BRANCHED DERIVATIVES

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Abstract. We investigate local behavior of branched covering maps at their branching points and introduce a notion of a branched derivative, similar to a derivative for diffeomorphisms. Then, under an additional assumption that the map is locally area preserving, we look at the dynamics in a neighborhood of a periodic branching point. The two stable (hyperbolic) cases are similar to the usual picture at a hyperbolic periodic point, with a few important differences. In particular, in the case analogous to the saddle behavior one gets one expanding direction and a Cantor set of contracting directions.

1. Introduction

One of the very basic notions of Mathematical Analysis is the notion of a derivative. In particular, it is used all the time when investigating the dynamics of smooth maps. Among other things, it allows to replace the actual map in a neighborhood of a periodic orbit by a linear map, whose dynamics on one hand is much easier to study, and on the other hand approximates well the dynamics of the original system. While this approach works very well for invertible systems (diffeomorphisms), there are quite natural situations for noninvertible systems when differentiability breaks down at some points. In one (real) dimension the remedy is to look at one-sided derivatives at such points. This is not so easy in dimension 2.

There is a well developed theory of dynamics of rational (complex) maps on the Riemann sphere. They are examples of *branched covering* maps, but the assumption of differentiability everywhere forces a very special behavior at the branching points. One would like to be able to admit much broader types of behavior without loosing completely differentiability at those points. For instance, in [BSTV] and [BN] the authors consider branched coverings of the complex plane of the form

$$f(z) = |z|^{2\alpha-2}z^2 + c,$$

which are not differentiable at the branching point $z = 0$.

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In a series of recent papers [BCM1, BCM2] we suggested a class of maps called *expanding polymodials* which seem to exhibit interesting phenomena similar to those of complex polynomials and piecewise expanding polymodal interval maps (whence the name for this class of maps). Expansion is a part of our definition, but is not essential for the discussion here. These maps cannot be made smooth in the usual sense exactly because of their behavior at branching points, where expansion, not contraction, has to take place.

While for expanding polymodials we could deduce many results without assuming any smoothness and using expansion instead, this is not so for other branched covering maps. Another important case occurs when the map locally preserves area, and there differentiability is indispensable. Since area preservation is only local, such maps can have attractors. Here are several examples.

1) c-tent maps, in complex notation $z \mapsto cz^2/|z| + 1$, where $|c| = \sqrt{2}/2$,
2) similar maps, where complex multiplication by c is replaced by a real linear map with determinant $1/2$ and norm less than 1,
3) $D \circ g \circ D$, where $D(z) = z^2/|z|$ and $g(x, y) = (\frac{125}{256}x + \frac{1}{2}, \frac{64}{125}y)$ (see Figure 1).

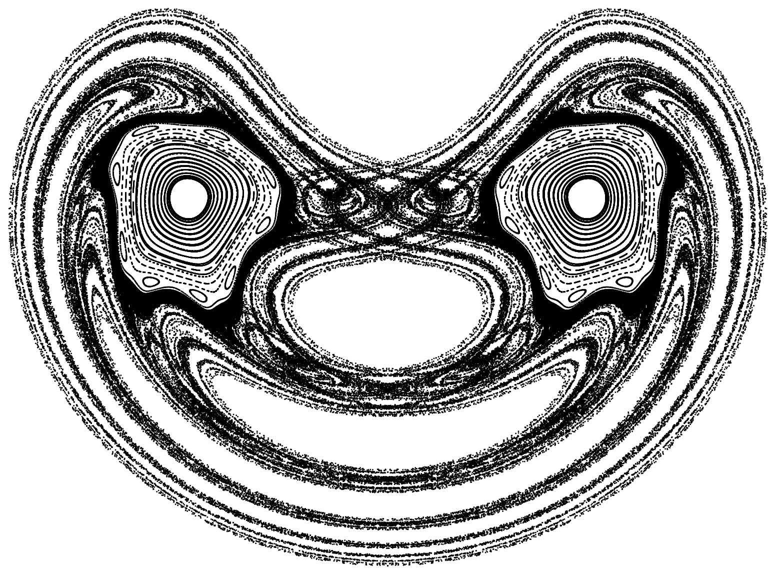

FIGURE 1. Attractor from the third example (rotated by 90 degrees)

Again, all such maps are not smooth in the usual sense because of their behavior at branching points. Let us describe a dynamical phenomenon that can occur for such maps. Suppose that our branched covering map F has a periodic branching point a of period $n > 1$. Suppose also that this point is attracting in the topological sense (a small neighborhood U of a is mapped by F^n into itself, so that all points of U converge to a under the iterates of F^n). Let G be a small perturbation of F. The neighborhood U may still be mapped back into itself by G. However, it may happen that much smaller neighborhoods of a are not mapped into themselves. Then the intersection of all images of U under the iterates of G^n, which used to consist of just one point a before we made the perturbation, may be an attractor with a complicated structure. Moreover, it may well happen that inside this attractor a new branching topologically attracting periodic point of higher period is created.

Observe that the period of the new periodic point for G must be a multiple of n. This procedure may be repeated again and again, similarly to renormalizations for unimodal interval maps. Such phenomena are possible because we allow for branching periodic points at which the map is not smooth in the usual sense.

Since we cannot dismiss such branching points, we want to tame them. Thus, we define *branched derivatives* and study their properties. Since the class of maps that can serve as usual derivatives are linear maps, we call the class of maps that can serve as branch derivatives *branched linear* maps. However, we are mostly interested in maps similar to the examples given earlier in this section. For the iterates of those maps branched derivatives at branching points are of a special form. Thus, we will define a class of maps narrower than the class of branched linear maps. We will call them *basic branched linear* maps and will study them closer.

The paper is arranged as follows. In Section 2 we define branched linear maps and branched derivatives, motivating our definitions and showing that they behave well under compositions. In Section 3 we study in great detail the basic branched linear maps. In Section 4 we discuss which results of Section 3 can be extended to the class of branched linear maps.

2. Branched linear maps and branched differentiation

Our question is, how can one define a "not-so-restrictive" version of smoothness at branching points? The literal analogies between diffeomorphisms and branched covering maps do not work here. Still, certain properties of the usual differentiation motivate our approach. Note that we are working with open maps, so we should draw analogies with diffeomorphisms.

At a branching point it really makes sense to look separately at what the map does with the rays emerging from it. That is, it makes sense to look at the map in polar coordinates, where the origin is placed at that point. We will speak then of two directions, the *radial* direction, along the rays, and the *angular* direction, perpendicular to the radial one. We also identify the two-dimensional Euclidean plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, and use the complex notation whenever possible.

Usual derivatives (provided they are nondegenerate) should be included in the class of branched derivatives. Therefore let us take a 2×2 nondegenerate matrix A (we will identify it with the corresponding linear map of the plane) and analyze it in polar coordinates (r, ϑ). In fact, instead of the angle ϑ it is much easier to use the point z of the unit circle $\mathbb{S}^1$ with argument ϑ. Then we can define the map $\varphi_A : \mathbb{S}^1 \to \mathbb{S}^1$ by the formula $\varphi_A(z) = A(z)/|A(z)|$. It describes how A acts in the angular direction. We do not worry too much what A does in the radial direction, except that it is homogeneous, that is, $A(tz) = tA(z)$ for $t \in \mathbb{R}$.

Our next step is to investigate the properties of the map φ_A. It turns out that a very natural way to get more information about it is to consider its 2-to-1 factor $\psi : \mathbb{S}^1 \to \mathbb{S}^1$ under the map $z \mapsto z^2$ (that is, $\psi_A(z^2) = (\varphi_A(z))^2$). It is well defined, since $\varphi(-z) = -\varphi(z)$. The following fact is known, although perhaps less popular than it deserves.

Lemma 2.1. *The map ψ_A extends to a Möbius map on the Riemann sphere.*

Proof. We have

$$\psi_A(z^2) = (\varphi_A(z))^2 = \frac{(A(z))^2}{|A(z)|^2} = \frac{A(z)}{\overline{A(z)}} .$$

If $z = u + vi$ then for some $a, b, c, d \in \mathbb{R}$ (the entries of the matrix A) we have

$$A(z) = (au + bv) + (cu + dv)i = (a + ci)u + (b + di)v.$$

In other words, $A(z)$ is a linear combination (with complex coefficients) of u and v. However, u and v are linear combinations of z and $\overline{z}$, so $A(z)$ is a linear combination of z and $\overline{z}$. Although the precise formulas are not necessary at this moment, we need them if we want to check the behavior of ψ_A for a concrete matrix A. We have $u = \frac{1}{2}(z + \overline{z})$ and $v = \frac{1}{2i}(z - \overline{z})$. Thus, $A(z) = \alpha z + \beta\overline{z}$ for

$$\alpha = \frac{1}{2}(a + d + ci - bi) \qquad \text{and} \qquad \beta = \frac{1}{2}(a - d + bi + ci). \tag{2.1}$$

Then we have

$$\psi_A(z^2) = \frac{\alpha z + \beta\overline{z}}{\overline{\alpha}\overline{z} + \overline{\beta}z}.$$

Since $z \in \mathbb{S}^1$, we have $\overline{z} = 1/z$, and thus

$$\psi_A(z^2) = \frac{\alpha z + \beta/z}{\overline{\alpha}/z + \overline{\beta}z} = \frac{\alpha z^2 + \beta}{\overline{\alpha} + \overline{\beta}z^2} = M(z^2), \tag{2.2}$$

where M is the Möbius map given by the formula

$$M(z) = \frac{\alpha z + \beta}{\overline{\alpha} + \overline{\beta}z}. \tag{2.3}$$

Since (2.2) holds for all $z \in \mathbb{S}^1$, we get $\psi_A = M$ on $\mathbb{S}^1$. ■

This motivates the following definition. We will call a map f from the plane to itself *branched linear* if the following five conditions are satisfied.

$$f \text{ is Lipschitz continuous,} \tag{B1}$$

$$f(z) = 0 \text{ if and only if } z = 0, \tag{B2}$$

$$f(tz) = tf(z) \text{ if } t > 0, \tag{B3}$$

$$f(-z) = \pm f(z). \tag{B4}$$

To state the fifth condition, we define the function $\varphi_f : \mathbb{S}^1 \to \mathbb{S}^1$ by

$$\varphi_f(z) = \frac{f(z)}{|f(z)|}$$

and take its 2-to-1 factor $\psi_f : \mathbb{S}^1 \to \mathbb{S}^1$ under the map $z \mapsto z^2$. It is well defined by (B4). Now, the fifth condition is

$$\psi_f \text{ can be extended to a rational map for which } \mathbb{S}^1 \text{ is fully invariant.} \tag{B5}$$

Note that this condition implies in particular that ψ_f has no critical points on $\mathbb{S}^1$.

By Lemma 2.1, every linear map of the plane with nonzero determinant is branched linear.

Suppose that U is an open subset of $\mathbb{R}^2 (= \mathbb{C})$. Consider a map $F : U \to \mathbb{R}^2$ and a point $z \in U$. We will say that F is *branched differentiable* at z if there is a branched linear map f such that

$$\lim_{w \to 0} \frac{|F(z+w) - F(z) - f(w)|}{|w|} = 0. \tag{2.4}$$

We will call such f the *branched derivative* of F at z.

Lemma 2.2. *If f is a branched derivative of F at z then $f(0) = 0$ and if $w \neq 0$ then*

$$f(w) = \lim_{t \searrow 0} \frac{F(z+tw) - F(z)}{t} . \tag{2.5}$$

Consequently, the branched derivative (if it exists) is determined uniquely.

Proof. We have $f(0) = 0$ by (B2). Assume that $w \neq 0$. Since the limit in (2.4) is 0, we can remove the absolute value signs, and by replacing w by tw and taking into account (B3) we get

$$\lim_{t \searrow 0} \frac{F(z+tw) - F(z)}{tw} = \frac{f(w)}{w} ,$$

and (2.5) follows. ∎

Clearly, if the map is a local diffeomorphism at z then its branched derivative coincides with the usual one.

We are looking at our maps from the Dynamical Systems point of view. That is, we are treating them really as maps, not as functions. Therefore we do not add them etc., but we compose them. This means that while we do not care about the branch derivative of a sum, we should demand that the branch derivative of a composition is the composition of branched derivatives (the chain rule). Let us start by checking that the class of branched linear maps is closed with respect to compositions.

Lemma 2.3. *If f, g are branched linear maps then $f \circ g$ is a branched linear map.*

Proof. Assume that f, g are branched linear maps and denote $h = f \circ g$. Conditions (B1)-(B4) for h are clearly satisfied. To check (B5) note that by (B3) for f we have

$$f(g(z)) = f(|g(z)|\varphi_g(z)) = |g(z)| f(\varphi_g(z)),$$

so

$$\varphi_h(z) = \frac{f(g(z))}{|f(g(z))|} = \frac{f(\varphi_g(z))}{|f(\varphi_g(z))|} = \varphi_f(\varphi_g(z)).$$

Therefore $\varphi_h = \varphi_f \circ \varphi_g$. This implies that $\psi_h = \psi_f \circ \psi_g$, and (B5) for h follows. ∎

Proposition 2.4. *If g is the branched derivative of G at z and f is the branched derivative of F at $G(z)$ then $f \circ g$ is the branched derivative of $F \circ G$ at z.*

Note that this Proposition does not follow immediately from the formula (2.4) for F, G and $F \circ G$. For instance, if we set (on $\mathbb{R}$) $F(t) = t + t^2$, $f(t) = t$, $G(t) = g(t) = \sqrt{t}$ and $z = 0$ then (2.4) holds for F and G, but not for $F \circ G$.

Proof of Proposition 2.4. If $w \neq 0$ then

$$\frac{|F(G(z+w)) - F(G(z)) - f(g(w))|}{|w|} \leq c_1(w)c_2(w) + c_3(w)c_4(w),$$

where

$$\begin{aligned}
c_1(w) &= \frac{|F(G(z+w)) - F(G(z)) - f\big(G(z+w) - G(z)\big)|}{|G(z+w) - G(z)|}\,, \\
c_2(w) &= \frac{|G(z+w) - G(z)|}{|w|}\,, \\
c_3(w) &= \frac{|f\big(G(z+w) - G(z)\big) - f(g(w))|}{|G(z+w) - G(z) - g(w)|}\,, \\
c_4(w) &= \frac{|G(z+w) - G(z) - g(w)|}{|w|}\,.
\end{aligned}$$

Since G is continuous and f is the branched derivative of F at $G(z)$, we have $c_1(w) \to 0$ as $w \to 0$. Since g is the branched derivative of G at z, we have

$$\lim_{w \to 0} \left| c_2(w) - \frac{g(w)}{|w|} \right| = 0.$$

However, $g(w)/|w| = g(w/|w|)$, $w/|w| \in \mathbb{S}^1$ and g is bounded on $\mathbb{S}^1$, so $c_2(w)$ stays bounded as $w \to 0$. This shows that $c_1(w)c_2(w) \to 0$ as $w \to 0$.

Since g is the branched derivative of G at z, we have $c_4(w) \to 0$ as $w \to 0$. Since f is Lipschitz continuous, $c_3(w)$ stays bounded. This shows that also $c_3(w)c_4(w) \to 0$ as $w \to 0$. ■

3. BASIC BRANCHED LINEAR MAPS

Let us now define a class of branched linear maps which is the most interesting for us. Let us start by defining for each integer $n \geq 2$ the map D_n of the plane to itself that multiplies the angle (argument) by n. This map can be described in the polar coordinates as $(r, \vartheta) \mapsto (r, n\vartheta)$, or in the complex notation as $D_n(z) = z^n/|z|^{n-1}$.

We will call a map f of the plane to itself *basic branched linear* if $f = A \circ D_n$, where A is linear and $\det A = 1/n$. Note that the Jacobian of D_n is n, so the Jacobian of f is 1, at any point $z \neq 0$. Hence, f locally preserves the area and the orientation, except at 0. However, since every point has n distinct preimages under f (except for 0), we see that f can globally strongly contract area.

Throughout this section f will be a basic branched linear map.

Consider f in polar coordinates. With this notation, we have

$$f(r, \vartheta) = (r\sigma(\vartheta), \varphi(\vartheta)).$$

Here φ is equal to φ_f if we identify the angle ϑ with the point $z \in \mathbb{S}^1$ with the same argument. The function σ measures stretching in the radial direction.

We are interested, similarly as for linear maps, in stable and unstable directions for f. This means that we have to investigate the dynamics of φ and the values of σ at dynamically interesting values of ϑ. Fortunately, the values of σ are closely connected to the derivative of φ. The simplest way to see it is to consider a thin sector in the unit disk, that is a region $R = \{(r, \vartheta) : \vartheta_0 \le \vartheta \le \vartheta_0 + \Delta\vartheta,\ 0 \le r \le 1\}$. The set $f(R)$ has the same area as R. On the other hand, the area of R is $\Delta\vartheta/2$, while the area of $f(R)$ is approximately $\varphi'(\vartheta_0)\,\Delta\vartheta\,(\sigma(\vartheta_0))^2/2$. Therefore for all ϑ we get

$$\varphi'(\vartheta)(\sigma(\vartheta))^2 = 1. \tag{3.1}$$

Observe that this property immediately extends to the iterates of φ. Formula (3.1) means that expanding in the angular direction implies contraction in the radial direction and vice versa (which is not surprising, since f locally preserves area). Both observations will be useful later.

We have $\psi_{D_n}(z) = \varphi_{D_n}(z) = z^n$, so by Lemma 2.1 and Proposition 2.4, $\psi_f(z) = M(z^n)$ for some Möbius map M preserving $\mathbb{S}^1$. Since A preserves orientation, so does ψ_f and therefore M preserves the unit disk. With the notation of Lemma 2.1, this means that $|\alpha| > |\beta|$.

To simplify notation, for the rest of this section we will write ψ for ψ_f, and treat it as a map of the whole Riemann sphere to itself. Similarly, we will write φ for φ_f.

Since both M and $z \mapsto z^n$ map the complement of $\mathbb{S}^1$ to itself, so does ψ, and by the Montel Theorem this complement is contained in the Fatou set of ψ. Hence the Julia set of ψ is contained in $\mathbb{S}^1$. The two critical points of ψ are 0 and ∞.

Our map ψ has $n+1$ fixed points in the complex plane, counting multiplicities. We will distinguish four cases, depending on the location and nature of those points. Together with the whole map ψ we will also look at $\psi|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$. This is an analytic map of degree n, preserving orientation and without critical points. An elementary analysis of how the graph of its lifting to the real line can intersect the diagonal(s) (in particular, there have to be at least $n-1$ topologically repelling fixed points) reveals four possibilities for the fixed points of $\psi|_{\mathbb{S}^1}$ (see Figure 2):

1) there are $n-1$ simple fixed points, all repelling,
2) there are $n+1$ simple fixed points, one attracting and the rest repelling,
3) there are $n-1$ repelling simple fixed points and one neutral double fixed point that is topologically attracting from one side and topologically repelling from the other side,
4) there are $n-2$ repelling simple fixed points, and one neutral triple fixed point, topologically repelling from both sides.

We will speak accordingly of the Cases 1, 2, 3 and 4. We will later see that all four cases really occur in our family of maps. Observe that since the critical points of ψ are 0 and ∞, every Fatou domain of ψ contains a critical point, and thus there are no Siegel disks. This implies that a neutral periodic point (if it exists) has to be contained in the Julia set.

Let us prove two lemmas that will be useful when considering various cases. By $J(\psi)$ we will denote the Julia set of ψ. Recall that Julia sets cannot contain isolated points.

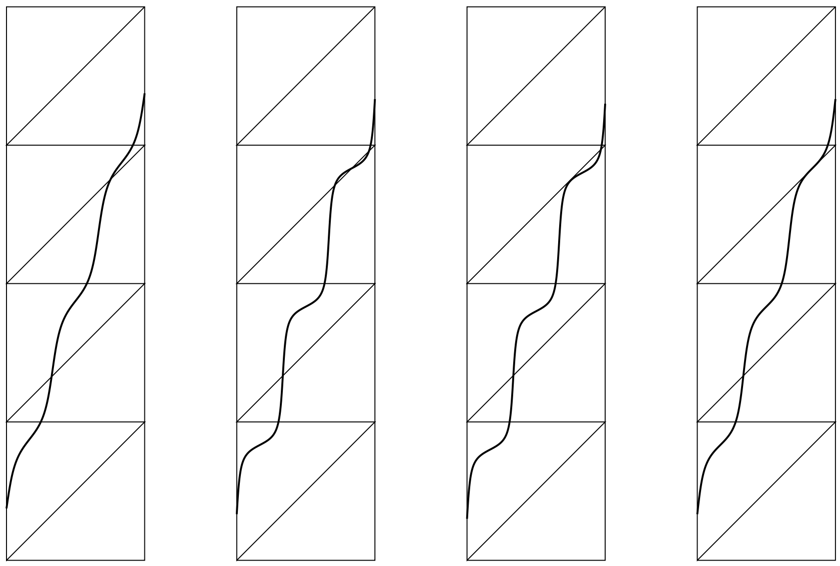

FIGURE 2. Four cases for $n = 3$

Lemma 3.1. *If $J(\psi)$ is not the whole circle then it is a Cantor set.*

Proof. If $J(\psi)$ is not the whole circle and not a Cantor set then $J(\psi)$ contains an arc S and $\mathbb{S}^1 \setminus J(\psi)$ contains an arc T. Choose an interior point z of S and its neighborhood U such that $U \cap \mathbb{S}^1 \subset S$. Since both $J(\psi)$ and the complement of $\mathbb{S}^1$ are fully invariant, all images of U miss T. Thus, by the Montel Theorem, $z \notin J(\psi)$, a contradiction. ■

Lemma 3.2. *For a positive constant t the set $\{z \in \mathbb{S}^1 : |\psi'(z)| \le t\}$ is the union of n arcs (perhaps degenerate or empty), with the same image under D_n. Moreover, the only points of those arcs at which $|\psi'(z)| = t$ are their endpoints.*

Proof. Since the modulus of the derivative of the map $z \mapsto z^n$ is constant on $\mathbb{S}^1$, it is enough to prove that for the Möbius map M (from (3.3)), when z moves around $\mathbb{S}^1$, then $|M'(z)|$ increases on one half of the circle and decreases on the other one (unless it is constant). We may replace M by M composed with a rotation (since it has the derivative of modulus 1). Thus, it is sufficient to consider maps $N : \mathbb{S}^1 \to \mathbb{S}^1$ of the form $N(z) = (z - \zeta)/(\overline{\zeta} z - 1)$, where $|\zeta| < 1$.

This map is a perspectivity with center a (see [FM]), that is, for all $z \in \mathbb{S}^1$ the points z, ζ and $N(z)$ are collinear (in other words, for $z \in \mathbb{S}^1$ its image $N(z)$ can be found by connecting z and ζ and extending this line beyond ζ until it intersects $\mathbb{S}^1$ at a point which actually is $N(z)$). By the definition of the derivative, $N'(z)$ is the limit of ratios of $(N(z') - N(z))/(z' - z)$ as $z' \to z$, and we may assume that $z' \in \mathbb{S}^1$. Since the triangle with vertices z', z, ζ is similar to that with vertices $N(z')$, $N(z)$, ζ, we see that in the limit $|N'(z)| = |N(z) - \zeta|/|z - \zeta|$.

If $\zeta = 0$ then of course $|N'|$ is constant. Assume that $\zeta \ne 0$. The point F of $\mathbb{S}^1$ furthest from ζ is that point of intersection of $\mathbb{S}^1$ with the diameter of $\mathbb{S}^1$ passing through ζ, which is further from ζ. Similarly, the point C of $\mathbb{S}^1$ closest to ζ is that point of intersection of $\mathbb{S}^1$ with the diameter of $\mathbb{S}^1$ passing through ζ, which is closer to ζ. As we move around $\mathbb{S}^1$, starting from F, the distance $|N(z) - \zeta|$ increases, while $|z-\zeta|$ decreases, so $|N'|$ increases. This continues until we reach the point C. Then, on the other half of the circle, $|N'|$ decreases by a similar reason. ■

Let us now consider the possible cases in a more detailed way. We will say that ψ is *expanding* on a set Q, if there exist constants $c>0$ and $\lambda>1$ such that for any $z\in Q$ and any $m\geq 0$ we have $|(g^m)'(z)|>c\lambda^m$. Note that if m is sufficiently large then $c\lambda^m>1$.

Case 1. There are $n-1$ simple fixed points in $\mathbb{S}^1$, all repelling. The other two fixed points are in the complement of $\mathbb{S}^1$. By the Symmetry Principle, they are symmetric with respect to $\mathbb{S}^1$. Thus, one of them is in the unit disk, while the other one is in the complement of the closed unit disk. This means that the first one attracts 0, while the second one attracts ∞. The points of $\mathbb{S}^1$ belong to the closure of two different Fatou domains, so they all belong to the Julia set of ψ. Thus, $J(\psi)$ is the whole unit circle. Since all critical points are attracted to the attracting fixed points, ψ is expanding on $J(\psi)$ (see, e.g., [B], Theorem 9.7.5).

Case 2. There are $n+1$ simple fixed points in $\mathbb{S}^1$, one attracting z_0 and the rest repelling. Then a neighborhood of z_0 is contained in the Fatou set, so there is only one Fatou domain, containing both critical points. By Lemma 3.1, $J(\psi)$ is a Cantor set. The trajectory of any point of $\mathbb{S}^1\setminus J(\psi)$ converges to z_0.

Let us prove that $|\psi'(z)|>1$ for every $z\in J(\psi)$. Let T be the open arc of $\mathbb{S}^1$ whose endpoints are adjacent repelling fixed points and which contains z_0. By Lemma 3.2, the set $\{z\in\mathbb{S}^1 : |\psi'(z)|\leq 1\}$ is the union of n arcs with the same image under D_n. One of them contains the attracting fixed point, but none of the repelling ones, so it is contained in T. Thus, for every $z\in\mathbb{S}^1\setminus(T')$, where $T'=D_n^{-1}(D_n(T))$, we have $|\psi'(z)|>1$. On the other hand, all points of T' are attracted to the attracting fixed point, so $J(\psi)$ is contained in the complement of T'. This completes the proof.

Note that since $J(\psi)$ is compact, there is a constant $\lambda>1$ such that $|\psi'(z)|\geq\lambda$ for $z\in J(\psi)$. Observe also, that since all critical points are attracted to the attracting fixed points, ψ is expanding on $J(\psi)$ (as in Case 1). However, applying Lemma 3.2 above we were able to obtain more specific information.

Case 3. There are $n-1$ repelling simple fixed points in $\mathbb{S}^1$, and one neutral double fixed point z_0 that is topologically attracting from one side and topologically repelling from the other side. By Lemma 3.1, $J(\psi)$ is a Cantor set. The trajectory of any point of $\mathbb{S}^1\setminus J(\psi)$ converges to z_0. Denote the fixed point closest to z_0 on the attracting side by a.

We want to prove that $|\psi'(z)|>1$ everywhere on the Julia set except for z_0 and its preimages under D_n. Let T be the open arc connecting z_0 and a and containing a small attracting semineighborhood U of z_0. By Lemma 3.2, the set $\{z\in\mathbb{S}^1 : |\psi'(z)|\leq 1\}$ is the union of n arcs with the same image under D_n. One of them contains U, but none of the repelling or neutral points, so it is contained in $\overline{T}$. Thus, for every $z\in\mathbb{S}^1\setminus T'$, where $T'=D_n^{-1}(D_n(T))$, we have $|\psi'(z)|\geq 1$. On the other hand, all points of T' are attracted to z_0, so $J(\psi)$ is contained in the complement of T'. The only points of the Julia set at which $|\psi'|$ can be 1 are the endpoints of T and their preimages under D_n. However, for every $b\in D_n^{-1}(a)$ we have $|\psi'(b)|=|\psi'(a)|>1$, so only z_0 and its preimages are the points in $J(\psi)$ at which the derivative is of absolute value 1. This completes the proof.

Case 4. There are $n-2$ repelling simple fixed points in $\mathbb{S}^1$, and one neutral triple fixed point z_0, topologically repelling from both sides. Consider the component K of the set of all points z with $|\psi'(z)| \le 1$ which contains z_0. Since this time $|\psi'(z)| > 1$ arbitrarily close to z_0 on both sides we see that $K = \{z_0\}$. Hence by Lemma 3.2 $|\psi'(z)| > 1$ except at the n points of $D_n^{-1}(z_0)$, at each of which the absolute value of the derivative of ψ is 1. This in particular implies that the repelling periodic points are dense in $\mathbb{S}^1$, so $J(\psi) = \mathbb{S}^1$.

Note that in Cases 3 and 4 we have $\psi'(z_0) = 1$.

Now we can interpret the results in Cases 1-4 in terms of the original map f. We have to use a slightly different language than for linear maps. Namely, for a linear map the image of a linear subspace is a linear subspace. The same is true in our case if n is odd. However, if n is even, then the image under f of a straight line through 0 is a ray emerging from 0. Thus, the word *direction* will apply to the whole line through 0 or to a ray from 0, depending on the parity of n.

If we know something about the trajectory of a point of $\mathbb{S}^1$ for ψ, we get the similar information about the trajectories of the two antipodal points for φ, that is a one-dimensional subspace of $\mathbb{R}^2$ for f. The absolute values of the derivatives of ψ and φ at the corresponding points are the same, since the absolute value of the derivative of $z \mapsto z^2$ on $\mathbb{S}^1$ is everywhere 2. Therefore by (3.1), the expansion of ψ corresponds to contraction in the radial direction of f and vice versa.

For any fixed point p of ψ there is the corresponding fixed point p_1 of φ if n is even; and if n is odd then either there are two antipodal fixed points p_1 and p_2, or there is a period two orbit consisting of p_1 and p_2. If p is neutral for ψ then so is p_1 (and p_2) for φ, and hence by (3.1) the entire corresponding direction consists of fixed points of f^2. This leads to the following interpretation of the four cases.

Case 1. The origin is attracting. This may seem strange, since f locally preserves area, but we have to remember that the map is n-to-1. There is m such that any closed disk centered at 0 is mapped into its interior by f^m; this is so for any m for which $|(\psi^m)'(z)|$ is bounded away from 1 on the unit circle. Therefore there is also a closed neighborhood of 0 that is mapped into its interior by f.

Case 2. There is one invariant expanding direction and an invariant Cantor set of contracting directions. This is analogous to the usual saddle case, except that one contracting direction is replaced by the whole Cantor set of them. All points whose directions belong to the Cantor set of contracting directions converge to the origin exponentially. All other points are eventually mapped into a wedge of directions around the expanding direction within which they exponentially converge to infinity.

Case 3. This is a degenerate Case 2. The expanding direction merged with one direction from the Cantor set of contracting directions and become neutral (that is, we have a direction of fixed or periodic of period 2 points of f). Any point whose direction belongs to this Cantor set is attracted to the origin except for the points from the direction of fixed or periodic of period 2 points and their preimages under the iterates of f. To see what is the behavior of other points observe that for any point z in the Fatou set of ψ, attracted to the neutral fixed point z_0, the derivative of $\psi^n(z)$ converges to 0 by the Weierstrass theorem (which is applicable because the iterations of ψ on a neighborhood of z converge uniformly to a constant z_0). By the formula (3.1) this implies that all points not from the Cantor set of contracting or fixed directions converge to infinity, getting closer and closer to the direction of fixed or periodic of period 2 points.

Case 4. This is a degenerate Case 2. The origin is attracting except the direction of fixed or periodic of period 2 points of f. It also can be considered as a degenerate Case 3; the Cantor set grew to the whole circle. The image under f of a closed disk centered at 0 is contained in this closed disk, and except two points, in its interior. The reason why we have two such points is the following. We have $|\psi'(z)| = 1$ at n points, so $|\varphi_f'(z)| = 1$ at $2n$ points. However, the image under φ of those points is two points.

Now we are going to analyze the four cases in a more quantitative way, in order to see how typical they are and how to check which case occurs for a given matrix A. Clearly, Cases 1 and 2 occur for open sets of matrices A, since they correspond to all fixed points being hyperbolic.

Let us apply the Singular Value Decomposition to the matrix A. Since its determinant is positive, we get $A = UBV$, where U and V are matrices of rotations, and B is a diagonal matrix. In complex notation, the actions of U and V are multiplication by complex numbers u and v of modulus 1, respectively. Thus, we get

$$\varphi_f(z) = \frac{A(z^n)}{|A(z^n)|} = u \cdot \frac{B(z^n v^n)}{|B(z^n v^n)|}.$$

Therefore

$$\psi(z) = u^2 \cdot M_B(z^n v^n),$$

where M_B is obtained from B in the same way as M for A. Let s and t be the singular values of A (square roots of the eigenvalues of $A^T A$, or, in geometric terms, the semi-axes of the image of the unit circle under A). Then the diagonal entries of B are s and t (we may assume that $s \geq t$), so by (2.1) and (2.3) we get

$$M_B(z) = \frac{(s+t)z + (s-t)}{(s+t) + (s-t)z}.$$

Hence, taking into account that $1/v = \overline{v}$, we get

$$\psi(z) = -u^2 v^n \cdot N(z^n),$$

where

$$N(z) = \frac{z - \zeta}{\overline{\zeta} z - 1} \qquad \text{and} \qquad \zeta = -\overline{v}^n \cdot \frac{s-t}{s+t}.$$

In such a way the matrix A determines the complex numbers ζ as above and $\lambda = -u^2 v^n$, with $|\zeta| < 1$ and $|\lambda| = 1$. On the other hand, if we know ζ and λ, then, using the additional information that the determinant of A is $1/n$, we can easily determine (via elementary functions) u^2, v^n, s^2 and t^2. This means that locally we can parametrize the set of all 2×2 matrices by the variables ζ and λ, and this is a smooth parametrization.

Now fix ζ and vary λ. When we look at the graph of the lifting g of $\psi|_{\mathbb{S}^1}$ to the real line, this corresponds to shifting the graph up or down (cf. Figure 2). There are 3 possibilities.

The first possibility is that $|N'| > 1/n$ everywhere, so $g' > 1$ everywhere. Then for all shifts all fixed points of $\psi|_{\mathbb{S}^1}$ are repelling, and we have Case 1. The geometric interpretation of N from the proof of Lemma 3.2 tells us that this happens if $(1+|\zeta|)/(1-|\zeta|) < n$, that is, $|\zeta| < (n-1)/(n+1)$. Since $|\zeta| = (s-t)/(s+t)$, this corresponds to $s/t < n$.

The second possibility is that there are points at which $|N'| > 1/n$ and points at which $|N'| < 1/n$. Thus, there are points at which $g' > 1$ and points at which $g' < 1$. Then there are some shifts for which all fixed points of $\psi|_{\mathbb{S}^1}$ are repelling, and we have Case 1, and there are some shifts for which one of the fixed points of $\psi|_{\mathbb{S}^1}$ is attracting and we have Case 2. There is also a discrete set of shifts (finitely many values of λ) for which one fixed point of $\psi|_{\mathbb{S}^1}$ is neutral and double. Then we have Case 3. This shows that Case 3 is a codimension 1 phenomenon. This possibility corresponds to $|\zeta| > (n-1)/(n+1)$, that is, $s/t > n$.

The third possibility is that the infimum of $|N'|$ is $1/n$, so the infimum of g' is 1. Then there is a discrete set of shifts (finitely many values of λ) for which one fixed point of $\psi|_{\mathbb{S}^1}$ is neutral and triple, and we have Case 4. For the rest of shifts we have Case 1. This shows that Case 4 is a codimension 2 phenomenon. This possibility corresponds to $|\zeta| = (n-1)/(n+1)$, that is, $s/t = n$.

The discussion above shows also that all Cases 1-4 occur.

If one needs to check for a matrix A which of the Cases 1-4 occurs, it is possible to use the following procedure. First, ζ and λ can be determined, using the standard methods. Then, if the possibility 2 above occurs, the arcs of $z \in \mathbb{S}^1$ for which $|N'(z)| < 1/n$ can be found by noticing that this condition is equivalent to

$$\mathrm{Re}(\overline{\zeta} z) < \frac{1}{2}\big((n+1)|\zeta|^2 - (n-1)\big).$$

After that, an application of the Intermediate Value Theorem (one has to be careful, because it is applied to the lifting of $\psi|_{\mathbb{S}^1}$) should determine whether there is a fixed point of ψ in those arc. In the borderline case, when the possibility 3 occurs, instead of the arcs one gets finitely many points that have to be checked whether they are fixed points of ψ.

4. General branched linear maps

Consider now the general case of branched linear maps. We will relax various assumptions that we made for basic branched linear maps and observe how the results change.

Assume first that in the definition of the basic branched linear maps we do not assume that A preserves orientation. If A reverses orientation, so does $\psi|_{\mathbb{S}^1}$. For topological reasons, $\psi|_{\mathbb{S}^1}$ has $n+1$ fixed points, so all fixed points of ψ belong to $\mathbb{S}^1$. By looking only at $\psi|_{\mathbb{S}^1}$, we cannot tell whether they are repelling, attracting or neutral. There is only one or two Fatou domains, and if there are two, one is mapped to the other and vice versa. Therefore there may be at most one attracting or neutral fixed point. If there are none, by the same argument as in Case 1 of Section 3 (applied to ψ^2), the Julia set is the whole circle, and the rest of the description and interpretation of Case 1 applies. If there is an attracting fixed point then we have the same situation as in Case 2. If there is a neutral fixed point, since this time the derivative of ψ at it is -1, the behavior is the same on both sides. It cannot be attracting from both sides, since then it would have a neighborhood contained in the Fatou set and would be an isolated point of the Julia set, which is impossible. Therefore it has to be repelling from both sides on $\mathbb{S}^1$. Then the description and interpretation of Case 4 applies.

We will not discuss what happens if $n = 1$, since then the point at which we are taking a branched derivative is not a branching point, so it is natural to assume that at this point the usual derivative exists. The case $n = 0$ is impossible, because then ψ would be constant, so $\mathbb{S}^1$ would not be fully invariant.

Assume now that f is branched linear and locally preserves area, but is not necessarily basic branched linear. For the reasons given in the preceding paragraph we also assume that f is not one-to-one. In our investigation of the properties of the map ψ, the only place where we used a specific information on the form of ψ, rather than just condition (B5), was Lemma 3.2 (and its consequences). Thus, in the situation considered now, we still have 4 cases described in the preceding section if f locally preserves orientation, and 3 cases described above if f locally reverses orientation. Cases 1 and 2 are generic, while Cases 3 and 4 are not. In Case 2, rather than $|\psi'(z)| \geq \lambda > 1$ for all $z \in J(\psi)$, we only get, as in Case 1, that ψ is expanding on $J(\psi)$. In Cases 3 and 4 we cannot say that $|\psi'(z)| > 1$ for every point of $J(\psi)$, except the n points of $D_n^{-1}(z_0)$. Instead we just know that there is some weak form of expansion, that follows from the general theory of complex dynamics. For instance, if K is a compact invariant subset of $J(\psi)$ (e.g., a periodic orbit) disjoint from z_0, then ψ is expanding on K.

The interpretation of the results for the original map f in Cases 1 and 2 (that is, in the generic, or "hyperbolic" cases) does not change.

At last, let us discuss briefly what happens if we have $f = A \circ D_n$, but $|\det A| \neq 1/n$. Then in (3.1) at the right-hand side instead of 1 we have some number different than 1. This means that to see whether there is contraction or expansion in specific radial directions, we have to compare $|\psi'(z)|$ (or more generally, the Lyapunov exponent of ψ at z) not with 1, but with a different number. If this number is less than 1 (that is, f locally contracts area), this may result in switching to Case 1. If this number is greater than 1 (that is, f locally expands area), this may result in "thinning" the set of contracting directions and adding a lot of expanding ones, for instance corresponding to mildly repelling periodic points of ψ.

An interesting picture emerges if we start with a basic branched linear map $A \circ D_n$ for which Case 1 occurs and then multiply A by a parameter $\mu > 1$. Then the right-hand side in (3.1) is μ and we have contraction in the directions corresponding to z with $|\psi'(z)| > \mu$ and expansion in the directions corresponding to z with $|\psi'(z)| < \mu$. If we look at the long-term behavior of the points in those directions, we have to compare the Lyapunov exponents of ψ with μ. If the exponent is larger than μ, the trajectories of the points in this direction go to 0; if the exponent is smaller than μ, the trajectories go to infinity. However, according to [W] and [Z], unless ψ is conjugate to $z \mapsto z^n$, the set of Lyapunov exponents contains an open interval.

If ψ is conjugate to $z \mapsto z^n$, then the finite critical point of ψ, that is 0, is a fixed point, that is M is a rotation. By (2.3) $\beta = 0$, so $A(z) = \alpha z$. This shows that if A is not a rotation multiplied by a real constant, the set of Lyapunov exponents contains an open interval. Since the Lyapunov exponent is constant on grand orbits and the grand orbit of every point of the Julia set is dense in the Julia set, we get the following picture. There is an open interval of values of the parameter μ such that for the map $\mu A \circ D_n$ the trajectories of the points from a dense set of directions converge to 0, while the trajectories of the points from another dense set of directions converge to infinity.

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