SURVEY ON BRAIDS

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Abstract. In the paper we give a survey on braid groups, their properties and generalizations.

Contents

1. Introduction

Braid groups describe intuitive concept of classes of continuous deformations of braids, which are collections of interwining strands whose endpoints are fixed. Mathematically they can be considered from various points of view. The first intuitive approach is formalized naturally as isotopy classes of a collection of n connected curves (strings) in 3-dimensional space. This point

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of view is connected with the definition of braid group as the fundamental group of configuration space of n points on a plane. Also braids can be interpreted as a mapping class group of a punctured disc and as a subgroup of automorphism group of a free group (subsection 3.3).

The present survey is organized as follows. In Section 2 we make some historical remark. Definition and general properties are considered in Section 3. Configuration spaces appear in Subsection 3.2. Connections with groups of automorphisms of free groups are given in Subsection 3.3 Presentations of the braid group which appeared quite recently are observed in Subsection 3.4. Section 4 is devoted to F. A. Garside's classical work [51] and Section 5 to that of P. Dehornoy on ordering for braids. Representations and in particular linearity are discussed in Section 6. In Section 7 various generalization of braids are presented. Homological properties are observed in Section 8.

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2. Historical Remarks

Braids were rigorously defined by E. Artin [6] in 1925, although the roots of this natural concept are seen in the works of A. Hurwitz ([59], 1891), R. Fricke and F. Klein ([49], 1897) and even in the notebooks of K.-F. Gauss. E. Artin [6] gave the presentation of the braid group (see formulas (3.2) in Section 3) which is common now. Already in the book of Felix Klein [63] published in 1926 there appeared a chapter about braids. Essential topics about braids were also presented in the Reidemeister's Knotentheorie [85] publisheded in 1932.

In 30-ies there appeared a series of papers of Werner Burau [20], [21], [22] where he in particular gave the presentation of the pure braid group (see subsection 3.5) and introduced the Burau representation (subsection 6.1). In his work [69] published in 1934 Wilhelm Magnus established relations between braid groups and the mapping class groups. At the same time there appeared the work of A. A. Markov [74], which together with Alexander Theorem [2] builds a bijection between links and equivalence classes of braids. It became an essential ingredient in study of links and knots (in the work of V. F. R. Jones [62], for example). In 1936-37 were published the works of O. Zariski [96], [97], where he discovered connections between braid groups and the fundamental group of the complement of discriminant of the general polynomial

$$f_n(t) = a_0t^n + a_1t^{n-1} + ... + a_{n-1}t + a_n,$$

a point of view later rediscovered by V. I. Arnold [4]. Zariski also understood connections between braids and configuration spaces, gave the presentation of the braid group of the sphere, and studied the braid groups of Riemann surfaces. Amasingly and unfortunately these works of Zariski were not noticed by the specialists on braids and are not mentioned even in books and papers where the presentations of braids of surfaces are discussed.

In the begining of 60-ies R. Fox and L. Neuwirth [48] and E. Fadell and L. Neuwirth [42] studied configuration spaces. They turned out to be $K(\pi, 1)$-spaces and so give natural geometrical model of the classifying spaces for the braid groups. Later, V. I. Arnold [4] in this direction proved the first results on cohomologies of braids. The motivation for his study was a connection (which he discovered) with the problem of representing algebraic functions in several variables by superposing algebraic functions in fewer variables. Also, in 1969 V. I. Arnold completely described the cohomologies of pure braid groups [3].

In 1969 there appeared the publication of F. A. Garside's work [51] where he suggests a new normal form of elements in the braid group and with its help gives a new solution of the word problem and also solves the conjugacy problem. In 1968 was published a two-page note of G. S. Makanin [73] where he sketches his algorithm for the solution of the conjugacy problem. The complete publication of Makanin's work didn't appear (as far as the author is aware).

In 70-ies the study of cohomologies of braids were continued independently and by different methods by D. B. Fuks [50] who determined mod 2 cohomologies, and F. R. Cohen [26], [27], [28] who described these homologies with coefficients in $\mathbb{Z}$ and in $\mathbb{Z}/p$ as modules over the Steenrod algebra.

In 1984-85 independently N. V. Ivanov [60] and J. McCarthy [76] proved the "Tits-alternative" for the mapping class groups of surfaces and as a consequence it is true for the braid groups. Namely, they proved that every subgroup of the mapping class group either contains an abelian subgroup of finite index, or contains a nonabelian free group.

The question of whether braid groups are linear attracted significant attention. It was realized that the Burau representation is faithful for Br_3 [52], [72]. Then after a long break in 1991 J. A. Moody [78] proved that Burau representation is unfaithful for $n \geq 9$. This bound was improved to $n \geq 6$ by D. D. Long and M. Paton [68] and to $n = 5$ by S. Bigelow [9]. In 1999-2000 there appeared preprints of papers of D. Krammer [64], [65] and S. Bigelow [10] who proved that Br_n is linear for all n.

At the begining of nineties P. Dehornoy [34], [35], [36] proved that there exits a left order in braid groups.

Interesting generalizations of braids were introduced in the work of E. Brieskorn [16]. The configuration space can also be considered as the orbit space of the complement of the complexification of the arrangement of hyperplanes corresponding to the Coxeter group $A_{n-1} = \Sigma_n$. Generalizing this approach for any finite Coxeter group, E. Brieskorn defined the so-called generalized braid groups which are also called Artin groups.

Another way of generalization is to consider braid groups in 3-manifolds, possibly with a boundary. The simplest examples are braid groups in handlebodies. A. B. Sossinsky [88] was the first who studied them. Such a group can be interpreted as the fundamental group of the configuration space of a plane without g points where g is the genus of the handlebody. The generalized braid group of type C is isomorphic to the braid group in the solid torus.

Under the theory of Vassiliev invariants singular braids were introduced. The corresponding algebraic structures are the Baez–Birman monoid [8], [12] and the braid-permutation group by R. Fenn, R. Rimányi and C. Rourke [45], [46]. Various properties of these objects were studied [47], [98], [53], [54], [33], [61], [31], [55].

3. Definitions and General Properties

3.1. **Systems of n curves in three-dimensional space and braid groups.** First of all as it was already mentioned braids naturally arise as objects in 3-space. Let us consider two parallel planes P_0 and P_1 in $\mathbb{R}^3$, which contain two ordered sets of points A_1, ..., $A_n \in P_0$ and B_1, ..., $B_n \in P_1$. These points are lying on parallel lines L_A and L_B respectively. The space between the planes P_0 and P_1 we denote by Π. Suppose that the point B_i is lying under the point A_i. Let us connect the set of points A_1, ..., A_n with the set of points B_1, ..., B_n by simple nonintersecting curves C_1, ..., C_n lying in the space Π and such that each curve meets only once each parallel plane P_t lying in the space Π (see Figure 3.1). This object is called a *braid* and the curves are called the *strings* of a braid. Usually braids are depicted by projections on

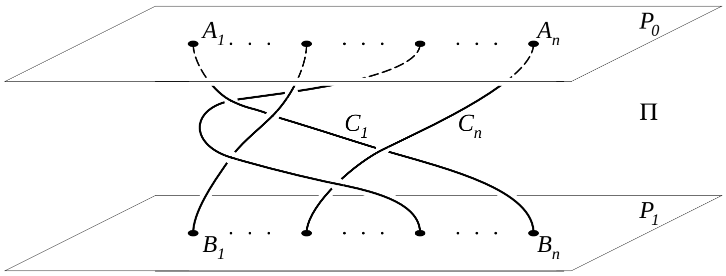

FIGURE 3.1

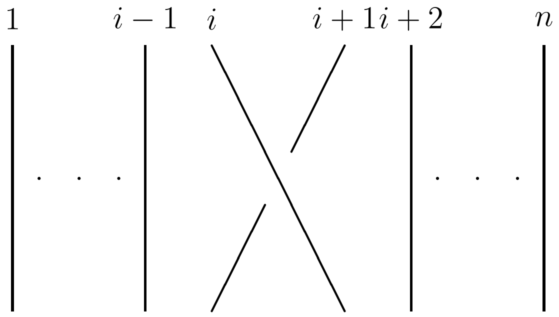

FIGURE 3.2

the plane passing through the lines L_A and L_B. This projection is supposed to be in general position so that there is only finite number of double points of intersection which are lying on pairwise different levels and intersections are transversal. The simplest braid σ_i (Fig. 3.2) corresponds to the transposition $(i, i+1)$.

Let us introduce the following equivalence relation on the set of all braids with n strings and with fixed P_0, P_1, A_i and B_i. It is defined by homeomorphisms $h : \Pi \to \Pi$, identical on $P_0 \cup P_1$ and such that $h(P_t) = P_t$. Braids β and β' are equivalent if there exists such a homeomorphism h that $h(\beta) = \beta'$. On the set Br_n of equivalence classes under the considered relation the structure of a group introduces as follows. We put a copy Π' of the domain Π under the Π in such a way that P_0' coincides with P_1 and each A_i coincides with B_i and it is possible to glue braids β and β'. This gluing gives a composition of braids $\beta\beta'$ (Fig. 3.3). Unit element is the equivalence class containing a braid of n parallel intervals, the braid β^{-1} inverse to β is defined by reflection of β with respect to the plane $P_{1/2}$. A string C_i of a braid β connects the point A_i with the pont B_{k_i} defining the permutation S^β. If this permutation is identical then the braid β is called *pure*. The map $\beta \to S^\beta$ defines an epimorphism τ_n of the braid group Br_n on the permutation group $\Sigma_n$with the kernel consisting of all pure braids:

$$1 \to P_n \to Br_n \xrightarrow{\tau_n} \Sigma_n \to 1. \tag{3.1}$$

The following presentation of the braid group Br_n with generators σ_i, $i = 1, ..., n-1$ and two types of relations:

$$\begin{cases} \sigma_i\sigma_j &= \sigma_j\,\sigma_i, \quad \text{if} \quad |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \end{cases} \tag{3.2}$$

is the algebraic expression of the fact that any isotopy of braids can be broken down into "elemntary moves" of two types that correspond to two types of relations.

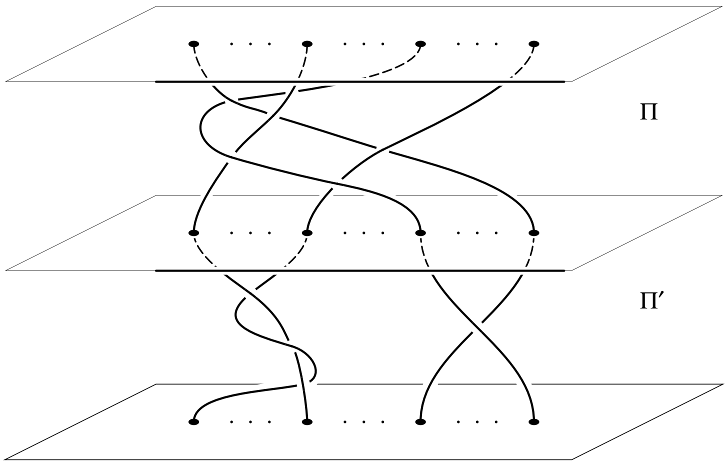

FIGURE 3.3

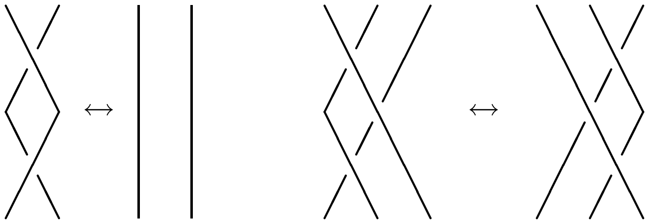

FIGURE 3.4

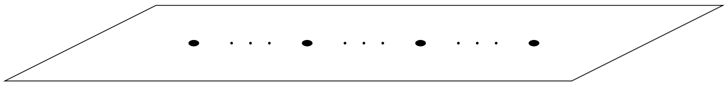

FIGURE 3.5

If the symmetric group Σ_n is given by its canonical presentation with generators s_i, $i = 1, ..., n-1$ and relations:

$$
\begin{cases} s_i s_j & = s_j\, s_i, \quad \text{if} \quad |i-j| > 1, \\ s_i s_{i+1} s_i & = s_{i+1} s_i s_{i+1} \\ s_i^2 = 1, \end{cases} \tag{3.3}
$$

then the homomorphism τ_n is given by the formula

$$\tau_n(\sigma_i) = s_i, \quad i = 1, \dots, n-1.$$

It is possible to consider brads as classes of equivalence of *braid diagrams* which are generic projections of three dimensional braids on a plane. The classes of equivalence are defined by the *Reidemeister moves* depicted at Figure 3.4.

3.2. **Braid groups and configuration spaces.** If we look at the Figure 3.1, then this picture can be interpreted as a graph of a loop in the *configuration space* of n points on a plane, that is the space of unordered sets of n points on a plane, see Figure 3.5. So, it is possible to interpret the braid group as the fundamental group of the configuration space. Formally it is done as follows. The symmetric group Σ_m acts on the Cartesian power $(\mathbb{R}^2)^m$ of the space $\mathbb{R}^2$:

$$w(y_1, ..., y_m) = (y_{w^{-1}(1)}, ..., y_{w^{-1}(m)}), \quad w \in \Sigma_m. \tag{3.4}$$

Denote by $F(\mathbb{R}^2, m)$ the space of m-tuples of pairwise different points in $\mathbb{R}^2$:

$$F(\mathbb{R}^2, m) = \{p_1, ..., p_m) \in (\mathbb{R}^2)^m : p_i \neq p_j \text{ for } i \neq j\}.$$

This is the space of regular points of this action. We call the orbit space of this action $B(\mathbb{R}^2, m) = F(\mathbb{R}^2, m)/\Sigma_m$ the *configuration space of n points on a plane*. The braid group Br_m is the fundamental group of configuration space

$$Br_m = \pi_1(B(\mathbb{R}^2, m)).$$

The pure braid group P_m is the is the fundamental group of the space $F(\mathbb{R}^2, m)$. The covering

$$p : F(\mathbb{R}^2, m) \to B(\mathbb{R}^2, m)$$

defines the exact sequence:

$$1 \to \pi_1(F(\mathbb{R}^2, m)) \xrightarrow{p_*} \pi_1(B(\mathbb{R}^2, m)) \to \Sigma_n \to 1. \tag{3.5}$$

which is equivalent to sequence (3.1).

It can be used for proving the canonical presentation of the braid group (3.2) as it is done, for example in the book of J. Birman [11].

Such considerations were made by R. Fox and L. Neuwirth [48].

3.3. **Braid groups as automorphism groups of free groups and the word problem.** Another important approach to the braid group bases on the fact that this group may be considered as a subgroup of the automorphism group of a free group.

Let F_n be the free group of rank n with the set of generators $\{x_1, \dots, x_n\}$. Assume further that $\operatorname{Aut} F_n$ is the automorphism group of F_n.

We have the standard inclusions of the symmetric group Σ_n and the braid group Br_n into $\operatorname{Aut} F_n$. These inclusions may be described as follows. Let $\overline{\sigma}_i \in \operatorname{Aut} F_n, i = 1, 2, \dots, n-1$, be given by the formula which describes its action on generators:

$$\begin{cases} x_i & \mapsto x_{i+1}, \\ x_{i+1} & \mapsto x_{i+1}^{-1} x_i x_{i+1}, \\ x_j & \mapsto x_j, j \neq i, i+1. \end{cases} \tag{3.6}$$

Let us define a map i of the generators σ_i, $i = 1, \dots, n-1$ of the braid group Br_n to these automorphisms:

$$i(\sigma_i) = \overline{\sigma}_i. \tag{3.7}$$

Theorem 3.1. *Formulas (3.7) define correctly a homomorphism*

$$i : Br_n \to \operatorname{Aut} F_n.$$

which is a monomorphism.

Theorem 3.1 gives a solution of the word problem for the braid groups. It was done first by E. Artin [6].

The free group F_n is a fundamental group of a disc D_n without n points and the generator x_i corresponds to a loop going around the i-th point. The braid group Br_n is the mapping class group of a disc D_n with its boundary fixed [11] and so it acts on the fundamental group of D_n.

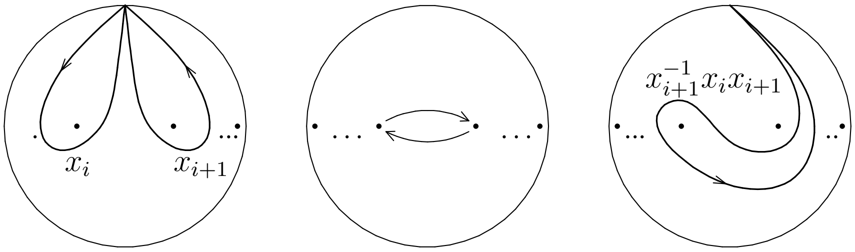

FIGURE 3.6

This action is described by the formulas (3.6) where x_i correspond to the canonical loops on D_n which form the generators of the fundamental group. Geometrically this action is depicted in the Figure 3.6.

3.4. **Commutator subgroup and other presentations.** Let us define a homomorphism from braid group to integers by taking the sum of exponents of the entries of the generators σ_i in the expression of any element of the group through these canonical generators:

$$\deg : Br_n \to \mathbb{Z},\ \deg(b) = \sum_j m_j,\ \text{ where } b = (\sigma_{i_1})^{m_1} \dots (\sigma_{i_k})^{m_k}.$$

Proposition 3.1. *The homomorphism*

$$\deg : Br_n \to \mathbb{Z}$$

gives the abelianization of the braid group and the commutator subgroup Br_n' is characterized by the condition

$$b \in Br_n' \ \text{ if and only if } \ \deg(b) = 0.$$

Proof. Let $a : Br_n \to A$ be a homomorphism to any other abelian group A, then from the relations (3.2) we have:

$$a(\sigma_i)a(\sigma_{i+1})a(\sigma_i) = a(\sigma_{i+1})a(\sigma_i)a(\sigma_{i+1}).$$

the commutativity of A gives that $a(\sigma_{i+1}) = a(\sigma_i)$. This means that the homomorphism deg is universal. □

Of course, there exist another presentations of the braid group. Let

$$\sigma = \sigma_1\sigma_2 \dots \sigma_{n-1},$$

then the group Br_n is generated by σ_1 and σ because

$$\sigma_{i+1} = \sigma^i\sigma_1\sigma^{-i}, \quad i = 1, \dots n-2.$$

The relations for the generators σ_1 and σ are the following

$$\begin{cases} \sigma_1\sigma^i\sigma_1\sigma^{-i} & = \sigma^i\sigma_1\sigma^{-i}\sigma_1 \ \text{ for } \ 2 \le i \le n/2, \\ \sigma^n & = (\sigma\sigma_1)^{n-1}. \end{cases} \tag{3.8}$$

This was observed by Artin in the initial paper [6].

An interesting series of presentations was given by V. Sergiescu [87]. For every planar graph he constructed a presentation of the group Br_n, where n is the number of vertices of the graph, with generators corresponding to edges and relations reflecting the geometry of the graph. Artin's presentation in this context corresponds to the graph consisting of the interval from 1 to n with the natural numbers (from 1 to n) as vertices and with segments between them as edges.

J. S. Birman; K. H. Ko; S. J. Lee [13] introduced the presentation with the generators a_{ts} with $1 < s < t \leq n$ and relations

$$\begin{cases} a_{ts}a_{rq} &= a_{rq}a_{ts} \ \text{ for } \ (t-r)(t-q)(s-r)(s-q) > 0, \\ a_{ts}a_{sr} &= a_{tr}a_{ts} = a_{sr}a_{tr} \ \text{ for } \ 1 \leq r < s < t \leq n. \end{cases} \tag{3.9}$$

The generators a_{ts} are expressed by the canonical generators σ_i in the following form:

$$a_{ts} = (\sigma_{t-1}\sigma_{t-2}\cdots\sigma_{s+1})\sigma_s(\sigma_{s+1}^{-1}\cdots\sigma_{t-2}^{-1}\sigma_{t-1}^{-1}) \ \text{ for } \ 1 < s < t \leq n.$$

The set of generators for braid groups were even enlarged in the work of Jean Michel [77] as follows. Let $|\ | : \Sigma_n \to \mathbb{Z}$ be the length function on the symmetric group with respect to the generators s_i: for $x \in \Sigma_n$, $|x|$ is the smallest natural number k such that x is a product of k elements of the set $\{s_1, ..., s_{n-1}\}$. It is known ([15], Sect. 1, Ex. 13(b)) that two minimal expressions for an element of Σ_n are equivalent by using only of the relations (3.2). This implies that the canonical projection $\tau_n : Br_n \to \Sigma_n$ has a unique set-theoretic section $r : \Sigma_n \to Br_n$ such that $r(s_i) = \sigma_i$ for $i = 1, ..., n-1$ and $r(xy) = r(x)\, r(y)$ whenever $|xy| = |x| + |y|$. Then the group Br_n admits a presentation by generators $\{r(x) \,|\, x \in \Sigma_n\}$ and relations $r(xy) = r(x)\, r(y)$ for all $x, y \in \Sigma_n$ such that $|xy| = |x| + |y|$.

3.5. **Presentaion of the pure braid group and Markov normal form.** Let $f(y_1, ..., y_m)$ be a word with (possibly empty) entries of y_i^{ϵ}, where y_i are some letters and ϵ may be ± 1. If y_i are elements of a group G then $f(y_1, ..., y_m)$ will be considered as the corresponding element of G.

Let us define the elements $s_{i,j}$, $1 \leq i < j \leq m$, of the braid group Br_m by the formula:

$$s_{i,j} = \sigma_{j-1}...\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}...\sigma_{j-1}^{-1}.$$

These elements satisfy the following Burau relations ([20], [75], see also [67]):

$$\begin{aligned} s_{i,j}s_{k,l} &= s_{k,l}s_{i,j} \text{ for } i < j < k < l \text{ and } i < k < l < j, \\ s_{i,j}s_{i,k}s_{j,k} &= s_{i,k}s_{j,k}s_{i,j} \text{ for } i < j < k, \\ s_{i,k}s_{j,k}s_{i,j} &= s_{j,k}s_{i,j}s_{i,k} \text{ for } i < j < k, \\ s_{i,k}s_{j,k}s_{j,l}s_{j,k}^{-1} &= s_{j,k}s_{j,l}s_{j,k}^{-1}s_{i,k} \text{ for } i < j < k < l. \end{aligned} \tag{3.10}$$

W. Burau and later A. A. Markov proved that the elements $s_{i,j}$ with the relations (3.10) give a presentation of the pure braid group P_m [75]. The following formula is a consequence of the Burau relations and also belongs to A. A. Markov:

$$[s_{i,l}, s_{j,k}^{\epsilon}] = f(s_{1,l}, ..., s_{l-1,l}), \quad \epsilon = \pm 1, \quad k < l. \tag{3.11}$$

Let us define the elements $\sigma_{k,l}$, $1 \leq k \leq l \leq m$ by the formulas

$$\sigma_{k,k} = e,$$

$$\sigma_{k,l} = \sigma_k^{-1}...\sigma_{l-1}^{-1}.$$

Let P_m^k be the subgroup of P_m generated by the elements $s_{i,j}$ with $k < j$.

Theorem 3.2. (A. A. Markov) *(i) Every element of the group Br_m can be uniquely written in the form*

$$f_m(s_{1,m}, ..., s_{m-1,m})...f_j(s_{1,j}, ..., s_{j-1,j})...f_2(s_{1,2})\sigma_{i_m,m}...\sigma_{i_j,j}...\sigma_{i_2,2}. \tag{3.12}$$

(ii) The factor group P_m^k/P_m^{k-1} is the free group on free generators $s_{i,k+1}$, $1 \leq i \leq k$.

The form (3.12) is called the *Markov normal form*, it also gives the solution of the word problem for the braid groups. of the word problem for the braid groups. ———- X-Sun-Data-Type: default X-Sun-Data-Description: default X-Sun-Data-Name: sec3.tex X-Sun-Content-Lines: 99 X-Sun-Charset: iso-8859-1

4. Garside Normal Form, Center and Conjugacy Problem

Essential role in Garside work [51] plays the monoide of *positive* braids, that is the monoide which has a presentation with generators σ_i, $i = 1, ..., n$ and relations (3.2). In other words each element of this monoide can be represented as a word on the elements σ_i, $i = 1, ..., n$ with no entrances of σ_i^{-1}. The *fundamental word* Δ in the braid group Br_{n+1} is defined by the formula:

$$\Delta = \sigma_1 \dots \sigma_n \sigma_1 \dots \sigma_{n-1} \dots \sigma_1 \sigma_2 \sigma_1.$$

For a positive word W in σ_i, $i = 1, ..., n$ we say that Δ is a *factor* of W or simply W contains Δ, if $W \doteq A\Delta B$ with A and B being arbitrary positive words, probably empty, the symbol $\doteq$ means that words are positively equal, i.e. as elements in the positive braid monoide. If W does not contain Δ we shall say W is *prime to* Δ.

Among positive words on the alphabet $\{\sigma_1 \dots \sigma_n\}$ let us introduce a lexicographical ordering with the condition that $\sigma_1 < \sigma_2 < \dots < \sigma_n$. For a positive word W the *base* of W is the smallest positive word which is positively equal to W. The base is uniquely determined. If a positive word A is prime to Δ, then for the base of A the notation $\overline{A}$ will be used.

Theorem 4.1. (F. A. Garside) *Every word W in Br_{n+1} can be uniquely written in the form $\Delta^m \overline{A}$, where m is an integer.*

The form of this theorem is called the *Garside normal form* and the index m is called the *power* of W. Garside normal form also gives a solution to the word problem in the braid group.

Theorem 4.2. (F. A. Garside) *The necessary and sufficient condition that two words in Br_{n+1} are equal is that their Garside normal forms are identical.*

The center of the braid group was fistly found by W.-L. Chow [25]. Namely, as it follows from the presentation of braid groups with two generators σ_1 and σ and relations (3.8) given in the subsection 3.1 the element σ^n commutes with $\sigma\sigma_1$ and so with σ_1. Chow proved that it generates the center.

Garside normal form gives an elegant proof of the following theorem.

Theorem 4.3. *(i) When $n = 1$ the center of the group Br_{n+1} is generated by Δ.*
(ii) When $n > 1$ the center of the group Br_{n+1} is generated by Δ^2.

Let α be a positive word such that $\Delta \doteq \alpha X$, where X is an arbitrary positive word, probably empty. For any word W in Br_{n+1}, the word $\alpha^{-1}W\alpha$, reduced to Garside normal form is called an *α-transformation of W*.

For any word W in Br_{n+1} with the Garside normal form $\Delta^m \overline{A} \equiv W_1$ consider the following chains of α-transformations: take all the α-transformations of W_1 and let those which are of power $\geq m$ and which are distinct from each other be $W_2, W_3, \dots, W_t$. Now repeat the process for each of the words W_2, W_3, $\dots$, W_t in turn, denoting successively by W_{t+1}, W_{t+2}, $\dots$, any new words occurring, the condition being always that each new word must be of power $\geq m$. Continue to repeat the process for every new distinct word arrising, as the sequence W_1, W_2, W_{t+2}, $\dots$, expands.

Proposition 4.1. *The set W_1, W_2, W_{t+2}, $\dots$, is finite.*

Suppose that in the set W_1, W_2, W_{t+2}, $\dots$, the highest power reached is s and that the words of power s form the subset V_1, V_2, $\dots$. Then this set V_1, V_2, $\dots$ is called the *summit set* of W.

Theorem 4.4. (F. A. Garside) *Two elements A and B of the group Br_{n+1} are conjugate if and only if their summit sets are identical.*

In subsequent works of S. I. Adyan [1], W. Thurston [40], E. El-Rifai and H. R. Morton [38], J. S. Birman, K. H. Ko and S. J. Lee [13] this direction in the theory of braids was developed and Garside normal form was precised. In particular Thurston introduced so called *right-greedy* normal form and with the help of it proved that the braid group is biautomatic. J. S. Birman, K. H. Ko and S. J. Lee considered the word $\delta = a_{n(n-1)} \cdots a_{32}a_{21} = \sigma_{n-1} \cdots \sigma_2\sigma_1$, as a fundamental and proved that every element in Br_n has a representative $W = \delta^j A_1 A_2 \cdots A_k$ with positive A_i in a unique way in some sense. Based on this form they gave an algorithm for the word problem in B_n which runs in time (nm^2) for a given word of length m.

5. Ordering of Braids

A group G is said *totally* (or *linearly*) *left* (correspondingly *right*) *ordered* if it has a total order $<$ invariant by left (right) multiplication, i.e. if $a < b$, then $ca < cb$ for any $a, b, c \in G$. If this order is also invariant by right (left) multiplication, then the group G is called *ordered*.

For any left ordered group G denote by P the set of positive elements $\{x \in G : x > 1\}$, then the set of negative elements is defined by the formula: $P^{-1} = \{x \in G : x \in P\}$. The total character of an order on G is expressed by the partition

$$G = P \coprod \{1\} \coprod P^{-1}.$$

The invariance of multiplication is expressed by the inclusion $P^2 \subset P$, where P^2 is formed by products of couples of elements of P. Conversely, if there exist a subset P of a group G with the properties:

$$G = P \coprod \{1\} \coprod P^{-1}, \quad P^2 \subset P,$$

then G is left ordered by the order defined by: $x < y$ if and only if $x^{-1}y \in P$. A group G then is ordered if and only if $xPx^{-1} \subset P$ for all $x \in G$.

Let $i \in \{1, \dots, n\}$ and a word w on the alphabet $\{\sigma_1, \dots, \sigma_n\}$ is expressed in the form

$$w_0\sigma_i w_1\sigma_i \dots \sigma_i w_r,$$

where the subwords $w_0, \dots, w_r$ are the words on the letters $\sigma_j^{\pm 1}$ with $j > i$. Then such a word is called *σ_i-positive*. This means that all entries of $\sigma_i^{\pm 1}$ in the word w with i minimal must be positive. If all such entries are negative then a word w is called *σ_i-negative*. A braid of Br_{n+1} is called *σ_i-positive* (*σ_i-negative*) if there exists its expression as a word on the standard generators which is *σ_i-positive* (*σ_i-negative*). A braid is called *σ-positive* (*σ-negative*) if it exists a number i, such that it is *σ_i-positive* (*σ_i-negative*).

Theorem 5.1. (P. Dehornoy) *Every braid in Br_{n+1} different from 1 is either σ-positive or σ-negative.*

Corollary 5.1. *For all n the braid group Br_{n+1} is left ordered.*

6. Representations

6.1. **Burau representation.** Let us map the generators of the braid group Br_n to the following elements of the group $GL_n\mathbb{Z}[t,t^{-1}]$

$$\sigma_i \mapsto \begin{pmatrix} E_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & E_{n-i-1} \end{pmatrix}, \tag{6.1}$$

where E_i is the unit $i \times i$ matrix. The formula (6.1) define correctly the representation of the braid group in $GL_n\mathbb{Z}[t,t^{-1}]$:

$$r : Br_n \to GL_n\mathbb{Z}[t,t^{-1}],$$

which is called *Burau representation* [22].

Theorem 6.1. *Burau representation is faithful for $n = 3$.*

Theorem 6.2. *(*J. A. Moody; D. D. Long and M. Paton; S. Bigelow) Burau representation is unfaithful for $n \geq 5$.

The case $n = 4$ remains open.

6.2. **Lawrence-Krammer representation.** Consider the ring $K = \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ of Laurent polynomials in two variables q, t, and the free K-module

$$V = \bigoplus_{1 \leq i < j \leq n} R\, x_{i,j}.$$

For $k \in \{1, 2, \ldots, n-1\}$, define the action of the braid generators σ_k on the basis of V by the formula:

$$\sigma_k(x_{i,j}) = \begin{cases} x_{i,j}, & k < i-1 \text{ or } j < k; \\ x_{i-1,j} + (1-q)x_{i,j}, & k = i-1; \\ tq(q-1)x_{i,i+1} + qx_{i+1,j}, & k = i < j-1; \\ tq^2 x_{i,j}; & k = i = j-1 \\ x_{i,j} + tq^{k-i}(q-1)^2 x_{k,k+1}, & i < k < j-1; \\ x_{i,j-1} + tq^{j-i}(q-1)x_{j-1,j}, & k = j-1; \\ (1-q)x_{i,j} + qx_{i,j+1}, & k = j. \end{cases} \tag{6.2}$$

Direct comutation shows that this defines a representation

$$\rho_n : Br_n \to GL(V),$$

which was firstly defined by R. Lawrence [66] in topological terms and in the explicit form (6.2) by D. Krammer [65].

Theorem 6.3. (S. Bigelow [10], D. Krammer [65]) *The representation*

$$\rho_n : Br_n \to GL\,(V)$$

is faithful for all $n \geq 1$.

Remark 6.1. *Actually, S Bigelow* [10] *proved this theorem for the representation ρ_n characterized in homological terms and D. Krammer* [65] *proved the following. Let $K = \mathbb{R}[t^{\pm 1}]$, $q \in \mathbb{R}$, and $0 < q < 1$. Then the representation ρ_n defined by (6.2) is faithful for all $n \geq 1$. This result implies Theorem 6.3: if a representation over $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ becomes faithful after assigning a real value to q, then it is faithful itself.*

M. G. Zinno [99] established connection between Birman-Murakami-Wenzl algebra [14], [80] and the Lawrence-Krammer representation. Namely, he proved that the Lawrence-Krammer representation is identical to the irreducible representations of the Birman-Murakami-Wenzl algebra parametrized by Young diagrams of shapes $(n-2)$ and (1^{n-2}). This means that the Young diagram in the case considered consists of one row (respectively of one column) only, with $n-2$ boxes. It follows that Lawrence-Krammer representation is irreducible. X-Sun-Data-Name: sec6.tex X-Sun-Content-Lines: 374 X-Sun-Charset: us-ascii

7. Generalizations of Braids

7.1. **Configuration spaces of manifolds.** The notion of configuration space of Subsection 3.2 can be naturally generalized for a manifold as follows.

Let Y be a connected topological manifold and let W be a finite group acting on Y. A point $y \in Y$ is called *regular* if its stabilizer $\{w \in W : wy = y\}$ is trivial, i.e., consists only of the unit of the group W. The set $\tilde{Y}$ of all regular points is open. Suppose that it is connected and nonempty. The subspace $ORB(Y, W)$ of the space of all orbits $Orb(Y, W)$ consisting of the orbits of all regular points is called the *space of regular orbits*. There is a free action of W on $\tilde{Y}$ and the projection $p : \tilde{Y} \to \tilde{Y}/W = ORB(Y, W)$ defines a covering. Let us consider the initial segment of the long exact sequence of this covering:

$$1 \to \pi_1(\tilde{Y}, y_0) \xrightarrow{p_*} \pi_1(ORB(Y, W), p(y_0)) \to W \to 1. \tag{7.1}$$

The fundamental group $\pi_1(ORB(Y, W), p(y_0))$ of the space of regular orbits is called the *braid group of the action of W on Y* and is denoted by $Br(Y, W)$. The fundamental group $\pi_1(\tilde{Y}, y_0)$ is called the *pure braid group of the action of W on Y* and is denoted by $P(Y, W)$. The spaces $\tilde{Y}$ and $ORB(Y, W)$ are path connected, so the pair of these groups is defined uniquely up to isomorphism and we may omit mentioning the base point y_0 in the notations.

For any space Y the symmetric group Σ_m acts on the Cartesian power Y^m of the space Y by the formulas (3.4) We denote by $F(Y, m)$ the space of m-tuples of pairwise different points in Y:

$$F(Y, m) = \{p_1, ..., p_m) \in Y^m : p_i \neq p_j \text{ for } i \neq j\}.$$

This is the space of regular points of this action. In the case when Y is a connected topological manifold M without boundary and $\dim M \geq 2$, the space of regular orbits $ORB(M^m, \Sigma_m)$ is open, connected and nonempty. We call $ORB(M^m, \Sigma_m)$ the *configuration space of the manifold M* and denote by $B(M, m)$. The braid group $Br(M^m, \Sigma_m)$ is called the *braid group on m strings of the manifold M* and is denoted by $Br(m, M)$. Analogously, we call the group $P(M^m, \Sigma_m)$ the *pure braid group on m strings of the manifold M* and denote it by $P(m, M)$. These definitions of braid groups were given by R. Fox and L. Neuwirth [48].

7.2. **Brieskorn-Artin braid groups.** The braid groups are included in the series of so called generalized braid groups (this was their name in the work of E. Brieskorn of 1971 [16]), or Artin groups (so they we called by E. Brieskorn and K. Saito in the paper of 1972 [19]). They were defined by E. Brieskorn [16], so we call them Brieskorn–Artin groups.

Let V be a finite dimensional real vector space ($\dim V = n$) with Euclidean structure. Let W be a finite subgroup of $GL(V)$ generated by reflections. Let $\mathcal{M}$ be the set of hyperplanes such that W is generated by orthogonal reflections with respect to $M \in \mathcal{M}$. We suppose that for every $w \in W$ and every hyperplane $M \in \mathcal{M}$ the hyperplane $w(M)$ belongs to $\mathcal{M}$.

The group W is generated by the reflections $w_i = w_i(M_i), i \in I$, satisfying only the following relations

$$(w_i w_j)^{m_{i,j}} = e, \ \ i, j \in I,$$

where the natural numbers $m_{i,j} = m_{j,i}$ form the *Coxeter matrix* of W from which the *Coxeter graph* $\Gamma(W)$ of W is constructed [15]. We use the following notation of P. Deligne [37]: $\mathrm{prod}(m; x, y)$ denotes the product $xyxy...$ (m factors). The *generalized braid group* (or *Brieskorn–Artin group*) $Br(W)$ *of* W [16], [37] is defined as the group with generators $\{s_i, i \in I\}$ and relations:

$$\mathrm{prod}(m_{i,j}; s_i, s_j) = \mathrm{prod}(m_{j,i}; s_j, s_i).$$

From this we obtain the presentation of the group W by adding the relations:

$$s_i^2 = e; \quad i \in I.$$

We see in Theorem 7.1 that this definition of the generalized braid group agrees with our general definition of a braid group of an action of a group W (Subsection 7.1). We denote by τ_W the canonical homomorphism from $Br(W)$ to W. The classical braids on k strings Br_k are obtained by this construction if W is the symmetric group on $k+1$ symbols. In this case $m_{i,i+1} = 3$, and $m_{i,j} = 2$ if $j \neq i, i+1$.

Classification of *irreducible* (with connected Coxeter graph) Coxeter groups is well known (see for example Theorem 1 , Chapter VI, §4 of [Bo]). It consists of the three infinite series: A, C (which is also denoted by B because in the corresponding classification of simple Lie algebras two different series B and C have this group as a Weyl group) and D as well as the exceptional groups E_6, E_7, E_8, F_4, G_2, H_3, H_4 and $I_2(p)$.

Now let us consider the complexification V_C of the space V and the complexification M_C of $M \in \mathcal{M}$. Let $Y_W = V_C - \bigcup_{M \in \mathcal{M}} M_C$. The group W acts freely on Y_W. Let $X_W = Y_W/W$ then Y_W is a covering over X_W corresponding to the group W. Let $y_0 \in A_0$ be a point in some chamber A_0 and let x_0 stand for its image in X_W. We are in the situation described in Subsection 7.1 in the definition of the braid group of the action of the group W. This braid group is defined as the fundamental group of the space of regular orbits of the action of W. In our case $ORB(V_C, W) = X_W$. So, the generalized braid group is $\pi_1(X_W, x_0)$. For each $j \in I$, let ℓ'_j be the homotopy class of paths in Y_W starting from y_0 and ending in $w_j(y_0)$ which contains a polygon line with successive vertices: $y_0, y_0 + iy_0, w_j(y_0) + iy_0, w_j(y_0)$. The image ℓ_j of the class ℓ'_j in X_W is a loop with base point x_0.

Theorem 7.1. *The fundamental group $\pi_1(X_W, x_0)$ is generated by the elements ℓ_j satisfying the following relations:*

$$\mathrm{prod}(m_{j,k}; \ell_j, \ell_k) = \mathrm{prod}(m_{k,j}; \ell_k, \ell_j).$$

This theorem was proved by E. Brieskorn [17].

The word problem and the conjugacy problem for Brieskorn-Artin groups were solved by E. Brieskorn and K. Saito [19] and P. Deligne [37]. The biautomatic structure of these groups was established by R. Charney [23].

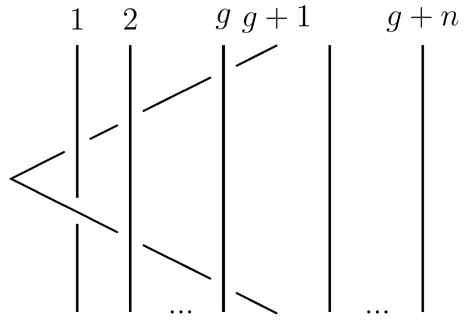

FIGURE 7.1

7.3. **Braid groups of surfaces.** Braid groups of a sphere $Br_n(S^2)$ also have simple geometric interpretation as a group of isotopy classes of braids lying in a layer between two concentric spheres. It has the presentation with generators δ_i, $i = 1, ..., n-1$, and relations:

$$\begin{cases} \delta_i \delta_j & = \delta_j\, \delta_i, \quad \text{if} \quad |i-j| > 1, \\ \delta_i \delta_{i+1} \delta_i & = \delta_{i+1} \delta_i \delta_{i+1}, \\ \delta_1 \delta_2 \dots \delta_{n-2} \delta_{n-1}^2 \delta_{n-2} \dots \delta_2 \delta_1 & = 1. \end{cases} \tag{7.2}$$

This presentation was find by O. Zariski [96] in 1936 and then rediscovered by E. Fadell and J. Van Buskirk [43] in 1961.

Presentations of braid groups of all closed surfaces were obtained by G. P. Scott [86] but look rather complicated.

7.4. **Braid groups in handlebodies.** The subgroup $Br_{1,n+1}$ of the braid group Br_{n+1} consisting of braids with the fixed first string can be interpreted also as the braid group in a solid torus. Here we study braids in a handlebody of the arbitrary genus g.

Let H_g be a handlebody of genus g. The braid group Br_n^g on n strings in H_g was first considered by A. B. Sossinsky [88]. Let Q_g denote a subset of the complex plain $\mathbb{C}$, consisting of g different points, $Q_g = \{z_1^0, ..., z_g^0\}$, say, $z_i^0 = i$. The interior of the handlebody H_g may be interpreted as the direct product of the complex plain $\mathbb{C}$ without g points: $\mathbb{C} \setminus Q_g$, and the open interval, for example, $(-1, 2)$:

$$\dot{H}_g = (\mathbb{C} \setminus Q_g) \times (-1, 2).$$

The space $F(\mathbb{C} \setminus Q_g, n)$ can be interpreted as the complement of the arrangement of hyperplanes in $\mathbb{C}^{g+n}$ given by the formulas:

$$H_{j,k} : z_j - z_k = 0 \text{ for all } j, k;$$

$$H_j^i : z_j = z_i^0 \text{ for } i = 1, ..., g; \; j = 1, ..., n.$$

The braids in Br_n^g are considered as lying between the planes with coordinates $z = 0$ and $z = 1$ and connecting the points $((g+1, 0), ..., (g+n, 0))$. So Br_n^g can be considered as a subgroup of the classical braid group Br_{g+n} on $g+n$ strings such that the braids from Br_n^g leave the first g strings unbraided. In this subsection we denote by $\bar{\sigma}_j$ the standard generators of the group Br_{g+n}. Let $\tau_k, \; k = 1, 2, ..., g$, be the following braids:

$$\tau_k = \bar{\sigma}_g \bar{\sigma}_{g-1} ... \bar{\sigma}_{k+1} \bar{\sigma}_k^2 \bar{\sigma}_{k+1}^{-1} ... \bar{\sigma}_{g-1}^{-1} \bar{\sigma}_g^{-1}.$$

Such a braid is depicted in Figure 7.1. The elements $\tau_k, \; k = 1, 2, ..., g$, generate a free subgroup F_g in the braid group Br_{g+n}. It follows for example from the Markov normal form that the

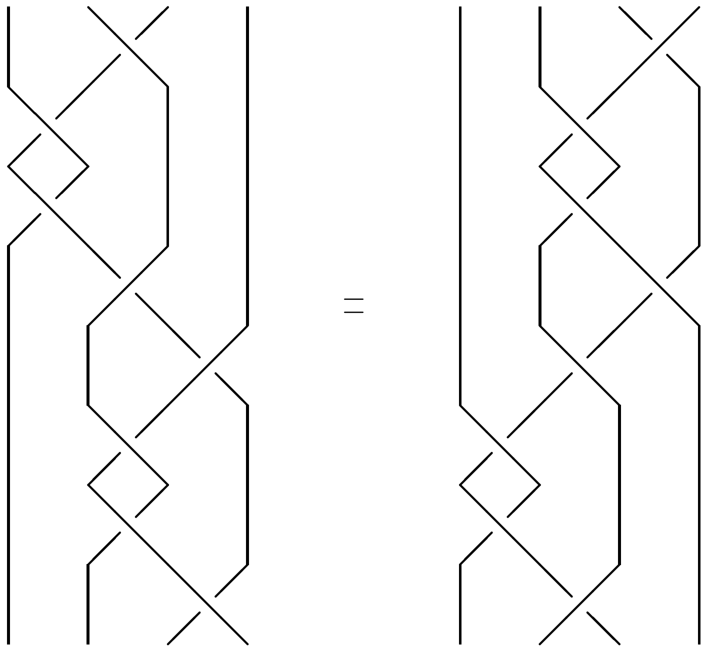

FIGURE 7.2

elements $\tau_k,\ k = 1, 2, ..., g$, together with the standard generators $\bar\sigma_{g+1}, ..., \bar\sigma_{g+n-1}$ generate the group Br_n^g. So, the braid group in the handlebody Br_n^g can be considered as a subgroup of Br_{g+n}, generated by two subgroups: F_g and Br_n. Denote by $\sigma_1, ..., \sigma_{n-1}$ the standard generators of Br_n considered as the elements of Br_n^g, $\sigma_i = \bar\sigma_{g+i}$, $i = 1, \dots, n-1$. So we have the presentation of Br_n^g with the generators τ_k and σ_i and relations [88], [92], [95]:

$$\begin{cases} \sigma_i\sigma_j & = \sigma_j\sigma_i \ \text{ if } \ |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i & = \sigma_{i+1}\sigma_i\sigma_{i+1}, \\ \tau_k\sigma_i & = \sigma_i\tau_k \ \text{ if } \ k \geq 1, \ i \geq 2, \\ \tau_k\sigma_1\tau_k\sigma_1 & = \sigma_1\tau_k\sigma_1\tau_k, \ k = 1, 2, ..., g; \\ \tau_k\sigma_1^{-1}\tau_{k+l}\sigma_1 & = \sigma_1^{-1}\tau_{k+l}\sigma_1\tau_k, \ k = 1, 2, ..., g-1; \ l = 1, 2, ..., g-k. \end{cases} \tag{7.3}$$

The relation of the fourth type in (7.3) is the relation of the braid group of type B (C). The relations of the fifth type in (7.3) describe the interaction between the generators of the free group and their closest neighbor σ_1. Geometrically this is seen in Figure 7.2. If we introduce the new generators θ_k, $k = 1, 2, ..., g-1$; by the formulas:

$$\theta_k = \sigma_1^{-1}\tau_k\sigma_1$$

we obtain the "positive" presentation of the group B_n^g with generators of the types σ_i, τ_k, θ_k and relations:

$$
(7.4)\qquad \begin{cases} \sigma_i\sigma_j & = \sigma_j\sigma_i \ \text{ if } \ |i-j|>1, \\ \sigma_i\sigma_{i+1}\sigma_i & = \sigma_{i+1}\sigma_i\sigma_{i+1}, \\ \tau_k\sigma_i & = \sigma_i\tau_k \ \text{ if } \ k\geq 1, \ i\geq 2, \\ \tau_k\sigma_1\tau_k\sigma_1 & = \sigma_1\tau_k\sigma_1\tau_k, \ k=1,2,...,g; \\ \tau_k\theta_{k+l} & = \theta_{k+l}\tau_k, \ k=1,2,...,g-1; \ l=1,2,...,g-k. \\ \sigma_1\theta_k & = \tau_k\sigma_1, \ k=1,2,...,g-1. \end{cases}
$$

There is an analog of the Markov Theorem 3.2 for the group Br_n^g [92], [95].

7.5. **Braids with singularities.** Let BP_n be the subgroup of $\operatorname{Aut} F_n$, generated by both sets of the automorphisms σ_i of (3.6) and ξ_i of the following form:

$$
\begin{cases} x_i & \mapsto x_{i+1}, \\ x_{i+1} & \mapsto x_i, \\ x_j & \mapsto x_j, j\neq i, i+1, \end{cases}
$$

This is the *braid-permutation group.* R. Fenn, R. Rimányi and C. Rourke proved [45], [46] that this group is given by the set of generators: $\{\xi_i, \sigma_i, \quad i=1,2,...,n-1\}$ and relations:

$$
\begin{cases} \xi_i^2 & = 1, \\ \xi_i\xi_j & = \xi_j\xi_i, \ \text{ if } \ |i-j|>1, \\ \xi_i\xi_{i+1}\xi_i & = \xi_{i+1}\xi_i\xi_{i+1}. \end{cases}
$$

Thesymmetric group relations

$$
\begin{cases} \sigma_i\sigma_j & = \sigma_j\sigma_i, \ \text{if } |i-j|>1, \\ \sigma_i\sigma_{i+1}\sigma_i & = \sigma_{i+1}\sigma_i\sigma_{i+1}. \end{cases}
$$

The braid group relations

$$
\begin{cases} \sigma_i\xi_j & = \xi_j\sigma_i, \ \text{if } |i-j|>1, \\ \xi_i\xi_{i+1}\sigma_i & = \sigma_{i+1}\xi_i\xi_{i+1}, \\ \sigma_i\sigma_{i+1}\xi_i & = \xi_{i+1}\sigma_i\sigma_{i+1}. \end{cases}
$$

The mixed relations

R. Fenn, R. Rimányi and C. Rourke also gave a geometric interpretation of BP_n as a group of *welded braids.* At first they defined a *welded braid diagram* on n strings as a collection of n monotone arcs starting from n points at a horizontal line of a plane (the top of the diagram) and going down to n points at another horizontal line (the bottom of the diagram). The diagrams may have crossings of two types: 1) the same as usual braids as for example on the Figure 3.2 or 2) welds as depicted in Figure 7.3

Composition of welded braid diagrams on n strings is defined by stacking one diagram under the other. The diagram with no crossings or welds is the identity with respect to composition. So the set of welded braid diagrams on n strings forms a semigroup which is denoted by WD_n.

R. Fenn, R. Rimányi and C. Rourke defined the allowable moves on welded braid diagrams. They consist of usual Reidemester moves (Figure 3.4) and the specific moves depicted of Figures 7.4, 7.5, 7.6 The automorphisms of F_n which lie in BP_n can be characterized as follows.

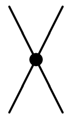

Figure 7.3

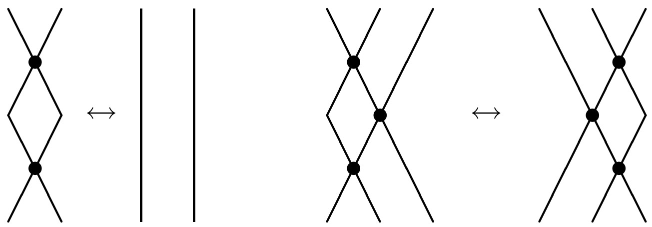

Figure 7.4

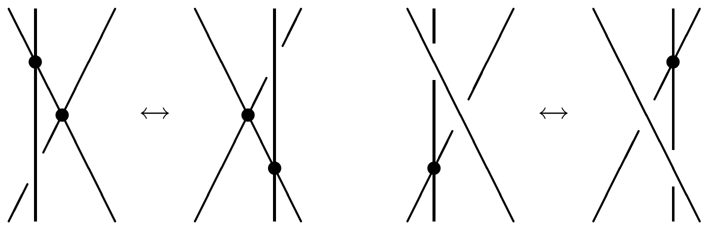

Figure 7.5

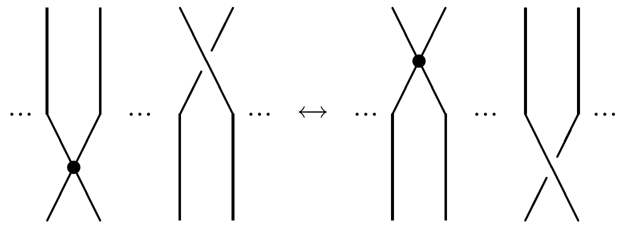

Figure 7.6

Let $\pi \in \Sigma_n$ be a permutation and $w_i, \quad i = 1, 2..., n$, be words in F_n. Then the mapping

$$x_i \mapsto w_i^{-1} x_{\pi(i)} w_i$$

determines an injective endomorphism of F_n. If it is also surjective, we call it an automorphism of *permutation-conjugacy type*. The automorphisms of this type comprise a subgroup of $\mathrm{Aut}\, F_n$ which is precisely BP_n.

The *Baez–Birman monoid* SB_n or *singular braid monoid* [8], [12] is defined as monoid with generators g_i, g_i^{-1}, a_i, $i = 1, \dots, n-1$, and relations

$$g_i g_j = g_j g_i, \text{ if } |i - j| > 1,$$

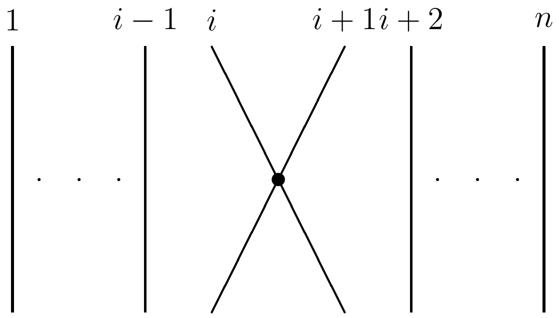

FIGURE 7.7

$$a_i a_j = a_j a_i, \text{ if } |i-j|>1,$$
$$a_i g_j = g_j a_i, \text{ if } |i-j| \neq 1,$$
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1},$$
$$g_i g_{i+1} a_i = a_{i+1} g_i g_{i+1},$$
$$g_{i+1} g_i a_{i+1} = a_i g_{i+1} g_i,$$
$$g_i g_i^{-1} = g_i^{-1} g_i = 1.$$

In pictures g_i corresponds to canonical generator of the braid group and a_i represents an intersection of the ith and $(i+1)$st strand as in Figure 7.7. More detailed geometric interpretation of the Baez–Birman monoid can be found in the article of J. Birman [12]. R. Fenn, E. Keyman and C. Rourke proved [44] that the Baez–Birman monoid embeds in a group SG_n which they called the *singular braid group*:

$$SB_n \to SG_n.$$

So, in SG_n the elements a_i become invertible and all relations of SB_n remain true.

8. Homological Properties

8.1. **Configuration spaces and $K(\pi,1)$-spaces.** Let $(q_i)_{i\in\mathbb{N}}$ be a fixed sequence of distinct points in the manifold M and put $Q_m = \{q_1, ..., q_m\}$. We use

$$Q_{m,l} = (q_{l+1}, ..., q_{l+m}) \in F(M \setminus Q_l, m)$$

as the standard base point of the space $F(M \setminus Q_l, m)$. If $k < m$ we define the projection

$$\mathrm{proj} : F(M \setminus Q_l, m) \to F(M \setminus Q_l, k)$$

by the formula: $\mathrm{proj}(p_1, ..., p_m) = (p_1, ..., p_k)$. The following theorems were proved by E. Fadell and L. Neuwirth [42].

Theorem 8.1. *The triple* $\mathrm{proj} : F(M \setminus Q_l, m) \to F(M \setminus Q_l, k)$ *is a locally trivial fiber bundle with fiber* $\mathrm{proj}^{-1} Q_{k,l}$ *homeomorphic to* $F(M \setminus Q_{k+l}, m-k)$.

Consideration of the sequence of fibrations

$$F(M \setminus Q_{m-1}, 1) \to F(M \setminus Q_{m-2}, 2) \to M \setminus Q_{m-2},$$
$$F(M \setminus Q_{m-2}, 2) \to F(M \setminus Q_{m-3}, 3) \to M \setminus Q_{m-3},$$
$$\dots ,$$
$$F(M \setminus Q_1, m-1) \to F(M, m) \to M$$

leads to the following theorem.

Theorem 8.2. *For any manifold M*

$$\pi_i(F(M \setminus Q_1, m-1)) = \oplus_{k=1}^{m-1} \pi_i(M \setminus Q_k)$$

for $i \geq 2$. If $\mathrm{proj} : F(M, m) \to M$ *admits a section then*

$$\mathrm{proj}_i\, \pi_i(F(M, m)) = \oplus_{k=0}^{m-1} \pi_i(M \setminus Q_k), \ i \geq 2.$$

Corollary 8.1. *If M is the Euclidean r-space, then*

$$\pi_i(F(M, m)) = \oplus_{k=0}^{m-1} \pi_i(\underbrace{S^{r-1} \vee \ldots \vee S^{r-1}}_{k}), \ i \geq 2.$$

Corollary 8.2. *If M is the Euclidean 2-space, then $F(\mathbb{R}^2, m)$ is the $K(P_m, 1)$-space and $B(\mathbb{R}^2, m)$ is the $K(Br_m, 1)$-space.*

Let X_W be the space defined in Subsection 7.2.

Theorem 8.3. *The universal covering of X_W is contractible, and so X_W is a $K(\pi; 1)$-space.*

This theorem for the groups of types C_n, G_2 and $I_2(p)$, was proved by E. Brieskorn [16] in much the same way as E. Fadell and L. Neuwirth [42] proved Theorems 8.1, 8.2 and Corollary 8.2. For the groups of types D_n and F_4 E. Brieskorn used this method with minor modifications. In general case Theorem was proved by P. Deligne [37].

It follows from Theorem 8.2 that $F(\mathbb{C} \setminus Q_g, n)$ and $B(\mathbb{C} \setminus Q_g)$ are $K(\pi, 1)$-spaces, $\pi_1 B(\mathbb{C} \setminus Q_g) = Br_n^g$, so, $B(\mathbb{C} \setminus Q_g)$ can be considered as the classifying space of Br_n^g.

8.2. **Cohomology of pure braid groups.** Cohomologies of pure braid groups were first calculated by V. I. Arnold [3]. The map

$$\phi : S^{n-1} \to F(\mathbb{R}^n, 2),$$

described by the formula $\phi(x) = (x, -x)$, is a Σ_2-equivariant homotopy equivalence. Denote by A the generator of $H^{n-1}(F(\mathbb{R}^n, 2), \mathbb{Z})$ that is mapped by ϕ^* to the standard generator of $H^{n-1}(S^{n-1}, \mathbb{Z})$. For i and j, such that $1 \leq i, j \leq m$, $i \neq j$, specify $\pi_{i,j} : F(\mathbb{R}^n, m) \to F(\mathbb{R}^n, 2)$ by the formula $\pi_{i,j}(p_1, \ldots, p_m) = (p_i, p_j)$. Put

$$A_{i,j} = \pi_{i,j}^*(A) \in H^{n-1}(F(\mathbb{R}^n, m), \mathbb{Z}).$$

It follows that $A_{i,j} = (-1)^n A_{j,i}$ and $A_{i,j}^2 = 0$. For $w \in \Sigma_m$ there is an action $w(A_{i,j}) = A_{w^{-1}(i), w^{-1}(j)}$, since $\pi_{i,j} w = \pi_{w^{-1}(i), w^{-1}(j)}$. Note also that under restriction to

$$F(\mathbb{R}^n \setminus Q_k, m-k) \cong \pi^{-1}(Q_k) \subset F(\mathbb{R}^n, m),$$

the classes $A_{i,j}$ with $1 \leq i, j \leq k$ go to zero since in this case the map $\pi_{i,j}$ is constant on $\pi^{-1}(Q_k)$.

Theorem 8.4. *The cohomology group $H^*(F(\mathbb{R}^n \setminus Q_k, m-k), \mathbb{Z})$ is the free Abelian group with generators*

$$A_{i_1,j_1} A_{i_2,j_2} \ldots A_{i_s,j_s},$$

where $k < j_1 < j_2 < \ldots < j_s \leq m$ and $i_r < j_r$ for $r = 1,\ dots, s$.

The multiplicative structure and the Σ_m-algebra structure of $H^*(F(\mathbb{R}^n, m), \mathbb{Z})$ are given by the following theorem which is proved using the Σ_3-action on $H^*(F(\mathbb{R}^n, 3), \mathbb{Z})$.

Theorem 8.5. *The cohomology ring $H^*(F(\mathbb{R}^n, m), \mathbb{Z})$ is multiplicatively generated by the square-zero elements*

$$A_{i,j} \in H^{n-1}(F(\mathbb{R}^n, m), \mathbb{Z}),\ 1 \le i < j \le m,$$

subject only to the relations

$$A_{i,k}A_{j,k} = A_{i,j}A_{j,k} - A_{i,j}A_{i,k} \text{ for } i < j < k. \tag{8.1}$$

The Poincaré series for $F(\mathbb{R}^n, m)$ is the product $\prod_{j=1}^{m-1}(1 + jt^{n-1})$.

Remark 8.1. *In the case of $\mathbb{R}^2 = \mathbb{C}$ the cohomology classes $A_{j,k}$ can be interpreted as the classes of cohomology of differential forms*

$$\omega_{j,k} = \frac{1}{2\pi i}\frac{dz_j - dz_k}{z_j - z_k}.$$

E. Brieskorn calculated the cohomologies of pure generalized braid groups [16] using ideas of V. I. Arnold for the classical case. Let $\mathcal{V}$ be a finite-dimensional complex vector space and $H_j \in \mathcal{V},\ j \in I$ be the finite family of complex affine hyperplanes given by linear forms l_j. E. Brieskorn proved the following fact.

Theorem 8.6. *The cohomology classes, corresponding to the holomorphic differential forms*

$$\omega_j = \frac{1}{2\pi i}\frac{dl_j}{l_j},$$

generate the cohomology ring $H^*(\mathcal{V} \setminus \cup_{j\in I} H_j, \mathbb{Z})$. Moreover, this ring is isomorphic to the $\mathbb{Z}$-subalgebra generated by the forms ω_j in the algebra of meromorphic forms on $\mathcal{V}$.

The cohomologies of pure generalized braid groups are described as follows.

Theorem 8.7. *(i) The cohomology group $H^k(P(W), \mathbb{Z})$ of the pure braid group $P(W)$ with integer coefficients is a free abelian group, and its rank is equal to the number of elements $w \in W$ of length $l(w) = k$, where l is the length considered with respect to the system of generators consisting of all reflections of W.*

(ii) The Poincaré series for $H^*(P(W), \mathbb{Z})$ is the product $\prod_{j=1}^{n}(1 + m_j t)$, where m_j are the exponents of the group W.

(iii) The multiplicative structure of $H^*(P(W), \mathbb{Z})$ coincides with the structure of algebra, generated by 1-forms described in the previous theorem.

8.3. **Homology of braid groups.** To study the cohomologies of the classical braid groups $H^*(Br_n, \mathbb{Z})$, V. I. Arnold [4] interpreted the space $K(Br_n, 1) \cong B(\mathbb{R}^2, n)$ as the space of monic complex polynomials of degree n without multiple roots

$$P_n(t) = t^n + z_1 t^{n-1} + ... + z_{n-1}t + z_n.$$

Using this idea he proved Theorems of finiteness, of recurrence and of stabilization. Homologies with coefficients in $\mathbb{Z}/2$ were calculated by D. B. Fuks in the following theorems [50].

Theorem 8.8. *The homology of the braid group on the infinite number of strings with coefficients in $\mathbb{Z}/2$ as a Hopf algebra is isomorphic to the polynomial algebra on infinitely many generators $a_i, i = 1, 2, ...$; $\deg a_i = 2^i - 1$:*

$$H_*(Br_\infty, \mathbb{Z}/2) \cong \mathbb{Z}/2[a_1, a_2, ..., a_i, ...]$$

with the coproduct given by the formula:

$$\Delta(a_i) = 1 \otimes a_i + a_i \otimes 1.$$

Theorem 8.9. *The canonical inclusion $Br_n \to Br_\infty$ induces a monomorphism in homology with coefficients in $\mathbb{Z}/2$. Its image is the subcoalgebra of the polynomial algebra $\mathbb{Z}/2[a_1, a_2, ..., a_i, ...]$ with $\mathbb{Z}/2$-basis consisting of monomials*

$$a_1^{k_1}...a_l^{k_l} \text{ such that } \sum_i k_i 2^i \leq n.$$

Theorem 8.10. *The canonical homomorphism $Br_n \to BO_n,\ 1 \leq n \leq \infty$ induces a monomorphism (of Hopf algebras if $n = \infty$)*

$$H_*(Br_n, \mathbb{Z}/2) \to H_*(BO_n, \mathbb{Z}/2).$$

F. R. Cohen calculated the homology of braid groups with coefficients $\mathbb{Z}/p,\ p > 2$ also as modules over the Steenrod algebra [26], [27], [28].

Later V. V. Goryunov [56], [57] applied the methods of Fuks and expressed the cohomologies of the generalized braid groups of types C and D in terms of the cohomologies of the classical braid groups.

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