# Straightening and bounded cohomology of hyperbolic groups

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# STRAIGHTENING AND BOUNDED COHOMOLOGY OF HYPERBOLIC GROUPS (PRELIMINARY VERSION)

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ABSTRACT. It was stated by M. Gromov [4] that, for any (word) hyperbolic group $G$, the map from bounded cohomology $H_b^*(G,\mathbb{R})$ to $H^*(G,\mathbb{R})$ induced by inclusion is surjective in dimensions 2 and higher. The present paper proves that the map $H_b^*(G,V) \to H^*(G,V)$ is surjective in those dimensions for any normed vector space $V$ over $\mathbb{Q}$. Rather than using quasigeodesic flows for the proof, we introduce a homological analog of straightening simplices, which works for any hyperbolic group.

## 1. INTRODUCTION

It is a simple consequence of the Gauss-Bonnet formula that the areas of geodesic triangles in the hyperbolic plane are uniformly bounded. This is also true in higher dimensions, i.e. the volumes of straight higher dimensional simplices are also bounded. The idea of straightening in $\mathbb{H}^n$ is that each map of a standard simplex to the hyperbolic space can be deformed to the straight one, that is, the convex hull of finitely many points. In homological terms, this implies that the map $H_b^i(G,\mathbb{R}) \to H^i(G,\mathbb{R})$ is surjective for $i \geq 2$, when $G$ is the fundamental group of a hyperbolic manifold ([3]).

The main result of the present paper is Theorem 7 which is essentially saying that straightening, in an appropriate homological context, works for a more general class of groups, namely, hyperbolic groups. The argument is combinatorial. Boundedness of areas of "straight triangles" in this setting reduces simply to summation of a converging geometric series (see Lemma 6). This is a combinatorial analog of exponential convergence of geodesic paths in $\mathbb{H}^n$. Boundedness of volumes for "straight simplices" in higher dimensions follows by induction using the fact that hyperbolic groups satisfy linear higher dimensional (homological) isoperimetric inequalities ([7]). This implies surjectivity of the maps $H_b^i(G,V) \to H^i(G,V)$ for $i \geq 2$ for any normed vector space over $\mathbb{Q}$. W. D. Neumann and L. Reeves [8] showed this for $i = 2$ when $V$ is any finitely generated abelian group.

The author is very thankful to Steve Gersten who suggested working on this problem, and also to Andrejs Treibergs for finding a gap in an earlier version of the paper.

*Date*: October 1998

## 2. Definitions

Let $X$ be a cellular complex with a cellular $G$-action. The result of the action of $g \in G$ on a cell $a$ in $X$ will be denoted by $g \cdot a$. We always equip the 1-skeleton of $X$ with the path metric $d$ induced by assigning length 1 to each edge.

All chains in $X$ will be assumed to be with $\mathbb{Q}$ coefficients.

If $G$ is a group, $\mathcal{U}_\infty(G)$ will denote the set of all cellular complexes $X$ equipped with a free cellular $G$-action which is cocompact on the $i$-skeleton $X^{(i)}$ for each $i$. This means that the quotient of $X$ by the $G$-action has only finitely many cells in each dimension. In this paper, $X$ will always stand for an element of $\mathcal{U}_\infty(G)$. Such a complex $X$ exists for each hyperbolic (or, more generally, combable) group (see [2, Theorem 10.2.6]), i.e. $\mathcal{U}_\infty(G)$ is non-empty.

Given a vertex $v$ in $X$ and a number $r$, a *sphere* $S(v,r)$ in $X$ is the set of all vertices $w$ in $X$ satisfying $d(v,w) = r$. A *ball* $B(v,r)$ in $X$ is the set of all vertices $w$ in $X$ satisfying $d(v,w) \leq r$. If $S$ is a subset of $X^{(1)}$, then the *r-neighborhood* of $S$, $N(S,r)$, is the set of all points $x \in X^{(1)}$ such that $d(x,s) \leq r$ for some $s \in S$.

A *geodesic path* in $X$ is a shortest edge path connecting two vertices. Abusing notations we will view each edge path as a map of an interval, as the image of this map, and also as a 1-chain over $\mathbb{Z}$. A *bicombing* $p$ in $X$ is a function assigning to each ordered pair $(a,b)$ of vertices in $X$ an oriented edge-path $p[a,b]$ from $a$ to $b$. A bicombing $p$ is called *quasigeodesic* if there exist constants $\lambda$ and $K$ such that each $p[a,b]$ is $(\lambda, K)$-quasigeodesic.

A *homological bicombing* $q$ in $X$ is an function which assigns a 1-chain $q[a,b]$ to each ordered pair $(a,b)$ of vertices in $X$, so that $\partial q[a,b] = b - a$. A homological bicombing is called *quasigeodesic* if there exists a constant $r \geq 0$ and a quasigeodesic bicombing $p$ such that *supp* $q[a,b] \subseteq N(p[a,b], r)$ for each $a, b \in X^{(0)}$. A homological bicombing $q$ is *G-equivariant* if $q[g \cdot a, g \cdot b] = g \cdot q[a,b]$ for each $a, b \in X$ and each $g \in G$.

A finitely generated group $G$ is called *(word) hyperbolic* if, for any graph $\Gamma$ with a free cocompact cellular $G$-action there exists a constant $\delta \geq 0$ such that all the geodesic triangles in $\Gamma$ are $\delta$-fine in the following sense: if $a$, $b$, and $c$ are vertices in $\Gamma$, $[a,b]$, $[b,c]$, and $[c,a]$ are geodesics from $a$ to $b$, from $b$ to $c$, and from $c$ to $a$, respectively, and points $\bar{a} \in [b,c]$, $v, \bar{c} \in [a,b]$, $w, \bar{b} \in [a,c]$ satisfy

$$d(b,\bar{c}) = d(b,\bar{a}), \quad d(c,\bar{a}) = d(c,\bar{b}), \quad d(a,v) = d(a,w) \leq d(a,\bar{c}) = d(a,\bar{b}),$$

then $d(v,w) \leq \delta$.

Given $G$, $\Gamma$, and $\delta$ as above and vertices $a$, $b$, $c$ in $\Gamma$, then a vertex $z$ is called *a center* of the triple $\{a,b,c\}$ if there exist geodesics $[a,b]$, $[b,c]$, $[c,a]$, points $\bar{a} \in [b,c]$, $\bar{b} \in [c,a]$, and $\bar{c} \in [a,b]$ satisfying

$$d(b,\bar{c}) = d(b,\bar{a}), \qquad d(c,\bar{a}) = d(c,\bar{b}), \qquad d(a,\bar{c}) = d(a,\bar{b}),$$

and such that $d(z,\bar{a}) \leq \delta$, $d(z,\bar{b}) \leq \delta$, and $d(z,\bar{c}) \leq \delta$. Such a center always exists since one can take $z$ to be $\bar{a}$.

Suppose that a vector space $W$ over $\mathbb{Q}$ has a preferred basis $\{w_i, i \in I\}$. The $\ell_1$-*norm* on $W$ (with respect to this basis) is given by

$$\left|\sum_{i\in I} \alpha_i w_i\right|_1 := \sum_{i\in I} |\alpha_i|.$$

For a linear map $\varphi : W \to W'$ between two vector spaces equipped with $\ell_1$-norms, the $\ell_\infty$-*norm* of $\varphi$, $|\varphi|_\infty$, is the operator norm of $\varphi$, i.e. $|\varphi|_\infty$ is the smallest number $K$ (possibly infinity) such that $|\varphi(w)|_1 \leq K|w|_1$ for each $w \in W$. One checks that

$$|\varphi|_\infty = \sup_{i\in I} \left|\varphi(w_i)\right|_1,$$

where $\{w_i,\ i \in I\}$ is the preferred basis of $W$. The preferred basis on the space of cellular $i$-chains, $C_i(X, \mathbb{Q})$, will always be the set of $i$-cells in $X$ and we always equip $C_i(X, \mathbb{Q})$ with the $\ell_1$-norm.

There are various definitions for bounded cohomology of a group, $H_b^*(G, \mathbb{Q})$ (see [5]). The one we will use in the paper is by (homogeneous) bar-construction.

## 3. Auxiliary statements.

For the rest of the paper, fix a hyperbolic group $G$ and some $X \in \mathcal{U}_\infty(G)$. Let $\delta$ be an integer such that all geodesic triangles in $X^{(1)}$ are $\delta$-fine. Increase $\delta$ if needed so that $\delta \geq 1$. For vertices $a$, $b$, and $c$ in $X$, the *Gromov product* is defined by

$$(b|c)_a := \frac{1}{2}\Big[d(a,b) + d(a,c) - d(b,c)\Big].$$

Note that, by the triangle inequality, this product always satisfies

$$(b|c)_a \leq d(a,b), \quad (b|c)_a \leq d(a,c), \quad (b|c)_a \geq 0, \quad d(a,b) = (b|c)_a + (a|c)_b,$$

and analogously for any permutation of letters $a$, $b$, and $c$.

The following lemma immediately follows from the fine-triangles definition of hyperbolic groups.

**Lemma 1** (fine-triangles property). *Let $G$ be a hyperbolic group, $X \in \mathcal{U}_\infty(G)$, and $z$, $x$, $y$, $x'$, and $y'$ be vertices in $X$ such that $x'$ and $y'$ lie on geodesics connecting $z$ to $x$ and $y$, respectively. Suppose also that*

$$d(z,x') = d(z,y') \leq (x|y)_z.$$

*Then $d(x',y') \leq \delta$.*

For the rest of the paper, fix some $G$-equivariant geodesic bicombing $p$ in $X$, i.e. for each pair of vertices $a$ and $b$ in $X$, pick a geodesic path $p[a,b]$, viewed as a 1-chain, with $\partial p[a,b] = b - a$, and such that $g \cdot p[a,b] = p[g\cdot a, g\cdot b]$ for any $g \in G$. Abusing the notation we will also view $p[a,b]$ as a geodesic path, i.e. an isometric embedding $p[a,b] : [0, d(a,b)] \to X^{(1)}$ with $p[a,b](0) = a$ and $p[a,b](d(a,b)) = b$. So $p[a,b](r)$ stands for the image of $r \in [0, d(a,b)]$ via the map $p[a,b]$. In the same way, vertices in $X$ will also be viewed as 0-chains.

A *convex combination* is a (cellular) 0-chain with non-negative coefficients which sum up to 1. For $v, w \in X^{(0)}$, the *flower* at $w$ with respect to $v$ is the set

$$Fl(v,w) := S(v, d(v,w)) \cap B(w, \delta) \subseteq X^{(0)}.$$

**Proposition 2** (cancelling convex combinations). *There exists a function $f : X^{(0)} \times X^{(0)} \to C_0(X, \mathbb{Q})$ mapping each pair $(a,b)$ to a 0-chain $f(a,b)$ with the following properties:*

(1) *$f(a,b)$ is a convex combination.*
(2) *If $d(a,b) \geq 10\delta$, then $\operatorname{supp} f(a,b) \subseteq Fl(a, p[a,b](10\delta))$.*
(3) *If $d(a,b) \leq 10\delta$, then $f(a,b) = b$.*
(4) *$f$ is $G$-equivariant, i.e. $f(g \cdot a, g \cdot b) = g \cdot f(a,b)$ for any $a, b \in X^{(0)}$ and $g \in G$.*
(5) *There exist constants $L \geq 0$ and $0 \leq \lambda < 1$ such that, for any $a, b, c \in X^{(0)}$,*

$$\left| f(a,b) - f(a,c) \right|_1 \leq L\lambda^{(b|c)_a}.$$

The proof of this and later statements may look a bit cumbersome, but the main point should be clear: use the fine-triangles property whenever possible. It is probably also worth mentioning that the number $10\delta$ in the statement is not essential for the proof and can be replaced by any "sufficiently large" integer.

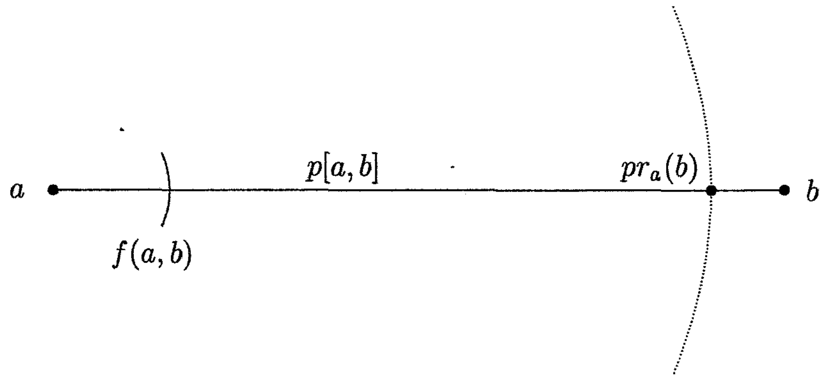


FIGURE 1. Convex combination $f(a,b)$.

*Proof of Proposition 2.* The proof uses the same idea as the dandelion construction in [7], though in a different context. For each vertex $a$ in $X$, define the "one-level-lower projection toward $a$" $pr_a : X^{(0)} \to X^{(0)}$ as follows.

- $pr_a(b) := p[a,b](r)$, where $r$ is the largest (integral) multiple of $10\delta$ which is strictly less than $d(a,b)$, provided $a \neq b$, and

- $pr_a(a) := a$.

Now the convex combination $f(a,b)$ is defined inductively on the distance $d(a,b)$. For vertices $a$ and $b$ with $d(a,b) \le 10\delta$, put $f(a,b) := b$. If $d(a,b) > 10\delta$ and $d(a,b)$ is not a multiple of $10\delta$, let $f(a,b) := f(a, pr_a(b))$. If $d(a,b) > 10\delta$ and $d(a,b)$ is a multiple of $10\delta$, let

$$f(a,b) := \frac{1}{\#Fl(a,b)} \sum_{x \in Fl(a,b)} f(a, pr_a(x)).$$

It is clear from the definition that $f(a,b)$ is a convex combination and it is $G$-equivariant because the the definition uses only metric properties of $X^{(0)}$, which are preserved under the $G$-action. So properties (1) and (4) are satisfied. Property (3) follows directly from the definition.

To finish the proof of Proposition 2 it only remains to show parts (2) and (5). Let

$$\omega := \max\{\#B(v,\delta) \mid v \in X^{(0)}\}.$$

Obviously, $\omega \ge 1$, and also $\omega < \infty$ because, up to the $G$-action, there are only finitely many balls of radius $\delta$ in $X^{(0)}$. Note that the cardinality of each flower $Fl(a,b)$ does not exceed $\omega$.

Fist we need the following lemma.

**Lemma 3.** (a) *Let $a, b \in X^{(0)}$ and let $m$ be any integer satisfying $10\delta \le 10\delta m \le d(a,b)$. Put $v := p[a,b](10\delta m)$. Then*

$$f(a,b) = \sum_{x \in Fl(a,v)} \alpha_x f(a,x),$$

*where $\alpha_x$ are some non-negative coefficients with $\sum_{x \in Fl(a,v)} \alpha_x = 1$.*

(b) *Suppose that vertices $a$, $b$, $c$ in $X$ and an integer $n \ge 1$ satisfy $d(b,c) \le \delta$ and $d(a,b) = d(a,c) = 10\delta n$. Then*

$$\left| f(a,b) - f(a,c) \right|_1 \le 2\left(1 - \frac{1}{\omega^2}\right)^{n-1}.$$

*Proof.* **(a)** follows almost immediately from the definition of $f$. The main tools here are the fine-triangles property and the fact that "a convex combination of convex combinations is again a convex combination". Fix an arbitrary pair of vertices $a$ and $b$ in $X$. We prove the assertion by the *inverse* induction on $m$. Let $m_{max}$ be the maximal integer among all $m$ satisfying $10\delta m \le d(a,b)$. Since $10\delta \le d(a,b)$ by the hypotheses of the lemma, then $m_{max} \ge 1$.

<u>$\boldsymbol{m = m_{max}}$</u> If $10\delta m_{max} = d(a,b)$, then $b = v$ and the 0-chain $f(a,b) = f(a,v)$ can be represented as the *trivial* linear combination $\sum_{x \in Fl(a,v)} \alpha_x f(a,x)$, where $\alpha_v = 1$ and $\alpha_x = 0$ for all $x \ne v$.

If $10\delta m_{max} < d(a,b)$, then, by the definition of $f$,

$$f(a,b) = f(a, pr_a(b)) = f(a,v),$$

which is again the trivial linear combination.
$\underline{\boldsymbol{m+1 \mapsto m}}$ If an integer $m$ satisfies $1 \le m < m_{max}$, then $10\delta \le 10\delta(m+1) \le d(a,b)$, so, by induction hypotheses,

$$f(a,b) = \sum_{x \in Fl(a,v')} \alpha_x f(a,x),$$

where $v' := p[a,b](10\delta(m+1))$ and $\alpha_x$ are some non-negative coefficients satisfying $\sum_{x \in Fl(a,v')} \alpha_x = 1$. By definition, each $f(a,x)$ in the last sum has the presentation

$$f(a,x) = \frac{1}{\#Fl(a,x)} \sum_{y \in Fl(a,x)} f(a, pr_a(y)),$$

therefore

$$f(a,b) = \sum_{x \in Fl(a,v')} \alpha_x \left[ \frac{1}{\#Fl(a,x)} \sum_{y \in Fl(a,x)} f(a, pr_a(y)) \right] = \tag{1}$$

$$= \sum_{x \in Fl(a,v')} \sum_{y \in Fl(a,x)} \frac{\alpha_x}{\#Fl(a,x)} f(a, pr_a(y))$$

Now collect like terms in the last double sum. It amounts to grouping the coefficients $\frac{\alpha_x}{\#Fl(a,x)}$. We have

$$\sum_{x \in Fl(a,v')} \sum_{y \in Fl(a,x)} \frac{\alpha_x}{\#Fl(a,x)} = \sum_{x \in Fl(a,v')} \left[ \frac{\alpha_x}{\#Fl(a,x)} \sum_{y \in Fl(a,x)} 1 \right] = \sum_{x \in Fl(a,v')} \alpha_x = 1,$$

and after grouping the coefficients will still sum up to 1. We have

$$d(v',y) \le d(v',x) + d(x,y) \le 2\delta,$$

then

$$d(a, pr_a(y)) = 10\delta m \le \frac{1}{2}\left[10\delta(m+1) + 10\delta(m+1) - 2\delta\right] \le$$

$$\le \frac{1}{2}\left[d(a,v') + d(a,y) - d(v',y)\right] = (v'|y)_a,$$

hence, by the fine-triangles property, $d(pr_a(y), p[a,b](10\delta)) \le \delta$. This implies that all the points $pr_a(y)$ mentioned in formula (1) belong to $Fl(a, p[a,b](10\delta m))$. Part (a) is proved.
**(*b*)** Induction on $n$.
$\underline{\boldsymbol{n=1}}$ In this case $d(a,b) = d(a,c) = 10\delta$, so, by Proposition 2(3),

$$\left| f(a,b) - f(a,c) \right|_1 = |b-c|_1 \le |b|_1 + |c|_1 = 2 = 2\left(1 - \frac{1}{\omega^2}\right)^{1-1}.$$

**$n-1 \mapsto n$** Suppose $d(b,c) \leq \delta$ and $d(a,b) = d(a,c) = 10\delta n$, where $n \geq 2$. Then

$$
\begin{aligned}
&\Big|f(a,b) - f(a,c)\Big|_1 = \\
(2) \qquad &= \left|\frac{1}{\#Fl(a,b)} \sum_{x \in Fl(a,b)} f(a, pr_a(x)) - \frac{1}{\#Fl(a,c)} \sum_{y \in Fl(a,c)} f(a, pr_a(y))\right|_1 = \\
&= \left|\frac{1}{\#Fl(a,b) \cdot \#Fl(a,c)} \sum_{x \in Fl(a,b)} \sum_{y \in Fl(a,c)} \Big[f(a, pr_a(x)) - f(a, pr_a(y))\Big]\right|_1 \leq \\
&\leq \frac{1}{\#Fl(a,b) \cdot \#Fl(a,c)} \sum_{x \in Fl(a,b)} \sum_{y \in Fl(a,c)} \Big|f(a, pr_a(x)) - f(a, pr_a(y))\Big|_1.
\end{aligned}
$$

By the hypotheses, $d(b,c) \leq \delta$, so $b \in Fl(a,b) \cap Fl(a,c)$, and therefore there is a term in the last double sum corresponding to $x := y := b$. This term is obviously zero. The remaining $\#Fl(a,b) \cdot \#Fl(a,c) - 1$ terms in this double sum can be bounded as follows. Since

$$d(x,y) \leq d(x,b) + d(b,c) + d(c,y) \leq \delta + \delta + \delta = 3\delta,$$

then we have

$$
\begin{aligned}
&d(a, pr_a(x)) = d(a, pr_a(y)) = 10\delta(n-1) \leq \\
&\leq \frac{1}{2}\Big[10\delta n + 10\delta n - 3\delta\Big] \leq \frac{1}{2}\Big[d(a,x) + d(a,y) - d(x,y)\Big] = (x|y)_a,
\end{aligned}
$$

so, by the fine-triangles property, $d(pr_a(x), pr_a(y)) \leq \delta$. The induction hypotheses now apply to the vertices $a$, $pr_a(x)$, and $pr_a(y)$ giving the bound

$$\Big|f(a, pr_a(x)) - f(a, pr_a(y))\Big|_1 \leq 2\left(1 - \frac{1}{\omega^2}\right)^{(n-1)-1}$$

for each $x \in Fl(a,b)$ and $y \in Fl(a,c)$. Continuing inequality (2) we have

$$
\begin{aligned}
&\Big|f(a,b) - f(a,c)\Big|_1 \leq \\
&\leq \frac{1}{\#Fl(a,b) \cdot \#Fl(a,c)} \sum_{x \in Fl(a,b)} \sum_{y \in Fl(a,c)} \Big|f(a, pr_a(x)) - f(a, pr_a(y))\Big|_1 \leq \\
&\leq \frac{1}{\#Fl(a,b) \cdot \#Fl(a,c)} \Big(\#Fl(a,b) \cdot \#Fl(a,c) - 1\Big) \cdot 2\left(1 - \frac{1}{\omega^2}\right)^{(n-1)-1} = \\
&= \left(1 - \frac{1}{\#Fl(a,b) \cdot \#Fl(a,c)}\right) \cdot 2\left(1 - \frac{1}{\omega^2}\right)^{(n-1)-1} \leq \\
&\leq \left(1 - \frac{1}{\omega^2}\right) \cdot 2\left(1 - \frac{1}{\omega^2}\right)^{(n-1)-1} = 2\left(1 - \frac{1}{\omega^2}\right)^{n-1}.
\end{aligned}
$$

Lemma 3 is proved. □

Now part (2) in Proposition 2 can be proved. If $d(a,b) \geq 10\delta$, then, by taking $m := 1$ in Lemma 3(a), we obtain $v = p[a,b](10\delta)$ and

$$f(a,b) = \sum_{x \in Fl(a,v)} \alpha_x f(a,x) = \sum_{x \in Fl(a,v)} \alpha_x x,$$

so part (2) follows.

Now we finish the proof of Proposition 2(5). Pick any triple of vertices $a$, $b$, $c$ in $X$. Let

$$\lambda := \left(1 - \frac{1}{\omega^2}\right)^{\frac{1}{10\delta}} \qquad \text{and} \qquad L := 2\left(1 - \frac{1}{\omega^2}\right)^{-3}.$$

Recall that $1 \leq \omega < \infty$, hence $0 \leq 1 - \frac{1}{\omega^2} < 1$, $L \geq 0$, and $0 \leq \lambda < 1$.

If $(b|c)_a \leq 20\delta$, then

$$\begin{aligned} \left|f(a,b) - f(a,c)\right|_1 &\leq \left|f(a,b)\right|_1 + \left|f(a,c)\right|_1 = 2 = \\ &= 2\left(1 - \frac{1}{\omega^2}\right)^{-3} \cdot \left(1 - \frac{1}{\omega^2}\right)^3 = L\lambda^{30\delta} \leq L\lambda^{20\delta} \leq L\lambda^{(b|c)_a}. \end{aligned}$$

We can now assume $(b|c)_a > 20\delta$. Let $m$ be the maximal integer among those satisfying $10\delta m \leq (b|c)_a$. It easily follows that

$$\frac{(b|c)_a}{10\delta} - 1 \leq m, \tag{3}$$

and

$$20\delta \leq 10\delta m \leq (b|c)_a \leq d(a,b),$$

hence, by Lemma 3(a),

$$f(a,b) = \sum_{x \in Fl(a,v)} \alpha_x f(a,x),$$

where $v := p[a,b](10\delta m)$ and $\alpha_x$ are some non-negative coefficients summing up to 1. A similar argument yields

$$f(a,c) = \sum_{y \in Fl(a,w)} \beta_y f(a,y),$$

where $w := p[a,c](10\delta m)$ and $\beta_y$ are some non-negative coefficients summing up to 1 (see Fig 2).

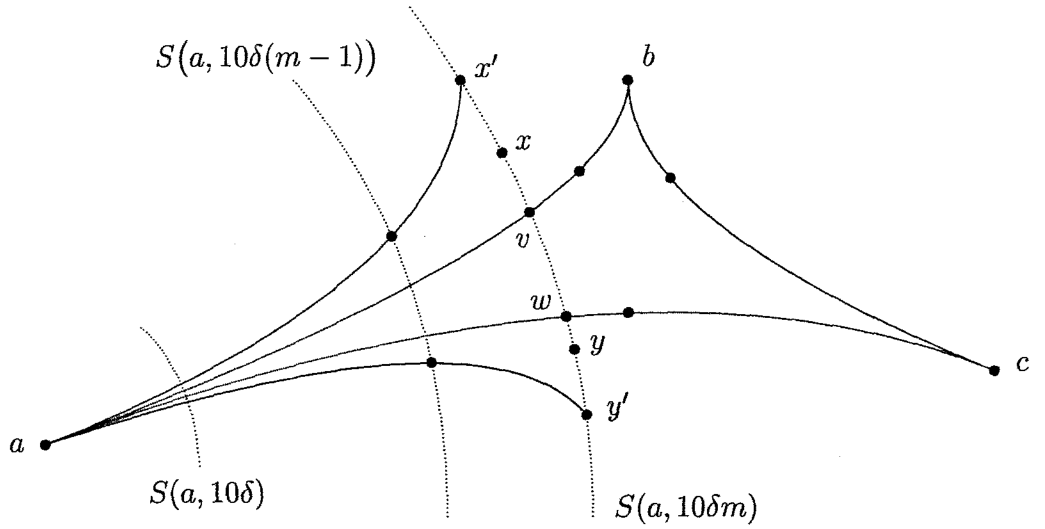


FIGURE 2. Proof of Lemma 3.

$$
\begin{aligned}
\left|f(a,b) - f(a,c)\right|_1 &= \left|\sum_{x\in Fl(a,v)} \alpha_x f(a,x) - \sum_{y\in Fl(a,w)} \beta_y f(a,y)\right|_1 = \\
&= \left|\sum_{x\in Fl(a,v)} \alpha_x f(a,x)\cdot \sum_{y\in Fl(a,w)} \beta_y - \sum_{x\in Fl(a,v)} \alpha_x \cdot \sum_{y\in Fl(a,w)} \beta_y f(a,y)\right|_1 = \\
&\doteq \left|\sum_{x\in Fl(a,v)} \sum_{y\in Fl(a,w)} \alpha_x\beta_y \Big[f(a,x) - f(a,y)\Big]\right|_1 \le \\
&\le \sum_{x\in Fl(a,v)} \sum_{y\in Fl(a,w)} \alpha_x\beta_y \Big|f(a,x) - f(a,y)\Big|_1 .
\end{aligned} \tag{4}
$$

Since, by the choice of $v$, $w$, and $m$, $d(a,v) = d(a,w) = 10\delta m \le (b|c)_a$, then the fine-triangles property yields $d(v,w) \le \delta$. If $x \in Fl(a,v)$, $y \in Fl(a,w)$, $x' \in Fl(a,x)$, and $y' \in Fl(a,y)$, then

$$
d(x',y') \le d(x',x) + d(x,v) + d(v,w) + d(w,y) + d(y,y') \le \delta + \delta + \delta + \delta + \delta = 5\delta
$$

and

$$
\begin{aligned}
d(a, pr_a(x')) = d(a, pr_a(y')) = 10\delta(m-1) &\le \frac{1}{2}\Big[10\delta m + 10\delta m - 5\delta\Big] \le \\
\le \frac{1}{2}\Big[d(a,x') + d(a,y') - d(x',y')\Big] &= (x'|y')_a,
\end{aligned}
$$

hence, by the fine-triangle property again,

$$d(pr_a(x'), pr_a(y')) \leq \delta.$$

Then Lemma 3(b) applies to the vertices $a$, $pr_a(x')$, and $pr_a(y')$ giving

$$\Big|f(a, pr_a(x')) - f(a, pr_a(y'))\Big|_1 \leq 2\left(1 - \frac{1}{\omega^2}\right)^{(m-1)-1} = 2\left(1 - \frac{1}{\omega^2}\right)^{m-2}. \tag{5}$$

Using inequalities (4), (5), (3), and the definition of $f$, we obtain

$$\Big|f(a,b) - f(a,c)\Big|_1 \leq \sum_{x\in Fl(a,v)} \sum_{y\in Fl(a,w)} \alpha_x\beta_y \Big|f(a,x) - f((a,y)\Big|_1 =$$

$$= \sum_{x\in Fl(a,v)} \sum_{y\in Fl(a,w)} \alpha_x\beta_y \left|\frac{1}{\#Fl(a,x)} \sum_{x'\in Fl(a,x)} f(a,x') - \frac{1}{\#Fl(a,y)} \sum_{y'\in Fl(a,y)} f(a,y')\right|_1 =$$

$$= \sum_{x\in Fl(a,v)} \sum_{y\in Fl(a,w)} \frac{\alpha_x\beta_y}{\#Fl(a,x)\cdot\#Fl(a,y)} \left|\sum_{x'\in Fl(a,x)} \sum_{y'\in Fl(a,y)} \Big[f(a,x') - f(a,y')\Big]\right|_1 \leq$$

$$\leq \sum_{x\in Fl(a,v)} \sum_{y\in Fl(a,w)} \frac{\alpha_x\beta_y}{\#Fl(a,x)\cdot\#Fl(a,y)} \sum_{x'\in Fl(a,x)} \sum_{y'\in Fl(a,y)} \Big|f(a,x') - f(a,y')\Big|_1 \leq$$

$$\leq \sum_{x\in Fl(a,v)} \sum_{y\in Fl(a,w)} \frac{\alpha_x\beta_y}{\#Fl(a,x)\cdot\#Fl(a,y)} \sum_{x'\in Fl(a,x)} \sum_{y'\in Fl(a,y)} 2\left(1 - \frac{1}{\omega^2}\right)^{m-2} =$$

$$= 2\left(1 - \frac{1}{\omega^2}\right)^{m-2} \leq 2\left(1 - \frac{1}{\omega^2}\right)^{\left[\frac{(b|c)_a}{10\delta} - 1\right] - 2} = 2\left(1 - \frac{1}{\omega^2}\right)^{-3} \cdot \left(1 - \frac{1}{\omega^2}\right)^{\frac{(b|c)_a}{10\delta}} =$$

$$= L\lambda^{(b|c)_a}.$$

Proposition 2 is proved. □

Now we use the function $f$ to construct another function $\bar{f}$ having additional properties.

**Proposition 4.** *There exists a function $\bar{f} : X^{(0)} \times X^{(0)} \to C_0(X, \mathbb{Q})$ mapping each pair $(a,b)$ to a 0-chain $\bar{f}(a,b)$ with the following properties:*

1. *$\bar{f}(a,b)$ is a convex combination.*
2. *If $d(a,b) \geq 10\delta$, then $\operatorname{supp} \bar{f}(a,b) \subseteq B\big(p[a,b](10\delta), 8\delta\big)$.*
3. *If $d(a,b) \leq 10\delta$, then $\operatorname{supp} \bar{f}(a,b) \subseteq B(b, 7\delta)$.*
4. *$\bar{f}$ is $G$-equivariant, i.e. $\bar{f}(g\cdot a, g\cdot b) = g\cdot \bar{f}(a,b)$ for any $a, b \in X^{(0)}$ and $g \in G$.*
5. *There exist constants $L \geq 0$ and $0 \leq \lambda < 1$ such that, for any $a, b, c \in X^{(0)}$,*

$$\Big|\bar{f}(a,b) - \bar{f}(a,c)\Big|_1 \leq L\lambda^{(b|c)_a}.$$

(6) *There exists a constant* $0 \le \lambda' < 1$ *such that if* $a, b, c \in X^{(0)}$ *satisfy* $(a|b)_c \le 10\delta$ *and* $(a|c)_b \le 10\delta$, *then*

$$\left|\bar{f}(b,a) - \bar{f}(c,a)\right|_1 \le 2\lambda'.$$

(7) *Let* $a, b, c \in X^{(0)}$, $\gamma$ *be a geodesic path from* $a$ *to* $b$, *and let* $c \in N(\gamma, 9\delta)$. *Then* $supp\, \bar{f}(c,a) \subseteq N(\gamma, 9\delta)$.

*Proof.* For each $a \in X^{(0)}$ in $X$ we define a 0-chain $star(a)$ by

$$star(a) := \frac{1}{\#B(a, 7\delta)} \sum_{x \in B(a,7\delta)} x.$$

In other words, $star(a)$ is "the uniform spread" of $a$ to all the vertices that are $7\delta$-close to $a$. Also, $star(a)$ makes sense if $a$ is any 0-chain, by linearity:

$$star\left(\sum_{x\in X^{(0)}} \alpha_x x\right) := \sum_{x \in X^{(0)}} \alpha_x star(x).$$

One easily checks the following properties.

- If $a$ is a convex combination, then $star(a)$ is as well.
- $supp\, star(a)$ lies in the $7\delta$-neighborhood of $supp\, a$, for any 0-chain $a$.
- $star$ is a linear operator $C_0(X, \mathbb{Q}) \to C_0(X, \mathbb{Q})$, i.e.
$$star(a) + star(b) = star(a+b)$$
for any 0-chains $a$ and $b$.
- This operator is of norm 1, i.e.
$$|star(a)|_1 \le |a|_1$$
for any 0-chain $a$.
- $star$ is $G$-equivariant, i.e.
$$star(g \cdot a) = g \cdot star(a)$$
for any 0-chain $a$ and any $g \in G$.

Now let $f$ be the function from Proposition 2 and for $a, b \in X^{(0)}$ define

$$\bar{f}(a,b) := star\big(f(a,b)\big).$$

The properties of $star$ above and parts (1), (2), (3), (4), (5) of Proposition 2 imply parts (1), (2), (3), (4), (5) of Proposition 4. We show Proposition 4(6) now.

Let $\omega_7 := \max\{\#B(v, 7\delta) \mid v \in X^{(0)}\}$, and $\lambda' := 1 - \frac{1}{\omega_7^2}$. We have $1 \le \omega_7 < \infty$, and hence $0 \le \lambda' < 1$. Let us assume that $a, b, c \in X^{(0)}$ satisfy the hypotheses

$$(a|b)_c \le 10\delta \quad \text{and} \quad (a|c)_b \le 10\delta.$$

This implies that

$$d(b,c) = (a|b)_c + (a|c)_b \le 20\delta. \tag{6}$$

Without loss of generality, $d(a,b) \leq d(a,c)$ (interchange $b$ and $c$ otherwise). Additionally we assume for the moment that

$$d(a,b) \geq 10\delta.$$

Let $v := p[b,a](10\delta)$. By Proposition 2(2),

$$supp\, f(b,a) \subseteq Fl(b,v),$$

hence

$$f(b,a) = \sum_{x \in Fl(b,v)} \alpha_x x,$$

where $\alpha_x$ are some non-negative coefficients summing up to 1. Analogously,

$$f(c,a) = \sum_{y \in Fl(b,w)} \beta_y y,$$

where $w := p[c,a](10\delta)$ and $\beta_y$ are some non-negative coefficients summing up to 1. (See Fig. 3)

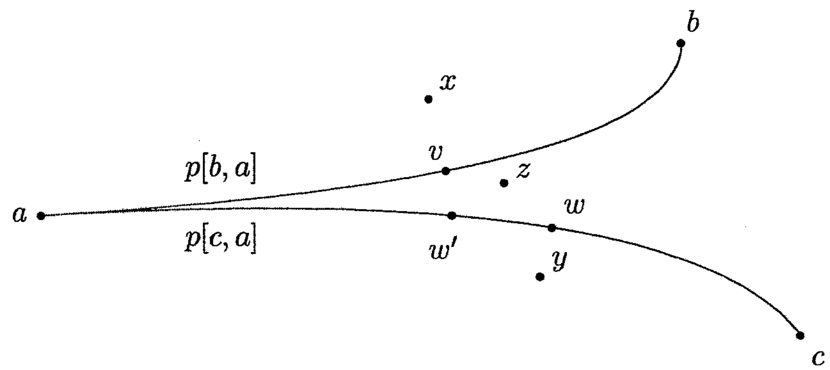


FIGURE 3. Proof of Proposition 4(6).

Then we have

$$\begin{aligned}
&\left|\bar{f}(b,a)-\bar{f}(c,a)\right|_1 = \left|star(f(b,a))-star(f(c,a))\right|_1 = \\
= \quad &\left|star\left(\sum_{x\in Fl(b,v)}\alpha_x x\right)-star\left(\sum_{y\in Fl(c,w)}\beta_y y\right)\right|_1 = \\
(7) \quad = \quad &\left|\sum_{x\in Fl(b,v)}\alpha_x star(x)-\sum_{y\in Fl(c,w)}\beta_y star(y)\right|_1 = \\
= \quad &\left|\sum_{x\in Fl(b,v)}\alpha_x star(x)\cdot\sum_{y\in Fl(c,w)}\beta_y-\sum_{x\in Fl(b,v)}\alpha_x\cdot\sum_{y\in Fl(c,w)}\beta_y star(y)\right|_1 = \\
= \quad &\left|\sum_{x\in Fl(b,v)}\sum_{y\in Fl(c,w)}\alpha_x\beta_y\Big[star(x)-star(y)\Big]\right|_1 \leq \\
\leq \quad &\sum_{x\in Fl(b,v)}\sum_{y\in Fl(c,w)}\alpha_x\beta_y\left|star(x)-star(y)\right|_1.
\end{aligned}$$

Let $w'$ be the vertex on the geodesic $p[c,a]$ satisfying $d(a,w')=d(a,v)$. We have

$$d(a,w')=d(a,v)=d(a,b)-d(b,v)=d(a,b)-10\delta\leq d(a,c)-10\delta=d(a,w),$$

hence, using inequality (6),

$$\begin{aligned}
d(a,w')=d(a,v)&\leq\frac{1}{2}\Big[\big(d(a,b)-10\delta\big)+\big(d(a,c)-10\delta\big)\Big]= \\
&=\frac{1}{2}\Big[d(a,b)+d(a,c)-20\delta\Big]\leq\frac{1}{2}\Big[d(a,b)+d(a,c)-d(b,c)\Big]=(a|b)_c,
\end{aligned}$$

therefore, by the fine-triangles property, $d(v,w')\leq\delta$. Also

$$\begin{aligned}
d(w',w)&=d(a,c)-d(c,w)-d(a,w')=d(a,c)-10\delta-d(a,v)= \\
&=d(a,c)-d(a,b)=\Big[(a|b)_c+(b|c)_a\Big]-\Big[(a|c)_b+(b|c)_a\Big]= \\
&=(a|b)_c-(a|c)_b\leq(a|b)_c\leq 10\delta.
\end{aligned}$$

So we have

$$d(v,w)\leq d(v,w')+d(w',w)\leq\delta+10\delta=11\delta.$$

If $x\in Fl(b,v)$ and $y\in Fl(c,w)$, then using the last formula we get

$$d(x,y)\leq d(x,v)+d(v,w)+d(w,y)\leq\delta+11\delta+\delta=13\delta.$$

This implies that, for each such a pair of vertices $x$ and $y$, there is a vertex $z\in B(x,7\delta)\cap B(y,7\delta)$. (Take $z$ to be a vertex on a geodesic edge path between $x$ and $y$ nearest to the midpoint.) Then we have

$$\begin{aligned}
&\left|star(x) - star(y)\right|_1 = \\
&= \left|\frac{1}{\#B(x,7\delta)} \sum_{x'\in B(x,7\delta)} x' - \frac{1}{\#B(y,7\delta)} \sum_{y'\in B(y,7\delta)} y'\right|_1 \leq \\
&\leq \left|\frac{1}{\#B(x,7\delta)\cdot\#B(y,7\delta)} \sum_{x'\in B(x,7\delta)} \sum_{y'\in B(y,7\delta)} [x'-y']\right|_1 = \\
&= \frac{1}{\#B(x,7\delta)\cdot\#B(y,7\delta)} \sum_{x'\in B(x,7\delta)} \sum_{y'\in B(y,7\delta)} |x'-y'|_1 \leq \\
&\leq \frac{1}{\#B(x,7\delta)\cdot\#B(y,7\delta)} \cdot 2\Big(\#B(x,7\delta)\cdot\#B(y,7\delta) - 1\Big) = \\
&= 2\left(1 - \frac{1}{\#B(x,7\delta)\cdot\#B(y,7\delta)}\right) \leq 2\left(1 - \frac{1}{\omega_7^2}\right) = 2\lambda'.
\end{aligned} \tag{8}$$

Combining inequalities (7) and (8) we obtain

$$\left|\bar{f}(b,a) - \bar{f}(c,a)\right|_1 \leq \sum_{x\in Fl(b,v)} \sum_{y\in Fl(c,w)} \alpha_x\beta_y \cdot 2\lambda' = 2\lambda'.$$

This was proved assuming that $d(a,b) \geq 10\delta$. Also recall that $d(a,b) \leq d(a,c)$ holds.

If $d(a,b) \leq d(a,c) \leq 10\delta$, then take $v := w := a$. If $d(a,b) \leq 10\delta \leq d(a,c)$, then take $v := a$ and $w := p[c,a](10\delta)$. In the latter case we have

$$d(v,w) = d(a,w) = d(a,c) - d(c,w) = d(a,c) - 10\delta \leq d(a,c) - (a|b)_c = (b|c)_a \leq 10\delta.$$

Therefore $d(v,w) \leq 10\delta$ in either case, hence, for any $x \in Fl(b,v)$ and any $y \in Fl(c,w)$,

$$d(y,x) \leq 10\delta + 2\delta = 12\delta,$$

so the same argument using formulas (7) and (8) works. Part (6) is proved.

Part (7) of Proposition 4 is almost immediate. If $d(a,c) \leq 10\delta$, then $supp\,\bar{f}(c,a) \subseteq B(a,7\delta) \subseteq N(\gamma,9\delta)$ by Proposition 4(3). Suppose now $d(a,c) > 10\delta$. Let $b'$ be the vertex on $\gamma$ with $d(b',c) \leq 9\delta$. Let also $v := p[c,a](10\delta)$ and $w$ be the vertex on $\gamma$ with $d(a,w) = d(a,v)$. Such a vertex $w$ always exists because

$$d(a,b') \geq d(a,c) - d(c,b') \geq d(a,c) - 9\delta \geq d(a,c) - 10\delta = d(a,v)$$

(See Fig 4). Then

$$\begin{aligned}
&d(a,w) = d(a,v) = d(a,c) - 10\delta = \frac{1}{2}\Big[d(a,c) + d(a,c) - 20\delta\Big] \leq \\
&\leq \frac{1}{2}\Big[d(a,c) + d(a,b') + d(b',c) - 20\delta\Big] \leq \frac{1}{2}\Big[d(a,c) + d(a,b') - d(b',c)\Big] = (b|c)_a,
\end{aligned}$$

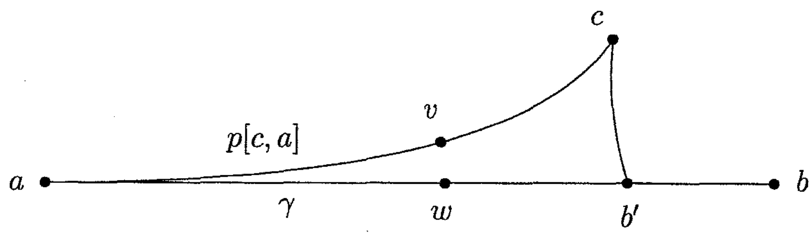


FIGURE 4. Proof of (7).

and, by fine-triangles property, $d(v,w) \leq \delta$. Since $supp\, \bar{f}(c,a) \subseteq B(v, 8\delta)$, then $supp\, \bar{f}(c,a) \subseteq B(w, 9\delta) \subseteq B(\gamma, 9\delta)$. Proposition 4 is proved. □

## 4. STRAIGHTENING

First we will construct a homological bicombing $q'$ in $X$ having certain properties. Recall that $p$ was a choice of a geodesic bicombing in $X$. The notation $p[a,b]$ makes sense not only when $a$ and $b$ are vertices in $X$, but it also can be defined when $a$ is any 0-chain, by linearity:

$$p\left[\sum_{x \in X^{(0)}} \alpha_x x,\ b\right] := \sum_{x \in X^{(0)}} \alpha_x p[x,b].$$

One easily checks that $\partial[a,b] = b - a$ if $a$ is a convex combination.

Fix a vertex $a$ in $X$. The 1-chain $q'[a,b]$ is defined inductively on $d(a,b)$. If $b$ is a vertex with $d(a,b) \leq 10\delta$, put $q'[a,b] := p[a,b]$. Assume now that $d(a,b) > 10\delta$. By Proposition 4(2),

$$supp\, \bar{f}(b,a) \subseteq B(p[b,a](10\delta), 8\delta),$$

hence, for each vertex $x \in supp\, \bar{f}(b,a)$,

$$d(a,x) \leq d(a, p[b,a](10\delta)) + d(p[b,a](10\delta), x) \leq \left[d(a,b) - 10\delta\right] + 8\delta < d(a,b),$$

so $q'[a,x]$ is defined by the induction hypotheses. Now we define $q'[a, \bar{f}(b,a)]$ by linearity over the second variable, and put

$$q'[a,b] := q'[a, \bar{f}(b,a)] + p[\bar{f}(b,a), b].$$

One easily checks that $\partial q'[a,b] := b - a$, so $q'$ is a homological bicombing in $X$.

**Proposition 5.** *The $\mathbb{Q}$-bicombing $q'$ constructed above satisfies the following conditions.*

(1) *$q'$ is $G$-equivariant.*

(2) *$q'$ is quasigeodesic.*
(3) *There exist constants $M \geq 0$ and $N \geq 0$ such that*

$$\left|q'[a,b] - q'[a,c]\right|_1 \leq M\, d(b,c) + N$$

*for all $a, b, c \in X^{(0)}$.*

*Proof.* **(1)** is obvious because the definition of $q'$ used $p$ and $\bar{f}$, and they are $G$-equivariant.

**(2)** First we define a sequence of sets of vertices $V_i(a,b)$ for each pair $a, b \in X^{(0)}$. Put $V_0(a,b) := \{b\}$ and

$$V_{i+1}(a,b) := V_i(a,b) \cup \bigcup_{c \in V_i(a,b)} supp\, \bar{f}(c,a).$$

This sequence is increasing and stabilizes at a certain vertex set which we denote by $V(a,b)$. Tracing the definitions of $q'[a,b]$ and $V(a,b)$ we see that $q'[a,b]$ is a linear combination of geodesic paths of length at most $10\delta$ whose endpoints lie in $V(a,b)$. Hence, to show that $q'$ is quasigeodesic, it is enough to show that $V(a,b)$ lies close to $p[a,b]$.

We prove that $V_i(a,b) \subseteq N(p[a,b], 9\delta)$ inductively on $i$. Firstly, $V_0(a,b) = \{b\} \subseteq N(p[a,b], 9\delta)$. Secondly, if $V_i(a,b) \subseteq N(p[a,b], 9\delta)$, then, by Proposition 4(7),

$$V_{i+1}(a,b) = V_i(a,b) \cup \bigcup_{c \in V_i(a,b)} supp\, \bar{f}(c,a) \subseteq N(p[a,b], 9\delta).$$

This implies $V(a,b) \subseteq N(p[a,b], 9\delta)$, so part (2) is proved.

**(3)** Up to the $G$-action, there are only finitely many triples of vertices $a$, $b$, $c$, satisfying $d(a,b) + d(a,c) \leq 60\delta$, hence there exists a uniform bound $N'$ for the norms

$$\left|q'[a,b] - q'[a,c]\right|_1$$

for such vertices $a$, $b$, and $c$. Let

$$M := 18\delta \quad \text{and} \quad N := \max\left\{N', \frac{56\delta M}{1-\lambda'}\right\},$$

where $\lambda'$ is the constant from Proposition 4(6). We prove the statement by induction on $d(a,b) + d(a,c)$.

If $d(a,b) + d(a,c) \leq 60\delta$, then

$$\left|q'[a,b] - q'[a,c]\right|_1 \leq N' \leq N \leq M\, d(b,c) + N$$

by the choice of $N'$ and $N$. We assume now that $d(a,b) + d(a,c) > 60\delta$. Consider the following two cases.

<u>*Case 1.*</u> $(a|c)_b > 10\delta$ *or* $(a|b)_c > 10\delta$.
Assume, for example, that $(a|c)_b > 10\delta$. Then, in particular, $d(a,b) > 10\delta$, hence, by definition,

$$q'[a,b] = q'[a, \bar{f}(b,a)] + p[\bar{f}(b,a), b]$$

and $supp\,\bar{f}(b,a) \subseteq B(v, 8\delta)$, where $v := p[b,a](10\delta)$. Also, $d(b,c) \geq (a|c)_b > 10\delta$, so there exists a geodesic $\gamma$ between $b$ and $c$, and a vertex $v'$ on $\gamma$ with $d(b,v') = d(b,v) = 10\delta$. By the fine-triangles property, $d(v,v') \leq \delta$. If $x \in supp\,\bar{f}(b,a)$, then

$$d(x,b) \leq d(x,v) + d(v,b) \leq 8\delta + 10\delta = 18\delta, \tag{9}$$

$$d(x,c) \leq d(x,v) + d(v,v') + d(v',c) \leq 8\delta + \delta + \big[d(c,b) - 10\delta\big] \leq d(c,b) - 1,$$

and

$$d(a,x) \leq d(a,v) + d(v,x) \leq \big[d(a,b) - 10\delta\big] + 8\delta < d(a,b),$$

therefore $d(a,x) + d(a,c) < d(a,b) + d(a,c)$, so the induction hypotheses apply to the vertices $a$, $x$, and $c$, giving

$$\Big|q'[a,x] - q'[a,c]\Big|_1 \leq M\,d(x,c) + N \leq M\big(d(b,c) - 1\big) + N = M\,d(b,c) - M + N. \tag{10}$$

For some non-negative coefficients $\alpha_x$ summing up to 1,

$$\bar{f}(b,a) = \sum_{x \in B(v,8\delta)} \alpha_x x.$$

Then, by the definition of $q'[a,b]$ and inequalities (9) and (10),

$$\begin{aligned}
&\Big|q'[a,b] - q'[a,c]\Big|_1 = \Big|q'[a,\bar{f}(b,a)] + p[\bar{f}(b,a),b] - q'[a,c]\Big|_1 = \\
&= \Bigg|\sum_{x\in B(v,8\delta)} \alpha_x q'[a,x] + \sum_{x\in B(v,8\delta)} \alpha_x p[x,b] - q'[a,c]\Bigg|_1 \leq \\
&\leq \Bigg|\sum_{x\in B(v,8\delta)} \alpha_x\big(q'[a,x] - q'[a,c]\big)\Bigg|_1 + \Bigg|\sum_{x\in B(v,8\delta)} \alpha_x p[x,b]\Bigg|_1 \leq \\
&\leq \sum_{x\in B(v,8\delta)} \alpha_x \Big|q'[a,x] - q'[a,c]\Big|_1 + \sum_{x\in B(v,8\delta)} \alpha_x \big|p[x,b]\big|_1 \leq \\
&\leq \sum_{x\in B(v,8\delta)} \alpha_x \cdot \big(M\,d(b,c) - M + N\big) + \sum_{x\in B(v,8\delta)} \alpha_x\, d(x,b) \leq \\
&\leq M\,d(b,c) - M + N + 18\delta = M\,d(b,c) + N.
\end{aligned}$$

*Case 2.* $(a|c)_b \leq 10\delta$ *and* $(a|b)_c \leq 10\delta$.
In this case Proposition 4(6) applies. Since $d(a,b) + d(a,c) > 60\delta$ and $d(b,c) = (a|c)_b + (a|b)_c \leq 20\delta$, then $d(a,b) > 10\delta$ and $d(a,c) > 10\delta$. Then, by the definition of $q'[a,b]$ and

$q'[a,c]$,

$$\text{(11)}\quad \left|q'[a,b]-q'[a,c]\right|_1 = \left|q'[a,\bar f(b,a)] + p[\bar f(b,a),b] - q'[a,\bar f(c,a)] - p[\bar f(c,a),c]\right|_1 \le$$
$$\le \left|q'[a,\bar f(b,a)] - q'[a,\bar f(c,a)]\right|_1 + \left|p[\bar f(b,a),b]\right|_1 + \left|p[\bar f(c,a),c]\right|_1 =$$
$$= \left|q'[a,\bar f(b,a) - \bar f(c,a)]\right|_1 + \left|p[\bar f(b,a),b]\right|_1 + \left|p[\bar f(c,a),c]\right|_1 .$$

The 0-chain $\bar f(b,a) - \bar f(c,a)$ (as any other) can be represented in the form $f_+ - f_-$, where $f_+$ and $f_-$ are 0-chains with non-negative coefficients and disjoint supports. By Proposition 4(6),

$$|f_+|_1 + |f_-|_1 = |f_+ - f_-|_1 = \left|\bar f(b,a) - \bar f(c,a)\right|_1 \le 2\lambda'.$$

The coefficients of the 0-chain

$$f_+ - f_- = \bar f(b,a) - \bar f(c,a)$$

sum up to 0, because $\bar f(b,a)$ and $\bar f(c,a)$ are convex combinations. It follows that

$$|f_+|_1 = |f_-|_1 \le \lambda'.$$

Also,

$$supp\, f_+ \subseteq supp\, \bar f(b,a) \subseteq B(p[b,a](10\delta), 8\delta)$$

and

$$supp\, f_- \subseteq supp\, \bar f(c,a) \subseteq B(p[c,a](10\delta), 8\delta),$$

hence, for each $x \in supp\, f_+$ and $y \in supp\, f_-$, we have

$$d(x,y) \le d(x,b) + d(b,c) + d(c,y) \le 18\delta + 20\delta + 18\delta = 56\delta.$$

Also $d(a,x) < d(a,b)$ and $d(a,y) < d(a,c)$, so, by the induction hypotheses for the vertices $a$, $x$, and $y$,

$$\left|q'[a,x] - q'[a,y]\right|_1 \le M\, d(x,y) + N \le M\big[d(b,c) + 36\delta\big] + N = M\, d(b,c) + 36\delta M + N$$

for each $x \in supp\, f_+$ and $y \in supp\, f_-$. Then we continue inequality (11):

$$\left|q'[a,b]-q'[a,c]\right|_1 \le \left|q'[a,\bar f(b,a) - \bar f(c,a)]\right|_1 + \left|p[\bar f(b,a),b]\right|_1 + \left|p[\bar f(c,a),c]\right|_1 =$$
$$= \left|q'[a,f_+] - q'[a,f_-]\right|_1 + \left|p[\bar f(b,a),b]\right|_1 + \left|p[\bar f(c,a),c]\right|_1 \le$$
$$\le \lambda' \cdot \big[M \cdot 56\delta + N\big] + 18\delta + 18\delta =$$
$$= \big[M \cdot 56\delta + N\big] + (\lambda' - 1)\big[M \cdot 56\delta + N\big] + 36\delta \le$$
$$\le M \cdot 56\delta + N + (\lambda' - 1)N \le N \le M\, d(b,c) + N.$$

Proposition 5 is proved. □

**Lemma 6.** *There exist constants $K \geq 0$ and $0 \leq \lambda < 1$ such that if $a', a, b, c \in X^{(0)}$, $z \in X^{(1)}$ is a center of the triple $\{a, b, c\}$, and $a' \in N(p[z, a], 10\delta)$, then*

$$\left| q'[b, a'] - q'[c, a'] - q'[b, z] + q'[c, z] \right|_1 \leq K(1 + \lambda + \lambda^2 + \dots + \lambda^{d(z,a')}).$$

*Proof.* Let $L$ and $\lambda$ be the constants from Proposition 4(5) and $M$ and $N$ be the constants from Proposition 5(3). We take $K$ to be sufficiently large, namely

$$K := \max\{44\delta M + 2N,\ L\lambda^{-4\delta}(26\delta M + N + 18\delta)\}.$$

Note that $K$ and $\lambda$ are universal constants, i.e. they depend only on the choice of $X$.

We prove the lemma by induction on $d(z, a')$. If $d(z, a') \leq 22\delta$, then, by Proposition 5(3),

$$\left| q'[b, a'] - q'[c, a'] - q'[b, z] + q'[c, z] \right|_1 \leq \left| q'[b, a'] - q'[b, z] \right|_1 + \left| q'[c, a'] - q'[c, z] \right|_1 \leq$$
$$\leq [M\, d(z, a') + N] + [M\, d(z, a') + N] \leq 44\delta M + 2N \leq K \leq K(1 + \dots + \lambda^{d(z,a')}).$$

We now assume that $d(z, a') \geq 22\delta$. Since $z$ is a center of $\{a, b, c\}$, there exist a geodesic $\gamma$ from $b$ to $a$ and a point $\bar{c} \in \gamma$ with $d(a, \bar{c}) = (b|c)_a$ and $d(z, \bar{c}) \leq \delta$ (see Fig. 5). Denote $v := p[a', b](10\delta)$ and pick an arbitrary $x \in B(v, 8\delta)$. We want to use the induction hypotheses for the vertex $x$, so our first goal is to show that $d(z, x) < d(z, a')$ and $x \in N(p[z, a], 10\delta)$. This will be possible to do because $d(z, a')$ is large enough.

Let $u$ be a vertex on $p[z, a]$ with $d(a', u) \leq 10\delta$. Then

$$d(z, u) \geq d(z, a') - d(a', u) \geq 22\delta - 10\delta \geq \delta \geq d(z, \bar{c}) \geq (a|\bar{c})_z,$$

hence

$$d(a, u) = d(a, z) - d(z, u) = [(z|\bar{c})_a + (a|\bar{c})_z] - d(z, u) \leq (z|\bar{c})_a.$$

The last inequality implies that there is a vertex $u'$ on $\gamma$ with $d(a, u') = d(a, u)$ and, by the fine-triangles property, $d(u, u') \leq \delta$. This implies that

$$d(a', u') \leq d(a', u) + d(u, u') \leq 10\delta + \delta = 11\delta,$$
$$d(a, \bar{c}) \geq d(a, z) - d(z, \bar{c}) \geq [d(a, u) + d(u, z)] - \delta \geq d(a, u) + [d(z, a') - d(a', u)] - \delta \geq$$
$$\geq d(a, u) + 22\delta - 10\delta - \delta = d(a, u) + 11\delta = d(a, u') + 11\delta.$$

This means that $\bar{c}$ lies between $b$ and $u'$ on the geodesic $\gamma$ and

$$d(u', \bar{c}) = d(a, \bar{c}) - d(a, u') \geq 11\delta.$$

Further,

$$d(b, \bar{c}) = d(b, u') - d(u', \bar{c}) \leq d(b, u') - 11\delta \leq$$
$$\leq d(b, u') - d(a', u') \leq d(b, u') - (a'|b)_{u'} = (a'|u')_b.$$

Therefore there exists a point $r$ on $p[a', b]$ with $d(b, r) = d(b, \bar{c})$ and, by the fine-triangles property, $d(r, \bar{c}) \leq \delta$, so

$$d(z, r) \leq d(z, \bar{c}) + d(\bar{c}, r) \leq \delta + \delta = 2\delta. \tag{12}$$

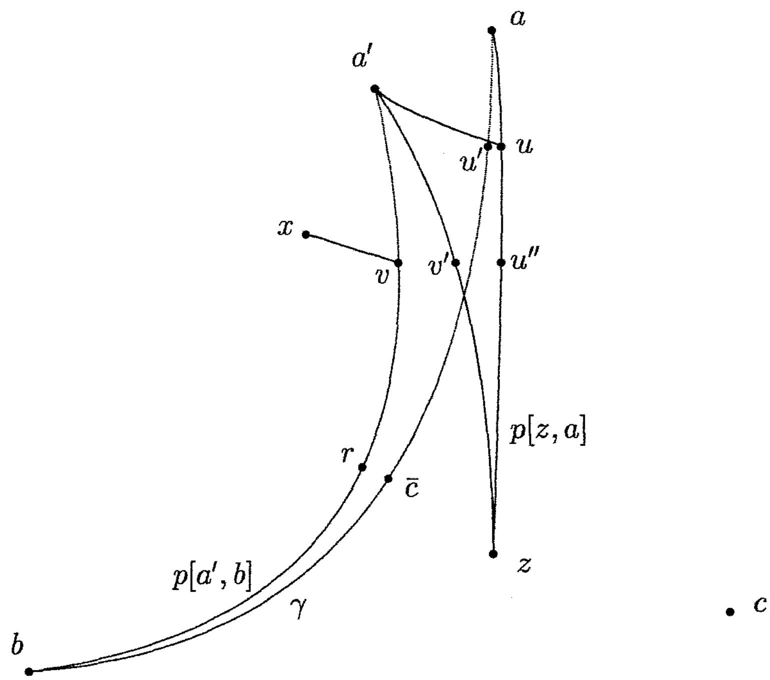


FIGURE 5. Proof of Lemma 6.

Recall that $v$ was defined as $p[a', b](10\delta)$, then we have

$$d(a', v) = 10\delta \leq 22\delta - 2\delta \leq d(a', z) - d(z, r) \leq d(a', z) - (a'|r)_z = (z|r)_{a'},$$

so there exists a vertex $v'$ on $p[z, a']$ with $d(a', v') = d(a', v) = 10\delta$ and, by the fine-triangles property, $d(v, v') \leq \delta$.

$$(z|u)_{a'} = \frac{1}{2}\Big[d(z, a') + d(a', u) - d(z, u)\Big] \leq \frac{1}{2}\Big[d(z, a') + d(a', u) - d(z, a') + d(a', u)\Big] =$$
$$= d(a', u) \leq 10\delta = d(a', v'),$$

then

$$d(z, v') = d(z, a') - d(a', v') \leq d(z, a') - (z|u)_{a'} = (a'|u)_z,$$

hence there exists a vertex $u'' \in p[z, a]$ with $d(z, u'') = d(z, v')$, and, by the fine-triangles property, $d(v', u'') \leq \delta$.

For any vertex $x \in B(v, 8\delta)$,

$$d(u'', x) \leq d(u'', v') + d(v', v) + d(v, x) \leq \delta + \delta + 8\delta = 10\delta,$$

so

$$x \in N(u'', 10\delta) \subseteq N(p[z, a], 10\delta)$$

and

$$d(z,x) \leq d(z,v') + d(v',x) = [d(z,a') - 10\delta] + d(v',x) \leq$$
$$\leq [d(z,a') - 10\delta] + 9\delta \leq d(z,a') - 1.$$

The last two formulas say that each vertex $x \in B(v, 8\delta)$ satisfies the induction hypotheses, so

$$\left| q'[b,x] - q'[c,x] - q'[b,z] + q'[c,z] \right|_1 \leq K(1 + \lambda + \lambda^2 + \ldots + \lambda^{d(z,x)}) \leq \tag{13}$$
$$\leq K(1 + \lambda + \lambda^2 + \ldots + \lambda^{d(z,a')-1}).$$

The convex combinations $\bar{f}(a',b)$ and $\bar{f}(a',c)$ have form

$$\bar{f}(a',b) = \sum_{x \in X^{(0)}} \alpha'_x x \quad \text{and} \quad \bar{f}(a',c) = \sum_{x \in X^{(0)}} \alpha''_x x$$

for some coefficients $\alpha'_x$ and $\alpha''_x$. Define a 0-chain $f_0$ by

$$f_0 := \sum_{x \in X^{(0)}} \alpha_x x,$$

where $\alpha_x := \min\{\alpha'_x, \alpha''_x\}$. Put

$$f_+ := \bar{f}(a',b) - f_0 \quad \text{and} \quad f_- := \bar{f}(a',c) - f_0.$$

Then we have

$$supp\, f_0 = supp\, \bar{f}(a',b) \cap supp\, \bar{f}(a',c),$$

and $f_+$ and $f_-$ are with non-negative coefficients and disjoint supports. Also

$$\bar{f}(a',b) - \bar{f}(a',c) = f_+ - f_-,$$

hence the coefficients of $f_+ - f_-$ sum up to 0, so $|f_+|_1 = |f_-|_1$. By Proposition 4(5),

$$|f_+|_1 + |f_-|_1 = |f_+ - f_-|_1 = \left| \bar{f}(a',b) - \bar{f}(a',c) \right|_1 \leq L\lambda^{(b|c)_{a'}},$$

We recently proved the existence of a vertex $r \in p[a',b]$ which is $2\delta$-close to $z$ (see inequality (12)). The same argument with $c$ in place of $b$ shows that there exists a vertex

$s \in p[a', c]$ which is $2\delta$-close to $z$. It follows that

$$
\begin{aligned}
(b|c)_{a'} &= \frac{1}{2}\big[d(a',b) + d(a',c) - d(b,c)\big] \geq \\
&\geq \frac{1}{2}\Big[d(a',b) + d(a',c) - \big(d(b,r) + d(r,z) + d(z,s) + d(s,c)\big)\Big] = \\
&= \frac{1}{2}\Big[\big(d(a',b) - d(b,r)\big) + \big(d(a',c) - d(c,s)\big) - d(r,z) - d(z,s)\Big] = \\
&= \frac{1}{2}\big[d(a',r) + d(a',s) - d(r,z) - d(z,s)\big] \geq \\
&\geq \frac{1}{2}\Big[\big(d(a',z) - d(z,r)\big) + \big(d(a',z) - d(z,s)\big) - d(r,z) - d(z,s)\Big] \geq \\
&\geq \frac{1}{2}\big[d(a',z) - 2\delta + d(a',z) - 2\delta - 2\delta - 2\delta\big] = d(a',z) - 4\delta.
\end{aligned} \tag{14}
$$

Thus,

$$
|f_+|_1 = |f_-|_1 \leq \frac{1}{2}L\lambda^{(b|c)_{a'}} \leq \frac{1}{2}L\lambda^{d(z,a')-4\delta}. \tag{15}
$$

Since $d(z, a')$ is large enough, then $d(a', b) > 10\delta$ and $d(a', c) > 10\delta$, so, by the definition of $q'$, we have

$$
\begin{aligned}
&\Big|q'[b,a'] - q'[c,a'] - q'[b,z] + q'[c,z]\Big|_1 = \\
&= \Big|\big(q'[b,\bar{f}(a',b)] + p[\bar{f}(a',b),a']\big) - \big(q'[c,\bar{f}(a',c)] + p[\bar{f}(a',c),a']\big) - q'[b,z] + q'[c,z]\Big|_1 \leq \\
&\leq \Big|q'[b,\bar{f}(a',b)] - q'[c,\bar{f}(a',c)] - q'[b,z] + q'[c,z]\Big|_1 + \Big|p[\bar{f}(a',b),a'] - p[\bar{f}(a',c),a']\Big|_1 = \\
&= \Big|q'[b,f_0+f_+] - q'[c,f_0+f_-] - q'[b,z] + q'[c,z]\Big|_1 + \Big|p[\bar{f}(a',b),a'] - p[\bar{f}(a',c),a']\Big|_1 \leq \\
&\leq \Big|q'[b,f_0] - q'[c,f_0] - |f_0|_1 \cdot q'[b,z] + |f_0|_1 \cdot q'[c,z]\Big|_1 + \\
&+ \Big|q'[b,f_+] - q'[c,f_-] - (1-|f_0|_1)\cdot q'[b,z] + (1-|f_0|_1)\cdot q'[c,z]\Big|_1 + \\
&+ \Big|p[\bar{f}(a',b) - \bar{f}(a',c),a']\Big|_1.
\end{aligned}
$$

We are going to bound each of the three terms in the last sum, let us call them $A_1$, $A_2$, and $A_3$, respectively.

The 0-chain $f_0$ is supported in the ball $B(v, 8\delta)$, so

$$
f_0 = \sum_{x \in B(v,8\delta)} \alpha_x x,
$$

and for each $x \in B(v, 8\delta)$, inequality (13) holds, hence

$$\begin{aligned}
A_1 &\leq \left| q'\left[b, \sum_{x\in B(v,8\delta)} \alpha_x x\right] - q'\left[c, \sum_{x\in B(v,8\delta)} \alpha_x x\right] - |f_0|_1 \cdot q'[b,z] + |f_0|_1 \cdot q'[c,z]|_1 \right|_1 = \\
&= \left| \sum_{x\in B(v,8\delta)} \alpha_x \big(q'[b,x] - q'[c,x] - q'[b,z] + q'[c,z]\big) \right|_1 \leq \\
&\leq \sum_{x\in B(v,8\delta)} \alpha_x \cdot K(1+\lambda+\lambda^2+...+\lambda^{d(z,a')-1}) = \\
&= |f_0|_1 \cdot K(1+\lambda+\lambda^2+...+\lambda^{d(z,a')-1}).
\end{aligned}$$

For the second term, pick any $x_0 \in B(v, 8\delta)$, so inequality (13) holds for $x_0$ as well:

$$\begin{aligned}
&\left| q'[b,x_0] - q'[c,x_0] - q'[b,z] + q'[c,z] \right|_1 \leq K(1+\lambda+\lambda^2+...+\lambda^{d(z,x)}) \leq \\
&\leq K(1+\lambda+\lambda^2+...+\lambda^{d(z,a')-1}).
\end{aligned}$$

Informally speaking, we are going to move both $f_+$ and $f_-$ to $x_0$. As before, $v = p[a',b](10\delta)$, and we denote $w := p[a',c](10\delta)$. The 0-chains $f_+$ and $f_-$ have forms

$$f_+ = \sum_{x\in B(v,8\delta)} \beta'_x x \qquad \text{and} \qquad f_- = \sum_{y\in B(w,8\delta)} \beta''_y y.$$

Note that

$$|f_+|_1 = |f_-|_1 = |\bar{f}(a',c)|_1 - |f_0|_1 = 1 - |f_0|_1.$$

Also, for each $x \in B(v, 8\delta)$,

$$d(x,x_0) \leq d(x,v) + d(v,x_0) \leq 8\delta + 8\delta = 16\delta$$

and, for each $y \in B(w, 8\delta)$,

$$d(y,x_0) \leq d(y,w) + d(w,a') + d(a',v) + d(v,x_0) \leq 8\delta + 10\delta + 10\delta + 8\delta = 36\delta.$$

Using these observations, Proposition 5(3), and formula (15), we obtain a bound for the second term:

$$
\begin{aligned}
A_2 &= \Big|q'[b,f_+] - q'[c,f_-] - |f_+|_1\cdot q'[b,z] + |f_-|_1\cdot q'[c,z]\Big|_1 \le \\
&\le \Big|q'[b,f_+] - q'[c,f_-] - |f_+|_1\cdot q'[b,x_0] + |f_-|_1\cdot q'[c,x_0]\Big|_1 + \\
&+\Big||f_+|_1\cdot q'[b,x_0] - |f_-|_1\cdot q'[c,x_0] - |f_+|_1\cdot q'[b,z] + |f_-|_1\cdot q'[c,z]\Big|_1 = \\
&= \left|\sum_{x\in B(v,8\delta)} \beta'_x\big(q'[b,x]-q'[b,x_0]\big) - \sum_{y\in B(w,8\delta)} \beta''_y\big(q'[c,y]-q'[c,x_0]\big)\right|_1 + \\
&+|f_+|_1\cdot\Big|q'[b,x_0]-q'[c,x_0]-q'[b,z]-q'[c,z]\Big|_1 \le \\
&\le \sum_{x\in B(v,8\delta)} \beta'_x\Big|q'[b,x]-q'[b,x_0]\Big|_1 + \sum_{y\in B(w,8\delta)} \beta''_y\Big|q'[c,y]-q'[c,x_0]\Big|_1 + \\
&+|f_+|_1\cdot\Big|q'[b,x_0]-q'[c,x_0]-q'[b,z]+q'[c,z]\Big|_1 \le \\
&\le \sum_{x\in B(v,8\delta)} \beta'_x\cdot\big(M\cdot 16\delta+N\big) + \sum_{y\in B(w,8\delta)} \beta''_y\cdot\big(M\cdot 36\delta+N\big) + \\
&+|f_+|_1\cdot K(1+\lambda+\lambda^2+\ldots+\lambda^{d(z,a')-1}) = \\
&= |f_+|_1\big(M\cdot 52\delta+2N\big) + |f_+|_1\cdot K(1+\lambda+\lambda^2+\ldots+\lambda^{d(z,a')-1}) \le \\
&\le \frac{1}{2}L\lambda^{d(z,a')-4\delta}\cdot\big(M\cdot 52\delta+2N\big) + |f_+|_1\cdot K(1+\lambda+\lambda^2+\ldots+\lambda^{d(z,a')-1}) = \\
&= L\lambda^{d(z,a')-4\delta}\cdot\big(M\cdot 26\delta+N\big) + |f_+|_1\cdot K(1+\lambda+\lambda^2+\ldots+\lambda^{d(z,a')-1}).
\end{aligned}
$$

To bound the third term, note that, for any vertex $x \in supp\, \bar{f}(a',b) \cup supp\, \bar{f}(a',c)$,

$$\big|p[a',x]\big|_1 = d(a',x) \le 18\delta,$$

then, using Proposition 4(5) and formula (14),

$$
\begin{aligned}
A_3 &= \Big|p[\bar{f}(a',b)-\bar{f}(a',c),a']\Big|_1 \le \Big|\bar{f}(a',b)-\bar{f}(a',c)\Big|_1\cdot 18\delta \le \\
&\le L\lambda^{(b|c)_{a'}}\cdot 18\delta \le L\lambda^{d(z,a')-4\delta}\cdot 18\delta.
\end{aligned}
$$

Combining the three bounds and the definition of $K$, we obtain

$$\begin{aligned}
&\Big|q'[b,a'] - q'[c,a'] - q'[b,z] + q'[c,z]\Big|_1 \le A_1 + A_2 + A_3 \le \\
&\le |f_0|_1 \cdot K\big(1+\lambda+\lambda^2+\ldots+\lambda^{d(z,a')-1}\big) + \\
&+L\lambda^{d(z,a')-4\delta}\cdot\big(M\cdot 26\delta + N\big) + |f_+|_1\cdot K\big(1+\lambda+\lambda^2+\ldots+\lambda^{d(z,a')-1}\big) + \\
&+L\lambda^{d(z,a')-4\delta}\cdot 18\delta = \\
&= K\big(1+\lambda+\lambda^2+\ldots+\lambda^{d(z,a')-1}\big) + L\lambda^{-4\delta}(26\delta M + N + 18\delta)\lambda^{d(z,a')} \le \\
&\le K\big(1+\lambda+\lambda^2+\ldots+\lambda^{d(z,a')-1}\big) + K\lambda^{d(z,a')} = K\big(1+\lambda+\lambda^2+\ldots+\lambda^{d(z,a')}\big).
\end{aligned}$$

Lemma 6 is proved. □

Now we can state one of the main results of the paper.

**Theorem 7.** *Given a hyperbolic group $G$ and $X \in \mathcal{U}_\infty(G)$, there exists a $\mathbb{Q}$-bicombing $q$ in $X$ with the following properties.*

(1) *$q$ is quasigeodesic.*
(2) *$q$ is $G$-equivariant.*
(3) *$q$ is anti-symmetric, i.e. $q[a,b] = -q[b,a]$ for any $a, b \in X^{(0)}$.*
(4) *There exists a constant $T$ such that, for any $a, b, c \in X^{(0)}$,*

$$\Big|q[a,b] + q[b,c] + q[c,a]\Big|_1 \le T.$$

*Proof.* Define $q$ by "anti-symmeterizing" $q'$, namely,

$$q[a,b] := \frac{1}{2}\Big(q'[a,b] - q'[b,a]\Big).$$

The first three properties follow directly from the definition of $q$ and the fact that $q'$ is quasigeodesic and $G$-equivariant. Now we prove property (4).

Let $a$, $b$, and $c$ be arbitrary vertices in $X$, and $z$ be a center of the triple $\{a,b,c\}$. Then, by Lemma 6, taking $a' := a$,

$$\Big|q'[b,a] - q'[c,a] - q'[b,z] + q'[c,z]\Big|_1 \le K\big(1+\lambda+\lambda^2+\ldots+\lambda^{d(z,a')}\big) \le K\sum_{i=0}^{\infty}\lambda^i = \frac{K}{1-\lambda}.$$

The same argument for cyclic permutations of the vertices $a$, $b$, and $c$ yields

$$\Big|q'[c,b] - q'[a,b] - q'[c,z] + q'[a,z]\Big|_1 \le \frac{K}{1-\lambda}$$

and

$$\Big|q'[a,c] - q'[b,c] - q'[a,z] + q'[b,z]\Big|_1 \le \frac{K}{1-\lambda}.$$

The three inequalities above provide just what is needed:

$$\begin{aligned}
&\Big|q[a,b]+q[b,c]+q[c,a]\Big|_1 = \\
&= \frac{1}{2}\Big|\Big(q'[a,b]-q'[b,a]\Big)+\Big(q'[b,c]-q'[c,b]\Big)+\Big(q'[c,a]-q'[a,c]\Big)\Big|_1 = \\
&= \frac{1}{2}\Big|-\Big(q'[b,a]-q'[c,a]-q'[b,z]+q'[c,z]\Big)-\Big(q'[c,b]-q'[a,b]-q'[c,z]+q'[a,z]\Big)- \\
&-\Big(q'[a,c]-q'[b,c]-q'[a,z]+q'[b,z]\Big)\Big|_1 \leq \\
&\leq \frac{1}{2}\Big(\Big|q'[b,a]-q'[c,a]-q'[b,z]+q'[c,z]\Big|_1+\Big|q'[c,b]-q'[a,b]-q'[c,z]+q'[a,z]\Big|_1+ \\
&+\Big|q'[a,c]-q'[b,c]-q'[a,z]+q'[b,z]\Big|_1\Big) \leq \frac{1}{2}\cdot 3\frac{K}{1-\lambda},
\end{aligned}$$

so we put $T := \dfrac{3K}{2(1-\lambda)}$. Theorem 7 is proved. □

## 5. Bounded cohomology.

Recall that $X \in \mathcal{U}_\infty(G)$ and let $Y$ be the geometric realization of the homogeneous bar-construction for $G$. This means that $Y$ is the simplicial complex whose $k$-simplices are labeled by ordered $(k+1)$-tuples $(x_0, ..., x_k)$ of elements of the group $G$, and each simplex labeled $(x_0, ...x_{i-1}, \widehat{x_i}, x_{i+1}, ..., x_k)$ is identified with the $i$-th face of $(x_0, ..., x_k)$ in the obvious way. The action of $G$ on $Y$ is diagonal:

$$g\cdot(x_0,...,x_k) := (g\cdot x_0, ..., g\cdot x_k).$$

Let $\mathcal{C}^X$ and $\mathcal{C}^Y$ be the augmented chain complexes of cellular $\mathbb{Q}$-chains on $X$ and $Y$, respectively. This means that the complexes have $C_i(X,\mathbb{Q})$ and $C_i(Y,\mathbb{Q})$, respectively, in dimensions $i \geq 0$, $\mathbb{Q}$ in dimension $-1$, zeros in all the lower dimensions, and the boundary homomorphisms $\mathcal{C}^X_0 = C_0(X,\mathbb{Q}) \to \mathcal{C}^X_{-1} = \mathbb{Q}$ and $\mathcal{C}^Y_0 = C_0(Y,\mathbb{Q}) \to \mathcal{C}^Y_{-1} = \mathbb{Q}$ are the linear operators taking each 0-chain to the sum of its coefficients. Both $X$ and $Y$ are contractible, hence $\mathcal{C}^X$ and $\mathcal{C}^Y$ are acyclic. Both $\mathcal{C}^X$ and $\mathcal{C}^Y$ have free $\mathbb{Q}G$-modules in each non-negative dimension. Once again, $\mathcal{C}^X$ and $\mathcal{C}^Y$ are normed vector spaces with respect to the $\ell_1$-norm.

**Proposition 8.** *Given a hyperbolic group $G$ and chain complexes $\mathcal{C}^X$ and $\mathcal{C}^Y$ as above, there exist $G$-equivariant chain maps $\varphi_*: \mathcal{C}^Y \to \mathcal{C}^X$ and $\psi_*: \mathcal{C}^X \to \mathcal{C}^Y$ such that*

(1) *$\varphi_*$ and $\psi_*$ are identities in each negative dimension, and*
(2) *$\varphi_*$ is bounded in each dimension at least 2.*

*Remark.* The existence of $\varphi$ and $\psi$ satisfying condition (1) follows from standard arguments for *any* group $G$ (see below). Since $X \in \mathcal{U}_\infty(G)$, then it follows automatically that

$\psi_*$ is bounded in each dimension. Property (2) is what requires a new argument, and hyperbolicity of $G$ is essential here.

We give a formal homological proof, but the main idea is that, when $G$ is hyperbolic, it is possible to represent $k$-simplices of the bar-construction as $k$-chains in $X$ of bounded $\ell_1$-norm.

Recall the following standard fact from homological algebra (see [1, Lemma I.7.4] for the proof).

**Lemma 9.** *Suppose that $(\mathcal{C}, \partial)$ is a free chain complex, $(\mathcal{C}', \partial')$ is an acyclic chain complex, and homomorphisms $\psi_i : \mathcal{C}_i \to \mathcal{C}'_i$ are defined for $i \leq -1$ such that $\partial'_i \circ \psi_i = \psi_{i-1} \circ \partial_i$ for each $i \leq -1$. Then the maps $\psi_i$ extend to a chain map $\psi_* : \mathcal{C} \to \mathcal{C}'$. This extension is unique up to a chain homotopy.*

The following theorem was proved in [7] using [6, Theorem 5.4].

**Theorem 10.** *Hyperbolic groups satisfy linear isoperimetric inequalities in all positive dimensions (over $\mathbb{Q}$ and over $\mathbb{R}$). More precisely, For each hyperolic group $G$, each $X \in \mathcal{U}_\infty(G)$, and each $i \geq 1$, there exists a constant $S_i$ such that, for any cellular $i$-cycle $b$ in $X$, there exists an $(i+1)$-chain $a$ with $\partial a = b$ and $|a|_1 \leq S_i |b|_1$.*

It was shown by S. Gersten that, for finitely presented groups, linearity of the isoperimetric inequalities for 1-cycles is equivalent to hyperbolicity.

*Proof of Proposition 8.* Define $\varphi_i$ and $\psi_i$ to be the identity maps in all dimensions $i \leq -1$. Let $\psi_*$ be an arbitrary extension of the maps $\psi_i$ guaranteed by Lemma 9.

The chain map $\varphi_*$ is constructed inductively on dimension as follows. $C_0(Y, \mathbb{Q})$ is a one-generated free $\mathbb{Q}G$-module, so we can define $\varphi_0 : C_0(Y, \mathbb{Q}) \to C_0(X, \mathbb{Q})$ by mapping the unit element of $G$ to some vertex in $X$ and extending by $G$-equivariance and by linearity over $\mathbb{Q}$. Define $\varphi_1 : C_1(Y, \mathbb{Q}) \to C_1(X, \mathbb{Q})$ on the 1-simplices $(x_0, x_1)$ by

$$\varphi_1(x_0, x_1) := q\big[\varphi_0(x_0), \varphi_0(x_1)\big],$$

and extending to $C_1(Y, \mathbb{Q})$ by linearity over $\mathbb{Q}$. In other words, each 1-simplex in $Y$ maps to an element of the homological bicombing $q$. Since $q$ and $\varphi_0$ are $G$-equivariant, then $\varphi_1$ is a homomorphism of $\mathbb{Q}G$-modules. For each 2-simplex $(x_0, x_1, x_2)$ of $Y$,

$$\begin{aligned}\varphi_1(\partial(x_0, x_1, x_2)) &= \varphi_1\big((x_1, x_2) - (x_0, x_2) + (x_0, x_1)\big) = \\ &= q\big[\varphi_0(x_1), \varphi_0(x_2)\big] - q\big[\varphi_0(x_0), \varphi_0(x_2)\big] + q\big[\varphi_0(x_0), \varphi_0(x_1)\big] = \\ &= q\big[\varphi_0(x_1), \varphi_0(x_2)\big] + q\big[\varphi_0(x_2), \varphi_0(x_0)\big] + q\big[\varphi_0(x_0), \varphi_0(x_1)\big],\end{aligned}$$

hence, by Theorem 7(4),

$$\Big|\varphi_1(\partial(x_0, x_1, x_2))\Big|_1 \leq \Big|q\big[\varphi_0(x_1), \varphi_0(x_2)\big] + q\big[\varphi_0(x_2), \varphi_0(x_0)\big] + q\big[\varphi_0(x_0), \varphi_0(x_1)\big]\Big|_1 \leq T,$$

where the constant $T$ is independent of the choice of the triple $(x_0, x_1, x_2)$. Since $\varphi_1(\partial(x_0, x_1, x_2))$ is a 1-cycle, then, by Theorem 10, there exists a 2-chain $c = c(x_0, x_1, x_2)$ in $X$ with $\partial c = \varphi_1(\partial(x_0, x_1, x_2))$ and

$$|c|_1 \leq S_1 \big|\varphi_1(\partial(x_0, x_1, x_2))\big|_1 \leq S_1 \cdot T.$$

This 2-chain $c(x_0, x_1, x_2)$ can be chosen $G$-equivariantly so that the map $\varphi_2 : C_2(Y, \mathbb{Q}) \to C_2(X, \mathbb{Q})$ defined by

$$\varphi_2(x_0, x_1, x_2) := c(x_0, x_1, x_2)$$

is a homomorphism of $\mathbb{Q}G$-modules and it is bounded, by the inequality above.

The further inductive steps are similar. If a $\mathbb{Q}G$-module homomorphism

$$\varphi_i : C_i(Y, \mathbb{Q}) \to C_i(X, \mathbb{Q})$$

is constructed for some $i \geq 2$ and has norm bounded by some constant $R_i$, then we define $\varphi_{i+1}(x_0, ..., x_{i+1})$ to be an equivariant choice of a filling for the $i$-cycle $\varphi_i(\partial(x_0, ..., x_{i+1}))$. Since

$$\left|\varphi_i(\partial(x_0, ..., x_{i+1}))\right|_1 = \left|\sum_{k=0}^{i+1}(-1)^k \varphi_i(x_0, ..., x_{k-1}, \widehat{x_k}, x_{k+1}, ..., x_{i+1})\right|_1 \leq$$

$$\leq \sum_{k=0}^{i+1}\left|\varphi_i(x_0, ..., x_{k-1}, \widehat{x_k}, x_{k+1}, ..., x_{i+1})\right|_1 \leq (i+2)R_i,$$

then, by Theorem 10, the filling can be chosen to satisfy

$$\left|\varphi_{i+1}(x_0, ..., x_{i+1})\right|_1 \leq S_i\left|\varphi_i(\partial(x_0, ..., x_{i+1}))\right|_1 \leq S_i(i+2)R_i,$$

i.e. the norm of the map $\varphi_{i+1} : C_{i+1}(Y, \mathbb{Q}) \to C_{i+1}(X, \mathbb{Q})$ is bounded by

$$R_{i+1} := S_i(i+2)R_i.$$

One easily checks that the maps $\varphi_i$ constructed above form a chain map $\varphi_* : \mathcal{C}^Y \to \mathcal{C}^X$. Proposition 8 is proved. □

Now we can prove the following theorem stated by M. Gromov [4] for $\mathbb{R}$-coefficients.

**Theorem 11.** *Let $G$ be a hyperbolic group and $V$ be a normed vector space over $\mathbb{Q}$. Then the map $H_b^n(G, V) \to H^n(G, V)$ induced by inclusion is surjective for each $n \geq 2$.*

*Remark.* Of course, here $V$ is considered a $\mathbb{Q}G$-module with the trivial $G$-action.

*Proof of Theorem 11.* Apply functor $Hom_{\mathbb{Q}G}(\cdot, V)$ to the chain maps $\psi_*$ and $\varphi_*$ from Proposition 8. With the notations

$$\mathcal{C}_X := Hom_{\mathbb{Q}G}(\mathcal{C}^X, V), \qquad \mathcal{C}_Y := Hom_{\mathbb{Q}G}(\mathcal{C}^Y, V),$$

$$\varphi^* := Hom_{\mathbb{Q}G}(\varphi_*, V), \qquad \psi^* := Hom_{\mathbb{Q}G}(\psi_*, V),$$

we have two cochain complexes, $\mathcal{C}_X$ and $\mathcal{C}_Y$, and two cochain maps, $\varphi^* : \mathcal{C}_X \to \mathcal{C}_Y$ and $\psi^* : \mathcal{C}_Y \to \mathcal{C}_X$. In the positive dimensions, the homologies of the cochain complexes $\mathcal{C}_X$ and $\mathcal{C}_Y$ are equal to the cohomology of $G$, $H^*(G, V)$, and in these dimensions both $\varphi^*$ and $\psi^*$ induce endomorphisms of $H^*(G, V)$.

The map $\varphi^* \circ \psi^* : \mathcal{C}_Y \to \mathcal{C}_Y$ and the identity map $id^* : \mathcal{C}_Y \to \mathcal{C}_Y$ coincide in all negative dimensions, hence, by Lemma 9, they are chain homotopic, so $\varphi^* \circ \psi^*$ induces the identity maps on $H^*(G, V)$ in each positive dimension.

Let $i \geq 2$. Given any element of $H^i(G,V)$, we represent it by an $i$-cocycle $\alpha \in C^i_Y$. Then the cocycle

$$(\varphi^i \circ \psi^i)(\alpha) = \varphi^i(\psi^i(\alpha)) \in C^i_Y$$

represents the same element of $H^i(G,V)$. It remains to show that $\varphi^i(\psi^i(\alpha))$ is bounded. Since

$$\psi^i(\alpha) \in C^i_X = Hom_{\mathbb{Q}G}(C^X_i, V) = Hom_{\mathbb{Q}G}(C_i(X,\mathbb{Q}), V),$$

then $\psi^i(\alpha)$ is a $G$-invariant homomorphism $C_i(X,\mathbb{Q}) \to V$, i.e. it takes the same values on the $i$-cells in the same $G$-orbit. There are only finitely many such orbits in $X$, hence $\left|\psi^i(\alpha)\right|_\infty < \infty$. Also

$$\left|\varphi^i(\psi^i(\alpha))\right|_\infty = \left|\psi^i(\alpha) \circ \varphi_i\right|_\infty \leq |\psi^i(\alpha)|_\infty \cdot |\varphi_i|_\infty,$$

and, by Lemma 8(2), $|\varphi_i|_\infty < \infty$, so the map $\varphi^i(\psi^i(\alpha))$ is bounded. This shows that each element of $H^i(G,V)$, for $i \geq 2$, is represented by a bounded cocycle in the bar-construction. Theorem 11 is proved. □

It was not needed for the proof, but (using the explicit cone-off procedure from [6]) it is possible to refine the argument above so that each $k$-simplex $\sigma$ in the bar-construction maps to a "quasi-straight" $k$-chain in $X$, in the sense that the support of this $k$-chain lies uniformly close to a union of geodesics connecting the images of the vertices in $\sigma$. Again, this is a combinatorial analog of the fact that straight simplices in $\mathbb{H}^n$ lie close to their 1-skeleta.

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