# Some examples of torsion in the Griffiths group


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**Contents.**



**0.Introduction.**

Let $K$ be a field, $\bar{K}$ a separable closure, and $W$ a smooth, projective, geometrically integral $K$-variety. The "Griffiths group"in codimension $r$, $Gr^r(W_K) :=$ nullhomologous cycles modulo algebraic equivalence, is an important but mysterious invariant associated to $W$. So far the study of $Gr^r(W_K) \otimes \mathbf{Q}$ has led to much interesting work on the Abel-Jacobi map and some fascinating conjectures in the case $[K : \mathbf{Q}] < \infty$. Yet an understanding of the structure of this vector space does not seem close at hand. One missing ingredient is a cycle class map which is known to be injective. For $Gr^2(W_{\bar{K}})_{tors}$ such a map exists thanks to the work of Merkuriev and Suslin [**M-S**,§18] and of Bloch [**Bl2**] (at least if we ignore $p$-torsion in characteristic $p > 0$). This fact combined with the important role played by torsion in the Chow group of codimension one cycles motivates the study of $Gr^r(W_{\bar{K}})_{tors}$.

As next to nothing is known about $Gr^r(W_{\bar{K}})_{tors}$, our main task will be to produce some non-zero elements. When $K = \mathbf{C}$ this is not so simple as one might hope. In this paper we produce a few elements of small order for some rather special projective varieties. This shows at least that the torsion subgroup of the Griffiths group is not always zero. Our first examples involve the quotient of a complete intersection by a finite group acting freely. The torsion cycles which we construct are Chern classes of vector bundles arising from representations of the fundamental group. In §3 we show that the torsion in the Griffiths group of simply connected varieties can also be non-zero. In fact we find two-torsion for some rather special hypersurfaces, $W_{\mathbf{C}} \subset \mathbf{P}^{2m}_{\mathbf{C}}$. The idea here is to begin with a cycle, $\mathcal{T}$, on a hypersurface, $V_{\mathbf{C}} \subset \mathbf{P}^{2m+1}_{\mathbf{C}}$, with isolated singularities. We arrange that the homology class of $\mathcal{T}$ is 2-torsion and then restrict to the smooth hyperplane section, $W_{\mathbf{C}}$.

Although our experience is that torsion in the Griffiths group for varieties over $\mathbf{C}$ is difficult to find, we have no compelling evidence that it is always finite or usually zero. When $K$ is a finite field, it seems to be easier to produce torsion elements. In §4 we consider the case $W = E^3$ with $E$ an elliptic curve. By considering essentially only a single cycle, we find that the order of

$$(Gr^2(E^3_{\bar{\mathbf{F}}_p})_{tors})^{Gal(\bar{\mathbf{F}}_p/\mathbf{F}_p)}$$

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is not bounded when $p$ varies through all primes and $E_{\mathbf{F}_p}$ varies through the various reductions of a certain elliptic curve over $\mathbf{Q}$. The method here is inspired by Bloch's paper [**Bl1**]. We hope that it will be developed further to give more information about $Gr^2(E^3_{\bar{\mathbf{F}}_p})_{tors}$.

The first section is devoted to summarizing important points concerning the torsion in the Chow group of a smooth projective variety over an algebraicly closed field. Most of the results have appeared elsewhere although (1.3.2) appears to be new.

The various techniques used here to show that a given nullhomologous cycle $z$ is not algebraicly equivalent to zero have the following in common: The variety $W$ is regarded as a fiber in a family over a positive dimensional base. The cycle $z$ is the specialization of a global cycle $Z$ which is actually the more tractible object. From its properties we deduce what we know about $z$. This approach was introduced in the original paper on the Griffiths' group [**Gri**] and is motivated by the difficulty in directly evaluating the existing cycle class maps at $z$. Such an indirect method is however not without its disadvantages. An attempt to study $Gr^r(W)$ without invoking auxillary families is made in [**Sch2**].

I wish to thank Spencer Bloch for helpful discussions pertaining to §4 and Uwe Jannsen for explaining relationships between certain cycle class maps. The material in (3.2) and (3.3) was worked out at MPI in 1990. The results of §4 achieved their present form during a visit to the University of Chicago in 1989. Much of the work was done during the special year in arithmetic algebraic geometry at MSRI (1986-87). I thank these institutions for their hospitality and the NSF and MPI for support.

**1. Notations and Preliminaries.** We shall adopt the following notations:

$K$ is a field. $\bar{K}$ is a separable closure of $K$

$G_K = Gal(\bar{K}/K)$

$W_K$ = a smooth, projective, geometrically integral $K$-variety

$l$ = a prime not equal to the charachteristic of $K$

$Z^r(W_K)$ = the group of codimension $r$ algebraic cycles on $W_K$

$CH^r(W_K) = CH_{dim(W)-r}(W_K)$ codimension $r$ algebraic cycles modulo rational equivalence [**Fu**,§1]

$NS(W_{\bar{K}})$ = Neron-Severi group of $W_{\bar{K}}$ [**Mi**,p. 215]

$H^1(G_K, N) = \varprojlim H^1(G_K, N/l^n)$. Here $N$ will always be a finitely generated $\mathbf{Z}_l$ module. Thus this group may be identified with continuous crossed homomorphisms modulo coboundaries where $N$ is given the usual $l$-adic topology [**Ta1**,2.2].

Given an abelian group $A$,

$A[m]$ = denotes the kernel of multiplication by $m$

$A_l = \varinjlim A[l^n]$

$\tau_l A := \varprojlim A[l^n]$ denotes the Tate module of $A$.

**(1.1) Subquotients of the Chow group.** In this subsection we recall definitions and important facts about torsion in the Chow group of a smooth, projective variety over an algebraicly closed field. To begin we recall from [**Mi**,VI.9] the existence of the classical cycle class map

$$cl^r_{W_{\bar K}} : CH^r(W_{\bar K}) \to H^{2r}(W_{\bar K}, \mathbf{Z}_l(r)).$$

Define

$$CH^r(W_K)_{hom} := \text{Ker}\ [CH^r(W_K) \to CH^r(W_{\bar K}) \xrightarrow{\prod_l cl^r_{W_{\bar K}}} \prod_{l \neq charK} H^{2r}(W_{\bar K}, \mathbf{Z}_l(r))\ ].$$

The torsion subgroup of $CH^r(W_K)_{hom}$ will be denoted $T^r(W_K)$. Write $CH^r(W_K)_{alg}$ for the subgroup of $CH^r(W_K)_{hom}$ generated by all subgroups

$$\Gamma_* CH^1(C_K)_{hom} \subset CH^r(W_K)_{hom},$$

where $C_K$ is a non-singular projective curve over $K$ and $\Gamma \subset C \times_K W$ is an integral codimension $r$ subscheme flat over $C$. As usual $\Gamma_*$ denotes the map on Chow groups induced by the correspondence $\Gamma$ [**Fu**,§16]. The torsion subgroup of $CH^r(W_K)_{alg}$ is denoted $T^r_{alg}(W_K)$.

LEMMA 1.1.1. *$CH^r(W_{\bar K})_{alg} \otimes \mathbf{Z}_l$ and $T^r_{alg}(W_{\bar K})_l$ are divisible groups.*

PROOF: Evidently $CH^r(W_{\bar K})_{alg}$ is generated by $Pic^0$ of smooth curves over $\bar K$. A finite, etale morphism of irreducible varieties over a separably closed field is surjective on rational points. Applying this to multiplication by $l$ on $Pic^0$ yields that $CH^r(W_{\bar K})_{alg}$ is an $l$-divisible group. Its torsion subgroup must also be $l$-divisble. The lemma follows easily.

The Griffiths group in codimension $r$ is defined by

$$Gr^r(W_{\bar K}) := CH^r(W_{\bar K})_{hom}/CH^r(W_{\bar K})_{alg}.$$

LEMMA 1.1.2. *For $l \neq char(K)$*

(1) $Gr^d(W_{\bar K}) \simeq 0$
(2) $Gr^1(W_{\bar K}) \otimes \mathbf{Z}_l \simeq 0$

PROOF: The first assertion is straight foward from the definition. When $\bar K = \mathbf{C}$, $Gr^1(W_{\bar K}) = 0$ by the exponential sequence. To treat the general case recall that a representability result of Grothendieck [**Gr**,4.2] implies $CH^1(W_{\bar K})_{hom} \simeq CH^1(W_{\bar K})_{alg} \times F$, where $F$ is a finite abelian group. By the Kummer sequence, each element $y \in CH^1(W_{\bar K})_{hom}$ is divisible by arbitrarily high powers of $l$ in $CH^1(W_{\bar K})$. To show that it is actually divisible in $CH^1(W_{\bar K})_{hom}$ recall that $NS(W_{\bar K}) = CH^1(W_{\bar K})/CH^1(W_{\bar K})_{alg}$ is finitely generated [**Mi**,V.3.25]. If $x_n \in CH^1(W_{\bar K})$ satisfies $l^n x_n = y$ and if $l^n$ annihilates $NS(W_{\bar K})_l$, then there is $x'_n \in CH^1(W_{\bar K})_{alg}$ such that $l^n x'_n = y$. Thus $CH^1(W_{\bar K})_{hom}$ is $l$-divisible, whence $F_l = 0$.

LEMMA 1.1.3. *There is an exact sequence*

$$0 \to T^r_{alg}(W_{\bar{K}})_l \to T^r(W_{\bar{K}})_l \to Gr^r(W_{\bar{K}})_l \to 0.$$

PROOF: One need only check surjectivity of the right hand map. Recall that $T^r_{alg}(W_{\bar{K}})_l$ is a divisible group. The result now follows from the short exact sequence

$$0 \to CH^r(W_{\bar{K}})_{alg} \to CH^r(W_{\bar{K}})_{hom} \to Gr^r(W_{\bar{K}}) \to 0$$

by taking the kernel-cokernel sequence for multiplication by $l^n$ and passing to the direct limit.

For varieties of dimension $d \geq 3$ and non-negative Kodaira dimension the prospect of computing $CH^r(W_{\bar{K}})_{hom}$ for $2 \leq r \leq d-1$ has appeared remote. The following remarkable result of Soulé is perhaps a first step in this direction.

THEOREM 1.1.4. *[**So**,Thm 3] Let $K$ be a finite field and $W$ a product of curves.Then for $r \in \{1, d, d-1\}$, the natural inclusion $T^r(W_{\bar{K}}) \to CH^r(W_{\bar{K}})_{hom}$ is an isomorphism.*

THEOREM 1.1.5. *If $\bar{K} \subset \bar{L}$ is an extension of algebraicly closed fields, the induced maps $CH^r(W_{\bar{K}})_l \to CH^r(W_{\bar{L}})_l$ and $T^r(W_{\bar{K}})_l \to T^r(W_{\bar{L}})_l$ are isomorphisms.*

PROOF: The first result is proved in [**Le**]. The second is an immediate consequence of the functoriality of the cycle class map and base change theorems in étale cohomology [**Mi**,VI ].

**(1.2) Cycle class maps.** Write $\mathbf{D}^r_l(W_{\bar{K}})$ for $H^{2r-1}(W_{\bar{K}}, \mathbf{Q}_l(r))/H^{2r-1}(W_{\bar{K}}, \mathbf{Z}_l(r))$ which is the maximal divisible subgroup of $H^{2r-1}(W_{\bar{K}}, \mathbf{Q}_l/\mathbf{Z}_l(r))$ and is the kernel of the coboundary map [**C-S-S**, 1.2 (13)]

$$H^{2r-1}(W_{\bar{K}}, \mathbf{Q}_l/\mathbf{Z}_l(r)) \to H^{2r}(W_{\bar{K}}, \mathbf{Z}_l(r)).$$

In this subsection we recall briefly the existence of three natural cycle class maps $T^r(W_{\bar{K}}) \to \mathbf{D}^r_l(W_{\bar{K}})$. We also state without proof some of their properties and the relationships between them.

DEFINITION 1.2.1: Bloch [**Bl2**] has defined a cycle class map

$$\lambda^r : CH^r(W_{\bar{K}})_{tors} \to H^{2r-1}(W_{\bar{K}}, \mathbf{Q}_l/\mathbf{Z}_l(r)).$$

Restriction to $T^r(W_{\bar{K}})$ gives a map, also denoted $\lambda^r$,

$$\lambda^r : T^r(W_{\bar{K}}) \to \mathbf{D}^r_l(W_{\bar{K}}).$$

THEOREM 1.2.2.

(1) *$\lambda^1$ agrees with the usual map from Kummer theory.*
(2) *The restriction of $\lambda^2$ to $CH^2(W_{\bar{K}})_l$ is injective.*
(3) *$\lambda^{\cdot}$ is functorial with respect to correspondences.*
(4) *When $\bar{K} = \mathbf{C}$, $\lambda^r$ may be identified with the Abel-Jacobi map restricted to $T^r(W_{\mathbf{C}})$ followed by projection to the $l$-primary subgroup of the torsion in the intermediate Jacobian.*

PROOF: For (1), (3), and (4) see [Bl2]. The second assertion is [M-S,18.4].

DEFINITION 1.2.3: A second map

$$\alpha^r : T^r(W_{\bar{K}}) \to \mathbf{D}_l^r(W_{\bar{K}})$$

is constructed in [Ra,§2] (see also [Bl1,§1] and [J2,§9]). One begins with the cycle class map [Gr-D]

$${}_n cl^r_{W_K} : CH^r(W_K) \to H^{2r}(W_K, \mathbf{Z}/l^n(r)).$$

The image of $CH^r(W_K)_{hom}$ lies in the kernel of the restriction homomorphism

$$H^{2r}(W_K, \mathbf{Z}/l^n(r)) \to H^{2r}(W_{\bar{K}}, \mathbf{Z}/l^n(r))^{G_K}.$$

Thus the Hochschild-Serre spectral sequence gives rise to a map

$${}_n cl^r_{W_K,0} : CH^r(W_K)_{hom} \to H^1(G_K, H^{2r-1}(W_{\bar{K}}, \mathbf{Z}/l^n(r))).$$

Passing to the limit gives a map

$$cl^r_{W_K,0} : CH^r(W)_{hom} \to \varprojlim H^1(G_k, H^{2r-1}(W_{\bar{K}}, \mathbf{Z}/l^n(r))) \simeq H^1(G_k, H^{2r-1}(W_{\bar{K}}, \mathbf{Z}_l(r))).$$

The cohomology group on the right is computed with continuous cochains where $H^{2r-1}(W_{\bar{K}}, \mathbf{Z}_l(r))$ has the inverse limit topology [Ta1]. The short exact sequence

$$0 \to H^{2r-1}(W_{\bar{K}}, \mathbf{Z}_l(r))/tors \to H^{2r-1}(W_{\bar{K}}, \mathbf{Q}_l(r)) \to \mathbf{D}_l^r(W_{\bar{K}}) \to 0$$

gives rise to a long exact sequence of continuous Galois cohomology modules [Ta1, §2]. The first coboundary map gives a surjection

$$\mathbf{D}_l^r(W_{\bar{K}})^{G_K} \to H^1(G_K, H^{2r-1}(W_{\bar{K}}, \mathbf{Z}_l(r))/tors)_{tors}. \tag{1.2.4}$$

Assume now that $K$ is finitely generated over the prime field. Then a specialization argument involving the Weil conjectures shows that (1.2.4) is in fact an isomorphism. [C-R, Thm 1.5]. Thus $cl^r_{W_K,0}$ gives rise to a map $T^r(W_K) \to \mathbf{D}_l^r(W_{\bar{K}})^{G_K}$. Now pass to the limit over finite separable extensions $K'/K$ to define

$$\alpha^r : T^r(W_{\bar{K}}) \to \mathbf{D}_l^r(W_{\bar{K}}).$$

DEFINITION 1.2.5: There is a third cycle class map [J2,§9] (see also [Sch2,§1]). Suppose given a nullhomologous cycle $Z \in Z^r(W_K)$. Write $|Z|$ for the support of $Z$ and define

$$H^{2r}_{|Z|}(W_{\bar{K}}, \mathbf{Z}/l^n(r))_0 = \mathrm{Ker}\,[H^{2r}_{|Z|}(W_{\bar{K}}, \mathbf{Z}/l^n(r)) \to H^{2r}(W_{\bar{K}}, \mathbf{Z}/l^n(r))].$$

By purity [**Mi**,VI.9.1],

$$H^{2r-1}_{|Z|}(W_{\bar{K}}, \mathbf{Z}/l^n(r)) = 0.$$

There results a short exact sequence of $G_K$-modules,

$$0 \to H^{2r-1}(W_{\bar{K}}, \mathbf{Z}/l^n(r)) \to H^{2r-1}((W - |Z|)_{\bar{K}}, \mathbf{Z}/l^n(r)) \to H^{2r}_{|Z|}(W_{\bar{K}}, \mathbf{Z}/l^n(r))_0 \to 0.$$

Write $[Z] \in H^{2r}_{|Z|}(W_{\bar{K}}, \mathbf{Z}/l^n(r))_0^{G_K}$ for the fundamental class of $Z$. Applying the first coboundary map in the long exact $G_K$-cohomology sequence associated with this short exact sequence gives an element ${}_nc^r_{W_K,0}(Z) \in H^1(G_K, H^{2r-1}(W_{\bar{K}}, \mathbf{Z}/l^n(r)))$. This class depends only on the rational equivalence class of $Z$ [**B-S-T**]. Taking the inverse limit yields a map

$$c^r_{W_K,0} : CH^r(W_K)_{hom} \to H^1(G_K, H^{2r-1}(W_{\bar{K}}, \mathbf{Z}_l(r))). \tag{1.2.6}$$

The proceedure used above to construct the map $\alpha^r$ beginning with $cl^r_{W_K,0}$ may be used to construct a map $\beta^r : T^r(W_{\bar{K}}) \to \mathbf{D}^r_l(W_{\bar{K}})$ beginning with $c^r_{W_K,0}$.

THEOREM 1.2.7. *The maps $cl^r_{W_K,0}, c^r_{W_K,0} : CH^r(W_{\bar{K}})_{hom} \to H^1(G_K, H^{2r-1}(W_{\bar{K}}, \mathbf{Z}_l(r)))$ as well as $\alpha^r, \lambda^r, \beta^r : T^r(W_{\bar{K}}) \to \mathbf{D}^r_l(W_{\bar{K}})$ agree up to sign. Furthermore $c^{\cdot}_{W_K,0}$ is functorial with respect to correspondences.*

PROOF: The homomorphisms $cl^r_{W_K,0}$ and $c^r_{W_K,0}$ are compared in [**J2**,§9]. That $\lambda^r$ and $\alpha^r$ should be related is mentioned in [**Ra**,§2]. I believe that a complete proof will appear in [**J1**]. For the functoriality of $c^{\cdot}_{W_K,0}$ see [**B-S-T**,§1].

The cycle class maps may sometimes be used to detect non-trivial elements in the Griffiths group, $Gr^r(W_{\bar{K}})$. We shall need this observation for two different sorts of base field. First, consider the case that $K \subset \mathbf{C}$. Write $d$ for the dimension of $W$. Let $P \in Z^d(W \times_K W)$ and consider the Hodge structure $U = P_*H^{2r-1}(W(\mathbf{C}))(r)$.

LEMMA 1.2.8. *If $U$ contains no non-trivial Hodge substructure, $U'$, with $F^1U' = 0$, then the cycle class map $\lambda^r \circ P_* : T^r(W_{\bar{K}})_l \to \mathbf{D}^r_l(W_{\bar{K}})$ factors through $Gr^r(W_{\bar{K}})_l$.*

PROOF: Let $C_{\bar{K}}$ be any smooth projective curve and $\Gamma \in Z^r((C \times W)_{\bar{K}})$ a correspondence. The map $P_* \circ \Gamma_* : H^1(C_{\bar{K}}, \mathbf{Q}_l(1)) \to H^{2r-1}(W_{\bar{K}}, \mathbf{Q}_l(r))$ is zero by Hodge theory. This is the map on Tate modules tensored with $\mathbf{Q}_l$ associated to $P_* \circ \Gamma_* : \mathbf{D}^1_l(C_{\bar{K}}) \to \mathbf{D}^r_l(W_{\bar{K}})$. Thus this map is also zero. By the functoriality of $\lambda$ with respect to correspondences (1.2.2), $\lambda^r \circ P_*(T^r_{alg}(W_{\bar{K}})) = 0$.

Suppose now that $\bar{K}$ is the algebraic closure of a finite field, $K$. Write $\phi \in G_K$ for the Frobenius element. Let $P \in Z^d(W \times_K W)$ and define $U = P_*H^{2r-1}(W_{\bar{K}}, \mathbf{Q}_l(r))$.

LEMMA 1.2.9. *If no eigenvalue of $\phi^{-1}$ acting on $U(-1)$ is an algebraic integer, then the map $P_* \circ \alpha^r : T^r(W_{\bar{K}})_l \to \mathbf{D}^r_l(W_{\bar{K}})$ factors through $Gr^r(W_{\bar{K}})_l$.*

PROOF: [**Sch2**,2.5].

DEFINITION 1.2.10: Let $W/K$ be a smooth projective variety of dimension $d$. In this paper $P \in Z^d(W \times_K W)$ will be called a transcendental correspondence for codimension $r$ cycles if $P_*T^r_{alg}(W_{\bar{K}})_l = 0$ for all $l \neq char(K)$.

EXAMPLE 1.2.11: Suppose that $r = 2$ and the conditions of (1.2.8) or (1.2.9) are satisfied. Then $P$ is transcendental. In fact this follows from the injectivity of $\lambda^2 : T^2(W_{\bar{K}})_l \to \mathbf{D}_l^2(W_{\bar{K}})$ and the compatibility of $\lambda^{\cdot}$ with correspondences.

**(1.3) Conjectural description of $\alpha^r(T^r_{alg}(W_{\bar{K}})_l)$.** Suppose first that $K \subset \mathbf{C}$. It suffices to describe $\alpha^r(T^r_{alg}(W_{\bar{K}})_l) \subset \mathbf{D}_l^r(W_{\mathbf{C}})$. Since these are divisible groups it suffices to describe the Tate module tensored with $\mathbf{Q}_l$, $\tau_l(\alpha^r(T^r_{alg}(W_{\bar{K}})_l)) \otimes \mathbf{Q}_l \subset H^{2r-1}(W_{\mathbf{C}}, \mathbf{Q}_l(r))$. Let $V \subset H^{2r-1}(W_{\mathbf{C}}, \mathbf{Z}(r))$ denote the largest Hodge substructure contained in the $-1^{st}$ level of the Hodge filtration.

CONJECTURE 1.3.1. $\tau_l(\alpha^r(T^r_{alg}(W_{\bar{K}})_l) \otimes \mathbf{Q}_l = V \otimes \mathbf{Q}_l$.

The inclusion $\subset$ follows from an argument analogous to the proof of (1.2.8). The opposite inclusion would follow from the ordinary Hodge conjecture for all varieties of the form $C \times W_{\mathbf{C}}$, where $C$ is a smooth, projective complex curve [**Gr2**,§2]. Indeed the polarized Hodge structure $V$ corresponds to an abelian variety which is generated by a smooth, projective complex curve, $C$. By the Hodge conjecture there would be a correspondence $\Gamma \subset C \times W$ such that the image of $\Gamma_* : H^1(C, \mathbf{Q}(1)) \to H^{2r-1}(W_{\mathbf{C}}, \mathbf{Q}(r))$ is $V_{\mathbf{Q}}$. Thus

$$\tau_l(\Gamma_*\alpha^1(T^1(C_{\mathbf{C}})_l) \otimes \mathbf{Q}_l = V \otimes \mathbf{Q}_l,$$

which would establish the conjecture when $\bar{K} = \mathbf{C}$. For general $K \subset \mathbf{C}$ note that $C$ and $\Gamma$ are in fact defined over a finitely generated extension field of $\bar{K}$. We view this field as the generic point of a smooth $\bar{K}$-variety, $B$. By passing to a non-empty open subset of $B$ if necessary we may spread out $C$ to a smooth relative curve $\mathcal{C} \to B$ and $\Gamma$ to a subscheme $\Gamma \subset \mathcal{C} \times W_{\bar{K}}$ which is flat over $B$. Let $0 \in B$ be a closed point. By the base change theorems in étale cohomology and the compatibility of the cycle class with respect to specialization [**Fu** ], the cycle class $cl^r_{(C_0 \times W)_{\bar{K}}}(\Gamma_0) \in H^{2r}(C_0 \times_{\bar{K}} W, \mathbf{Z}_l(r))$ may be identified with $cl^r_{(C \times W)_{\mathbf{C}}}(\Gamma) \in H^{2r}((C \times W)_{\mathbf{C}}, \mathbf{Z}_l(r))$. Since the map, $\Gamma_{0*}$ (respectively $\Gamma_*$) induced by the correspondence $\Gamma_0$ (respectively $\Gamma$) on cohomology depends only on the cohomology class $cl^r_{(C_0 \times W)_{\bar{K}}}(\Gamma_0)$ (respectively $cl^r_{(C \times W)_{\mathbf{C}}}(\Gamma)$),

$$\tau_l(\Gamma_{0*}\alpha^1(T^1(C_{0\bar{K}})_l)) \otimes \mathbf{Q}_l = V \otimes \mathbf{Q}_l$$

as desired.

Now suppose that $K$ is a finite field with $q$ elements. Let $\phi \in G_K$ be the Frobenius. Write $V_{\mathbf{Q}_l} \subset H^{2r-1}(W_{\bar{K}}, \mathbf{Q}_l(r))$ for the largest $G_K$-submodule such that the eigenvalues of $\phi^{-1}|_{V_{\mathbf{Q}_l}(-1)}$ are algebraic integers.

PROPOSITION 1.3.2. *Suppose*

1. $\phi^{-1}$ *acts semi-simply on* $V_{\mathbf{Q}_l}$,
2. *The Tate conjecture holds for all varieties of the form* $C \times W$ *where* $C/K$ *is a smooth projective curve.*

*Then* $\tau_l(\alpha^r(T^r_{alg}(W_{\bar{K}})_l) \otimes \mathbf{Q}_l = V_{\mathbf{Q}_l}$.

PROOF: By Deligne's Theorem [**De**] the eigenvalues of $\phi^{-1}$ acting on $V_{\mathbf{Q}_l}(-1)$ are algebraic integers, which have absolute value $q^{1/2}$ for any complex embedding. By

(1) and the theorem of Honda and Tate [**Ta2**, Thm.1], there is an abelian variety $A/K$ such that $V_{\mathbf{Q}_l}$ is isomorphic to a $G_K$-submodule of $H^1(A_{\bar{K}}, \mathbf{Q}_l(1))$. Let $C/K$ be the normalization of a curve $\bar{C}_K \subset A_K$ which generates $A$ as an abelain variety. By pullback $V_{\mathbf{Q}_l}$ becomes a direct summand of the semi-simple $G_K$-module $H^1(C_{\bar{K}}, \mathbf{Q}_l(1))$. Projection to this summand gives an element

$$pr \in Hom_{G_K}(H^1(C_{\bar{K}}, \mathbf{Q}_l(1)), V_{\mathbf{Q}_l}) \simeq (H^1(C_{\bar{K}}, \mathbf{Q}_l) \otimes V_{\mathbf{Q}_l})^{G_K}.$$

By the Tate conjecture a non-zero multiple of $pr$ is the class of an algebraic cycle $\Gamma \subset Z^r(C \times_K W) \otimes \mathbf{Z}_l$. The image of the map on cohomology, $\Gamma_* H^1(C_{\bar{K}}, \mathbf{Q}_l(1))$, is $V_{\mathbf{Q}_l}$. This establishes the inclusion $\supset$. The opposite inclusion is clear from the proof of (1.2.9) and is not dependent on hypotheses (1) and (2).

**2.Quotients of complete intersections by fixed point free group actions.** In this section we construct some examples of torsion in the Griffiths group of 1-cycles. We work with threefolds which are quotients of complete intersections by finite groups acting freely. The cycles are simply second Chern classes of rank two vector bundles associated to representations of the finite fundamental group. All varieties in this section are defined over $\mathbf{C}$.

Let $G$ be an arbitrary finite group and $n > 1$ an integer. It is possible to find an $n+1$-dimensional complete intersection $\tilde{V} \subset \mathbf{P}^N$ in a suitably large projective space where $G$ acts without fixed points [**Se3**,§20]. Write $\rho : \tilde{V} \to V$ for the canonical quotient map.

LEMMA 2.1. *$H_i(G, \mathbf{Z})$ is canonically a direct summand of $H_i(V, \mathbf{Z})$ when $i \leq n$.*

PROOF: [**A-H**,Proposition 6.6].

Associated to a representation $\kappa : G \to GL(r, \mathbf{C})$ we have an algebraic vector bundle $E := \tilde{V} \times_G \mathbf{C}^r$ on $V$. The Chern classes of $E$ in either the Chow ring or the cohomology are annihilated by $\rho^*$ hence by $\rho_* \circ \rho^*$ which is multiplication by $|G|$. Of course $E$ is obtained by pulling back a universal vector bundle $\mathcal{E}$ on the classifying space $BG$. To show that these torsion classes do not always vanish we give two elementary examples:

(1) $G \simeq \mathbf{Z}/m, r = 2, \kappa$ corresponds to a direct sum of two primitive characters. Then $\mathcal{E} \simeq L_1 \oplus L_2$ is a direct sum of line bundles. Since $H^{even}(BG, \mathbf{Z}) \simeq \mathbf{Z}[t]/mt$ [**Br**, p.114 Ex.3], $c_2(\mathcal{E}) = c_1(L_1) \cdot c_1(L_2)$ is a generator of $H^4(BG, \mathbf{Z}) \simeq \mathbf{Z}/m$.
(2) $G$ is the binary icosohedral group, $r = 2, \kappa$ is the standard representation in $SU(2)$. Let $B'$ denote the quotient of $EG$ by a Sylow-5- subgroup of $G$. The restriction of $\kappa$ to such a subgroup splits as the direct sum of a primitive chatacter and its complex conjugate. By (1), $c_2(\mathcal{E}_{|B'}) \in H^4(B', \mathbf{Z})$ is non-zero. Now functoriality of Chern classes implies that $c_2(\mathcal{E}) \in H^4(BG, \mathbf{Z})$ is non-zero.

To see how these Chern classes look when pulled back to $V$ use the universal coefficient theorem which says that the pull back map on $H^4()_{tors}$ is given by

$$Ext(H_3(BG, \mathbf{Z}), \mathbf{Z}) \to Ext(H_3(V, \mathbf{Z}), \mathbf{Z}).$$

This map is injective when $n \geq 3$ by (2.1).

Suppose now that $n = 3$ and that $W \subset V$ is a non-singular very ample divisor. In either of the cases (1) or (2) above we have $H^4(W, \mathbf{Z})_{tors} = 0$. In fact using Poincaré duality, the Lefschetz hyperplane theorem, and [**A-H**,Proposition 6.6] one gets the first three isomorphisms in

$$H^4(W, \mathbf{Z})_{tors} \simeq H_2(W, \mathbf{Z})_{tors} \simeq H_2(V, \mathbf{Z})_{tors} \simeq H_2(BG, \mathbf{Z}) \simeq 0.$$

The last isomorphism is well known for cyclic groups. In the case of the binary icoshedral group it is an easy consequence of the famous fact that $S^3/G$ is a homology sphere. In either case it is clear that the codimension two cycle $c_2(E_{|W})$ is homologous to zero on the hyperplane section $W$.

PROPOSITION 2.2. *With notation as above, the image of the torsion cycle $c_2(E_{|W})$ under the Abel-Jacobi homomorphism is not zero.*

PROOF: Fix $m > 0$ such that $c_2(E) \in H^4(V, \mathbf{Z})[m]$. Via the coboundary map associated to the coefficient sequence

$$0 \to \mathbf{Z} \xrightarrow{m} \mathbf{Z} \to \mathbf{Z}/m \to 0$$

and the fact that $H^3(V, \mathbf{Z}) \simeq H_2(V, \mathbf{Z})_{tors} \simeq 0$, $c_2(E)$ gets identified with an element of $H^3(V, \mathbf{Z}/m)$. This element is easliy seen to coincide with $\lambda^2(c_2(E))$, where $\lambda^2$ is the cycle class map introduced by Bloch [**Bl2**,3.7]. Write $i_{W/V} : W \to V$ for the inclusion. By the Lefschetz hyperplane theorem [**Mi**,VI.7] restriction $i^*_{W/V} : H^3(V, \mathbf{Z}/m) \to H^3(W, \mathbf{Z}/m)$ is injective. By [**Bl2**,3.5], $i^*_{W/V}(\lambda^2(c_2(E))) = \lambda^2(i^*_{W/V}(c_2(E)))$. Furthermore

$$\lambda^2 : (CH^2(W)_{hom})_{tors} \to H^3(W, \mathbf{Q})/H^3(W, \mathbf{Z})$$

may be identified with the Abel-Jacobi map [**Bl2**,3.7]. The proposition follows.

By fixing the degree, $d >> 0$, of the hypersurface $W$ to be sufficiently large we arrange $F^3H^3(W) \neq 0$. Let $\mathcal{M}$ denote the parameter space for smooth degree $d$ hypersurface sections of $V$. For $t \in \mathcal{M}$, $W_t$ denotes the corresponding hypersurface.

COROLLARY 2.4. *For a sufficiently general choice of $t \in \mathcal{M}$, $c_2(E_{|W_t})$ is not algebraicly equivalent to zero.*

PROOF: There is a natural action of $\pi_1(\mathcal{M}, t_0)$ on $H^3(W_{t_0}, \mathbf{Q})$ which is known to be irreducible by Lefschetz theory. By a standard argument (cf. eg. the paragraphs prior to (3.2.2)) there is a countable union of proper analytic subsets of $\mathcal{M}$ with the property that for any $t$ in the complement, the largest Hodge substructure $H_{\mathbf{Q}} \subset H^3(W_t, \mathbf{Q})$ of pure Hodge type $(2,1) + (1,2)$ is zero. Since the Abel-Jacobi image of $c_2(E|_{W_t})$ is not zero, $c_2(E_{|W_t})$ is not algebraicly equivalent to zero by applying (1.2.8) with $P = Id$.

VARIANT 2.5. *Suppose that $V$ in is defined over a number field $K$. Then (2.4) holds for infinitely many $W$ defined over $K$.*

PROOF: Let $U \subset \mathbf{P}^1_K$ parametrize the smooth fibers in a Lefschetz pencil of $V$ which is defined over $K$. According to Terasoma [**Te**] there is an infinite set of rational points $u \in U(K)$ with the property that the image of the decomposition group $G_{\bar{K}/K} \subset \pi_1(U_K, \bar{u})$ in $Aut(H^3(W_{\bar{u}\ \bar{K}}, \mathbf{Q}_l(2))$ contains the image of the monodromy representation $\pi_1(U_{\bar{K}}, \bar{u}) \to Aut(H^3(W_{\bar{u}\ \bar{K}}, \mathbf{Q}_l(2)))$. The latter is known to be an open subgroup of the group of all linear transformations preserving the intersection form [**Te**]. Thus if $[L : K] < \infty$, $H^3(W_{\bar{u}\ \bar{K}}, \mathbf{Q}_l(2))$ is an irreducible $G_{\bar{K}/L}$-module. Given a curve $C$ and a correspondence $\Gamma \subset C \times W_{\bar{u}\ \bar{K}}$ defined over $L$, the image $\Gamma_* H^1(C_{\bar{K}}, \mathbf{Q}_l(1)) \subset H^3(W_{\bar{u}\ \bar{K}}, \mathbf{Q}_l(2))$ is a $G_{\bar{K}/L}$-submodule. By hypothesis $F^3H^3(W_{\bar{u}}(\mathbf{C}), \mathbf{C}) \neq 0$, so this sub-module is proper and hence zero. As in the proof of (1.2.8) we conclude that

$$\Gamma_* : \mathbf{D}_l^1(C_{\bar{K}}) \to \mathbf{D}_l^2(W_{\bar{u}\ \bar{K}})$$

is 0. As $C$ and $\Gamma$ are arbitrary, $\lambda^2 T^2_{alg}(W_{\bar{u}\ \bar{K}})_l = 0$ follows from the compatibility of $\lambda^2$ with correspondences. Thus $c_2(E|_{W_u})$ is not algebraicly equivalent to zero even after extending scalars to $\bar{K}$.

**3. Special hypersurfaces in $\mathbf{P}^{2m}_{\mathbf{C}}$.** In this section we prove

THEOREM 3.0. *There exist smooth hypersurfaces $W \subset \mathbf{P}^{2m}_{\mathbf{C}}$ $(m \geq 2)$ for which $Gr^m(W)[2] \neq 0$.*

PLAN OF PROOF: The argument is inspired by Griffiths' original construction of non-trivial elements in the Griffiths group [**Gri**]. There are three steps.

1. (1) Construct a hypersurface $V \subset \mathbf{P}^{2m+1}_{\mathbf{C}}$ with isolated singularities and a cycle class $\tau \in CH_m(V)[2]$ whose homology class in $H_{2m}(V, \mathbf{Z})[2]$ is not zero.
2. (2) Deduce from (1), that if $W \subset V$ is a smooth hyperplane section, then the Abel-Jacobi image of the restricted cycle, $\lambda^m(i^*_{W/V}\tau) \in J^m(W)[2]$, is not zero.
3. (3) For a general choice of $V$ and a general hyperplane section $W$, the largest Hodge substructure of $H^{2m-1}(W)$ having pure Hodge type $(m, m-1) + (m-1, m)$ is 0.

The theorem follows from these three steps by applying (1.2.8) with $P = Id$. Perhaps the third step deserves special comment. We use an argument based on infinitesimal variation of Hodge structure. This technique is however poorly suited for dealing with families of varieties with small parameter space. Thus the argument is complicated and the statement of the final result (3.3.4) contains restrictive hypotheses, which one might hope to eventually eliminate. A possible alternative approach is mentioned in (3.4.3). As this paper goes to press, I have learned from S. Mueller-Stach of a method for establishing (3) when $W$ is a hypersurface section of large degree [**Mue**].

**3.1 Certain singular hypersurfaces $V \subset \mathbf{P}^{2m+1}$.** Fix a positive, even, integer $d$ and consider a subset $\{g_{-m}, ..., g_m\} \subset \mathbf{C}[x_0, ..., x_{2m+1}]$ of homogeneous polynomials having positive degrees $d_{-m}, ..., d_m$ such that $d_i + d_{-i} = d$ for all $i, 0 \leq i \leq m$. The hypersurface $V \subset P := \mathbf{P}^{2m+1}_{\mathbf{C}}$ defined by

$$g_0^2 + \sum_{1 \leq i \leq m} g_i g_{-i} = 0 \tag{3.1.1}$$

is singular along the locus $R$ defined by the ideal $(g_{-m},...,g_m)$. By choosing sufficiently general $g_j$ 's we arrange that $S = V_{sing}$ consists entirely of isolated ordinary double points (3.4.1). (One can of course arrange that $R = S$, but for present purposes it is not necessary to exclude the possiblity $S - R \neq \emptyset$.) Assume furthermore that $d_i = 0$ mod 2 for some $i > 0$. Consider the codimension $m$ subvariety $Z \subset V$ (respectively $Y \subset P$) defined by the homogeneous ideal $(g_0,...,g_m)$ (respectively $(g_1,...,g_m)$). Under the pullback map defined by intersection theory $j^* : CH^m(P) \to CH_m(V), j^*Y = 2Z$. Let $L \subset P - S$ be a codimension $m$ linear space which meets $V$ properly. Set $r = d_1...d_m/2$. Set $\tau = Z - j^*rL$. Then

$$2(\tau) = j^*(Y - 2rL) = 0 \in CH_m(V) \simeq CH^m(V - S).$$

PROPOSITION 3.1.2. *The cohomology class $c(Z - j^*rL) \in H^{2m}(V - S)[2]$ is not zero.*

NOTE: Homology and cohomology groups in this subsection have $\mathbf{Z}$ coefficients unless the contrary is specificly indicated. The familiar cycle class map to cohomology with $\mathbf{Z}$ coefficients will be denoted by $c$.

PROOF: Write $\sigma_V : \tilde{V} \to V$ for the blow up of $V$ along $S$. Set $E = \sigma_V^{-1}(S)$. Recall that $E \simeq \amalg_{s \in S} E_s$ with $E_s \subset \mathbf{P}^{2m}$ a smooth quadric hypersurface and $\mathcal{N}_{E/\tilde{V}}|_{E_s} \simeq \mathcal{O}_P(-1)|_{E_s}$. The map $\cdot c_1(\mathcal{N}_{E/\tilde{V}}) : CH^{m-1}(E) \to CH^m(E)$ has image $2CH^m(E)$. An analogous statement holds for cohomology. We need

LEMMA 3.1.3. *There is a commutative diagram*

$$\begin{array}{ccc} CH^m(V-S) & \xrightarrow{\epsilon} & CH^m(E)/2CH^m(E) \\ c\downarrow & & \downarrow\simeq \\ H^{2m}(V-S) & \xrightarrow{\epsilon'} & H^{2m}(E)/2H^{2m}(E). \end{array}$$

PROOF: Use the localization sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & CH^{m-1}(E) & \xrightarrow{i_{E/\tilde{V}*}} & CH^m(\tilde{V}) & \longrightarrow & CH^m(V-S) & \longrightarrow & 0 \\ & & & & \downarrow i^*_{E/\tilde{V}} & & & & \\ & & & & CH^m(E) & & & & \end{array} \tag{3.1.4}$$

and the formula $i^*_{E/\tilde{V}} \circ i_{E/\tilde{V}*} = \cdot c_1(\mathcal{N}_{E/\tilde{V}})$ to define the map $\epsilon$. There is an analogous diagram of cohomology groups which gives rise to $\epsilon'$. Compatibility with cycle class maps is clear so (3.1.3) follows.

To verify $\epsilon(Z - j^*rL) \neq 0$, observe that the hypotheses on the singularities of $V$ imply that $g_{-m},...,g_m$ are local coordinates on $P$ at each $s \in R$. The $g_i$'s become homogeneous coordinates on the exceptional $\mathbf{P}^{2m}$'s in the blow up of $P$ along $S$. The strict transform of $Z$ meets $E_s$ in the linear space $g_0 = ... = g_m = 0$, which is a generator of $CH^m(E_s)$. Hence (3.1.2).

PROPOSITION 3.1.5. *If $\epsilon(z) \neq 0$ for $z \in CH^m(V-S)[2]$, then $i^*_{W/V-S}(z) \in CH^m(W)_{hom}$ is not zero.*

PROOF: Consider the composition

$$(3.1.6) \qquad \begin{aligned} CH^m(V-S)[2] &\xrightarrow{c} H^{2m}(V-S,\mathbf{Z})[2] \simeq H^{2m-1}(V-S,\mathbf{Z}/2) \\ &\xrightarrow{i^*_{W/V-S}} H^{2m-1}(W,\mathbf{Z}/2) \simeq J^m(W)[2], \end{aligned}$$

in which the second map is the inverse of the coboundary in the long exact cohomology sequence associated to the coefficient sequence $\mathbf{Z} \xrightarrow{2} \mathbf{Z} \to \mathbf{Z}/2$. We claim that $i^*_{W/V-S}$ is injective. Indeed, $H^{2m-1}(V-S,W;\mathbf{Z}/2)$ maps onto the kernel. By duality (relative Alexander duality for the pair $(V-S,W)$ in $\tilde{V}$ [**Gr-H**,27.6])

$$H^{2m-1}(V-S,W;\mathbf{Z}/2) \simeq H_{2m+1}(\tilde{V}-W,E;\mathbf{Z}/2) \simeq H_{2m+1}(V-W,\mathbf{Z}/2),$$

which is zero, since $V-W$ is affine of dimension $2m$ [**Hamm**].

It remains to show that (3.1.6) takes $z$ to the Abel-Jacobi image of $i^*_{W/V-S}(z)$. It is convenient to know that $H_{2m+1}(V) \simeq 0$. For this, consider $V_t$ a nearby smooth hypersurface and $\mathcal{S} \subset V_t$ the collection of vanishing $2m$-spheres for $V$. Then $H_{2m+1}(V) \simeq H_{2m+1}(V_t,\mathcal{S}) \simeq 0$ follows from the known homology of $V_t$ (cf. [**Cl**,p. 119-120]). It follows that the definition of the cycle class map $\lambda_m : CH_m(V)[2] \to H_{2m+1}(V,\mathbf{Z}/2)$ given in [**Bl2**,3.7] makes sense inspite of the singularities of $V$. By duality we get a map $\lambda^m : CH^m(V-S)[2] \simeq CH_m(V)[2] \to H^{2m-1}(V-S,\mathbf{Z}/2)$, which is easily seen to coincide with the composition of the first two maps in (3.1.6).

LEMMA 3.1.7. *The following diagram commutes*

$$\begin{array}{ccc} CH_m(V)[2] & \xrightarrow{i^*_{W/V-S}} & CH_{m-1}(W)[2] \\ \lambda_{m,V-S}\downarrow & & \lambda_{m-1,W}\downarrow \\ H_{2m+1}(V,\mathbf{Z}/2) & \xrightarrow{\cap c(W)} & H_{2m-1}(W,\mathbf{Z}/2). \end{array}$$

SKETCH OF PROOF: The vertical maps are the cycle class maps defined in [**Bl2**,p. 116]. The commutativity is then proved as in [**Bl2**, 3.5]. One must however carry through the various steps in the language of homology (not cohomology) due to the singularites of $V$.

By duality the bottom row in (3.1.7) may be identified with the map $i^*_{W/V-S}$ in (3.1.6). It is well known that $H^{2m}(W,\mathbf{Z}) \simeq \mathbf{Z}$. Thus $CH^m(W)[2] \subset CH^m(W)_{hom}$. By [**Bl2**,3.7], $\lambda_{m-1,W}$ may be identified with the Abel-Jacobi map on $CH_{m-1}(W)[2]$. Now (3.1.5) follows.

**3.2 Infinitesimal variation of Hodge structure.** Now that it has been verified that the Abel-Jacobi map sends $i^*_{W/V-S}(Z-j^*rL)$ to a non-zero element in the intermediate Jacobian, $J^m(W)$, one can hope to apply Hodge theoretic techniques to show that for a general choice of $V$ and $W$, $i^*_{W/V-S}(Z-j^*rL)$ is not algebraicly equivalent to zero. By (1.2.8) it would suffice to show that among all possible $V$'s there is one with a smooth hypersurface section $W$ for which $H^{2m-1}(W)$ has no non-zero Hodge substructure of pure type $(m, m-1)+(m-1, m)$. Roughly speaking, we wish to show that the hypersurfaces under consideration, although quite special, nonetheless exhibit generic Hodge structures. This is tricky and we shall only pursue it in the case that $W$ is the hyperplane section $x_{2m+1}=0$ of the varying hypersurface $V$.

Consider the map

$$\xi_{\mathbf{P}^{2m}} : \bar{\mathcal{M}} := \prod_{-m\le i\le m} \mathbf{P}H^0(\mathbf{P}^{2m}, \mathcal{O}(d_i)) \to \mathbf{P}H^0(\mathbf{P}^{2m}, \mathcal{O}(d)),$$

$$\xi_{\mathbf{P}^{2m}}((h_{-m},...,h_m)) = H := h_0^2 + \sum_{1\le i\le m} h_i h_{-i}. \tag{3.2.1}$$

The subset $\mathcal{M} \subset \bar{\mathcal{M}}$ consisting of those $\overrightarrow{h} := (h_{-m},...,h_m)$ such that $W = \mathcal{W}_{\overrightarrow{h}}$ defined by (3.2.1) is non-singular and $\mathcal{Z}_{\overrightarrow{h}} \subset \mathcal{W}_{\overrightarrow{h}}$ defined by the ideal $(h_0,...,h_{-m})$ has codimension $m$ is open and dense. One may note that a general point of $\mathcal{M}$ corresponds to a hyperplane section of a variety $V$ defined by (3.1.1).

Pullback the universal degree $d$ hypersurface via $\xi_{\mathbf{P}^{2m}}$ to obtain a smooth family $\pi : \mathcal{W} \to \mathcal{M}$ with a cycle $\mathcal{Z} \subset \mathcal{W}$ flat over $\mathcal{M}$. Furthermore, $\mathcal{T} := (\mathcal{Z} - ri^*_{\mathcal{W}/\mathbf{P}^{2m}}L)$ gives a non-zero 2-torsion section of the relative intermediate Jacobian (3.1.5).

Base change $\pi$ to the universal cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$ and consider the variation of Hodge structure associated to the resulting family $\tilde{\pi} : \tilde{\mathcal{W}} \to \tilde{\mathcal{M}}$. The condition that a section of the constant sheaf $H^{2m-1}(W, \mathbf{Z}) \otimes \underline{\mathbf{Z}} \simeq R^{2m-1}\tilde{\pi}_*\mathbf{Z}$ be in the $(m-1)$-st level of the Hodge filtration is an analytic condition on points in $\tilde{\mathcal{M}}$. As $\tilde{\mathcal{M}}$ is not the union of a countable number of proper analytic subsets, the only way for $(R^{2m-1}\pi_*\mathbf{Z})_\mu$ to admit a non-trivial Hodge substructure of pure type $(m-1, m)+(m, m-1)$ for all $\mu \in \mathcal{M}$, is for $H^{2m-1}(W, \mathbf{Z})$ to admit a sublattice $H_{\mathbf{Z}}$ satisfying $H_{\mathbf{Z}} \subset F^{m-1}_{\tilde{\mu}}$ for all $\tilde{\mu} \in \tilde{\mathcal{M}}$. In other words, for all $\tilde{\mu}$, $F^{m+1}_{\tilde{\mu}}$ must be contained in the orthogonal complement of $H_{\mathbf{Z}}$ under the cup product pairing. The largest such sublattice, $H_{\mathbf{Z}}$, would be invariant under the action of $\pi_1(\mathcal{M})$ and would give rise to a subvariation of Hodge structure $\underline{H}_{\mathbf{Z}} \subset R^{2m-1}\tilde{\pi}_*\mathbf{Z}$.

If the action of $\pi_1(\mathcal{M})$ on $H^{2m-1}(W, \mathbf{Z})$ were irreducible, we would have $H_{\mathbf{Z}} = 0$ or $H^{2m-1}(W, \mathbf{Z})$. When $F^{m+1}_{\tilde{\mu}} \neq 0$, the second possibility is ruled out. Unfortunately, we do not presently know if the monodromy representation is irreducible. We thus resort to the method of infinitesimal variation of Hodge structure to bound the rank of $H_{\mathbf{Z}}$. This involves studying the map

$$T_{\overrightarrow{h}}\mathcal{M} \otimes H^{m+1,m-2}(W) \to H^{m,m-1}(W) \tag{3.2.2}$$

constructed by composing the Kodaira-Spencer map $T_{\overrightarrow{h}}\mathcal{M} \to H^1(W, \mathcal{T}_W)$ with the cup product on cohomology. This map coincides with the derivative of the variation of Hodge substructure $\underline{H}_{\mathbf{Z}}^{\perp} \subset R^{2m-1}\tilde{\pi}_*\mathbf{Z}$ at the point $\overrightarrow{h}$ [**Do**,§2]. Thus the image of (3.2.2) is contained in $H_{\mathbf{Z}}^{\perp}$. If (3.2.2) turns out to be surjective, then we have succeeded in showing $H_{\mathbf{Z}} = 0$. We thus turn now to computing the image of (3.2.2).

The computation of (3.2.2) is based on the interpretation of the differential of the period map of a projective hypersurface in terms of multiplication in the Jacobian ring. It would take up too much space to summarize this by now standard technique here. We refer instead to [**Do**,§2] and references therein for background material. Write $\mathbf{h}$ for the graded ideal $(h_{-m}, ..., h_m)$. The homogenous elements of degree $a$ in a graded module $M$ will be denoted by $M_a$. An identification of $\mathbf{h}_d$ with $T\xi_{\mathbf{P}^{2m}}(T_{\overrightarrow{h}}\mathcal{M})$ is obtained by differentiating

$$(3.2.3) \qquad (h_0 + t\hat{h}_0)^2 + \sum_{-m \le i \le m} (h_i + t\hat{h}_i)(h_{-i} + t\hat{h}_{-i})$$

with respect to $t$. Write $J \subset \mathbf{C}[x_0, ..., x_{2m}]$ for the ideal generated by the partial derivatives of $H$, and set $A = \mathbf{C}[x_o, ..., x_{2m}]/J$. Observe that $J \subset \mathbf{h}$ and that (3.2.3) identifies $\mathbf{h}_d/J_d$ with the image of the Kodaira Spencer map $T_{\overrightarrow{h}}\mathcal{M} \to H^1(W, \mathcal{T}_W)$. Set $t_a = (2m - a)d - (2m + 1)$. Via residues one obtains an identification

$$(3.2.4) \qquad H^{a,2m-1-a}(W) \simeq A_{t_a}$$

such that multiplication in $A$

$$(3.2.5) \qquad \mathbf{h}_d/J_d \times A_{t_{m+1}} \to A_{t_m}$$

gets identified with a modified form of (3.2.2) in which $T_{\overrightarrow{h}}\mathcal{M}$ has been replaced by its image in $H^1(W, \mathcal{T}_W)$ [**Do**,§2].

LEMMA 3.2.6. *Suppose $H^{m+1,m-2}(W) \neq 0$ (ie. $t_{m+1} \geq 0$). Then the image of (3.2.5) is $\mathbf{h}_{t_m}/J_{t_m} \neq A_{t_m}$. In particular (3.2.2) is not surjective.*

PROOF: Since all generators of $\mathbf{h}$ have degree less than $d$, the image of (3.2.5) is $\mathbf{h}_{t_m}/J_{t_m}$ unless $H^{m+1,m-2}(W) = 0$ (ie. unless $t_{m+1} < 0$). Since $W$ is non-singular, $h_{-m}, ..., h_m$ is a regular sequence in $\mathbf{C}[x_o, ..., x_{2m}]$. The dimension of $(\mathbf{C}[x_o, ..., x_{2m}]/\mathbf{h})_t$ is computable from the Koszul complex of this regular sequence. It is zero exactly when $t > \sum_{-m \le i \le m}(d_i - 1) = (m + 1/2)d - (2m + 1)$. As $t_m$ does not satisfy this inequality, $\mathbf{h}_{t_m}/J_{t_m} \neq A_{t_m}$ follows from the short exact sequence

$$0 \to \mathbf{h}_{t_m}/J_{t_m} \to A_{t_m} \to (\mathbf{C}[x_0, ..., x_{2m}]/\mathbf{h})_{t_m} \to 0.$$

We may summarize (3.2.6) by saying that an understanding of the maps among the vector spaces $F^pH^{2m-1}(W, \mathbf{C})/F^{p+1}H^{2m-1}(W, \mathbf{C})$ arrising from infinitesimal variation of Hodge structure, is not by itself sufficient to show $\underline{H}_{\mathbf{Z}} = 0$. In the next section we will choose with considerable care a point $\overrightarrow{h} \in \mathcal{M}$ so that the Hodge structure of the fiber $H^{2m-1}(W_{\overrightarrow{h}})$ may be understood in detail. This additional information will, in certain cases allow us to conclude $\underline{H}_{\mathbf{Z}} = 0$.

**3.3 A hypersurface with many automorphisms.** For $d = 0 \mod 2$ consider the non-singular hypersurface $W_m \subset \mathbf{P}^{2m}$ defined by the polynomial

$$H_m = x_0^d + x_1^d + x_1 x_2^{d-1} + x_2 x_3^{d-1} + \dots + x_{2m-1} x_{2m}^{d-1}. \tag{3.3.1}$$

One can write $H_m$ in the form (3.2.1) by choosing

$$h_0 = x_0^{d/2}, h_1 = x_1(x_1 + x_2), h_{-1} = \sum_{0 \le i \le d-2} (-1)^i x_1^{d-2-i} x_2^i \tag{3.3.2}$$

and for $i > 1$, $h_i = x_{2i-1}, h_{-i} = x_{2i-2} x_{2i-1}^{d-2} + x_{2i}^{d-1}$. Set $D_m = d(d-1)^{2m-1}$, fix a primitive $D_m$-root of unity $\zeta_{D_m}$ and observe that $H_m$ is fixed by the action of

$$\mu_{D_m} \simeq < \gamma_m >, \text{ where } \gamma_m = diag.(1, \zeta_{D_m}^{(1-d)^{2m-1}}, \zeta_{D_m}^{(1-d)^{2m-2}}, \zeta_{D_m}^{(1-d)^{2m-3}}, \dots, \zeta_{D_m}).$$

Thanks to the work of Shioda [**Sh1**], [**Sh2**], [**Sh3**], the hypersurfaces (3.3.1) are among the few for which one can explicitly describe the Hodge structure on the middle cohomology. Using the explicit description of the image of the differential of the period map (3.2.2) we will show

PROPOSITION 3.3.3. *Let $H_{\mathbf{Q},m}$ denote the largest rational Hodge substructure of $H^{2m-1}(W_m)$ of pure type $(m, m-1)+(m-1, m)$ which is orthogonal to the image of (3.2.2) under the cup product pairing. If $d-1$ is prime, $d \ge min.\{4m, 18\}$, and $m > 1$ , then $H_{\mathbf{Q},m} = 0$.*

COROLLARY 3.3.4. *Suppose that $d$ satisfies the hypotheses of (3.3.3). If $d_1 = 2, d_i = 1$ for $i > 1$, then the hypersurface $W$ defined by a general polynomial $H$ of the form (3.2.1) satifies $Gr^m(W)[2] \neq 0$.*

PROOF OF 3.3.4: The algebraic cycle $T_{\vec{h}}$ is two torsion for rational equivalence and has non-trivial image in the intermediate Jacobian $J^m(W_{\vec{h}})$. By (3.3.3) $H^{2m-1}(W_{\vec{h}})$ has no non-zero Hodge substructure of pure type $(m, m-1)+(m-1, m)$ for a general $\vec{h} \in \mathcal{M}$. Now apply (1.2.8) with $P = Id$ to conclude that for such an $\vec{h}$, $T_{\vec{h}} \in Gr^m(W_{\vec{h}})[2]$ is non-zero.

PROOF OF 3.3.3: We will explicitly describe the largest rational Hodge substructure $U \subset H^{2m-1}(W_m)$ of pure type $(m, m-1)+(m-1, m)$ as a direct sum of pairwise non-isomorphic irreducible Hodge structures $U_i$. For each of these $U_i$ we shall verify that the image of (3.2.2) is not contained in $U_i^{\perp}$. The proposition will then follow immediately.

Begin with the disection of the Hodge structure $H^{2m-1}(W_m)$. This is done by induction on $m$ with the help of the following correspondence. The hyperplane section defined by $x_{2m-1} = 0$ is the cone, $C_m$, over $W_{m-1}$ with vertex $p_m = (0 : \dots : 0 : 1)$. Using the notation $\tilde{}$ for the blow up at $p_m$, we describe a correspondence, $\beta_m$,

$$W_{m-1} \xleftarrow{p} \tilde{C}_m \xrightarrow{q} \tilde{W}_m \xrightarrow{\sigma} W_m,$$

where $p$ is the projection, $q$ is the desingularization of $C_m$, and $\sigma$ is the blow up of $W_m$ at $p_m$. This induces a map $\beta_{m*} : H^{2m-3}(W_{m-1}) \to H^{2m-1}(W_m)$. Define the composed correspondence $\eta = \beta_m \circ \dots \circ \beta_1 : W_1 \to W_m$. To simplify notation we will frequently write $\beta_*$ for $\beta_{m*}$.

LEMMA 3.3.5.

(1) $\beta_*$ *is injective.*

(2) *Assume* $d \geq 4m$ *and* $d-1$ *is prime. Write* $A$ *for the augmentation representation of* $\mathbf{Z}/d$ *and* $B_m$ *for the "primitive " representation of* $\mathbf{Z}/(d-1)^{2m-1}$ *(ie. the* $\mathbf{Q}$*-irreducible representation which over* $\mathbf{C}$ *decomposes as the direct sum of all primitive characters, each occurring with multiplicity 1). Then* $Coker(\beta_*)$ *is isomorphic to* $A \otimes B_m$ *as a representation of* $< \gamma_m >$.

(3) *For* $d$ *as in (2) any non-zero Hodge substructure,* $U$*, of* $Coker(\beta_*)$ *satisfies* $F^{2m-1}U_{\mathbf{C}} \neq 0$.

(4) *For* $d$ *as in (2) and* $m \geq 2$ *the largest rational Hodge substructure,* $U \subset H^{2m-1}(W_m)$ *of Hodge type* $(m, m-1) + (m-1, m)$ *is* $\eta_*(H^1(W_1))$.

PROOF: (1) Write $E_m \subset \tilde{C}_m$ for the exceptional divisor and $L_m \subset \tilde{C}_m$ for the strict transform of a hyperplane section of $C_m$. Observe that $\mathcal{N}_{\tilde{C}_m/\tilde{W}_m} \simeq \mathcal{O}_{\tilde{C}_m}(L_m - dE_m)$. From the excess intersection formula $q^*q_*(\eta) = \eta \cdot c_1(\mathcal{N}_{\tilde{C}_m/\tilde{W}_m})$ we deduce that $p_*q^*q_*p^*$ is multiplication by $1-d$ on $H^{2m-3}(W_{m-1})$. The injectivity follows since $\sigma$ induces an isomorphism on odd dimensional cohomology.

$$(2) \qquad F^{2m-1}H^{2m-1}(W_m) \simeq F^{2m}H^{2m}(\mathbf{P}^{2m} - W_m) \simeq \mathbf{C}[x_0, ..., x_{2m}]_{d-2m-1}\xi_m,$$
$$\text{where, } \xi_m = \Big(\sum_{0 \leq i \leq 2m} (-1)^i x_i dx_0 ... \widehat{dx_i} ... dx_{2m}\Big)/H_m.$$

Now $\gamma_m$ acts on $\xi_m$ by multiplication by $\zeta_{D_m}^a$, where $a = (2-d)\sum_{0 \leq j \leq m-1}(d-1)^{2j}$. Since $a = 1 \mod (d-1)$ all characters of the group $< \gamma_m^d >$ which appear in the decomposition of the representation $F^{2m-1}H^{2m-1}(W_m)$ are primitive. Observe that $\gamma_m^{(d-1)^{2m-1}}$ acts by multiplication by $\zeta_{D_m}^{d-2m-1-a_0}$ on $x_0^{a_0}...x_{2m}^{a_{2m}}$. Since $d - 2m - 1 \geq d/2 - 1$, and $a = 2m \mod d$ every non-trivial character of $< \gamma_m^{(d-1)^{2m-1}} >$ (or its inverse) appears with positive multiplicity in $F^{2m-1}H^{2m-1}(W_m)$. This shows that $A \otimes B_m \subset Coker(\beta_*)$. Now $dim.(A \otimes B_j) = (d-2)(d-1)^{2j-1}$. The well known formula for the betti number of a smooth degree $d$ hypersurface gives

$$h^{2m-1}(W_m) = (d-2)\sum_{1 \leq j \leq m}(d-1)^{2j-1} = \sum_{1 \leq j \leq m} dim(A \otimes B_j).$$

Induction on $m$ shows $dim.(Coker(\beta_{m*})) = dim.(A \otimes B_m)$.

(3) If $F^{2m-1}U_{\mathbf{C}} = 0$, the same holds when $U$ is replaced by the smallest $\gamma_m$-stable Hodge substructure containing it. But if $U$ is a non-zero subrepresentation of $A \otimes B_m$ which is defined over $\mathbf{Q}$ then we have seen in the proof of (2) that $F^{2m-1}U_{\mathbf{C}} \neq 0$.

(4) This follows from (1), (2), and (3) by induction on $m$.

Let $J$ (respectively $J'$) denote the Jacobian ideal of $H_m$ (respectively $H_{m-1}$), $A = \mathbf{C}[x_0, ..., x_{2m}]/J$, and $A' = \mathbf{C}[x_0, ..., x_{2m-2}]/J'$. Also define

$$\nu_m = x_4^{d-2}x_6^{d-2}...x_{2m}^{d-2}\xi_m/H_m^{m-1}.$$

LEMMA 3.3.6.

(1) *For $m \geq 2$ the natural isomorphism $H^{m,m-1}(W_m) \simeq A_{t_m}\xi_m/H_m^{m-1}$ [Do,§2], gives rise to $\beta_*(H^{m-1,m-2}(W_{m-1})) \simeq A'_{(m-1)d-(2m-1)}x_{2m}^{d-2}\xi_m/H_m^{m-1}$.*

(2) *Similarly we get the identification of $\eta_*(H^{1,0}(W_1))$ with*

$$\mathbf{C}[x_0, x_1, x_2]_{d-3}\nu_m.$$

PROOF: (1) Let $\kappa = \gamma_m^{d(1-d)^{2m-2}}$ and write $\zeta_{d-1} = \zeta_{D_m}^{d(1-d)^{2m-2}}$ which is a primitive $d-1$-st root of unity. By (3.3.5(2))

$$\beta_*(H^{m-1,m-2}(W_{m-1})) = H^{m,m-1}(W_m)^{<\kappa>}.$$

Observe that $x_0^{a_0}...x_{2m}^{a_{2m}}\xi_m/H_m^{m-1}$ is an eigenvector for $\kappa$ with eigenvalue $\zeta_{d-1}^{a_{2m}+1}$. This will be invariant when $a_{2m} = d-2 \mod (d-1)$. The surjectivity of

$$(\mathbf{C}[x_0, ..., x_{2m-2}]_{(m-1)d-(2m-1)}x_{2m}^{d-2}/J_{md-2m-1})\xi_m/H_m^{m-1} \to (A_{t_m}\xi_m/H_m^{m-1})^{<\kappa>}$$

follows from $x_{2m-1}x_{2m}^{d-2}, x_{2m}^{d-2+(d-1)} \in J$. Use again the fact $x_{2m-1}x_{2m}^{d-2} \in J$ to see that the obvious map

$$A'_{(m-1)d-(2m-1)}x_{2m}^{d-2} \to (\mathbf{C}[x_0, ..., x_{2m-2}]_{(m-1)d-(2m-1)}x_{2m}^{d-2}/J_{md-2m-1})$$

is well defined and an isomorphism. As the dimension of this space is $h^{m-1,m-2}(W_{m-1})$, (1) follows.

(2) Recall that $J_{d-3} = 0$ and that $\beta_*$ is injective. Now (2) follows from (1) by induction on $m$.

LEMMA 3.3.7. *With $d$ as in (3.3.3) the irreducible Hodge substructures of $H^1(W_1)$ are precisely the $\mathbf{Q}[\gamma_m]$-irreducible submodules of $H^1(W_1, \mathbf{Q})$. Furthermore, no two irreducible Hodge substructures are isomorphic.*

PROOF: The proof is based on Aoki's detailed analysis of the factors of the Jacobians of Fermat curves. Indeed, the degree $D_1$ Fermat curve, $X : y_0^{D_1} + y_1^{D_1} + y_2^{D_1} = 0$, maps to $W_1$ via $x_0 = y_0^{d-1}, x_1 = y_1^{d-1}, x_2 = y_2^d y_1^{-1}$. We write the character of $\mu_{D_1}^3/\Delta_{\mu_{D_1}} \subset Aut(X)$ corresponding to a monomial $y_0^{a_0}y_1^{a_1}y_2^{a_2}$ of degree 0 mod $D_1$ as $(a_0, a_1, a_2) \in (\mathbf{Z}/D_1)^3$. View $H^1(W_1)$ as a Hodge substructure of $H^1(X)$. The element

$$x_0^{b_0-1}x_1^{b_1-1}x_2^{b_2-1}\xi_1 \in \mathbf{C}[x_0, x_1, x_2]_{d-3}\xi_1 \simeq F^2H^2(\mathbf{P}^2 - W_1) \simeq F^1H^1(W_1)$$

corresponds to the character $a(d-1, 1, -d) \in (\mathbf{Z}/D_1)^3$ where $a = (d-1)b_1 - b_2$. As we run through monomials in $x_0, x_1, x_2$ of degree $d-3$, $a$ and $-a$ run through $\mathbf{Z}/D_1 - (d\mathbf{Z}/D_1 \cup (d-1)\mathbf{Z}/D_1)$, each element being hit once. The irreducible $\mathbf{Q}[\mathbf{Z}/D_1]$-submodules of $H^1(W_1, \mathbf{Q})$ are in bijective correspondence with the divisors $d'$ of $d$ with $d' > 1$ via $d' \to H(d') =$ the "primitive"representation of $\mathbf{Z}/d'(d-1)$. In Aoki's language the Hodge substructure $H(d') \subset H^1(X)$ is denoted by the unordered 3-tuple $\{d-1, 1, -d\}$ of elements of $\mathbf{Z}/d'(d-1)$. By [**Ao**,Thm 0.2] $H(d')$ is an irreducible Hodge structure. (It is at this point that the hypothesis that the prime $(d-1) \geq 17$ enters. One must be sure that $d'(d-1)$ is not in Aoki's exceptional set $\mathcal{E}$).

It remains to check that the Hodge structures assigned to distinct divisors $d'$ of $d$ are non-isomorphic. Associate to each $d'$ the unordered tuple $\{(d/d')(d-1), (d/d'), -d(d/d')\}$ of elements in $\mathbf{Z}/D_1$ and apply [**Ao**,Thm 0.1].

CONTINUATION OF THE PROOF OF 3.3.3: Write $\mathbf{n}'$ for the intersection of the image of (3.2.2) with $U^{m,m-1} = \eta_* H^{1,0}(W_1)$. By (3.2.6) and (3.3.6) we have the identification

$$\mathbf{n}' = \mathbf{h}_{t_m} \cap \mathbf{C}[x_0, x_1, x_2]_{d-3}\nu_m.$$

Write $\mathbf{n}$ for the $\mathbf{C}[<\gamma_m>]$-submodule of $H^{m,m-1}(W_m)$ generated by $\mathbf{n}'$. Consider the ideal $I = (x_0^{d/2}, x_1^2, x_1x_2) \subset \mathbf{C}[x_0, x_1, x_2]$.

LEMMA 3.3.8. $I_{d-3}\nu_m \subset \mathbf{n}$.

PROOF: Set $\zeta_{D_1} = \zeta_{D_m}^{2m-2}$. Then $\gamma_m$ operates on $x_0$, $x_1$, $x_2$, $\nu_m$ by multiplication with 1, $\zeta_{D_1}^{1-d}$, $\zeta_{D_1}$, $\zeta_{D_1}^{2-d}$ respectively. Now $x_0^{d/2}$ and $x_1^2 + x_1x_2 \in \mathbf{h}$. For each monomial $f \in \mathbf{C}[x_0, x_1, x_2]_{d-5}$ there is $b \in \mathbf{Z}/D_1$ such that $\gamma_m$ operates on $x_1^2 f\nu_m$ by multiplication by $\zeta_{D_1}^b$ and on $x_1x_2f\nu_m$ by multiplication by $\zeta_{D_1}^{b+d}$. The element

$$\sum_{t\in\mathbf{Z}/D_1} \zeta_{D_1}^{-bt}\gamma_m^t, \qquad \text{(respectively } \sum_{t\in\mathbf{Z}/D_1} \zeta_{D_1}^{-(b+d)t}\gamma_m^t)$$

of the group ring $\mathbf{C}[<\gamma_m>]$ annihilates $x_1x_2f\nu_m$ (respectively $x_1^2 f\nu_m$) and multiplies $x_1^2 f\nu_m$ (respectively $x_1x_2f\nu_m$) by $D_1$. Since $(x_1^2 + x_1x_2)f\nu_m \in \mathbf{n}'$, $x_1^2 f\nu_m$ and $x_1x_2f\nu_m \in \mathbf{n}$. The lemma follows.

As $\{x_0^j x_2^{d-3-j} : 0 \leq j < d/2\}$ is a basis for $(\mathbf{C}[x_0, x_1, x_2]/(x_1^2, x_1x_2, x_0^{d/2}))_{d-3}$, the codimension of $\mathbf{n} \subset U^{m,m-1}$ is at most $d/2$. Note that when $d' \neq 2$, $dim(H(d')) = \phi(d')(d-2) > d$, where $\phi$ is Euler's function. Thus $\eta_* H(d')^{0,1} \subset U^{m-1.m}$ has dimension $> d/2$, whence $\mathbf{n} \not\subset (\eta_* H(d'))^{\perp}$. As $<\gamma_m>$ is in the orthogonal group for the intersection form, $\mathbf{n}' \not\subset (\eta_* H(d'))^{\perp}$.

The same argument works with a little modification in the case $d' = 2$. Indeed, a simple computation shows that the subspace of $H^{1,0}(W_1)$,

$$\mathbf{n}_0 := span\{x_0^j x_2^{d-3-j}\xi_1 : (0 \leq j < d/2 - 1)\},$$

is contained in $H(2)^{\perp}$. Thus $\mathbf{n}' \subset (\eta_* H(2))^{\perp}$ implies $\mathbf{n} \subset (\eta_* H(2))^{\perp}$, which implies $\mathbf{n} + \eta_*\mathbf{n}_0 \subset (\eta_* H(2))^{\perp}$, which contradicts $dim(H(2))^{0,1} = (d-2)/2 > 1$ since $\mathbf{n} + \eta_*\mathbf{n}_0$ has codimension one in $U^{m,m-1}$. This completes the proof of (3.3.3).

LEMMA 3.3.9. *The cycle $\mathcal{T}_{\vec{h}}$ ($\vec{h}$ as in (3.3.2)) on the hypersurface $W_m$ may be seen to have non-trivial Abel-Jacobi image by direct computation. The class of $\mathcal{T}_{\vec{h}}$ in $Gr^m(W_m)[2]$ is zero.*

PROOF: The first statement in the lemma offers an alternative way to establish item (2) in the proof of 3.0 in a limited number of cases. I have emphasized the methodological approach of §3.1, since the present lemma is based on the fortunate accident that the variety $H_m = 0$ is remarkably easy to analyze.

Begin with the second assertion of the lemma. Note that the codimension $m-1$ linear space $x_3 = x_5 = ... = x_{2m-1} = 0$ cuts $W_m$ in the cone,$K_m$ , over $W_1$ with vertex $\mathbf{P}^{m-2}$ defined by $x_0 = x_1 = x_2 = x_3 = x_5 = ...x_{2m-1} = 0$. The cycle $\mathcal{Z}_{\vec{h}}$ on $W_m$ consists of the two rulings in $K_m$ over the points $q_1 = (0:0:1), q_2 = (0:-1:1)$ of $W_1$ each taken with multiplicity $d/2$. As a linear space section $i^*_{W_m/\mathbf{P}^{2m}}L$ we may choose the ruling over $q_1$ taken with multiplicity $d$. Thus $\mathcal{T}_{\vec{h}} = \mathcal{Z}_{\vec{h}} - i^*_{W_m/\mathbf{P}^{2m}}L$ is the difference of the rulings over $q_1$ and $q_2$ taken with multiplicity $d/2$. Write $E$ for the exceptional divisor of the $\mathbf{P}^{m-1}$-bundle over $W_1$, $\tilde{K}_m$, obtained by blowing up $K_m$ along the vertex. The obvious intersection computation shows that the strict transform of $\mathcal{T}_{\vec{h}}$ in $\tilde{K}_m$ is numerically and hence algebraicly equivalent to zero. This proves the second assertion.

In effect we have just shown that the cycle $\mathcal{T}_{\vec{h}}$ may be described as $\eta_*((d/2)(q_1 - q_2))$. It is apparent from the proof of (3.3.5(1)) that $p_*q^*\sigma^*\sigma_*q_*p^*$ is multiplication by the odd number $(1-d)$ on the intermediate Jacobian $J^{m-1}(W_{m-1})$. Iterating gives that $\eta^* \circ \eta_*$ acts by multiplication by $(1-d)^{m-1}$ on $J^1(W_1)$. To show that the Abel-Jacobi image of $\mathcal{T}_{\vec{h}}$ is not zero, it suffices to show $(1-d)^{m-1}(d/2)(q_1 - q_2)$ is not zero in $J^1(W_1)$.

Since the canonical quotient map

$$W_1 \to C := < \gamma_1^{2(d-1)} > \backslash W_1$$

is totally ramified at the points $p_i \in C$ below $q_i$, we conclude that the cycle, $(d/2)(q_1 - q_2)$, is the pull back of the cycle, $(p_1 - p_2)$, and that the pullback map on Jacobians is injective. But $C$ is the a double cover of $< \gamma_1^{d-1} > \backslash W_1 \simeq \mathbf{P}^1$ branched at $d \geq 4$ distinct points (including the two points $p_1$ and $p_2$). Thus $p_1 - p_2$ gives rise to a non-trivial two torsion point of $J^1(C)$, whence $(1-d)^{m-1}(d/2)(q_1 - q_2)$ is not zero in $J^1(W_1)$.

### 3.4 Further remarks and open problems.

REMARK 3.4.1: We prove the assertions about the singularities of (3.1.1) when the $g_i$'s are in general position. We may assume that $d_i \leq d_{-i}$ for $i \geq 0$. By general position we arrange that the subscheme of $P$, $Z$ (respectively $R$), defined by the ideal $(g_0, ..., g_m)$ (respectively $\mathbf{g} := (g_{-m}, ..., g_m)$) is non-singular of codimension $m+1$ (respectively $2m+1$). A general member, $G$, of the linear system $|\mathcal{I}_R\mathcal{I}_Z(d)| \subset |\mathcal{I}_R^2(d)| = (\mathbf{g}^2)_d$ [**Schl**,2.4] has only ordinary double points and is non-singular away from $R$ [**Schl**,2.5]. We may assume that the sequence $d_0, ..., d_m$ is decreasing. Let $n \geq 0$ be maximal such that $d_n = d/2$. Write

$$G = Q(g_{-n}, ..., g_n) + \sum_{n+1 \leq i \leq m} g_i p_{-i}$$

with $p_{-i} \in \mathbf{g}_{d_{-i}}$ and $Q$ a non-degerate quadratic form. A linear transformation brings the quadratic form into the normal form $Q = y_0^2 + \sum_{1\le i\le n} y_i y_{-i}$. Thus $G$ has the form (3.1.1) as desired.

OPEN PROBLEM 3.4.2: Suppose that a hypersurface $V \subset \mathbf{P}^{2m}$ has only ordinary double point singularities. It would be interesting to compute the two torsion group $H_{2m}(V,\mathbf{Z})_{tors}$, and if the degree of $V$ is even, to relate it to $H_{2m+2}(X,\mathbf{Z})$ where $X$ is the double cover of $\mathbf{P}^{2m+1}$ branched along $V$. It would also be interesting to know if $H_{2m}(V,\mathbf{Z})_{tors}$ is generated by the image of $CH_m(V)[2]$. Related questions may be posed in the context of double covers of $\mathbf{P}^{2m}$ branched along a nodal hypersurface $X$.

OPEN PROBLEM 3.4.3: It would be interesting to know if the monodromy representation in the middle cohomology with $\mathbf{Q}$-coefficients for a general pencil of hypersurface sections on an even dimensional nodal hypersurface $V \subset \mathbf{P}^{2m+1}$ is irreducible. If so, this would immediately yield stronger results than obtained in §3.2-3.3.

OPEN PROBLEM 3.4.4: The question of whether or not $Gr^m(W)_{tors} = 0$ for a sufficiently general hypersurface $W \subset \mathbf{P}^{2m}_{\mathbf{C}}$ of large degree is especially interesting in light of the results of M. Green [**Gre**]. Unfortunately, I have no results about general hypersurfaces.

## 4. Varieties over finite fields.

The purpose of this section is to produce examples of smooth, projective varieties $W_{\mathbf{F}_\nu}$, over finite fields $\mathbf{F}_\nu$, with $Gr^r(W_{\bar{\mathbf{F}}_\nu})_l \neq 0$. The idea is to begin with a variety $W_K$ defined over a global field $K$ and a cycle $z_K \in Z^r(W_K)_{hom}$. Suppose that the cycle class

$$c^r_{W_K,0}(z_K) \in H^1(G_K, H^{2r-1}(W_{\bar{K}}, \mathbf{Z}_l(r)))$$

defined in (1.2.6) has infinite order. Let $\nu$ be a place of good reduction and let $z_\nu$ denote the specialization of $z_K$ to the fiber $W_{\mathbf{F}_\nu}$. Under certain hypotheses we verify that there is a set of places $\mathcal{V}$, having positive Dirichlet density, for which $\nu \in \mathcal{V}$ implies $c^r_{W_{\mathbf{F}_\nu},0}(z_\nu) \neq 0$. By bringing correspondences into play and applying (1.2.9) we find examples where the class $z_\nu \in Gr^r(W_{\bar{\mathbf{F}}_\nu}) \otimes \mathbf{Z}_l$ does not vanish. We will concentrate on the case that $W$ is a threefold self-product of an elliptic curve and $r = 2$. In this case $Gr^r(W_{\bar{\mathbf{F}}_\nu}) \otimes \mathbf{Z}_l \simeq Gr^r(W_{\bar{\mathbf{F}}_\nu})_l$ (1.1.4).

**(4.1) Specialization.** Fix a smooth, projective, geometrically integral variety $W$ over a global field $K$. Let $l$ be a prime, $l \neq char(K)$. Let $S$ denote a finite set of places of $K$ which includes all places of bad reduction of $W_K$ as well as any places which divide $l$ and any archimedean places. Denote by $\mathcal{O}_\nu$ the local ring of $K$ at a place $\nu \notin S$ and by $W_{\mathcal{O}_\nu}$ a smooth model of $W$ over $\mathcal{O}_\nu$. A codimension $r$ subvariety $z_K \subset W_K$ has closure $\tilde{z} \subset W_{\mathcal{O}_\nu}$ flat over $\mathcal{O}_\nu$ with special fiber $z_\nu \subset W_{\mathbf{F}_\nu}$ also of codimension $r$. Thus there is a specialization map $sp : Z^r(W_K) \to Z^r(W_{\mathbf{F}_\nu})$. It is compatible with rational equivalence [**Fu**] and with the cycle class map to cohomology [**Gr-D**,2.3.8]. Furthermore if $K_\nu$ denotes the completion of $K$ at $\nu$ there is a canonical map [**Se1**,II1.1]

$$\epsilon_\nu : H^1(G_K, H^{2r-1}(W_{\bar{K}}, \mathbf{Z}_l(r))) \to H^1(G_{K_\nu}, H^{2r-1}(W_{\bar{K}}, \mathbf{Z}_l(r))).$$

Write $I_\nu$ for the inertia group of $K_\nu$. Since $\nu \notin S$, $H^1(I_\nu, H^{2r-1}(W_{\bar{K}}, \mathbf{Z}_l(r)))^{G_{\mathbf{F}_\nu}} \simeq 0$ [**Bl1**,§1]. By the inflation restriction sequence the natural injection

$$\beta : H^1(G_{\mathbf{F}_\nu}, H^{2r-1}(W_{\bar{\mathbf{F}}_\nu}, \mathbf{Z}_l(r))) \to H^1(G_{K_\nu}, H^{2r-1}(W_{\bar{K}}, \mathbf{Z}_l(r)))$$

is an isomorphism. Define $\gamma_\nu = \beta^{-1} \circ \epsilon_\nu$.

We now turn to the compatibility between the cycle class map and specialization. In the examples of this section $W$ is always an Abelian variety. Hence the following statement will suffice for our purposes.

PROPOSITION 4.1.1. *Suppose that $W_{\mathcal{O}_\nu}/\mathcal{O}_\nu$ is an Abelian scheme of relative dimension $d > 2r$. Let $P$ be a cycle on $W_{\mathcal{O}_\nu} \times_{\mathcal{O}_\nu} W_{\mathcal{O}_\nu}$ which is a linear combination of graphs of Abelian scheme endomorphisms. Let $z_K \in Z^r(W_K)_{hom}$ be a cycle with the property that the closure in $W_{\mathcal{O}_\nu}$ of each irreducible component of the support of $z_K$ is smooth over Spec $\mathcal{O}_\nu$. Then $\gamma_\nu(c^r_{W_K,0}(P_* z_K)) = c^r_{W_{\mathbf{F}_\nu},0}(P_* z_\nu)$.*

PROOF: [**B-S-T**, Prop. 3.7]

It is often the case that $c^r_{W_K,0}(P_* z_K) \in H^1(G_K, H^{2r-1}(W_{\bar{K}}, \mathbf{Z}_l(r)))$ has infinite order. The following lemma shows that $c^r_{W_{\mathbf{F}_\nu},0}(P_* z_\nu)$ is a torsion class.

LEMMA 4.1.2. *$H^1(G_{\mathbf{F}_\nu}, H^{2r-1}(W_{\bar{\mathbf{F}}_\nu}, \mathbf{Z}_l(r))$ is a finite group.*

PROOF: Write $\phi \in G_{\mathbf{F}_\nu}$ for the Frobenius element. $H^1(G_{\mathbf{F}_\nu}, H^{2r-1}(W_{\bar{\mathbf{F}}_\nu}, \mathbf{Z}_l(r)) \simeq H^{2r-1}(W_{\bar{\mathbf{F}}_\nu}, \mathbf{Z}_l(r)/(\phi - 1)H^{2r-1}(W_{\bar{\mathbf{F}}_\nu}, \mathbf{Z}_l(r)$. By the Riemann hypothesis $(\phi - 1)$ acts invertibly on $H^{2r-1}(W_{\bar{\mathbf{F}}_\nu}, \mathbf{Z}_l(r) \otimes \mathbf{Q}_l$. The assertion follows.

Set $N_{\bar{K}} = P_* H^{2r-1}(W_{\bar{K}}, \mathbf{Z}_l(r))$ and $N_{\bar{\mathbf{F}}_\nu} = P_* H^{2r-1}(W_{\bar{\mathbf{F}}_\nu}, \mathbf{Z}_l(r))$.

PROPOSITION 4.1.3. *Suppose that the image of the Galois representation $\rho : G_K \to Aut(N_{\bar{K}})$ contains an open subgroup of the homotheties. Fix an integer $m > 0$. If $f \in H^1(G_K, N_{\bar{K}})$ has infinite order, then the set of places $\nu \notin S$, for which $l^m \gamma_\nu(f) \in H^1(G_{\mathbf{F}_\nu}, N_{\bar{\mathbf{F}}_\nu})$ does not vanish, contains a subset of positive Dirichlet density.*

PROOF: For each positive integer, $n$, write $K_n$ for the fixed field of the kernel of the Galois representation, $\rho_n : G_K \to Aut(N_{\bar{K}}/l^n N_{\bar{K}})$. Define $f_n$ to be the image of $f$ under the map $H^1(G_K, N_{\bar{K}}) \to H^1(G_K, N_{\bar{K}}/l^n)$. Define $L_n$ to be the fixed field of $Ker(f_n|_{G_{K_n}})$.

LEMMA 4.1.4. *$L_n/K$ is Galois.*

PROOF: From the exact sequence

$$(4.1.5) \quad 0 \to H^1(G_{K_n/K}, N_{\bar{K}}/l^n) \to H^1(G_K, N_{\bar{K}}/l^n) \to Hom(G_{K_n}, N_{\bar{K}}/l^n)^{G_{K_n/K}},$$

we see that for $\gamma \in G_K$ and $\sigma \in G_{K_n}$ we have $f_n(\gamma\sigma\gamma^{-1}) = \gamma f_n(\sigma)$. The assertion follows.

Given a place $\nu$ of $K$, unramified in $L_n/K$, write $Frob_\nu \subset G_{L_n/K}$ for the Frobenius conjugacy class. Suppose that $\nu$ splits completely in $K_n$. Now $l^m f_n(\sigma)$ is either 0 for all $\sigma \in Frob_\nu$ or is non-zero for all $\sigma \in Frob_\nu$. In the latter case we write $l^m f_n(Frob_\nu) \neq 0$. Let $\mathcal{V}_n$ denote the set of places $\nu$ of $K$ which satisfy

(1) $\nu$ does not divide $l$
(2) $W_K$ has good reduction at $\nu$
(3) $\nu$ is unramified in $L_n$

(4) $\nu$ splits completely in $K_n$
(5) Any Frobenius element, $Frob_\mu$, at a place $\mu$ of $L_n$ above $\nu$ satisfies $l^m f_n(Frob_\mu) \neq 0$.

LEMMA 4.1.6. *Assume that the image of $\rho$ contains an open subgroup of the homotheties. Then for large $n$, the set, $\mathcal{V}_n$, has positive Dirichlet density.*

PROOF: We may ignore the first three conditions in the definition of $\mathcal{V}_n$ since they exclude only finitely many places. By the Tchebotarev density theorem applied to the extension $L_n/K$, it would suffice to show that for large $n$ the image of $l^m f_n$ under the right hand arrow in (4.1.5) is not zero. By hypothesis there is $u \in \mathbf{Z}_l^*$, a positive integer $n_0$, and a homothety $\eta \in im(\rho)$ with $\eta - Id = ul^{n_0} Id$. For each $n \geq 1$ there is an element $\sigma_n$ of the center of $G_{K_n/K}$ with $\rho_n(\sigma_n) = \eta \mod l^n End(N_{\bar{K}})$. A standard lemma in group cohomology now implies $\eta - Id \in End_{G_K}(N_{\bar{K}})$ annihilates $H^1(G_{K_n/K}, N_{\bar{K}}/l^n)$ [**La**,V Thm. 5.1]. Hence multiplication by $l^{n_0}$ annihilates $H^1(G_{K_n/K}, N_{\bar{K}}/l^n)$. Since $f$ has infinite order, the image, $f_n$, of ($f \mod l^n$) under the inclusion $H^1(G_K, N_{\bar{K}})/l^n \to H^1(G_K, N_{\bar{K}}/l^n)$ is not annihilated by the homothety $l^{m+n_0}$ when $n >> 0$. By applying the endomorphism, multiplication by $l^{m+n_0}$, to the exact sequence (4.1.5), one deduces that $l^m f_n|_{G_{K_n}} \neq 0$ for large $n$.

Define the subgroup of cohomology unramified away from $S$ by

$$H^1(G_K, N_{\bar{K}}/l^n)_S := \text{Ker } : H^1(G_K, N_{\bar{K}}/l^n) \to \prod_{\nu \notin S} H^1(I_\nu, N_{\bar{K}}/l^n).$$

Write $S_n$ for the places of $K_n$ over $S$ and $\mu$ for a place of $K_n$ over $\nu$.

LEMMA 4.1.7.
(1) *The image of $H^1(G_K, N_{\bar{K}})/l^n \to H^1(G_K, N_{\bar{K}}/l^n)$ is contained in $H^1(G_K, N_{\bar{K}}/l^n)_S$.*
(2) *There is a commutative diagram*

$$(4.1.8)\qquad \begin{array}{ccccc} H^1(G_K, N_{\bar{K}})/l^n & \longrightarrow & H^1(G_K, N_{\bar{K}}/l^n)_S & \longrightarrow & Hom(G_{K_n}, N_{\bar{K}}/l^n)_{S_n} \\ {\scriptstyle \gamma_\nu}\downarrow & & {\scriptstyle \vartheta_\nu}\downarrow & & {\scriptstyle \vartheta_\mu}\downarrow \\ H^1(G_{\mathbf{F}_\nu}, N_{\bar{\mathbf{F}}_\nu})/l^n & \longrightarrow & H^1(G_{\mathbf{F}_\nu}, N_{\bar{\mathbf{F}}_\nu}/l^n) & \longrightarrow & Hom(G_{\mathbf{F}_\mu}, N_{\bar{\mathbf{F}}_\nu}/l^n). \end{array}$$

PROOF: (1) [**Ra**,§1].
(2) The map $\vartheta_\nu$ is defined as the composition of the restriction map

$$H^1(G_K, N_{\bar{K}}/l^n)_S \to [\text{Ker } : H^1(G_{K_\nu}, N_{\bar{K}}/l^n) \to H^1(I_\nu, N_{\bar{K}}/l^n)]$$

with the inverse of the tautological isomorphism,

$$\beta_n : H^1(G_{\mathbf{F}_\nu}, N_{\bar{\mathbf{F}}_\nu}/l^n) \to [\text{Ker } : H^1(G_{K_\nu}, N_{\bar{K}}/l^n) \to H^1(I_\nu, N_{\bar{K}}/l^n)].$$

The map $\vartheta_\mu$ is defined analgously. It is clear that the diagram commutes.

To prove the proposition it suffices to show that the image of $l^m\gamma_\nu(f)$ in $Hom(G_{\mathbf{F}_\mu}, N_{\bar{\mathbf{F}}_\nu}/l^n)$ is not zero for each $\nu \in \mathcal{V}_n$. Now $Frob_\mu \in G_{L_n/K_n}$ is an element in the conjugacy class $Frob_\nu \subset G_{K_n/K}$. By the definition of $\mathcal{V}_n$, $l^m f_n(Frob_\mu) \neq 0$. The existence of an element $\phi \in G_{K_n}$ which maps to the Frobenius in $G_{\mathbf{F}_\mu}$ gives a splitting, $s$, of the canonical map $G_{K_n} \to G_{\mathbf{F}_\nu}$. Thus $f_n \circ s = \vartheta_\mu$. Now $l^m f_n(Frob_\mu) \neq 0$ implies $l^m f_n \circ s = \vartheta_\mu(l^m f_n)$ is not zero. By (4.1.8) $l^m\gamma_\nu(f) \neq 0$.

**(4.2) The threefold product of a CM elliptic curve.** Consider the smooth plane curves over $\mathbf{Z}[1/2]$,

$$E : x_1^2 x_2 = x_0^3 + x_0 x_2^2,$$
$$C : t_0^4 + t_1^4 + t_2^4 = 0,$$

the threefold $W = E^3$, and the map $\varrho : C \to W$ defined by

$$\varrho(t_0 : t_1 : t_2) = [(-t_2 t_1^2 : t_0^2 t_1 : t_2^3), (-t_0 t_2^2 : t_1^2 t_2 : t_0^3), (-t_1 t_0^2 : t_2^2 t_0 : t_1^3)].$$

View $E$ as an abelian scheme with neutral element $(0:1:0)$ and let $\iota \in Aut(W)$ denote inversion in the obvious group law. Define $z = \varrho(C) - \iota_* \varrho(C)$. For any separably closed field, $L$, of characteristic not equal to 2, the cycle $z_L$ on $W_L$ is homologous to zero. This situation (actually a slight variant) was studied by Bloch [**Bl1**]. From his results we shall deduce

PROPOSITION 4.2.1. *Let $l \in \{3, 5, 7, 11, 17\}$. Fix an arbitrary positive integer $m$. The set of places $p$ of $\mathbf{Q}$ for which the class $l^m z_p \in Gr^2(W_{\bar{\mathbf{F}}_p})_l$ does not vanish contains a set of positive Dirichlet density.*

PROOF: The first step is to construct an approriate self-correspondence of $W$. Consider the automorphism $i \in Aut(E_{\mathbf{Z}[1/2,\sqrt{-1}]})$ given by $(x_0 : x_1 : x_2) \to (-x_0 : \sqrt{-1}x_1 : x_2)$. Define $\sigma_0 = (i^2, i, i)$ and $\sigma_1 = (i, i^2, i) \in Aut(W)$. Set

$$P = \sum_{(a,b) \in \mathbf{Z}/4 \times \mathbf{Z}/4} \sigma_0^a \sigma_1^b$$

and $N_L = P_* H^3(E_L^3, \mathbf{Z}_l(2))$ for any separably closed field $L$ of characteristic different from 2.

LEMMA 4.2.2.

(1) *$P$ is defined over $\mathbf{Z}[1/2]$.*
(2) *$P^2 = 16P \in \mathbf{Z}[\mathbf{Z}/4 \times \mathbf{Z}/4]$.*
(3) *$P_* z = 16z$.*
(4) *If $p = 1 \mod 4$, $P_{\mathbf{F}_p} \in Z^3(W_{\mathbf{F}_p} \times W_{\mathbf{F}_p})$ is transcendental for codimension 2 cycles in the sense of (1.2.10).*

PROOF: (1) and (2) are evident. The third assertion follows from $P \circ \iota = \iota \circ P$ and $\varrho \circ \tilde{\sigma}_j = \sigma_j \circ \varrho$, where $\tilde{\sigma}_j \in Aut(C_{\mathbf{Z}[1/2,\sqrt{-1}]})$ is defined for $j \in \{0,1\}$ by $t_i \circ \tilde{\sigma}_j = (\sqrt{-1})^{\delta_{ij}} t_i$.

To prove (4) we note that the curve $E$ has complex multiplication from $\mathbf{Z}[\sqrt{-1}]$ and good reduction away from 2. By the theory of complex multiplication [**La2**,§10], if $p = 1 \mod 4$, $E$ has good, ordinary reduction. Write $\gamma_p$ and $\bar{\gamma}_p$ for the eigenvalues of $Frob_p^{-1} \in G_{\mathbf{Q}}$ acting on $H^1(E_{\bar{\mathbf{F}}_p}, \bar{\mathbf{Q}}_l)$. Now $\gamma_p$ and $\bar{\gamma}_p$ generate distinct principal ideals of $\mathbf{Z}[\sqrt{-1}]$ into which the prime $p$ splits. One checks easily that $\gamma_p^3/p$ and $\bar{\gamma}_p^3/p$ are the eigenvalues for $Frob_p^{-1}$ acting on $N_{\bar{\mathbf{F}}_p} \otimes \bar{\mathbf{Q}}_l(-1)$. Since these are not algebraic integers, (4) follows from (1.2.11).

LEMMA 4.2.3. *For* $l \in \{3, 5, 7, 11, 17\}$, $c^2_{W_{\mathbf{Q}},0}(P_* z_{\mathbf{Q}}) \in H^1(G_{\mathbf{Q}}, N_{\bar{\mathbf{Q}}})$ *is not zero.*

PROOF: By (4.1.1) it suffices to check that $c^2_{W_{\mathbf{F}_p},0}(P_* z_p) \neq 0$ for a prime $p$ of good reduction. It is simplest to do this when $p = -1 \mod 4$, since in this case $E_{\mathbf{F}_p}$ is supersingular and $P_{\mathbf{F}_p}$ is not transcendental for codimension 2 cycles. Write $\pi \in End(E_{\mathbf{F}_p})$ for the Frobenius endomorphism, define $h : E^2_{\mathbf{F}_p} \to E^3_{\mathbf{F}_p}$ by $h(a, b) = (a, \pi a, b)$, and write $pr_2 : E^2_{\mathbf{F}_p} \to E_{\mathbf{F}_p}$ for projection on the second factor. It would suffice to show that $pr_{2*} \circ h^*(c^2_{W_{\mathbf{F}_p},0}(P_* z_p)) \neq 0$. By (4.2.2 (3)) one may replace $P_* z_p$ by $z_p$. By the functoriality of the cycle class map one is reduced to computing $c^1_{E_{\mathbf{F}_p},0}(pr_{2*} \circ h^*(z_p))$. To show that this is non-zero we need only show that the class of $pr_{2*} \circ h^*(z_p)$ in $Alb_E(\bar{\mathbf{F}}_p) \otimes \mathbf{Z}_l$ does not vanish [**B-S-T**,1.9(7)]. Since $z_p$ is defined over the prime field and since a supersingular elliptic curve over $\mathbf{F}_p$ has $p + 1$ rational points this class will vanish unless $l | (p + 1)$. Note that the prime $l = 3, 5, 7, 11$, respectivley 17 divides $p + 1$ when $p = 11, 19, 83, 43$, respectively 67. In these cases computations of Bloch [**Bl1**, p.103] and Top [**B-S-T**, ] show that the class of $pr_{2*} \circ h^*(z_p)$ in $Alb_E(\bar{\mathbf{F}}_p) \otimes \mathbf{Z}_l$ is non-zero.

REMARK 4.2.4: When $p = 1 \mod 4$ the argument in the previous lemma cannot be pushed through because the map $pr_{2*} \circ h^* : N_{\bar{\mathbf{F}}_p} \to H^1(E_{\bar{\mathbf{F}}_p}, \mathbf{Z}_l(1))$ is zero.

For the remainder of §4.2 $K = \mathbf{Q}(\sqrt{-1})$.

LEMMA 4.2.5. *For* $l > 2$, $H^1(G_K, N_{\bar{K}})_{tors} = 0$.

PROOF: The exact sequence

$$0 \to N_{\bar{K}} \to N_{\bar{K}} \otimes \mathbf{Q}_l \to N_{\bar{K}} \otimes \mathbf{Q}_l/\mathbf{Z}_l \to 0,$$

gives rise to a surjective map $(N_{\bar{K}} \otimes \mathbf{Q}_l/\mathbf{Z}_l)^{G_K} \to H^1(G_K, N_{\bar{K}})_{tors}$. For any finite place $\mu$ of $K$

$$(N_{\bar{K}} \otimes \mathbf{Q}_l/\mathbf{Z}_l)^{G_K} \subset (N_{\bar{K}} \otimes \mathbf{Q}_l/\mathbf{Z}_l)^{Frob_\mu}.$$

Consider the case that $\mu = q\mathbf{Z}[\sqrt{-1}]$, where $q = -1 \mod 4$ is a rational prime. Then $Frob_\mu$ acts by multiplication by the scalar $-q$ on $H^1(E_{\bar{K}}, \mathbf{Z}_l(1))$ because $E_{\mathbf{F}_q}$ is supersingular. The action of $Frob_\mu$on $N_{\bar{K}}$ is by multiplication by $-q^3/N^K_{\mathbf{Q}} q = -q$. Thus $(N_{\bar{K}} \otimes \mathbf{Q}_l/\mathbf{Z}_l)^{Frob_3} = 0$ for $l \notin \{2, 3\}$ and $(N_{\bar{K}} \otimes \mathbf{Q}_l/\mathbf{Z}_l)^{Frob_7} = 0$ for $l \notin \{2, 7\}$.

In order to apply (4.1.3) we note that the image of $\rho : G_K \to Aut(N_{\bar{K}})$ contains an open subgroup of the homotheties. Indeed, this follows from the fact that the image of $\kappa$ contains an open subgroup of the homotheties [**Se4**,§4.5]. As a by-product we see that $N_{\bar{K}}^{G_K} \simeq 0$. The inflation-restriction sequence then implies that the restriction map

$$H^1(G_{\mathbf{Q}}, N_{\bar{K}}) \to H^1(G_K, N_{\bar{K}})$$

is injective. By (4.2.3) and (4.2.5) the cycle class $c^2_{W_K,0}(P_* z_K) \in H^1 G_K, N_{\bar{K}})$ has infinite order. Now (4.1.1) and (4.1.3) imply that there is a set of places , $\mathcal{V}$, of $K$, having positive Dirichlet density, with the property that $l^m c^2_{W_{\mathbf{F}_\nu},0}(P_* z_\nu) \in H^1(G_{\mathbf{F}_\nu}, N_{\bar{\mathbf{F}}_\nu})$ does not vanish for any $\nu \in \mathcal{V}$. Write $\mathcal{U}_0$ for the set of rational primes, $r = 1 \mod 4$, which lie below $\mathcal{V}$. The places $\mathcal{V}_0 \subset \mathcal{V}$ lying above $\mathcal{U}_0$ consists of the unramified places of $\mathcal{V}$ of degree one. This subset has positive Dirichlet density, since the set of all primes of $K$ of degree greater than one has Dirichlet density zero [**Co**,§19]. Thus the subset of rational primes, $\mathcal{U}_0$, also has positive Dirichlet density.

Pullback gives an obvious isomorphism of pairs $(z_r, W_{\mathbf{F}_r}) \simeq (z_\nu, W_{\mathbf{F}_\nu})$. By Soulé's theorem (1.1.4), $z_r$ gives a class in $T^2(W_{\bar{\mathbf{F}}_r})$. As $P_*$ is transcendental by (4.2.2), the non-vanishing of $l^m c^2_{W_{\mathbf{F}_r},0}(P_* z_r)$ implies that the class $l^m z_r \in Gr^2(W_{\bar{\mathbf{F}}_r})$ does not vanish for any $r \in \mathcal{U}_0$. This proves (4.2.1).

**(4.3) The threefold product of an elliptic curve without CM.** It is natural to ask if the method of §4.2 extends to the case that $E$ does not have complex multiplication. In this section we show that this is indeed the case. Unfortunately, there is no longer a useful, globally defined correspondence which is transcendental for codimension 2 cycles on $E^3_{\bar{\mathbf{F}}_\nu}$. Consequently, the technical problems become more involved. Inspite of this, the non-CM case is important, because most elliptic curves over global fields fall into this class.

Let $E/K$ be an elliptic curve over a global field. Let $l > 3$ be a prime distinct from the characteristic of $K$. Assume that the image of the Galois representation $\kappa : G_K \to Aut(E_{\bar{K}}[l^\infty]) \simeq GL(2, \mathbf{Z}_l)$ is an open subgroup. This will be the case if $K$ is a number field and $E_{\bar{K}}$ does not have complex multiplication [**Se2**], or if $K$ is the function field of a curve over a finite field and the $j$-invariant of $E$ is transcendental. Suppose that $z_K$ is a nullhomologous 1-cycle on $W = E^3_K$ whose support is a union of smooth curves. Under certain additional hypotheses we shall show that there is a set of places, $\mathcal{V}$, having positive Dirichlet density, for which the specialization $z_\nu$ is not algberaicly equivalent to zero for any $\nu \in \mathcal{V}$.

A preliminary step is to eliminate certain 'uninteresting submotives'of $H^3(W)$. This is done with a globally defined correspondence $P \in Z^3(E^3_K \times E^3_K)$. The construction of $P$ is in part inspired by some work of B. Gross. $P$ will have the following properties:

(1) $P$ is a linear combination of Abelain scheme endomorphisms;
(2) $P \circ P = 3P$;
(3) $P_* H^3(E^3_{\bar{K}}, \mathbf{Z}_l) \simeq Sym^3 H^1(E_{\bar{K}}, \mathbf{Z}_l)$.

The construction of $P$ is as follows. For each subset $T \subset \{1,2,3\}$ let $p_T : E^3 \to E^{|T|}$ denote the projection obtained by omitting the factors not in $T$. For example, $p_{13}(x_1, x_2, x_3) = (x_1, x_3)$. $p_\emptyset : E^3 \to \mathrm{Spec}(K)$ is the structure map. Define inclusions $q_T : E^{|T|} \to E^3$ using the neutral element $e$ to fill in the missing coordinate. For example, $q_{13}(x_1, x_2) = (x_1, e, x_2)$. Let $P'_T$ denote the graph of the morphism $q_T \circ p_T : W \to W$, and define

$$P' = P'_{123} - P'_{12} - P'_{13} - P'_{23} + P'_1 + P'_2 + P'_3 - P'_\emptyset$$

as a 3-cycle on $W \times W$. Then $P'$ gives rise to the endomorphism $(q_{123} \circ p_{123})_* - (q_{12} \circ p_{12})_* - ... + (q_\emptyset \circ p_\emptyset)_*$ of $Z_\cdot(W)$ respectively $H^\cdot(W, \mathbf{Z}_l(\cdot))$.

Denote by $\tau : W \to W$ the map $(x_1, x_2, x_3) \to (x_2, x_3, x_1)$. Define two new self-correspondences of $W$ by

$$\begin{aligned} P'' &= \mathrm{Id} + \tau + \tau^2 \\ P &= P'' \circ P'. \end{aligned}$$

It is clear from the construction that $P$ specializes well to places of $K$ where $E_K$ has good reduction. The properties of $P$ listed above are straight foward to verify [**B-S-T**, 2.3].

Set $N_{\bar{K}} = Sym^3 H^1(E_{\bar{K}}, \mathbf{Z}_l)(2)$. The action of $G_K$ on $N_{\bar{K}}$ factors through $\kappa : G_K \to Aut(H^1(E_{\bar{K}}, \mathbf{Z}_l)) \simeq GL(2, \mathbf{Z}_l)$. The relevant action of $GL(2, \mathbf{Z}_l)$ is by twisting the third symmetric power representation with the character $det^{-2}$. Set $f = c^2_{W_K,0}(P_* z_K) \in H^1(G_K, N_{\bar{K}})$.

PROPOSITION 4.3.1. *If $f$ has infinite order, then the set of places, $\nu$ of $K$, where the specialization, $l^m P_* z_\nu$, is not algebraicly equivalent to zero contains a subset of positive Dirichlet density.*

PROOF: Let $K_n$ denote the fixed field of the kernel of the Galois representation, $\kappa_n : G_K \to Aut(H^1(E_{\bar{K}}, \mathbf{Z}/l^n))$, and let $L_n$ denote the fixed field of the kernel of $f_n|_{G_{K_n}} \in Hom(G_{K_n}, N/l^n)^{G_{K_n/K}}$. Write $\mathcal{V}_n$ for the set of places $\nu$ of $K$ which satisfy

(1) $\nu$ does not divide $l$
(2) $E_K$ has good reduction at $\nu$
(3) $\nu$ is unramified in $L_n$
(4) $\nu$ splits completely in $K_n$
(5) Any Frobenius element, $Frob_\mu$, at a place $\mu$ of $L_n$ above $\nu$ satisfies $l^m f_n(Frob_\mu) \neq 0$.

For $n$ large, $\mathcal{V}_n$ has positive Dirichlet density (4.1.6). It follows from (4.1.3) and (4.1.1) that $\nu \in \mathcal{V}_n$ implies $c^2_{W_{\mathbf{F}_\nu},0}(l^m P_* z_\nu) \neq 0$. Consequently, $l^m P_* z_\nu$ is not rationally equivalent to zero. It is more difficult to discribe a set of places, $\mathcal{V}'_n \subset \mathcal{V}_n$ having positive Dirichlet density, with the property that $l^m P_* z_\nu$ is not algebraicly equivalent to zero for any $\nu \in \mathcal{V}'_n$. In order to do this we begin with a cohomological lemma in the style of Bashmakov [**Ba**].

Write $K_\infty$ for the fixed field of $\mathrm{Ker}(\kappa)$ and $L_\infty$ for the union of the $L_n$'s. A choice of continuous crossed homomorphism $\tilde{f} \in Z^1(G_K, N_{\bar{K}})$ representing $f$ gives rise to a continuous homomorphism to the semi-direct product,

$$\varphi : G_K \to N_{\bar{K}} \cdot GL(2, \mathbf{Z}_l), \ \varphi(g) = (\tilde{f}(g), \kappa(g)),$$

where the action of $GL(2, \mathbf{Z}_l)$ on $N_{\bar{K}}$ has been discribed above. Changing the choice of $\tilde{f}$ by a coboundary amounts to composing $\varphi$ with conjugation by an element of $N_{\bar{K}}$.

LEMMA 4.3.2. *The image of $\varphi$ is open.*

PROOF: Since $G_K$ is compact, the image is closed. It suffices to show that the image has finite index. By hypothesis, the image of $\kappa$ has finite index. Thus there is a positive integer $n_0$ and an element $g$ of the center of $G_{K_\infty/K}$ with the property that $\eta = \kappa(g)$ is a homothety and the valuation of $\eta - Id$ is $n_0$. A lemma in group cohomology shows that $\eta - Id$ annihilates $H^1(G_{K_\infty/K}, N_{\bar{K}})$ [**La**,V Thm 5.1]. By the inflation-restriction sequence, the kernel of

$$H^1(G_K, N_{\bar{K}}) \rightarrow Hom(G_{K_\infty}, N_{\bar{K}})^{G_{K_\infty/K}}$$

is annihilated by $\eta - Id$, and hence by multiplication by $l^{n_0}$. Thus $f|_{G_{K_\infty}} \in Hom(G_{K_\infty}, N_{\bar{K}})^{G_{K_\infty/K}}$ has infinite order. Consequently $f(G_{K_\infty}) \subset N_{\bar{K}}$ is a non-torsion $G_{K_\infty/K}$-submodule. Now $N_{\bar{K}} \otimes \mathbf{Q}_l$ is an irreducible module for any finite index subgroup of $GL(2, \mathbf{Z}_l)$. In particular $N_{\bar{K}} \otimes \mathbf{Q}_l$ is an irreducible $G_{K_\infty/K}$-module. Thus $f(G_{K_\infty}) \otimes \mathbf{Q}_l = N_{\bar{K}} \otimes \mathbf{Q}_l$. Hence $N_{\bar{K}}/f(G_{K_\infty})$ is annihilated by $l^r$ for some positive integer $r$. The lemma follows.

The fact that the image of $\varphi$ is large should give us enough freedom to choose places $\nu$ of $K$ where $f(Frob_\nu)$ is not annihilated by a transcendental correspondence on $W_{\mathbf{F}_\nu}$. To realize this idea requires some rather intricate considerations to which we now proceed.

For $n > r$ consider the map induced by $\varphi$,

$$\varphi_{2n,r} : G_{L_{2n}/L_r} \longrightarrow [\text{Ker} : N_{\bar{K}}/l^{2n} \cdot GL(2, \mathbf{Z}/l^{2n}) \rightarrow N_{\bar{K}}/l^r \cdot GL(2, \mathbf{Z}/l^r)].$$

By the lemma, it is an isomorphism if $r$ is sufficently large. Note also that the map

$$\kappa_{2n,n} : G_{K_{2n}/K_n} \rightarrow [\text{Ker} : GL(2, \mathbf{Z}/l^{2n}) \rightarrow GL(2, \mathbf{Z}/l^n)]$$

is an isomorphism for $n$ sufficiently large.

We now fix a positive integer $m$ as in the statement of the proposition and integers $n$ and $r$ for which $\kappa_{2n,n}$ and $\varphi_{2n,r}$ are isomorphisms and $n > r + m$.

Given $\zeta \in G_{K_{2n}/K_n}$ define a matrix $\xi \in M_2(\mathbf{Z}/l^n)$ by the equation

$$\kappa_{2n}(\zeta^{-1}) - Id = \xi l^n \cdot Id$$

in $M_2(\mathbf{Z}/l^{2n})$. Write $\xi \rightarrow \bar{\xi}$ for the anti-involution of $M_2(\mathbf{Z}/l^n)$ characterized by $\xi\bar{\xi} = det(\xi)Id$ and $\xi + \bar{\xi} = tr(\xi)Id$. In terms of matrices the involution is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Since $l > 3$ and $\kappa_{2n,n}$ is an isomorphism, we may and will chose $\zeta \in G_{K_{2n}/K_n}$ so that $\xi, \xi - \bar{\xi}, \xi + \bar{\xi}$ are units. Define $N = Sym^3(\mathbf{Z}_l^2)$. We view $\xi$ as an endomorphism of $N/l^n = Sym^3((\mathbf{Z}/l^n)^2)$ by the third symmetric power of the matrix $\xi$. Define $d = \xi\bar{\xi}, t = \xi + \bar{\xi}, b = \xi^3 + \bar{\xi}^3$ in $\mathbf{Z}/l^n$. The endomorphisms

$$Q_1 = \xi^2 - dt\xi + d^3 \qquad \text{and} \qquad Q_2 = \xi^2 = b\xi + d^3$$

of $N/l^n$ give rise to a splitting

$$N/l^n \simeq \text{ Ker } Q_1 \oplus \text{ Ker } Q_2,$$

(see (4.3.7) below).

Now choose $\theta \in l^r N/l^{2n}N$ such that the image of $l^m\theta$ in $l^r N/l^n N$ is a non-zero element of $l^r$Ker $Q_2$. As $\varphi_{2n,r}$ is an isomorphism, there is a unique $\sigma \in G_{L_{2n}/L_r}$ with $\varphi_{2n,r}(\sigma) = (\theta, \zeta)$.

DEFINITION 4.3.3: Let $\mathcal{V}'_n$ denote the set of all places of $K$ which satisfy

(1) $\nu$ is unramified in $L_{2n}/K$
(2) The Frobenius conjugacy class $Frob_\nu \subset G_{L_{2n}/K}$ contains $\sigma$
(3) $\nu$ does not divide $l$
(4) $\nu$ is a place of good reduction for $E_K$
(5) The elliptic curve $E_{\mathbf{F}_\nu}$ is not supersingular.

LEMMA 4.3.4. *$\mathcal{V}'_n$ has positive Dirichlet density.*

PROOF: (1), (3) and (4) exclude only finitely many places. (5) excludes only a set of Dirichlet density 0 [**Se2**,IV-13 exer.1]. By Tchebotarev's theorem the set of places for which (2) holds has positive Dirichlet denstiy.

The proof of (4.3.1) reduces to the following asserton.

PROPOSITION 4.3.5. *Let $l > 3$ and let $m$, $n$, and $r$ be as above. If $\nu \in \mathcal{V}'_n$ then the class $l^m P_* z_\nu \in Gr^2(W_{\bar{\mathbf{F}}_\nu})_l$ is non-zero.*

PROOF: Let $\nu \in \mathcal{V}'_n$. We may identify $\sigma \in G_{L_{2n}/L_r}$ with the Frobenius element at a place of $L_{2n}$ above $\nu$. Define $R = End(E_{\bar{\mathbf{F}}_\nu}) \simeq End(E_{\mathbf{F}_\nu})$. Write $\pi \in R$ for the geometric Frobenius. Modulo $l^{2n}$ we may identify $\pi$ with $\kappa_{2n}(\sigma^{-1})$. Since $\sigma$ maps to the neutral element in $G_{K_n/K}$, there exists $\xi \in R$ with $\pi - 1 = \xi l^n$. Now $\sigma$ was constructed so that $\xi, \bar{\xi}, \xi + \bar{\xi}, \xi - \bar{\xi}$ have invertible images in $R/l$.

The graph of the diagonal action of $\xi$ on $E^3_{\bar{K}}$, denoted $\Gamma_\xi \in Z^3(E^3_{\bar{\mathbf{F}}_\nu} \times E^3_{\bar{\mathbf{F}}_\nu})$, gives rise to an endomorphism of $Sym^3 H^1(E_{\bar{K}}, \mathbf{Z}_l) \subset H^3(E^3_{\bar{\mathbf{F}}_\nu}, \mathbf{Q}_l)$. Define integers $d = \xi\bar{\xi}, t = \xi + \bar{\xi}, b = \xi^3 + \bar{\xi}^3$. Consider the action of

$$(4.3.6) \qquad Q_1 = \Gamma_\xi^2 - dt\Gamma_\xi + d^3\Delta, \qquad Q_2 = \Gamma_\xi^2 - b\Gamma_\xi + d^3\Delta \quad \in CH^3(E^3_{\bar{\mathbf{F}}_\nu} \times E^3_{\bar{\mathbf{F}}_\nu})$$

on $N_{\bar{\mathbf{F}}_\nu} := Sym^3 H^1(E_{\bar{\mathbf{F}}_\nu}, \mathbf{Z}_l)(2)$. Define $N_i = \text{Ker } Q_i \subset N_{\bar{\mathbf{F}}_\nu}$.

LEMMA 4.3.7.

(1) *$Q_1|_{N_2} \in End(N_2)$ and $Q_2|_{N_1} \in End(N_1)$ are invertible endomorphisms.*
(2) *$N_{\bar{\mathbf{F}}_\nu} \simeq N_1 \oplus N_2$.*

PROOF: Note that $\xi \otimes 1$ acts on $H^1(E_{\bar{K}}, \mathbf{Z}_l) \otimes R \simeq R \oplus R$ by $(\xi, \bar{\xi})$. Thus $\Gamma_\xi \otimes 1$ acts on $N_{\bar{\mathbf{F}}_\nu} \otimes R \simeq Sym^3_R(H^1(E_{\bar{\mathbf{F}}_\nu}, \mathbf{Z}_l) \otimes R)(2) \simeq R^{\oplus 4}$ by $(\xi^3, \xi^2\bar{\xi}, \xi\bar{\xi}^2, \bar{\xi}^3)$. Now $Q_1$, respectively $Q_2$, act on $N_{\bar{\mathbf{F}}_\nu} \otimes R \simeq R^{\oplus 4}$ by

$$\xi^3\delta \;\oplus\; 0 \;\oplus\; 0 \;\oplus\; \bar{\xi}^3\delta, \quad \text{respectively} \quad 0 \;\oplus\; -\xi^2\bar{\xi}\delta \;\oplus\; -\xi\bar{\xi}^2\delta \;\oplus\; 0,$$

where $\delta = (\xi+\bar{\xi})(\xi-\bar{\xi})^2$. One may make the identifications $N_1 \otimes R \simeq 0 \oplus R \oplus R \oplus 0$ and $N_2 \otimes R \simeq R \oplus 0 \oplus 0 \oplus R$. Now (2) follows since $R$ is faithfully flat over $\mathbf{Z}$. Furthermore (1) is a consequence of $\xi, \xi+\bar{\xi}$, and $\xi - \bar{\xi}$ all being units in $R \otimes \mathbf{Z}_l$.

LEMMA 4.3.8. *$Q_1 \circ P \in CH^3(E^3_{\bar{\mathbf{F}}_\nu} \times E^3_{\bar{\mathbf{F}}_\nu})$ is a transcendental correspondence for codimension 2 cycles.*

PROOF: Write $\phi \in G_{\mathbf{F}_\nu}$ for the Frobenius automorphism. The action of $\pi$ and $\phi^{-1}$ on $H^{\cdot}(E_{\bar{\mathbf{F}}_\nu}, \mathbf{Q}_l)$ coincide [**Mi**,VI13.5]. The action of $\phi^{-1}$ is affected by Tate twists; the action of $\pi$ is not. Set $F = R \otimes \mathbf{Q}$. The action of the torus $F^*$ on $H^1(E_{\bar{K}}, \bar{\mathbf{Q}}_l)$ is described by the two characters $\psi, \bar{\psi}$ corresponding to the distinct field homomorphisms $F \to \bar{\mathbf{Q}}_l$. The induced action of $F^*$ on $N_{\bar{\mathbf{F}}_\nu} \otimes \bar{\mathbf{Q}}_l$ commutes with $Q_i$. The $F^*$ action on $N_2 \otimes \bar{\mathbf{Q}}_l$ is the sum of weight spaces for the characters $\psi^3$ and $\bar{\psi}^3$. Since $E$ is not supersingular, $\psi(\pi) + \bar{\psi}(\pi)$ is an integer prime to $p = char(\mathbf{F}_\nu)$ [**Wa**,§4.1]. Thus $\psi(\pi)$ and hence $\psi(\pi)^3$ may be viewed as algebraic integers not divisble by $p$. Write $p^k = N_{F/\mathbf{Q}}\pi$. The eigenvalues of $\phi^{-1}$ acting on $N_2(-1) \otimes \mathbf{Q}_l$, $\psi(\pi)^3/p^k$ and $\bar{\psi}(\pi)^3/p^k$, are not algebraic integers. The lemma follows from (1.2.11).

In order to prove (4.3.5) we shall show that $l^m Q_{1*}(c^2_{W_{\mathbf{F}_\nu}}(P_* z_\nu)) \in H^1(G_{\mathbf{F}_\nu}, N_{\bar{\mathbf{F}}_\nu})$ does not vanish when $\nu \in \mathcal{V}'_n$. By (4.1.1) it suffices to show $l^m Q_{1*}\gamma_\nu(f) \neq 0$. Since $\sigma$ is a Frobenius element at a place above $\nu$ one reduces, as in the last paragraph of the proof of (4.1.3), to showing $l^m Q_1 f_n(\sigma) \neq 0$. We deduce $f_n(\sigma) = \theta$ from

$$\varphi_{2n,r}(\sigma) = (f_r(\sigma), \kappa_{2n}(\sigma)) = (\theta, \zeta) \qquad \text{and} \qquad f_r|_{G_{K_n}} = f_n.$$

By construction, $l^m\theta \in l^{r+m}N_2/l^nN_2$ is not zero. Since $Q_1$ acts invertibly on $N_2$, $l^m Q_1 f_n(\sigma) \neq 0$, as required.

EXAMPLE 4.3.9: Consider the elliptic curve

$$E_a : zy^2 = (a^2-4)x^3 + (2a^2-4a)x^2z + (a^2-4)xz^2$$

with

$$j(E_a) = -16\frac{(a^2-12a-12)^3}{(a+2)^4(a+1)}.$$

Set $W_a = E^3_a$. By generalizing the construction of $z_{\mathbf{Q}}$ in §4.2 Top [**Top**] constructs a cycle $z_a \in Z^3(W_a)_{hom}$. In [**B-S-T**] it is shown that there are infinitely many values for $a \in \mathbf{Q}$ such that $c^2_{W_a,0}(P_* z_a) \in H^1(G_{\mathbf{Q}}, H^3(W_{a\bar{\mathbf{Q}}}, \mathbf{Z}_l(2)))$ has infinite order. Furthermore $E_a$ has complex multiplication only for finite, known set of values of $a$ [**Top**, Lemma 3.4.1]. This gives many examples where the hypotheses of (4.3.1) hold.

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