{2}(p-5), \frac{1}{2}(p-1))$			
$(3, p-3)$	$(p-3, 1)$	$(p-3, 1)$	$(p-5, 1)$	$(p-5, 1)$
	$(p-4, 1)$	$(p-2, 0)$	$(p-4, 1)$	$(p-6, 1)$
	$(p-4, 2)$	$(p-1, 0)$	$(p-4, 0)$	$(p-3, 0)$
	$(p-5, 2)$		$(p-7, 2)$	
$(1, p-2)$	$(p-2, 0)$	$(p-2, 0)$	$(p-3, 0)$	$(p-3, 0)$
	$(p-1, 0)$	$(p-3, 0)$	$(p-2, 0)$	$(p-4, 0)$
	$(p-3, 1)$	$(p-4, 1)$	$(p-4, 1)$	$(p-5, 1)$
$(p-2, s)$,	$(2p-3s-3, s)$		$(-p+3s+1, p-s-2)$	
$\frac{1}{3}(p+2) \le s$,	$(-p+3s, p-2s-2)$		$(-p+3s+1, p-s-2)$	
$s \le \frac{1}{2}(p-3)$	$(-p+3s-1, p-2s-2)$		$(-p+3s+2, p-s-2)$	
	$(-p+3s+1, p-2s-3)$		$(-p+3s, p-s-2)$	
	$(-p+3s-1, p-2s-1)$		$(-p+3s+2, p-s-3)$	
	$(-p+3s+2, p-2s-3)$		$(-p+3s+3, p-s-3)$	
	$(-p+3s-2, p-2s-1)$		$(-p+3s-1, p-s-1)$	
	$(0, s-1)$	$(1, s-1)$	$(1, s-2)$	$(2, s-2)$
$(p-2, \frac{1}{3}(p+1))$	$(p-4, \frac{1}{3}(p+1))$		$(2, \frac{1}{3}(2p-7))$	
	$(1, \frac{1}{3}(p-8))$		$(2, \frac{1}{3}(2p-7))$	
	$(0, \frac{1}{3}(p-8))$		$(3, \frac{1}{3}(2p-7))$	
	$(2, \frac{1}{3}(p-11))$		$(3, \frac{1}{3}(2p-10))$	
	$(0, \frac{1}{3}(p-5))$		$(4, \frac{1}{3}(2p-10))$	
	$(3, \frac{1}{3}(p-11))$		$(0, \frac{1}{3}(2p-4))$	
	$(1, \frac{1}{3}(p-2))$			
	$(2, \frac{1}{3}(p-5))$			
$(p-1, s)$,	$(-p+3s+2, p-s-2)$		$(-p+3s+2, p-2s-3)$	
$\frac{1}{3}(p+1) \le s$,	$(-p+3s+2, p-s-2)$		$(-p+3s+1, p-2s-3)$	
$s \le \frac{1}{2}(p-3)$	$(-p+3s+1, p-s-2)$		$(-p+3s+3, p-2s-4)$	
	$(-p+3s+3, p-s-3)$		$(-p+3s, p-2s-2)$	
	$(-p+3s+4, p-s-3)$			

$\lambda = (r, s)$	$\mu = (u, v)$ with $2u + 3v \leq 2r + 3s$, and $Ext^1_{G_2(p)}(L(\lambda), L(\mu)) \neq 0$			
$(p-1, \frac{1}{3}(p-2))$	$(0, \frac{1}{3}(2p-4))$	$(1, \frac{1}{3}(2p-7))$	$(1, \frac{1}{3}(p-5))$	
	$(2, \frac{1}{3}(2p-7))$		$(1, \frac{1}{3}(p-8))$	
$(p-1, \frac{1}{3}(p-1))$	$(1, \frac{1}{3}(2p-5))$		$(1, \frac{1}{3}(p-7))$	
	$(1, \frac{1}{3}(2p-5))$		$(0, \frac{1}{3}(p-7))$	
	$(0, \frac{1}{3}(2p-5))$		$(2, \frac{1}{3}(p-10))$	
	$(2, \frac{1}{3}(2p-8))$		$(0, \frac{1}{3}(p-4))$	
	$(3, \frac{1}{3}(2p-8))$			
$r + 2s > 2p - 2$,	$(4p-2r-3s-6, -2p+r+2s+2)$		$(r, 2p-r-s-3)$	
$r + 3s < 3p - 5$,	$(p-r-2, -p+r+s+1)$		$(-2p+r+3s+3, p-s-2)$	
$r < p - 2$	$(p-r-2, -p+r+s+1)$		$(-2p+r+3s+4, p-s-2)$	
	$(p-r-1, -p+r+s+1)$		$(-2p+r+3s+2, p-s-2)$	
	$(p-r-3, -p+r+s+1)$		$(-2p+r+3s+4, p-s-3)$	
	$(p-r-1, -p+r+s)$		$(-2p+r+3s+2, p-s-1)$	
	$(p-r-3, -p+r+s+2)$		$(-2p+r+3s+5, p-s-3)$	
	$(p-r, -p+r+s)$		$(-2p+r+3s+1, p-s-1)$	
	$(p-r-4, -p+r+s+2)$			
$r + 3s = 3p - 5$,	$(p-r-1, p-s-3)$		$(r, 2p-r-s-3)$	
$7 \leq r \leq p - 3$	$(p-r-2, -p+r+s+1)$		$(p-2, p-s-2)$	
	$(p-r-2, -p+r+s+1)$		$(p-1, p-s-2)$	
	$(p-r-3, -p+r+s+1)$		$(p-3, p-s-2)$	
	$(p-r-1, -p+r+s)$		$(p-1, p-s-3)$	
	$(p-r-3, -p+r+s+2)$		$(p-3, p-s-1)$	
	$(p-r, -p+r+s)$		$(p-4, p-s-1)$	
	$(p-r-4, -p+r+s+2)$			
$r + 2s = 2p - 2$,	$(s-2, 0)$		$(r, s-1)$	
$6 \leq r \leq p - 3$	$(p-r-2, p-s-1)$		$(s+1, p-s-2)$	
	$(p-r-2, p-s-1)$		$(s+2, p-s-2)$	
	$(p-r-1, p-s-1)$		$(s+2, p-s-3)$	
	$(p-r-3, p-s)$		$(s, p-s-1)$	
	$(p-r, p-s-2)$		$(s+3, p-s-3)$	
	$(p-r-4, p-s)$			
$(4, p-3)$	$(p-5, 0)$	$(p-6, 2)$	$(4, p-4)$	$(p-2, 1)$
	$(p-6, 2)$	$(p-7, 3)$	$(p-1, 1)$	$(p-1, 0)$
	$(p-4, 1)$	$(p-8, 3)$	$(p-3, 2)$	

$\lambda = (r,s)$	$\mu = (u,v)$ with $2u+3v \leq 2r+3s$, and $Ext^1_{G_2(p)}(L(\lambda),L(\mu)) \neq 0$	
$(p-2,s)$,	$(2p-3s-2,-p+2s)$	$(-p+3s+1,p-s-2)$
$\frac{1}{2}(p+1) \leq s$,	$(p-2,p-s-1)$	$(-p+3s+2,p-s-2)$
$s \leq \frac{1}{3}(2p-4)$	$(0,s-1)$	$(-p+3s,p-s-2)$
	$(1,s-1)$	$(-p+3s+2,p-s-3)$
	$(1,s-2)$	$(-p+3s+3,p-s-3)$
	$(2,s-2)$	$(-p+3s-1,p-s-1)$
$(p-1,s)$,	$(-p+3s+2,p-s-2)$	$(p-1,p-s-2)$
$\frac{1}{2}(p+1) \leq s$,	$(-p+3s+1,p-s-2)$	
$s \leq \frac{1}{3}(2p-5)$	$(-p+3s+3,p-s-3)$	
	$(-p+3s+4,p-s-3)$	
$(p-1,\frac{1}{2}(p-1))$	$(\frac{1}{2}(p+1),\frac{1}{2}(p-3))$	$(p-1,\frac{1}{2}(p-3))$
	$(\frac{1}{2}(p+3),\frac{1}{2}(p-5))$	
	$(\frac{1}{2}(p+5),\frac{1}{2}(p-5))$	
$(p-1,\frac{1}{3}(2p-4))$	$(p-2,\frac{1}{3}(p-2))$	$(p-1,\frac{1}{3}(p-2))$
	$(p-3,\frac{1}{3}(p-2))$	$(p-1,\frac{1}{3}(p-5))$
$r+3s=3p-4$,	$(p-r-2,p-s-2)$	$(p-r-2,-p+r+s+1)$
$3 \leq r \leq p-4$	$(p-1,p-s-2)$	$(p-r-2,-p+r+s+1)$
	$(p-2,p-s-2)$	$(p-r-3,-p+r+s+1)$
	$(p-2,p-s-1)$	$(p-r-1,-p+r+s)$
	$(p-3,p-s-1)$	$(p-r-4,-p+r+s+2)$
$(2,p-2)$	$(p-4,0)$ $(p-1,0)$	$(p-4,1)$ $(p-4,1)$
	$(p-2,0)$ $(p-2,1)$	$(p-3,1)$ $(p-5,1)$
	$(p-3,1)$	$(p-3,0)$ $(p-2,0)$
		$(p-6,2)$
$(p-3,\frac{1}{3}(2p-1))$	$(1,\frac{1}{3}(p-5))$	$(1,\frac{1}{3}(2p-7))$
	$(p-1,\frac{1}{3}(p-5))$	$(1,\frac{1}{3}(2p-7))$
	$(p-2,\frac{1}{3}(p-5))$	$(0,\frac{1}{3}(2p-7))$
	$(p-2,\frac{1}{3}(p-2))$	$(0,\frac{1}{3}(2p-4))$
	$(p-3,\frac{1}{3}(p-2))$	$(2,\frac{1}{3}(2p-10))$
$(p-2,\frac{1}{3}(2p-2))$	$(0,\frac{1}{3}(p-4))$	$(0,\frac{1}{3}(2p-5))$
	$(p-1,\frac{1}{3}(p-4))$	$(1,\frac{1}{3}(2p-5))$
	$(p-2,\frac{1}{3}(p-4))$	$(1,\frac{1}{3}(2p-8))$
	$(p-3,\frac{1}{3}(p-1))$	

$\lambda = (r,s)$	$\mu = (u,v)$ with $2u+3v \leq 2r+3s$, and $Ext^1_{G_2(p)}(L(\lambda), L(\mu)) \neq 0$	
$r+3s > 3p-2$,	$(4p-2r-3s-6, -2p+r+2s+2)$	$(-3p+r+3s+3, p-s-2)$
$2r+3s < 4p-7$,	$(-3p+2r+3s+4, 3p-r-2s-4)$	$(-3p+r+3s+3, p-s-2)$
$s < p-1$	$(p-r-2, -p+r+s+1)$	$(-3p+r+3s+3, p-s-2)$
	$(p-r-1, -p+r+s+1)$	$(-3p+r+3s+3, p-s-1)$
	$(p-r-3, -p+r+s+1)$	$(-3p+r+3s+3, p-s-3)$
	$(p-r-1, -p+r+s)$	$(-3p+r+3s+4, p-s-2)$
	$(p-r-3, -p+r+s+2)$	$(-3p+r+3s+2, p-s-2)$
	$(p-r, -p+r+s)$	$(-3p+r+3s+4, p-s-3)$
	$(p-r-4, -p+r+s+2)$	$(-3p+r+3s+2, p-s-1)$
		$(-3p+r+3s+5, p-s-3)$
		$(-3p+r+3s+1, p-s-1)$
		$(-3p+r+3s+6, p-s-3)$
		$(-3p+r+3s, p-s-1)$
		$(-3p+r+3s+6, p-s-4)$
		$(-3p+r+3s, p-s)$
$2r+3s = 4p-7$,	$(1, 2p-r-s-5)$	$(p-r-4, p-s-2)$
$\frac{1}{2}(p-3) \leq r$,	$(p-3, -p+r+s+3)$	$(p-r-4, p-s-2)$
$r \leq p-6$	$(p-r-2, -p+r+s+1)$	$(p-r-4, p-s-2)$
	$(p-r-1, -p+r+s+1)$	$(p-r-4, p-s-3)$
	$(p-r-3, -p+r+s+1)$	$(p-r-3, p-s-2)$
	$(p-r-1, -p+r+s)$	$(p-r-5, p-s-2)$
	$(p-r-3, -p+r+s+2)$	$(p-r-3, p-s-3)$
	$(p-r, -p+r+s)$	$(p-r-5, p-s-1)$
	$(p-r-4, -p+r+s+2)$	$(p-r-2, p-s-3)$
		$(p-r-6, p-s-1)$
		$(p-r-7, p-s-1)$
		$(p-r-1, p-s-4)$
		$(p-r-7, p-s)$
$2r+3s = 4p-6$,	$(0, 2p-r-s-4)$	$(p-r-3, p-s-2)$
$\frac{1}{2}(p-1) \leq r$,	$(p-2, -p+r+s+2)$	$(p-r-3, p-s-2)$
$r \leq p-5$,	$(p-r-2, -p+r+s+1)$	$(p-r-3, p-s-2)$
	$(p-r-1, -p+r+s+1)$	$(p-r-3, p-s-3)$
	$(p-r-3, -p+r+s+1)$	$(p-r-4, p-s-2)$
	$(p-r-1, -p+r+s)$	$(p-r-2, p-s-3)$
	$(p-r, -p+r+s)$	$(p-r-4, p-s-1)$
	$(p-r-4, -p+r+s+2)$	$(p-r-1, p-s-3)$
	$(p-r-5, p-s-1)$	$(p-r-6, p-s-1)$
	$(p-r, p-s-4)$	$(p-r-6, p-s)$

$\lambda = (r, s)$	$\mu = (u, v)$ with $2u + 3v \leq 2r + 3s$, and $Ext^1_{G_2(p)}(L(\lambda), L(\mu)) \neq 0$	
$(p-4, \frac{1}{3}(2p+2))$	$(0, \frac{1}{3}(p-2))$	$(1, \frac{1}{3}(p-8))$
	$(p-2, \frac{1}{3}(2p-4))$	$(1, \frac{1}{3}(p-8))$
	$(2, \frac{1}{3}(2p-7))$	$(1, \frac{1}{3}(p-8))$
	$(3, \frac{1}{3}(2p-7))$	$(1, \frac{1}{3}(p-5))$
	$(3, \frac{1}{3}(2p-10))$	$(1, \frac{1}{3}(p-11))$
	$(4, \frac{1}{3}(2p-10))$	$(2, \frac{1}{3}(p-11))$
	$(0, \frac{1}{3}(2p-4))$	$(3, \frac{1}{3}(p-11))$
		$(4, \frac{1}{3}(p-14))$
$r + 3s = 3p - 2$,	$(p-r-4, p-s)$	$(1, p-s-2)$
$2 \leq r \leq p-6$	$(r+2, s-2)$	$(1, p-s-2)$
	$(p-r-2, -p+r+s+1)$	$(1, p-s-2)$
	$(p-r-1, -p+r+s+1)$	$(1, p-s-1)$
	$(p-r-1, -p+r+s)$	$(1, p-s-3)$
	$(p-r-3, -p+r+s+2)$	$(2, p-s-2)$
	$(p-r, -p+r+s)$	$(2, p-s-3)$
	$(p-r-4, -p+r+s+2)$	$(3, p-s-3)$
		$(4, p-s-3)$
		$(4, p-s-4)$
$r + 3s = 3p - 3$,	$(p-r-3, p-s-1)$	$(0, p-s-2)$
$3 \leq r \leq p-5$	$(r+1, s-1)$	$(0, p-s-2)$
	$(p-r-1, -p+r+s+1)$	$(0, p-s-1)$
	$(p-r-3, -p+r+s+2)$	$(0, p-s-3)$
	$(p-r, -p+r+s)$	$(2, p-s-3)$
		$(3, p-s-3)$
		$(3, p-s-4)$
$(p-4, \frac{1}{3}(2p+1))$	$(1, \frac{1}{3}(p-4))$	$(0, \frac{1}{3}(p-7))$
	$(p-3, \frac{1}{3}(2p-2))$	$(0, \frac{1}{3}(p-4))$
	$(3, \frac{1}{3}(2p-8))$	$(0, \frac{1}{3}(p-10))$
	$(1, \frac{1}{3}(2p-5))$	$(0, \frac{1}{3}(p-10))$
	$(4, \frac{1}{3}(2p-11))$	$(2, \frac{1}{3}(p-13))$
		$(3, \frac{1}{3}(p-7))$
$(p-3, \frac{1}{3}(2p+1))$	$(1, \frac{1}{3}(p-7))$ $(1, \frac{1}{3}(p-7))$	$(p-1, \frac{1}{3}(2p-5))$
	$(1, \frac{1}{3}(p-10))$	$(1, \frac{1}{3}(2p-5))$
	$(2, \frac{1}{3}(p-7))$	$(0, \frac{1}{3}(2p-5))$
	$(2, \frac{1}{3}(p-10))$	$(2, \frac{1}{3}(2p-8))$
	$(4, \frac{1}{3}(p-13))$	$(3, \frac{1}{3}(2p-8))$

$\lambda = (r, s)$	$\mu = (u, v)$ with $2u + 3v \leq 2r + 3s$, and $Ext^1_{G_2(p)}(L(\lambda), L(\mu)) \neq 0$	
$(p-2, \frac{1}{3}(2p-1))$	$(0, \frac{1}{3}(p-5))$ $(0, \frac{1}{3}(p-8))$ $(2, \frac{1}{3}(p-8))$ $(3, \frac{1}{3}(p-11))$	$(p-1, \frac{1}{3}(2p-4))$ $(1, \frac{1}{3}(2p-7))$ $(2, \frac{1}{3}(2p-7))$
$2r + 3s = 4p - 5$, $\frac{1}{2}(p+1) \leq s$, $r \leq p - 5$	$(p-r-2, p-s-2)$ $(p-r-2, p-s-2)$ $(p-r-2, p-s-2)$ $(p-r-2, p-s-3)$ $(p-r-3, p-s-2)$ $(p-r-1, p-s-3)$ $(p-r-4, p-s-1)$ $(p-r-5, p-s-1)$ $(p-r+1, p-s-4)$ $(p-r-5, p-s)$	$(p-1, -p+r+s+1)$ $(p-r-2, -p+r+s+1)$ $(p-r-3, -p+r+s+2)$ $(p-r-1, -p+r+s)$ $(p-r, -p+r+s)$
$(r, p-1)$, $2 \leq r$, $r \leq \frac{1}{2}(p-5)$	$(p-r-2, r)$ $(p-r-3, r)$ $(p-r-1, r-1)$ $(p-r-3, r+1)$ $(p-r-4, r+1)$	$(p-2r-3, r)$
$(0, p-1)$	$(p-2, 0)$ $(p-1, 0)$ $(p-3, 0)$ $(p-3, 1)$ $(p-4, 1)$	$(p-3, 0)$
$(1, p-1)$	$(p-3, 1)$ $(p-4, 1)$ $(p-2, 0)$ $(p-4, 2)$ $(p-1, 0)$ $(p-5, 2)$	$(p-5, 1)$
$(\frac{1}{2}(p-3), p-1)$	$(\frac{1}{2}(p-1), \frac{1}{2}(p-3))$ $(\frac{1}{2}(p-3), \frac{1}{2}(p-3))$ $(\frac{1}{2}(p+1), \frac{1}{2}(p-5))$ $(\frac{1}{2}(p-5), \frac{1}{2}(p-1))$	$(0, \frac{1}{2}(p-3))$
$2r + 3s > 4p - 2$, $r < p - 3$, $s < p - 2$	$(-3p+r+3s+3, p-s-2)$ $(-3p+r+3s+3, p-s-2)$ $(-3p+r+3s+3, p-s-2)$ $(-3p+r+3s+3, p-s-1)$ $(-3p+r+3s+3, p-s-3)$	$(-4p+2r+3s+4, 3p-r-2s-4)$ $(-4p+2r+3s+5, 3p-r-2s-4)$ $(-4p+2r+3s+3, 3p-r-2s-4)$ $(-4p+2r+3s+5, 3p-r-2s-5)$ $(-4p+2r+3s+3, 3p-r-2s-3)$

$\lambda = (r, s)$	$\mu = (u, v)$ with $2u + 3v \leq 2r + 3s$, and $Ext^1_{G_2(p)}(L(\lambda), L(\mu)) \neq 0$	
$2r + 3s > 4p - 2$,	$(-3p+r+3s+4, p-s-2)$	$(-4p+2r+3s+6, 3p-r-2s-5)$
$r < p - 3$,	$(-3p+r+3s+2, p-s-2)$	$(-4p+2r+3s+2, 3p-r-2s-3)$
$s < p - 2$	$(-3p+r+3s+4, p-s-3)$	$(p-r-2, -p+r+s+1)$
	$(-3p+r+3s+2, p-s-1)$	$(p-r-1, -p+r+s+1)$
	$(-3p+r+3s+5, p-s-3)$	$(p-r-3, -p+r+s+1)$
	$(-3p+r+3s+1, p-s-1)$	$(p-r-1, -p+r+s)$
	$(-3p+r+3s+6, p-s-3)$	$(p-r-3, -p+r+s+2)$
	$(-3p+r+3s, p-s-1)$	$(p-r, -p+r+s)$
	$(-3p+r+3s+6, p-s-4)$	$(p-r-4, -p+r+s+2)$
	$(-3p+r+3s, p-s)$	$(4p-r-3s-5, s)$
$2r + 3s = 4p - 2$,	$(r-3, s)$	$(2, -p+r+s-2)$
$\frac{1}{2}(p+7) \leq r$,	$(p-r+1, p-s-2)$	$(3, -p+r+s-2)$
$r \leq p - 4$	$(p-r+1, p-s-2)$	$(1, -p+r+s-2)$
	$(p-r+1, p-s-2)$	$(3, -p+r+s-3)$
	$(p-r+1, p-s-1)$	$(1, -p+r+s-1)$
	$(p-r+2, p-s-2)$	$(4, -p+r+s-3)$
	$(p-r, p-s-2)$	$(0, -p+r+s-1)$
	$(p-r+2, p-s-3)$	$(p-r-2, -p+r+s+1)$
	$(p-r, p-s-1)$	$(p-r-1, -p+r+s+1)$
	$(p-r+3, p-s-3)$	$(p-r-3, -p+r+s+1)$
	$(p-r-1, p-s-1)$	$(p-r-1, -p+r+s)$
	$(p-r+4, p-s-3)$	$(p-r-3, -p+r+s+2)$
	$(p-r+4, p-s-4)$	$(p-r, -p+r+s)$
	$(p-r-2, p-s)$	$(p-r-4, -p+r+s+2)$
$2r + 3s = 4p - 3$,	$(r-2, s)$	$(1, -p+r+s-1)$
$\frac{1}{2}(p+3) \leq r$,	$(p-r, p-s-2)$	$(2, -p+r+s-1)$
$r \leq p - 4$	$(p-r, p-s-2)$	$(0, -p+r+s-1)$
	$(p-r, p-s-2)$	$(2, -p+r+s-2)$
	$(p-r, p-s-1)$	$(0, -p+r+s)$
	$(p-r+1, p-s-2)$	$(3, -p+r+s-2)$
	$(p-r-1, p-s-1)$	$(p-r-2, -p+r+s+1)$
	$(p-r+2, p-s-3)$	$(p-r-1, -p+r+s+1)$
	$(p-r+3, p-s-3)$	$(p-r-3, -p+r+s+1)$
	$(p-r+3, p-s-4)$	$(p-r-3, -p+r+s+2)$
	$(p-r-3, p-s)$	$(p-r, -p+r+s)$
		$(p-r-4, -p+r+s+2)$

$\lambda = (r,s)$	$\mu = (u,v)$ with $2u+3v \leq 2r+3s$, and $Ext^1_{G_2(p)}(L(\lambda), L(\mu)) \neq 0$	
$2r+3s = 4p-4$,	$(p-r-1, p-s-2)$	$(1, -p+r+s)$
$\frac{1}{2}(p+5) \leq r$,	$(p-r-1, p-s-2)$	$(1, -p+r+s-1)$
$r \leq p-4$	$(p-r-1, p-s-1)$	$(2, -p+r+s-1)$
	$(p-r, p-s-2)$	$(p-r-1, -p+r+s+1)$
	$(p-r+2, p-s-3)$	$(p-r-3, -p+r+s+2)$
	$(p-r+2, p-s-4)$	$(p-r-4, -p+r+s+2)$
	$(p-r-4, p-s)$	$(r-1, s)$
$(p-2, s)$,	$(3p-3s-3, s)$	$(-2p+3s, 2p-2s-2)$
$\frac{1}{3}(2p+4) \leq s$,	$(-2p+3s+1, p-s-2)$	$(-2p+3s-1, 2p-2s-2)$
$s \leq p-3$	$(-2p+3s+1, p-s-2)$	$(-2p+3s+1, 2p-2s-3)$
	$(-2p+3s+1, p-s-2)$	$(-2p+3s-1, 2p-2s-1)$
	$(-2p+3s+1, p-s-3)$	$(-2p+3s+2, 2p-2s-3)$
	$(-2p+3s+2, p-s-2)$	$(-2p+3s-2, 2p-2s-1)$
	$(-2p+3s, p-s-2)$	$(1, s-1)$
	$(-2p+3s+2, p-s-3)$	$(1, s-2)$
	$(-2p+3s+3, p-s-3)$	$(2, s-2)$
	$(-2p+3s-1, p-s-1)$	
	$(-2p+3s+4, p-s-3)$	
	$(-2p+3s-2, p-s-1)$	
	$(-2p+3s+4, p-s-4)$	
$(p-2, p-2)$	$(3, p-2)$ $(p-5, 0)$	$(p-6, 2)$ $(p-7, 3)$
	$(p-5, 0)$ $(p-4, 0)$	$(p-4, 1)$ $(p-8, 3)$
	$(p-6, 0)$ $(p-5, 1)$	$(1, p-2)$ $(2, p-3)$
	$(p-7, 1)$ $(p-8, 1)$	$(3, p-3)$
$(p-2, \frac{1}{3}(2p+2))$	$(3, \frac{1}{3}(p-8))$	$(2, \frac{1}{3}(2p-10))$
	$(3, \frac{1}{3}(p-8))$	$(1, \frac{1}{3}(2p-10))$
	$(3, \frac{1}{3}(p-8))$	$(3, \frac{1}{3}(2p-13))$
	$(4, \frac{1}{3}(p-8))$	$(1, \frac{1}{3}(2p-7))$
	$(2, \frac{1}{3}(p-8))$	$(4, \frac{1}{3}(2p-13))$
	$(4, \frac{1}{3}(p-11))$	$(0, \frac{1}{3}(2p-7))$
	$(5, \frac{1}{3}(p-11))$	$(1, \frac{1}{3}(2p-1))$
	$(1, \frac{1}{3}(p-5))$	$(1, \frac{1}{3}(2p-4))$
	$(6, \frac{1}{3}(p-11))$	$(2, \frac{1}{3}(2p-4))$
	$(6, \frac{1}{3}(p-14))$	$(p-5, \frac{1}{3}(2p+2))$

$\lambda = (r, s)$	$\mu = (u, v)$ with $2u + 3v \leq 2r + 3s$, and $Ext^1_{G_2(p)}(L(\lambda), L(\mu)) \neq 0$	
$(p-2, \frac{1}{3}(2p+1))$	$(p-4, \frac{1}{3}(2p+1))$	$(1, \frac{1}{3}(2p-8))$
	$(2, \frac{1}{3}(p-7))$	$(0, \frac{1}{3}(2p-8))$
	$(2, \frac{1}{3}(p-7))$	$(2, \frac{1}{3}(2p-11))$
	$(2, \frac{1}{3}(p-7))$	$(0, \frac{1}{3}(2p-5))$
	$(3, \frac{1}{3}(p-7))$	$(3, \frac{1}{3}(2p-11))$
	$(1, \frac{1}{3}(p-7))$	$(1, \frac{1}{3}(2p-2))$
	$(4, \frac{1}{3}(p-10)$	$(2, \frac{1}{3}(2p-5))$
	$(5, \frac{1}{3}(p-10))$	
	$(5, \frac{1}{3}(p-13))$	
$(p-3, s)$,	$(3p-3s-2, s)$	$(-2p+3s-2, 2p-2s-1)$
$\frac{1}{3}(2p+5) \leq s$,	$(-2p+3s, p-s-2)$	$(-2p+3s-1, 2p-2s-1)$
$s \leq p-2$	$(-2p+3s, p-s-2)$	$(-2p+3s-3, 2p-2s-1)$
	$(-2p+3s, p-s-2)$	$(-2p+3s-1, 2p-2s-2)$
	$(-2p+3s, p-s-3)$	$(-2p+3s-3, 2p-2s)$
	$(-2p+3s+1, p-s-2)$	$(-2p+3s, 2p-2s-2)$
	$(-2p+3s-1, p-s-2)$	$(-2p+3s-4, 2p-2s)$
	$(-2p+3s+1, p-s-3)$	$(1, s-2)$
	$(-2p+3s-1, p-s-1)$	$(2, s-2)$
	$(-2p+3s+2, p-s-3)$	$(0, s-2)$
	$(-2p+3s-2, p-s-1)$	$(2, s-3)$
	$(-2p+3s+3, p-s-3)$	$(0, s-1)$
	$(-2p+3s-3, p-s-1)$	$(3, s-3)$
	$(-2p+3s+3, p-s-4)$	
$(p-3, \frac{1}{3}(2p+4))$	$(p-6, \frac{1}{3}(2p+4))$	$(2, \frac{1}{3}(2p-11))$
	$(4, \frac{1}{3}(p-10))$	$(3, \frac{1}{3}(2p-11))$
	$(4, \frac{1}{3}(p-10))$	$(1, \frac{1}{3}(2p-11))$
	$(4, \frac{1}{3}(p-10))$	$(3, \frac{1}{3}(2p-14))$
	$(5, \frac{1}{3}(p-10))$	$(1, \frac{1}{3}(2p-8))$
	$(3, \frac{1}{3}(p-10))$	$(4, \frac{1}{3}(2p-14))$
	$(5, \frac{1}{3}(p-13))$	$(0, \frac{1}{3}(2p-8))$
	$(3, \frac{1}{3}(p-7))$	$(1, \frac{1}{3}(2p-2))$
	$(6, \frac{1}{3}(p-13))$	$(2, \frac{1}{3}(2p-2))$
	$(2, \frac{1}{3}(p-7))$	$(0, \frac{1}{3}(2p-2))$
	$(7, \frac{1}{3}(p-13))$	$(2, \frac{1}{3}(2p-5))$
	$(7, \frac{1}{3}(p-16))$	$(0, \frac{1}{3}(2p+1))$
		$(3, \frac{1}{3}(2p-5))$

$\lambda = (r, s)$	$\mu = (u, v)$ with $2u + 3v \leq 2r + 3s$, and $Ext^1_{G_2(p)}(L(\lambda), L(\mu)) \neq 0$	
$(p-3, \frac{1}{3}(2p+2))$	$(p-4, \frac{1}{3}(2p+2))$ $(2, \frac{1}{3}(p-8))$ $(2, \frac{1}{3}(p-8))$ $(3, \frac{1}{3}(p-8))$ $(0, \frac{1}{3}(p-5))$ $(5, \frac{1}{3}(p-11))$ $(5, \frac{1}{3}(p-14))$	$(1, \frac{1}{3}(2p-7))$ $(1, \frac{1}{3}(2p-10))$ $(2, \frac{1}{3}(2p-10))$ $(2, \frac{1}{3}(2p-4))$ $(0, \frac{1}{3}(2p-1))$
$(p-1, s)$, $\frac{1}{3}(2p+1) \leq s$, $s \leq p-3$	$(-2p+3s+2, p-s-2)$ $(-2p+3s+2, p-s-2)$ $(-2p+3s+2, p-s-2)$ $(-2p+3s+2, p-s-3)$ $(-2p+3s+1, p-s-2)$ $(-2p+3s+3, p-s-3)$ $(-2p+3s+4, p-s-3)$ $(-2p+3s+5, p-s-3)$ $(-2p+3s-1, p-s-1)$ $(-2p+3s+5, p-s-4)$	$(-2p+3s+2, 2p-2s-3)$ $(-2p+3s+1, 2p-2s-3)$ $(-2p+3s+3, 2p-2s-4)$ $(-2p+3s, 2p-2s-2)$
$(p-1, p-2)$	$(p-4, 0)$ $(p-4, 0)$ $(p-4, 1)$ $(p-3, 0)$ $(p-5, 0)$ $(p-5, 1)$ $(p-7, 1)$ $(p-7, 2)$	$(p-4, 1)$ $(p-3, 1)$ $(p-5, 1)$ $(p-3, 0)$ $(p-2, 0)$ $(p-6, 2)$
$(p-1, \frac{1}{3}(2p-1))$	$(1, \frac{1}{3}(p-5))$ $(1, \frac{1}{3}(p-5))$ $(1, \frac{1}{3}(p-5))$ $(1, \frac{1}{3}(p-8))$ $(3, \frac{1}{3}(p-8))$ $(4, \frac{1}{3}(p-8))$ $(4, \frac{1}{3}(p-11))$	$(1, \frac{1}{3}(2p-7))$ $(0, \frac{1}{3}(2p-7))$ $(0, \frac{1}{3}(2p-4))$ $(2, \frac{1}{3}(2p-10))$
$(p-1, \frac{1}{3}(2p-2))$	$(0, \frac{1}{3}(p-4))$ $(0, \frac{1}{3}(p-4))$ $(0, \frac{1}{3}(p-7))$ $(3, \frac{1}{3}(p-7))$ $(3, \frac{1}{3}(p-10))$	$(1, \frac{1}{3}(2p-5))$ $(1, \frac{1}{3}(2p-8))$

$\lambda = (r, s)$	$\mu = (u, v)$ with $2u + 3v \leq 2r + 3s$, and $Ext^1_{G_2(p)}(L(\lambda), L(\mu)) \neq 0$	
$(r, p-1)$, $\frac{1}{2}(p+1) \leq r$, $r \leq p-4$	$(p-r-2, r)$ $(p-r-3, r)$ $(p-r-1, r-1)$ $(p-r-3, r+1)$ $(p-r-4, r+1)$	$(p-r-2, p-1)$
$(p-3, p-1)$	$(1, p-3)$ $(2, p-3)$ $(0.p-3)$ $(2, p-4)$ $(0, p-2)$	$(1, p-1)$
$(p-2, p-1)$	$(1, p-2)$ $(1, p-3)$ $(2, p-3)$	$(0, p-1)$
$(\frac{1}{2}(p-1), p-1)$	$(\frac{1}{2}(p-5), \frac{1}{2}(p+1))$ $(\frac{1}{2}(p-7), \frac{1}{2}(p+1))$	$(\frac{1}{2}(p-3), (p-1))$

Table 2

$Q_1(0,0)$	$U_1(0,0) \oplus U_1(0, p-1) \oplus U_1(2, p-1) \oplus U_1(3, p-1)$ $\oplus U_1(p-3, p-1) \oplus U_1(p-1, p-1) \oplus U_1(p-1, 0)$ $\oplus U_1(p-1, 1) \oplus U_1(p-2, 1) \oplus U_1(3, p-2)$ $\oplus U_1(3, p-3) \oplus U_1(5, p-4)$,
$Q_1(1,0)$	$U_1(1,0) \oplus U_1(1, p-1) \oplus U_1(2, p-1) \oplus U_1(3, p-1)$ $\oplus U_1(4, p-1) \oplus U_1(4, p-2) \oplus U_1(5, p-4)$ $\oplus U_1(7, p-5) \oplus U_1(p-1, 1)$,
$Q_1(r,0)$	$U_1(r,0) \oplus U_1(r, p-1) \oplus U_1(r+1, p-1) \oplus U_1(r+2, p-1)$ $\oplus U_1(r+3, p-1) \oplus U_1(r+3, p-2) \oplus U_1(2r+3, p-r-3)$ $\oplus U_1(2r+5, p-r-4)$ $\quad 2 \leq r \leq \frac{1}{2}(p-9)$,

$Q_1(\frac{1}{2}(p-7),0)$	$U_1(\frac{1}{2}(p-7),0)\oplus U_1(\frac{1}{2}(p-7),p-1)\oplus U_1(\frac{1}{2}(p-5),p-1)$ $\oplus 2U_1(\frac{1}{2}(p-3),p-1)\oplus U_1(\frac{1}{2}(p-1),p-2)\oplus U_1(p-4,4)$ $\oplus U_1(p-2,3)$,
$Q_1(\frac{1}{2}(p-5),0)$	$U_1(\frac{1}{2}(p-5),0)\oplus U_1(\frac{1}{2}(p-5),p-1)\oplus U_1(\frac{1}{2}(p-3),p-1)$ $\oplus U_1(\frac{1}{2}(p+1),p-2)\oplus U_1(p-2,2)$,
$Q_1(\frac{1}{2}(p-3),0)$	$U_1(\frac{1}{2}(p-3),0)\oplus U_1(\frac{1}{2}(p-3),p-1)\oplus U_1(\frac{1}{2}(p+1),p-1)$ $\oplus U_1(\frac{1}{2}(p+3),p-1)\oplus U_1(\frac{1}{2}(p+3),p-2)$,
$Q_1(r,0)$	$U_1(r,0)\oplus U_1(r,p-1)\oplus U_1(r+1,p-1)\oplus U_1(r+2,p-1)$ $\oplus U_1(r+3,p-1)\oplus U_1(r+3,p-2)$ $\frac{1}{2}(p-1)\le r\le p-6$,
$Q_1(p-5,0)$	$U_1(p-5,0)\oplus U_1(p-5,p-1)\oplus U_1(p-4,p-1)$ $\oplus 2U_1(p-3,p-1)\oplus U_1(p-2,p-1)$ $\oplus U_1(p-2,p-2)$,
$Q_1(p-4,0)$	$U_1(p-4,0)\oplus U_1(p-4,p-1)\oplus U_1(p-3,p-1)$ $\oplus U_1(p-2,p-1)\oplus 5U_1(p-1,p-1)$ $\oplus 2U_1(p-1,p-2)$,
$Q_1(p-3,0)$	$U_1(p-3,0)\oplus U_1(p-3,p-1)\oplus 4U_1(p-2,p-1)$ $\oplus 10U_1(p-1,p-1)$,
$Q_1(p-2,0)$	$U_1(p-2,0)\oplus 2U_1(p-2,p-1)\oplus 8U_1(p-1,p-1)$,
$Q_1(p-1,0)$	$U_1(p-1,0)\oplus U_1(p-1,p-1)$,
$Q_1(0,1)$	$U_1(0,1)\oplus U_1(3,p-1)\oplus U_1(p-1,1)\oplus U_1(p-1,2)$ $\oplus U_1(p-2,2)$,
$Q_1(0,s)$	$U_1(0,s)\oplus U_1(p-1,s)\oplus U_1(p-1,s+1)\oplus U_1(p-2,s+1)$ $s\notin\{\frac{1}{3}(p-5),\frac{1}{3}(2p-5),\frac{1}{3}(p-4),\frac{1}{3}(2p-4),\frac{1}{2}(p-3)\}$ **and** $2\le s\le p-3$
$Q_1(0,\frac{1}{3}(p-5))$	$U_1(0,\frac{1}{3}(p-5))\oplus U_1(p-1,\frac{1}{3}(p-5))\oplus 2U_1(p-2,\frac{1}{3}(p-2))$,
$Q_1(0,\frac{1}{3}(2p-5))$	$U_1(0,\frac{1}{3}(2p-5))\oplus U_1(p-1,\frac{1}{3}(2p-5))$ $\oplus 2U_1(p-2,\frac{1}{3}(2p-2))$,

$Q_1(0, \frac{1}{3}(p-4))$	$U_1(0, \frac{1}{3}(p-4)) \oplus U_1(p-1, \frac{1}{3}(p-4))$,
$Q_1(0, \frac{1}{3}(2p-4))$	$U_1(0, \frac{1}{3}(2p-4)) \oplus U_1(p-1, \frac{1}{3}(2p-4))$,
$Q_1(0, \frac{1}{2}(p-3))$	$U_1(0, \frac{1}{2}(p-3)) \oplus U_1(p-1, \frac{1}{2}(p-3)) \oplus U_1(p-2, \frac{1}{2}(p-1))$,
$Q_1(0, p-2)$	$U_1(0, p-2) \oplus U_1(p-1, p-2) \oplus 5U_1(p-1, p-1)$ $\oplus 2U_1(p-2, p-1)$,
$Q_1(0, p-1)$	$U_1(0, p-1) \oplus U_1(p-1, p-1)$,
$Q_1(1,1)$	$U_1(1,1) \oplus U_1(4, p-1) \oplus U_1(p-1, 2)$,
$Q_1(r,1)$	$U_1(r,1) \oplus U_1(r+3, p-1) \quad 2 \leq r \leq p-5$,
$Q_1(p-4,1)$	$U_1(p-4,1) \oplus U_1(p-3, p-1) \oplus U_1(p-2, p-1)$ $\oplus 3U_1(p-1, p-1)$
$Q_1(1,s)$	$U_1(1,s) \oplus U_1(p-1, s+1) \quad s \notin \{\frac{1}{3}(p-5), \frac{1}{3}(2p-5), \frac{1}{2}(p-3)\}$ and $2 \leq s \leq p-3$,
$Q_1(1, \frac{1}{2}(p-3))$	$U_1(1, \frac{1}{2}(p-3)) \oplus U_1(p-1, \frac{1}{2}(p-1)) \oplus U_1(\frac{1}{2}(p-1), p-1)$ $\oplus U_1(\frac{1}{2}(p+1), p-1)$,
$Q_1(1, p-2)$	$U_1(1, p-2) \oplus 3U_1(p-1, p-1)$,
$Q_1(r,s)$	$U_1(r,s) \oplus U_1(r+s, p-1) \oplus U_1(r+s+1, p-1)$ $3 \leq r \leq p-6$ and $r + 2s = p-2$,
$Q_1(r,s)$	$U_1(r,s) \quad r, s$ are not listed above.

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