Analytic Properties of Zeta Functions and Subgroup Growth.

Marcus du Sautoy*and Fritz Grunewald†

September 10, 1999

1 Introduction

It has become somewhat of a cottage industry over the last fifteen years to understand the rate of growth of the number of subgroups of finite index in a group G. Although the story began much before, the recent activity grew out of a paper by Dan Segal in the proceedings of Groups St. Andrews 1985 [20]. The story so far has been well-documented in Lubotzky's survey paper in the proceedings of the subsequent Groups Galway 1993 [16].

In [13] the second author, Segal and Smith introduced the zeta function of a group as a tool for understanding this growth of subgroups. Let $a_n(G)$ be the number of subgroups of index n in the finitely generated group G and $s_n(G) = a_1(G) + \cdots + a_n(G)$ be the number of subgroups of index n or less. The zeta function is defined as the Dirichlet series with coefficients $a_n(G)$ and has a natural interpretation as a non-commutative generalization of the Dedekind zeta function of a number field:

$$\begin{aligned} \zeta_G(s) &= \sum_{n=1}^{\infty} a_n(G) n^{-s} \qquad (1) \\ &= \sum_{H \leq G} |G : H|^{-s}. \end{aligned}$$

*DPMMS, 16 Mill Lane, Cambridge UK. dusautoy@dpmms.cam.ac.uk

†Math. Institut, Heinrich Heine Universität, D-40225 Düsseldorf, Germany. fritz@math.uni-duesseldorf.de

For example, without such a tool it would be difficult to prove that the number of subgroups in the rank two free abelian group $G = \mathbb{Z}^2$ grows as follows:

$$s_n(\mathbb{Z}^2) \sim \left(\pi^2/12\right) n^2$$

as n tends to infinity. (Here $f(n) \sim g(n)$ means $f(n)/g(n)$ tends to 1 as n tends to infinity.) This is a consequence of the expression for the zeta function of the free abelian group:

$$\zeta_{\mathbb{Z}}^d(s) = \zeta(s)...\zeta(s-d+1)$$

where $\zeta(s)$ is the Riemann zeta function.

The zeta function (1) defines an analytic function on some right half of the complex plane $\Re(s) > \alpha(G)$ precisely when the coefficients $a_n(G)$ are bounded by a polynomial. A characterization of such groups, groups of polynomial subgroup growth, was provided by Lubotzky, Mann and Segal [17]. They are the finitely generated groups which have a subgroup of finite index that is soluble of finite rank.

In this paper we consider the analytic behaviour of the zeta function of the subclass of finitely generated nilpotent groups. This class of groups has the added bonus that their zeta functions satisfy an Euler product (see [13]):

$$\zeta_G(s) = \prod_p \zeta_{G,p}(s)$$

where the local factors for each prime p are defined as:

$$\zeta_{G,p}(s) = \sum_{n=0}^{\infty} a_{p^n}(G) p^{-ns}.$$

It was also proved in [13] that if the nilpotent group is torsion-free then these local factors are all rational functions in p^{-s}. However the proof gave little understanding of how these rational functions varied as p varied and was not sufficient to understand the global behaviour of $\zeta_G(s)$.

In this paper we introduce some new methods to understand the zeta function of a group and prove the following:

Theorem 1 *(1) Let G be a finitely generated nilpotent infinite group. Then there exist a rational number $\alpha(G) \in \mathbb{Q}$, a non-negative integer $b \in \mathbb{N}_0$ and some real number $c \in \mathbb{R}$ such that*

$$s_n(G) \sim c \cdot n^{\alpha(G)} (\log n)^b .$$

(2) The abscissa of convergence $\alpha(G)$ of $\zeta_G(s)$ is a rational number and $\zeta_G(s)$ can be meromorphically continued to $\Re(s) > \alpha(G) - \delta$ for some $\delta > 0$.

Part (1) follows from (2) by a suitable application of the Tauberian Theorem. Whether the abscissa of convergence was a rational number was raised as one of the major open problems in the field in Lubotzky's survey article [16].

Note that the integer $b+1$ is the multiplicity of the pole at $s = \alpha(G)$. In [10] several examples are given where this multiplicity is greater than one. For example, the zeta function of the discrete Heisenberg group

$$G = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$$

has the following expression:

$$\zeta_G(s) = \zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-3) \cdot \zeta(3s-3)^{-1}.$$

The double pole at $s = 2$ implies that the growth of subgroups is:

$$s_n(G) \sim \frac{\zeta(2)^2}{2\zeta(3)} n^2 \log n.$$

This was first observed in Smith's thesis [21]. This example has meromorphic continuation to the whole complex plane. In [8] it is shown that this is also true for any finite extension of a free abelian group. In general, though, these functions have natural boundaries as discussed in [10]. However we have introduced in a separate paper [7] the concept of the ghost zeta function which does tend to have meromorphic continuation.

The proof of (2) depends on showing a more general result which holds for any zeta function which can be defined as an Euler product over primes p of *cone integrals over* $\mathbb{Q}$.

Definition 2 *(1) Call a formula $\psi(\mathbf{x})$ in the first order language for the valued field $\mathbb{Q}_p$ a cone condition over $\mathbb{Q}$ if there exist polynomials $f_i(\mathbf{x}), g_i(\mathbf{x})(i = 1, ..., l)$ over $\mathbb{Q}$ in the variables $\mathbf{x} = x_1, ..., x_m$ such that $\psi(\mathbf{x})$ is a conjunction of formulas*

$$v(f_i(\mathbf{x})) \leq v(g_i(\mathbf{x}))$$

for $i = 1, ..., l$.

(2) Given a cone condition $\psi(\mathbf{x})$ over $\mathbb{Q}$ and polynomials f_0 and g_0 with coefficients in $\mathbb{Q}$, we call an integral

$$Z_{\mathcal{D}}(s,p) = \int_{V_p = \left\{\mathbf{x} \in \mathbb{Z}_p^m : \psi(\mathbf{x}) \text{ is valid}\right\}} |f_0(\mathbf{x})|^s \, |g_0(\mathbf{x})| \, |dx|$$

a cone integral defined over $\mathbb{Q}$, where $|dx|$ is the normalized additive Haar measure on $\mathbb{Z}_p^m$ and $\mathcal{D} = \{f_0, g_0, f_1, g_1, ..., f_l, g_l\}$ is called the cone integral data.

(3) We say that a function $Z(s)$ is defined as an Euler product of cone integrals over $\mathbb{Q}$ with cone integral data $\mathcal{D}$ if

$$Z(s) = Z_{\mathcal{D}}(s) = \prod_{p \text{ prime}} \left(a_{p,0}^{-1} \cdot Z_{\mathcal{D}}(s,p)\right)$$

where $a_{p,0} = Z_{\mathcal{D}}(\infty, p)$ is the constant coefficient of $Z_{\mathcal{D}}(s,p)$, i.e. we normalize the local factors to have constant coefficient 1.

It is clear from the defintion of a cone integral that $a_{p,0}$ is non-zero so the normalization in (3) is well defined.

In section 5 we show that for a nilpotent group G, $\zeta_G(s) = Z_{\mathcal{D}}(s-d)$ where $Z_{\mathcal{D}}(s)$ is defined as an Euler product of cone integrals over $\mathbb{Q}$ and d is the Hirsch length of G. (The Hirsch length is the number of infinite cyclic factors in a composition series for G.)

We adapt some ideas of Denef introduced in [3] to give an explicit expression for a cone integral valid for almost all primes p in terms of the resolution of singularities (Y, h) of the polynomial $F(\mathbf{x}) = \prod_{i=0}^{l} f_i(\mathbf{x}) g_i(\mathbf{x})$. In particular we show:

Theorem 3 *Let (Y, h) be a resolution over $\mathbb{Q}$ for $F(\mathbf{x}) = \prod_{i=0}^{l} f_i(\mathbf{x}) g_i(\mathbf{x})$ and let $E_i, i \in T$, be the irreducible components defined over $\mathbb{Q}$ of the reduced* *scheme* $(h^{-1}(D))_{\mathrm{red}}$ *where* $D = \mathrm{Spec}\left(\frac{\mathbb{Q}[\mathbf{x}]}{(F)}\right)$. *Then there exist rational functions* $P_I(x,y) \in \mathbb{Q}(x,y)$ *for each* $I \subset T$ *with the property that for almost all primes* p

$$Z_{\mathcal{D}}(s,p) = \sum_{I \subset T} c_{p,I} P_I(p, p^{-s}) \tag{2}$$

where

$$c_{p,I} = \mathrm{card}\{a \in \overline{Y}(\mathbb{F}_p) : a \in \overline{E_i} \text{ if and only if } i \in I\}$$

and $\overline{Y}$ *means the reduction mod* p *of the scheme* Y.

We can then use the Lang-Weil estimates for the number of points on varieties mod$p$ to identify the abscissa of convergence of the global zeta function $Z_{\mathcal{D}}(s)$.

However just knowing the shape of the zeta function from the expression (2) is not sufficient to infer that the Euler product of these expressions can be meromorphically continued beyond its region of convergence. For example

$$\prod_{p \text{ prime}} \left(1 + \frac{p^{-1}}{(1-p^{-s})}\right) \tag{3}$$

converges for $\Re(s) > 0$ but has $\Re(s) = 0$ as a natural boundary. We give instead a subtler expression for the cone integrals. Rather than a sum over the subsets of T, the indexing set of the irreducible components, this second expression is a sum over the open simplicial pieces of a natural polyhedral cone that one associates to the cone condition ψ.

Theorem 4 *There exists a closed polyhedral cone* $\overline{D}$ *in* $\mathbb{R}^t_{\geq 0}$ *where* $t = \mathrm{card}T$ *whose open simplicial pieces we denote by* R_k *where* $k \in \{0,1,...,w\}$. *We arrange that* $R_0 = (0,...,0)$ *and* $R_1,...,R_q$ *are the one-dimensional pieces. For each* $k \in \{0,1,...,w\}$ *let* $M_k \subset \{1,...,q\}$ *denote those one-dimensional pieces in the closure* $\overline{R_k}$ *of* R_k. *Then there exist positive integers* A_j, B_j *for* $j \in \{1,...,q\}$ *such that*

$$Z_{\mathcal{D}}(s,p) = \sum_{k=0}^{w} (p-1)^{I_k} p^{-m} c_{p,I_k} \prod_{j \in M_k} \frac{p^{-(A_j s + B_j)}}{\left(1 - p^{-(A_j s + B_j)}\right)} \tag{4}$$

where c_{p,I_k} *is defined in Theorem 3 and* I_k *is the subset of* T *defined so that* $i \in T \backslash I_k$ *if and only if the ith coordinate is zero for all elements of* R_k.

This expression (4) motivates the name *cone integral.* An explicit expression is given for the integers A_j and B_j in terms of the numerical data of the resolution. This is contained in the proof of this Theorem which appears in section 3.

With this combinatorial expression to hand, we can show that the pathologies of examples like (3) do not arise. In particular, we show that the abscissa of convergence of the global zeta function is determined by the terms in the expression (4) corresponding to the one dimensional edges $R_1, ..., R_q$. We then show how to use Artin L-functions to analytically continue a function like

$$\prod_{p \text{ prime}} \left(1 + c_{p,I_k} \frac{p^{-s}}{(1-p^{-s})}\right)$$

beyond its region of convergence. We can then use the Tauberian Theorem to estimate the growth of the coefficients in the Dirichlet series expressing $Z_{\mathcal{D}}(s)$. In particular we prove the following:

Theorem 5 *Let $Z_{\mathcal{D}}(s)$ be defined as an Euler product of cone integrals over $\mathbb{Q}$. Then $Z(s)$ is expressible as a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ with non-negative coefficients a_n.*

(1) The abscissa of convergence α of $Z(s)$ is a rational number and $Z(s)$ has a meromorphic continuation to $\Re(s) > \alpha - \delta$ for some $\delta > 0$.

(2) Let the pole at $s = \alpha$ have order w. Then for every β with $\beta + \alpha > 0$, there exists some real number $c \in \mathbb{R}$ such that

$$a_1 + \cdots + a_n n^{\beta} \sim c \cdot n^{\beta+\alpha} (\log n)^{w-1}.$$

One of the key problems in the area was to link zeta functions of groups up to questions in some branch of more classical number theory. We restate Theorem 3 explicitly for groups as it provides just such a path from zeta functions of groups to the more classical question of counting points mod p on a variety. The path is quite explicit. We define in section 5 a polynomial F_G over $\mathbb{Q}$ associated to each nilpotent group G.

Theorem 6 *Let G be a finitely generated nilpotent group. Let (Y, h) be a resolution over $\mathbb{Q}$ for the polynomial F_G. Let $E_i, i \in T$ be the irreducible components of the reduced scheme $(h^{-1}(D))_{red}$ associated to $h^{-1}(D)$ where*

$D=\mathrm{Spec}\left(\frac{\mathbb{Q}[\mathbf{x}]}{(F_G)}\right)$. *Then there exist rational functions $P_I(x,y)\in\mathbb{Q}(x,y)$ for each $I\subset T$ with the property that for almost all primes p*

$$\zeta_{G,p}(s)=\sum_{I\subset T}c_{p,I}P_I(p,p^{-s})$$

where

$$c_{p,I}=\mathrm{card}\{a\in\overline{Y}(\mathbb{F}_p):a\in\overline{E_i}\text{ if and only if }i\in I\}$$

and $\overline{Y}$ means the reduction mod p of the scheme Y.

The behaviour of the local factors as we vary p is one of the other major problems in the field. For example in the Heisenberg group with entries from a quadratic number field, the behaviour of the local factors depends on how p behaves in the number field [13]. Our explicit formula however takes the subject away from the behaviour of primes in number fields to the problem of counting points mod p on a variety, a question which is in general wild and far from the uniformity predicted by all previous examples (see [13] and [6]). In a future paper [12] we explain an example constructed by the first author of a class two nilpotent group of Hirsch length 9 whose zeta function depends on counting points on the elliptic curve $y^2=x^3-x$ mod p.

Given a nilpotent group G it is possible to construct and analyse the polynomial F_G in question. For example, in the free abelian group or the Heisenberg group, the polynomial does not require any resolution of singularities as D in this case only involves normal crossings. Hence the $E_i, i\in T$ in this case are just the irreducible components of the algebraic set $F_G=0$. However this is not true in general. For example the class two nilpotent group defined using the elliptic curve mentioned above has an F_G whose singularities are not normal crossings and which therefore require some resolution.

We have put the emphasis in this introduction on applying these cone integrals to the question of counting subgroups in nilpotent groups, however our results extend in a number of other directions.

(1) Variants of our zeta functions have been considered which count only subgroups with some added feature, for example normal subgroups. Define

$$\begin{aligned}a_n^{\triangleleft}(G)&=\mathrm{card}\{H:H\text{ is normal subgroup of }G\text{ and }|G:H|=n\}\\ \zeta_G^{\triangleleft}(s)&=\sum a_n^{\triangleleft}(G)n^{-s}.\end{aligned}$$

Our Theorems hold for this normal zeta function and many of the other variants.

(2) Let L be a ring additively isomorphic to $\mathbb{Z}^d$. Define

$$\begin{aligned} a_n(L) &= \operatorname{card}\{H : H \text{ is a subring of } L \text{ and } |L:H| = n\} \\ a_n^{\triangleleft}(L) &= \operatorname{card}\{H : H \text{ is an ideal of } L \text{ and } |L:H| = n\}. \end{aligned}$$

Zeta functions of L were also defined in [13] as the Dirichlet series

$$\begin{aligned} \zeta_L(s) &= \sum a_n(L) n^{-s} \\ \zeta_L^{\triangleleft}(s) &= \sum a_n^{\triangleleft}(L) n^{-s} \end{aligned}$$

It was pointed out in [13] that these zeta functions have an Euler product, as for the case of nilpotent groups:

$$\begin{aligned} \zeta_L(s) &= \prod_{p \text{ prime}} \zeta_{L\otimes\mathbb{Z}_p}(s) \\ \zeta^{\triangleleft}_L(s) &= \prod_{p \text{ prime}} \zeta^{\triangleleft}_{L\otimes\mathbb{Z}_p}(s) \end{aligned}$$

Unlike for groups, there is no need to make an assumption of nilpotency in the case of rings. We can therefore consider examples like $L = \mathfrak{sl}_2(\mathbb{Z})$ or the $\mathbb{Z}$-points of any simple Lie algebra of classical type. We get then the following:

Theorem 7 *Let L be a ring additively isomorphic to $\mathbb{Z}^d$. Then there exists some rational number $\alpha(L) \in \mathbb{Q}$, a non-negative integer $b \in \mathbb{N}_0$ and some real number $c \in \mathbb{R}$ such that $\zeta_L(s)$ has abscissa of convergence $\alpha(L)$ and*

$$s_n(L) = a_1(L) + \cdots + a_n(L) \sim c \cdot n^{\alpha(L)} (\log n)^b .$$

There is a similar theorem for the invariant $a_n^{\triangleleft}(L)$ counting ideals.

We actually prove this Theorem as part of our proof of Theorem 1. We make use of the fact that for a nilpotent group G there is a Lie algebra $L(G)$ defined over $\mathbb{Z}$ with the property that for almost all primes p

$$\zeta_{G,p}(s) = \zeta_{L(G),p}(s).$$

This fact was established in [13].

In [9] the example of $\zeta_{\mathfrak{sl}_2(\mathbb{Z})}(s)$ is calculated using some previous work of Ishai Ilani [14]. There is a rational function $P(T)$ such that

$$\zeta_{\mathfrak{s}l_2(\mathbb{Z})}(s) = P(2^{-s}) \cdot \zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-1)\zeta(3s-1)^{-1}.$$

Note that this example has a single pole at $s = 2$ (provided the uncalculated local factor $P(T)$ at $p = 2$ does not have a pole at $s = 2$ which is unlikely since it would mean that most additive lattices in $\mathbb{Z}_2^3$ were subalgebras in $\mathfrak{sl}_2(\mathbb{Z}_2)$). This means then that the subalgebra growth, in contrast to the 3-dimensional Heisenberg Lie algebra, is $s_n(\mathfrak{s}l_2(\mathbb{Z})) \sim c \cdot n^2$ where $c = P(2^{-2}) \cdot \zeta(2)^2\zeta(3)/2\zeta(5)$.

(3) Let G be a linear algebraic group over $\mathbb{Q}$. Let $\rho : G \to \mathrm{GL}_n$ be a $\mathbb{Q}$-rational representation. Define the '*local zeta function of the algebraic group G at the representation ρ and the prime p* ' to be

$$Z_{G,\rho,p}(s) = \int_{G^+} |\det \rho(g)|^s \, \mu_G(g)$$

where $G^+ = \rho^{-1}\left(\rho\left(G\left(\mathbb{Q}_p\right)\right) \cap \mathrm{M}_n\left(\mathbb{Z}_p\right)\right)$ and μ_G denotes the right Haar measure on $G(\mathbb{Q}_p)$ normalized such that $\mu_G\left(G\left(\mathbb{Z}_p\right)\right) = 1$.

We define the '*global zeta function of G at the representation ρ*' to be Euler product

$$Z_{G,\rho}(s) = \prod_p Z_{G,\rho,p}(s).$$

Such zeta functions were first studied by Hey and Tamagawa in the case that $G = \mathrm{GL}_{l+1}$ where $Z_{G,\rho}(s) = \zeta(s)...\zeta(s-l)$. Note that this zeta function is precisely the zeta function $\zeta_{\mathbb{Z}^{l+1}}(s)$. More generally in [13] the zeta functions $Z_{G,\rho}(s)$ are shown to count subgroups H in a nilpotent group Γ with the property that the profinite completions are isomorphic, i.e. $\widehat{H} \cong \widehat{\Gamma}$. In this case the algebraic group G is the automorphism group of Γ. A result of Bryant and Groves shows that any algebraic group can be realised modulo a unipotent group as the automorphism group of a nilpotent group. In [6] an explicit expression is given for the local factors of a class of nilpotent groups in terms of the combinatorics of the building of the algebraic group. The local zeta functions $Z_{G,\rho,p}(s)$ can be expressed in terms of cone integrals. Hence our results apply to these zeta functions.

Although our results imply we can meromorphically continue the zeta function $Z_{G,\rho}(s)$ past its abscissa of convergence, this zeta function in general has a natural boundary, except for the case of $G = \mathrm{GL}_{l+1}$ (see [10]).

However we have discovered a procedure which produces something we call the *ghost zeta function* associated to $Z_{G,\rho}(s)$ which turns out often to have a meromorphic continuation to the whole complex plane (see [7]).

(4) Let $g(n,c,d)$ be the number of finite nilpotent groups of size n of class bounded by c and generated by at most d elements. In [11] the zeta function $\zeta_{\mathcal{N}(c,d)} = \sum_{n=1}^{\infty} g(n,c,d)n^{-s}$ is shown to be expressible as the Euler product of p-adic cone integrals. Hence the results of this paper imply that asymptotically $g(n,c,d)$ behaves as follows:

$$g(1,c,d) + \cdots + g(n,c,d) \sim c \cdot n^{\alpha} (\log n)^{b}$$

where $\alpha \in \mathbb{Q}$, $b \in \mathbb{N}_0$ and $c \in \mathbb{R}$. The details are explained in [11].

(5) The Igusa zeta function of a polynomial $f(\mathbf{x})$ is defined as

$$Z(s) = \int_{\mathbb{Z}_p^m} |f(\mathbf{x})|^s \, |d\mathbf{x}| \, .$$

Hence it is a particular example of a cone integral where the cone condition ψ is empty. The global zeta function that one can define as the Euler product of these Igusa zeta functions (normalized to have constant coefficient 1) is a special case of our analysis.

Notation

$\mathbb{N}$ denotes the set $\{1, 2, ...\}$.

$\mathbb{N}_0$ denotes the set $\{0, 1, 2, ...\}$.

$\mathbb{R}_{>0}$ denotes the set $\{s \in \mathbb{R} : s > 0\}$.

$\mathbb{R}_{\geq 0}$ denotes the set $\{s \in \mathbb{R} : s \geq 0\}$.

$\mathbb{Z}_p^*$ denotes the units of $\mathbb{Z}_p$.

$f(n) \sim g(n)$ means $f(n)/g(n)$ tends to 1 as n tends to infinity.

Acknowledgements We should like to thank Jürgen Elstrodt for discussions concerning the Tauberian Theorem. The first author would like to thank the Royal Society, the Max-Planck-Institute in Bonn and the Heinrich Heine Universität in Düsseldorf for support and hospitality during the preparation of this paper.

2 An explicit formula for cone integrals

In this section we give a proof of Theorem 3. We recall from the introduction the definition of a cone integral:

Definition 8 *(1) Call a formula $\psi(\mathbf{x})$ in the first order language for the valued field $\mathbb{Q}_p$ a cone condition over $\mathbb{Q}$ if there exist polynomials $f_i(\mathbf{x}), g_i(\mathbf{x})(i = 1, ..., l)$ over $\mathbb{Q}$ in the variables $\mathbf{x} = x_1, ..., x_m$ such that $\psi(\mathbf{x})$ is a conjunction of formulas*

$$v(f_i(\mathbf{x})) \leq v(g_i(\mathbf{x}))$$

for $i = 1, ..., l$.

(2) Given a cone condition $\psi(\mathbf{x})$ over $\mathbb{Q}$ and polynomials f_0 and g_0 with coefficients in $\mathbb{Q}$, we call an integral

$$Z_{\mathcal{D}}(s,p) = \int_{V_p=\left\{\mathbf{x}\in\mathbb{Z}_p^m:\psi(\mathbf{x}) \text{ is valid}\right\}} |f_0(\mathbf{x})|^s\,|g_0(\mathbf{x})|\,|dx|$$

a cone integral defined over $\mathbb{Q}$, where $|dx|$ is the normalized additive Haar measure on $\mathbb{Z}_p^m$ and $\mathcal{D} = \{f_0, g_0, f_1, g_1, ..., f_l, g_l\}$ is called the cone integral data.

We are going to use resolution of singularities to get an explicit formula for such cone integrals valid for almost all primes p. We follow section 5 of [3].

Definition 9 *A resolution (Y, h) for a polynomial F over $\mathbb{Q}$ consists of a closed integral subscheme Y of $\mathbf{P}^k_{X_\mathbb{Q}}$ (where $X_\mathbb{Q} = \mathrm{Spec}\mathbb{Q}[\mathbf{x}]$ and $\mathbf{P}^k_{X_\mathbb{Q}}$ denotes projective k-space over the scheme $X_\mathbb{Q}$) and the morphism $h : Y \to X$ which is the restriction to Y of the projection morphism $\mathbf{P}^k_{X_\mathbb{Q}} \to X_\mathbb{Q}$, such that*

(i) Y is smooth over $\mathrm{Spec}\mathbb{Q}$;

(ii) the restriction $h : Y\backslash h^{-1}(D) \to X\backslash D$ is an isomorphism (where $D = \mathrm{Spec}\left(\frac{\mathbb{Q}[\mathbf{x}]}{(F)}\right) \subset X_\mathbb{Q}$) ; and

(iii) the reduced scheme $(h^{-1}(D))_{red}$ associated to $h^{-1}(D)$ has only normal crossings (as a subscheme of Y).

Let E_i, $i \in T$, be the irreducible components of $(h^{-1}(D))_{red}$. For $i \in T$, let N_i be the multiplicity of E_i in the divisor of $F \circ h$ on Y and let $\nu_i - 1$ be the multiplicity of E_i in the divisor of $h^*(dx_1 \wedge ... \wedge dx_m)$. The (N_i, ν_i) $i \in T$ are called the *numerical data* of the resolution (Y, h) for F.

Let us recall some necessary facts about reduction of varieties mod$p$. When $X = X_\mathbb{Q} = \mathrm{Spec}\mathbb{Q}[\mathbf{x}]$ one defines the reduction mod$p$ of a closed integral subscheme Y of $\mathbf{P}^k_{X_\mathbb{Q}}$ as follows : let $\widetilde{X} = \mathrm{Spec}\mathbb{Z}[\mathbf{x}]$ and $\widetilde{Y}$ be the scheme-theoretic closure of Y in $\mathbf{P}^k_{\widetilde{X}}$. Then the reduction mod$p$ of Y is the scheme $\widetilde{Y}\times_{\mathbb{Z}}\mathrm{Spec}\mathbb{F}_p$ and we denote it by $\overline{Y}$. Let $\widetilde{h}:\widetilde{Y}\rightarrow\widetilde{X}$ be the restriction to $\widetilde{Y}$ of the projection morphism $\mathbf{P}^k_{\widetilde{X}}\rightarrow\widetilde{X}_{\mathbb{Q}}$ and $\overline{h}:\overline{Y}\rightarrow\overline{X}$ be obtained from $\widetilde{h}$ by base extension.

Definition 10 *A resolution (Y,h) for F over $\mathbb{Q}$ has good reduction modp if*
(1) $\overline{Y}$ is smooth over $\mathrm{Spec}\mathbb{F}_p$;
(2) $\overline{E_i}$ is smooth over $\mathrm{Spec}\mathbb{F}_p$, *for each $i\in T$, and $\bigcup_{i\in T}\overline{E_i}$ has only normal crossings as a subscheme of $\overline{Y}$; and*
(3) $\overline{E_i}$ and $\overline{E_j}$ have no common irreducible components, when $i\neq j$.

Note that a resolution over $\mathbb{Q}$ has good reduction for almost all primes p (see Theorem 2.4 of [3]).

Let (Y^o,h^o) be a resolution for the polynomial $F=f_0.g_0.\prod_{i=1}^{l}f_ig_i$ over $\mathbb{Q}$, and p any prime such that (Y^o,h^o) has good reduction mod$p\mathbb{Z}_p$ and $\overline{f_0.g_0.\prod_{i=1}^{l}f_ig_i}\neq 0$. Here $\bar{}$ means reduction mod$p$. Let (Y,h) be the resolution over $\mathbb{Q}_p$ obtained from (Y^o,h^o) by base extension.

Let $a\in\overline{Y}(\mathbb{F}_p)$. Since we consider $\overline{Y}$ as a closed subscheme of $\widetilde{Y}$, a is also a closed point of $\widetilde{Y}$. Let $T_a=\left\{i\in T:a\in\overline{E_i}\right\}=\left\{i\in T:a\in\widetilde{E_i}\right\}$. Let $r=\mathrm{card}T_a$ and $T_a=\{i_1,...,i_r\}$. Then in the local ring $O_{\widetilde{Y},a}$ we can write

$$F\circ\widetilde{h}=uc_1^{N_{i_1}}...c_r^{N_{i_r}}$$

where $c_i\in O_{\widetilde{Y},a}$ generates the ideal of $\widetilde{E_{i_j}}$ in $O_{\widetilde{Y},a}$ and u is a unit in $O_{\widetilde{Y},a}$. Since f_i and g_i divide F we can also write for $i=0,...,l$

$$\begin{aligned} f_i\circ\widetilde{h} &= u_{f_i}c_1^{N_{i_1}(f_i)}...c_r^{N_{i_r}(f_i)} \\ g_i\circ\widetilde{h} &= u_{g_i}c_1^{N_{i_1}(g_i)}...c_r^{N_{i_r}(g_i)} \end{aligned}$$

Put

$$J_a(s,p)=\int_{\theta^{-1}(a)\cap h^{-1}(V_p)}|f_0\circ h|^s\,|g_0\circ h|\,|h^*(dx_1\wedge...\wedge dx_m)|$$

where we define θ as follows: let $H=\{b\in Y(\mathbb{Q}_p):h(b)\in\mathbb{Z}_p^m\}$. A point $b\in H\subset Y(\mathbb{Q}_p)$ can be represented by its coordinates $(x_1,...,x_m,y_0,...,y_k)$ in $\mathbb{Q}_p^m\times\mathbf{P}^k_X(\mathbb{Q}_p)$ where $(x_1,...,x_m)\in\mathbb{Z}_p^m$ and $y_0,...,y_k$ are homogeneous coordinates which can therefore be chosen such that $\min_{i=0,...,k}\mathrm{ord}y_i=0$. The map $\theta : H \to \overline{Y}(\mathbb{F}_p)$ is then defined as follows: $\theta(b) = (\overline{x_1}, ..., \overline{x_m}, \overline{y_0}, ..., \overline{y_k}) \in \overline{Y}(\mathbb{F}_p) \subset \mathbf{P}^k_{\overline{X}}(\mathbb{F}_p)$.

Then $Z_{\mathcal{D}}(s,p) = \sum_{a \in \overline{Y}(\mathbb{F}_p)} J_a(s,p)$. Now we have

$$J_a(s,p) = \int_{\theta^{-1}(a) \cap h^{-1}(V_p)} |c_1|^{N_{i_1}(f_0)s + N_{i_1}(g_0) + \nu_{i_1} - 1} \dots |c_r|^{N_{i_r}(f_0)s + N_{i_r}(g_0) + \nu_{i_r} - 1} |dc_1 \wedge \dots \wedge dc_m|$$

Since $\overline{c_1}, ..., \overline{c_m}$ belong to the maximal ideal of $O_{\overline{Y},a}$, we have that $c_1(b), ..., c_m(b) \in p\mathbb{Z}_p$ for all $b \in \theta^{-1}(a)$. The map

$$\begin{array}{rcl} c & : & \theta^{-1}(a) \to (p\mathbb{Z}_p)^m \\ b & \mapsto & (c_1(b), ..., c_m(b)) \end{array}$$

is a bijection. Hence

$$J_a(s,p) = \int_{V'_p} |y_1|^{N_{i_1}(f_0)s + N_{i_1}(g_0) + \nu_{i_1} - 1} \dots |y_r|^{N_{i_r}(f_0)s + N_{i_r}(g_0) + \nu_{i_r} - 1} |dy_1| \dots |dy_m| \tag{5}$$

where V'_p is the set of all $y = (y_1, ..., y_m) \in (p\mathbb{Z}_p)^m$ satisfying for $i = 1, ..., l$

$$\sum_{j=1}^{r} N_{i_j}(f_i)\mathrm{ord}(y_j) \leq \sum_{j=1}^{r} N_{i_j}(g_i)\mathrm{ord}(y_j).$$

Let $A_{j,a} = N_{i_j}(f_0)$ and $B_{j,a} = N_{i_j}(g_0) + \nu_{i_j}$ for $j = 1, ..., r$ and $A_{j,a} = 0, B_{j,a} = 1$ for $j > r$. Then

$$\begin{aligned} J_a(s,p) &= \sum_{(k_1,...,k_m) \in \Lambda} p^{-\sum_{j=1}^{m} k_j(A_{j,a}s + B_{j,a} - 1)}(p^{-k_1} - p^{-k_1 - 1}) \dots (p^{-k_m} - p^{-k_m - 1}) \\ &= (1 - p^{-1})^m \sum_{(k_1,...,k_m) \in \Lambda} p^{-\sum_{j=1}^{m} k_j(A_{j,a}s + B_{j,a})} \end{aligned}$$

where $A_{j,a} \in \mathbb{N}_0$ and $B_{j,a} \in \mathbb{N}_0$ and

$$\Lambda = \left\{ (k_1, ..., k_m) \in \mathbb{N}^m : \sum_{j=1}^{r} N_{i_j}(f_i)k_j \leq \sum_{j=1}^{r} N_{i_j}(g_i)k_j \text{ for } i = 1, ..., l \right\}.$$

Thus Λ is the intersection of $\mathbb{N}^m$ and a rational convex polyhedral cone C in $\mathbb{R}^m_{>0}$. We can write this cone as a disjoint union of simplicial cones $C_1, ..., C_w$ of the form:

$$C_i = \{\alpha_1 v_{i1} + \cdots + \alpha_{m_i} v_{im_i} : \alpha_j \in \mathbb{R}, \alpha_j > 0 \text{ for } j = 1, ..., m_i\}$$

where $\{v_{i1},...,v_{im_i}\}$ is a linearly independent set of vectors in $\mathbb{R}^m$. Then Λ can be written as the disjoint union of the following sets

$$\Lambda_i = \{l_1 v_{i1} + \cdots + l_{m_i} v_{im_i} : l_j \in \mathbb{N} \text{ for } j = 1,...,m_i\}.$$

Now $v_{jk} = (q_{jk1},...,q_{jkm}) \in \mathbb{R}^m_{>0}$ for $k = 1,...,m_j$. Hence

$$J_a(s,p) = (1-p^{-1})^m \sum_{j=1}^{w} \prod_{k=1}^{m_j} \frac{p^{-(A_{k,a,j}s + B_{k,a,j})}}{1 - p^{-(A_{k,a,j}s+B_{k,a,j})}}$$

where $A_{k,a,j} = \sum_{i=1}^m q_{jki} A_{i,a} \in \mathbb{R}_{>0}$ and $B_{k,a,j} = \sum_{i=1}^m q_{jki} B_{i,a} \in \mathbb{R}_{>0}$.

Notice that the above calculations just depended on which components $\overline{E_i}$ contained a. If $T_{a_1} = T_{a_2}$ then $J_{a_1}(s,p) = J_{a_2}(s,p)$. So for each $I \subset T$ let

$$c_{p,I} = \text{card}\{a \in \overline{Y}(\mathbb{F}_p) : a \in \overline{E_i} \text{ if and only if } i \in I\}$$

and put $A_{k,I,j} = A_{k,a,j}$ and $B_{k,I,j} = B_{k,a,j}$ for any $a \in \{a \in \overline{Y}(\mathbb{F}_p) : a \in \overline{E_i}$ if and only if $i \in I\}$ where $j = 1,...,w_I$ and w_I is the number of simplicial cones defined by the linear inequalities corresponding to I. Then we have a final formula for $Z_{\mathcal{D}}(s,p)$

$$Z_{\mathcal{D}}(s,p) = (1-p^{-1})^m \sum_{I \subset T} c_{p,I} \sum_{j=1}^{w_I} \prod_{k=1}^{m_j} \frac{p^{-(A_{k,I,j}s+B_{k,I,j})}}{1-p^{-(A_{k,I,j}s+B_{k,I,j})}}. \tag{6}$$

Note that if $A_{k,I,j} = 0$ and $B_{k,I,j} = 1$, which will correspond to a bit of the integral like $\int_{p\mathbb{Z}_p} |dy_m|$, then we get $(1-p^{-1}) \cdot \frac{p^{-1}}{(1-p^{-1})}$ which is correct.

This completes the proof of Theorem 3. Note that this expression (6) for $Z_{\mathcal{D}}(s,p)$ holds for all primes for which the resolution (Y,h) had good reduction.

We could also consider a cone integral defined over $\mathbb{Q}_p$ whose cone data $\mathcal{D}$ consisted of polynomials in $\mathbb{Q}_p[\mathbf{x}]$. Our formula (6) would still be valid for such integrals provided that the resolution had good reduction mod$p$.

Notice that, as we vary p, the only things in this formula which depend on p are the terms $c_{p,I}$.

We should note that there is one term which is always a constant term in the expression for our final formula (6) corresponding to the subset $I = \emptyset$; then $w_\emptyset = 1$, $m_1 = m$ and $A_{k,\emptyset,1} = 0$ and $B_{k,\emptyset,1} = 1$ for $k = 1,...,m$. Hence the term corresponding to the subset $I=\emptyset$ has the following form:

$$(1-p^{-1})^m c_{p,\emptyset}\sum_{j=1}^{w_\emptyset}\prod_{k=1}^{m_j}\frac{p^{-(A_{k,\emptyset,j}s+B_{k,\emptyset,j})}}{1-p^{-(A_{k,\emptyset,j}s+B_{k,\emptyset,j})}}=c_\emptyset(1-p^{-1})^m\frac{p^{-m}}{(1-p^{-1})^m}=c_\emptyset p^{-m}. \tag{7}$$

Since the restriction $h: Y\backslash h^{-1}(D)\to X\backslash D$ is an isomorphism (where $D=\mathrm{Spec}\left(\frac{\mathbb{Q}[\mathbf{x}]}{(F)}\right)\subset X_{\mathbb{Q}}$)

$$\begin{aligned} c_{p,\emptyset} &= \mathrm{card}\{a\in\overline{Y}(\mathbb{F}_p): a\notin\overline{E_i} \text{ for all } i\in T\} \\ &= \mathrm{card}\overline{X}(\mathbb{F}_p)-\mathrm{card}\overline{D}(\mathbb{F}_p). \end{aligned}$$

The term (7) is part of the constant term of the rational function $Z_{\mathcal{D}}(s,p)$. The other parts of the constant term come from those $I\subset T$ and $j\in\{1,...,w_I\}$ such that $A_{k,I,j}=0$ for all $k=1,...,m_j$.

3 A second explicit expression for cone integrals

The explicit expression (6) determined in the previous section has a number of advantages. It expresses the function as a sum over the subsets of I which identifies precisely the bits $c_{p,I}$ which depend on p. This form of the sum is also more amenable to Denef and Meuser's proof that the Igusa local zeta function (where ψ is the empty condition) satisfies a functional equation (see [4]).

However for the analysis of the analytic properties of the global zeta function, as we explained in the introduction, it is preferable to work with a second explicit formula we can establish which writes the cone integrals as a sum over open simplicial pieces of a single cone defined in card$T$ dimensions where each open simplicial piece of the cone gets a weight according to the size of I and $c_{p,I}$.

We give a proof in this section of Theorem 4 where all the data in the formula e.g. A_j and B_j are identified explicitly in terms of the numerical data of the resolution and the underlying cone.

The cone is defined as follows:

$$\overline{D_T}=\left\{(x_1,...,x_t)\in\mathbb{R}^t_{\geq 0}:\sum_{j=1}^t N_j(f_i)x_j\leq\sum_{j=1}^t N_j(g_i)x_j \text{ for } i=1,...,l\right\}$$

where card$T = t$ and $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$, so this is a closed cone. Denote the lattice points in $\overline{D_T}$ by $\overline{\Delta_T}$, i.e. $\overline{\Delta_T} = \overline{D_T} \cap \mathbb{N}_0^t$. We can write this cone as a disjoint union of open simplicial pieces that we shall call R_k $k = 0, 1, ..., w$. We shall assume that $R_0 = (0, ...0)$ and that the next q pieces are all the open one-dimensional edges in the cone $\overline{D_T}$: for $k = 1, ..., q$

$$R_k = \{\alpha \mathbf{e}_k = \alpha(q_{k1}, ..., q_{kt}) : \alpha > 0\}.$$

Since these are all the edges, for any $k \in \{0, ..., w\}$ there exists some subset $M_k \subset \{1, ..., q\}$ such that

$$R_k = \left\{ \sum_{j \in M_k} \alpha_j \mathbf{e}_j : \alpha_j > 0 \text{ for all } j \in M_k \right\}.$$

Note that $m_k :=$ card$M_k \leq t$.

Define for each $k = 1, ..., q$ the following constants:

$$\begin{aligned} A_k &= \sum_{j=1}^{t} q_{kj} N_j(f_0) \qquad (8) \\ B_k &= \sum_{j=1}^{t} q_{kj} \left(N_j(g_0) + \nu_j\right). \end{aligned}$$

For each subset $I \subset T$ we previously defined a rational convex polyhedral cone C_I with lattice points Λ_I which we broke down into simplicial cones $C_1^I, ..., C_{w_I}^I$ with corresponding lattice points $\Lambda_1^I, ..., \Lambda_{w_I}^I$. These were cones in the open positive quadrant $\mathbb{R}_{>0}^m$. We are going to define a new cone and lattice points inside C_T which we can use to express the same rational function that we associated to C_I.

For each $I \subset T$ define:

$$\begin{aligned} D_I &= \left\{(k_1, ..., k_t) \in \overline{D_T} : k_i > 0 \text{ if } i \in I \text{ and } k_i = 0 \text{ if } i \in T \backslash I\right\} \\ \Delta_I &= D_I \cap \mathbb{N}_0^t. \end{aligned}$$

So $\overline{D_T} = \bigcup_{I \subset T} D_I$, a disjoint union. The reason we chose the notation $\overline{D_T}$ is of course because it is the closure then of D_T defined as above.

For each $I \subset T$ there is then a subset $W_I \subset \{0, ..., w\}$ so that

$$D_I = \bigcup_{k \in W_I} R_k.$$

We now have the following:

Lemma 11

$$(1-p^{-1})^m c_{p,I}\sum_{j=1}^{w_I}\prod_{k=1}^{m_j}\frac{p^{-(A_{k,I,j}s+B_{k,I,j})}}{1-p^{-(A_{k,I,j}s+B_{k,I,j})}}$$

$$=c_{p,I}\sum_{k\in W_I}(1-p^{-1})^{|I|}p^{-(m-|I|)}\prod_{j\in M_k}\frac{p^{-(A_js+B_j)}}{1-p^{-(A_js+B_j)}}.$$

Corollary 12 *Set $c_{p,k}=c_{p,I}$ and $I_k=I$ if $k\in W_I$. Then for all primes p for which the resolution has good reduction we get*

$$Z_{\mathcal{D}}(s,p)=\sum_{k=0}^{w}(p-1)^{|I_k|}p^{-m}c_{p,k}\prod_{j\in M_k}\frac{p^{-(A_js+B_j)}}{1-p^{-(A_js+B_j)}}$$

Proof We go back to the integral expression (5) for $J_a(s,p)$ where $a\in\{a\in\overline{Y}(\mathbb{F}_p):a\in\overline{E_i}$ if and only if $i\in I\}$. The calculations of the previous section gave:

$$J_a(s,p)=(1-p^{-1})^m\sum_{j=1}^{w_I}\prod_{k=1}^{m_j}\frac{p^{-(A_{k,I,j}s+B_{k,I,j})}}{1-p^{-(A_{k,I,j}s+B_{k,I,j})}}.$$

We can rewrite the expression for $J_a(s,p)$ as

$$J_a(s,p)=p^{-(m-|I|)}\int_{V_p'}\prod_{i\in I}|z_i|^{N_i(f_0)s+N_i(g_0)+\nu_i-1}\prod_{i\in I}|dz_i|$$

where V_p' is now the set of $(z_i)_{i\in I}\in(p\mathbb{Z}_p)^{|I|}$ satisfying for $j=1,...,l$

$$\sum_{i\in I}N_i(f_j)\mathrm{ord}(z_i)\leq\sum_{i\in I}N_i(g_j)\mathrm{ord}(z_i).$$

Looking at the definition of $\Delta_I=D_I\cap\mathbb{N}_0^t\subset\overline{D_T}$ this then reduces to

$$\begin{aligned}J_a(s,p)&=p^{-(m-|I|)}(1-p^{-1})^{|I|}\sum_{(k_1,\dots,k_t)\in\Delta_I}p^{-\sum_{j=1}^t k_j(N_j(f_0)s+N_j(g_0)+\nu_j)}\\&=\sum_{k\in W_I}p^{-(m-|I|)}(1-p^{-1})^{|I|}\sum_{(k_1,\dots,k_t)\in R_k\cap\mathbb{N}_0^t}p^{-\sum_{j=1}^t k_j(N_j(f_0)s+N_j(g_0)+\nu_j)}\end{aligned}$$

since $D_I = \bigcup_{k \in W_I} R_k$. Since

$$R_k \cap \mathbb{N}_0^t = \left\{ \sum_{j \in M_k} \alpha_j \mathbf{e}_j : \alpha_j \in \mathbb{N} \text{ for all } j \in M_k \right\}$$

by making a change of variable as in the previous section and using the definitions (8) of A_k and B_k for $k = 1, ..., q$ we get

$$J_a(s,p) = \sum_{k \in W_I} p^{-(m-|I|)}(1-p^{-1})^{|I|} \prod_{j \in M_k} \frac{p^{-(A_j s + B_j)}}{1 - p^{-(A_j s + B_j)}}.$$

This completes the proof of the lemma.

Recall that even if the dimension of the simplicial piece has gone down (i.e. $m_k < |I|$), we will still get a $(1-p^{-1})$ for each variable. For example, the integral $\int_{\mathbb{Z}_p^2} |x|^s |y|^s$ over $v(x) = v(y)$ is $(1-p^{-1})^2(1-p^{-2(s+1)})^{-1}$. The second point to note is that $c_{p,I} = 0$ for any $I \subset T$ for which card$I > m$, where recall m is the number of variables in the original integral.

4 Zeta functions defined as Euler products of cone integrals

We now turn to analysing the global behaviour of a product of these cone integrals over all primes p.

We make some normalisation of the cone integrals so that the constant coefficient of the local factors is 1.

Definition 13 *Let $Z_{\mathcal{D}}(s,p)$ be a cone integral defined over $\mathbb{Q}$. Then denote by $a_{p,0}$ the constant coefficient of the power series in p^{-s} representing the rational function $Z_{\mathcal{D}}(s,p)$.*

We should point out that even for primes p for which the resolution (Y,h) has bad reduction, the cone integral $Z_{\mathcal{D}}(s,p)$ is a rational function in p^{-s}. We just don't get the nice expressions of the previous two sections. The rationality follows from Denef's proof that definable p-adic integrals are always rational functions in p^{-s} (see [2]).

Definition 14 *We say that a function $Z(s)$ is defined as an Euler product of cone integrals over $\mathbb{Q}$ with cone integral data $\mathcal{D}$ if*

$$Z(s) = Z_{\mathcal{D}}(s) = \prod_{p \text{ prime}} \left(a_{p,0}^{-1} \cdot Z_{\mathcal{D}}(s,p) \right).$$

Theorem 15 *A function $Z(s)$ defined as an Euler product of cone integrals over $\mathbb{Q}$ has a rational abscissa of convergence $\alpha \in \mathbb{Q}$.*

Proof We know that for each prime p, $Z_{\mathcal{D}}(s,p)$ is a rational function with rational abscissa of convergence. From our explicit formula for $Z_{\mathcal{D}}(s,p)$ which holds for almost all primes it will suffice to prove that for some finite set of primes S

$$\prod_{p \notin S} \left(a_{p,0}^{-1} \cdot \sum_{k=0}^{w} (p-1)^{|I_k|} p^{-m} c_{p,k} \prod_{j \in M_k} \frac{p^{-(A_j s + B_j)}}{1 - p^{-(A_j s + B_j)}} \right)$$

has a rational abscissa of convergence.

The following Lemma collects some relevant information about the constant $a_{p,0}$ with which we are normalising our integrals:

Lemma 16 *(1) $a_{p,0} = \lim_{s \to \infty} Z_{\mathcal{D}}(s,p)$.*

(2)

$$a_{p,0} = \int_{\left\{\mathbf{x} \in \mathbb{Z}_p^m : \psi(\mathbf{x}) \text{ holds in } \mathbb{Z}_p \text{ and } f_0(\mathbf{x}) \in \mathbb{Z}_p^*\right\}} |g_0(\mathbf{x})|\, |dx|.$$

(3) For each $I \subset T$, let W_I' denote the set of those k for which $\sum_{j \in M_k} A_j \neq 0$. Then for almost all primes p

$$a_{p,0} = \sum_{I \subset T} p^{-m} c_{p,I} \sum_{k \in W_I \setminus W_I'} (p-1)^{|I|} \prod_{j \in M_k} \frac{p^{-B_j}}{1 - p^{-B_j}}.$$

Let T_0 be the subset of T consisting of those i for which $N_i(f_0) = 0$. Then note that $W_I' = \emptyset$ for $I \subset T_0$.

(4) For almost all primes p

$$1 \geq a_{p,0} \geq (1 - p^{-1})^m.$$

(5) There exists $M \in \mathbb{N}$ such that for almost all primes p

$$1 \leq a_{p,0}^{-1} \leq 1 + M p^{-1/2}.$$

Proof Most of this is obvious. For (4) note that for almost all p

$$\mathbb{Z}_p^m \supseteq \left\{\mathbf{x} \in \mathbb{Z}_p^m : \psi(\mathbf{x}) \text{ holds in } \mathbb{Z}_p \text{ and } f_0(\mathbf{x}) \in \mathbb{Z}_p^*\right\} \supseteq \left(\mathbb{Z}_p^*\right)^m$$

and $|g_0(\mathbf{x})| \leq 1$ on $\mathbb{Z}_p^m$ whilst $|g_0(\mathbf{x})| = 1$ on $\left(\mathbb{Z}_p^*\right)^m$. Combining this with the description of $a_{p,0}$ in (2) yields the estimate in (4). For (5) we use $(1-p^{-1})^{-1} = 1 + p^{-1/2}(p^{1/2} - p^{-1/2})^{-1} \leq 1 + p^{-1/2}$.

Put $W' = \bigcup_{I \subset T} W_I'$. Then we get:

$$Z_{\mathcal{D}}(s) = \prod_p \left(1 + \sum_{k \in W'} \frac{c_{p,k}}{a_{p,0}} p^{-m} (p-1)^{|I|} \prod_{j \in M_k} \frac{p^{-(A_j s + B_j)}}{1 - p^{-(A_j s + B_j)}}\right). \tag{9}$$

We explain now a few facts about counting points on the reduction of varieties mod$p$.

We shall be interested in counting points on $\overline{X}$, the reduction mod$p$ of a variety X defined over $\mathbb{Q}$ which is irreducible over $\mathbb{Q}$. Let d be the dimension of X. The reduction $\overline{X}$ need not be irreducible as a variety over $\mathbb{F}_p$.

Let $\widehat{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ and $\mathbb{G}$ its Galois group. Consider the decomposition

$$X = X_1 \cup ... \cup X_n$$

of X into irreducible components over $\widehat{\mathbb{Q}}$. The Galois group $\mathbb{G}$ acts transitively on the set of components $\{X_1, ..., X_n\}$ because X is defined over $\mathbb{Q}$. Since the action is transitive all the X_i also have dimension d. Let $U \leq \mathbb{G}$ be the kernel of this action and put $L = \widehat{\mathbb{Q}}^U$. Then L is a finite Galois extension of $\mathbb{Q}$ with Galois group $\mathcal{G} = \mathbb{G}/U$ and every X_i $(i = 1, ..., n)$ is defined over L.

For every prime number p we choose a prime ideal $\mathfrak{p}$ in L which divides p. We write $I_\mathfrak{p} \leq D_\mathfrak{p} \leq \mathbb{G}$ for the corresponding inertia, respectively decomposition group. If p is unramified in L, that is $I_\mathfrak{p} = \{1\}$, we denote by Frob_p the conjugacy class in $\mathcal{G}$ consisting of the Frobenius elements.

We choose a finite set of primes S such that for every $p \notin S$ the following are satisfied:

(1) the reduction $\overline{X}$ mod$p$ of X is smooth; and

(2) p is unramified in L.

We define a function associated to the variety X which will be an essential tool in analysing our zeta functions.

Definition 17 *Let X be a smooth projective variety defined over $\mathbb{Q}$ which is irreducible over $\mathbb{Q}$. Define $l_p(X)$ to be the number of irreducible components (defined over $\mathbb{F}_p$) of $\overline{X}$, the reduction modp of X, which are absolutely irreducible. Define*

$$V_X(s) = \prod_p \left(1 - l_p(X)p^{-s}\right).$$

We prove the following important properties of this function $V_X(s)$:

Lemma 18 *(1) The abscissa of convergence of $V_X(s)$ is* 1.

(2) There is a $\delta > 0$ such that $V_X(s)$ has a meromorphic continuation to the half-plane $\Re(s) > 1 - \delta$.

Proof We apply the set up introduced above. The finite group $\mathcal{G}$ acts on $\{X_1, ..., X_n\}$ and we write M for the corresponding (complex) permutation module. We obtain a complex finite dimensional representation $\rho : \mathcal{G} \to \mathrm{GL}(M)$. Denote by $\mathrm{Tr}(\rho(\mathrm{Frob}_p))$ the trace of a representation of the conjugacy class Frob_p where $p \notin S$. We claim that

$$\mathrm{Tr}(\rho(\mathrm{Frob}_p)) = l_p(X) \tag{10}$$

for all $p \notin S$. To prove this we let $\overline{X_1}, ..., \overline{X_n}$ be the reductions of $X_1, ..., X_n$ mod$\mathfrak{p}$. These are absolutely irreducible, smooth projective varieties defined over the residue field $\mathbb{F}_\mathfrak{p}$ corresponding to the prime ideal $\mathfrak{p}$. Let $\mathcal{G}_\mathfrak{p}$ be the Galois group of $\mathbb{F}_\mathfrak{p}$ over $\mathbb{F}_p$. Then $\mathcal{G}_\mathfrak{p}$ acts on the set of components $\overline{X_1, ..., X_n}$ and the reduction mod$\mathfrak{p}$ is an equivariant map from $\{X_1, ..., X_n\}$ to $\left\{\overline{X_1}, ..., \overline{X_n}\right\}$ with respect to the natural isomorphism from $D_\mathfrak{p}$ to $\mathcal{G}_\mathfrak{p}$. Hence the number of fixed points of $\mathcal{G}$ on $\{X_1, ..., X_n\}$ is also equal to $l_p(X)$ and formula (10) follows.

We are now ready to prove statement (1) of Lemma 18. Clearly $l_p(X)$ is bounded by the number n of absolutely irreducible components of X. This implies that the abscissa of convergence of $V_X(s)$ is less than or equal to 1. On the other hand the set of primes $p \notin S$ such that $\mathrm{Frob}_p = \{1\}$ has non-zero arithmetic density by Čebotarov's theorem (see [18]). This establishes (1).

Finally we prove (2). We define

$$L_X(s) = \prod_{p \notin S} \det\left(1 - \rho(\mathrm{Frob}_p) \cdot p^{-s}\right)^{-1}.$$

This Euler product is up to finitely many Euler factors equal to the Artin L-function of ρ. It is well known that $L_X(s)$ has abscissa if convergence 1 and also has a meromorphic continuation to all of $\mathbb{C}$ (see [18]). From formula (10) we infer that

$$\det\left(1-\rho(\mathrm{Frob}_p)\cdot p^{-s}\right)=1-l_p(X)\cdot p^{-s}+\sum_{k=2}^{n}a_k p^{-ks}$$

with suitable a_k. It follows that

$$\left(1-l_p(X)\cdot p^{-s}\right)\left(\det\left(1-\rho(\mathrm{Frob}_p)\cdot p^{-s}\right)^{-1}\right)=1+\sum_{k=2}^{\infty}b_k p^{-ks}$$

with $b_k\in\mathbb{C}$. Since the eigenvalues of $\rho(\mathrm{Frob}_p)$ are roots of unity, the b_k can be bounded independently of $p\notin S$. This proves (2).

Lemma 19 *Let X be a smooth projective variety defined over $\mathbb{Q}$ which is irreducible over $\mathbb{Q}$. Let d be the dimension of X. There exists $\delta\in\mathbb{R}$ such that for almost all primes p*

$$|\mathrm{card}(\overline{X}(\mathbb{F}_p))-l_p(X)p^d|\leq\delta p^{d-1/2}$$

and $l_p(X)>0$ for a dense set of primes.

Proof Let Y be an smooth projective irreducible variety over $\mathbb{F}_p$ which is not absolutely irreducible. Then $Y(\mathbb{F}_p)$ is empty. To prove this let $Y=Y_1\cup\ldots\cup Y_r$ be the decomposition into irreducible components over $\widehat{\mathbb{F}_p}$. The Galois group of $\widehat{\mathbb{F}_p}$ over $\mathbb{F}_p$ acts transitively on the set of components $Y_1,\ldots,Y_r$. So an $\mathbb{F}_p$-point of Y would be contained in every component. If there is more than one component then it is a singular point of Y. But Y was assumed to be a smooth variety. Hence no such $\mathbb{F}_p$-point can exist.

Let Y be now a smooth projective, not necessarily irreducible variety over $\mathbb{F}_p$ and let $Y=Y_1\cup\ldots\cup Y_r$ be the decomposition into the disjoint union of the irreducible components over $\mathbb{F}_p$. Let them be ordered such that $Y_1,\ldots,Y_l$ are absolutely irreducible and $Y_{l+1},\ldots,Y_r$ are not. Then the above argument shows that

$$\mathrm{card}(Y(\mathbb{F}_p))=\mathrm{card}(Y_1(\mathbb{F}_p))+\cdots+\mathrm{card}(Y_l(\mathbb{F}_p)).$$

The statement of the lemma follows for all primes p such that reduction mod$p$ of X is smooth by the Lang-Weil estimate (see [15]). Note the irreducible components of $\overline{X}$ all have dimension d, the dimension of X, because the irreducible components over $\widehat{\mathbb{F}_p}$ can be obtained from the irreducible components of X over $\widehat{\mathbb{Q}}$ by reduction mod$\mathfrak{p}$ as explained above.

The fact that $l_p(X) > 0$ for a dense set of primes follows from the proof of the previous Lemma.

We can use Lemma 19 to estimate the size of $c_{p,I}$. Recall that this is the number of points mod p on $E_I \setminus \bigcup_{j\in T\setminus I} E_j$ where $E_I = \bigcap_{i\in I} E_i$.

Let $F_{I,k}$, $k \in C_I$, be the irreducible components over $\mathbb{Q}$ of E_I with maximal dimension d_I say. We show later (see Proposition 24) that $d_I = m - |I|$.

Lemma 20 *The dimension of $F_{I,k} \cap \bigcup_{j\in T\setminus I} E_j$ is strictly smaller than d_I. (If F_j has dimension zero then this means that $F_I \cap \bigcup_{j\in T\setminus I} E_j$ is the empty set, which by convention has dimension $-\infty$.)*

Proof If not then since $F_{I,k}$ is a variety it means that $F_{I,k} \subset \bigcup_{j\in T\setminus I} E_j$ and hence for some $j \notin I$, $E_j = F_{I,k} \subset E_I$, which can't happen.

Since the dimension of $E_I \cap \bigcup_{j\in T\setminus I} E_j$ is strictly smaller than d_I, this implies that for some $\delta' > 0$

$$\left| c_{p,I} - \sum_{k\in C_I} \mathrm{card}\overline{F_{I,k}}(\mathbb{F}_p) \right| \le \delta' p^{d_I - 1}.$$

But now this together with the Lang-Weil estimate in Lemma 19 implies:

Proposition 21 *There exists $\delta \in \mathbb{R}$ such that for almost all primes p*

$$|c_{p,I} - \sum_{k\in C_I} l_p(F_{I,k}) p^{d_I}| \le \delta p^{d_I - 1/2}$$

and $l_p(F_{I,k}) > 0$ for a dense set of primes p.

We make the following definitions for $k \in W'$:

$$\begin{aligned} Z_k(s) &= \frac{c_{p,k}}{a_{p,0}} p^{-m}(p-1)^{|I|} \prod_{j\in M_k} \frac{p^{-(A_j s + B_j)}}{1 - p^{-(A_j s + B_j)}} \\ d_k &= d_I \text{ and } I_k = I \text{ if } k \in W_I. \end{aligned}$$

If $d_k \geq 0$ then put

$$\alpha_k = \max \left\{ \frac{1 + d_k - m + |I_k| - \sum_{j \in M_k} B_j}{\sum_{j \in M_k} A_j}, \frac{-B_j}{A_j} \text{ for } j \in M_k \text{ and } A_j \neq 0 \right\}.$$

If $d_k = -\infty$, i.e. $c_{p,k} = 0$ for almost all p, then put $\alpha_k = -\infty$.

To analyse the convergence of our Euler product we use the following basic facts:

(A) An infinite product $\prod_{n \in J}(1 + a_n)$ converges absolutely if and only if the corresponding sum $\sum_{n \in J} |a_n|$ converges.

(B) $\sum_{p \text{ prime}} |p^{-s}|$ converges if and only if $\{s \in \mathbb{C} : \Re(s) > 1\}$. In fact if $\mathcal{P}$ is a dense set of primes we also have that $\sum_{p \in \mathcal{P}} |p^{-s}|$ converges if and only if $\{s \in \mathbb{C} : \Re(s) > 1\}$.

(C) The Lang-Weil estimate in Proposition 21 means that a sum of the form

$$\sum_p c_{p,I} r_p$$

converges absolutely if and only

$$\sum_p p^{d_I} r_p$$

converges absolutely. Note that we are using here that $l_p(F_{I,k})$ is non-zero with positive density.

Lemma 22 *For $k \in W'$, α_k is the abscissa of convergence of*

$$\prod_p (1 + Z_k(s)).$$

Proof First note that each term $\frac{p^{-(A_j s + B_j)}}{1 - p^{-(A_j s + B_j)}}$ converges if and only if $\Re(s) > \frac{-B_j}{A_j}$.

Suppose that for some $l \in M_k$ for which $A_l \neq 0$, $\frac{-B_l}{A_l} \geq \frac{1 + d_k - m + |I_k| - \sum_{j \in M_k} B_j}{\sum_{j \in M_k} A_j}$. To prove that α_k is the abscissa of convergence it will suffice in this case to show that

$$\sum_p (p-1)^{|I_k|} \frac{c_{p,k}}{a_{p,0}} p^{-m} \prod_{j \in M_k} \frac{p^{-(A_j s + B_j)}}{1 - p^{-(A_j s + B_j)}} \tag{11}$$

converges for $\Re(s) > \alpha_k$. Since $\left(1 - p^{-(A_j s + B_j)}\right)^{-1}$ is a positive decreasing sequence as p increases for $\Re(s) > \frac{-B_j}{A_j}$ if $A_j \neq 0$ it suffices to show that

$$\sum_p (p-1)^{|I_k|} \frac{c_{p,k}}{a_{p,0}} p^{-m} p^{-\left(\sum_{j \in M_k} A_j s + B_j\right)} \tag{12}$$

converges.

From Lemma 16 (5) $a_{p,0}^{-1} \leq 1 + Mp^{-1/2}$. Hence (12) converges provided

$$\sum_p (p-1)^{|I_k|} c_{p,k} p^{-m} p^{-\left(\sum_{j \in M_k} A_j s + B_j\right)}$$

converges. This sum converges (using our Lang-Weil argument (C)) if

$$\Re(s) > \frac{1 + d_k - m + |I_k| - \sum_{j \in M_k} B_j}{\sum_{j \in M_k} A_j}.$$

This is the case since $\Re(s) > \alpha_k \geq \frac{1 + d_k - m + |I_k| - \sum_{j \in M_k} B_j}{\sum_{j \in M_k} A_j}$.

Suppose now that $\alpha_k = \frac{1 + d_k - m + |I_k| - \sum_{j \in M_k} B_j}{\sum_{j \in M_k} A_j} > \frac{-B_j}{A_j}$ for all $j \in M_k$.

Now there exist $\varepsilon > 0$, $\delta_1, \delta_2 > 0$ such that $\alpha_k - \varepsilon > \frac{-B_j}{A_j}$ for all $j \in M_k$, and for $\Re(s) > \alpha_k - \varepsilon$ and all primes p:

$$1 < \prod_{j \in M_k} \left(1 - p^{-(A_{k,I,j} s + B_{k,I,j})}\right)^{-1} < 1 + \delta_1 p^{-\delta_2}.$$

Hence, combining this with Lemma 16 (5) and our Lang-Weil argument (C), for $\Re(s) > \alpha_k - \varepsilon$ the sum (11) converges if and only if the sum

$$\sum_p (p-1)^{|I_k|} p^{d_k} p^{-m} p^{-\left(\sum_{j \in M_k} A_j s + B_j\right)}$$

i.e. if and only if $\Re(s) > \alpha_k$. This completes our proof that α_k is the abscissa of convergence of $\prod_p (1 + Z_k(s))$.

Corollary 23 *The abscissa of convergence of $Z_{\mathcal{D}}(s)$, $\alpha_{\mathcal{D}}$, is equal to* $\max\{\alpha_k : k \in W'\}$. *In particular, $\alpha_{\mathcal{D}}$ is a rational number.*

The next task is to show that we can meromorphically continue $Z_{\mathcal{D}}(s)$ a little beyond $\Re(s) > \alpha_{\mathcal{D}}$ to allow an application of the Tauberian Theorem. The first step is to show that the maximal value of α_k as k varies over the set $W' \subset W$ (where recall W indexes the open simplicial pieces R_k of $\overline{D_T}$) is realised only by the one dimensional simplicial edges, i.e. $k \in \{1, ..., q\}$.

The essential fact that we shall use here is the following:

Proposition 24 *For $k \in W$, $d_k = m - |I_k|$.*

Proof Each E_i is of dimension $m-1$. The E_i intersect with normal crossings hence the irreducible components of $\bigcap_{i\in I_k} E_i$ have dimension $d_k = m - |I_k|$. For further details see the proof of Theorem 2.4 of [3] or 17.F of [19].

This already gives us a much simpler description of α_k :

Corollary 25 *(1) For $k \in W'$*

$$\alpha_k = \max\left\{ \frac{1-\sum_{j\in M_k} B_j}{\sum_{j\in M_k} A_j}, \frac{-B_j}{A_j} \textit{ for } j \in M_k \textit{ and } A_j \neq 0 \right\}.$$

(2) If $k \in W' \cap \{1, ..., q\}$ then

$$\alpha_k = \frac{1-B_k}{A_k}.$$

Lemma 26

$$\alpha_{\mathcal{D}} = \max\{\alpha_k : k \in W'\} = \max\{\alpha_k : k \in W' \cap \{1, ..., q\}\}$$

and $\alpha_k < \alpha_{\mathcal{D}}$ if $k > q$.

Proof This follows once we can show that for $k > q$ and $k \in W'$

$$\frac{1-\sum_{j\in M_k} B_j}{\sum_{j\in M_k} A_j} < \max\left\{ \frac{1-B_j}{A_j} : j \in M_k \cap W' \right\} \tag{13}$$

Recall the definition of W' from Lemma 16 that $k \in W'$ if $\sum_{j\in M_k} A_j \neq 0$. We may suppose that if $j \in M_k$ then $j \in W'$ since

$$\frac{1-\sum_{j\in M_k} B_j}{\sum_{j\in M_k} A_j} \leq \frac{1-\sum_{j\in M_k\cap W'} B_j}{\sum_{j\in M_k\cap W'} A_j}.$$

We may suppose without loss of generality that $|M_k| = 2$, $\frac{1-B_2}{A_2} \leq \frac{1-B_1}{A_1}$ and that A_1 and $A_2 > 0$. Then

$$\frac{1-(B_1+B_2)}{A_1+A_2} < \frac{1-B_1}{A_1}$$

if and only if

$$A_1 - A_1B_1 - A_1B_2 < A_1 + A_2 - A_1B_1 - A_2B_1.$$

But our assumptions that $\frac{1-B_2}{A_2} \leq \frac{1-B_1}{A_1}$ and $A_2 > 0$ imply this second inequality. This confirms statement (13) and hence the lemma.

Theorem 27 $\prod_p \left(1 + \sum_{j=1}^{w} Z_j(s)\right)$ *has a meromorphic continuation to* $\Re(s) > \alpha_{\mathcal{D}} - \delta$ *for some* $\delta > 0$.

Proof Define

$$\begin{aligned} R &= \left\{ k \in W' \cap \{1, ..., q\} : \frac{1-B_j}{A_j} = \alpha_{\mathcal{D}} \right\} \\ V_p(s) &= \prod_{k \in R} \left(1 - c_{p,k} p^{-m+|I_k|} p^{-(A_k s + B_k)}\right). \end{aligned}$$

We introduce the following notation which will be convenient during the course of the proof. Write $\prod_p F_p(s) \equiv \prod_p G_p(s)$ if there exists $\delta > 0$ such that $\sum_p (F_p(s) - G_p(s))$ converges for $\Re(s) > \alpha_{\mathcal{D}} - \delta$. To prove our lemma it will suffice to prove the following:

(1) $\prod_p V_p(s)$ is a meromorphic function on $\Re(s) > \alpha_{\mathcal{D}} - \delta$ for some $\delta > 0$.

(2) $\prod_p \left(1 + \sum_{j=1}^{w} Z_j(s)\right) W_p(s) V_p(s) \equiv 1$, i.e. the Euler product converges on $\Re(s) > \alpha_{\mathcal{D}} - \delta$ for some $\delta > 0$.

We prove (2). It will be convenient to note the following fact: suppose that for $s > \alpha_{\mathcal{D}} - \delta$, the function $X_p(s)$ as $p \to \infty$ is a positive decreasing sequence. If $\prod_p F_p(s) \equiv \prod_p G_p(s)$ then $\prod_p F_p(s) X_p(s) \equiv \prod_p G_p(s) X_p(s)$.

By Lemma 26 $\prod_p \left(1 + \sum_{j=1}^{w} Z_j(s)\right) \equiv \prod_p \left(1 + \sum_{k \in R_1 \cup R_0} Z_k(s)\right)$. Recall the definition of $Z_k(s)$ for $k \in \{1, ..., q\}$:

$$Z_k(s) = \frac{c_{p,k}}{a_{p,0}} p^{-m} (p-1)^{|I_k|} \frac{p^{-(A_k s + B_k)}}{1 - p^{-(A_k s + B_k)}}.$$

Note that $\sum_p c_{p,k} p^{-m+|I_k|} \frac{p^{-2(A_k s+B_k)}}{1-p^{-(A_k s+B_k)}}$ converges on $\Re(s) > \alpha_{\mathcal{D}} - \delta$ for some $\delta > 0$. This follows because $\left(1 - p^{-(A_k s+B_k)}\right)^{-1}$ is a positive decreasing sequence and $\frac{(1-2B_k)}{2A_k} < \frac{1-B_k}{A_k}$. Putting this together with our analysis on the estimate for $a_{p,0}$ means that $\prod_p \left(1 + \sum_{k\in R_1\cup R_0} Z_k(s)\right) \equiv \prod_p \left(1 + \sum_{k\in R_1\cup R_0} \overline{Z_k(s)}\right)$ where

$$\overline{Z_k(s)} = c_{p,k} p^{-m+|I_k|} p^{-(A_k s+B_k)}.$$

Hence we have

$$\prod_p \left(1 + \sum_{j=1}^{w} Z_j(s)\right) \equiv \prod_p \left(1 + \sum_{k\in R} \overline{Z_k(s)}\right). \tag{14}$$

Now

$$\prod_p V_p(s) \equiv \prod_p \left(1 - \sum_{k\in R} c_{p,k} p^{-m+|I_k|} p^{-(A_k s+B_k)}\right) = \prod_p \left(1 - \sum_{k\in R} \overline{Z_k(s)}\right) \tag{15}$$

since $\frac{\left(1-B_{k_1}-\ldots-B_{k_i}\right)}{A_{k_1}+\ldots+A_{k_i}} < \frac{1-B_k}{A_k}$ for $i > 1$. Now from (14) and (15) we get that

$$\begin{aligned}
\prod_p \left(1 + \sum_{j=1}^{w} Z_j(s)\right) V_p(s) &\equiv \prod_p \left(1 + \sum_{j=1}^{w} Z_j(s)\right)\left(1 - \sum_{k\in R} \overline{Z_k(s)}\right) \\
&\equiv \prod_p \left(1 + \sum_{k\in R} \overline{Z_k(s)}\right)\left(1 - \sum_{k\in R} \overline{Z_k(s)}\right).
\end{aligned}$$

Here we are using that $\sum_{j=1}^{w} Z_j(s)$ and $\sum_{k\in R} c_{p,k} p^{-m+|I_k|} p^{-(A_k s+B_k)}$ are both decreasing positive sequences. Finally the fact that $\frac{\left(1-B_{k_1}-B_{k_2}\right)}{A_{k_1}+A_{k_2}} < \frac{1-B_{k_i}}{A_{k_i}}$ implies that

$$\prod_p \left(1 + \sum_{k\in R} \overline{Z_k(s)}\right)\left(1 - \sum_{k\in R} \overline{Z_k(s)}\right) \equiv 1.$$

Hence we have proved (2).

For the proof of (1) note that by Proposition 21

$$\begin{aligned}
\prod_{p \text{ prime}} \left(1 - c_{p,k} p^{-m+|I_k|} p^{-(A_k s+B_k)}\right) &\equiv \prod_{p \text{ prime}} \left(1 - \sum_{j\in C_I} l_p(F_{I_k,j}) p^{d_k} p^{-m+|I_k|} p^{-(A_k s+B_k)}\right) \\
&\equiv \prod_{j\in C_I} \prod_{p \text{ prime}} \left(1 - l_p(F_{I_k,j}) p^{-(A_k s+B_k)}\right)
\end{aligned}$$

Hence the fact that $\prod_p V_p(s)$ is meromorphic on $\Re(s) > \alpha_{\mathcal{D}} - \delta$ for some $\delta > 0$ follows from Lemma 18. This completes the proof of Theorem 27.

To prove our main Theorem about the growth of the coefficients in a Dirichlet series defined as the Euler product of cone integrals over $\mathbb{Q}$ we now can apply the Tauberian Theorem. We take the following statement from page 62 of [1]:

Theorem 28 *Suppose $F(s) = \frac{1}{s}\int_0^\infty e^{-st}dA(t)$ for $\Re(s) > a > 0$, $A(t) > 0$, and that for $\Re(s) > a$*

$$F(s) = g(s)(s-a)^{-w} + h(s)$$

where g and h are holomorphic in a neighbourhood of a with $g(a) \neq 0$ and $w > 0$. Then

$$A(t) \sim \frac{g(a)}{\Gamma(w)} \cdot e^{at}t^{w-1}.$$

In our case we take $A(t) = \sum_{\log n < t} a_n$ and $f(s) = \sum_{n=1}^\infty a_n n^{-s} = sF(s)$ then substituting $t = \log x$ we get:

Theorem 29 *Let the Dirichlet series $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ with non-negative coefficients be convergent for $\Re(s) > a > 0$. Assume in a neighbourhood of a, $f(s) = g(s)(s-a)^{-w} + h(s)$, holds where $g(s), h(s)$ are holomorphic functions, $g(a) \neq 0$ and $w > 0$. Then for x tending to infinity, we have*

$$\sum_{n \leq x} a_n \sim \left(\frac{g(a)}{a\Gamma(w)}\right) \cdot x^a (\log x)^{w-1}.$$

We collect together the results of this section in:

Corollary 30 *Let $Z(s) = \sum_{n=1}^\infty a_n n^{-s}$ be defined as an Euler product of cone integrals over $\mathbb{Q}$.*

(1) The abscissa of convergence α of $Z(s)$ is a rational number and $Z(s)$ has a meromorphic continuation to $\Re(s) > \alpha - \delta$ for some $\delta > 0$.

(2) Let the pole at $s = \alpha$ have order w. Then for every β with $\beta + \alpha > 0$, there exists some real number $c \in \mathbb{R}$ such that

$$a_1 + \cdots + a_n n^\beta \sim c \cdot n^{\beta+\alpha} (\log n)^{w-1}.$$

Note the we do not know anything about the possibility of other poles in the region $\alpha = \Re(s) > \alpha - \delta$ other than the one at $s = \alpha$. This is because we have used Artin L-functions to do our meromorphic continuation. However it is conjectured that these Artin L -functions actually have this one pole and no others (see [18]).

5 Nilpotent Groups

In this section we show that the zeta functions of nilpotent groups are Euler products of cone integrals over $\mathbb{Q}$.

To simplify our analysis of subgroups and normal subgroups we use the following notation: $\zeta_G^{\leq}(s) = \zeta_G(s)$.

The following result established in [13] means we can concentrate on counting in rings rather than groups:

Theorem 31 *Let G be a finitely generated nilpotent group. Then there is a Lie algebra $L(G)$ over $\mathbb{Z}$ constructed as the image under* log *of some subgroup of finite index in G such that for $* \in \{\leq, \triangleleft\}$ and almost all primes p*

$$\zeta^*_{G,p}(s) = \zeta^*_{L(G),p}(s).$$

The finitely many primes will not worry us thanks to the following result, proved first for torsion-free nilpotent groups in [13] and more generally by the first author in [5]:

Theorem 32 *Let G be a finitely generated group of finite rank then, for $* \in \{\leq, \triangleleft\}$ and for each prime p, $\zeta^*_{G,p}(s)$ is a rational function with rational abscissa of convergence $\alpha^*(G)_p \in \mathbb{Q}$.*

Recall that a group G has finite rank if there is a bound on the number of generators of finitely generated subgroups; for example this holds if G is a finitely generated nilpotent group or more generally a polycyclic group.

The proof of this rationality of the local factors depended on expressing them as "definable" p-adic integrals. We recall the description of these integrals in the case of a ring L which is additively isomorphic to $\mathbb{Z}^d$. Fix a basis for L which identifies it with $\mathbb{Z}^d$. The multiplication in L is given by a bilinear mapping

$$\beta : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}^d,$$

which extends to a bilinear map $\mathbb{Z}_p^d \times \mathbb{Z}_p^d \to \mathbb{Z}_p^d$ for each prime p, giving the structure of the $\mathbb{Z}_p$-algebra $L_p = L \otimes \mathbb{Z}_p$.

Define

$$V_p^{\leq} = \left\{ \begin{array}{c} (m_{kl}) \in \mathrm{Tr}_d\,(\mathbb{Z}_p) : \text{for } 1 \leq i,j \leq d, \exists Y_{ij}^1, ..., Y_{ij}^d \in \mathbb{Z}_p \\ \text{such that } \beta(\mathbf{m}_i, \mathbf{m}_j) = \sum_{k=1}^d Y_{ij}^k \mathbf{m}_k \end{array} \right\}$$

and

$$V_p^{\triangleleft} = \left\{ \begin{array}{c} (m_{kl}) \in \mathrm{Tr}_d(\mathbb{Z}_p) : \text{for } 1 \leq i, j \leq d, \exists Y_{ij}^1, ..., Y_{ij}^d \in \mathbb{Z}_p \\ \text{such that } \beta(\mathbf{m}_i, \mathbf{e}_j) = \sum_{k=1}^d Y_{ij}^k \mathbf{m}_k \end{array} \right\}$$

where $\mathrm{Tr}_d(\mathbb{Z}_p)$ denotes the set of upper triangular matrices. Each subset consists of matrices whose rows $\mathbf{m}_i$ generate a lattice in $\mathbb{Z}_p^d$ which is either a subalgebra or an ideal. The d tuple $\mathbf{e}_j$ denotes the standard unit vector with 1 in the jth entry and zeros elsewhere and corresponds to the jth basis element.

Proposition 33 *([13] Proposition 3.1) For $* \in \{\leq, \triangleleft\}$ and each prime p*

$$\zeta_{L,p}^{*} = (1 - p^{-1})^{-d} \int_{V_p^*} |m_{11}|^{s-1} \dots |m_{dd}|^{s-d} |dx|$$

where v is the valuation on $\mathbb{Z}_p$, $|m| = p^{-v(m)}$ and $|dx|$ is the normalized additive Haar measure on $\mathbb{Z}_p^{d(d+1)/2} \equiv \mathrm{Tr}_d(\mathbb{Z}_p)$.

The proof of the rationality of these p-adic integrals relied on observing that V_p^* are definable subsets in the language of fields. One can then apply a Theorem of Denef's [2] which establishes the rationality of definable p-adic integrals. Denef's proof relies on an application of Macintyre's quantifier elimination for the theory of $\mathbb{Q}_p$ which simplifies in a generally mysterious way the description of definable subsets like V_p^*. However in our case it is possible to do this elimination by hand since it involves solving linear equations. This removes the necessity for the model theoretic black box in the proof of the rationality.

Theorem 34 *Let L be a ring additively isomorphic to R^d where $R = \mathbb{Z}$ or $\mathbb{Z}_p$. For $* \in \{\leq, \triangleleft\}$ there exist homogeneous polynomials $g_{ijk}^*(\mathbf{X}) \in R[X_{rs} : 1 \leq r \leq s \leq d]$ $(i, j, k \in \{1, ..., d\})$ of degree i for $* = \triangleleft$ and $i + 1$ for $* = \leq$ such that*

$$V_p^* = \left\{ (m_{rs}) \in \mathrm{Tr}_d(\mathbb{Z}_p) : v(m_{11} ... m_{kk}) \leq v(g_{ijk}^*(m_{rs})) \text{ for } i, j, k \in \{1, ..., d\} \right\}.$$

Proof We can express the defining conditions for V_p^* in matrix form which makes things quite transparent. Let C_j denote the matrix whose rows are $\mathbf{c}_i = \beta(\mathbf{e}_i, \mathbf{e}_j)$.

$M \in V_p^{\triangleleft}$ if we can solve for each $1 \leq i, j \leq d$ the equation

$$\mathbf{m}_i C_j = \left(Y_{ij}^1, ..., Y_{ij}^d\right) M$$

with $\left(Y_{ij}^1, ..., Y_{ij}^d\right) \in \mathbb{Z}_p^d$. Let M' denote the adjoint matrix and

$$M^{\natural} = M' diag(m_{22}^{-1}...m_{dd}^{-1}, ..., m_{dd}^{-1}, 1).$$

Then since the matrix M is upper triangular, the ikth entry of $M^{\natural}$ is a homogeneous polynomial of degree $k-1$ in the variables m_{rs} with $1 \leq r \leq s \leq k-1$. Then we can rewrite the above equation as:

$$\mathbf{m}_i C_j M^{\natural} = \left(m_{11} Y_{ij}^1, ..., m_{11}...m_{dd} Y_{ij}^d\right).$$

Let $g_{ijk}^{\triangleleft}(m_{rs})$ denote the kth entry of the d-tuple $\mathbf{m}_i C_j M^{\natural}$ which is a homogeneous polynomial of degree k in m_{rs}. (In fact we can see that it is homogenous of degree 1 in m_{is} ($s = 1, ..., d$) and degree $k-1$ in m_{rs} with $1 \leq r \leq s \leq k-1$.) Then $V_p^{\triangleleft}$ has the description detailed in the statement of the Theorem.

For $M \in V_p^{\leq}$ we are required to solve for each $1 \leq i, j \leq d$ the equation

$$\mathbf{m}_i \left(\sum_{l=j}^{d} m_{jl} C_l\right) M^{\natural} = \left(m_{11} Y_{ij}^1, ..., m_{11}...m_{dd} Y_{ij}^d\right)$$

with $\left(Y_{ij}^1, ..., Y_{ij}^d\right) \in \mathbb{Z}_p^d$. Let $g_{ijk}^{\leq}(m_{rs})$ denote the kth entry of the d-tuple $\mathbf{m}_i \left(\sum_{l=j}^{d} m_{jl} C_l\right) M^{\natural}$ which is a homogeneous polynomial of degree $k+1$ in m_{rs}. Again with these polynomials, $V_p^{\leq}$ has the description detailed in the statement of the Theorem.

So our Theorem has the following corollary that the local zeta functions of the ring L can be represented by cone integrals over $\mathbb{Q}$:

Corollary 35 *Let ψ^* be the cone condition defining V_p^*. Set $f_0 = m_{11}...m_{dd}$ and $g_0 = m_{11}^{d-1}...m_{d-1,d-1}$. Let $\mathcal{D}^*$ be the associated cone integral data. Then*

$$\zeta_{L,p}^*(s) = (1-p^{-1})^{-d} Z_{\mathcal{D}^*}(s-d, p).$$

We know that the constant term of $Z_{\mathcal{D}^*}(s-d,p)$ must be $(1-p^{-1})^d$ since $\zeta_{L,p}^*(s)$ has constant term 1. So the global zeta function $\zeta_G^*(s)$ is the Euler product of the cone integrals. Hence we get as a corollary of the work we have done on the zeta functions defined as Euler products of cone integrals the following:

Theorem 36 *(1) Let G be a finitely generated nilpotent infinite group. Then there exist finitely many varieties $E_i, i \in T$ defined over $\mathbb{Q}$ sitting in some $\mathbb{Q}$-scheme Y and rational functions $P_I(x,y) \in \mathbb{Q}[x,y]$ for each $I \subset T$ with the property that for almost all primes p*

$$\zeta^*_{G,p}(s) = \sum_{I \subset T} c_{p,I} P_I(p, p^{-s})$$

where

$$c_{p,I} = \mathrm{card}\{a \in \overline{Y}(\mathbb{F}_p) : a \in \overline{E_i} \textit{ if and only if } i \in I\}$$

and $\overline{Y}$ means the reduction mod p of the scheme Y.

(2) The abscissa of convergence $\alpha^*(G)$ of $\zeta^*_G(s)$ is a rational number and $\zeta^*_G(s)$ has a meromorphic continuation to $\Re(s) > \alpha^*(G) - \delta$ for some $\delta > 0$.

(3) There exist a non-negative integer $b \in \mathbb{N}_0$ and some real number $c \in \mathbb{R}$ such that

$$s_n(G) \sim c \cdot n^{\alpha^*(G)} (\log n)^b.$$

Recall that in (1) the irreducible varieties arise from the resolution of singularities of F, the product of all the polynomials defining the cone integral. Let us identify this $F = F^*_L$ explicitly for the cone integrals defining the zeta function of the Lie algebra $L = L(G)$ of G. The polynomial depends on a choice of basis for L which identifies it with $\mathbb{Z}^d$. Let multiplication in L be given by the bilinear mapping $\beta : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}^d$. Let C_j denote the matrix whose rows are $\mathbf{c}_i = \beta(\mathbf{e}_i, \mathbf{e}_j)$. Let $M = (m_{ij})$ be a $d \times d$ upper triangular matrix and $M^\natural = M' \mathrm{diag}(m_{22}^{-1}...m_{dd}^{-1}, ..., m_{dd}^{-1}, 1)$ where M' denotes the adjoint matrix of M.

Definition 37 *(1) $F^{\triangleleft}_L(m_{ij})$ is defined to be the product of $m_{11}^{(d^2+1)d}...m_{ii}^{(d^2+1)(d-i+1)}...m_{dd}^{(d^2+1)}$ with all the entries in the d matrices $MC_jM^\natural$ for $j = 1, ..., d$.*

(2) $F^{\leq}_L(m_{ij})$ is defined to be the product of $m_{11}^{(d^2+1)d}...m_{ii}^{(d^2+1)(d-i+1)}...m_{dd}^{(d^2+1)}$ with all the entries in the d matrices $M\left(\sum_{k=j}^d m_{jk}C_k\right)M^\natural$ for $j = 1, ..., d$.

The expression $m_{11}^{2d}...m_{ii}^{2(d-i+1)}...m_{dd}^2$ arises from the product of the two monomials in the integrand $f_0 = m_{11}...m_{dd}$ and $g_0 = m_{11}^{d-1}...m_{dd}^d$ and the monomials $f_{ijk} = m_{11}...m_{kk}$ for $i,j,k \in \{1,...,d\}$.

Definition 38 *For a finitely generated nilpotent group G we define the polynomials $F^*_G = F^*_{L(G)}$ for $* \in \{\leq, \triangleleft\}$, where $L(G)$ is the associated Lie algebra of G.*

Again there is some choice involved in the definition of the Lie algebra $L(G)$ as we have written it as it is the image under log of some subgroup of finite index in G. However up to finitely many primes, the explicit formulas associated to the cone integrals will be the same for polynomials $F_G^* = F_{L(G)}^*$ arising from different choices of Lie algebra and basis for the Lie algebra.

Although we know that the constant term of the cone integrals is $(1-p^{-1})^d$ it is instructive to see what the analysis of the resolution corresponding to $F_L^*(m_{ij})$ tells us. The constant term corresponds to when the integrand is constant which occurs when $f_0 = m_{11}...m_{dd}$ is a unit, i.e. for those basis for the whole group G. Let $T_0 \subset T$, (where recall T is the set indexing the irreducible components E_i), be defined as follows: if $i \in T_0$ then $N_i(f_0) \neq 0$. Note that this also implies that $N_i(g_0) \neq 0$ since $f_0 = m_{11}...m_{dd}$ and $g_0 = m_{11}^{d-1}...m_{d-1d-1}$. So those $I \subset T$ with $I \cap T_0$ non-empty are precisely those that don't contribute to the constant term. So the constant term comes from those $I \subset T \backslash T_0$. We analyse now the data in our formula associated to such I. If $I \subset T \backslash T_0$ then for $i \in I$, $N_i(f_0) = 0$ and since $f_{ijk} = m_{11}...m_{kk}$ this also implies that $N_i(f_{ijk}) = 0$. Hence there are no conditions on our subset since the conditions are

$$\sum_{l=1}^{r} N_{i_l}(f_{ijk})\mathrm{ord}(y_l) \leq \sum_{l=1}^{r} N_{i_l}(g_{ijk})\mathrm{ord}(y_l).$$

This implies then that $w_I = 1$, $m_1 = m$ and $A_{k,I,1} = 0$ and $B_{k,I,1} = 1$ for $k = 1, ..., m$ as before in the case that $I = \emptyset$ considered at the end of section 2. Hence the constant term is

$$p^{-m} \sum_{I \subset T \backslash T_0} c_I.$$

We know of course what the constant term should be for this integral, namely $(1 - p^{-1})^d$. Hence we get that

$$\sum_{I \subset T \backslash T_0} c_I = p^{m-d}(p-1)^d.$$

This is the same size as the number of points over $\mathbb{F}_p$ in affine space of dimension m minus the hyperplanes defined by $m_{ii} = 0$. This means that the number of points has not changed outside of the union of hyperplanes $m_{ii} = 0$ after we have done the resolution. Does this mean that our variety is non-singular outside this union of hyperplanes?

We collect together here the present knowledge we have about the possible values for $\alpha^*(G)$. Let G be a torsion-free, finitely nilpotent group. Let h denote the Hirsch length of the nilpotent group G and h_{ab} denote the Hirsch length of the abelianisation, $h(G/G')$.

(1) $h_{\mathrm{ab}} \leq \alpha^{\triangleleft}(G) \leq h$.

(2) $(3 - 2\sqrt{2})h - \frac{1}{2} \leq \alpha^{\leq}(G) \leq h$. The lower bound, which for large h ($h \geq 17$) exceeds $h/6$, is due to Segal and a proof will appear in [12].

(3) The paper [13] contains a number of examples of zeta functions of class two groups calculated by Geoff Smith. There are some other examples calculated by Dermott Grenham which are recorded in [10]. In all these examples $\alpha^{\leq}(G) = \alpha^{\triangleleft}(G) = h_{\mathrm{ab}}$. However this reflects the paucity and small nature of the nilpotent groups so far considered rather than a general feature. It is true that for class two nilpotent groups one can prove that $\alpha^{\triangleleft}(G) = h_{\mathrm{ab}}$ (see [12]). However in the same paper [12] we shall show that the zeta function counting all subgroups in the free class two nilpotent group F_2^3 on three generators has abscissa of convergence at $\alpha^{\leq}(F_2^3) = 7/2$. In particular this is the first example of a group for which the abscissa of convergence is not an integer.

At present we do not know anything about the order of the poles of $\zeta_G(s)$. Our analysis above might help in giving a bound in terms of the Hirsch length since the pole at the abscissa of convergence has order given by the number of times we need to multiply by the Artin L-function. This is determined by the number of irreducible components of E_I as $I \subset T$.

If the nilpotent group is not torsion-free then we do not know very much about the abscissa of convergence. Extending even the free abelian group by a finite group can have quite a dramatic effect on the movement of the poles of $\zeta_G(s)$. As an example compare the infinite cyclic group $\mathbb{Z}$ where $\zeta_{\mathbb{Z}}(s) = \zeta(s)$ and the infinite dihedral group D_∞ where $\zeta_{D_\infty}(s) = 2^{-s}\zeta(s) + \zeta(s-1)$. So extending by a finite group (here C_2) therefore has quite a subtle effect on the lattice of subgroups even to the extent of changing the rate of polynomial growth.

In [8] explicit examples calculated by John McDermott are recorded for the wallpaper groups which show the effect of extending $\mathbb{Z}^2$ by a finite group. It reveals how sensitive the zeta function is as the nature of the poles varies dramatically.

References

[1] H. Delange, Théorèmes taubériens et applications arithmétiques, Mém. Soc. Roy. Sci. Liège, **16** (1955), 5-87.

[2] J. Denef, The rationality of the Poincaré series associated to the p-adic points on a variety, Invent. Math., **77**, 1-23 (1984).

[3] J. Denef, On the degree of Igusa's local zeta function, Amer. J. Math., **109** (1987), 991-1008.

[4] J. Denef and D. Meuser, A functional equation of Igusa's local zeta function, Amer. J. Math., **113** (1991), 1135-1152.

[5] M.P.F. du Sautoy, Finitely generated groups, p-adic analytic groups and Poincaré series. Annals of Math., **137** (1993), 639-670.

[6] M.P.F. du Sautoy and A. Lubotzky, Functional equations and uniformity for local zeta functions of nilpotent groups, Amer. J. Math., **118** (1996), 39-90.

[7] M.P.F. du Sautoy and F.J. Grunewald, Zeta functions of classical groups and their friendly ghosts, C. R. Acad. Sci. Paris, **327**, Série 1, 1-6 (1998).

[8] M.P.F. du Sautoy, J. J. McDermott and G. C. Smith, Zeta Functions of Crystallographic Groups and analytic continuation., preprint (to appear in Proc. Lon. Math. Soc.).

[9] M.P.F. du Sautoy, Zeta functions of $\mathfrak{sl}_2(\mathbb{Z})$, preprint (to appear in Forum Mathematica).

[10] M.P.F. du Sautoy, Natural boundaries for zeta functions of groups, preprint.

[11] M.P.F. du Sautoy, Counting finite p-groups and nilpotent groups, preprint.

[12] M.P.F. du Sautoy and F.J. Grunewald, Counting subgroups of finite index in nilpotent groups of class two. In preparation.

[13] F.J. Grunewald, D. Segal and G.C. Smith, Subgroups of finite index in nilpotent groups, Invent. Math., **93** (1988), 185-223.

[14] I. Ilani, Zeta functions related to the group $\mathrm{SL}_2(\mathbb{Z}_p)$. To appear in the Israel J. of Math.

[15] S. Lang and A. Weil, Numbers of points of varieties in finite fields, Amer. J. Math., **76** (1954), 819-827.

[16] A. Lubotzky, Counting finite index subgroups. In Groups '93 Galway/St. Andrews, L.M.S. Lecture Note Series **212** C.U.P. (1995), 368-404.

[17] A. Lubotzky, A. Mann and D. Segal, Finitely generated groups of polynomial subgroup growth, Israel J. Math. **82** (1993), 363-371.

[18] F. Martinet, Character theory and Artin L-functions. In Algebraic Number Fields, edited by A. Fröhlich, Academic Press (1979).

[19] H. Matsumura, Commutative Algebra, W.A. Benjamin (1970).

[20] D. Segal, Subgroups of finite index in soluble groups I. In Proceedings of Groups-St. Andrews 1985, L.M.S. Lecture Note Series **121** C.U.P (1986), 307-314.

[21] G. C. Smith, Zeta functions of torsion free finitely generated nilpotent groups, Ph.D. thesis, (Manchester (UMIST), 1983).