On a method of exact linearization ordinary differential equations and on a new class of higher order nonlinear evolutionary equations

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The last 30 years or so have seen an explosion of activity in the study of nonlinear phenomena. The motivation has largely come from the physical sciences and engineering, and also from problems in mathematical physics and mechanics.

Mathematical progress in the treatment of nonlinear differential equations has been swift and broadly based. The most spectacular developments have been in the field of exact analytic solutions of nonlinear equations.

Formulation of problem.

For a linear Ordinary Differential Equations (ODE), or a system of such equations, a *linear superposition principle* (LSP) allows us to represent the general solution as a linear combination in terms of a finite number of particular solutions.

It makes sense to ask whether this property is restricted to linear ODE's or whether it can be generalized. This question was posed by several mathematicians in the end of XIX century (S. Lie, [1], E.Vessiot [2], V. Ermakov [3] and others).

The problem can by formulated as follows. Consider a system of first-order ODE's

$$\frac{du}{dt} = \eta(u,t), \ \text{i.e.} \ \frac{du^k}{dt} = \eta^k(u,t), \ 1 \le k \le n, \tag{1}$$

where $u, \eta \in \mathbf{R}^n$ or $\mathbf{C}^n$.

Question: when can the general solution of this system be expressed as a function of a finite number m of particular solutions

$$u_1(t), u_2(t), \cdots, u_m(t) \tag{2}$$

and a sufficient number n of arbitrary constants $c_1, c_2, \cdots, c_n$, i.e.

$$u(t) = F(u_1(t), u_2(t), \cdots, u_m(t), c_1, c_2, \cdots c_n). \tag{3}$$

Theorem (S.Lie [1]). The system of equations (1) has a fundamental set of solutions (FSS) (2) such that its general solution is given by nonlinear superposition rule (3), if and only if the following conditions hold:

1) The right-hand side of (1) can be (possibly after a transformation) factorized to give

$$\frac{du}{dt} = Z_1(t)\xi_1(u) + Z_2(t)\xi_2(u) + \cdots + Z_r(t)\xi_r(u).$$

2) The vector fields

$$X_i = \sum_{s=1}^{n} \xi_i^s(u)\frac{\partial}{\partial u^s}$$

generate a finite-dimensional Lie algebra

$$[X_a, X_b] = \sum_{c=1}^{r} C_{ab}^c X_c, \quad 1 \leq a, b, c \leq r.$$

The number m of solutions in the fundamental set (2) satisfies the inequality

$$n \cdot m \geq r.$$

Example 1. The Riccati equation. The superposition principle for the Riccati equation is well known. Indeed, any four solutions of

$$\frac{du}{dt} = Z_1(t) + Z_2(t)u + Z_3(t)u^2,$$

e.g. u, u_1, u_2, u_3, satisfy the property that their anharmonic ratio is constant:

$$\frac{u-u_1}{u-u_2} : \frac{u_3-u_1}{u_3-u_2} = c. \tag{4}$$

Considering $u(t)$ to be the general solution and u_1, u_2, u_3 to be particular solutions, we can solve (4) for u and obtain the superposition law

$$u(t) = \frac{(u_1-u_3)u_2c + u_1(u_3-u_2)}{(u_1-u_3)c + (u_3-u_2)},$$

where $c \in \mathbf{R}$ is a constant related to the initial value $u(t_0)$ of the solution $u(t)$.

This direction is being developed by Winternitz et all (see, for example [4]).

Ermakov's nonlinear superposition principle (NSP) (see also [5,6]).

Consider a nonlinear equation

$$y^{(n)} = f(x, y, y', \cdots, y^{(n-1)}) \tag{5}$$

be given. Let also the associated linear equation be given by

$$Y^{(m)} + a_1(x)Y^{(m-1)} + \cdots + a_{m-1}(x)Y' + a_m(x)Y = 0 \tag{6}$$

and

$$Y_1(x), Y_2(x), \cdots, Y_m(x)$$

be particular linear independent solutions of (6).

Definition 1. Equation (5) admits NSP, if its general solution has the form

$$y = F(Y_1(x), \cdots, Y_m(x); c_1, c_2, \cdots, c_n). \tag{5'}$$

Example 2. Ermakov–Pinney equation (Ermakov (1880), Pinney (1951)).
Consider the equation

$$y'' + a_0(x)y + b_0y^{-3} = 0,\ b_0 = const. \tag{7}$$

and the associated equation

$$Y'' + a_0(x)Y = 0. \tag{8}$$

General solution of (7) has the form:

$$y(x) = \sqrt{AY_1^2 + BY_1Y_2 + CY_2^2}, \quad B^2 - 4AC = const,$$

where $Y_1(x), Y_2(x)$ are FSS of (8).

Example 3. Kummer-Schwarz equation

$$\frac{1}{2}\frac{y''}{y}-\frac{3}{4}(\frac{y'}{y})^2+b_0y^2=a_0(x),$$

It has the general solution

$$y(x)=(AY_1^2(x)+BY_1Y_2+CY_2^2)^{-1},$$

where $Y_1(x), Y_2(x)$ are FSS of (8).

Example 4.

$$\frac{y'''}{y}-\frac{3}{2}(\frac{y''}{y})^2\equiv\{y,x\}=0,$$

where $\{,\}$ is the Schwarz derivative. Let $Y''(x)=0$ be the associated equation, and $Y_1=1, Y_2=x$ is a FSS. Then

$$y(x)=\frac{ax+b}{cx+d},\ ad-bc=const.$$

NSP: generalized definition.

Definition 2. The equation (5) admits NSP, if its general solution is (5′), where $c_1, c_2, \cdots, c_n$ are arbitrary const, and a set of particular solutions is either

1) particular solutions of (5), or
2) partial linear independent solutions of associated linear equation (6) , or
3) partial solutions of associated nonlinear equation

$$Y^{(m)}=g(x,Y,Y',\cdots,Y^{(m-1)});$$

4) general solutions fo (5) is an nonlinear function of arbitrary const.

NSP may be extended also to partial differential equations.

Example 5. *Burgers equation*:

$$u_t=-2\nu+uu_x-\nu u_{xx}=0. \tag{9}$$

By the transformation (Cole, 1951, and Hopf, 1950)

$$u=-2\nu\frac{\theta_x}{\theta},$$

the equation (9) is reduced to the linear *heat equation*

$$\theta_t-\nu\theta_{xx}=0.$$

Factorization of nonlinear ordinary differential equations and linearization

Theorem 1 [7,8]. Consider the equation

$$y^{(n)}=f(y,y',\cdots,y^{(n-1)}),\ (')=d/dx. \tag{10}$$

The invertible transformation

$$y=v(y)z,\quad dt=u(y)dx, \tag{11}$$

where v, u are sufficiently smooth nonvanishing functions in the domain $G(x,y)$, transforms (10) into the linear autonomous form (12),

$$z^{(n)}(t)+\sum_{k=1}^{n}\binom{n}{k}b_kz^{(n-k)}(t)=0, \tag{12}$$

if and only if (10) admits the following factorization through first-order nonlinear differential operators:

$$\prod_{k=1}^{n}\left(\frac{1}{u}D-\frac{v_y y'}{vu}-r_k\right)y=0,\quad D=\frac{d}{dx},$$

(commutative factorization), or

$$\prod_{k=n}^{1}\left[D-\frac{v_y y'}{v}-(k-1)\frac{u_y y'}{u}-r_k u\right]y=0$$

(noncommutative factorization).

Theorem 2. The transformation (11) reduces the equation (10) to the linear form

$$z^{(n)}(t)+\sum_{k=1}^{n}\binom{n}{k}b_k z^{(n-k)}(t)+c=0,\ \ b_k, c=const. \tag{13}$$

if and only if (10) can factorized as

$$\prod_{k=1}^{n}\left[\frac{1}{\varphi}D-\left(\frac{1}{y\varphi}-\frac{\varphi^{\frac{n^2-3n+2}{2n}}\exp(\int \mathrm{fdy})}{\int\varphi^{\frac{n^2-n+2}{2n}}\exp(\int \mathrm{fdy})\mathrm{dy}}\right)\frac{\partial y}{\partial x}-r_k\right]y+$$

$$+\varphi^{\frac{-n^2+n-2}{2n}}\exp(-\int \mathrm{fdy})\left(\mathrm{b_n}\int\varphi^{\frac{\mathrm{n}^2-\mathrm{n}+2}{2\mathrm{n}}}\exp(\int \mathrm{fdy})\mathrm{dy}+\frac{\mathrm{c}}{\beta}\right)=0, \tag{14}$$

where β is normalizing factor.

The lexicographic form of equation (14) is the following:

$$y^{(n)}+nf(y)y'y^{(n-1)}+\cdots+nb_1\varphi(y)y^{(n-1)}+\cdots+$$

$$\varphi^{\frac{n^2+n-2}{2n}}\exp(-\int \mathrm{fdy})\left(\mathrm{b_n}\int\varphi^{\frac{\mathrm{n}^2-\mathrm{n}+2}{2\mathrm{n}}}\exp(\int \mathrm{fdy})\mathrm{dy}+\frac{\mathrm{c}}{\beta}\right)=0. \tag{15}$$

The linearizing transformation is

$$z=\beta\int\varphi^{\frac{n^2-n+2}{2n}}\exp(\int \mathrm{f(y)dy})\mathrm{dy},\ \ \mathrm{dt}=\varphi(\mathrm{y})\mathrm{dx}.$$

At $c=0$, the equation has one-parameter families of solutions

$$\int\frac{\varphi^{\frac{n^2-3n+2}{2n}}\exp(\int \mathrm{fdy})\mathrm{dy}}{\int\varphi^{\frac{n^2-n+2}{2n}}\exp(\int \mathrm{fdy})\mathrm{dy}}=r_k x+c_k,$$

where r_k are different roots of equation

$$r^n+\sum_{k=1}^{n}\binom{n}{k}b_k r^{n-k}=0.$$

NSP for autonomous equations.

Corollary. Equation (10), represented in the form (14), (15), admits the following general solution:

$$\beta\int\varphi^{\frac{n^2-n+2}{2n}}\exp(\int \mathrm{fdy})\mathrm{dy}=\sum_{\mathrm{k}=1}^{\mathrm{n}}\mathrm{c_k z_k(t)}+\tilde{\mathrm{z}}(\mathrm{t}),$$

$$dt=\varphi(y)dx,$$

where $z_k(t),\ k=1,...,n$, are linear independent solutions of linear homogeneous equation (12) and $\tilde z$ is a particular solution of nonhomogeneous equation (13).

A new class of nonlinear evolutionary equations (NEE).

Theorem 3. [9] *In order for NEE*

$$\frac{\partial y}{\partial t}=F(y,\frac{\partial y}{\partial x},\ldots,\frac{\partial^n y}{\partial x^n}) \tag{16}$$

to be transformed by substitution

$$y=v(y)z,\quad ds=u(y)dx,$$

where

$$z=\beta\int\varphi^{\frac{n^2-n+2}{2n}}\mathrm{exp}\Big(\int \mathrm{fdy}\Big)\mathrm{dy},\ \mathrm{ds}=\varphi\mathrm{dx},\ \beta \text{ is normalizing factor}$$

to the linear evolutionary equation

$$\frac{\partial z}{\partial t}=\frac{\partial^n z}{\partial s^n}+\sum_{k=1}^{n}\binom{n}{k}b_k\frac{\partial^{n-k}z}{\partial s^{n-k}},\ b_k=const, \tag{17}$$

it is nessesary and sufficient that (16) can be represented in the form

$$\frac{\varphi^{\frac{n^2-n+2}{2n}}\mathrm{exp}(\int\mathrm{fdy})}{\int\varphi^{\frac{n^2-n+2}{2n}}\mathrm{exp}(\int\mathrm{fdy})\mathrm{dy}}\frac{\partial y}{\partial t}=$$

$$=\prod_{k=1}^{n}\left[\frac{1}{\varphi}D-\left(\frac{1}{y\varphi}-\frac{\varphi^{\frac{n^2-3n+2}{2n}}\mathrm{exp}(\int\mathrm{fdy})}{\int\varphi^{\frac{n^2-n+2}{2n}}\mathrm{exp}(\int\mathrm{fdy})\mathrm{dy}}\right)\frac{\partial y}{\partial x}-r_k\right]y. \tag{18}$$

Nonlinear superposition principle for NEE.

Let $z_1(s,t),\ldots,z_n(s,t)$ be linearly independent partial solutions of (17). Then the total integral of (17) is

$$z=c_1z_1+\cdots+c_nz_n,$$

where $c_1,c_2,\ldots,c_n$ are arbitrary constants, and this formula is a LSP. We obtain a NSP according to the formulas

$$z=\beta\int\varphi^{\frac{n^2-n+2}{2n}}\mathrm{exp}(\int\mathrm{fdy})\mathrm{dy}=\sum_{\mathrm{k=1}}^{\mathrm{n}}\mathrm{c_kz_k},\ \mathrm{ds}=\varphi\mathrm{dx},$$

where $\beta=const$ is a normalizing factor.

Let us demonstrate NEE of orders $n=2$ and $n=3$ of general form belonging to the class (16), (18):

$$\varphi^2y_t=y_{xx}+fy_x^2+2b_1\varphi y_x+b_2\varphi\mathrm{exp}(-\int\mathrm{fdy})\int\varphi\mathrm{exp}(\int\mathrm{fdy})\mathrm{dy};$$

$$\varphi^3y_t=y_{xxx}+3fy_xy_{xx}+\left(\frac{1}{3}\frac{\varphi_{yy}}{\varphi}-\frac{5}{9}\frac{\varphi_y^2}{\varphi^2}-\frac{1}{3}f\frac{\varphi_y}{\varphi}+f^2+f_y\right)y_x^3+$$

$$+3b_1\varphi\left(y_{xx}+(f+\frac{1}{3}\frac{\varphi_y}{\varphi})y_x^2\right)+3b_2\varphi^2y_x+b_3\varphi^{5/3}\mathrm{exp}(-\int\mathrm{fdy})\int\varphi^{4/3}\mathrm{exp}(\int\mathrm{fdy})\mathrm{dy}.$$

Example 6. (Maslov et al., see, for example, [10]) The equation

$$y_t=\frac{h}{2}y_{xx}-\frac{1}{2}y_x^2 \tag{19}$$

by the transformation

$$z = \exp(-\frac{\mathrm{y}}{\mathrm{h}}),\ \ \mathrm{ds} = \sqrt{\frac{2}{\mathrm{h}}}\mathrm{dx}$$

is linearized to

$$z_t = z_{ss}.$$

Remark. *The correspondence principle (in the sense [8]) is a NSP .*

The equation (19) by transformation $z = \exp(-\mathrm{y}/\mathrm{h})$ takes the form

$$z_t = -hz_{xx}. \tag{20}$$

Let $z_1,\ z_2$ be linearly independed solutions of (20). Then the total integral $z = c_1z_1 + c_2z_2$, where $c_1,\ c_2$ are arbitrary constants, is a LSP, and the formula

$$y = -h\ln\left[\exp\left(-\frac{1}{\mathrm{h}}\mathrm{y}_1 - \frac{\lambda_1}{\mathrm{h}}\right) + \exp\left(-\frac{1}{\mathrm{h}}\mathrm{y}_2 - \frac{\lambda_2}{\mathrm{h}}\right)\right],$$

where $\lambda_1,\ \lambda_2$ are arbitrary constants, and $y(x,t)$ is a total integral, and $y_1(x,t),\ y_2(x,t)$ are particular solutions of (19), is a NSP.

Concluding remarks

1) *A FSS does not determine the nonlinear equation completely; to determine the equation one has to know both the FSS and the NSP.*

Example 7. Equation

$$y'' + \left(-\frac{2}{y} + \frac{1}{y^2}\right)y'^2 - y^2 = 0$$

by transformation $z = \exp(-1/\mathrm{y})$ is reduced to $z'' - z = 0$.

$FSS:$

$$y_1 = \frac{1}{x},\ y_2 = \frac{-1}{x},$$

$NSP:$

$$y = -\frac{1}{\ln(c_1e^x + c_2e^{-x})} = -\frac{1}{\ln\left(\exp(-\frac{1}{\mathrm{y}_1} + \alpha) + \exp(-\frac{1}{\mathrm{y}_2} + \beta)\right)},$$

where $c_1,\ c_2,\ \alpha,\ \beta$ are integration constants. The following equation

$$y'' - \frac{1}{y}y'^2 - y^3 = 0$$

can be linearized to the form

$$\ddot{y} - y = 0$$

by the substitution $dt = ydx$. It has the same FSS as before, but different from the NSP above. We omit the NSP in this case, since it very bulky.

2) *One can linearise an equation by means of different transformations.*

Example 8. Consider the equation

$$y'' + 3yy' + y^3 = 0 \tag{21}$$

Equation (21) can be linearized by the following substitution

$$z = y^2,\quad dt = ydx$$

to the form

$$\ddot{z} + 3z\dot{z} + z^3 = 0.$$

Applying the point transformations

$$X = x - \frac{1}{y}, \quad Y = \frac{1}{2}x^2 - \frac{x}{y},$$

we get another linear equation

$$Y''(X) = 0.$$

3)*Factorization may be regarded as a possible method of linearization.*

Let equation

$$y^{(n)} = f(y, y', \cdots, y^{(n-1)})$$

admit a factorization

$$\prod_{k=1}^{n} (\gamma D - \beta y' - \alpha - r_k) y = 0, \ \gamma = \gamma(y), \ \beta = \beta(y), \ \alpha = \alpha(y), \ r_k = const.$$

This factorization generetes Bäcklund transformation:

$$\frac{y'}{y}(1 - \frac{\beta}{\gamma} y) = \frac{\alpha}{\gamma} + \frac{z'}{z}$$

that reduces to a linear equation

$$\prod_{k=1}^{n} (D - r_k) z = 0$$

Open problem.

Can one obtain new non-linear compositions laws making use of the following invertible transformations

$$y = v(x, y) z, \quad dt = u_1(x, y) dx + u_2(x, y) dy$$

for nonlinear and nonautonomous equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})?$$

Acknowledgment: the author expresses his gratitude to Sergei Kuleshov and Victor Protsak for their help in preparing the text for publication.

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