# A CHARACTERIZATION OF TITS BUILDINGS BY METRICAL PROPERTIES

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## 1. Introduction

The basic combinatorial theory of "abstract" buildings has been developed by Tits in [7], Chapters 1 to 4 and [8], Sections 1,2,3. The book [7] is primarily concerned with the classification of buildings of spherical type. To that end, Tits uses a definition of buildings which is closely related to that of a BN-pair in a group. He introduces buildings as structured simplicial complexes, that is, complexes with a family of subcomplexes called apartments. The hardest result by far in the general part of [7] is the "reduction theorem" 4.1.2 which states that certain locally defined maps between buildings of spherical type can be extended to isomorphisms. Using this purely combinatorial result, Tits is able to prove that any building of spherical type and rank at least 3 is isomorphic to one of the "known" buildings.

The main ingredient in the proof of the reduction theorem is the existence of certain "projection maps" from the whole building onto the star of any fixed simplex, that is, the set of simplices containing this simplex. The projection maps induce partial automorphisms of the two buildings in question which have to commute with the desired isomorphism and conversely allow the construction of this isomorphism. This fact suggests the possibility of developing the theory of buildings by taking the existence of projection maps as one main axiom.

In this paper, we shall make this idea precise by showing that the existence of projections together with a rather weak kind of homogeneity implies that a geometry actually is a building. (For the purpose of this introduction, the term "geometry" stands for any of the three notions "geometry" in the sense of [6], [8], "numbered complex", [2] Chap. 4, Exerc. 15 ff.,[7], [8], or "chamber system" [3].) The additional homogeneity assumption is the following: the diameter of any rank 2 star only depends on the "type" of that star. This holds trivially if the geometry is of type $M$ for some

Coxeter diagram M , or if it has a chamber transitive automorphism group.

Our result is not restricted to any particular class of diagrams. It is independent of the proof of the above mentioned reduction theorem of [7]. Instead of [7], we have to use the results of [8], where Tits presents the basic notions and theorems concerning a characterization of buildings distinc from that in [7]. He characterizes in several ways, for a fixed Coxeter diagram M , the buildings of type M among the larger class of all geometries of type M . Our main theorem uses the "first characterization" Theorem 2 of [8]. If one is willing to take this as a definition, our paper is more or less independent of [8].

Apart from the reduction theorem, our interest in the projection maps comes from the observation that their existence is sufficient to imply the bouquet-property (Cohen-Macaulay'ness) of buildings ([5], proof of Lemma 4, [1], Appendix and [4]). This fact, however, is not sufficient to prove our main result , because a geometry belonging to a Coxeter diagram may possess the bouquet-property without being a building. This is shown by the well-known "$A_7$-geometry" with diagram $C_3$ , consisting of the "points" of a 7-element set X , all 3-subsets of X as "lines" and one $A_7$-orbit of projective-plane structures on X as "planes".

In a final section we show that the conclusion of our theorem becomes false if the assumption of homogeneity (in the sense explained above) ist dropped. To clarify this failure, we consider a certain "exchange property" for galleries, introduced by A. Dress in [3], which generalizes the well-known exchange property of Coxeter groups. This property is intermediate between the building property and the existence of projection maps, so in the homogeneous case they are all equivalent, by our theorem. We construct one class of examples which show that in the non-homogeneous case the existence of projection maps does not imply the exchange property. A second (fairly obvious) class of examples shows that the exchange property does not imply homogeneity, in particular it does not imply the building property.

I would like to thank Andreas Dress for valuable discussions on the topics of this paper.

We shall now collect the main definitions and results of Sections 2 and 3 of [8], with some minor complements.

A chamber system over an (index) set I is an object $(\mathcal{C}, \overset{i}{\sim}, i \in I)$ , where $\mathcal{C}$ is a set and the $\overset{i}{\sim}$ are equivalence relations on $\mathcal{C}$ . The elements of $\mathcal{C}$ are called chambers, we write $C \overset{i}{—} D$ if $C, D \in \mathcal{C}$ , $C \overset{i}{\sim} D$ , $C \neq D$ and say that C and D are i-adjacent in this case. The rank of $\mathcal{C}$ is by definition the cardinality of I .

The principal examples are given by the sets $\mathcal{C}(\Delta)$ of chambers, that is maximal simplices of a building $\Delta$ with type set I , or more generally any numbered complex over I . "Numbered" means that the vertex set X of the complex is partitioned as $X = \biguplus_{i \in I} X_i$ ($X_i$ the "vertices of type i") such that every chamber has exactly one vertex of each type. Two chambers C and D are by definition i-adjacent if they differ exactly in the vertex of type i.

There is also a construction which conversely associates a numbered complex $\Delta(\mathcal{C})$ over I to any chamber system $\mathcal{C}$ over I . One has a canonical surjective morphism $\Delta(\mathcal{C}(\Delta)) \to \Delta$ resp. $\mathcal{C} \to \mathcal{C}(\Delta(\mathcal{C}))$ for any given $\Delta$ resp. $\mathcal{C}$ , which is an isomorphism under well-known conditions. See [3] for details.

We shall say that a chamber system $\mathcal{C}$ corresponds to a building if it is isomorphic to a system $\mathcal{C}(\Delta)$ , where $\Delta$ is a building. This holds if and only if $\Delta(\mathcal{C})$ is a building and $\mathcal{C} \to \Delta(\mathcal{C}(\Delta))$ is an isomorphism. We include the case of "non-thick" buildings, e. g. the "Coxeter chamber system" of a given Coxeter matrix $M = (m_{ij})_{i,j \in I}$ over I . Here the set of chambers is by definition the Coxeter group (Weyl group)

$$W(M) := \langle i \in I \mid (ij)^{m_{ij}} = 1 \rangle \quad ,$$

and w and w' are i-adjacent if $w = w'i$ (where i is identified with the corresponding involution in $W(M)$). This chamber system can be identified with $\mathcal{C}(\Delta)$ , where $\Delta$ is the Coxeter complex of $W(M)$ .

By a slight deviation from the terminology of [8], we define a gallery of length n of a chamber system $(\mathcal{C}, \overset{i}{\sim}, i \in I)$ as a sequence

$$G = (C_o, C_1, \ldots, C_n; i_1, \ldots, i_n) ,$$

where $C_o, \ldots, C_n \in \mathcal{C}$ , $i_1, \ldots, i_n \in I$ and $C_{t-1} \overset{i_t}{\sim} C_t$ for $t = 1, \ldots, n$ . The word $i_1 \ldots i_n$ is called the type of $G$ . The gallery is simple if $C_{t-1} \neq C_t$ and $i_{t-1} \neq i_t$ for all $t$ . A simple closed gallery (i. e. $C_o = C_n$) is called a circuit if furthermore $C_t \neq C_{t'}$ for all $t, t'$ such that $0 \leq t < t' < n$ . If any two chambers can be joined by a gallery, $\mathcal{C}$ is connected.

A gallery of minimal length for given extremeties $C$ and $D$ is called geodesic, its length is the distance $d(C,D)$ between $C$ and $D$ .

The relations $\overset{i}{\sim}$ are extended to equivalence relations $\overset{J}{\sim}$ for all subsets $J \subseteq I$ as follows:

$C \overset{J}{\sim} D \Leftrightarrow$ there exists a gallery $(C = C_o, C_1, \ldots, C_n = D , i_1, \ldots, i_n)$ such that $i_t \in J$ for all $t$ .

(Thus $\overset{J}{\sim}$ is the smallest upper bound of the $\overset{i}{\sim}$ , $i \in J$ , in the lattice of all equivalence relations $R \subseteq \mathcal{C} \times \mathcal{C}$ .)

We call J-star (in $\mathcal{C}$) an equivalence class of $\overset{J}{\sim}$ , because in case $\mathcal{C}$ corresponds to a complex, it consists of all chambers in the star of a simplex $A$ . A J-star is naturally a chamber system over $J$ , in particular it possesses a rank $|J|$ .

Tits considers the following condition on a chamber system $\mathcal{C}$ of rank 2, say over $I = \{i,j\}$ , for $m$ a natural number $\geq 2$ or $\infty$ .

(CS 1): For any $C \in \mathcal{C}$ and $k \in I$ there exists a chamber $C' \in \mathcal{C}$ which is k-adjacent to $C$ .

($CS_m 2$): $\mathcal{C}$ contains no circuit of length $< 2m$ .

($CS_m 3$): If $C$ and $C'$ can be joined by a gallery of type $iji\ldots$ ($m$ factors), they can also be joined by a gallery of type $jij\ldots$ ($m$ factors).

(In case $m = \infty$ , this means that $\mathcal{C}$ , regarded as a graph, is a tree.)

It is easily seen that for given $m$, a chamber system of rank 2 satisfies (CS 1), ($CS_m2$) and ($CS_m3$) if and only if it corresponds to a building of rank 2 and diameter $m$, that is, a generalized m-gon. This fact is formally contained in the criterion below, because the condition (P) of that criterion is automatically true in the rank 2 case. The various well known characterizations of generalized polygons suggest the following remark on the corresponding chamber systems.

Remark. Consider the following property, for $m \neq \infty$.

($CS_m4$): For any two chambers $C$ and $D$, there exists a circuit of length $2m$ containing $C$ and $D$.

For given $m$, a chamber system $\mathcal{C}$ of rank 2 satisfies (CS 1), ($CS_m2$) and ($CS_m3$) if and only if it satisfies ($CS_m2$) and ($CS_m4$) if and only if $2m$ is the minimal length of a circuit and ($CS_m4$) holds. Here, ($CS_\infty 4$) means (CS 1).

The proof is straightforward, using the following consequences of ($CS_m2$): Any simple gallery of length $\leq m$ is geodesic, any closed simple gallery of length $2m$ is a circuit.

If $\mathcal{C}$ has the properties (CS 1) to ($CS_m4$), then in particular $m$ is the diameter

$$\operatorname{diam} \mathcal{C} := \sup_{C,D \in \mathcal{C}} d(C,D)$$

of $\mathcal{C}$, considered as a graph.

It seems convenient in our context to call a chamber system of rank 2 a generalized m-gon if it satisfies (CS 1), ($CS_m2$) and ($CS_m3$), and of course a generalized polygon if it is a generalized m-gon for some $m$. We include the case $m = 1$ in this definition. The remark above remains trivially true in this case, however, for $m = 1$ the corresponding complex does not exist, i. e. $\mathcal{C}$ is not isomorphic to $\mathcal{C}(\Delta(\mathcal{C}))$.

Returning to the case of arbitrary rank, we can now state the definition in Section 3.2 of [8] as follows.

Let $M = (m_{ij})_{i,j \in I}$ be a symmetric matrix with entries in $\{1,2,\ldots,\infty\}$. A chamber system $\mathcal{C}$ over $I$ is of type $M$ or has diagram $M$, if each

$\{i,j\}$-star in $C$ is a generalized $m_{ij}$-gon, for any subset $\{i,j\}$ of cardinality 2 of $I$.

Thus $C$ is of type $M$ for some $M$ if all rank 2 stars in $C$ are generalized polygons, and their diameter only depends on their type.

The following statement is essentially Theorem 2 of [8].

Let $M = (m_{ij})_{i,j \in I}$ be a Coxeter matrix, i. e. $m_{ii} = 1$, $m_{ij} \in \{2,3,\ldots,\infty\}$ for $i \neq j$. A chamber system $C$ over $I$ corresponds to a building with diagram $M$ if and only if it has diagram $M$ and the following condition holds.

(P) Consider simple galleries $G$ and $G'$ with common origin and common extremity, suppose that their types $f$ and $f'$ are reduced words (with repsect to $M$), i. e. of minimal length among all words representing the same element in the Weyl group $W(M)$. Then the images of $f$ and $f'$ in $W(M)$ coincide.

We finally recall the notion of "projection maps", following [3]. For the moment, let $(C, d : C \times C \to \mathbb{R}_{\geq 0})$ be any metric space. A subset $A \subseteq C$ is called <u>gated</u>, more precisely gated inside $C$, if the following holds:

For any $C \in C$, there exists $D \in A$ such that

$$d(C,A) = d(C,D) + d(D,A) \quad \text{for all } A \in A.$$

Pictorially, $D$ is the gate of $A$ with respect to $C$.

Obviously, $D$ only depends on $C$ and $A$, we call $D$ the <u>projection</u> of $C$ onto $A$.

If $C$ is a chamber system corresponding to a building and $A \subseteq C$ any star, then $A$ is gated, by [7], 3.19.6. See also Section 3 below.

## 2. The result

We are now prepared to formulate and prove our main theorem which in particular says that the buildings are characterized among all the geometries with some given Coxeter diagram by the existence of projection maps onto all stars of simplices. It is even sufficient to require the existence for stars of rank 1 and rank 2 only.

Although we have recalled some generalities in Section 1, some familiarity with Coxeter groups and with Sections 2 and 3 of [8] will be required in the proof.

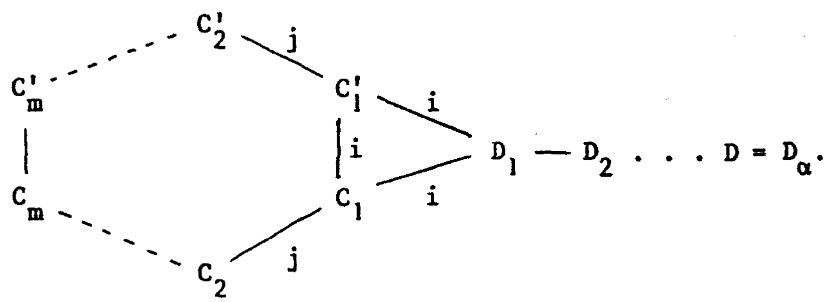
Theorem. Let $C$ be a chamber system such that every star $A$ of rank 1 or 2 is gated inside $C$. Then the following is true.

a) Any two star $A$ of $C$ is a generalized polygon. In particular, if diam $A$ only depends on the type of $A$, then $C$ is of type $M = (m_{ij})_{i,j \in I}$, where $m_{ij}$ is the diameter of any $\{i,j\}$-star.

b) Suppose that $C$ is of type $M$ for some Coxeter matrix $M$. Then $C$ corresponds to a building.

Proof of a): We may suppose that $C$ is of rank 2, and that $C$ is not a tree. Let $2m$ be the length of a shortest circuit. We have to show that property $(CS_m 4)$ holds.
Consider a circuit of minimal length $2m$ and a fixed chamber $C$ of that circuit. We will show the following.
(*) For any chamber $D$, there is a circuit of length $2m$ containing $C$ and $D$.

This will be proved by induction on $d := d(C,D)$.
Let $(C,D_1,D_2,\dots,D_d=D;ij\dots)$ be a geodesic.
Let $(C=C_1,C_2,\dots,C_m,C'_m,C'_{m-1},\dots,C'_1,C_1;\dots)$ be the given circuit, where the notation is such that $i$ is the last symbol in its type.




We will first show the following:

(**) For $t \leq m$, $t \leq d$, the gallery $G_t = (D_t, D_{t-1}, \ldots, D_1, C_1, \ldots, C_{m+1-t}; \ldots)$ of length $m$ (and the appropriate type, depending on the parity of $t$) is geodesic, and there also exists a gallery $G_t^v$ joining $D_t$ to $C_{m+1-t}$ of the opposite type ($i$ and $j$ interchanged).

Note that the closed gallery obtained by composing $G_t$ and the inverse of $G_t^v$ necessarily is a circuit, for otherwise $m$ would not have been minimal.

If (**) is proved for some $t$ and we want to prove it for $t+1$, we can replace $C$ by $D_t$, the original circuit by the circuit consisting of $G_t$ and $G_t^v$, and can then apply the case $t = 1$ to get the result. Thus it is sufficient to treat the case $t = 1$.

We first show that $d(D_1, C_m) = m$. Because of the equalities $d(C_m, C_1) = m-1$ and $d(C_m, C_1') = m$, the chamber $C_1$ is the projection of $C_m$ onto the i-star of $C_1$, and $d(C_m, D_1) = d(C_m, C_1) + d(C_1, D_1) = m - 1 + 1 = m$. Analogously, we have $d(D_1, C_m') = m$.

Now let $D'$ denote the projection of $D_1$ onto the $\{k\}$-star of $C_m$ (or $C_m'$), where $k = i$ if $m$ is even and $k = j$ if $m$ is odd, in particular $C_m \overset{k}{—} C_m'$. The equations $d(D_1, C_m) = d(D_1, C_m') = m$ imply that $d(D_1, D') = m-1$. A geodesic of the form $(D_1, \ldots, D', C_m; \ldots k)$ has the required opposite type.

For the proof of (*), it is now sufficient to show that $d \leq m$. If $d$ were greater than $m$, we could apply the following observation to $D_m$, $C_1$ and $D_{m+1}$ to get a contradiction:

Let $E, F$ be chambers which for either type $k \in \{i,j\}$ can be joined by a simple gallery $(E, \ldots, F; k \ldots)$ of length $m$. Then $d(E', F) \leq m$ for any chamber $E'$ adjacent to $E$.

The property $(CS_m 4)$ now readily follows. Consider any two chambers $D_1$ and $D_2$. Apply (*) to the $C$ considered above and $D_1$, then replace $C$ by $D_1$ and $D_1$ by $D_2$ and apply (*) once again.

We finally want to point out that the proof includes the case $m = 1$ : If there exists a pair of distinct chambers which are simultaneously i-adjacent and j-adjacent, the same holds for any two distinct chambers, i. e. $(CS_1 4)$ holds.

Proof of b): Let $M = (m_{ij})_{i,j \in I}$, thus $m_{ij} = \operatorname{diam} A$ for any $\{i,j\}$-star $A$, and $m_{ii} = 1$. As in Section 1, denote by $W(M)$ the Weyl group corresponding to $M$,

$$W(M) = \langle i \in I \mid (ij)^{m_{ij}} = 1 \rangle ,$$

and by $f \to w(f)$ the canonical map from the set of words over $I$ to $W(M)$. We will write $CfD$ for short, if there exists a gallery of type $f$ joining $C$ to $D$. Pictorially, we represent this as

$$C \succ\!\!-\, f \longrightarrow D .$$

Consider the following condition $(A_n)$, for a natural number $n$ :

$(A_n)$ Let $C, C', D, D' \in \mathcal{C}$, $i, k \in I$ such that $C \overset{i}{\sim} C'$, $D \overset{k}{\sim} D'$, let $f$ be the type of a geodesic from $C$ to $D$, so $CfD$. Suppose

$$d(C,D) = d(C',D') =: n$$
$$d(C,D') = d(C',D) = n+1$$

$$\begin{array}{ccc} C & \succ\!\!-\, f \longrightarrow & D \\ i \Big| & & \Big| k \\ C' & & D' \end{array} .$$

Then the relations

$$C'fD' \quad \text{and} \quad w(if) = w(fk)$$

hold.

We first show, by induction on $n$, that $(A_m)$ for all $m \leq n-1$ implies the following Property $(B_n)$.

$(B_n)$ Let $(C_o,\ldots,C_n;i_1,\ldots,i_n)$ and $(D_o,\ldots,D_n;j_i,\ldots,j_n)$ be geodesics such that $C_o = D_o$ and $C_n = D_n$ . Then

$$w(i_1,\ldots,i_n) = w(j_1,\ldots,j_n) .$$

To prove this, let the index $m$ be such that

$$d(C_{m-1},D_{n-1}) < d(C_{m-1},C_n)$$

$$d(C_m,D_{n-1}) \geq d(C_m,C_n)$$

$$\begin{array}{ccc} C_m & \rightarrowtail i_{m+1}\ldots i_n \longrightarrow & C_n \\ \Big| i_m & & \Big| j_n \\ C_{m-1} & & D_{n-1} \end{array} .$$

We claim that the second inequality is strict. Assume the contrary, and let $C'$ be the projection of $C_m$ onto the $\{j_n\}$-star of $C_n$ . It satisfies

$$d(C_m,C_n) = d(C_m,D_{n-1}) = d(C_m,C') + 1 .$$

Together with the inequality $d(C_o,C') \leq d(C_o,C_m) + d(C_m,C') = n-1$ , this implies $n = d(C_o,C_n) = d(C_o,C') + 1$ . The last equality means that $C'$ is also the projection of $C_o$ onto the $\{j_n\}$-star of $C_n$ , therefore $d(C_o,D_{n-1}) = d(C_o,C') + 1 = n$ , a contradiction.

We now can apply the Property $(A_{n-m})$ to $C_m$, $C_n$, $C_{m-1}$, $D_{n-1}$ and conclude the relations

$$C_{m-1}i_{m+1}\ldots i_n D_{n-1} ,$$

$$w(i_m i_{m+1}\ldots i_n) = w(i_{m+1}\ldots i_n j_n) .$$

By the induction hypothesis, applied to $C_o$ and $D_{n-1}$ , we have

$$w(i_1\ldots i_{m-1}i_{m+1}\ldots i_n) = w(j_1\ldots j_{n-1}) ,$$

thus $$w(j_1\ldots j_n) = w(i_1\ldots i_{m-1}i_m\ldots i_n) .$$

We now turn to the proof of $(A_n)$ , again by induction on $n$ . For $n = 0$ , we necessarily have $C' = D'$ , $i = k$ . For $n > 0$ , let $f = f'j$ for a word $f'$ of length $n-1$ and $j \in I$ and let $D''$ the penultimate term of the geodesic in question:

$$\begin{array}{ccccc} C & \rightarrowtail f' \longrightarrow & D'' & \overset{j}{\text{---}} & D \\ \Big| i & & & & \Big| k \\ C' & & & & D' \end{array}$$

Consider the $\{j,k\}$-star $A$ of $D$ and the projections $E = pr_A C$, $E' = pr_A C'$. By the general properties of $pr_A$ we have

$$d(C,D) = d(C,E) + d(E,D)$$
$$d(C,D') = d(C,E) + d(E,D')$$
$$d(C',D) = d(C,E') + d(E',D)$$
$$d(C',D') = d(C',E') + d(E',D')$$
$$d(E,E') \leq 1$$

Together with our assumption this implies

$$d(E,D') = d(E,D) + 1$$
$$d(E',D) = d(E',D') + 1$$
$$d(C,E) + d(E,D) = d(C',E') + d(E',D') ,$$

in particular $\quad E \neq E'$

and therefore

$$d(C,E') = d(C,E) + d(E,E') = d(C,E) + 1$$
$$d(C',E) = d(C',E') + 1 .$$

On the other hand, $d(C,E') \leq 1 + d(C',E')$, so $d(C,E) \leq d(C',E')$, and by symmetry

$$d(C,E) = d(C',E') .$$

We finally conclude

$$d(E,D) = d(E',D') ,$$

so altogether we have the following picture

$$\begin{array}{ccccccc} C & - \dots - & E & - \dots - & D'' & \overset{j}{\text{---}} & D \\ i\Big| & & \Big|\ell & & & & \Big|k \\ C' & - \dots - & E' & - \dots\dots\dots & & \text{---} & D' \end{array}$$

for the appropriate $\ell \in \{j,k\}$.

Let $m = d(E',D) = d(E,D')$ and

$$G = (E=E_1, E_2, \dots, E_{m-1}=D'', E_m=D; \dots)$$
$$G' = (E'=E'_1, E'_2, \dots, E'_m=D'; \dots)$$

be geodesics joining $E$ and $D$ resp. $E'$ and $D'$. Then

$$d(E_s,D) < d(E_s,D')$$
$$\text{and} \quad d(E_s',D) > d(E_s',D')$$

holds for all $s$, so $E_s \neq E_t$, for all $s,t$. That is, type $G$ = type $G' =: h =: h'j$, and $m$ is the diameter of the generalized polygon $A$, i. e., the entry $m_{ij}$ of our Coxeter matrix defining $W$. This implies

$$w(\ell h) = w(hk)$$

$$\begin{array}{ccc} E & \rightarrowtail h \longrightarrow & D \\ \ell \,\big| & & \big|\, k \\ E' & \rightarrowtail h \longrightarrow & D' \end{array}$$

Now let $g$ be the type of a geodesic from $C$ to $E$. By the induction hypothesis the following relations hold:

$$C'gE', \; w(ig) = w(g\ell) .$$

By the Property $(B_{n-1})$, applied to $C$ and $D''$, we have $w(gh') = w(f')$, thus

$$w(gh) = w(f) .$$

Together with the relation $C'ghD'$ and the reducedness of $gh$ and $f$, this implies the desired relation

$$C'fD'.$$

Also,

$$w(if) = w(igh) = w(g\ell h) = w(ghk) = w(fk) .$$

Having proved the Properties $(A_n)$ and $(B_n)$ for all $n$, we finally show that the Property (P) of Section 1 holds. Let $C,D \in \mathcal{C}$ and $f,g$ be reduced words such that $CfD$ and $CgD$. By $(B_n)$, it is sufficient to show that any gallery $(C_o,C_1,\dots;i_1,i_2,\dots)$ of reduced type is geodesic. Assume the contrary and let the index $t$ be such that

$$d(C_o,C_t) = t$$

$$d(C_o,C_{t+1}) \leq t ,$$

write $j := i_{t+1}$. If $D$ denotes the projection of $C_o$ onto the $\{j\}$-star of $C_t$, we have $d(C_o,D) = t-1$. (Of course, $E = C_{t+1}$ may occur.) Choose any geodesic $(C_o,\dots,D;g)$, thus the word $g$ has length $t-1$. Then, $(C_o,\dots,D,C_t;gj)$ is geodesic as well. From the property $(B_t)$ we conclude $w(gj) = w(i_1\dots i_t)$, thus $w(i_1\dots i_{t+1}) = w(g)$. This contradicts the fact that $i_1\dots i_{t+1}$ is reduced.

The following corollary is a specialisation of our Theorem which is formulated entirely in terms of groups with a distinguished set of generators of order 2.

<u>Corollary</u>. Let $W$ be a group and $S \subset W$ a set of generators such that $s^2 = 1$ for all $s \in S$, denote by $\ell$ the length function on $W$ with respect to $S$. Suppose that $\ell(ws) \neq \ell(w)$ holds for all $s \in S$, $w \in W$, and that $\ell(wx) = \ell(w) + \ell(x)$ holds for all subsets $\{s,s'\} \subseteq S$ of cardinality 2, all $x \in \langle s,s' \rangle$ and all $w \in W$ which are of smallest length in the coset $w\langle s,s' \rangle$.
Then $(W,S)$ is a Coxeter system.

For the proof, consider $W$ as a chamber system over $S$ by defining $w \overset{s}{\text{---}} ws$ for all $w \in W$ and $s \in S$. This chamber system is of type $M$, where $M = (\mathrm{ord}(s,s'))_{s,s' \in S}$. The assumptions of the corollary express the fact that the projection of 1 onto the $\{s\}$-star, resp. $\{s,s'\}$-star containing w, exists. Therefore, by the transitive action of $W$ on the chamber system, the projection of any chamber onto any star of rank 1 or 2 exists. By the theorem, $W$ corresponds to a building, i. e. the Property (P) holds. This means that the canonical map $W(M) \to W$ is an isomorphism.

## 3. A general construction and some counterexamples

For the rest of this paper, we want to give a name to the property of chamber systems that appeared as a condition in part b) of our theorem. We call a chamber system $C$ over $I$ homogeneous if for all subsets $J$ of $I$ of cardinality 2, the diameter of any J-star only depends on $J$. We shall show that the assumption of homogeneity cannot be dropped in our theorem. More precisely, we shall naturally break up the conclusion of the theorem into two consecutive implications, and we shall see that neither of them remains true in the general case.

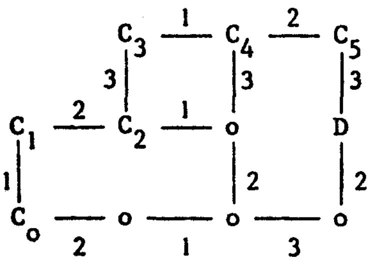
For this purpose, we shall formulate a condition which is intermediate between the building property and the gate property. (Here and in the sequel, we say that a chamber system $C$ has the "gate property" for short if all stars of $C$ are gated inside $C$.) The intermediate property is that the following condition on certain galleries, introduced by A. Dress in [3], is always fulfilled.

Exchange condition. Let $(C_o,\ldots,C_n;i_1,\ldots,i_n)$ be a geodesic and $D$ a chamber, $i \in I$ such that $C_m \overset{i}{\text{---}} D$. If $(C_o,\ldots,C_m,D;i_1,\ldots,i_n,i)$ is not a geodesic, then there exists a gallery of the form $(C'_o,\ldots,C'_{n-1};i_1,\ldots,\hat{i}_\nu,\ldots,i_n)$ ($i_\nu$ omitted) such that $C'_o = C_o$ and $C'_{n-1} \overset{i}{\text{---}} C_n$.

We observe that this condition is an immediate generalization of the usual exchange condition which holds for a Coxeter group. One merely has to formulate the latter condition as a condition on the corresponding Coxeter chamber system. Note that the Property $(A_n)$ used in the proof of our theorem is a special case of the exchange property. Conversely, if $(A_n)$ holds for all $n$ and if in addition all rank 1 stars are gated, the exchange condition is satisfied.

The important point about the exchange property is that it easily implies the gate property, see [3] for a proof. Furthermore, it is easily seen that a chamber system corresponding to a building, i. e. of type M for some Coxeter matrix $M$ and possessing the Property (P) of Section 1, satisfies the exchange condition. One reduces the assertion to the ordinary exchange condition for the Weyl group. Combining these two facts with our theorem we see that for homogeneous chamber systems, the building property, the exchange property and the gate property are all equivalent. We shall now show that in general, neither the gate property implies the exchange property, nor the exchange property implies the building property.

<u>Example</u>. Let C be the following chamber system over the type set $\{1,2,3\}$ .




It is easily checked that all stars are gated. On the other hand, for the geodesic $(C_o,C_1,C_2,C_3,C_4,C_5;123123)$ and the chamber D , the exchange condition is violated because the unique geodesic joining $C_o$ and D has the type 2132 .

This chamber system lacks the property (CS 1) that every rank 1 star contains at least two chambers. We shall improve our example by using the following general construction of "free thin extensions".

A chamber system is called <u>thin</u> if every rank 1 star consists of exactly two chambers. The <u>free thin</u> chamber system over I consists of all words f over I , with $f \overset{i}{—} fi$ for all f and i . This is the Coxeter chamber system corresponding to the Coxeter matrix over I with all off-diagonal entries equal to $\infty$ . As a graph, it is a tree with all valencies equal to the cardinality of I .

Now let C be a chamber system over I such that each rank 1 star contains at most two chambers. For each pair (C,i) such that the {i}-star of C consists of C alone, let C(C,i) be the set of all (C,f) , where f is a word over I beginning with i . This is a chamber system by defining $(C,f) \overset{j}{—} (C,fj)$ for all (C,f) and j . It may be regarded as "one half" of the free chamber system over I . We construct a new chamber system by attaching C(C,i) to C via an edge of type i , for each (C,i) as above. More precisely, let

$$\bar{C} = C \cup \bigcup_{(C,i)} C(C,i)$$

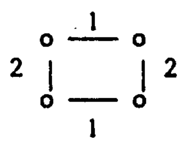
where (C,i) ranges over all pairs as above and where the edges $\overset{i}{—}$ of $\bar{C}$ are defined to be the old edges inside C or the C(C,i) together with the edges $C \overset{i}{—} (C,i)$ . In this way, $\bar{C}$ becomes a thin chamber system over I . We call $\bar{C}$ the free thin extension of C .

It is easy to describe all geodesics in $\bar{C}$ in terms of the geodesics of C . We omit the proof of the following proposition, which is lengthy but straightforward, from the description of the geodesics in $\bar{C}$ .

Proposition. Let C be a chamber system such that every rank 1 star consists of at most two chambers, and let $\bar{C}$ be the free thin extension of C .

a) If C has the gate property, the same holds for $\bar{C}$ .
b) If C satisfies the exchange condition, the same holds for $\bar{C}$ .

Applying part a) to the above example one obtains the required improvement of that example, because $\bar{C}$ also does not satisfy the exchange condition. Part b) of the proposition supplies examples of chamber systens satisfying the exchange condition which are not homogenous and therefore do not correspond to buildings. For this purpose, let us start with any Coxeter chamber system C over some I , excluding only the free thin chamber systems. The most simple case is




Now add one further symbol k to the type set I and consider C as a chamber system over $I \uplus \{k\}$ . The corresponding extension $\bar{C}$ is not homogenous because for some subset J of I of cardinality 2 , it contains finite as well as infinite J-stars.

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