A refined Kodaira dimension and its canonical fibration

Dedicated to Professor Hirzebruch on his 75th Birthday

Steven S. Y. Lu *

October 28, 2002

1 Introduction

From many earlier works on the classification theory of compact complex manifolds and their intrinsic geometric structures, it has been clear that "positivity" or "nonpositivity" properties of subsheaves of exterior powers of the tangent or cotangent bundle provide some of the most important global information concerning the manifold in question. The programs of Iitaka and Mori on a general classification scheme for algebraic varieties testify to this with a special focus on the top exterior power of the cotangent bundle, the canonical bundle. This paper is motivated by some of our earlier studies to bring the other subsheaves of the cotangent bundle, such as those defined by foliations and fibrations (studied by Miyaoka and Bogomolov for example), into focus as important objects of study for a general classification theory in birational geometry. Especially relevant here are the line subsheaves of exterior powers of the cotangent bundle that correspond to the canonical "bundles" of the orbifold bases of fibrations.

Our main motivation comes from the papers [BL1, BL2] and [Lu01]. In these papers, it was shown that the properties of having identically vanishing Kobayashi pseudometric (or holomorphic connectedness), having a Zariski dense entire holomorphic curve and being dominated by a holomorphic map from $\mathbb{C}^n$ (in the sense of having maximal rank somewhere) are the same for most algebraic varieties in low dimensions as well as for rationally connected fibrations and abelian fibrations over curves and that this property is characterized by birational invariants. In [BL1], we studied the compact complex surface case, among others, and found that the Kodaira dimension and the fundamental group are sufficient to characterize these properties. [BL2] showed that this characterization fails for quasiprojective surfaces but a more intrinsic birational invariant related to Campana's κ^+ in [Ca] was introduced in [Lu01] that completely characterized the above properties, even for the quasiprojective case, for the above mentioned examples and more (namely, those for which any of the above properties can be verified via the techniques we have introduced in these papers).

*Partially supported by an NSERC grant and the Max Planck Institute of Mathematics in Bonn.

Our invariant in the projective category is defined simply as follows. Given a complex projective manifold X of dimension n, we define a bimeromorphic invariant $\kappa'_+(X)$ as the maximum p for which there is a saturated line subsheaf L of the sheaf of holomorphic p forms whose Kodaira dimension $\kappa(L)$ equals p. Such a line sheaf were first studied, and quite extensively, by Bogomolov in [Bo]. Saturated subsheaves corresponding to foliations were also studied by Miyaoka in connection with classification theory, see [MP].

We call X special if $\kappa'_+(X) = 0$ and call a fibration from X special if its general fibers are. If $\dim X \leq 3$ and X is not birationally equivalent to a Calabi-Yau threefold, we have verified in [Lu01] that X is special if and only if it is holomorphically chain connected in the sense that there is a holomorphic family of entire holomorphic curves in X such that any general pair of points in X are connected by a chain of holomorphic curves in this family. Also any holomorphically connected X is special and hence so are the rationally connected ones. An important fact about special varieties, first obtained by Campana for a slightly different definition from above but more recently for the definition above ([Ca02], see also [Lu02]), is that a projective manifold of Kodaira dimension zero is special. As rationally connected projective manifolds are also special, the two basic types of fibrations in the classification program of algebraic varieties, namely rationally connected fibrations and Iitaka fibrations are examples of special fibrations. Thus, special objects are basic building blocks in birational geometry. We mention in this connection that Campana, Kollar, Miyaoka and Mori (see [KoMM, Ca, Kol1, Debarre]) have defined (and developed extensively the theory of) the maximal rationally connected (meromorphic) fibration (or MRC-fibration [Kol1, IV.5]) of a smooth projective variety. The structure of this MRC-fibration, which is a proper fibration when restricted to some open subset (i.e., almost holomorphic), has just been resolved completely in [GHS] using the theory developed by the above mentioned people and the theory of compactification of the appropriate space of stable curves à la Kontsevich:

THEOREM 1.1 (Graber-Harris-Starr) *If a smooth projective variety has a rationally connected (meromorphic) fibration over a smooth rationally connected base, then it is rationally connected. Hence, the base of the MRC-fibration of a projective manifold is not uniruled, or equivalently, the base admits no rationally connected (meromorphic) fibrations with positive dimensional fibers. Equivalently, every rationally connected fibration over a curve admits a section with prescribed behavior (in the sense given by [BL2] for example) over any finite number of points of the curve.*

Campana has also constructed in [Ca02] (from a view point that is independent from ours) a canonical special fibration for any projective variety and made a number of conjectures about its geometric structure and relevance to the classification program. (His original construction [Ca01] was slightly different, compare also [Lu02].) Here, we give an alternative more direct construction of this special fibration, generalizing the theorem above and resolve, among others, most of his conjectures on it relating to their structure and classification outside of the main conjectures in Mori's Program. We obtain an analog of the above theorem for our canonical fibration in the sense that our fibration has an orbifold base (see the next sections for the definitions) of general type. It follows in particular (see Corollary 5.9) that this orbifold base has no orbifold special fibrations with positive dimensional fibers. Another analog we obtain (of no lesser importance) is that if a manifold has a special fibration over a special orbifold base, then it is itself special, the content of Proposition 6.5 which is a consequence of the easy addition law of Kodaira dimension. Without the notion of orbifold, our main theorem can only be stated in part as follows (see Theorem 10.3 for the full statement).

THEOREM 1.2 *Let X be a complex projective manifold. Then a saturated subsheaf L of Ω_X^p with $\kappa(L) = \kappa'_+(X) = p$ is unique and gives rise, via its Iitaka fibration, to a special meromorphic fibration $bc_X : X \dashrightarrow Y$ that is holomorphic above a smooth open set U of Y and Y is p-dimensional. Moreover, $bc_X^* K_U \hookrightarrow \Omega_X^p$ factors through (i.e., is contained in) L and bc_X factors through any special fibration from X. In particular, if X is not of general type (i.e., $p \neq \dim Y$), then $\dim Y = p < dim X$ and so bc_X has positive dimensional fibers.*

A word on notation: Line sheaves L with the above properties were extensively studied by Bogomolov in [Bo] and are now known as Bogomolov sheaves ([Ca02]). The notation bc_X is an adaptation of Campana's $c_X : X \dashrightarrow C(X)$, which is called the core of X in [Ca02]. Although our construction of bc_X is different from that of c_X in [Ca02], it follows directly from our main theorem that they are in fact the same fibration. The terminology "core" is very appropriate for our context except for the ambiguity in the choice of the definition of the orbifold base. The ambiguity disappears once it is defined by Bogomolov sheaves as we amply demonstrates in the next section. Thus our notation bc_X, which we heartily thank Bogomolov for his agreement for adopting, is meant to indicate that the fibration is defined by Bogomolov sheaves and can be called the Bogomolov-Campana fibration or the Bogomolov-Campana core of X (or more simply the basic fibration (or reduction) of X). We then call $\kappa'_+(X)$ the Bogomolov-Campana dimension or simply the basic (or base core) dimension of X.

We actually resolve the orbifold version of our claims and hence of this theorem in full generality, including Zariski open subsets of compact complex manifold, where we have an alternative simple definition of κ'_+ and special. We believe that the notion of special in this generality is not artificial since all examples of compact complex manifold known to us that do not have vanishing Kobayashi pseudodistance are non-special. In this generality, we show for example that the algebraic reduction is a special fibration (irrespective of the orbifold structure imposed). Using this, we show via Proposition 6.5 that the fibration given above in the general compact complex category is nothing but the algebraic reduction composed with iterations of the orbifold Iitaka and Mori's fibrations assuming that the latter fibrations exist in the negative Kodaira dimension range. This last assumption on the existence of a particular kind of special fibrations in this situation is in fact one of the central problems in Mori's classification program and we give a solution to this problem here in the sense that any variety with negative Kodaira dimension has a special fibration with positive dimensional fibers.

A major aim of this paper is to introduce a framework for a theory of the birational geometry of orbifolds in an elementary and self-contained way as an accessible extension of an important part Mori's program. We do this in the most general setting of orbifolds as compact complex manifolds endowed with $\mathbb{Q}$-divisors. Such an extension does not seem to exist in any coherent or consistent fashion in the literature especially with respect to the birational geometry. This lack is the topic of Section 3.30 of the book [KM] by Kollar and Mori. We tackle this problem in the most general compact complex category for the purpose of further studies on them in complex geometry. This is in fact a tall task for general orbifolds for which this paper only hopes to have done some initial justice (our original motivation to do this was to resolve some confusion in the literature about their birational geometry) even though the notions are fairly straightforward geometrically for smooth orbifolds (see Remark 3.1 and the paragraph preceding it for the definition of smooth orbifolds and Appendix A). A possibly more natural but less elementary category (or notion) for this is that of "stacks" which has recently become popular and which the paper of S. Mori and S. Keel ([KeM]) and its followups seem to suggest. Still another possible direction would be Alan Conne's theory of noncommutative geometry, which parallels much of that of stacks, for our birational geometry of orbifolds since our approach is essentially that of foliations (compare the suggestive terminology used for the title of [McQ]).

Our notion of orbifold is based on subsheaves of the cotangent sheaf corresponding to fibrations, which as far as we know includes all known notions of orbifolds in the complex analytic setting besides possibly stacks. This allows us to deduce easily for example that bc_X is the iterated compositions of orbifold Iitaka and Mori's fibrations, see our discussion right after Conjecture 6.7 where we also reduce a key conjecture in Mori's program on uniruled orbifolds to "rationally connected" orbifolds. Also, by allowing all rational multiplicities not greater than one, it allows us to "approximate" an orbifold by neighboring orbifold structures that are easier to handle in some way, whether it be an arithmetic, geometric or analytic problem. A particular instance of this is our solution to the conjecture of Campana in [Ca02] (and therefore also of Demailly-Peternell-Schneider in [DPS, DPS0] concerning the Albanese map) that projective manifolds with nef anticanonical bundle are "nearly" special (see Theorem 11.2), of importance to their fundamental group and the structure of their universal cover, see also [DPS1, DPS2]. A corollary of this is an extension to all dimensions of the result of [DPS] that such a manifold of dimension not greater than three is nearly special, with a different proof from that of [DPS], see Theorem 11.2, and showing as a consequence that the fundamental group is almost abelian. The latter is a recent result of Mihai Paun in [P] derived using a strong differential geometric result of Cheeger and Colding ([CC]), a result of Qi Zhang ([ZQ]) concerning the Albanese map that employs characteristic $p > 0$ techniques from [MiMo] and a result of Campana ([Ca3]). Our result refines that of Qi Zhang but without involving characteristic $p > 0$ thereby showing that this result of Mihai Paun can be derived more directly and more in line with complex geometry with improved understanding on the structure of such varieties. Indeed, one of Campana's main conjectures in [Ca02] is that a special projective manifold has almost abelian fundamental group from which one can easily deduce Iitaka's conjecture that

a projective manifold whose universal cover is $\mathbb{C}^n$ has a finite cover by an abelian variety.

Our orbifold perspective, such as our derivation of the above mentioned results, do yield some concrete insights and results in this direction which we will come back to.

The key ingredients that goes into the proofs of our main results, apart from a wellsuited generalized theory of orbifolds and their “birational geometry”, are generalizations of the theorems of Kawamata, Viehweg and Kollar on the subadditivity of Kodaira dimension for a fiber space (general type case) to the orbifold category. We believe that our proofs for these in the orbifold category, which are natural modifications of their original proofs, nevertheless shed some light on the full subadditivity conjecture of Iitaka and hence to the classification program.

Many of the facts derived here about orbifolds are natural generalizations of facts and resolutions of conjectures stated in [Ca01] (also generalized to the full orbifold situation here). The perspective in that paper is broader than ours and was motivated completely independently from ours. We urge the reader to consult this very motivating paper for a different perspective and many more facts and questions concerning the geometry of orbifolds. The preliminary sections are elementary in nature but we decided to include them as we cannot find any satisfactory reference in the literature.

The author would like to thank J. P. Demailly, A. Fujiki, B. Shiffman, E. Viehweg and M. Zaidenberg for their keen interest and guidance in this problem, especially to B. Shiffman to whom this paper owes its initiation and existence. A very special thanks is owed to F. Campana for communicating his inspiring preprint in November of 2001. It has served as an important guiding force behind many of our results here. He also expresses his thanks to Professors Y. I. Manin, F. Hirzebruch and Dr. Y. Holla for some useful discussions at the Max Planck Institute and finally to Y. Kawamata and F. Bogomolov for their suggestions and encouragements. The first version of this paper was written in Osaka University while this revised version is written at the Max Planck Institute for Mathematics in Bonn, the author would like to thank these respective places for their very generous hospitality and support. Finally the author would like to thank a most wonderful couple Barbara and Spyros for their tolerance and kindness in making the place like a home to him.

2 The birational geometry of fibrations and orbifolds

All objects in this paper are compact complex manifold unless clear from the context or otherwise specified. In this category, we adopt the terminology of identifying holomorphic maps with morphisms, meromorphic maps with rational maps, and bimeromorphic maps with birational morphisms. We also identify vector bundles with locally free sheaves. We assume that the reader is familiar with the rudiments of the theory of Kodaira dimension as found in Ueno [Ue], Mori [Mori] or Iitaka [Ii] for example. The analytic version of the resolution of singularity theorem of Hironaka ([Hi0]) proved by more elementary methods recently in [BM] is used freely throughout this paper.

Let $f : X \to Y$ be a fibration, i.e., f is a surjective morphism with connected fibers. Then f imposes an orbifold structure on Y a part of which can be visualized by putting in the discriminant (branched) $\mathbb{Q}$-divisor $D(f)$ on Y inherited from f so that Y becomes a branched orbifold as follows.

Let $\mathrm{Div}'(X)$ be the set of codimension one subvarieties of X (as Cartier divisors), i.e.,

$$\mathrm{Div}'(X) = \{\ D \in \mathrm{Div}(X) \mid D \text{ a reduced irreducible divisor on } X,\ D \neq 0\ \}.$$

Given $D \in \mathrm{Div}'(Y)$, we may write $f^*D = \sum_i m_i D_i$ for $m_i \in \mathbb{N}$ and $D_i \in \mathrm{Div}'(X)$. Then we define the (minimum) multiplicity of f over D by

$$m(D, f) = \min\{\ m_i \mid f(D_i) = D\ \}.$$

We note that the classical multiplicity $m_f(D)$ is defined by replacing min above by gcd and is the more natural multiplicity to consider for questions concerning the fundamental group, see [Ca01]. We define the $\mathbb{Q}$-divisor

$$D(f) = \sum \Big\{\ \Big(1 - \frac{1}{m(D, f)}\Big) D\ \Big|\ D \in \mathrm{Div}'(Y)\ \Big\}.$$

which is a finite sum supported on the discriminant locus $\Delta(f)$ of f. We call the pair $(Y; D(f))$ an orbifold which, in order to avoid ambiguity of notation, we will now always write as $Y \backslash D(f)$ and, in the case no ambiguity arises, also write as Y^∂ for simplicity. We define the canonical $\mathbb{Q}$ bundle and the Kodaira dimension of $Y^\partial = Y \backslash D(f)$ by

$$K_{Y^\partial} := K_Y(D(f)) \text{ and } \kappa(Y^\partial) = \kappa(Y \backslash D(f)) := \kappa(K_Y(D(f)))$$

respectively. We also write $K_{Y^\partial} = K_Y + D(f)$ as a $\mathbb{Q}$-divisor by abuse of notation.

Note that all of these definitions above are clearly unchanged if we change the birational model of X and therefore these definitions make sense for meromorphic fibrations, i.e., surjective meromorphic maps whose desingularizations have connected fibers.

Now, $Y^\partial = Y \backslash D(f)$ captures only part of the orbifold structure from f and ignores, for example, all codimension two structures imposed on Y, the most serious being that the divisors on X that project to codimension two subsets of Y are being ignored. To remedy this, we consider a related object on X which is one of the most basic geometric invariants of f.

DEFINITION 2.1 *Let $f : X \to Y$ be a fibration. We define $L_{(Y,f)}$ (or L_f for short) to be the saturation of f^*K_Y in Ω^r_X where K_Y is the canonical (or dualizing) sheaf of Y and $r = \dim Y$. We set $\kappa(Y, f) = \kappa(L_f)$ and call it the Kodaira dimension of the orbifold (Y, f) defined by f.*

We recall that the saturation of a rank r subsheaf L in a coherent sheaf Ω over X means taking the largest rank r subsheaf of Ω containing L. If Ω is locally free and L is of rank one, then the saturation of L in Ω is locally free (but not necessarily a vector subbundle except outside a codimension two subset) and is uniquely determined by any nontrivial meromorphic section of Ω, see [MP] and the references therein. Furthermore, the saturation remains saturated on any open subset.

Since nontrivial meromorphic differential forms are preserved by birational maps, a saturated line subsheaf L of Ω^i_X defines a unique such subsheaf $L' \hookrightarrow \Omega^i_{X'}$ on any birational model X' of X and we have $\kappa(L) = \kappa(L')$. So we may define L_f for any meromorphic fibration $f : X \dashrightarrow Y$. In fact, we may even allow Y here to be singular and then this notion of orbifolds, as far as we know, includes all previously known notions of orbifolds in complex geometry, see however [KeM]. Note that $\kappa(Y, f) := \kappa(L_f)$ depends only on the birational class of f in this sense. Moreover, $L_f \hookrightarrow \Omega^\bullet_X$ continues to make sense for any rational map $f : X \dashrightarrow Y$ as it is determined uniquely by its associated foliation on X and one can still make sense of (Y, f) as an "orbifold" or at least of $\kappa(Y, f) := \kappa(L_f)$ as its Kodaira dimension (in line with Grothendieck's philosophy that every map is a "fibration").

LEMMA 2.2 (Birational Invariance) *Consider a commutative diagram of mermorphic maps (i.e., where compositions are defined as meromorpic maps and the diagram commutes as meromorphic maps):*

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd} \tag{2.1}$$

where u is generically finite and surjective. Then there is a natural inclusion

$$v^\dagger : H^0(X, L_f^m) \hookrightarrow H^0(X', L_{f'}^m)$$

for all $m \in \mathbb{N}$ and thus

$$\kappa(Y', f') \geq \kappa(Y, f).$$

If moreover u and v are birational, then $v^\dagger$ is an isomorphism and so

$$\kappa(Y', f') = \kappa(Y, f).$$

Proof: By taking the Stein factorization of f and f', we may assume that they are meromorphic fibrations. Note that $f \circ v = u \circ f'$ is a well defined surjective rational map by commutativity of the diagram. Hence there is a Zariski open set U in X' where v is defined and where $L_{f'} = v^* L_f$. Let V be the open set where v is defined. Then, for all $m \in \mathbb{N}$, we have

$$L_{f'}^m \supset v^* L_f^m \tag{2.2}$$

as sheaves on V by the commutativity of diagram 2.1. Let $r : \tilde{X} \to X$ be a birational modification so that $\tilde{f} = f \circ r$ is a morphism. Then sections of powers of L_f corresponds to that of $L_{\tilde{f}}$. Since $L_{\tilde{f}}$ is trivial on the general fibers of $\tilde{f}$ and $f \circ v$ is surjective, we have an injection $H^0(X, L_f^m) \hookrightarrow H^0(X', v^* L_f^m)$. As $X' \setminus V$ is codimension two or more, the Hartog extension theorem gives the first part of the lemma. The second part follows similarly. ■

If the condition of the last part of the lemma is satisfied, we say that f and f' are **birationally equivalent fibrations**.

Now given an f' as in Lemma 2.2, it is clear from the Hartog extension theorem applied to Z' that $\kappa(Y, f) = \kappa(Y', f') \leq \kappa(Y' \backslash D(f'))$. It is in fact always possible to achieve equality here by choosing an appropriate f' as follows:

DEFINITION 2.3 *A fibration $f' : X' \to Y'$ is called* **admissible** *if there is a birational morphism $v : X' \to X$ with X smooth such that every reduced divisor E in X' with $f'(E)$ of codimension two or higher is exceptional with respect to v. The union of such divisors will be denoted by $E(f')$.*

LEMMA 2.4 *Let $f : X \to Y$ be a fibration. Then there is a natural inclusion*

$$H^0(X, L_f^m) \hookrightarrow H^0(Y, K_{Y^\partial}^m)$$

for $Y^\partial = Y \backslash D(f)$ and all $m > 0$ divisible by the multiplicities in Y^∂. In particular,

$$\kappa(Y, f) \leq \kappa(Y \backslash D(f)).$$

Also, one can find a commutative diagram as in diagram 2.1 with u, v birational and onto so that the birationally equivalent fibration f' is admissible. If $f' : X' \to Y'$ is an admissible fibration and m is divisible by the multiplicities of $D(f')$, then

$$H^0(X', L_{f'}^m) = H^0(Y, K_{Y' \backslash D(f')}^m) \quad \textit{and} \quad \kappa(Y', f') = \kappa(Y' \backslash D(f')).$$

Proof: The construction of a birationally equivalent admissible fibration is a basic starting point in the usual classification theory via Kodaira dimension and the Iitaka fibration. It is achieved by resolving the singularities of the flattening of f (by grace of [Hi] and, in the projective category, also by [GR]). For the last statement, we have by definition (with $r = \dim Y'$) the inclusion

$$f'^* K_{Y'}(D(f'))^m \hookrightarrow L_{f'}^m \left(\hookrightarrow (\Omega_{X'}^r)^{\otimes m} \right) \tag{2.3}$$

outside $O \cup E(f')$ where O is a subset of X' of codimension two or higher contained above $\Delta(f')$ and m is a positive multiples of all relevant multiplicities. Moreover, this inclusion is an isomorphism on an open subset of X' that surjects to the complement of a subset of Y' of codimension two or higher. Hence

$$H^0(Y', K_{Y'}(D(f'))^m) \hookrightarrow H^0(X, L_f^m) = H^0(X', L_{f'}^m)$$

by the Hartog extension theorem applied to X and the reverse inclusion by the Hartog extension theorem applied to Y. The first two statement now follows from the inclusion of (2.3) but for the fibration f and applying the Hartog extension theorem to Y since the said inclusion is an isomorphism on an open subset of X that surjects to the complement of a comdimension two subset of Y. ■

The inclusion (2.3) comes from the fact, true outside the inverse image of a codimension two subset S of Y, that $L_{f'} = f'^* K_Y((df'))$ where (df') is a divisor on X given by (c.f. [Re])

$$\sum \{ \ (f'^* D) - (f'^* D)_{red} \mid D \in \mathrm{Div}'(Y) \ \}$$

and the easily verified fact that $(df') \geq f'^* D(f')$ with equality achieved on an open set that surjects to $Y \smallsetminus S$, c.f. Lemma 3.19.

Remark 2.5 • *An f' satisfying the conclusion of Lemma 2.4 is called admissible by Campana in [Ca01]. (His definition of $\kappa(Y, f)$ is different, but is equivalent to ours by our result above.) In that paper, he posed the problem of finding an explicit condition satisfying his notion of admissibility. Our lemmas above give a solution to this problem.*
• *Readers not interested in the full orbifold version of our results can safely skip to Section 5 by ignoring the ∂'s on the total space (of fibrations) and some related terminologies.*

3 Log pairs as orbifolds and their fibrations

We now generalize our notion of orbifolds to the full log version as in Mori's Program by allowing the multiplicities to be rational numbers as well as ∞. That is, $Y\backslash B$ is a log orbifold (also known as a log pair or log variety, see [KM, KMM]) or simply an orbifold if B is a $\mathbb{Q}$-divisor of the form $\sum_i (1 - 1/m_i)D_i$ where $0 < m_i \in \mathbb{Q} \cup \{\infty\}$ and $D_i \in \mathrm{Div}'(Y)$ for all i. It is called a standard pair or a **standard orbifold** if $m_i \in \mathbb{N} \cup \{\infty\}$ for all i. We call $Y\backslash B$ an **effective orbifold** if B is an effective (possibly the trivial) divisor. Note that standard orbifolds are effective. Although many (if not most) of the results in this paper are worked out for arbitrary orbifolds, the important results in this paper only deals with effective orbifolds in the end.

It is also possible to consider real multiplicities $m_i > 0$ as in Mori's program at the cost of introducing more technical machinery for real divisors in additions to notations such as the round up. We will restrict to $\mathbb{Q}$-divisors in this paper for simplicity but the $\mathbb{R}$-divisor version is obtained similarly.

Given an orbifold structure B on Y, we call B the boundary divisor of $Y^\partial = Y\backslash B$ and write it also as

$$B = \partial(Y^\partial) = \sum \left\{ \left(1 - \frac{1}{m(D \cap B)}\right) D \;\middle|\; D \in \mathrm{Div}'(Y) \right\}.$$

We also write $m(D) = m(D, Y^\partial) = m(D \cap B)$. Note that for $D \in \mathrm{Div}'(Y)$, we have

$$0 < m(D \cap B) \leq \infty.$$

Also $m(D \cap B) = 1$ if $D \not\subset B_{red}$ and $m(D \cap B) = \infty$ if $D \leq B$ as a $\mathbb{Q}$-divisor. We define

$$\mu(D \cap B) = 1 - \frac{1}{m(D \cap B)} \leq 1$$

In general, if B' is another $\mathbb{Q}$-divisor on Y, we define

$$B \cap B' = \sum \left\{ \Big(\min\{\mu(D \cap B), \mu(D \cap B')\}\Big) D \;\middle|\; D \in \mathrm{Div}'(Y) \right\}.$$

We also set, as before,

$$K_{Y^\partial} := K_Y(B) = K_Y(\partial(Y^\partial)) \;\text{ and }\; \kappa(Y^\partial) = \kappa(Y\backslash B) := \kappa(K_Y(B))$$

respectively. If $m_i = \infty$ for all i and $B = \sum_i D_i$ is a (simple) normal crossing divisor, then Y^∂ is a **log-manifold** in the usual sense and one can define the log-tangent sheaf $T(Y, -\log B)$

(i.e., the orbifold tangent sheaf) of Y^∂ as the sheaf of vector fields on Y leaving B invariant. It is locally free with dual the log-cotangent sheaf $\Omega_{Y^\partial} = \Omega(Y, \log B)$. We note $K_{Y^\partial} = \det \Omega_{Y^\partial}$. Hence, this notion of orbifolds includes open subsets in the sense of log-manifold.

To be consistent with the theory of log-manifolds, we say that $Y\backslash D$ is a **smooth orbifold** or simply smooth if D_{red} is simple normal crossing. This allows, among others, a good notion of orbifold tangent and cotangent sheaves. A smooth orbifold $X^\partial = X\backslash A$ gives rise to a log-manifold in a natural way, which we denote by

$$X^{|\partial|} := X\backslash A_{red}.$$

Remark 3.1 *The above definition of orbifolds via boundary with arbitrary singularities is only provisional since only smooth orbifolds make good sense for the purpose of geometry. This is because the tangent and cotangent sheaves for them exist (on some space) and are locally free in this case (see the next section for a view to this). Our purpose in allowing arbitrary singularity in the above definition is to clarify certain issues in the literature as well as to explore some limits of such a definition. However, it should be mentioned that these sheaves, in the case the orbifold boundary is reduced for example, can be locally free without being normal crossing in the case of "free boundaries" introduced and studied by K. Saito ([Saito]). The free boundary orbifolds, which allow for example ordinary triple points in the boundary divisor of a complex surface, do play a role in the search for a good model. But* **for all practical purposes, it is sufficient to restrict attention to smooth orbifolds** *whenever it comes to orbifold maps or morphism. This is what we will do implicitly from now on as it is certainly no loss of generality from the view point of birational geometry.*

Smooth orbifolds are important in considering the birational geometry of orbifolds, for which we first give a preliminary discussion.

DEFINITION 3.2 *Let $u : Y_1 \to Y$ be a generically finite surjective morphism where Y_1^∂ and Y^∂ are orbifolds. We then write $u^\partial : Y_1^\partial \dashrightarrow Y^\partial$ as an orbifold rational map. We say that u gives rise to a* K-map

$$u^\partial : Y_1^\partial \dashrightarrow Y^\partial$$

if $K_{Y_1^\partial} - (u^* K_{Y^\partial})$ as a $\mathbb{Q}$-divisor defined by (du) is effective. If further u is birational, we say that u^∂ is K-birational if

$$K_{Y_1^\partial} = u^* K_{Y^\partial} + \sum_i a_i E_i$$

where each E_i is an exceptional divisor of u and $a_i \geq 0$ is a rational number. If $a_i = 0$ for all i, then we say that u^∂ is strictly K-birational.

It should be understood above that the trivial divisor is effective.

K-birational maps are used in the characterization of log-terminal and of log-canonical singularities of log-pairs in Mori's Program (our notion of orbifold can be restricted to that of log-canonical pairs and most of our results for smooth orbifolds hold in this more general context), see [KM] for example. It is a good definition of orbifold birational morphisms outside the singularities of the boundary divisors of the target orbifold, see Lemma 12.4. If one is only interested in preserving the Kodaira dimension for birational maps, it is quite sufficient. However, further conditions must be imposed in general to ensure the morphism condition. The problem is already present in the absolute case since a K-map $f^\partial : X^\partial \dashrightarrow Y^\partial$ between log-manifolds need not be a log-morphism, even if it is K-birational. A simple example is the following:

Example 3.3 *Let $f : X \to Y$ be the blowup of $Y = \mathbb{C}^2$ at $0 \in \mathbb{C}^2$, B is the y-axis and A is the strict transform of B. Then $f^\partial : X\backslash A \dashrightarrow Y\backslash B$ is a K-map but not a log morphism.*

We now consider orbifold fibrations (as a prelude to orbifold morphisms).

DEFINITION 3.4 *Let $f : X \to Y$ be a fibration with $X^\partial = X\backslash A$ an orbifold. We denote this by $f^\partial : X^\partial \to Y$ and call it an orbifold fibration, or fibration for short. We define $L_{(Y,f^\partial)}$ (or L_{f^∂} for short) to be $L_f(A \cap f)$ where*

$$A \cap f = \sum \Big\{ \Big(1 - \frac{1}{m(D \cap A)}\Big) D \;\Big|\; f(D) \neq Y,\ D \in \mathrm{Div}'(X) \Big\}.$$

We set

$$\kappa(Y, f^\partial) = \kappa(L_{f^\partial}),$$

which we call the **Kodaira dimension of the orbifold** (Y, f^∂).

The f^∂ as given in the definition imposes a branched orbifold structure on Y as follows: Given $D \in \mathrm{Div}'(Y)$, we may write $f^*D = \sum_i m_i D_i$ for $m_i \in \mathbb{N}$ and $D_i \in \mathrm{Div}'(X)$. Then we define the (minimum) multiplicity of f^∂ over D by

$$m(D, f^\partial) = \min\{\ m_i\, m(D_i \cap A) \mid f(D_i) = D\ \}.$$

Then the $\mathbb{Q}$-divisor

$$D(f^\partial) = D(f, \partial(X^\partial)) = D(f, A) := \sum \Big\{ \Big(1 - \frac{1}{m(D, f^\partial)}\Big) D \;\Big|\; D \in \mathrm{Div}'(Y) \Big\}$$

defines the **orbifold base** $Y^\partial = Y\backslash D(f^\partial)$ of f^∂. If $B \leq D(f^\partial)$ as $\mathbb{Q}$-divisors, then we say that the orbifold fibration f^∂ (and $Y\backslash D(f^\partial)$) dominates $Y\backslash B$.

We remark that we could have arrived at the full log version of orbifolds by considering the branched orbifold base of a fibration from a log-manifold this way (see Definition 3.17). The following are immediate consequences of the definition.

LEMMA 3.5 *Let $f^\partial : X^\partial \to Y$ be a fibration with $X^\partial = X\backslash A$. Let $B = D(f^\partial)$ and $Y^\partial = Y\backslash B$. Then, above the complement U of a codimension two subset of Y, we have*

$$(df) + A \geq f^*B \quad \text{where} \quad (df) = f^*B - (f^*B)_{red}$$

as $\mathbb{Q}$-divisors on X with equality achieved everywhere on U (compare Lemma 12.4, the paragraph after Lemma 2.4 and the slightly more general equations 3.3 and 3.4).

LEMMA 3.6 *Let $B = D(f^\partial) = D(f, A)$ for a fibration $f^\partial : X\backslash A \to Y$, $g^\partial : Y\backslash B \to Z$ a fibration and $h = g \circ f$. Then $D(g^\partial) \geq D(h^\partial)$, i.e., $D(g, D(f, A)) \geq D(g \circ f, A)$. If the exceptional part of A with respect to f is reduced, then equality holds.*

Now, all the above definitions for an orbifold fibration actually make sense for an arbitrary (rational) map f from an orbifold, such as $A \cap f$ and L_{f^∂} for example, because f is a morphism outside a codimension two subset. We will adopt this more general usage of notation and, in the next definition, even allow f^∂ to have noncompact open sets as domains, where our definitions above generalize straightforwardly.

Observe that if $f^\partial : X^\partial \to Y$, $g^\partial : Y^\partial \to Z$ are surjective and $h^\partial = g \circ f^\partial$, then $f^*L_g \subset L_h$ is a nontrivial subsheaf so that $L_h - f^*L_g$ is an effective Cartier divisor on X, playing in some sense the role of "$(df) \cap h$." Hence $L_{h^\partial} - f^*L_{g^\partial}$ is uniquely given by a $\mathbb{Q}$-divisor. A $\mathbb{Q}$-divisor D is called effective, written $D \geq 0$, if it is a nonnegative linear combination of reduced divisors. More generally, we will drop the surjectivity condition on f, the compactness assumption on X and Y and even allow g to be meromorphic and Z to be singular. We only require that $h^\partial = g \circ f^\partial$ is defined on an open set and extends to a holomorphic map onto a non-empty open subset of Z and that the graph of g restricted to the image of f is irreducible. We then say that the map g **composes holomorphically** with f.

DEFINITION 3.7 *Let $u : Y_1 \to Y$ be a morphism where Y_1^∂ and Y^∂ are orbifolds. We then write $u^\partial : Y_1^\partial \dashrightarrow Y^\partial$ as an orbifold rational map. We say that u gives rise to an* **orbifold morphism**

$$u^\partial : Y_1^\partial \to Y^\partial$$

(or simply that u^∂ is a morphism) if for every open subset U of Y and a meromorphic map $g : U \dashrightarrow Z$ that composes holomorphically with $u_U := u|_{u^{-1}(U)}$, we have

$$L_{g_1^\partial} - (u^*L_{g^\partial}) \geq 0.$$

If equality is attained everywhere, then we say that Y_1^∂ has the **induced orbifold structure** *from Y^∂ via u, denoted by $u^\circ\partial Y^\partial$, and if further u is an inclusion, then we say that u^∂ is an* **orbifold inclusion**. *If we have equality outside a codimension two subset of Y independent of g above and u is birational, then we say that u^∂ is a* **birational morphism** *or simply that u^∂ is birational. An orbifold morphism $u^\partial : Y_1^\partial \to Y^\partial$ is called* **smooth** *at a point $p \in Y_1$, if p has a neighborhood in which u is equidimensional, the reduced fiber F through p is smooth and in which F is not contained in, but is transversal to D_{red} where*

$$D := \partial Y_1^\partial - u^\circ\partial Y^\partial = \partial Y_1^\partial - \big(u^*\partial Y^\partial - (L_u - u^*K_Y)\big)$$

in the sense that D_{red} is locally trivial with respect to u in a neighborhood of p. If u^∂ is a morphism and u is a fibration, then we call u^∂ a **fibration morphism** *and we denote its discriminant locus by $\Delta(u^\partial)$, the image of the nonsmooth locus of u^∂.*

We note that if $\partial Y^\partial \geq 0$ and Y_1^∂ and Y^∂ are smooth orbifolds, an orbifold morphism $u^\partial : Y_1^\partial \to Y^\partial$ must satisfy

$$u^{-1}(\partial Y^{|\partial|}) \subset \partial Y_1^{|\partial|}.$$

It is immediate that the discriminant locus $\Delta(u^\partial)$ of a fibration morphism $u^\partial : Y_1^\partial \to Y^\partial$ is always a proper algebraic subset outside of which the fibers are differentiably isomorphic as "differentiable" orbifolds. Also, if $i : F \hookrightarrow Y_1$ is the inclusion of a fiber of u outside $u^{-1}(\Delta(u^\partial))$, then the induced orbifold structure on f is just $i^*\partial Y_1^\partial$ and it has normal crossing support if Y_1^∂ does.

Of course, the above condition defining an orbifold morphism is really a local one and it is not necessary to consider all maps $g : Y \dashrightarrow Z$ but only a subset of maps each of which takes some components of $\partial(Y^\partial)$ as fibers, see Lemma 12.4 and Lemma 4.3 and its proof for further simplifications in particular circumstances. In fact, the notion of orbifold morphism for smooth orbifold is very simple geometrically and has much simpler characterizations, see Appendix A. However, the above definition is a more practical and direct approach to our main results.

A particular class of orbifold morphisms consists of log-morphisms between log-manifolds as we show below. In this case the characterization is particularly simple in sharp contrast to the general case. Hence given two log-manifolds $X\backslash A$ and $Y\backslash B$, a map $f^\partial : X\backslash A \dashrightarrow Y\backslash B$ is an orbifold morphism if and only if it is a log-morphism, namely $f^{-1}(B) \subset A$. This comes from the fact that we actually have an inclusion of the log-cotangent sheaves in this case.

LEMMA 3.8 *Let $f : X \to Y$ be a morphism where X^∂ and Y^∂ are log-manifolds. Then f^∂ with respect to these orbifold structures is an orbifold morphism if and only if it is a log morphism.*

Proof: We only prove the "if" direction since the other direction is not needed. The result then follows from the next self-evident lemma and the fact that a saturated line subsheaf of a vector bundle on X is locally free as it must be reflexive and is, outside a codimension two subset of X, a subbundle (see for example [MP]). ■

LEMMA 3.9 *Let L be a saturated line subsheaf of Ω^i_X and $X\backslash A$ a log-manifold. Then the saturation of L in $\Omega^i_X(\log A)$ is $L(A')$ where A' consists of components D of A whose normal bundles N_D satisfy $N^*_D \wedge L = 0$ in $\Omega^{i+1}_X(\log A)|_D$. Hence given a fibration $f^\partial : X\backslash A \dashrightarrow Y$, $L_{f^\partial} := L_f(A \cap f)$ is the saturation of L_f in $\Omega^r_X(\log A)$ where $r = \dim Y$.* ■

DEFINITION 3.10 *A fibration $f^\partial : X^\partial \to Y$ is called* **admissible** *if there is a birational orbifold morphism $v^\partial : X^\partial \to X_0^\partial$ such that every reduced divisor E in X with $f(E)$ of codimension two or higher is exceptional with respect to v. It is called* **very admissible** *if further X^∂ is smooth and* **adapted** *if furthermore $Y\backslash D(f^\partial)$ is smooth. If moreover $f^\partial : X^\partial \to Y\backslash D(f^\partial)$ is a morphism, then f^∂ is called* **well adapted**. *An orbifold fibration is called* **good** *if it is a morphism to its orbifold base. An orbifold Y^∂ is called an* **admissible orbifold** *if it is the orbifold base of an admissible fibration $f^\partial : X^\partial \to Y$ from a smooth orbifold. It is called effectively admissible or* **e-admissible** *if $\partial X^\partial \cap E(f)$ is effective.*

It is important to note that, by definition, *smooth orbifolds are e-admissible.* The following are key lemmas that allow for a good theory of birational geometry for smooth orbifolds.

Recall that the exceptional divisor $E(f)$ of a morphism f is the union of prime divisors whose images under f are codimension two or higher.

LEMMA 3.11 (1) *Let $f: X \to Y$ be a generically finite surjective morphism with $Y^\partial = Y \backslash B$ a smooth orbifold. Suppose $X\backslash\bar{A}$ is a smooth orbifold for $\bar{A} = (f^*B)_{red}$. Then X can be endowed with a smooth (and standard) orbifold structure A supported on $\bar{A}$ (not unique in general) so that $A = f^*B$ outside the exceptional divisor of f and so that $f^\partial : X^\partial \to Y^\partial$ is a morphism. In particular, one can choose f^∂ to be birational in the case f is birational.*

(2) *Let $f^\partial : X^\partial \to Y$ be a fibration with $Y^\partial = Y\backslash B$ smooth for $B = D(f^\partial)$. Assume that $f^{|\partial|}$ is a log morphism. Then the multiplicities of X^∂ along $E(f)$ can be increased if necessary so that the resulting orbifold $X^{\partial'}$, which gives rise via f to the same orbifold base B, gives rise to an orbifold morphism $f^{\partial'} : X^{\partial'} \to Y^\partial$ via f. If further f^∂ is admissible, then X^∂ and $X^{\partial'}$ are birationally equivalent as orbifolds.*

Proof: This follows directly by considering the absolute case $f^{|\partial|} : X\backslash\bar{A} \to Y\backslash B_{red}$, which is a log-morphism and hence an orbifold morphism by Lemma 3.8. Here, we take $\bar{A} = (f^*B)_{red}$ for (2) as well. ■

We note that the choice of multiplicities can always be taken to be standard and not ∞ on any component of $E(f)$ that does not map into a divisor with multiplicity ∞.

We generalize the notion of orbifold morphisms in the special case of maps between orbifold base of fibrations as follows.

DEFINITION 3.12 *Given a commutative diagram of morphisms*

$$\begin{array}{ccc} X_1^\partial & \xrightarrow{v^\partial} & X^\partial \\ {\scriptstyle f_1^\partial}\downarrow & & \downarrow{\scriptstyle f^\partial} \\ Y_1 & \xrightarrow[u]{} & Y \end{array} \tag{3.1}$$

where f_1^∂ and f^∂ are fibrations with f^∂ admissible and where v^∂ gives a morphism of orbifolds X_1^∂ and X^∂. Then we say that u gives rise to a **premorphism**

$$u^\partial : Y_1\backslash D(f_1) \to Y\backslash D(f).$$

If v^∂ and u are birational, then we say that u^∂ is **prebirational**. *A composition consisting of premorphisms and morphisms is called a* **promorphism** *and if the factors are birational, we say that it is* **probirational**.

LEMMA 3.13 *Given a fibration $f^\partial : X^\partial \to Y$ with $Y^\partial = Y\backslash D(f^\partial)$, we have for $m > 0$ divisible by all the relevant multiplicities that*

$$H^0(X, L^m_{f^\partial}) \hookrightarrow H^0(Y, K^m_{Y^\partial}) \quad \text{and so} \quad \kappa(Y, f^\partial) \leq \kappa(Y^\partial).$$

Moreover, if f^∂ is an admissible fibration, then

$$H^0(X, L^m_{f^\partial}) = H^0(Y, K^m_{Y^\partial}) \quad \text{and so} \quad \kappa(Y, f^\partial) = \kappa(Y^\partial).$$

***If** X^∂ **is smooth, one can find** a commutative diagram as in diagram (3.1) with v^∂ and u birational and onto so that the **birationally equivalent fibration** f_1^∂ is **well adapted and** hence **good**. Also*

$$\kappa(Y_1, f_1^\partial) = \kappa(Y, f^\partial).$$

The same holds for X^∂ admissible with v^∂ weakened to prebirational.

Proof: The proof is essentially the same as Lemma 2.4 and Lemma 2.2 using Lemma 3.11. The proof of the last part is also contained in the proof of Proposition 3.19. It is sufficient to observe here that an adapted fibration is well adapted if one choose the multiplicities of the components of the exceptional divisors of v as given in Definition 3.10 above to be ∞ and that

$$1 - \frac{1}{m} + \frac{1}{m}\left(1 - \frac{1}{n}\right) = 1 - \frac{1}{mn}. \quad \blacksquare$$

The cautious reader would note, in comparing this lemma with Lemma 2.4, that the birationally equivalent admissible fibration f' there cannot be chosen in general to be an orbifold morphism to its orbifold base, the crucial difference being that we are allowing orbifold structures on the total space of a fibration in the above lemma. We record here the following obvious but useful fact.

LEMMA 3.14 *Consider the commutative diagram (3.1) of morphism above with X^∂, X_1^∂ (smooth) orbifolds, f, f_1 fibrations and u, v^∂ birational. If f^∂ is admissible, then so is f_1^∂.* $\blacksquare$

Note that the class of premorphism includes that of orbifold morphisms and that of adapted fibrations, the orbifold analogue in some sense of strictly rational maps (c.f., [Ii]). In view of Lemma 3.11, the category of admissible orbifolds and promorphism may serve in considering the birational geometry of (not necessarily smooth) orbifolds. The use of this notion is not indispensable in this paper although it allows for a larger class for which most of the theorems in this paper hold. It would be of much interest to have a more intrinsic and direct characterization of a good class of maps that includes premorphisms for this purpose. It is easily seen from the definition that a composition of orbifold morphisms is an orbifold morphism. This means in particular that a (pro)morphism from Y^∂ remains so when restricted to the general fibers of a fibration $g : Y \to Z$. This is easily verified to be true also for a premorphism. We record these facts as follows.

LEMMA 3.15 *Let $f^\partial : X^\partial \to Y^\partial$, $g^\partial : Y^\partial \to Z^\partial$ and $h^\partial = g^\partial \circ f^\partial$. If f^∂ and g^∂ are (pre)morphisms, then so is h^∂. If f^∂ is a morphism and g^∂ a promorphism, then h^∂ is a promorphism. If g is a fibration and $f_z^\partial : X_z^\partial \to Y_z^\partial$ the restriction of f^∂ to the general fiber X_z^∂ of h, then f_z^∂ is admissible, a morphism, a premorphism or a promorphism respectively if so is f^∂.* $\blacksquare$

It is understood whenever we speak of a premorphism $u^\partial : Y_1^\partial \to Y^\partial$ from now on that we have a commutative diagram as above where Y_1^∂ and Y^∂ are the orbifold base of fibrations (in practice, from smooth orbifolds or from log-manifolds) X_1 and X respectively with the latter assumed to be admissible and where v^∂ is a morphism. We remark that an orbifold map is really properly understood by the fact that the differential of the map gives a (linear and holomorphic) map of the orbifold tangent bundles (which can be understood as the "tangent bundle" of the corresponding V-manifold for example) or alternatively as done in the next section by orbifold cotangent sheaves. But for our purpose here, it is sufficient to take this more indirect but perhaps more practical approach.

The connection between premorphisms, morphisms and adapted or good orbifold fibrations can be seen from the following lemma.

LEMMA 3.16 *In the situation of definition 3.12, suppose f^∂ is a good (or more strongly, a well adapted) fibration. Then u gives rise to an orbifold morphism*

$$u^\partial : Y_1 \backslash D(f_1^\partial) \to Y \backslash D(f^\partial).$$

Proof: Set $Y_1^\partial = Y_1 \backslash D(f_1^\partial)$, $Y^\partial = Y \backslash D(f^\partial)$ and $B = D(f^\partial)$. Let V be a neighborhood of a point $p \in B_{red} \cap u(Y_1)$ and $g : Y \cap V \dashrightarrow Z$ a meromorphic map that compose holomorphically with $u|_{u^{-1}(V)}$ which we still denote by u by abuse of notation. We need to show that the $\mathbb{Q}$-divisor $L_{g \circ u^\partial}$ arising from du is effective on an open subset of the form $W = u^{-1}(U)$ for a small neighborhood U of p in V on which L_{g^∂} is trivial. Now a nowhere vanishing section of $L^l_{g^\partial}|_U$ gives rise to a section of $L^l_{g \circ f^\partial}|_{f^{-1}(U)}$ as f^∂ is a morphism, which in turn gives rise to a section of $L^l_{g \circ f \circ v^\partial} = L^l_{g \circ u \circ f_1^\partial}$ above W as v^∂ is a morphism and finally to a section of $L_{g \circ u^\partial}$ on W by the Hartog extension theorem (c.f. Proposition 3.19), where l is chosen to divide all multiplicities so that the l-th tensor powers above makes sense as line bundles. So the desired conclusion follows. ∎

DEFINITION 3.17 *Let $f^\partial : X^\partial \to Y$ be an admissible fibration with $X^\partial = X \backslash A$ a log-manifold. Given a fibration $g^\partial : Y^\partial = Y \backslash D(f^\partial) \to Z$, we define $L(Z, g^\partial; f^\partial)$ to be the saturation of $f^* g^* K_Z$ in $\Omega^s(X, \log A)$ where $s = \dim Z$ and we let*

$$L(Z, g^\partial \circ f^\partial) = L(Z, g^\partial; f^\partial).$$

We set $\kappa'(Z, g^\partial; f^\partial) = \kappa(L(Z, g^\partial; f^\partial))$ and $\kappa'(Z, g^\partial) = \kappa(L(Z, g^\partial; \mathrm{id}))$.

A crucial point is the fact that $(Z, g^\partial; f^\partial)$ is independent of f^∂ in the sense that $\kappa'(Z, g^\partial; f^\partial)$ is and in fact we will identify $\kappa'(Z, g^\partial; f^\partial)$ with $\kappa(Z, g^\partial)$. We now show this fact by first warming up with the following special case.

LEMMA 3.18 *Assume that $f^\partial : X^\partial \to Y$ is an adapted fibration with $X^\partial = X \backslash A$ a log-manifold. Given a fibration $g : Y \to Z$, define L_Y (resp. L_X) to be the saturation of $g^* K_Z$ in $\Omega(Y, \log D(f^\partial))$ (resp. $f^* g^* K_Z$ in $\Omega(X, \log A)$). Then $L_X = L_{g \circ f^\partial}$ and $\kappa(L_X) = \kappa(L_Y)$.*

Proof: The first equality follows from Lemma 3.9. For the second, let $s = \dim Z$ and $v^\partial : X^\partial \to X_0^\partial$ be a birational morphism for which $E(f)$ is exceptional. For any $l \in \mathbb{N}$, any section of $\Omega^s(Y, \log D(f^\partial))^{\otimes l}$ gives rise to a section of $\Omega^s(X, \log A)^{\otimes l}$ outside $E(f)$ which therefore extends over $E(f)$ via Hartog extension on X_0. So $\kappa(L_X) \geq \kappa(L_Y)$. Conversely, there is an open subset U with codimension two complement in Y where $f|_{f^{-1}(U)}$ is smooth as a log-morphism, i.e., where $f_*(T(X, -\log A))$ has maximal rank in $T(Y, -\log D(f^\partial))$, since $D(f^\partial)$ is reduced. Hence, $L_X|_{f^{-1}(U)} = f^*(L_Y|_U)$ so that $\kappa(L_X) \leq \kappa(L_Y)$ via Hartog extension on Y. ∎

PROPOSITION 3.19 *Let $f^\partial : X^\partial \to Y$, $g : Y \dashrightarrow Z$ be fibrations and $B = \partial Y = D(f^\partial)$. Let $h^\partial = g^\partial \circ f^\partial$, where $g^\partial : Y \backslash B \dashrightarrow Z$ is the orbifold fibration that arises. Then there is an inclusion*

$$f^* L_{g^\partial}^N \hookrightarrow L_{h^\partial}^N \tag{3.2}$$

outside $f^{-1}(U) \cup E(f)$ where U is an open subset of Y with codimension two complement and N is a positive multiples of all relevant multiplicities. Moreover, this inclusion is an isomorphism on an open subset of X that surjects to the complement of a codimension two subset of Y. Thus, there is an inclusion

$$H^0(L_{h^\partial}^l) \hookrightarrow H^0(L_{g^\partial}^l)$$

for all $l > 0$ divisible by the relevant multiplicities and so

$$\kappa(Z, h^\partial) \leq \kappa(Z, g^\partial) := \kappa(L_{g^\partial}).$$

In addition, in the case X^∂ is a log-manifold, one can replace $L_{h^\partial}^N$ by $L(Z, g^\partial; f^\partial)^N$ in (3.2) without affecting any conclusion. Also, if f^∂ is admissible, then

$$H^0(L_{h^\partial}^l) = H^0(L_{g^\partial}^l) \quad \textit{and} \quad \kappa(Z, g^\partial) = \kappa(Z, h^\partial) := \kappa(L_{h^\partial})$$

and (in the case X^∂ is a log-manifold)

$$\kappa(Z, g^\partial) = \kappa'(Z, g^\partial; f^\partial) := \kappa(L(Z, h^\partial)).$$

In particular, if Y^∂ is a log-manifold (for f the identity morphism), we obtain

$$\kappa(Z, g^\partial) = \kappa'(Z, g^\partial) := \kappa(L(Z, g^\partial)).$$

Proof: By applying the Hartog extension theorem, the last four statements follow directly from the claim about (3.2), which we now prove. (In fact, the last two statements also follow directly from Lemma 3.9.)

Replacing Z by a modification, we may assume that all prime divisors in Y not dominating Z maps onto divisors in Z. By this, we may take an open set $U \subset Y$ with codimension two complement on which g is equidimensional (onto the new Z) and where g is smooth when restricted to each prime divisor.

Now, outside the discriminant locus $\Delta(f)$ of f, the first inclusion in (3.2) is an isomorphism. So it remains to look at a divisorial component of $\Delta(f)$. Let R be such a component. If $g(R) = Z$, then $L_g = g^*K_Z$ on a neighborhood of $R \cap U$ in U by definition and so $L_h = L_{h^\partial} = f^*L_{g^\partial} = h^*K_Z$ $(= L(Z, h^\partial)$ in the case X^∂ is a log-manifold) there. If not, let $S = g(R) \in \mathrm{Div}'(Z)$. From now on, we will only consider our problem over a neighborhood of $R \cap U$ in U and outside $E(f)$ in X. There, we have $g^*(S) = mR$ for some $m \in \mathbb{N}$. Let $f^*R = \sum m_i D_i$ for $D_i \in \mathrm{Div}'(X_1)$ and set $m(D_i) = m_i\, m(D_i \cap \partial X^\partial)$ and $m' = \min_i m(D_i) = m(R \cap \partial Y^\partial)$. Now, by a direct local computation, we have (compare [Re]) that

$$(L_{h^\partial}) = h^*K_Z + h^*S - (h^*S)_{red} + \partial X^\partial = h^*K_Z + \sum_i (m \cdot m_i - m_i/m(D_i))D_i \tag{3.3}$$

as divisors and similarly that $(L_g) = g^*K_Z + (m-1)R$. Hence

$$(f^*L_{g^\partial}) = (f^*L_g) + (1 - 1/m')f^*R = h^*K_Z + \sum_i (m - 1/m')m_i D_i. \tag{3.4}$$

We thus see that $(f^*L_{g^\partial}) \leq (L_{h_1^\partial})$ as a $\mathbb{Q}$-divisor above this neighborhood of R and outside $E(f)$ with equality achieved above every point of $R \cap U$ (on D_{i_0} where $m(D_{i_0}) = m'$). This coupled with the natural isomorphism of L_h with f^*L_g outside $\Delta(f)$ in U establishes our claim about (3.2), and so the lemma follows. ■

The following are straightforward generalizations of Lemma 2.2 and Lemma 2.4 to orbifold fibrations respectively.

LEMMA 3.20 (Birational Invariance) *Consider a commutative diagram of maps:*

$$\begin{array}{ccc} X_1^\partial & \xrightarrow{v^\partial} & X^\partial \\ {\scriptstyle f_1^\partial}\downarrow & & \downarrow{\scriptstyle f^\partial} \\ Y_1^\partial & \xrightarrow{u^\partial} & Y^\partial \\ {\scriptstyle g_1^\partial}\downarrow & & \downarrow{\scriptstyle g^\partial} \\ Z_1 & \xrightarrow{w} & Z \end{array} \tag{3.5}$$

where f_1^∂, f^∂ are orbifold fibrations with orbifold bases $Y_1^\partial = Y_1 \backslash D(f_1^\partial)$ and $Y^\partial = Y \backslash D(f^\partial)$, g_1^∂ and g^∂ are fibrations, v^∂ is a morphism and w is generically finite and surjective. If f^∂ is admissible (so that u^∂ is a premorphism), then we have

$$H^0(L^l_{g^\partial}) \hookrightarrow H^0(L^l_{g_1^\partial}) \quad \text{and so} \quad \kappa(Z_1, g_1^\partial) \geq \kappa(Z, g^\partial)$$

for all $l > 0$ divisible by the relevant multiplicities. If u^∂ is a morphism, then this remains true without the upper part of the diagram. If moreover u^∂ and thus w are birational, then we have (without the upper part of the diagram),

$$H^0(L^l_{g^\partial}) = H^0(L^l_{g_1^\partial}) \quad \text{and so} \quad \kappa(Z_1, g_1^\partial) = \kappa(Z, g^\partial).$$

If v^∂ is birational and onto and u is generically finite (hence birational), then the same conclusions hold (with the upper part of the diagram). Moreover, all the results remain valid if g, g_1 and w are only assumed to be meromorphic.

Proof of Lemma 3.20: The last part of the lemma without the last statement and without assuming that the upper part of the diagram exists, follows the same way as that of Lemma 2.2. If we assume that f^∂ is admissible, then the proof of the first part and the second last statement is also the same as that of Lemma 2.2 using Lemma 3.14 and Proposition 3.19. Hence the result follows. ∎

Note in particular that a prebirational morphism preserves orbifold Kodaira dimension (by taking g and g_1 to be the identity map).

LEMMA 3.21 *Let the second column of diagram 3.5 be given as in the previous lemma with X^∂ smooth. Namely, let $f^\partial : X^\partial \to Y$, $g : Y \to Z$ be fibrations with X^∂ smooth, $Y^\partial := Y \backslash D(f^\partial)$. Then one can complete this column to a diagram as in diagram 3.5 with u^∂, v^∂ birational and onto so that the fibrations $f_1^\partial, g_1^\partial$ and $h_1^\partial = g_1^\partial \circ f_1^\partial$ are all well adapted and all divisors in X (resp. Y) not mapping onto Y (resp. Z) push forward to divisors in Y_1 (resp. Z_1). f_1^∂ and h_1^∂ are birationally equivalent to f^∂ and $h^\partial = g^\partial \circ f^\partial$ resp., and if f^∂ is admissible, g_1^∂ is also birationally equivalent to g^∂. Hence if f^∂ is admissible, then there are birationally equivalent fibrations to f^∂, g^∂ and h^∂ such that equality is attained in Lemma 3.6. Moreover, there is an open subset U of Y with codimension two complement on which f and f_1 are isomorphic under this correspondence.*

If further $g^\partial \circ f^\partial : X^\partial \to Z$ is admissible, then

$$\kappa(Z, g^\partial; f^\partial) = \kappa(Z \backslash D(g^\partial)).$$

Proof of Lemma 3.21: The last statement follows from Lemma 2.4. So we only need to construct f_1 and g_1. For this purpose, we may first assume that f^∂ is well adapted by blowing up and replacing it with a birationally equivalent one using Lemma 3.13. Then we take a flattening of f and let the resulting flat fibration be $f' : X' \to Y'$ and the base change map be $u' : Y' \to Y$. Then we take a flattening of $g \circ u'$ and let the resulting flat fibration be $g_0 : Y_0 \to Z_0$ with map $u_0 : Y_0 \to Y'$. Then we base change f' by u_0 to give a flat fibration $f_0 : X_0 \to Y_0$. We now normalize X_0 and Y_0 and denote the resulting equidimensional fibrations by the same symbols f_0, g_0 and $g_0 \circ f_0$. Let $u_1 : Y_1 \to Y_0$ be a resolution of singularities of Y_0. Since f^∂ is well adapted, the resulting birational morphism from Y_1 to Y is an orbifold birational morphism by Lemma. 3.16. Then $g_1 := g_0 \circ u_1$ is admissible. Now let $f'' : X'' \to Y_1$ be the base change of f_0 by u_1. Let $v_1 : X_1 \to X''$ be a resolution of singularities of X'' so that the map from X_1 to X gives an orbifold birational morphism. Then $f_1 := f'' \circ v_1$ is admissible since f'' is equidimensional. Also $g_1 \circ f_1$ is admissible since it factors through the equidimensional morphism $g_0 \circ f_0$. Hence if we chosse the resolutions to be normal crossing and the multiplicity of the exceptional divisors on X_1 to be ∞, we see that f_1, g_1 and h_1 are all well adapted. ∎

We see in particular that $\kappa(Z, g^\partial) = \kappa(L_{g^\partial})$ is a birational invariant of orbifolds. In this sense, we may speak of $L(Z, g^\partial)$ (*as an equivalence class of sheaves*) as a well defined object for g^∂ and its birational class.

We summarize the main result of this section in the log-case: Given an adapted fibration $f^\partial : X^\partial \to Y$ where $X^\partial = X\backslash B$ is a log-manifold, there is an injective correspondence between birational equivalence classes of orbifold fibrations from $Y^\partial = Y\backslash D(f^\partial)$ and saturated subsheaves of $\Omega^i(X, \log B)$ which are trivial on the generic fibers of f for $i = 1, \ldots, \dim Y$. Furthermore, the Kodaira dimension of such a subsheaf is the Kodaira dimension of the orbifold $Z_1\backslash D(g_1^\partial)$ for an admissible birationally equivalent fibration $g_1^\partial : Y_1^\partial \to Z_1$, which one can always attain by flattening any orbifold fibration corresponding to the subsheaf and then resolving the singularities of the resulting family suitably.

4 The birational geometry of projective orbifolds

For simplicity of treatment, all orbifolds treated in this chapter are standard. The general non-standard case can be obtained by considering (in general) a morphism (or fibration) from a standard orbifold X^∂ to Y^∂ as the composition of a morphism (respectively fibration) from a smooth orbifold $\tilde{X}^\partial$ with the inverse of a finite morphism from $\tilde{X}^\partial$ to X^∂. Alternatively, if one is not strictly interested on line sheaves of holomorphic p-forms, on can simply regard the L_{g^∂} constructed in this section for the nonstandard case as line sheaves on $\tilde{Y}$ which are pullbacks of $\mathbb{Q}$-line sheaves on Y. All the results of this section are valid in the more general category this way.

In the projective category, we have the following basic technical covering lemma of Kawamata ([Ka, Theorem 17, 19], [KMM, Theorem 1-1-1]) for reducing a question on a smooth orbifold to a corresponding question concerning ordinary manifold. In general, the technical lemma is part of some fundamental results in the theory of log-schemes mentioned in [LM1], which we will discuss elsewhere.

LEMMA 4.1 (Kawamata Branched Covering Theorem) *Let $\gamma' : Y' \to Y$ be a finite surjective morphism from a normal variety Y' to a smooth variety Y and D a reduced divisor on Y. Suppose $D(\gamma')_{red} + D$ is a simple normal crossing divisor and $D = \sum_i D_i$, where $D_i \in \mathrm{Div}'(Y)$ for all i. Given $n_i \in \mathbb{N}$, there exist a finite Galois covering $\gamma : \tilde{Y} \to Y$ factoring through γ' with $\tilde{Y}$ smooth. Moreover, $\gamma^{-1}(D) \cup R$ is simple normal crossing where $R = (\det(d\gamma))_{red}$.*

Proof: The statement above is slightly more general than that stated in [Ka] but the proof is exactly the same. ■

LEMMA 4.2 *Let $Y^\partial = Y\backslash B$ be a smooth orbifold. Then*

(I) *There exist a finite Galois covering $\gamma : \tilde{Y} \to Y$ whose ramification divisor R on $\tilde{Y}$ is supported on a simple normal crossing divisor containing $\gamma^*(B_{red})$ and $\gamma^*(B)$ is a Cartier divisor on $\tilde{Y}$.*

(II) *There is a unique locally free subsheaf Ω_{Y^∂} of $\Omega_{\tilde{Y}}$ with determinant sheaf $\gamma^* K_{Y^\partial}$ such that Ω_{Y^∂} is the natural subsheaf $\gamma^*\Omega_Y \hookrightarrow \Omega_{\tilde{Y}}$ outside R_{red}. In particular, Ω_{Y^∂} is invariant under the Galois group G of γ.*

Proof: (I) follows from the previous lemma. (II) follows from (I) by the following observation: Each $p \in Y$ has a coordinate neighborhood U with coordinate $(y_1, \ldots, y_m)$ on which we may write $\gamma = \delta \circ \alpha$ for

$$\delta : \Delta^r \twoheadrightarrow U \subset Y; \quad (x_1, \ldots, x_r) \mapsto (x_1^{M_1}, \ldots, x_r^{M_r}),$$

where $B|_U$ is supported on the coordinate axis, $M_i = m(D_i, Y^\partial)$ or 1 according to whether $m(D_i, Y^\partial)$ is finite or not and where we have arranged the indices so that $D_i|_U = (y_i)$ for all i in $B_{red} = \sum_i D_i$. We then define $\Omega_{Y^\partial}|_{\gamma^{-1}(U)} = \alpha^* \Omega_{\Delta^r}(\log D) \hookrightarrow \Omega_{\tilde{Y}}|_U$ where $D = \sum\{(y_i) \mid m(D_i, Y^\partial) = \infty\}$. Hence, we may conclude the proof by noting that this definition is invariant under the Galois group of f and that $\det \Omega_{Y^\partial} = \gamma^* K_{Y^\partial}$ by construction. ■

We call the covering γ so constructed a **∂-covering** of $Y \backslash B$. It can also be written as a finite surjective orbifold "étale" morphism:

$$\gamma : \tilde{Y} \to Y^\partial.$$

In general, a finite surjective morphism $\gamma^\partial : \tilde{Y}^\partial \to Y^\partial$, is called **étale** if the induced structure from Y^∂ on $\tilde{Y}$, namely

$$\gamma^*(\partial Y^\partial) - (\det(d\gamma))$$

coincides with that of $\tilde{Y}^\partial$.

LEMMA 4.3 *Let $u : Y_1 \to Y$ be a generically finite surjective morphism with Y_1^∂ and Y^∂ smooth orbifolds. Let $\gamma : \tilde{Y} \to Y$ be a ∂-covering of $Y \backslash B$ (as given in Lemma 4.2). Then there is a finite Galois covering γ_1 as given by Lemma 4.2 for the orbifold Y_1^∂ where $u \circ \gamma_1 = \gamma \circ \tilde{u}$ for a generically finite morphism $\tilde{u} : \tilde{Y}_1 \to \tilde{Y}$. Furthermore, u^∂ is a morphism if and only if there is a natural inclusion $\iota : \tilde{u}^* \Omega_{Y^\partial} \hookrightarrow \Omega_{Y_1^\partial}$ which is an isomorphism on $\gamma_1^{-1}(W)$ for the open set W on which u is an isomorphism. In particular, if u^∂ is orbifold birational, then ι is an isomorphism outside the exceptional divisor of $\tilde{u}$.*

Proof: To construct γ_1, we first take the normalization Y_1' of the fiber product $Y_1 \times_Y \tilde{Y}$. Then Lemma 4.1 guarantees the existence of our required γ factoring through γ_1. If we have an inclusion ι as above, then it is easy to see that u^∂ is a morphism and we leave this verification to the reader. Suppose now that u^∂ is a morphism. To see the inclusion ι, we first observe that the natural inclusion $j : \tilde{u}^* \Omega_{Y^\partial} \hookrightarrow \Omega_{Y_1^{|\partial|}} = \Omega(Y_1, \log D)$ is an isomorphism outside $D = \gamma_1^{-1}((\partial Y_1)_{red})$. Now j factors through the natural inclusion $\Omega_{Y_1^\partial} \hookrightarrow \Omega(Y_1, \log D)$ outside the exceptional divisor E of $\tilde{u}$ by virtue of the Hartog extension theorem and the condition $K_{Y_1^\partial} \geq u^* K_{Y_1^\partial}$, which on $\tilde{Y}_1$ is the condition

$$\gamma_1^* K_{Y_1^\partial} \geq \tilde{u}^* \gamma^* K_{Y_1^\partial}. \tag{4.1}$$

To show the desired inclusion ι on E, it is equivalent to showing that the holomorphic map $\tilde{u}^* \Omega_{Y^\partial} \hookrightarrow \Omega_{Y_1^{|\partial|}}$ of vector bundles factors through $\Omega_{Y_1^\partial}$. Since $\tilde{u}^* \Omega_{Y^\partial}$ is trivial on E, we only have to show this for each vector v in the vector bundle Ω_{Y^∂} restricted to E. But this follows from the fact that one can find a map $g : Y \to C$ where C is a curve with $v \in L_{g \circ \tilde{\gamma}} \in \Omega_{Y^\partial}$ and that $\tilde{u}^* L_{g \circ \tilde{\gamma}}$ includes in $\Omega_{Y_1^\partial}$ as u^∂ is a morphism. If further u^∂ is birational, then (4.1) is an equality outside the exceptional divisor of $\tilde{u}$ forcing ι to be an equality there as well. ∎

It follows in particular that given two coverings γ_1 and γ_2 both as prescribed by Lemma 4.2 for a smooth orbifold Y^∂, we can create a third such covering γ factoring through γ_i and on which the $\Omega^p_{Y^\partial}$ is naturally isomorphic to the pull back of the $\Omega^p_{Y^\partial}$ defined by γ_i for each i. It is in this sense that $\Omega^s_{Y^\partial}$ is well defined independent of the covering γ that one can choose to satisfy Lemma 4.2.

The same proof as above gives us the following more standard characterization of an orbifold morphism for any map between manifolds with orbifold structures.

PROPOSITION 4.4 *Let $u : Y_1 \to Y$ be a morphism with $Y_1^\partial = Y_1 \backslash B_1$ and $Y^\partial = Y \backslash B$ smooth orbifolds such that $u^{-1}(B_{red}) \subset D := (B_1)_{red}$. Then there exists a commutative diagram*

$$\begin{array}{ccc} \tilde{Y}_1 & \xrightarrow{\tilde{u}} & \tilde{Y} \\ \gamma_1 \Big\downarrow & & \Big\downarrow \gamma \\ Y_1^\partial & \xrightarrow{u} & Y^\partial \end{array} \tag{4.2}$$

where γ_1 and γ are ∂-coverings and $\tilde{u}$ is a morphism. Moreover, setting $\tilde{D} = \gamma_1^{-1}(D)$, u gives rise to a morphism $u^\partial : Y_1^\partial \to Y^\partial$ if and only if the natural morphism $j : \tilde{u}^*\Omega_{Y^\partial} \to \Omega(Y_1, \log \tilde{D})$ factors through the inclusion $\Omega_{Y_1^\partial} \hookrightarrow \Omega(Y_1, \log \tilde{D})$. ∎

DEFINITION 4.5 *Given a fibration $g^\partial : Y^\partial = Y \backslash B \to Z$ with $\gamma : \tilde{Y} \to Y$ as given in Lemma 4.2, we define $L_{(Z,g^\partial)}$ (or L_{g^∂} for short) to be the saturation of $\gamma^* g^* K_Z$ in $\Omega^s_{Y^\partial}$ where $s = \dim Z$. We set*

$$\kappa(Z, g^\partial) = \kappa(L_{g^\partial})$$

and call it the Kodaira dimension of the orbifold (Z, g^∂) defined by g^∂.

Note that we have just given a new definition of $\kappa(Z, g^\partial)$ and of L_{g^∂} and that this L_{g^∂} depends on γ. However, our main object of interest $\kappa(Z, g^\partial)$ is the same as that of the old definition because of Lemma 4.7, which therefore allows us to keep this ambiguity in L_{g^∂} without harm. Note also that the L_{g^∂} defined previoiusly was only a $\mathbb{Q}$-line sheaf but its pullback to $\tilde{Y}$ is a line sheaf that we also denote by L_{g^∂}, by abuse of notation.

LEMMA 4.6 (Birational Invariance) *Consider the diagram of morphisms*

$$\begin{array}{ccc} \tilde{Y}_1 & \xrightarrow{\tilde{u}} & \tilde{Y} \\ \gamma_1 \Big\downarrow & & \Big\downarrow \gamma \\ Y_1^\partial & \xrightarrow{u^\partial} & Y^\partial \\ g_1^\partial \Big\downarrow & & \Big\downarrow g^\partial \\ Z_1 & \xrightarrow{w} & Z \end{array} \tag{4.3}$$

where g_1^∂ and g are fibrations from smooth orbifolds Y_1^∂ and Y^∂ respectively. Assume that the lower square of the diagram commutes and that w is generically finite and surjective. Then there exist ∂-coverings γ_1 for Y_1^∂ and γ for Y^∂ and a morphism $\tilde{u} : \tilde{Y}_1 \to \tilde{Y}$ as given by Lemma 4.4 making the above diagram commutative and as a result,

$$\kappa(Z_1, g_1^\partial) \geq \kappa(Z, g^\partial).$$

If moreover u^∂ and thus w are birational, then

$$\kappa(Z_1, g_1^\partial) = \kappa(Z, g^\partial).$$

Again, all the results hold if g, g_1 and w are only assumed to be meromorphic.

Proof: We restrict to the case where $\tilde{u}$ is surjective for simplicity. The existence of the upper diagram making the entire diagram commutative is given by Lemma 4.4. If w is generically finite and surjective, then we have

$$L^m_{g_1^\partial} \supset \tilde{u}^* L^m_{g^\partial}, \tag{4.4}$$

as sheaves by the commutativity of the above diagram. By the surjectivity of $\tilde{u}$, we have an injection $H^0(\tilde{Y}, L^m_{g^\partial}) \hookrightarrow H^0(\tilde{Y}_1, \tilde{u}^* L^m_{g^\partial})$ and so the first result follows. If u^∂ is birational, then the sheaf inclusion above is an equality on $\tilde{Y}_1 \setminus E$ where E is the exceptional divisor of $\tilde{u}$. Hence

$$H^0(\tilde{Y}_1, L^{Nm}_{g_1^\partial}) \geq H^0(\tilde{Y}, L^{Nm}_{g^\partial}) \geq H^0(\tilde{Y}_1, L^m_{g_1^\partial}),$$

where $N = \deg \tilde{u}$ and the last inequality is obtained by the following procedure: We may assume (by taking a finite covering if necessary) that $\tilde{u}$ is Galois for simplicity. Given a section s of $L^m_{g_1^\partial}$, we first identify s with a section of $\tilde{u}^* L^m_{g^\partial}$ on $\tilde{Y}_1 \setminus E$. Then we take the $\tilde{u}$-trace of s via

$$s \mapsto \prod \{\ g(s) \mid g \in G\ \}$$

where G is the Galois group of $\tilde{u}$ and apply the Hartog extension theorem on $\tilde{Y}$ as $v(E)$ is codimension two or higher in $\tilde{Y}$. (We remark here that we can define the trace of sections of the pull back of a line sheaf under any finite map similarly.) ■

LEMMA 4.7 *Any fibration $g^\partial : Y^\partial \to Z$ has an admissible birationally equivalent representative. Any admissible fibration $g^\partial : Y^\partial \to Z$ satisfies*

$$\kappa(Z \backslash D(Y^\partial)) = \kappa(Z, g^\partial) := \kappa(\tilde{Y}, L_{g^\partial}).$$

Proof: The proof is similar to the one above and that of Lemma 2.4 and thus left as an exercise. ■

This lemma shows that $\kappa(Z, g^\partial) = \kappa(L_{g^\partial})$ is not only independent of the choice of the finite covering $\gamma : \tilde{Y} \to Y^\partial$ used to define L_{g^∂} but it is also independent of the birational model of the orbifold fibration g^∂. As in the previous section, we will thus think of L_{g^∂} (*as an equivalence class of sheaves*) as a well defined object for g^∂.

We summarize a main part of this section, in parallel with the previous section, as follows. Given a smooth projective orbifold Y^∂, there is an injective map from the birational equivalence classes of orbifold fibrations and saturated line subsheaves of $\Omega^i_{Y^\partial}$, or equivalently, saturated subsheaves of $\Omega^i_{Y^\partial}$ of rank i for $i = 1, \ldots, r = \dim Y$. Furthermore, their Kodaira dimensions are the Kodaira dimensions of the "true" orbifold bases of the orbifold fibrations.

5 Bogomolov sheaves and general type fibrations

We have established in the previous two sections some elementary facts concerning (orbifold) fibrations from an admissible orbifold Y^∂, in particular, the fact that that their birationally equivalent classes are in one to one correspondence with certain saturated line subsheaves of exterior powers of the orbifold cotangent sheaf Ω_{Y^∂} (or Ω_{X^∂} in the non-projective category, where X^∂ is a log-manifold of which Y^∂ is the orbifold base). Hence such subsheaves are certainly worthy of study in birational geometry. A distinguished class of such is played by such a subsheaf $L \hookrightarrow \Omega^p_{Y^\partial}$ (resp. $L \hookrightarrow \Omega^p_{X^\partial}$) of **general type**, which we define by the condition

$$\kappa(L) = p.$$

We will allow $p = 0$ for which $\Omega^p_{Y^\partial} = \mathcal{O}_Y$. When Y^∂ is a projective manifold, such subsheaves were studied by Bogomolov in [Bo]. He also gave there an alternative characterization for them via a generalization of the lemma of Castelnuovo-De Franchis. To state it, we first give an important definition.

DEFINITION 5.1 *A meromorphic fibration $f : X \dashrightarrow Y$ is called* **almost holomorphic** *if f is holomorphic above a Zariski open subset of Y.*

THEOREM 5.2 (Bogomolov) *Let L be a saturated line subsheaf of Ω^p_X with X Kähler, or more generally of Fujiki class $\mathcal{C}$ (denoted by $X \in \mathcal{C}$). Then*

(I) $\kappa(L) \leq p$

(II) *If $\kappa(L) = p$, then the Iitaka fibration I_L of L defines an almost holomorphic fibration from X to a projective base B of dimension p and $I_L^* K_B$ saturates to L in Ω^p_X. In particular, $L = L_{I_L}$.*

Bogomolov in [Bo] only stated the theorem in the projective case and that, as far as we know, the above generalization (to the Kähler case) is due to Campana in [Ca01]. But the proof of this more general case is essentially the same and well known among experts including, of course, Bogomolov (see [Vi1] for example). In view of its importance, we indicate a derivation of a slightly more general version.

PROPOSITION 5.3 *Let L be a saturated line subsheaf of $\Omega^p(X, \log A)$ with X Kähler, or more generally of Fujiki class $\mathcal{C}$ (denoted by $X \in \mathcal{C}$), and A a normal crossing divisor in X. Then*

(I) $\kappa(L) \leq p$.

(II) *If $\kappa(L) = p$, then the Iitaka fibration I_L of L defines an almost holomorphic fibration of general type from X to a projective base B of dimension p and $I_L^* K_B$ saturates to L in $\Omega^p(X, \log A)$. In particular, $L = L_{I_L^\partial}$.*

Proof: Assume $\kappa(L) = r$. We then have, for some m, a fibration $f : X \to Y$ birationally equivalent to I_L defined by $H^0(X, L^m)$ such that $r = \dim Y$. By a well-known covering trick of taking roots of sections (see [BlG]), we can reduce the problem to the case $m = 1$ as was done by Bogomolov in [Bo] (see also [Vi1]). We now assume $m = 1$. We may then take $r + 1$ (logarithmic) p-forms $\omega_0, \ldots, \omega_s$ in L such that the r meromorphic functions $y_i = \omega_i/\omega_0$, $i = 1, \ldots, r$ are then pull backs of meromorphic functions on Y which serve as coordinates for Y generically. Since X is Kähler (or of Fujiki class $\mathcal{C}$), logarithmic forms are closed by Hodge theory (or by [Nog]) and so

$$0 = d\omega_i = (dy_i) \wedge \omega_0$$

for all i. Since the dy_i's are pointwise linearly independent over an open subset of Y, we see that $r \leq p$.

When $r = p$, linear algebra shows that $dy_1 \wedge \cdots \wedge dy_r$ is generically a nontrivial multiple of s_0. This forces $f^* K_Y \hookrightarrow \Omega_X^p(X, \log A)$ to factor through $L \hookrightarrow \Omega_X^p(X, \log A)$ genercally so that $L = L_{I_L^\partial}$. Hence $I_L^\partial : X \backslash A \dashrightarrow Y$ is a meromorphic orbifold fibration of general type. Now Proposition 6.8 concludes the proof. ■

The same proof gives us the following generalization:

PROPOSITION 5.4 *Let Y^∂ be an orbifold with a ∂-covering $\gamma : X \to Y^\partial$ (respectively an admissible fibration f^∂ from a log-manifold X^∂ with $Y^\partial = Y \backslash D(f^\partial)$). Let L be a subsheaf of $\Omega_{Y^\partial}^p$ (respectively $\Omega_{X^\partial}^p$) invariant under the Galois group of γ (respectively trivial on the general fibers of f). If $X \in \mathcal{C}$, then*

(I) *$\kappa(L) \leq p$*

(II) *If $\kappa(L) = p$, then γ (resp. f^∂) dominates the Iitaka fibration $I_L : X \dashrightarrow B$ of L giving an almost holomorphic fibration from Y to a projective base B of dimension p and $I_L^* K_B$ saturates to L in $\Omega_{Y^\partial}^p$ (resp. $\Omega_{X^\partial}^p$). In particular, $L = L_{I_L^\partial}$.*

We remark that these Bogomolov type theorems fail for more general compact complex manifolds. In view of the above theorems, we can make the following definition (compare also [Ca]).

DEFINITION 5.5 *A meromorphic fibration $g^\partial : Y^\partial \dashrightarrow Z$ is said to be of* **general type** *if*

$$\kappa(Z, g^\partial) := \kappa(L_{g^\partial}) = \dim Z.$$

Given an orbifold Y^∂ which is smooth and projective (resp. which is the orbifold base of an admissible fibration from a log-manifold or from a smooth and projective orbifold X^∂), a saturated line subsheaf of $\Omega_{Y^\partial}^p$ (resp. of $\Omega_{X^\partial}^p$) for $p \in \{1, \ldots, \dim X\}$ corresponding to a general type fibration is called a **Bogomolov sheaf** *(of dimension p on Y^∂). A meromorphic map from an orbifold is said to be of general type if its Stein factorization is so.*

Hence, if X is of Fujiki class $\mathcal{C}$, the above theorem of Bogomolov states that saturated line subsheaves L of Ω^p_X with $\kappa(L) = p$ for $p = 1, \dots, \dim X$ are precisely the Bogomolov sheaves on X, i.e., they are the sheaves corresponding to meromorphic general type fibrations. And this continues to hold for admissible orbifolds by Proposition 5.4 above. On the other hand, we note that for non-admissible X^∂, our notion of general type above should be used with care since it can no longer be made meaningful as a birational invariant and indeed is not a birational invariant in general.

The following basic lemma is a direct consequence of Lemma 3.20.

LEMMA 5.6 *Consider a fibration $g^\partial : Y^\partial \dashrightarrow Z$ and a premorphism $\phi^\partial : Y_1^\partial \to Y^\partial$ such that $\psi = g \circ \phi$ is surjective. If g^∂ is of general type, then so is ψ^∂ (or more properly, the birational class of its Stein factorization).* ■

We now establish an important technical lemma for this paper, Proposition 5.8.

LEMMA 5.7 *Consider the following commutative diagram*

$$\begin{tikzcd} X_t^\partial \arrow[r, "i^\partial"] \arrow[d, "f_t^\partial"'] & X^\partial \arrow[d, "f^\partial"] \arrow[dr, "h^\partial"] & \\ Y_t \arrow[r, "j"] & Y \arrow[r, "g"] \arrow[d, "u"] & Z \\ & T & \end{tikzcd} \tag{5.1}$$

where f^∂, g, $h^\partial = g \circ f^\partial$ and u are fibrations, i and j are the inclusion of the general fibers $Y_t := u^{-1}(t)$ and $X_t := f^{-1}(Y_t)$ over T (by which we mean that t is a representative point in the complement of some countably many closed proper subsets of T) and i^∂ is an orbifold inclusion. Let $h = g \circ f$. Assume that $g \circ j$ is generically finite. Then we have with $p = \dim Z$ and $q = \dim Y_t$ that

$$\kappa(Z, h^\partial) - (p - q) \le \kappa(i^* L_{(Z,h^\partial)}) \le \kappa(L_{f_t^\partial}) = \kappa(Y_t, f_t^\partial).$$

In particular, if h^∂ is of general type, then so is f_t^∂.

All the results remain valid if one only assumes that h, g f and hence f_t are meromorphic maps as long as $u \circ f$ is holomorphic.

Proof: We first ignore the last statement. We note that $X_t^\partial = X_t \backslash i^* A$ for $A = \partial X^\partial$, true for general $t \in T$. We have the following commutative diagram of sheaf morphisms

$$\begin{tikzcd} i^* h^* K_Z \arrow[r, hook] \arrow[d, equal] & i^* L_{(Z,h)} \arrow[r, hook] & i^* \Omega^p_X \arrow[r] & \Omega^q_{X_t} \otimes \wedge^{p-q} N^\vee_{X_t} \arrow[r] & \Omega^q_{X_t} \\ f_t^* j^* g^* K_Z \arrow[rr, hook] & & f_t^* j^* \Omega^p_Y \arrow[r] & f_t^* K_{Y_t} \otimes \wedge^{p-q} N^\vee_{X_t} \arrow[r] \arrow[u] & f_t^* K_{Y_t} \arrow[u] \end{tikzcd}$$

where the vertical arrows are inclusions, $N^\vee_{X_t} = \mathcal{O}^{\oplus(p-q)}_{X_t}$ is the conormal bundle of X_t and where the two right arrows on the very right are projections obtained by choosing a section s of $\wedge^{p-q} N^\vee_{X_t}$ so that the image of $i^*h^*K_Z$ in $f_t^*K_{Y_t}$, or equivalently, that the image of $j^*g^*K_Z$ in K_{Y_t} is nontrivial (such an s exists for general t by construction). Hence the saturations of $f_t^*K_{Y_t}$ and of $i^*L_{(Z,h)}$ in $\Omega^q_{X_t}$ are the same, namely $L_{(Y_t,f_t)}$. The above argument still works under the more general situation given by the last statement of the lemma since we only need to establish the equality of $f_t^*K_{Y_t}$ and $i^*L_{(Z,h)}$ generically on X_t and so we only need to work on the open set U on which f holomorphic as U and therefore its restriction to X_t has at most codimension two complement. Hence, the right hand side of the above inequality follows from the easily deduced fact that $i^*(A \cap h) = (i^*A) \cap h \circ i = (i^*A) \cap f_t$.

For the rest of the inequality and hence the lemma, we generalize the proof of the easy addition law: We may assume that $\kappa(Z, h^\partial) := \kappa(L_{h^\partial}) \geq 0$, since otherwise the result is clear. Then $\kappa(i^*L_{h^\partial}) \geq 0$ as well and so we may also assume that $\kappa(L_{h^\partial}) > p - q \geq 0$. We may then take the Iitaka fibration of L_{h^∂}, which we may take to be $\mu : X \dashrightarrow X'$ with X' projective. Since L_{h^∂} is trivial on the general fibers of h, μ factors through a meromorphic fibration $\gamma : Z \dashrightarrow X'$. The image X'_t of X_t in X', which is also the image of Y_t in X', is positive dimensional as $\dim X' > p - q$. Let the codimension of X'_t in X' be r. We then have by construction that $\kappa(i^*L_{h^\partial}) \geq \dim X'_t$ and $p - q \geq r$. It follows that

$$\kappa(L_{h^\partial}) = \dim X' = \dim X'_t + r \leq \kappa(i^*L_{h^\partial}) + p - q$$

as required. ■

We observe that the above proof does not use the fact that Z is smooth. This is because we can replace K_Z above by K_U for a smooth open subset of Z since we only need to consider rational sections of K_U because the saturation considered above in $\Omega^q_{X_t}$ is uniquely determined by their pullbacks. For the same reason, nor do we need f, g and h to be holomorphic.

PROPOSITION 5.8 *Consider the following commutative diagram*

$$\begin{array}{ccccc} X_t^\partial & \xrightarrow{i^\partial} & X^\partial & \xrightarrow{v^\partial} & X_0^\partial \\ f_t^\partial \downarrow & & \downarrow f^\partial & & \downarrow f_0^\partial \\ V_t^\partial & \xrightarrow{k^\partial} & V^\partial & \xrightarrow{u_0^\partial} & Y_0^\partial \\ & & \downarrow e & & \downarrow g_0^\partial \\ & & T & & Z_0 \end{array} \tag{5.2}$$

where the vertical arrows are fibrations with f_0^∂ admissible, v^∂ a morphism, $\partial Y_0 = D(f_0^\partial)$, $\partial V = D(f^\partial)$ and $\partial V_t = D(f_t^\partial)$, i^∂ and k^∂ are orbifold inclusions of the fibers at the general point $t \in T$ and u_0 is a morphism. Assume that $g_0 \circ u_0$ is surjective with Stein factorization given by $g : V \to Z$ and that $g_0 \circ u_0 \circ k$ is nontrivial with Stein factorization $g_t : V_t \to Z_t$. Then we have, with $h_0 = g_0 \circ f_0$, $h = g \circ f$, $h_t = g_t \circ f_t$ and $d_c = \mathrm{codim}_{Z_0}\, g_0(u_0(k(V_t)))$, that

$$\kappa(L_{g_0^\partial}) = \kappa(Z_0, h_0^\partial) \leq \kappa(Z, h^\partial) \leq \kappa(Z_t, h_t^\partial) + d_c \leq \kappa(L_{g_t^\partial}) + d_c.$$

In particular, if further g_0^∂ is a fibration of general type, then so is $g_t^\partial : V_t^\partial \to Z_t$. All results remain valid if g_0 is only meromorphic.

Proof: The left most inequality is a direct application of Lemma 3.20 while the above lemma gives the rest via the following construction. Set $\phi' = (e, g) : V \dashrightarrow T \times Z$ and let $\phi : V \dashrightarrow Y$ be its Stein factorization, where Y is a smooth model of the normalization of the image of ϕ' in the function field of V and so we may assume that the map from Y to $T \times Z$ is a morphism. Let $u : Y \to T$ and $w : Y \to Z$ be the projections to the factors, necessarily surjective, and $\phi_t = \phi|_{V_t} : V_t \twoheadrightarrow Y_t$. Then w restricted to the general fiber Y_t of u is finite by construction and $e = u \circ \phi$ is holomorphic. Hence the result follows from the above lemma and the fact that $\kappa(L_{\phi_t}) = \kappa(L_{g_t^\partial}) \geq \kappa(L_{h_t^\partial}) = \kappa(L_{\phi \circ f^\partial})$ via the first inequality in Proposition 3.19. ∎

COROLLARY 5.9 *Let $g^\partial : Y^\partial \dashrightarrow Z$ and $w : Y \to T$ be fibrations such that $g(Y_t)$ is nontrivial for the general fiber Y_t of w. If g^∂ is of general type, then so is its restriction $g_t^\partial : Y_t^\partial \to g(Y_t)$ (or, more properly, the Stein factorization of g_t^∂) to the general fiber Y_t.*

A meromorphic fibration of general type enjoys the following well-known but important property (c.f. [Re] for example) whose proof we delay to the next section.

PROPOSITION 5.10 *A meromorphic fibration of general type from a smooth orbifold is almost holomorphic.*

We conclude this section by recalling a very useful lemma of Fujita ([Fta], [Mori]) in the theory of Kodaira dimension:

LEMMA 5.11 *Let $f : X \to Y$ be a fibration with Y projective and L a line bundle on X. Then the following are equivalent:*

- $h^0(L^m f^* H^{-1}) \neq 0$ *for some $m \in \mathbb{N}$ and an ample line bundle H on Y.*
- $\kappa(L) = \kappa(L|_F) + \dim Y$*, where F is a general fiber of f.*
- *The Iitaka model $I_L(X)$ dominates Y, i.e., f factors through I_L.*

We say that L dominate Y in these cases, meaning that I_L dominates f in the sense that the base of I_L dominates that of f.

We remark that the lemma as stated in the reference cited is for the algebraic case, but it is easy to see that the lemma works for any fibration as long as Y is projective (even Moishezon if we take the H above to be big rather than ample). We refer the reader to the above references for the proof. The following is a generalization of a well-known lemma of Kodaira to the orbifold base of a fibration.

COROLLARY 5.12 *Let $f : X \to Y$ be a fibration. Then it is of general type if and only if there exist a big line bundle H on Y such that $h^0(L_f^m f^* H^{-1}) \neq 0$ for some $m \in \mathbb{N}$. We may take H to be ample if Y is projective.*

Proof: This follows from the proposition above since L_f is trivial on the generic fibers of f and hence of zero Kodaira dimension there. ■

6 Special orbifolds and special fibrations

The notion of special in our sense was originally motivated from our search of a purely algebro-geometric characterization of an algebraic manifold, including the quasiprojective case, which admits a generically surjective holomorphic map from $\mathbb{C}^n$ for some n, see [BL1, BL2]. Then we weakened the notion to manifolds which are holomorphically connected (generalization of rational connectedness) and even to manifolds whose Kobayashi metrics vanish identically. We have succeeded soon after in finding the characterization of $\kappa'_+ = 0$ in [Lu01], which works for all the cases where we have a characterization of any of the above properties even though we do not know if these properties are equivalent in general. We did not use the terminology of special at the beginning when we were working with κ'_+ and its associated Iitaka fibration as the iteration of the orbifold Iitaka and still conjectural Mori's fibration. But as soon as we realized that Campana has independently arrived at essentially the same notion in [Ca01] from a broader perspective but restricted to manifolds, we have adopted the very apt terminology he introduced, namely "special," for this notion. Although his original notion in [Ca01] was conjecturally equivalent to ours in the case of manifolds, i.e., without orbifold structure, he has later adopted our notion so that there is no ambiguity in this terminology (except for the easy misinterpretation of special to mean not of general type, which had a popular informal circulation at one time). Neverthless, his original notion of special should still be very useful for many of the current applications since the above mentioned conjectural equivalence (which is only supported by scanty evidence given in [Lu01] outside that given by Theorem 1.1) should be deep if true.

The terminology of special is also used in geometry to mean compact manifolds with special holonomy groups such as Calabi-Yau or holomorphic symplectic manifolds in complex geometry. But our usage of the term is in the context of complex birational geometry where our notion is more general.

DEFINITION 6.1 *Let Y^∂ be an admissible orbifold. Y^∂ is called* **special** *if it does not admit a meromorphic fibration of general type with positive dimensional base. A fibration $f^\partial : Y^\partial \to Z$ is called special if its orbifold general fibers are special and if f is only assumed to be meromorphic then f^∂ is special if it is prebirational (birational modifications of Y^∂ via premorphisms and of Z by morphism) to a special orbifold fibration.*

The above definition of a special fibration makes sense since the general orbifold fiber of a fibration from an admissible orbifold remains admissible, see Lemma 3.15.

Notice that in general we require Y^∂ to be an admissible orbifold above to make sense of a meromorphic special fibration. It is clear that the notion of special above would be void of meaning as a birational invariant otherwise as easy examples would show. We mention again the key fact that *smooth orbifolds are admissible.* In general however, admissibility can be dropped to allow for a possibly wider range of results even though birational invariance may no longer be present. A weaker notion of special also fits in this wider range: Given any orbifold Y^∂, we will call Y^∂ **nearly special** if it does not admit any holomorphic fibration of general type to a possibly singular variety.

As a direct corollary of Bogomolov's theorem, Proposition 5.3, we have the following alternative characterization in category $\mathcal{C}$.

PROPOSITION 6.2 *Let Y^∂ be a log-manifold or more generally a standard orbifold with a ∂-covering. If $Y \in \mathcal{C}$, then Y^∂ is special if and only if it does not admit any Bogomolov sheaf.* ■

The following theorem is simply a restatement of a typical theorem in [BL1, BL2] in light of our definitions here.

THEOREM 6.3 *Let X be an algebraic surface (not necessarily compact) not birational to a K3 surface nor (in the case X is not projective) birational to $\mathbb{P}^2$, then X admits a generically surjective holomorphic map from $\mathbb{C}^2$ if and only if X is special.* ■

This is an example relating to our original motivation for this paper for which we restrict to surfaces for simplicity. We have since been working with the notion of identically vanishing Kobayashi metric in connection with special for which Campana latter conjectures to be equivalent in the Kähler category. In this connection, we would like to mention that for all known examples of compact complex manifold X whose Kobayashi pseudometric vanishes identically (and we have a good understanding of this at least for the case of surfaces), X is a special manifold although the same is not true for the infinitesimal pseudometric (as observed in [BL1]).

In preparation for a discussion of examples, we state two results about special orbifolds and fibrations which are fundamental in the classification theory in terms of special objects.

PROPOSITION 6.4 *Let $f^\partial : X^\partial \to Y^\partial$ be a surjective premorphism between admissible orbifolds. If X^∂ is special, then Y^∂ is special.*

Proof: This follows directly from Lemma 5.6. ■

PROPOSITION 6.5 *Let X^∂ be admissible, $f^\partial : X^\partial \dashrightarrow Y$ a meromorphic special fibration and $h^\partial : X^\partial \dashrightarrow Z$ a meromorphic fibration of general type. Then $h^\partial = g \circ f^\partial$ for a meromorphic fibration $g : Y \dashrightarrow Z$. The same statement hold if we drop "meromorphic" from it. In particular, if f is holomorphic and $Y^\partial = Y \backslash D(f^\partial)$ is special, then X^∂ is special.*

Proof: This is a direct consequence of corollary 5.9 and the first inequality of Proposition 3.19. ■

This elementary result is the key to connecting the main result of this paper with Mori's Program and resolves in particular the conjectures given in section 6.1 of [Ca02] which are not already present in Mori's Program, see our discussion right after Conjecture 6.7.

A simple example of a special manifold is an abelian variety because any section of a positive representation of its cotangent bundle is constant. Similarly $(\mathbb{C}^*)^n = \mathbb{P}^n \backslash D$ where D is the sum of $n+1$ hyperplanes in general position is a special oribifold. Since these dominates $\mathbb{P}^n$, it follows that all unirational manifolds are special.

It is clear that a quasiprojective curve is special if and only if it has non-negative Euler characteristic so that $\mathbb{P}^1$ and elliptic curves are the only projective special curves. Also an orbifold curve is special if and only if it has non-negative orbifold Euler characteristic (see [BL2]), or equivalently, the orbifold anti-canonical $\mathbb{Q}$-bundle is semi-ample.

Another class of special orbifolds consists of rationally chain connected manifolds and smooth rationally chain connected orbifolds, or towers of them via fibration morphisms. An orbifold X^∂ is called **rationally chain connected** if any general pair of points is contained in a chain of rational curves belonging to an irreducible family Γ of **orbifold rational curves**, i.e, nontrivial morphism from $C = \mathbb{P}^1$ to X such that the induced orbifold anti-canonical $\mathbb{Q}$-bundle on C is ample. The fact that such an orbifold X^∂ is special is not difficult to see since for each fibration f^∂ from X with positive dimensional base, the general member in this covering family Γ of special orbifold rational curves is not contained in the fibers of f so that Proposition 6.5 shows that f is not of general type.

A more difficult example of a special projective orbifold is a smooth projective orbifold with trivial canonical $\mathbb{Q}$-bundle and even one with zero Kodaira dimension (see Theorem 9.5). The latter can also be deduced from the Miyaoka's generic semipositivity theorem generalized to the orbifold category as given in [LM, Lu0] assuming the standard conjecture in Mori's program that an orbifold with zero Kodaira dimension has a "good" model with trivial canonical $\mathbb{Q}$-bundle. We note that without orbifold structures, the first instance of this theorem concerning manifold with zero Kodaira dimension is due to Campana in [Ca01, Ca02] with a slightly different definition of special initially (see also [Lu02]) but neverthless a cornerstone result in this subject. It should be mentioned that these facts remains true for compact Kähler manifold (see section 9) but fails otherwise as the non-Kähler Calabi threefolds ([Cbi]) have trivial canonical bundle but dominate general type curves by construction.

It is not difficult to deduce from the above results that a smooth projective orbifold Y^∂ with semiample anticanonical $\mathbb{Q}$-bundle $K_{Y^\partial}^\vee$ is special since one can construct a dominating smooth orbifold with trivial canonical $\mathbb{Q}$-bundle. In fact, we show in Theorem 11.2 that if $K_{Y^\partial}^\vee$ is nef (see [DPS] for example concerning these objects), then Y^∂ is nearly special, which is a conjecture in the non-orbifold category by Campana in relation to the fundamental group and the Shafarevich conjecture concerning the universal cover, see also [DPS0, DPS1, DPS2] (he even conjectures that such a manifold is special). It follows that smooth **Fano orbifolds**, i.e., those with ample anticanonical $\mathbb{Q}$-bundle, are special. Hence, a generic hypersurface in $\mathbb{P}^n$ is special if and only if it is of degree not more than $n+1$, since such a hypersurface has an open subset with trivial logarithmic canonical bundle. These latter results can also be deduced directly from the fact that Fano manifolds are rationally connected, see [KMM, Ca2].

To prevent the wrong impression that Kodaira dimension may be strongly connected to being special, we emphasize that among the manifolds of dimension n, special ones exists for each Kodaira dimension less than n but only the Kodaira dimension zero class consists entirely of special ones. We illustrate this with some surfaces of Kodaira dimension one.

Example 6.6 *Consider the Kummer K3 surface X constructed from the product of two elliptic curves as groups by the blowup of the 16 ordinary double points that results from its quotient by the involution of its group structure. We obtain an elliptic fibration $f : X \to \mathbb{P}^1$ with four singular fibers of type I_0^* but otherwise holomorphically trivial. Therefore, $D(f) = 0$. If we base change the elliptic fibration by a finite map $\gamma : C \to \mathbb{P}^1$ of positive degree with the discriminant condition $\Delta(\gamma) \cap \Delta(f) = \emptyset$ and denote the resulting elliptic fibration by f', then $D(f') = 0$ also. Such a base change is easily constructed for any curve C. Hence if $C = \mathbb{P}^1$ or if C is an elliptic curve, then X' is special by Proposition 6.5. Otherwise, f' is of general type and X' is not special. It can be deduced (directly from Kodaira's canonical bundle formula for an elliptic fibration ([BPV]) for example) that all such X' constructed has Kodaira dimension one. In particular, if the orbifold base of the Iitaka fibration of X' has zero Kodaira dimension, $\kappa(X') = 1$ even though the general fibers have zero Kodaira dimension. This is in contrast to the corresponding properties of special and that of being rationally connected as given by Proposition 6.5 and Theorem 1.1 respectively.*

As for special fibrations, there is first the rationally connected fibrations, or more generally the orbifold Mori's fibration defined below. Since the general fibers of an orbifold Iitaka fibration have zero Kodaira dimension, the Iitaka fibration of a smooth (or more generally an admissible) orbifold is a special fibration. These two fibrations combine via their iteration in the projective category to form the canonical special fibration of general type by assuming the standard conjectures in Mori's program discussed in Conjecture 6.7 and its ensuing paragraphs below. This has been our approach to the subject in [Lu01] before [Ca] when we identified this canonical special fibration as the Iitaka fibration of the top Bogomolov sheaf. The latter, however, brought the general type case of the orbifold Cnm conjecture to the fore, which we solve completely in this paper. To include the compact complex category in this discussion, we will need the Algebraic reduction: Recall that if X is a compact complex manifold, then there is birational modification $v : X' \to X$ and a fibraion $Alg_X : X' \to Y = Alg(X)$ unique up to birational equivalence where Y is a projective manifold and the functional field of Y coincides with that of X via Alg_X^* (we identify Alg_X with the meromorpic map $Alg_X \circ v$ by abuse of notation). If $h : X' \to Z$ is a fibration, we also define the fibration

$$\tilde{a}_{X/Z} = (h, Alg_X) : X' \to Z \times Alg(X).$$

It is understood that a projective model is chosen whenever we speak of $Alg(\bullet)$. We will show that the Algebraic reduction is a special fibration in the next section.

We now give an important class of special fibrations that we obtain, for now, by a generalization of a well-known weak abundance conjecture in Mori's Program which we call Mori's orbifold ansatz:

CONJECTURE 6.7 (Mori's Oribifold ansatz) *Let X^∂ be a smooth projective orbifold with $\kappa(X^\partial) = -\infty$. Then X^∂ is birational to a smooth orbifold having an (orbifold) rationally chain connected fibration, called* **(orbifold) Mori's fibration***, in the sense that the general fibers are (orbifold) rationally chain connected as defined above.*

Coupled with the orbifold Iitaka fibration, the conjectures implies the following trichotomy for a projective smooth orbifold X^∂:

$$\begin{cases} \text{(1) } X^\partial \text{ is of general type, or} \\ \text{(2) Up to a birational modification, } X^\partial \text{ has a special fibration,} \\ \qquad \text{namely the Iitaka fibration of } X^\partial, \\ \qquad \text{whose general fibers are positive dimensional and of zero Kodaira dimension, or} \\ \text{(3) } X^\partial \text{ has an almost holomorphic special fibration,} \\ \qquad \text{namely the Mori's fibration of } X^\partial, \\ \qquad \text{whose general fibers are positive dimensional and rationally connected.} \end{cases}$$

Case (1) corresponds to $\kappa(X^\partial) = \dim X$, (2) to $\kappa(X^\partial) \geq 0$ and (3) to $\kappa(X^\partial) = -\infty$. A weaker but equivalent formulation of (3) is as follows (c.f., [Debarre, Kol1]):

(3)' X^∂ is **uniruled** in the sense that there is a countable union of subvarieties of X outside of which every point is contained in an orbifold rational curve.

Now if X^∂ is not of general type, then Conjecture 6.7 implies that it has a special fibration $f^\partial : X^\partial \to Y$, which we may assume to be well adapted by modifiying X. If $Y^\partial = Y \backslash D(f^\partial)$ is not of general type, then it would have a specical fibration $g^\partial : Y^\partial \to Z$ which we may again assume to be well adapted by modifying X and Y. Note that by Proposition 6.5, $h^\partial = g^\partial \circ f^\partial$ is then a special fibration with base orbifold $Z^\partial = Z \backslash D(h^\partial) = Z \backslash D(g^\partial)$, which is smooth. We can then iterate the process with Z^∂ and so forth until we reach a special fibration of general type from some smooth birational model of X^∂. This produces, along with Proposition 6.8, a canonical almost holomorphic special fibration of general type from X^∂. We summarize this as follows.

Fact: *Let X^∂ be a smooth projective orbifold. Then Mori's orbifold ansatz implies that X^∂ has a canonical special fibration of general type, which is then necessarily almost holomorphic.*

We will show that this statement is true without assuming any conjecture and thereby resolving in particular a part of Mori's orbifold ansatz, see Theorem 10.3.

We conclude this section with the following result promised in the last section.

PROPOSITION 6.8 *Let $f : X \dashrightarrow Y$ be a meromorphic fibration and $X^\partial = X \backslash A$ a smooth orbifold such that f^∂ is a fibration of general type. Then f is almost holomorphic.*

Proof: Let $v : X_1 \to X$ be the composition of a sequence of blowups v_i with smooth centers which resolves the indeterminancies of f. Let $f_1 = f \circ v$ and impose the infinity multiplicity on the exceptional fibers of v so that v^∂ is birational and f_1^∂ is of general type. Suppose f is not almost holomorphic. Then the exceptional divisor V of some v_i, say v', is mapped surjectively onto Y and such that f_1 restricted to the fibers $\mathbb{P}^l = V_t$ of v' is nontrivial. We impose the smooth orbifold structure $D = v_1^* A$ on V which we easily verify to be special at least when restricted to the general V_t since $V \backslash D_{red}$ is and since $D_{red}|_{V_t}$ can consists of no more than $l+1$ normal crossing hyperplanes by construction (one only need to verify this for point blowups of normal crossing configurations). It is easy to see that the inclusion of $V \backslash D$ is a morphism to X_1^∂ so that $f' := f_1|_V$ is a fibration of general type from V onto Y. As v' is a special fibration from V, the pair (v', f') then clearly contradict the first part of Proposition 6.5 and concludes our proof. ∎

It is clear from the proof that the smoothness assumption on X^∂ is only necessary at the points of indeterminancy of f. But the result is false otherwise.

7 The refined Kodaira dimension

We have deduced using Mori's ansatz that any smooth projective orbifold has a canonical special fibration of general type. It is clear from the construction that this fibration can be obtained by the Iitaka fibration of a line sheaf associated to the fibration (by Bogomolov theorem for example). Hence there is a natural Kodaira like dimension associated to each projective orbifold given the validity of the conjecture. We define this Kodaira dimension for any orbifold in the following way. We remark that the notation below is motivated from [Ca], which also contains parts of our results in the non-orbifold projective case below if coupled with Theorem 1.1.

DEFINITION 7.1 *Let X^∂ be an orbifold. If X^∂ is not special, we define*

$$\kappa'_+(X^\partial) = \kappa'(X^\partial) = \max\{\ \kappa(L_{f^\partial}) \mid f^\partial : X^\partial \dashrightarrow Y \ \ \textit{is of general type}\ \},$$

which is necessarily positive. If X^∂ is special, we define $\kappa'_+(X^\partial) = 0$ and

$$\kappa'(X^\partial) = \begin{cases} -\infty & \textit{if } \kappa(L_{f^\partial}) = -\infty \textit{ for all nonconstant fibrations } f^\partial : X^\partial \dashrightarrow Y, \\ 0 & \textit{otherwise.} \end{cases}$$

It follows from this definition that X^∂ is special if and only if $\kappa'_+(X^\partial) = 0$. There are alternative definitions which obviate the use of cases, especially in the projective (or $\mathcal{C}$) category. Of course, whether X^∂ is special or not, we have

$$\kappa'_+(X^\partial) = \max\{\ \kappa(L_{f^\partial}) \mid f^\partial : X^\partial \dashrightarrow Y \text{ is of general type }\},$$

and in the case X^∂ is a log-manifold and $X \in \mathcal{C}$, we have

$$\kappa'_+(X^\partial) = \max\{\ \kappa(L) \mid L \hookrightarrow \Omega^p_{X^\partial} \text{ is a line subsheaf with } \kappa(L) = p,\ p = 0, \ldots, \dim X\ \},$$

by Bogomolov Theorem, Proposition 5.3.

As for κ', we only offer the following alternative definitions which we deduce for now from Mori's orbifold ansatz. It is actually true without any assumption by the subadditivity of Kodaira dimension theorem that we obtain in the next section. It brings out (and reduces Mori's orbifold ansatz to) an important conjecture concerning rationally connected orbifold.

LEMMA 7.2 *Let X^∂ be a smooth projective orbifold. Assume Mori's orbifold ansatz. Then*

$$\begin{aligned}\kappa'(X^\partial) &= \max_{m>0}\ \min\{\ \kappa(L_{f^\partial}) \mid f^\partial : X^\partial \dashrightarrow Y \textit{ a special fibration with } 0 \le \dim Y \le m\ \} \\ &= \begin{cases} -\infty & \textit{if } \kappa(L_{f^\partial}) = -\infty \textit{ for all nonconstant fibrations } f^\partial : X^\partial \dashrightarrow Y. \\ \min\{\ \kappa(L_{f^\partial}) \mid f^\partial : X^\partial \dashrightarrow Y \textit{ a special fibration with } \dim Y > 0\ \} & \textit{otherwise.}\end{cases}\end{aligned}$$

Proof: The two expressions on the right hand side are easily seen to be the same, which we denote by $\bar{\kappa}$. We have seen that Mori's orbifold ansatz implies the existence of a canonical special fibration of general type $f^\partial : X^\partial \dashrightarrow Y$. If $m = \dim Y$ is positive, then it is clear by construction that $\bar{\kappa} = m = \kappa'(X^\partial)$. Hence we assume that $\dim Y = 0$, i.e. that X^∂ is special. This means we can take X^∂ to be a tower of (orbifold) Iitaka and Mori's fibrations over a point. If there is an (orbifold) Iitaka fibration in this tower, then we have again by construction that $\bar{\kappa} = 0 = \kappa'(X^\partial)$. Otherwise, this tower consists entirely of orbifold rationally connected fibrations. The result now follows from the following Lemma. ∎

LEMMA 7.3 *Suppose X^∂ is a smooth orbifold having a tower of Mori's fibration over a point. Then $\kappa'(X^\partial) = -\infty$. If further X^∂ is a log manifold, then $\kappa(L) = -\infty$ for all subsheaves L of $\Omega^p_{X^\partial}$ with $p > 0$.*

Proof: Since the Kodaira dimension of a fibration whose orbifold base is an orbifold rational curve is by definition $-\infty$, it follows by Proposition 5.8 that any fibration from X^∂ of nonnegative Kodaira dimension must factor through each stage of the tower and hence must have zero dimensional base. The last statement of the lemma without the log structure on X is a theorem of Campana in [Ca] whose proof is completed by Theorem 1.1 for which we refer to [Ca] for details. The log case is an easy generalization for which we leave to the reader. ∎

It follows from the above proofs that Mori's orbifold ansatz implies the following statement, which we state as a conjecture ([Ca] has a similar one in the non-orbifold case).

CONJECTURE 7.4 *A smooth projective orbifold X^∂ is birational to one with a tower of Mori's fibration if and only if*

$$\kappa'(X^\partial) = -\infty.$$

Conversely, this conjecture implies Mori's orbifold ansatz by our main results. We outline for now a proof of this implication below based on the subadditivity of Kodaira dimensions in the orbifold case for which we refer to the next section.

Remark 7.5 *Conjecture 7.4 implies Mori's orbifold ansatz, i.e., the orbifold weak abundance conjecture as stated in Conjecture 6.7.*

A proof of Remark 7.5: Let X^∂ be a smooth projective orbifold with $\kappa(X^\partial) = -\infty$ and $\dim X = n$. We need to prove that X^∂, up to a birational modification, has a Mori's fibration with positive dimensional fibers. We proceed by induction on n so assume that the result is true for such orbifolds of smaller dimension (the result being trivially true in dimension one). We may assume that there is a fibration $f^\partial : X^\partial \dashrightarrow Y$ with $\dim Y > 0$ and $\kappa(Y, f^\partial) \geq 0$. For otherwise $\kappa'(X^\partial) = -\infty$ so that X^∂ is birational to a smooth projective orbifold with a rationally connected fibration by Conjecture 7.4. We may assume that f is holomorphic and f^∂ is well adapted by modifying X^∂ and Y. If the general fibers of f^∂ has nonnegative orbifold Kodaira dimension, then the subadditivity theorem would give $\kappa(X^\partial) \geq 0$ which contradicts our assumption. Hence, there is an infinite set of fibers F^∂ of f^∂ above the complement of $\Delta(f^\partial) \cap \Delta(f)$ with $\kappa(F^\partial) = -\infty$. But this implies that these F^∂'s admits Mori's fibrations by our induction hypothesis. Since these fibrations are almost holomorphic, Proposition 8.4 and the standard argument of Fujiki in [Fujk] shows that the general fibers of f have Mori's fibrations and that these fibrations patch up to give a Mori's fibration for X^∂. ■

Conjecture 7.4 is known for projective threefold because Mori's ansatz is solved by Miyaoka in [Miya1] in this case. It is also known for some (but not yet all) cases of orbifold surfaces. We will address this conjecture for orbifold surfaces with some remarks for higher dimensions in a forthcoming paper. We remark that this conjecture implies a conjecture of Kollar in [Kol1] that a manifold X is rationally connected if and only if for all $l \in \mathbb{N}$,

$$H^0(\Omega_X^{\otimes l}) = 0.$$

Remark 7.6 *Lemma 7.2 and Proposition 6.5 shows that $\kappa'_+(X^\partial)$ is precisely the dimension of the base of the canonical special fibration of general type constructed in the last section, assuming Mori's orbifold ansatz. In fact, we show in Section 9 (Theorem 10.1) that if X^∂ is not special, then there is only one Bogomolov sheaf L_{f^∂} corresponding to $\kappa'_+(X^\partial)$ in Definition 7.1 and therefore only one general type fibration $f^\partial : X^\partial \dashrightarrow Y$ up to birational equivalence of fibrations where $\kappa'_+(X^\partial) = \dim Y$. We call this birational equivalence class of fibrations the* **top general type fibration of** *X^∂ (or simply the* **basic reduction of** *X^∂) and this line sheaf the* **top Bogomolov sheaf** *of X^∂.*

8 Proper behavior in an analytic family

We are interested here in the behavior of κ'_+, Bogomolov sheaves and general type fibrations in a family. We approach this here following partly the ideas of Campana in the construction of his relative general type reduction in [Ca02] but generalizing them to the compact complex full orbifold category. We actually only need to work out the required behavior in a family in the projective category rather than in the category $\mathcal{C}$ Campana chose to work out his relative general type reduction. But for the sake of the subject matter and for proper attribution, as he was the first to come up with the idea of the relative general type reduction, we will work out his ideas in category $\mathcal{C}$ clarifying some necessary points in our more general full orbifold context. However, we would like to mention that these ideas can be circumvented by more direct (and perhaps more elementary) methods (using for example Fujiki's theorem on the projectivity of the space of divisors in a family, especially [Fujk2, Theorem 7] and [Fujk3]).

We start with the following proposition which follows from the upper semicontinuity of the dimension in an analytic family or simply by quoting [LS] for example.

PROPOSITION 8.1 *Let $f : X \dashrightarrow Y$ be a meromorphic fibration and $h : X \to Z$ a fibration. Suppose X^∂ is a smooth or an admissible orbifold. Then $f_z^\partial : X_z^\partial \dashrightarrow Y_z = f(X_z)$ is a well defined meromorphic fibration for z in a Zariski open subset U of Z and $\kappa(Y_z, f_z^\partial)$ is constant for z in the complement V of a countable number of subvarieties in X with $V \subset U$, i.e., for general $z \in Z$.*

Proof: Consider the meromorphic fibration $\bar{f} = (h, f) : X \dashrightarrow \bar{Y}$ where $\bar{Y} = \bar{f}(X) \hookrightarrow Z \times Y$. It is clear that $\bar{f}_z = f_z$ is well defined as a meromorphic map for z in a Zariski open subset U of Z and hence so are $f(X_z)$ and f_z^∂. By blowing up X and $\bar{Y}$, we may assume that $\bar{f}$ is an admissible morphism and $\bar{Y}$ is smooth since this procedure does not change $\kappa(\bar{Y}_z, \bar{f}_z^\partial)$ for the general z (by choosing the ∞ multiplicity for the exceptional divisors of the blowups for example). Let $k = \dim \bar{Y} - \dim Z$. From the morphism (actually an inclusion modulo torsion) $\bar{f}^*\Omega_{\bar{Y}/Z} \to \Omega_{X/Z}$ of the sheaf of relative differentials, we take double dual of $\wedge^k$ of the saturation of the image of the above morphism of sheaves to obtain a line subsheaf L' of $\wedge^k \Omega_{X/Z}^{\vee\vee}$ over X. Let $i : X_z \hookrightarrow X$ be the inclusion of the generic fiber of h and let $()_z$ denote i^* or the restriction to X_z by abuse. By construction, $i^*\Omega_{X/Z}^{\vee\vee} = \Omega_{X_z}$ and K_{Y_z} are locally free and L'_z is the saturation of $\bar{f}_z^* K_{Y_z}$ in Ω_{X_z}, i.e., $L'_z = L_{\bar{f}_z}$. Let $L = L'(A \cap \bar{f})$. Then $L_z = L_{\bar{f}_z^\partial} = L_{f_z^\partial}$ and so the result follows by looking at the sequence of images of natural maps $X_{U_m} = h^{-1}(U_m) \dashrightarrow \mathbf{Proj}(h_*(L^m))$ over U_m where U_m is a Zariski open subset of Z on which h is smooth and $h_*(L^m)$ has constant rank. ■

Now the basic idea of Campana in [Ca02] is to look at the family of cycles given by the fibers of fibrations. For this we recall the theorem of Barlet ([Ba]) that for a compact complex space X, there exists a reduced complex space $B_q(X)$ and an analytic family $\{A_b \mid b \in B_q(X)\}$ of q-cycles on X such that every analytic family $\{A_s \mid s \in S\}$ of q-cycles on X is induced by a unique morphism $h : S \to B_q(X)$, i.e., $A_s = A_{h(s)}$ for all $s \in S$. Every analytic family S of cycles of X corresponds to an analytic subset $i : A_S \hookrightarrow S \times X$ defined by $A_S = \{(s, x) \mid x \in \mathrm{supp}(A_S)\}$, see [Ba, Théorème 1], called the graph of S. It is known ([Fujk1], for example) that $B_q(X)$ has a countable base for its topology if X is so. A terse acount of cycle space theory can be found in [CP].

For the next lemma, suppose $f : X \dashrightarrow Y$ is an almost holomorphic fibration where X is a compact complex manifold as usual. Let U be a Zariski open subset of Y above which f is defined and smooth. Then we have a map $k : U \to B_q(X)$ where $q = \dim X - \dim Y$.

LEMMA 8.2 *k extends to a birational map $k_f : Y \dashrightarrow B_q(X)$ onto a (necessarily compact) component S_f of $B_q(X)$. Conversely, every compact component S of $B_q(X)$ satisfying the following condition (*) gives rise to an almost holomorphic (rational) map $f_S : X \dashrightarrow S$ with $q = \dim X - \dim Y$.*

(*) *The general members of S as q-cycles of X are irreducible with multiplicity one and whose supports are disjoint and fills up an open subset of X.*

Proof: For the second statement, consider the graph $i : A_S \hookrightarrow S \times X$ of S. Let p_1 and p_2 be the projections of $S \times X$ onto the respective factors. Then $p_2 \circ i$ is a map onto X defined and one-to-one above an open subset of U of S by condition (*). Hence its inverse composed with p_1 is an almost holomorphic rational map by the compactness of S and we can identify this map as f_S.

For the first statement, we can flatten f via birational modifications $v : X' \to X$ and $u : Y' \to Y$ where by construction all fibers of $f' : X' \to Y'$ map to q cycles in X via v. Hence we obtain a morphism $r : Y' \to B_q(X)$. Consider now the component S of $B_q(X)$ containing $r(Y')$ and the map $h = (id, f) \circ i : A_S \dashrightarrow S \times Y$ where $i : A_S \hookrightarrow S \times X$ is the graph of S. By hypothesis, there exists $s \in S$ such that f (and hence h) is holomorphic in a neighborhood of A_S in X (resp. in A_S) and $h(A_s)$ is a point. So by upper semicontinuity of the dimension of the fibers of the image of h over S, we have that $h(A_s)$ is a point for s in an open subset U of S (on which $h(A_s)$ is defined). This forces $r(Y')$ to contain U. ■

COROLLARY 8.3 *The set of almost holomorphic (and hence of general type) fibrations from an orbifold X^∂ is at most countable and is discrete in the compact open topology.* ■

We now generalize Lemma 8.2 to the relative situation. Let $h : X \to Z$ be a fibration. Recall that by [Ba, Fujk0], there exists a relative Barlet space $B_q(X/Z) = \cup_{z \in Z} B_q(X_z)$ which is a reduced complex subspace of $B_q(X)$. There is a natural map $h_* : B_q(X/Z) \to Z$ which is holomorphic up to a normalization. Furthermore, Fujiki in [Fujk0, Fujk4] has shown using the result of of Harvey-Shiffman ([HS]) (itself a variation of a well known theorem of Bishop used in [Lieb] for the same purpose) that all connected (hence also irreducible) components of $B_q(X/Z)$ are compact if $X \in \mathcal{C}$ and more generally if $h \in \mathcal{C}/Z$. For our purpose, it is not necessary for us to know the precise meaning of $h \in \mathcal{C}/Z$ which roughly means that each fiber is in $\mathcal{C}$ where $X \in \mathcal{C}$ means that X is holomorphically dominated by a Kähler manifold. It is sufficient for us to know that $h : X \to Z$ is in $\mathcal{C}/Z$ (in fact, Kähler) if it is a projective morphism or if X is Kähler. Let $Z_0 = Z \setminus \Delta(h)$ and $X_0 = h^{-1}(Z_0)$. We will need to consider the subset $B_q^F(X_0/Z_0)$ of $B_q(X_0/Z_0) = h_*^{-1}(Z_0)$ defined to be the largest subset of cycles in $B_q(X_0/Z_0)$ whose intersection with $B_q(X_z)$ is either empty or fills up X_z for each $z \in Z_0$.

PROPOSITION 8.4 *Let $AH_q(X_0/Z_0)$ be the subset of $B_q^F(X_0/Z_0)$ defined by the union of S_f over all almost holomorphic rational maps f from X_z as z vary over Z_0. Assume that all components of $B_q(X/Z)$ are compact, which is the case, in particular, if $X \in \mathcal{C}$ or more generally if $h \in \mathcal{C}/Z$. Then there exist a Zariski closed subset $B_q^F(X/Z)$ of $B_q(X/Z)$ containing $B_q^F(X_0/Z_0)$ as a Zariski open subset and $B_q^F(X_0/Z_0) = B_q^F(X/Z) \cap B_q(X_0/Z_0)$. Moreover, $AH_q(X_0/Z_0)$ consists of Zariski open subsets of irreducible components of $B_q^F(X/Z)$.*

Proof: The first claim is clear by upper semicontinuity of dimension in a family. We may assume that the f considered above are all almost holomorphic fibration by Stein factorizing. Fix z and an almost holomorphic fibration $f : X_z \dashrightarrow Y_z$. Let S be a component of $B_q^F(X/Z)$ containing the general fibers of f. By Lemma 8.2, a component of $S \cap B_q(X_z)$ is S_f and so satisfies (*). As S is compact by our assumption on X, the result follows since the condition for a component of $S \cap B_q(X_{z'})$ to satisfy (*) is a Zariski open condition on the ramified cover of Z defined by S (again by upper semicontinuity of dimension in a family). ■

PROPOSITION 8.5 *Let X^∂ be a smooth or admissible orbifold and $h : X \to Z$ a fibration with Z projective or Moishezon. Let $GT_q(X^\partial/Z)$ be the subset of $B_q(X/Z)$ defined by the union of S_f over all general type fibrations f^∂ from X_z^∂ as z vary over Z. Then, above a countable intersection of Zariski open subset of Z, $GT_q(X^\partial/Z)$ coincides with a countable union of (Zariski opens of) compact complex analytic subvarieties of $B_q(X/Z)$.*

Proof: We only work out the case X^∂ is smooth as the admissible case would follow by the basic lemmas already established on premorphisms. We assume for simplicity that Z is projective. Let $\Omega = \Omega^s_{X/Z}$ with $s + q = \dim X/Z$ and $r : P \to \mathbf{Proj}\ \Omega$ be a resolution of singularity of $\mathbf{Proj}\ \Omega$ with $p : P \to X$ the projection composed with r. Let $\mathbb{I} = r^*\mathcal{O}_\Omega(1)$ and $\bar{h} = h \circ p$. We may choose a smooth projective $Alg(P)$ and r so that $p^{-1}(D_{red})$ is normal crossing and so that the algebraic reduction $Alg_P : P \to Alg(P)$ of P is a morphism by Hironaka's theorem. We focus our attention on the open set $U_0 := Z \smallsetminus (\Delta(h^\partial) \cup \Delta(\bar{h}^\partial))$ on which p has multiplicity one, i.e., $D(p)_{red} \subset \Delta(h^\partial)$. Note that P_z^∂ and X_z^∂ are smooth for $z \in U_0$. So Lemma 3.11, or its proof, shows that we can find an orbifold structure on P (by choosing the ∞ multiplicity on the exceptional fibers of r and choosing ∂p^∂ to be p^*D over U_0 otherwise for example) so that $p^\partial : P^\partial \to X^\partial$ is a well adapted special fibration over U_0. This means in particular that there is a Zariski open subset U of Z contained in U_0 where $p_z^\partial : P_z^\partial \to X_z^\partial$ is a well adapted special fibration for all $z \in U$. By blowing up further (again choosing the ∞ multiplicity on the exceptional fibers if one prefers), we may assume that $Alg_P^\partial : P^\partial \to Alg(P)$ and $\bar{a}^\partial : P^\partial \to \bar{Y}$ are well adapted fibrations (over U) where $r' : \bar{Y} \to \tilde{a}_{P/Z}(P)$ is a resolution of singularity of $\tilde{a}_{P/Z}(P) \subset Z \times Alg(P)$ and $\bar{a}$ is defined by $\tilde{a}_{P/Z} = r' \circ \bar{a}$ (see the definition just before Conjecture 6.7). Hence by shrinking U if necessary, we may assume that $p_z^\partial : P_z^\partial \to X_z^\partial$ and $\bar{a}_z^\partial : P_z^\partial \to \bar{Y}_z = (r')^{-1}(p_1^{-1}(z))$ are well adapted fibrations for all $z \in U$ and that $(p_1 \circ r')^\partial$ is smooth over U.

Now fix a point $z \in U$ and let $i : P_z \hookrightarrow P$ be the inclusion of the fiber over z. As $p_z^\partial : P_z^\partial \to X_z^\partial$ is a well adapted special fibration, it follows from Proposition 6.5 that every general type fibration $\bar{f}^\partial : P_z^\partial \dashrightarrow Y_z$ gives rise to a general type fibration $f^\partial : X_z^\partial \dashrightarrow Y_z$ such that $\bar{f} = f \circ p$ and conversely. Now fix such a $\bar{f}^\partial$ and its associated f^∂. Let $L = L_{f^\partial}$. Then we have an inclusion of line sheaves $L \hookrightarrow L_f(D_z)$. Since $p_z^* L_f$ is a subsheaf of $\mathbb{I}_z$, we see that $p_z^* L$ is a subsheaf of the line sheaf J_z where $J = \mathbb{I}(p^*D)$. Also there exist $l \in \mathbb{N}$ such that f^∂ (resp. $\bar{f}^\partial$) is defined by the linear system of L^l (resp. of $p_z^* L^l$) by definition. Let $V_l = \bar{h}_*(J^l)$ and H_l be an ample divisor on Z such that $V(l) := V_l(H) = \bar{h}_*(J^l(\bar{h}^*H))$ is spanned. Let U_l be a Zariski open subset of X contained in $U_0 \smallsetminus (H_l)_{red}$ over which $V(l)$ has constant rank and where the base change map $V(l)/\mathcal{I}_z V(l) \to H^0(J_z^l(\bar{h}^*H))$ is bijective (see [BPV, Theorem I.85]).

Suppose now and furthermore that $z \in U_l$. Then the natural restriction map

$$\epsilon : H^0(J^l(\bar{h}^*H)) = H^0(V(l)) \to V(l)/\mathcal{I}_z V(l) = H^0(J^l_z(\bar{h}^*H))$$

is surjective. Consider the subspace $V_z = H^0(p_z^* L^l)$ of $H^0(J_z^l) = H^0(J_z^l(\bar{h}^*H))$ and let $V = \epsilon^{-1}(V_z)$. Then the image of the natural map $f'_z : P_z \dashrightarrow \mathbb{P}(V_z^\vee) \subset \mathbb{P}(V^\vee)$ is by definition birational to Y_z so that this map to its image is birational to the almost holomorphic fibration $\bar{f}$ given by $\bar{f}^\partial$. On the other hand, the natural map $f' : P \dashrightarrow \mathbb{P}(V^\vee)$ factors through Alg_P and hence through the fibration given by $\bar{a} : P \to \bar{Y}$, i.e., there is a map $f'' : \bar{Y} \dashrightarrow \mathbb{P}(V^\vee)$ with $f' = f'' \circ \bar{a}$. Since $f'' \circ \bar{a} \circ i = f'_z$, there is a map $e : \bar{Y}_z = \bar{a}(i(P_z)) \dashrightarrow Y_z$ birationally equivalent $f''|_{\bar{Y}_z}$ with $e \circ \bar{a} \circ i = \bar{f}$. Note that e is an almost holomorphic fibration onto Y_z since $\bar{f}$ is. Hence the associated orbifold fibration $e^\partial : \bar{Y}_z^\partial = \bar{Y}_z \backslash D(\bar{a}_z^\partial) \dashrightarrow Y_z$ is of general type since $\bar{f}^\partial$ is and since $\bar{a}_z^\partial = \bar{a} \circ i^\partial$ is admissible. Note that the key to the argument of this paragraph is the inclusion $\mathbb{P}(V_z^\vee) \subset \mathbb{P}(V^\vee)$ induced by ϵ.

Finally let $z \in W = \cap_l U_l$. We have just shown that a general type fibration f^∂ from X_z^∂ gives rise to a general type fibration e^∂ from $\bar{Y}_z^\partial$ and conversely and that the correspondence is obtained on the level of cycles in the form $(p^*)^{-1}\bar{a}^* S_e = S_f$. Here the map $\bar{a}^*$ can be defined ad-hoc on a component S' of $B_q^F(\bar{Y}/Z)$ intersecting $(p_1 \circ r')_*^{-1}(W)$ by associating the cycle $k[\bar{a}^{-1}(\mathrm{supp} A_s)]$ to each A_s for the general s in S' where k is the multiplicity of A_s. We note that $B_q^F(\bar{Y}/Z)$ is well defined with compact components by Proposition 8.4 since $\bar{Y}$ is projective. As before, $\bar{a}^*$ extends to a meromorphic map from S' birationally into $B_q(P/Z)$ (via the usual flattening argument). Using $e \circ \bar{a} \circ i = f \circ p_z$, we see easily that $\bar{a}^*(S') = p^*(S)$ for some compact subvariety S in $B_q(X/Z)$ bimeromorphically equivalent to S' and on which p^* is bimeromorphic. Hence, the map $(p^*)^{-1}\bar{a}^*$ is bimeromorphic on the relevant components of $B_q(\bar{Y}/Z)$ and so preserves compactness and the result follows. ■

The question naturally arises as to whether the above proposition is true for $Z \in \mathcal{C}$ or more generally for Z arbitrary. The latter seems farfetched while the former should be possible as it is at least true for Z Moishezon and also true for $X \in \mathcal{C}$ for which we know by the definition of $\mathcal{C}$ that $Z \in \mathcal{C}$ in this case. By combining Propositions 8.1, 8.4 and 8.5, we obtain the following corollary:

COROLLARY 8.6 *Let $h : X \to Z$ be a fibration with $h \in \mathcal{C}/Z$ or with Z Moishezon. Then $\kappa'_+(X_z)$ is constant for general $z \in Z$.* ■

Remark 7.6 now allows us to deduce the following corollary, which gives a canonical almost holomorphic fibration

$$bc_{h^\partial} : X^\partial \dashrightarrow bc(h^\partial)$$

associated to a fibration $h^\partial : X^\partial \to Z$ known as the **relative basic reduction** of h^∂, from Theorem 10.1 and the above proposition and its proof.

PROPOSITION 8.7 *Let $h : X \to Z$ be a fibration such that either $h \in \mathcal{C}/Z$ or Z is Moishezon. Let X^∂ be a smooth or admissible effective orbifold. Then there exist a fibration $Y \to Z$ and a dominant almost holomorphic map $f : X \dashrightarrow Y$ such that $f_z^\partial : X_z^\partial \dashrightarrow Y_z$ is the top general type fibration of X_z^∂ for the general z in Z. Furthermore, Y is dominated by a projective fibration over Z.* ■

Hence, the basic reduction of h^∂ restricted to the general fibers F of h are the basic reductions of F^∂. Coupled with Theorem 10.1, also proved in [Ca02] for less general settings, this is also a result in [Ca02] for the case $X \in \mathcal{C}$ without orbifold structures.

Another important corollary, also with the help of Theorem 10.1, is the specialness of the algebraic reduction already mentioned and promised:

THEOREM 8.8 *Let X^∂ be a smooth or admissible effective orbifold and*

$$h = Alg_X : X \dashrightarrow Z = Alg(X),$$

the algebraic reduction of X. Then $h^\partial : X^\partial \dashrightarrow Z$ is a special meromorphic orbifold fibration.

Proof: By blowing up, we may assume that h is a morphism. The fact that h is a fibration follows from the definition and Stein factorization ([Ue, Proposition 3.4]). If h^∂ is not special, then $Y := bc(h^\partial)$ would have dimension greater than that of $Z = Alg(X)$ by the last proposition. As Y is dominated by a projective fibration over Z and Z is projective, Y is Moishezon by [Ue, Corollary 3.9] and so its algebraic dimension $\mathrm{a}(Y)$ is strictly greater than that of Z. But Y is dominated by X (via bc_{h^∂}) so that $\dim Z = \mathrm{a}(X) \geq \mathrm{a}(Y) > \dim Z$. This is a contradiction and hence h^∂ is special. ∎

9 The orbifold Cnm conjecture

We derive in this section the most important ingredient for our main theorems. This is obtained from the "positivity" theorems of Kawamata and Viehweg; The key, beside that given in [Lu02] (though already known to the author at the time), is a coupling of (1) Viehweg's famous (very deep but ingenious) proof that weak positivity (see below for the definition) of the direct images of dualizing sheaves implies the same for tensor powers of dualizing sheaves with (2) the base change property for dualizing sheaves (both of which are essential ingredients in all the proofs of Kawamata-Viehweg type theorems on additivity of Kodaira dimension). The author is extremely indepted to Eckart Viehweg for a chance to discuss the proof and for pointing out a careless gap in the original formulation during that discussion. He would also like to thank G. Dethloff for inviting him to Brest where these ingredients were dicussed and given in an algebraic geometry seminar talk in July 2001.

In fact the results in this section are essentially contained in the literature or can be deduced without difficulty from existing but now very standard techniques. These techniques were extensively developed by Kawamata and Viehweg with an important contribution by Kollar and some initial contributions by Fujita and Ueno among others (see the article of Mori [Mori] for an excellent thorough account of the theory and its role in classification). What we can perhaps claim to be new is our introduction of a new (orbifold) framework for which these results apply. Besides the main theorem offered here, we only offer few applications here one of which is an application to the Albanese map of certain special varieties via the original argumemt of Kawamata in Section 11. There are many more interesting applications but they are relegated elsewhere.

We restrict to the Kähler category in this section unless specified otherwise. The fact that the theorems used in this section treated in the literature in the projective category are actually valid in the Kähler case (even in Fujiki class $\mathcal{C}$) is believed to be well known among the experts but we first learned this from Viehweg, see related comments and discussions in [Vi2, Kol, Na0]. For our purpose however, there is no loss of generality by restricting to the projective category. Recall that an orbifold is called **effective** if its boundary divisor is. We first state an orbifold generalization of a well known conjecture of Iitaka concerning the Kodaira dimensions of an algebraic fiber space.

CONJECTURE 9.1 (Orbifold Iitaka's Cnm Conjecture) *Let Y^∂ be a smooth or an e-admissible effective orbifold. Let $g^\partial : Y^\partial \to Z$ be a fibration and Y_z^∂ its orbifold general fiber. Then*

$$\kappa(Y^\partial) \geq \kappa(Y_z^\partial) + \kappa(Z, g^\partial).$$

More generally, if X^∂ is an e-admissible orbifold, $f : X^\partial \to Y$ a morphism with effective orbifold base Y^∂, $g : Y \dashrightarrow Z$ meromorphic and $h = g \circ f$ almost holomorphic. Then we have with $f_z = f|_{Y_z}$ that

$$\kappa(Y, f^\partial) \geq \kappa(Y_z, f_z^\partial) + \kappa(Z, h^\partial).$$

The author thanks F. Bogomolov for calling this conjecture to the author's attention in connection to the canonical fibration. Various forms of this type of orbifold Iitaka conjecture can be found in [Ca01, Ca02, Lu02]. However, the first theorem of this type, with orbifold structure both on the total space and on the base, precedes these by two decades and is due to Kawamata in [Ka].

Without the orbifold ∂, the Cnm conjecture of Iitaka is known in the case of general type Z by theorems of Viehweg and Kawamata (see [En, Ka, Vi0, Vie, Mori]) and in the case of general type Y_z by a theorem of Kollar ([Kol]). We prove here the orbifold analog of their theorems. We begin with a special case which is also given in [Lu02] and in [Ca02].

THEOREM 9.2 *Let $f : X \to Y$ and $g : Y \to Z$ be fibrations with g of general type. Then $\kappa(Y^\partial) = \kappa(Y_z, f_z) + dim(Z)$ where Y_z is the general fiber and $f_z = f|_{Y_z}$.*

This is a direct corollary via Lemma 5.11 of the following theorem, which generalizes slightly the results of the people mentioned above.

THEOREM 9.3 *Assume that $f : X \to Y$ is a fibration. Set $K_{X/Y^\partial} = K_X \otimes f^* K_{Y^\partial}^{-1}$ as $\mathbb{Q}$-line bundles. Let m be a positive integer such that $mD(f)$ is a Cartier divisor on Y. Then there is an effective divisor E with $E_{red} \subset E(f)$ such that $f_*(K_{X/Y^\partial}^{\otimes m}(E))$ is weakly positive in the sense of [Vi0].*

Without the orbifold ∂, this is the Kawamata-Viehweg's theorem. We will not define weak positivity except to note the following two properties (see [En]).

(P1) A torsion free coherent sheaf is weakly positive if it contains a weakly positive subsheaf of the same rank.

(P2) Let Υ be a torsion free coherent sheaf on Y. Let $v : Y' \to Y$ be a flat base change. If $v^*\Upsilon$ is weakly positive, then so is Υ.

Proof of Theorem 9.3: We denote $D(f)$ by D^f for simplicity. Recall that $D = D^f_{red} = \sum_i D_i$ is a simple normal crossing divisor on Y. Write $f^*D_i = \sum_j m_{ij}D_{ij}$ and let M_i be the least common multiple of $\{ \ m_{ij} \mid f(D_{ij}) = D_i \ \}$. The Kawamata branched covering trick ([Ka]) guarantees the existence of a finite Galois cover $v : Y' \to Y$ with Y' smooth such that $v^*D_i = M_iD'_i$ for some effective divisor. We let $u : X' \to X$ be the composition $u = u_1 \circ d$, where $u_1 : X_1 \to X$ is the base change of X by v composed with the normalization map, and $d : X' \to X_1$ is a desingularization which is an isomorphism above the smooth locus of X_1. We also denote by $f_1 : X_1 \to Y'$ and $f' = f_1 \circ d : X' \to Y'$ the resulting pullbacks of f by v. Note that if $f(D_{ij}) = D_i$, then $u^*(m_{ij}D_{ij}) = M_iD'_{ij}$ for some effective divisor D'_{ij} by construction. Hence, applying u^* to the $\mathbb{Q}$-divisor f^*D^f renders it into an effective (integral) divisor on X'.

Now, a local computation shows the following natural inclusion of torsion free coherent sheaves of the same rank outside a codimension two subset of Y':

$$(f')_*\big(K^{\otimes m}_{X'/Y'}\big) \hookrightarrow v^*f_*\big(K_{X/Y}(-(df))^{\otimes m}\big) \hookrightarrow v^*f_*\big(K_{X/Y}(-f^*D^f)^{\otimes m}\big) = v^*f_*\big(K^{\otimes m}_{X/Y^\partial}\big). \qquad (*)$$

A round about way to see this is as follows: This fact without the $(-f^*D^f)$ is well-known (e.g., [Ka], [En, Lemma 13]). Hence, we only need to check $(*)$ above an analytic neighborhood V in Y of a generic point $p \in D'_i = v^{-1}D_i$. Let D'_{ij} be a component of $f_1^{-1}D'_i$ dominating D_{ij} and let U be a small neighborhood in X_1 of a generic point q on D'_{ij} above p. These neighborhoods as j varies cover $f_1^{-1}(V)$ outside a codimension two subset if we include also the open subset $f_1^{-1}(V \setminus D'_i)$, above which $K_{X_1/Y'}$ and $u_1^*K_{X/Y}$ are naturally identical. Since $d_*K_{X'/Y'} = K_{X_1/Y}$ on X_1, see the proof of Lemma 13 in [En], and since X_1 is normal, it is sufficient to show that $u_1^*K_{X/Y}(-u_1^*(df))$ is naturally identified with $K_{X_1/Y'}$ on U, where $(df) = \sum f^*(D_i) - f^*(D_i)_{red} \geq f^*D^f$ with equality attained everywhere on V as given in Lemma 3.5. This identification is easily seen from the fact that, on a neighborhood of q, we can write $u = u_1 = u_{ij} \circ u'_{ij}$ where $u_{ij} : X_{ij} \to X$ is obtained from the base change by an m_{ij}-cyclic cover Y_{ij} of Y branched precisely on D_i and u'_{ij} is obtained from the (remainder) cyclic cover of Y' over Y_{ij}. Let $f_{ij} : X_{ij} \to Y_{ij}$ be the map corresponding to f. Then by construction, $K_{X_{ij}} = u_{ij}^*K_X$ and f_{ij} is smooth on a neighborhood W of $u'_{ij}(q)$. The first implies $K_{X_{ij}/Y_{ij}} = u_{ij}^*K_{X/Y}(-(df))$ on W while the second implies that $K_{X_1/Y'} = (u'_{ij})^*K_{X_{ij}/Y_{ij}}$ on $U \cap u'^{-1}_{ij}(W)$ as required for $(*)$. Since $K_{X_1/Y'} \hookrightarrow u_1^*K_{X/Y}$ in X_1 by Lemma 13 of [En] and we have just shown that this inclusion factors through

$$K^m_{X_1/Y'} = u_1^*(K_{X/Y}(-(df)))^m \hookrightarrow u_1^*K^{\otimes m}_{X/Y^\partial} \qquad (\dagger)$$

outside a codimension two subset above V where equality is achieved at every point over V, it follows that $K^m_{X_1/Y'} \hookrightarrow u_1^*K^{\otimes m}_{X/Y^\partial}(E)$ on X_1 with equality achieved everywhere on V for some effective divisor E supported on the exceptional divisor of f where E dominates f^*D^f. Combining this with the theorem of Kawamata-Viehweg via (P1) and (P2) now gives us theorem 9.3 and hence theorem 9.2. ■

By taking $X = Y$ in Theorem 9.2, we get the following corollaries.

THEOREM 9.4 *Let $g : Y \to Z$ be a fibration of general type. Then*

$$\kappa(Y) = \kappa(Y_z) + dim(Z),$$

where Y_z is the general fiber of g. In particular, if Y_z is of general type, then Y is also. ■

THEOREM 9.5 *A projective (or Kähler) manifold X is special if $\kappa(X) = 0$.* ■

Remark 9.6 *Although we had written up the above theorems at around the same time in [Ca02] and in [Lu02] independently, Campana in [Ca01] had at least claimed similar but slightly simpler theorems and therefore they should be credited to him.*

The following is the key technical result of this section. We say that an orbifold fibration $f^\partial : X^\partial \to Y$ is effective if the horizontal orbifold divisors are, i.e., $A - (A \cap f)$ is effective for $A = \partial X^\partial$.

THEOREM 9.7 *Assume that $f^\partial : X^\partial \to Y$ is an effective fibration from a smooth or e-admissible orbifold. Set $K_{X^\partial/Y^\partial} = K_{X^\partial} \otimes f^* K_{Y^\partial}^{-1}$ as $\mathbb{Q}$-line bundles. Let m be a positive integer such that $mD(f^\partial)$ and $m\partial X^\partial$ are Cartier divisors. Then $f_*(K_{X^\partial/Y^\partial}^{\otimes m})$ is weakly positive.*

One way to obtain this theorem is to look directly at the proof of Theorem 9.3. We observe via $(*)$ and Lemma 3.5 that given any vertical orbifold structure on X, meaning that no component of ∂X^∂ surjects to Y, we have the following injection outside a codimension two subset of Y' of torsion free coherent sheaves of the same rank on Y':

$$(f')_*\big(K_{X'/Y'}^{\otimes m}\big) \hookrightarrow v^* f_*\big(K_{X/Y}(-(df))^{\otimes m}\big) \hookrightarrow v^* f_*\big(K_{X^\partial/Y^\partial}^{\otimes m}\big), \qquad (**)$$

where Y^∂ is the orbifold base of X^∂. Now the key point to obtaining this is the factorization through the inclusion (†). This is unaffected if we add an admissible (or smooth) horizontal (i.e., nonvertical) orbifold divisor to X which pulls back only to a horizontal orbifold divisor on X' and push forward to the original horizontal divisor. Hence $(**)$ holds for any admissible X^∂. Hence the theorem follows directly from the following proposition, which is essentially explicit in the literature and certainly well known (in various forms).

PROPOSITION 9.8 *Assume that $f^\partial : X^\partial \to Y$ is a fibration where $X^\partial = X \backslash A$ is a smooth effective orbifold (with A horizontal). Then $f_*(K_{X^\partial/Y}^{\otimes m})$ is weakly positive for any $m \in \mathbb{N}$ such that mA is Cartier.*

Proof: One direct way to do this is to quote Theorem 7.3.3 or its proof in [Na0] which can easily be modified to give the required result. We outline below our initial method for doing this with the (Hard Covering) trick of Viehweg [Vi0, Corollary 5.2] coupled with the base change property for dualizing sheaves as follows:

Let $\gamma : X' \to X$ be a fibration so that $\gamma^* A$ and thus $\gamma^* K$ where $K = K_{X^\partial/Y}$ are restrictions of Cartier divisors on an open set V of X' such that $X \setminus \gamma(V)$ is contained in the complement of a codimension two subset of X. Such a fibration always exists from the problem in practice but can be constructed in general by a Kummer covering of section 4 using for example an elliptic curve or an abelian variety as a general fiber. Let $f' = f \circ \gamma$. Let K' be a Cartier divisor on X' that is identical with $\gamma^* K$ on V and such that $D := m(K' - \gamma^* K)$ is an effective (and Cartier by construction) divisor. Let H_0 be an ample divisor on Y with $H = f^* H_0$ and $H' = \gamma^* H$. Note then for any integer l that $\gamma_* \gamma^*(mK + lH) = (mK + lH)$ is an invertible subsheaf of the torsion free line sheaf $\gamma_*(mK' + lH')$ and that these sheaves are identical outside a codimension two subset of X. Hence they give the same result after applying f_* (for the relevant properties such as weak positivity) and so we identify them by abuse of notation. As in [Vi0, Corollary 5.2], we only need to show that if $S^N(f_*(m(K + H))) = S^N(f'_*(m(K' + H')))$ is generated by global sections over an open set U for some N, then $f_*(mK + (m-1)H) = f'_*(mK' + (m-1)H')$ is weakly positive. The rest of the argument is the same as that of [Vi0] to give the weak positivity of $f_*(mK_{X^\partial/Y})$.

For this purpose, let $L = K + H$ and $L' = K' + H'$. We can regard L' as a line bundle on X'. Now L is only a line bundle outside a codimension one subset where $L' = \gamma^* L$. Still, L can be completed to a singular space $\bar{L}$ with a vector space structure for each fiber over X (this is clear if the orbifold is standard since one can use the "singular" canonical bundle of the corresponding V-manifold but in general the reader should have no problem in defining it via L').

Let M be the image of $f^*(f_*(mL))$ in the invertible sheaf mL. As in Lemma 5.1 of [Vi0], we may assume that M is spanned over U so that after suitably blowing up X outside U, we may assume that the effective divisors $mL - M$ and M together has normal crossing support. We may assume that the pull back of these divisors to X' and D are jointly normal crossing by blowing up. Then the sum of these divisors add up to mL' and so the standard construction of taking m-th root apply to obtain a subvariety in L' which we denote by T'. By construction, T' is the inverse image of a subvariety T in $\bar{L}$. Let $u : Z \to X$ be the finite map obtained from the normalization Z of T and $v : Z' \to X'$ the finite map obtained from the base change via u with the fibration morphism $g : T' \to T$. Let $\tilde{Z}$ be a resolution of singularity of Z and $\tilde{u} : \tilde{Z} \to X$ be the resulting morphism to X. From the base change property for dualizing sheaves we get $g_* v^! L' = u^! \gamma_* L' = u^!(K_{X/Y} + \bar{A} + H) = K_{Z/Y} + u^*(\bar{A} + H)$ where $\bar{A}$ is the reduced part of A. Using the proof of Lemma 5.1 in [Vi0], we see that the problem now reduces to establishing the weak positivity of $f_* u_*(K_{Z/Y} + u^* \bar{A})$ or, what is equivalent, of $f_* \tilde{u}_*(K_{\tilde{Z}/Y} + \tilde{u}^* \bar{A})$ which we know from [Ka, Theorem 30]. ■

We have as before the following corollaries by looking at the relevant birationally equivalent well-adapted fibrations:

THEOREM 9.9 *Let $g^\partial : Y^\partial \to Z$ be a fibration of general type with Y^∂ smooth or more generally e-admissible effective. Then*

$$\kappa(Y^\partial) \geq dim(Z) + \kappa(Y_z^\partial),$$

where Y_z^∂ is the general fiber of g^∂. Hence, Conjecture 9.1 is true if the g^∂ or the h^∂ there is of general type. In particular, if Y_z^∂ is of general type, then Y^∂ is also. ■

THEOREM 9.10 *A smooth effective orbifold or an e-admissible effective orbifold X^∂ is special if $\kappa(X^\partial) = 0$.* ■

We have an analog of a result of N. Nakayama [Na0, Thm 7.3.5] on the subadditivity of sectional Kodaira dimensions:

THEOREM 9.11 *Let $f^\partial : X^\partial \to Y^\partial$ be a surjective morphism of smooth effective orbifolds with connected fibers, i.e., a fibration morphism, and let F^∂ be a general fiber. Then*

$$\kappa_\sigma(K_{X^\partial} - f^*K_{Y^\partial} + f^*D) \geq \kappa_\sigma(K_{F^\partial}) + \kappa_\sigma(D),$$

for any $\mathbb{Q}$-divisor D of Y. In particular,

$$\kappa_\sigma(K_{X^\partial}) \geq \kappa_\sigma(K_{F^\partial}) + \kappa_\sigma(Y^\partial).$$

Proof: We refer the reader to [Na0] for the proof since the main additional steps are given above. ∎

We also refer the reader to Definition 6.2.7 in [Na0] for κ_σ. For our purpose, it is sufficient for us to know that $\kappa_\sigma(K_{X^\partial}) \geq 0$ if and only if K_{X^∂} is pseudoeffective by the definition. Finally N. Nakayama has a generalization of the additivity of Kodaira dimension theorem of Kollar to the compact complex category, see [Na1, Theorem 5.7]. In light of our propositions above we have the following corollary of this theorem, which we will not use but many of our main results can also be derived directly from it.

THEOREM 9.12 (Orbifold Kollar's additivity theorem) *Conjecture 9.1 is true if Y_z^∂ or (Y_z, f_z^∂) or both are of general type.* ∎

10 The canonical fibration

The following theorem resolves some conjectures of Campana on unramified coverings and on the structure of his general type reduction in [Ca02]. It also answers in the affirmative a vital question raised by Bogomolov concerning Bogomolov sheaves.

THEOREM 10.1 *Let $X^\partial = X \backslash A$ be a smooth or an e-admissible effective orbifold. Suppose that X^∂ is not special. Then there is one and only one Bogomolov $\mathbb{Q}$-sheaf $L \hookrightarrow \Omega_X^p(A)$ (where the inclusion is as $\mathbb{Q}$-sheaves) such that $\kappa(L) = \kappa'_+(X^\partial) = p > 0$. That is, there is a meromorpic fibration of general type $f^\partial : X^\partial \dashrightarrow Y$ which is unique up to birational modification of Y with the property that $\dim Y = \kappa'_+(X^\partial)$. Furthermore, given any meromorphic fibration of general type $h^\partial : X^\partial \dashrightarrow Z$, there exist a meromorphic map $g : Y \dashrightarrow Z$ such that $h^\partial = g \circ f^\partial$.*

Proof: Suppose $f^\partial : X^\partial \dashrightarrow Y$ and $h^\partial : X^\partial \dashrightarrow Z$ are meromorpic fibrations of general type with $\dim Y = \kappa'_+(X^\partial)$. It is sufficient to show that h factors through f. We may assume that f and h are morphisms by blowing up. Consider the product map $\psi = (f, h) : X \to Y \times Z$. Let $\psi : X \to \bar{Y}$ be obtained from the Stein factorization of (f, h) by blowing up so that it is a morphism with $\bar{Y}$ smooth. We have natural morphisms $p : \bar{X} \to Y$ and $q : \bar{X} \to Z$ which are obtained from the projections to the two factors of $X \times Y$. It is easily seen that these are fibrations since f and h are. It follows that the general fiber $F = \psi^{-1}(y, z) = f^{-1}(y) \cap h^{-1}(z)$ is well defined and smooth if nonempty. Let $d_c = \dim X - \dim Y$. If $\dim F \neq d_c$, then

$\dim \bar{Y} > \dim Y$ and so p is a fibrations with positive dimensional general fibers. The general orbifold fibers of this fibration must be of general type by Corollary 5.9 and so we have that $\bar{Y}^\partial$ must be of general type by Theorem 9.9. This means that ψ itself is of general type as it is admissible. This is a contradiction to the definition of $\kappa'_+(X^\partial)$. Hence $\dim F = d_c$, which means that $f^{-1}(y) \cap h^{-1}(z) = f^{-1}(y)$ if the intersection is nonempty. This, in turn, means that h factors through f ∎

It follows from this theorem that there is no ambiguity in calling the f^∂ above as the top (or the maximum) fibration of general type from X (up to a modification on the base), which we denote by bc_{X^∂}. The following was a conjecture of Campana in the non-orbifold case.

COROLLARY 10.2 *Let $X^\partial = X \backslash A$ be a special e-admissible effective orbifold and let $\pi : \bar{X} \to X$ be a finite étale covering and $\bar{X}^\partial = \bar{X} \backslash \pi^* A$. Then $\bar{X}^\partial$ is special.*

Proof: Clearly we may assume that π is a Galois covering. If $\bar{X}^\partial$ is not special, then $\bar{f} = bc_{\bar{X}^\partial}$ is a general type fibration with positive dimensional base. By uniqueness of $bc_{\bar{X}^\partial}$, $\bar{f}$ commutes with the covering group G and hence gives rise to a morphism $f : X \to Y$ and f^∂ is of general type by construction. This is a contradiction.

An easier argument is to look at the problem on the level of the top Bogomolov $\mathbb{Q}$-sheaf $\bar{L}$ of $\bar{X}$ which is invariant under G by its uniqueness. Hence, this sheaf descends to a Bogomolov $\mathbb{Q}$ sheaf by the fact that we can take trace of sections, i.e., the Kodaira dimension of $\bar{L}$ is unchanged after taking its quotient by G by the same argument as that for Lemma 4.6. ∎

The next theorem is the Main theorem of this paper which resolves in particular some conjectures of Campana in [Ca02].

THEOREM 10.3 *Let X^∂ be an e-admissible effective orbifold. Then, up to modifications on the base, there is a unique (almost holomorphic) fibration of general type*

$$h^\partial = bc_{X^\partial} : X^\partial \dashrightarrow Z = bc(X^\partial)$$

whose general fibers are special. We have $\kappa'_+(X^\partial) = \dim bc(X^\partial)$. Moreover, the general fibers of h^∂ are the maximal special subvarieties of X^∂ through the general points of X. In particular, in the case $\partial X^\partial = 0$ and $X \in \mathcal{C}$ the maximal special fibration c_X defined by Campana is birationally equivalent to bc_X and hence is of general type.

Proof: We have already seen in the previous theorem that there is a maximum fibration of general type bc_{X^∂} from X^∂. It remains to show that this fibration is special since the other claims follows from Proposition 6.5.

We will assume that $h^\partial = bc_{X^\partial} : X^\partial \to Z$ is a well adapted fibration by blowing up. Let $bc_{h^\partial} : X^\partial \to Y$ be the relative basic reduction (Proposition 8.7) of $h^\partial = bc_{X^\partial}$, which we again assume to be admissible by blowing up. We have a canonical projection $p : Y \to Z$. If p is not birational, then the general fiber Y_z of p has positive dimension and so we have again by Theorem 9.9 that Y^∂ is of general type. But this contradicts the definition of bc_{X^∂}. Hence p is birational, which means that the general orbifold fibers of bc_{X^∂} are special by

Proposition 8.7. ■

An easy corollary is the following theorem, which gives us in principle many possible special fibrations connected to a compact complex orbifold.

THEOREM 10.4 *Given a fibration $h : Z \to Z$ with either $h \in \mathcal{C}$ or Z Moishezon. If X^∂ is an e-admissible effective orbifold, in particular if X^∂ is smooth and effective, then bc_{h^∂} is a special fibration.* ■

11 Remarks on the Albanese map

There has been quite a number of focus on the structure of the Albanese map of a projective or Kähler manifold with nef anticanonical bundle, see for example [DPS]. We can approach the problem of the Albanese map as done in [Ca02] for special manifolds. We first recall:

DEFINITION 11.1 *Let Y^∂ be an orbifold. Y^∂ is called* **nearly special** *if it does not admit a general type holomorphic fibration with positive dimensional but possibly singular base. A fibration $f^\partial : Y^\partial \to Z$ is called nearly special if its orbifold general fibers are $*$-special.*

By definition, special orbifolds are nearly special. The following result was promised earlier.

THEOREM 11.2 *Let X_0^∂ be a smooth projective and effective orbifold such that $K_{X_0^\partial}^\vee$ is a nef $\mathbb{Q}$-line bundle or equivalently $-K_{X_0^\partial}$ is a nef $\mathbb{Q}$-divisor. Then X_0 is nearly special.*

Proof: Suppose $f_0^\partial : X_0^\partial \to Y_0$ is a holomorphic fibration of general type with Y_0 a positive dimensional normal compact complex space, which is necessarily Moishezon. We will reach a contradiction from its existence. By flattening and then resolving using the $-\infty$ multiplicity on the exceptional fiber, we obtain by Lemma 3.13 a well adapted birationally equivalent fibration $f^\partial : X^\partial \to Y$ via birational morphisms $v^\partial : X^\partial \to X_0^\partial$ and $u : Y \to Y_0$. We may assume Y is projective. Then we may write

$$K_{X^\partial} = K_X + A = K + E$$

where $A = \partial X^\partial$ has normal crossing support, $K = v^* K_{X_0^\partial}$ and E is an effective divisor supported on the exceptional divisor of v. We note that $-K$ is nef, $f(E_{red}) \neq Y$ and E is exceptional with respect to Y_0 in the sense that it maps to a codimension two subset S of Y_0. Let H_0 (resp. A') be an ample $\mathbb{Q}$-divisor on Y_0 (resp. on X). Now, there exist an $\epsilon > 0$ such that

$$E - f^* u^* H_0 + \epsilon A'$$

is not pseudoeffective (To see this, restrict to a complete intersection curve on Y_0 disjoint from S). As $-K + \epsilon A'$ is ample, it is linear equivalent to an effective $\mathbb{Q}$-divisor D such that $D \leq D_{red}$ and that $D_{red} \cup \bar{A}$ has normal crossing support. It follows that $X^{\partial_1} = X \backslash (A + D)$ is a smooth orbifold such that $f_1^\partial : X^{\partial_1} \to Y$ is still a fibration of general type dominating

$Y^\partial := Y \backslash D(f^\partial)$. We may choose H_0 such that $K_{Y^\partial} - 2u^* H_0$ is linearly equivalent to an effective $\mathbb{Q}$ divisor. Since $K_{X^{\partial_1}} = E + \epsilon A'$ restricts to an ample divisor on the general fibers F of f_1 by construction, we have

$$\kappa_\sigma(K_{X^{\partial_1}} - f^* K_{Y^\partial} + f^* u^* H_0) \geq \dim F + \kappa_\sigma(u^* H_0) = \dim X$$

by Theorem 9.11. Hence $K_{X^{\partial_1}} - f^* K_{Y^\partial} + f^* u^* H_0$ is big. Since

$$K_{X^{\partial_1}} - f^* K_{Y^\partial} + f^* u^* H_0 = E - f^* K_{Y^\partial} + f^* u^* H_0 + \epsilon A' \leq E - f^* u^* H_0 + \epsilon A',$$

this means that the right hand side is big and hence peudoeffective, a contradiction. ■

The above proof also applies to the case where $-K$ is only assumed to be almost nef with respect to certain holomorphic fibrations as defined in [DPS].

The same argument as that given in [Ca02], which follows that of Kawamata and Viehweg in [KV], gives:

THEOREM 11.3 *Suppose $-K_X$ is nef. Then the Albanese map of X,*

$$Alb_X : X \to Alb(X),$$

is surjective, connected, has no multiple fibers in codimension one and is a nearly special fibration. In particular, if the general fibers of Alb_X (which are thus nearly special) has almost abelian fundamental group, then so does X. ■

We now note the following result of Mihai Paun in [P] derived using a strong differential geometric result of Cheeger and Colding ([CC]), a result of Qi Zhang ([ZQ]) concerning the Albanese map that employs characteristic $p > 0$ techniques from [MiMo] and a result of Campana ([Ca3]). Our result refines that of Qi Zhang but without involving characteristic $p > 0$ thereby giving another proof (more direct for differential geometry) of this result of Mihai Paun, which we can now state with a slight strengthening at least in the direction of the Albanese map. See [P] for the proof. The last part of the theorem follows from the standard exact sequence of fundamental groups for a fibration, see for example [XG], whose theme in the full orbifold context will be taken up later.

THEOREM 11.4 (A. Paun) *Let X be a projective manifold with nef $-K_X$. Then X has almost abelian fundamental group and $\pi_1(F)$ is finite, where F is the generic fiber. Furthermore, the Albanese map of X is surjective, connected, has no multiple fibers in codimension one and induces $\pi_1(X)$ as an extension of $\pi_1(Alb(X))$ by a quotient of $\pi_1(F)$.* ■

This is a generalization in the projective category of a result of Demailly, Peternell and Schneider in [DPS] that projective threefolds with nef anticanonical bundle have almost abelian fundamental group whose statement and proof we also refine as above. It should be noted however, that their result is for Kähler threefolds.

12 Appendix A. Orbifold morphism

The most natural definition of an orbifold morphism is via orbifold metrics, which we did in [BL2] and in [Lu01] under the name of Z-orbifold maps. We only give this definition for standard smooth orbifolds in the following but it should be clear how it is generalized for an arbitrary smooth orbifold.

Given an n-dimensional complex manifold Y with a divisor $D^Y = \sum_i (1-1/m_i)D_i$ where $D^Y_{red} = \sum_i D_i$ is a simple normal crossing divisor and $m_i \in \mathbb{N}$. Then D^Y defines an orbifold structure on Y and Y so endowed is written as $Y^\partial := (Y; D^Y) = Y \backslash D^Y$. Locally, Y^∂ can be realized as a quotient of open subsets of $\mathbb{C}^n$ by a finite subgroup of $\mathbb{C}^{*n}$ of cyclic indices prescribed by the corresponding m_i's. Pushing down the flat metric on $\mathbb{C}^n$ by these local quotient maps and patching them up via a partition of unity on Y gives us an orbifold metric g for Y^∂ and we will define any other metric on $Y \setminus D^Y_{red}$ to be an orbifold metric for Y^∂ if its ratio with g extends to a continuous function on Y.

DEFINITION 12.1 *A holomorphic map $f : Y_1 \to Y_2$ between the complex manifolds Y_1 and Y_2 endowed with orbifold structures as above is called a Z-orbifold map if the natural pull back of an orbifold metric on Y_2^{orb} to Y_1 can be dominated by an orbifold metric on Y_1^{orb}.*

It is easy to see that these definitions are independent of the choices involved. It is not difficult to verify that orbifold morphism between smooth standard orbifolds are precisely the Z-orbifold maps.

We end with a few observations concerning orbifold morphisms.

For the next definition, if $B = \sum_{i\in I} a_i D_i$ is a $\mathbb{Q}$-divisor, we call $B' = \sum_{i \in I'} a_i D_i$ a partial sum in B for any $I' \subset I$.

DEFINITION 12.2 *Given a surjective morphism $f : X \to Y$ of manifolds X, Y endowed with smooth orbifold structures A, B respectively. We say that f gives rise to a* **good K-map** *$f^\partial : X^\partial \dashrightarrow Y^\partial$ if for $B^* = B \cap \Delta(f^\partial)$ and $\bar{B} := f^{-1}(B^*_{red})$, we have*

$$f^{-1}(B_{red}) \subset A_{red} \quad \text{and} \quad f^*(B - B^*_{red}) \le A \cap \bar{B} - \bar{B}$$

as $\mathbb{Q}$-divisors. If further

$$f^*(B' - B'_{red}) \le A \cap \bar{B}' - \bar{B}'$$

for $\bar{B}' := f^{-1}(B'_{red})$ for all partial sum B' in B^*, then we say that f^∂ is a **good orbifold morphism** *or simply a good morphism for short.*

Given log-manifolds $X \backslash A$ and $Y \backslash B$, we observe that a good morphism $f^\partial : X^\partial \to Y^\partial$ is precisely a surjective log-morphism of log-manifolds. The following is a straightforward generalization of generically finite surjective K-maps defined in Definition 3.2.

DEFINITION 12.3 *Given a surjective morphism $f : X \to Y$ of manifolds X, Y endowed with smooth orbifold structures A, B respectively. We say that $f^\partial : X^\partial \dashrightarrow Y^\partial$ is a* **K-map** *if*

$$f^* K_Y + B \le L_f + A \cap f.$$

LEMMA 12.4 *If $f^\partial : X^\partial \dashrightarrow Y^\partial$ between smooth orbifolds X^∂ and Y^∂ is a good K-map, then f^∂ is a K-map. If further f^∂ is a good morphism, then it is an orbifold morphism. A (good) K-map f^∂ is a (good) morphism outside its exceptional divisor $E(f)$.* ∎

Proof: The first two statements are direct consequences of the fact that $f^*(K_Y + B_{red}) \leq L_f + \bar{B}$ because $f^{|\partial|} : Y \backslash f^{-1}(B_{red}) \to Y \backslash B_{red}$ is a log-morphism. The second statement is clear from the local expression for f^∂. ∎

We have many different types of maps and "morphisms." We conclude with their easily observed behaviors under composition:

LEMMA 12.5 *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms, X^∂, Y^∂ and Z^∂ orbifolds and $h = g \circ f$. If f^∂ and g^∂ are both (good) K-maps and g is generically finite and surjective or if they are both (good) morphisms, then so is h^∂.*

References

[Ba] D. Barlet, Espace analytique réduit des cycles analytiques complexes de dimension finiem Seminaire F. Norguet, *Lect. Notes in Math.,* 482 (1975), 1–158.

[BL1] G. Buzzard and S. Lu, Algebraic surfaces holomorphically dominable by $\mathbb{C}^2$, *Inventiones Math.*, 139 Fasc 3 (2000), 617–659.

[BL2] G. Buzzard and S. Lu, Double sections, dominating maps, and the Jacobian fibration, *American Journal of Math.*, 122 (2000) no. 5, 1061–1084.

[BlG] S. Bloch, D. Gieseker, The positivity of the Chern classes of an ample vector bundle, *Invent. Math.* 12 (1971), 112–117.

[BM] E. Bierstone and P. Milman, Canonical desingularization in characteristic zero by blowing up the maximal strata of a local invariant, *Invent. Math.* 128 (1997), 207–302. 103 (1991), no. 1, 69–99.

[Bo] F. Bogomolov, Holomorphic Tensors and vector bundles on projective varieties, *Math. USSR Izv.*, vol 13 (1979), 499 – 555.

[BPV] W. Barth, C. Peters, A. Van de Ven, *Compact Complex Surfaces*, Springer-Verlag, Berlin, 1984.

[Cbi] E. Calabi, Construction and properties of some 6-dimensional almost complex manifolds. *Trans. A.M.S.*, 87 (1958), 407–438.

[Ca] F. Campana, Fundamental group and positivity of cotangent bundles of compact Kähler manifolds. *J. Algebraic Geom.* 4 (1995), no. 3, 487–502.

[Ca01] F. Campana, Special varieties and classification theory, Preprint arXiv:math.AG/0110051 v1 4 Oct 2001.

[Ca02] F. Campana, Special varieties and classification theory, Preprint arXiv:math.AG/0110051 v7 Sept 7, 2001.

[Ca2] F. Campana, Connexité rationnelle des variétés de Fano, *Ann. Sci. École Norm. Sup.* 25 (1992), 539–545.

[Ca3] F. Campana, Remarques sur les groupes de Kähler nilpotents, *C. R. Acad. Sci.* 317 (1993), 777–780.

[CP] F. Campana and T. Peternell, Cycle spaces. Several complex variables, VII, 319–349, *Encyclopaedia Math. Sci.,* 74, Springer, (1994).

[CC] J. Cheeger and T. Coding, Lower bounds on Ricci curvature and almost rigidity of warped products, *Ann. Math.* 144 (1996), 189–234

[CT] J. L. Colliot-Thélène, Arithmétique des variétés rationelles et problème birationels. *Proc. Intl. Conf. Math.* (1986), 641–653.

[Debarre] Olivier Debarre, *Higher Dimensional Algebraic Geometry*, Universitext, Springer (2000).

[DL] G. Dethloff and S. Lu, Logarithmic jet bundles and applications, *Osaka J. Math.* 38 (2001), 185–237.

[DPS] J.P. Demailly, T. Peternell and M. Schneider, Pseudo-effective line bundles on compact Kähler manifolds, *International Journal of Math.* vol. 12, no. 6 (2001), 689–741.

[DPS0] J.P. Demailly, T. Peternell and M. Schneider, Kähler mainfolds with numerically effective Ricci-class, *Compositio Math.* 89 (1993), 217–224.

[DPS1] J.P. Demailly, T. Peternell and M. Schneider, Compact complex manifolds with numerically effective tangent bundles, *J. Alg. Geom.* 3 (1994), 295–345.

[DPS2] J.P. Demailly, T. Peternell and M. Schneider, Compact Kähler manifolds with Hermitian semi-positive anticanonical bundle, *Compositio Math.* 101 (1996), 217-224.

[Dy] A. Douady, Le problème de modules pour les sous-espaces analytiques complexes d'un espace analytique donné, *Ann. Inst. Fourier, Grenoble*, 16 (1966), 1–95.

[En] H. Esnault, Classification des variétés de dimension 3 et plus, Séminaire Bourbaki, Exposé 568, *Lecture Notes in Math.*, 901 (1980/81), 111–131.

[Fujk0] A. Fujiki, Closedness of the Douady spaces of compact Kähler spaces, *Publications RIMS, Kyoto U.* 14 (1978), 1–52.

[Fujk] A. Fujiki, Deformation of uniruled manifold, *Publications RIMS, Kyoto U.* 17 no. 2 (1981), 687–702.

[Fujk1] A. Fujiki, Countability of the Douady space of a complex space, *Japan. J. Math. (N.S.)* 5 (1979), no. 2, 431–447.

[Fujk2] A. Fujiki, Projectivity of the space of divisors on a normal compact complex space, *Publications RIMS, Kyoto U.* 18 (1982), 1163–1173.

[Fujk3] A. Fujiki, Relative algebraic reduction and relative Albanese map for a fiber space in $\mathcal{C}$, *Publ. Res. Inst. Math. Sci.* 19 (1983), 207–236.

[Fujk4] A. Fujiki, On the Douady spaces of a compact complex space in the category $\mathcal{C}$, II, *Publications RIMS, Kyoto U.* 20 (1984), 461–489.

[Fujt] T. Fujita, Semipositive line bundles, *J. Fac. Sci., Tokyo U.* Sect IA, Math. 30 (1983), 353–378.

[Fta] T. Fujita, Some remarks on Kodaira dimensions of fiber spaces, *Proc. Japan Acad.* 53 (1977), 28–30.

[GHS] T. Graber, M. Harris and J. Starr, Families of rationally connected varieties, Preprint 2001.

[GR] L. Gruson and M. Raynaud, Critères de platitude et de projectivitè, *Invent. Math.* 13 (1971), 1–89.

[Ha] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math. 52, Springer Verlag, New York, 1977.

[HS] R. Harvey and B. Shiffman, A characterization of holomorphic chains, *Ann. of Math.*, 99 (1974), 553–587.

[Hi0] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, *Ann. Math.*, 79 (1964) 109–326.

[Hi] H. Hironaka, Flattening theorem in complex-analytic geometry, *Amer. J. Math.*, 97 (1975), 503–547.

[Ii] S. Iitaka, *Algebraic Geometry*, Graduate Texts in Math. 76, Springer Verlag, New York, 1982.

[Ka] Y. Kawamata, Characterization of Abelian varieties. *Comp.Math.*, 43 (1981), 253–276.

[Kod1] K. Kodaira, Pluricanonical systems on algebraic surfaces of general type, *J. Math. Soc. Japan* 20 (1968), 170–192.

[KM] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tract in Mathematics (1998).

[KeM] S. Keel and S. Mori, Quotients by groupoids, *Ann. of Math.* (2) 145 (1997), no. 1, 193–213.

[KMM] Y. Kawamata, K. Matsuda, K. Matsuki, Introduction to the Minimal Model Problem, *Adv. Studies in Mathematics* 10 (1987), 283–360.

[Kol] J. Kollár, Subadditivity of the Kodaira dimension: fibers of general type, *Adv. Studies in Pure Math.*, 10 (1989), 361–398.

[Kol1] J. Kollar, *Rational Curves on Algebraic Varieties* Ergebnisse der Math. und ihrer Grenzgebiete, 3 Folge, Bd. 32, Springer-Verlag, Berlin, 1996.

[Kob1] S. Kobayashi, *Hyperbolic Complex Spaces*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 318. Springer-Verlag, Berlin, 1998.

[KoMM1] J. Kollar, Y. Miyaoka, S. Mori, Rationally connectedness and boundedness of Fano manifolds. *J. Diff. Geom.* 36 (1992), 765–769.

[KoMM] J. Kollar, Y. Miyaoka, S. Mori, Rationally connected varieties. *J. Algebraic Geom.* 1 (1992), no. 3, 429–448.

[KV] Y. Kawamata and Viehweg, On a characterization of Abelian varieties, *Comp. Math.* 41 (1980, 355–359.

[Lang1] S. Lang, Hyperbolic and Diophantine analysis, *Bull. Amer. Math. Soc.* 14,(1986) no. 2, 159–205.

[Lang] S. Lang, *Introduction to Complex Hyperbolic Spaces.* Springer Verlag, 1987.

[Lieb] D. Lieberman, Compactness of the Chow scheme: applications to automorphisms and deformations of Khler manifolds, *Lecture Notes in Math.*, 670 (1978), 140–186.

[LS] D. Lieberman and E. Sernesi, Semicontinuity of L-dimension, *Math. Ann.* 225 (1977), 77–88.

[LM] S. Lu, Y. Miyaoka, Bounding codimension-one subvarieties and a general inequality between Chern numbers, *American Journal of Math.* 119 (1997), 487–502.

[LM1] S. Lu, Y. Miyaoka, Bounding curves in algebraic surfaces by genus and Chern numbers, *Math. Res. Lett.* 2 (1995), 663–676.

[Lu0] S. Lu, A refined Kodaira dimension and its canonical fibration 0, Preprint version from Osaka University (2001), 11 pp.

[Lu01] S. Lu, Multiply marked Riemann surface and the Kobayashi pseudometric on algebraic manifolds, Preprint 2001.

[Lu02] S. Lu, The Kobayashi pseudometric on algebraic manifold and a canonical fibration, Max Planck Institute (Bonn) and Math. ArXiv Web Preprint.

[Ma] K. Maehara, The weak 1-positivity of direct image sheaves, *J. Reine Angew. Math.* 364 (1986), 112–129.

[McQ] Noncommutative Mori's theory, *IHES Preprint* 2001.

[Miya0] Y. Miyaoka, Deformation of a morphism along a foliation and applications, *Proc. Sympos. Pure Math.* 46 (1987) Part I, 245–268.

[Miya] Y. Miyaoka, The Chern class and Kodaira dimension of a minimal variety, *Adv. Stud. Pure Math.* 10 (1987), 449–476.

[Miya1] Y. Miyaoka, On the Kodaira dimension of minimal threefolds. *Math. Ann.* 281 (1988), no. 2, 325–332.

[Mori] S. Mori, Classification of higher-dimensional varieties, *Proceeding of Symposia in Pure Math.* Vol. 46 (1987), 269–331. (Bowdoin 1985)

[MoMu] S. Mori and S. Mukai, The uniruledness of the moduli space of curves of genus 11, *Lecture Notes in Math.*, 1019 (1982), 334–353.

[MiMo] Y. Miyaoka and S. Mori, A numerical criterion for uniruledness, *Ann. Math.* 124 (1986), 65–69.

[MP] Y. Miyaoka and Peternell, *Geometry of Higher Dimensional Algebraic Varieties*, *DMV Seminar*, Band 26, Birkhäuser.

[Na0] N. Nakayama, Zariski-Decomposition and Abundance, it RIMS Preprint no. 1142 (1998) Kyoto University, 121pp.

[Na] N. Nakayama, Invariance of plurigenera of algebraic varieties, *RIMS Preprint no. 1191* (1998) Kyoto University, 35pp.

[Na1] N. Nakayama, Compact Kähler manifold whose universal covering spaces are biholomorphic to $\mathbb{C}^n$, *RIMS Preprint no. 1230* (1999) Kyoto University, 100pp.

[Nog] J. Noguchi, A short analytic proof of closedness of logarithmic forms, Kodai Math. J. 18 (1995), no. 2, 295–299

[P] M. Paun, Sur le groupe fondamental des variétés kähleriennes compactes à classe de Ricci numériquement effective, *C.R. Acad. Sci. Paris* t.324 Série I (1997), 1249–1254.

[Re] Miles Reid, Bogomolov's theorem $4c_2 \geq c_1^2$, Proc. Internat. Sympos. on Algebraic Geometry (Kyoto 1977), Kinokuniya, Tokyo, 623 – 642.

[Siu] Y. T. Siu Invariance of plurigenera. *Invent. Math.* 134 (1998), no. 3, 661–673.

[Saito] K. Saito, Theory of logarithmic differential forms and logarithmic vector fields, *J. Fac. Sci., Univ. of Tokyo* Sci. IA 27 (1980) no. 2, 265–291.

[Ue] K. Ueno, *Classification Theory of Algebraic Varieties and Compact Complex Manifolds*, *Lecture Notes in Math.* 439 (1975).

[Vie] E. Viehweg, Die Additivität der Kodaira Dimension für Projektive Faserräume über Varietäten des Allgemeinen Typs. *J.Reine.Angew.Math.*, 330 (1982), 132–142.

[Vi0] E. Viehweg, Weak positivity and the additivity of the Kodaira dimension for certain fiber spaces, *Adv. Studies in Pure Math.* 1 (1983), 329–353.

[Vi1] E. Viehweg, Vanishing theorems, *J. Reine. Angew. Math.* 335 (1982), 1–8.

[Vi2] E. Viehweg, Vanishing theorems and positivity of algebraic fiber spaces, *Proc. I.C.M.,* Vol 1,2 (Berkeley, Calif., 1986), 682–688.

[XG] Gang Xiao, π_1 of elliptic and hyperelliptic surfaces. *Internat. J. Math.*, 2 (1991), no. 5, 599–615.

[ZQ] Qi Zhang, On projective manifolds with nef anticanonical bundle, *J. Reine. Angew. Math* 478 (1996), 57–60.

Steven Shin-Yi Lu
Max Planck Institute of Mathematics
7 Vivatsgasse, 53111 Bonn
Germany

slu@math.sci.osaka-u.ac.jp
slu@mpim-bonn.de