ALMOST ETALE EXTENSIONS

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0. Introduction

The theory of almost *é*tale coverings goes back to Tate for the case of discrete valuation rings. The higher dimensional theory has been invented by the author, and used to prove conjectures about Hodge-Tate and crystalline structures on the *é*tale cohomology of varieties over p-adic fields. The main purpose of these notes is to extend the theory to cover toroidal singularities, especially all schemes with semistable reduction. This is possible because we have found a new method of proof for the main technical result, the purity theorem. This new method makes heavy use of Frobenius.

At the end we explain how to derive comparison maps to crystalline cohomology. However this is only sketched as all these ideas have already been presented in a number of papers ([Fa 3], [Fa 5], [Fa 6], [Fa 7], so that everybody interested in the matter can learn it there. We also add some (a little bit) incomplete results about cohomology of isocrystals, which we could not find in the literature.

All in all this suffices to prove the C_{st} -conjecture of Fontaine-Jansen. This had been done previously by T. Tsuji ([T]). However we can also treat the case of non-constant coefficients where some new phenomena occur. Anyway the author profited form his participation in the p-adic year at Institut Poincar*é* in Paris, first by giving lectures on a preliminary version and then by learning about T. Tsuji's technique.

1. Almost-Mathematics

Let $\bar{V}$ denote a (commutative with unit) ring together with a sequence of principal ideals $\mathfrak{m}_\alpha \subseteq \bar{V}$ parametrized by positive elements $\alpha \in \Lambda^+$, where $Y \subseteq \Lambda \subseteq \mathbb{Q}$ denotes a subgroup done in $\mathbb{R}$. Denote by π a generator of $\mathfrak{m}_1$, and π^α a generator of $\mathfrak{m}_\alpha$. Assume furthermore that $\pi^\alpha \pi^\beta = \text{unit} \cdot \pi^{\alpha+\beta}$.

Examples.

a) $\bar{V} = \bar{\mathbb{Z}}_p =$ integral closure of $\mathbb{Z}_p$ in $\bar{\mathbb{Q}}_p$, $\mathfrak{m}_\alpha =$ elements of valuation $\geq \alpha, \pi = p \quad (\Lambda = \mathbb{Q})$

b) Consider $R = \varprojlim(\bar{\mathbb{Z}}_p/p\bar{\mathbb{Z}}_p, \text{Frobenius}), \bar{V} = W(R),\ \pi = [\underline{p}, 0, \dots, 0] \in W(R)$, $\underline{p} = \varprojlim p^{1/p^n} \in R, \Lambda = \mathbb{Z}[\frac{1}{p}], \pi^\alpha =$ the obvious element.

If $\mathfrak{m} = \bigcup\limits_{\alpha > 0} \mathfrak{m}_\alpha$, then a $\bar{V}$-module is almost zero if it is annihilated by $\mathfrak{m}$. Denote this by $M \approx 0$. Assume R is a $\bar{V}$ -algebra and M an R-module.

1. Definition. M is almost projective if $Ext^i_R(M, N) \approx 0$, all R-modules N and all $i > 0$ (or only $i = 1$)

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M is almost flat if $Tor_i^R(M,N) \approx 0$, all R-modules N and all $i > 0$ (or only $i = 1$)
M is almost finitely generated if for each $\alpha > 0, \alpha \in \Lambda$, there exists a finitely generated R-module N and a π^α -isomorphism $\psi_\alpha : N \to M$ (i.e., there exists $\phi_\alpha : M \to N$ with $\phi_\alpha \circ \psi_\alpha = \pi^\alpha \cdot id_N$, $\psi_\alpha \circ \phi_\alpha = \pi^{\alpha_0} \circ id_M$).
M is almost finitely presented: dito with N finitely presented.

2. Remarks.

a) P almost projective
$\Leftrightarrow$ for all surjections $f : M \to N$ and maps $g : P \to N, \pi^\alpha \cdot g$ factors through $f (\alpha > 0, \alpha \in \Lambda)$
$\Leftrightarrow$ all $\alpha > 0, \alpha \in \Lambda$, there exists a free R-module L and maps $L \xrightarrow{g} P \xrightarrow{f} L$ with $g \circ f = \pi^\alpha \cdot id_p$.

b) If P is almost projective and almost finitely generated, then P is almost finitely presented. Then also $P^* = Hom_R(P, R), P \otimes_R P, \Lambda^i P$ etc. are almost finitely generated projective. Furthermore there is an almost isomorphism $P \otimes_R P^* \approx End_R(P)$, and thus a trace-map $tr : \mathfrak{m} \otimes_R End_R(P) \to R$, or equivalently $End_R(P) \to Hom(\mathfrak{m}, R)$. Thus for $f \in End_R(P)$ we can define a power-series

$$\det(1 + Tf) = \sum_{i=0}^{\infty} tr(\Lambda^i f) \cdot T^i \in Hom(\mathfrak{m}, R)[[T]].$$

One checks that $Hom(\mathfrak{m}, R)$ is a ring

$$f \circ g(\pi^{\alpha+\beta}) = f(\pi^\alpha) \cdot g(\pi^\beta))$$

and that

$$\det(1 + Tf) \det(1 + Tg) = \det(1 + T(f + g + Tfg))(= \sum_{i=0}^{\infty} tr(\Lambda^i(f + g + Tfg)) \cdot T^i$$

We say that P has rank $\leq r$ if $\Lambda^{r+1} P \approx 0$. Then letting $e \in Hom(\mathfrak{m}, R)$ denote the coefficient of T^r in $\det(1+T \cdot id_p)$, one checks that $e = e^2$ is an idempotent, and thus $R \approx R_1 \times R_2$ factors (up to almost isomorphism). Also $P \approx (P \otimes_R R_1) \times (P \otimes_R R_2)$.

Over one of the factors (say R_1)$e = 1$. One then checks that after base-change to R_1 $L = \Lambda^r P$ satisfies $L \otimes_R L^* \approx R$, i.e. L behaves like a line-bundle: There are the maps $tr : End_R(L) \approx L \otimes_R L^* \to R$ and the identity: $R \to End_R(L)$. We have $tr(id_p) = e = 1$. In fact for two $f, g \in End_R(P)$ we have $tr(\Lambda^r f) \cdot tr(\Lambda^r g) = tr(\Lambda^r(fg))$ = coefficient of T^{2r} in $\det((1 + Tf)(1 + Tg))$. Apply this to f and g of the form

$$f(x) = \sum_{i=1}^{r} \lambda_i(x)\mu_i, \lambda_i \in P^*, \mu_i \in P$$

$$g(x) = \sum_{i=1}^{r} \nu_i(x)\phi_i$$

to get $tr(\Lambda^r f) \cdot \det(\nu_i(\phi_j)) = \det(\nu_i(f(\phi_j)))$, i.e.
$tr(\lambda^r f)(\nu_1 \wedge \ldots, \wedge \nu_r \mid \phi_1 \wedge \ldots \phi_r) = (\nu_1 \wedge \ldots \wedge \nu_r \mid \Lambda^r f(\phi_1 \wedge \ldots \wedge \phi_r))$.
Varying ν's and g's gives $\Lambda^r f = tr(\Lambda^r f)$ is a scalar. As the $\Lambda^r f$ (given by $x \mapsto \langle \lambda_1 \wedge \ldots \wedge \lambda_r \mid x \rangle \mu_1 \wedge \ldots \wedge \mu_r$) generate $End_R(L)$, everything follows. The same argument gives:
If $e = 0$, then $tr(\Lambda^r f) = 0$ and $\Lambda^r f = 0$, and then $\Lambda^r P = 0$. Thus by induction we get an almost isomorphism

$$R \approx R_0 \times \ldots \times R_r,$$

such that over R_i $\Lambda^i P$ is invertible, $\lambda^{i+1} P = 0$. Also over R_i we have a determinant-function $\det(f) = tr(\Lambda^i f) : End_R(P) \to Hom(\mathfrak{m}, R_i)$ homogeneous of degree i with $\det(fg) = \det(f)\det(g), \det(1) = 1$.

3. Remark. I do not know whether for arbitrary finitely generated projective P's we have a decomposition $R \approx \prod_{i=0}^{\infty} R_i$ with $P \otimes_R R_i$ of rank i.

2. Almost étale coverings

1. Definition. A ring homomorphism $A \to B$ is called an almost étale covering if

i) B is almost finitely generated projective as an A-module, of finite rank
ii) B is almost finitely generated as a $B \otimes_A B$-module

2. Remarks.

a) There exists an idempotent $e_{B/A} \in Hom(\mathfrak{m}, B \otimes_A B)$ measuring where B has rank 1 as $B \otimes_A B$ -module. $e_{B/A}$ is annihilated by $(b \otimes_A 1 - 1 \otimes_A b)$ and maps to $1 \in Hom(\mathfrak{m}, B)$ under multiplication.
b) Replacing A by $Hom(\mathfrak{m}, A)$ we may reduce (in many cases) to the case where $rank_A B = r$ is constant. Then there exists an almost faithfully flat base-change $A \to A'$ (A' is almost flat, and $M \otimes_A A' \approx 0$ implies $M \approx 0$) with $B' = B \otimes_A A' \approx A'^r$

Proof. If $r \geq 1$, $A \to B$ is almost faithfully flat: If $\mathfrak{a} \subseteq A$ is an ideal with $\mathfrak{a}B \approx B$, then $\mathfrak{m}A \subseteq Norm_{B/A}(\mathfrak{a}B) \subseteq \mathfrak{a}\,Hom(\mathfrak{m}, A)$. Furthermore after base-change to B we may split of one factor B from $B \otimes_A B$, thus reducing r to $r-1$.

c) $A \to B$ is called a G-covering if a finite group G acts on B such that after almost faithfully flat base-change $A \to A'$ we have $B' = B \otimes_A A' \approx A'^G$. If B has constant rank r, there exists $A \to B \to C$ with C/A is a G-covering with $G = \gamma_r$ (symm. group in r letters) C/B is an H-covering with $H = \gamma_{r-1} \subseteq G$ ($C \subseteq B^{\otimes r}$ is an almost direct factor given by the idempotent $\prod_{i<j} \lambda_{ij}(1 - e_{B/A})$, $\lambda_{ij} : B^{\otimes 2} \to B^{\otimes r}$ the canonical maps).

Recall that for an A-algebra B (with unit) we can form a complex of B-bimodules ($B \otimes B^{op}$ -modules)

$$C(B/A) : B \otimes_A B \leftarrow B \otimes_A B \otimes_A B \leftarrow B \otimes_A B \otimes_A B \otimes -AB \leftarrow,$$

with differential

$$d_n : C_n(B/A) = B^{\otimes n+2} \to C_{n-1}(B/A) = B^{\otimes n+1}$$

$$d_n(b_0 \otimes b_1 \otimes \ldots \otimes b_{n+1}) = \sum_{i=0}^{n} (-1)^i b_0 \otimes \ldots \otimes b_i \cdot b_{i+1} \otimes \ldots \otimes b_{n+1}$$

This complex has an augmentation $C_0(B/A) \to B$ and is (with B in degree -1) nullhomotopic as complex of left respectively right B-modules. For a B-bimodule M define

$$H^*(B/A, M) = (Hom_{B \otimes B^{op}}(C_*(B/A), M))$$

Thus $H^0(B/A, M) = M^B = \{m \in M/bm = mb\, (\text{all}\, b \in B)\}$
$H^1(B/A, M) = \{$ biderivations $B \to M\}/\{$ inner derivations $b \to [b, m] = bm - mb\}$ etc.
For any B-module N the bimodule $M = Hom_A(B, N)$ has

$$H^i(B/A, M) = \begin{cases} 0, \; i > 0 \\ N, \; i = 0 \end{cases}$$

For example if B is a projective $B \otimes_A B$ -module then for any bimodule M we have that $M = Hom_B(B, M)$ (using say left - B -module structures) is a direct summand in $Hom_B(B \otimes_A B, M) = Hom_A(B, M)$, and thus Hochschild-cohomology vanishes in positive degrees. By the same argument B almost $B \otimes_A B$ -projective gives $H^i(B/A, M) \approx 0$ for $i > 0$.

Now suppose that $I \subseteq A$ is an ideal with square zero: $I^2 = 0$.

If B is an A-algebra which is almost unramified (B is an almost projective $B \otimes_A B$ -module), C an A-algebra, $\bar{f} : B \to \bar{C} = C/IC$ an A-algebra morphism (or almost morphism), then there is at most one way to lift $\bar{f}$ to $f : B \to C$ (up to almost equality): For two lifts f_1, f_2 the difference is an almost derivation $\bar{B} \to I \cdot C$, thus almost zero is $H^1(\bar{B}/\bar{A}, I \cdot C) \approx 0$. If B is also an almost projective A-module, then in fact an almost lifting exists: For each $\alpha > 0\, \pi^\alpha \bar{f}$ lifts to an A-linear map $f_\alpha : B \to C$. Consider $g_\alpha(x, y) = \pi^{2\alpha} f_\alpha(xy) - \pi^\alpha f_\alpha(x) f_\alpha(y) \in I \cdot C$. Then f_α, g_α defines a cochain in $Hom_{B \otimes_A B}(C_2(\bar{B}/\bar{A}, I \cdot C))$. We claim that it is a cocycle. To check that compute the boundary

$$dg_\alpha(x, y, z) = x g_\alpha(y, z) - g_\alpha(xy, z) + g_\alpha(x, yz) - g_\alpha(x, y) z$$

Using that multiplication by $f_\alpha(u)$ coincides with multiplication by $\pi^\alpha u$ in IC, this is:

$$f_\alpha(x)(\pi^\alpha f_\alpha(yz) - f_\alpha(y) f_\alpha(z)) - \pi^{2\alpha} f_\alpha(xyz) + \pi^\alpha f_\alpha(xy) f_\alpha(z)$$
$$+ \pi^{2\alpha} f_\alpha(xyz) - \pi^\alpha f_\alpha(x) f_\alpha(yz) - (\pi^\alpha f_\alpha(xy) - f_\alpha(x) f_\alpha(y)) f_\alpha(z) = 0.$$

Thus $\pi^{\alpha} g_{\alpha}$ is a coboundary,

$$\pi^{\alpha} g_{\alpha}(x, y) = x h_{\alpha}(y) - h_{\alpha}(xy) + h_{\alpha}(x) y,$$

and $\tilde{f}_{\alpha} = \pi^{3\alpha} f_{\alpha} + h_{\alpha}$ lifts $\pi^{4\alpha} \cdot \bar{f}$ and satisfies

$$\pi^{4\alpha} \cdot \tilde{f}_{\alpha}(xy) = \tilde{f}_{\alpha}(x) \tilde{f}_{\alpha}(y)$$

Furthermore for $0 < \alpha \leq \frac{\beta}{2} < \beta$

$$g_{\beta,\alpha} = \pi^{4\beta} f_{\beta} - \pi^{(\beta-4\alpha)} f_{\alpha}$$

satisfies

$$\begin{gathered} g_{\beta,\alpha}(xy) = f_{\beta}(x) f_{\beta}(y) - \pi^{8\beta-8\alpha} f_{\alpha}(x) f_{\alpha}(y) \\ = f_{\beta}(x)(f_{\beta}(y) - \pi^{4\beta-4\alpha} f_{\alpha}(y)) + (f_{\beta}(x) - \pi^{4\beta-4\alpha} f_{\alpha}(x)) \pi^{4\beta-4\alpha} f_{\alpha}(y) \end{gathered}$$

(using the terms in brackets are in IC)

$$\begin{gathered} = \pi^{4\beta} x(f_{\beta}(g) - \pi^{4\beta-4\alpha} f_{\alpha}(y)) + (f_{\beta}(x) - \pi^{4\beta-4\alpha} f_{\alpha}(x)) \pi^{4\beta} y \\ = x\, g_{\beta,\alpha}(y) + g_{\beta,\alpha}(x) y. \end{gathered}$$

Thus $g_{\beta,\alpha}$ is an almost derivation, hence almost zero. Multiplying by π^{β} gives

$$\pi^{5\beta} \cdot f_{\beta} = \pi^{g(\beta-\alpha)} \pi^{5\alpha} \cdot f_{\alpha}$$

Hence considering the $\pi^{5\alpha} \cdot f_{\alpha}$ as multiplicative maps from $\pi^{9_{\alpha}} \cdot V \otimes_V B$ to C (lifting $\bar{f}$), they glue to a map

$$f : \mathfrak{m} \otimes_V B = \varinjlim (\pi^{9_{\alpha}} \cdot V) \otimes_V B \to C$$

Also $f(1) \in Hom(\mathfrak{m}, C)$ is an idempotent lifting 1, thus $f(1) = 1$.

Finally we lift almost étale coverings. First we treat the underlying projective module: Let $\bar{P}$ be an almost projective $\bar{A} = A/I$ -module. For any $\alpha > 0$ choose a free $\bar{A}$ -module $\bar{L}$ and maps $\bar{f}_{\alpha} : \bar{L}_{\alpha} \to P, \bar{g}_{\alpha} : P \to \bar{L}_{\alpha}$ with $\bar{f}_{\alpha} \circ \bar{g}_{\alpha} = \pi^{\alpha} \cdot id$. Let $\bar{e}_{\alpha} \in End(\bar{L}_{\alpha})$ denote $\bar{g}_{\alpha} \circ \bar{f}_{\alpha}$, so $\bar{e}_{\alpha}^2 = \pi^{\alpha} \cdot \bar{e}_{\alpha}$.

Lift $\bar{e}_{\alpha}$ to an endomorphism f_{α} of L. Thus $f_{\alpha}^2 - \pi^{\alpha} f_{\alpha}$ takes values in $I \cdot L$, hence $(f_{\alpha}^2 - \pi^{\alpha} f_{\alpha})^2 = 0$. If $e_{\alpha} = 3\pi^{\alpha} f_{\alpha}^2 - 2f_{\alpha}^3$, then e_{α} lifts $\pi^{2\alpha} e_{\alpha}$ and satisfies $e_{\alpha}^2 = \pi^{3\alpha} e_{\alpha}$. Hence $P_{\alpha} = L_{\alpha}/(\pi^{3\alpha} - e_{\alpha}) L_{\alpha}$ is "projective up to $\pi^{3\alpha}$", i.e. $\pi^{3\alpha}$ annihilates all higher $Ext_A^i(P_{\alpha}, ?)$. Also $\bar{f}_{\alpha}$ and $\bar{g}_{\alpha}$ induce maps

$$\bar{P}_{\alpha} \xrightarrow{\bar{f}'_{\alpha}} \bar{P} \xrightarrow{\bar{g}'_{\alpha}} \bar{P}_{\alpha}$$

with

$$\bar{f}'_{\alpha} \circ \bar{g}'_{\alpha} = \pi^{\alpha} \cdot id_{\bar{P}}, \pi^{2\alpha} \circ \bar{g}'_{\alpha} = \pi^{3\alpha} \cdot id_{\bar{P}_{\alpha}}$$

For $\alpha \leq \beta/2 < \beta$,

$$\pi^{3\alpha+4\alpha-8\beta} \circ \bar{g}'_\beta \circ \bar{f}'_\alpha : P_\alpha \to \bar{P}_\alpha \to \bar{P} \to \bar{P}_\beta$$

lifts to a map $g_{\alpha,\beta} : P_\alpha \to P_\beta$ with $\bar{f}'_\beta \circ \overline{g_{\alpha,\beta}} = \pi^{7\alpha-7\beta} \circ \bar{f}'_\alpha$. Considering $g_{\alpha,\beta}$ as a map from $\pi^{7\alpha}V \otimes_V P_\alpha$ to $\pi^{7\beta}V \otimes_V P_\beta$, we form $P = \varinjlim_{\alpha} \pi^{7\alpha}V \otimes_V P_\alpha$, for a suitable sequence $\alpha = \alpha_n, \alpha_{n+1} \leq \frac{\alpha_n}{2}$. Then the $\bar{f}_\alpha$ define an almost isomorphism $P \otimes_A A/I \approx \bar{P}$. As coker $\bar{g}_{\alpha,\beta}$ is annihilated by $\pi^{7\alpha-7\beta+\alpha+3\beta}$ and hence by $\pi^{8\alpha}$, coker $g_{\alpha,\beta}$ is annihilated by $\pi^{16\alpha}$, and $\pi^{7\alpha}V \otimes_V P_\alpha \to P$ is an isomorphism up to $\pi^{16\alpha}$. Thus P is almost projective and lifts $\bar{P}$.

Remark. If $\bar{P}$ has rank $\leq r$, so has P.

Now assume $\bar{B}/\bar{A}$ is an almost étale covering, that $\bar{B}$ is almost projective over $\bar{A}$ as well as over $\bar{B} \otimes_{\bar{A}} \bar{B}$. We claim that there is an almost étale covering B/A with $B \otimes_A A/I \approx \bar{B}$:

We give a sketch, as the method follows the previous but the constants become more complicated. First lift $\bar{B}$ to an almost projective A-module B. Then for all $\alpha > 0$ the multiplication

$$\bar{m} : \bar{B} \otimes_A \bar{B} \to \bar{B}$$

lifts after multiplication by π^α to

$$m^0_\alpha : B \otimes_A B \to B.$$

Then $\pi^{c\alpha}(m^0_\alpha(m^\alpha_0(a,b),c) = m^0_\alpha(a, m^\alpha_0(b,c)))$ ($c =$ suitable constnat) defines a class in $H^3(\bar{B}/\bar{A}, I \cdot B) \approx 0$. After modifying m_α we can make it associative. Then the commutator with a fixed element defines a derivation, thus $\pi^{c\alpha} \cdot m_\alpha$ (c suitable) is commutative, and similarly $1 \in \bar{B}$ lifts to an idempotent which is a unit. For $\alpha \leq \frac{\beta}{2} < \beta$ a suitable multiple $\pi^{c\alpha} \cdot id_{\bar{B}}$ lifts to a multiplicative map

$$(\pi^{c\alpha}V \otimes_V B, m_\alpha) \to (\pi^{c\beta}V \otimes_V B, m_\beta).$$

Then $B = \varinjlim \pi^{c\alpha}V \otimes_V B$ with $\varinjlim m_\alpha$ does the job.

We have shown

3. Theorem. The functor sending B to $\bar{B} = B \otimes_A \bar{A}$ is an equivalence of categories (almost étale coverings of A) $\xrightarrow{\sim}$ (almost étale coverings of $\bar{A}$).

The previous theory is applied to the study of universal étale coverings. This was first done by Tate: Assume V is a ρ-adic discrete calculation-ring, with residue field $k = V/\pi V$ perfect of characteristic $p > 0$ (π a uniformiser) and fraction field K of characteristic zero. Assume furthermore that K_∞ is a totally ramified $\mathbb{Z}_p$ -extension of $K, V_\infty \subseteq K_\infty$ the normalisation of V. Then for any finite extension L_∞ of K_∞, the normalisation W_∞ of V_∞ in L_∞ is an almost étale covering of V_∞. This was generalised to non-perfect residue fields as follows: (see [F2], Theorem 1.2) Assume $[k : k^p] = p^d < \infty$. Assume given a tower $K = K_0 \subset K_1 \subset K_2 \subset \ldots \subset K_\infty = \bigcup\limits_{n\geq 0} K_n$. Let $V_n =$ normalisation of V in $K_n, \Omega_{V_n/V}$ the universal differential module. It is known that $\Omega_{V_n/V}$ can be generated by $\leq d+1$ elements, and that its length is the different of V_n/V. Now assume that for any $m \geq 0$ there exists an n such that $\Omega_{V_n/V}$ has as quotient $(V_n/\pi^m V_n)^{d+1}$. Then again for any finite $L_\infty \supseteq K_\infty, W_\infty =$ normalisation of V_∞ is an almost étale covering of V_∞.

Now suppose that over V we have an affine normal torus-embedding $T \cong \mathbb{G}_m^{d+1} \subseteq \bar{T}$ ([KKMS], § 1) If $L = Hom(\mathbb{G}_m, T)$ denotes the characters of T, we furthermore assume given a rational convex polyhedron $\sigma \subseteq L_\mathbb{R}$ not containing any line, such that $\bar{T} = Spec(V[L^* \cap \sigma^\vee])$. Finally we assume given an element $\lambda \in L^* \cap \sigma^\vee$ such that $\sigma \cap \ker(\lambda)$ is a simplex, i.e. consists of the positive linear combinations of elements $\rho_1, \ldots, \rho_r \in L \cap \sigma$ which are linearly independant over $\mathbb{Z}$. Let $\bar{T}_\lambda \subseteq \bar{T}$ denote the closed subscheme defined by the vanishing of $(\pi - \lambda)$. Assume given a noetherian ring R and a map $f : Spec(R) \to \bar{T}_\lambda$ which is flat, has geometrically regular fibres and is formally étale in charateristic p, i.e. for a prime $\mathfrak{p} \subseteq R (p \in \mathfrak{p})$ with image $x \in \bar{T}$ is $R_\mathfrak{p}$ formally étale over $O_{\bar{T}_{\lambda,x}}$. This is equivalent to the fact that Frobenius induces an isomorphism $(\hat{R}_\mathfrak{p}/p\hat{R}_\mathfrak{p}) \otimes O_{\bar{T}_{\lambda,x}} \xrightarrow{\sim} \hat{R}_\mathfrak{p}/p\hat{R}_\mathfrak{p}$. ($\hat{}$ = completion in $\mathfrak{p}$ -adic topology). Also for each residue-field κ of $\bar{O}_{T_\lambda}$ the fibre $R \otimes \kappa$ has dimension $\leq \log_p[\kappa : \kappa^p] \leq d$. Denote by $\bar{T}_n \to \bar{T}$ the covering induced by multiplication by $n!, Spec(R_n) \to Spec(R)$ its pullback via $Spec(R) \to \bar{T}_\lambda \subseteq \bar{T}$ (then R_n is a module over $V_n = V[\pi^{1/n!}]$ because of the relation $\lambda = \pi$). If $R_\infty = \bigcup\limits_{n>0} R_n$, and if $R_\infty \to S_\infty$ is the normalisation of R_∞ in a finite étale covering of the preimage of $T \subseteq \bar{T}$ in $R_\infty[\frac{1}{\pi}]$, we claim that S_∞ is an almost étale covering of R_∞.

4. Theorem. S_∞/R_∞ is almost étale.

Reduction-steps.

i) Instead of the $\bar{T}_n$ we may choose $Spec\, V[L_n^* \cap \sigma^\vee]$, where the $L_n \subseteq L$ form a decreasing sequence of sublattices such that for any integer $m, L_n \subseteq m \cdot L$ for $n >> 0$.

ii) We may start with a fixed L_n (instead of L) right away. Especially we may assume that S_∞ is defined by an R-algebra S such that $S[\frac{1}{\pi}]$ is finite étale over $R[\frac{1}{\pi}]$ over the preimage of $T \subseteq \bar{T}$.

iii) For a suitable choice of L_n we can achieve that

$$\bar{T}_{\lambda,n} = Spec(V[L_n^* \cap \sigma^\vee]/(\lambda - \pi)) \supseteq T_{\lambda,n} = Spec(V[L_n^*]/(\lambda - \pi))$$

is over $Spec(K)$ an embedding of the complement of a simple normal crossings divisor (use the properties of λ), and especially $\bar{T}_{\lambda,n} \otimes_V K$ is regular.

By Abhyankhar's lemma we then may assume that $S[\frac{1}{\pi}]$ is finite *é*tale over $R[\frac{1}{\pi}]$.

iv) From now on it will be sufficient to work with a sequence of lattices L_n cofinal with $p^n L$. The more general result then follows by base-extension (once we have that $S_\infty[\frac{1}{\pi}]$ extends to an almost *é*tale covering, this covering must be the normalisation of R_∞, and this remains true after base-change).

v) Next we consider what happens in the generic points in characteristic p. Suppose Z is an irreducible component $\bar{T}_\lambda \otimes_V k = Spec(k[L \cap \sigma^\vee]/(\lambda))$. Then σ corresponds to an extremal ray (halfline) in σ. If it is generated by an indivisible element $\rho \in L$, and if $\lambda(\rho) = e$, then in a neighbourhood of the generic point z of Z the torus-embedding $\bar{T}$ is isomorphic to $T' \times \mathbb{A}^1, T' \cong \mathbb{G}_m^d$, and $\bar{T}_\lambda$ has as local ring there

$$\bar{T}_{\lambda,z} \cong V'[t]/(t^e - u\pi) \cong V'[\sqrt[e]{u\pi}]$$

where V' is the lcoal ring corresponding to $T', u \in V'$ is a unit. As $u\pi$ is a uniformiser of V' it follows that $\bar{T}_{\lambda,z}$ is a discrete valuation-ring. It follows that $\bar{T}_\lambda$ satisfies R_1. As it is Cohen-Maccaulay ($\bar{T}$ is as all torus-embeddings are [KKMS], § 3), $\bar{T}_\lambda$ is even normal. Also R and R_n are normal and Cohen-Macaulay.

After that interlude let us pass to the sublattices L_n, and choose a projective system of points $z_n \in \bar{T}_{\lambda,n}$ lifting z. We claim that the local rings $\mathcal{O}_{\bar{T}_{\lambda,n,z_n}}$ satisfy the ramification-conditions of [Fa2], Th. 1.2: The integer d there coincides with our d. If we compute relative logarithmic differentials we obtain $L_n^*/L_n^* \otimes \mathcal{O}_{\bar{T}_{\lambda,n,z_n}}$. The usual differentials map into these, with cokernel annihilated by $\prod_{i=0}^{d+1} \mu_i$, if the $\mu_i \in \sigma^\vee \cap L_n^*$ form a basis of L_n^*. Now for $L_n = p^n L$ the valuation of this product decreases like $c \cdot p^{-n}$, and the claim follows for those L_n and thus also for any sequence.

By formal *é*taleness the same applies to the sequence $(R_n, \mathfrak{p}_n)$, where $\mathfrak{p}_n \subseteq R_n$ is a projective system of primes above a fixed $\mathfrak{p} \subseteq R$ which is a minimal prime divisor of pR or πR. It follows that $S_{\infty,p}$ is almost *é*tale over $R_{\infty,\mathfrak{p}}$.

vi) It now suffices to prove that S_∞ is almost flat over R_∞: There exists an idempotent $e_{S_\infty/R_\infty} \in S_\infty \otimes_{R_\infty} S_\infty[\frac{1}{\pi}]$ defining the diagonal (the direct summand $S_\infty[\frac{1}{\pi}]$). For each $\alpha > 0$ its product with π^α then lies in

$$\bigcap_{\mathfrak{p}} (S_\infty \otimes_{R_\infty} S_\infty)_{\mathfrak{p}} \cap S_\infty \otimes_{R_\infty} S_\infty[\frac{1}{\pi}] \approx S_\infty \otimes_{R_\infty} S_\infty$$

($\mathfrak{p}$ runs over all minimal prime-divisors of πR). Thus S_∞ is an almost direct summand in $S_\infty \otimes_{R_\infty} S_\infty$. Furthermore if

$$\pi^\alpha e_{S_\infty/R_\infty} = \sum_{i=1}^r x_i \otimes y_i \quad (x_i, y_i \in S_\infty),$$

then $\pi^\alpha x = \sum_{i=1}^r x_i \cdot Tr_{S_\infty/R_\infty}(y_i x)$ for all $x \in S_\infty$, which exhibits S_∞ as an up to π^α direct summand in R_∞^r. Then S_∞ is almost projective over R_∞.

By localization we may thus assume that R is local with maximal ideal $\mathfrak{m}$, and that S_∞ is almost flat over R_∞ on the punctured spectrum $Spec(R) -$

$\{\mathfrak{m}\}$. It also follows that $S_\infty \otimes_R \hat{R}(\hat{R} = \mathfrak{m})$ -adic completion) is almost equal to the normalisation of $R_\infty \otimes_R \hat{R}$, using that $\hat{R}$ is normal (f is formally smooth) and that the two can only differ almost by $\mathfrak{m}$ -torsion.

vii) Assume now that $\dim(R) = 2$: Let $z \in \bar{T}_\lambda$ denote the image of $\{\mathfrak{m}\}$, $z_n \in \bar{T}_{\lambda,n}$ a projective system of liftings. We claim that for a suitable choice of the lattices L_n the local rings of $\bar{T}_{\lambda,n}$ in z_n are regular. The same is then true for R_n. As S_n is normal of dimension two it is Cohen-Macaulay, and thus flat over R_n (depth + projective dimension = 2). Thus S_∞ is flat over R_∞.

To show the claim we may assume that $\pi \in \mathfrak{m}$. Thus $z \in \bar{T}_\lambda \otimes_N k \subseteq \bar{T} \otimes_V k$ corresponds two a codimension-two-point in $\bar{T} \otimes_V k$. If it is not the generic point of a codimension-two stratum each $\bar{T}_{\lambda,n,z_n}$ is already regular by the previous. If it is such a generic point then near z $\bar{T}_{\lambda,n}$ looks like the product of a $\mathbb{G}_m^{d-1}$ with a torus-embedding of rank 2. This torus-embedding is given by a 2-dimensional convex polyhedral cone which must be a simplex. For suitable choice of the L_n this simplex is spanned by two basis-vectors of L_n, and $\bar{T}_n$ is smooth over V in z_n. It then follows easily that $\bar{T}_{\lambda,n}$ is still regular in z_n. (The singularity looks like $V_n[T_1, T_2]/(T_1^e T_2^f - \pi_n)$).

So finally now $\dim(R) \geq 3$. The Frobenius-mapping is surjective on R_∞/pR_∞ (or $R_\infty/\pi R_\infty$):

This holds for $\mathcal{O}_{\bar{T}_\infty}$, and S_∞ is generated by $\mathcal{O}_{\bar{T}_\infty}$ and R^p.

Following Fontaine we form the projective limit $\mathcal{R} = \varprojlim(R_\infty/pR_\infty)$ (transition-maps are Frobenius) and $A_{\inf}(R) = W(\mathcal{R})$. Then $A_{\inf}(R)/p \cdot A_{\inf}(R) \xrightarrow{\sim} \mathcal{R}$, $A_{\inf}(R)$ admits a Frobenius-automorphism ϕ, and there is a surjection $A_{\inf}(R) \twoheadrightarrow \hat{R}_\infty$ (ρ-adic completion) with kernel generated by a single element $\xi = p - [p]$. Here $[p] = [p, 0, \ldots, 0]$ with $p = \varprojlim(p^{1/p^n}) \in \mathcal{R}$ a compatible system of p-power roots of p. Note that $\phi(\xi) = p - [\underline{p}]^p = p - (p - \xi)^p = \pm\xi^p + p \cdot (unit)$. The elements ρ and ξ form a regular sequence in $A_{\inf}(R)$, and $A_{\inf}(R)/\xi$ is complete in the ξ -adic topology. However this topology is not ϕ- invariant. We thus consider the finer topology where a fundamental system of neighbourhoods of zero is given by the ideals generated by finite products ($m_n \geq 0$, almost all zero) $\prod_{n\in\mathbb{Z}}(\phi^n(\xi))^{m_n}$. Note that any product $\prod_{n>0} \phi^n(\xi)^{m_n}$ with positive ϕ -powers forms together with ξ a regular sequence in $A_{\inf}(R)$, and that the ideal generated by those two elements defines the same topology as (p, ξ). Thus $A_{\inf}(R)$ with the topology defined by $\prod_{n=0}^N \rho^n(\xi)$ is the fibred product of (A, ξ -adic topology) with ($A, \prod_{n=1}^N \phi^n(\xi)$ -adic topology), over ($A, (p, \xi)$ -adic topology).

Now S_∞ defines an almost étale covering V_∞ of $U_\infty = Spec(R_\infty) - Spec(R_\infty/\mathfrak{m}R_\infty)$. It obviously has finite rank $[L_\infty : K_\infty]$. Furthermore if we reduce modulo p it is almost invariant under Frobenius, i.e. Frobenius on $\mathcal{O}_{V_\infty}/p\mathcal{O}_{V_\infty}$ induces an almost isomorphism

$$\mathcal{O}_{V_\infty}/p\mathcal{O}_{V_\infty} \otimes_{\mathcal{O}_{U_\infty}/p\mathcal{O}_{U_\infty}} \mathcal{O}_{U_\infty}/p\mathcal{O}_{U_\infty} \approx \mathcal{O}_{V_\infty}/p\mathcal{O}_{V_\infty} :$$

Reduce to the split case.

Let $\mathfrak{m}_0 \subseteq A_{\inf}(R)$ denote the ideal generated by p and suitable lifts of generators of $\mathfrak{m}$. Then we can lift V_∞ first to an almost étale covering of $Spec(A_{\inf}(R)/\xi^m) - \{\mathfrak{m}_0\}$, all m, which by transport of structure induces almost étale coverings on $Spec(A_{\inf}(R)/\phi^n(\xi)^m) - \{\mathfrak{m}_0\}(n \in \mathbb{Z})$. These are all isomorphic modulo p because of Frobenius-invariance, and thus by induction (and gluing, that is taking fibred products) we obtain almost $\acute{e}$tale coverings of

$$Spec(A_{\inf}(R)/(\prod_{n=0}^{N}\phi^n(\xi))^m) - \{\mathfrak{m}_0\}$$

and by applying ϕ of

$$Spec(A_{\inf}(R)/\prod_{n\in\mathbb{Z}}\phi^n(\xi)^{m_n}) - \{\mathfrak{m}_0\}.$$

That is if we consider the ind-scheme defined by quotients $A_{\inf}(R)/\prod_{n\in\mathbb{Z}}\phi^n(\xi)^{m_n}$, then we have defined a ϕ -invariant almost $\acute{e}$tale covering of its punctured spectrum.

Notations.

$$\begin{aligned}
&X_{\inf} = \text{"}\varinjlim\text{"}\, Spec(A_{\inf}(R)/\prod_{n\in\mathbb{Z}}\phi^n(\xi)^{m_n})\\
&U_{\inf} = X_{\inf} - \{\mathfrak{m}_0\}\\
&V_{\inf} \to U_{\inf} : \quad \text{the almost} \quad \acute{e}\text{tale covering derived from} \quad S_\infty.
\end{aligned}$$

Also "almost" should now be defined with respect to the sequence of ideals generated by $\underline{p}^{1/p^n} = \varphi^{-n}(\underline{p})$. Define the cohomology $H^*(V_{\inf}, \mathcal{O}_{V_{\inf}})$ as $\check{C}$ech-cohomology of the affine covering defined by a system of generators of $\mathfrak{m}$. This is easily seen to be independant of the choice of the covering, and one has long exact cohomology-sequences from short (topologically) exact sequences. I conjecture but have not checked that it coincides with the cohomology of the ind-scheme $V_{\inf}$. It certainly admits a Frobenius-automorphism (almost) ϕ.

For $i > 0$ the cohomology

$$H^i(V_{\inf}, \mathcal{O}_{V_{\inf}}/\xi\cdot\mathcal{O}_{V_{\inf}}) \cong H^{i+1}_{\mathfrak{m}}(S_\infty),$$

the local cohomology (SGA2). It satisfies a certain finiteness condition: Firstly for any n the cohomology $H^2_{\mathfrak{m}}(S_n)$ is a finitely generated S_n -module and thus of finite length, as for any prime $\mathfrak{p} \neq \mathfrak{m}$ $depth(S_{n,\mathfrak{p}}) + \dim(S_{n/\mathfrak{p}}) \geq 3$. Furthermore there are maps

$$H^2_{\mathfrak{m}}(S_n)\otimes_{R_n} R_\infty \to H^2_{\mathfrak{m}}(S_n\otimes_{R_n} R_\infty) \to H^2_{\mathfrak{m}}(S_\infty).$$

For any positive $\alpha > 0$ the kernels and cokernels of both maps are annihilated by p^α, for n big enough:

If R_∞ where flat over R_n the first map would be an isomorphism. This is not true in general, but in the appendix 2 it is shown that for a sequence α_n with $\alpha_n \to 0$ p^{α_n} annihilates all $Tor_i^{R_n}(R_m, M), (i > 0)$ $(m \geq n$ and M any R_n -module). This implies the assertion for the first map.

For the second one studies $S_n\otimes_{R_n} R_\infty \to S_\infty$. Its kernel consists of p-torsion and is annihilated by the p^{α_n} from above. For the cokernel use that for each height-one prime $\mathfrak{p}$ the trace-form on $S_{n,\mathfrak{p}}$ has discriminant p^{β_n} with $\beta_n \to 0$. Thus

$$p^{\beta_n}\cdot S_{\infty,\mathfrak{p}} \subseteq \bigcap (S_n\otimes_{R_n} R_\infty)_{\mathfrak{p}}$$

and (using annihilation of $Tor_1's$)

$$p^{\alpha_n+\beta_n}\cdot S_\infty \subseteq (\bigcap_{\mathfrak{p}} S_{n,\mathfrak{p}})\otimes_{R_n} R_\infty = S_n\otimes_{R_n} R_\infty.$$

It follows that for each $\alpha>0$ $p^\alpha\cdot H^2_{\mathfrak{m}}(S_\infty)$ is contained in a finitely generated $\mathfrak{m}$ -torsion S_∞ -module. In the appendix 2 it is also shown how to define an invariant (the normalized length)

$$\lambda(p^\alpha\cdot H^2_{\mathfrak{m}}(S_\infty))\in\mathbb{R}.$$

λ is ≥ 0, is additive in short exact sequences, and for an $\mathfrak{m}$ -torsion module which is a submodule of a finitely presented $\mathfrak{m}$ -torsion S_∞ -module, λ vanishes only if the module is zero. Also if M is annihilated by $p^{\frac{1}{p}}$, then

$$\lambda(M\otimes_{R_\varphi} R/pR)=p^{d+1}\lambda(M).$$

Now consider

$$M=H^1(V_{\mathrm{inf}},\mathcal{O}_{\mathrm{inf}}).$$

Then $M/\xi M$ injects into $H^1(V_\infty,\mathcal{O}_{V_\infty})=H^2_{\mathfrak{m}}(S_\infty)$. Thus for any $\alpha>0$ $p^\alpha(M/\xi M)$ as well as $p^\alpha(M/(\xi,p)M)$ have finite λ -invariant. Furthermore Frobenius induces an almost isomorphism

$$M/(\phi^{-1}(\xi),p)M\otimes_{R_\varphi} R/pR\approx M/(\xi,p)M$$

Thus for any $\beta>0$ (p^β takes care of the "almost")

$$p^{d+1}\lambda(p^\alpha M/(\phi^{-1}(\xi),p)M)\leq\lambda(p^{p\alpha-\beta}M/(\xi,p)M)$$

Choose $0<\alpha<\frac{1}{p^2},\beta\leq\frac{\alpha}{p-1}$. Then $p^{p\alpha-\beta}M/(\xi,p)M$ has a filtration by submodules $p^{\alpha-\beta}+i(\frac{1}{p}-\alpha)M/(\xi,p)M, 0\leq i\leq p$. Furthermore each subquotient is a quotient of $p^\alpha M/(\rho^{-1}(\xi),p)M$, via multiplication by $p^{(p-1)\alpha-\beta+i(\frac{1}{p}-\alpha)}$. Thus the right hand side is $\leq(p+1)\lambda(p^\alpha M)(\rho^{-1}(\xi),p)M)$. As $p^{d+1}\geq p^3>p+1$, we see that both sides vanishes, so $\lambda(p^\alpha\cdot M/(\xi,p)M)=0$ (all $\alpha>0$) and $\lambda(p^\alpha\cdot M/\xi M)=0$ (p is nilpotent in $M/\xi M$). Now for $n>>0$ there is a map

$$M/\xi M\rightarrow H^2_{\mathfrak{m}}(S_n)\otimes_{R_n} R_\infty$$

with kernel annihilated by p^α. As $\lambda(p^\alpha\cdot image)=0$ it follows that $p^\alpha\cdot iamge=0$, thus $p^{2\alpha}$ annihilates $M/\xi M$. As α was arbitrary, $M/\xi M\approx 0$, and by induction also any quotient $M/\prod_{n\in\mathbb{Z}}\phi^n(\xi)^{m_n}\cdot M\approx 0$ vanishes almost.

It follows that the map $\Gamma(V_{\mathrm{inf}},\mathcal{O}_{V_{\mathrm{inf}}})\rightarrow S_\infty=\Gamma(V_\infty,\mathcal{O}_{V_\infty})$ is almost surjective: Order the finite products $\prod_{n\in\mathbb{Z}}\phi^n(\xi)^{m_n}$ in a countable sequence $g_1=\xi,g_2,g_3,\ldots$, with g_{n+1}/g_n = a ϕ -power of ξ. Then $H^1(V_{\mathrm{inf}},g_n\cdot\mathcal{O}_{V_{\mathrm{inf}}}/g_{n+1}\cdot\mathcal{O}_{V_{\mathrm{inf}}})\approx 0$, as this φ -power induces an almost isomorphism with M. For $\alpha>0$ choose a strictly increasing sequence $\alpha_1=0<\alpha_n<\alpha_{n+1}\ldots<\alpha$. For $f_1\in S_\infty=\Gamma(V_{\mathrm{inf}},\mathcal{O}_{V_{\mathrm{inf}}}/g_1\cdot\mathcal{O}_{V_{\mathrm{inf}}})$ construct inductively $f_n\in\Gamma(V_{\mathrm{inf}},\mathcal{O}_{V_{\mathrm{inf}}}/g_n\cdot\mathcal{O}_{V_{\mathrm{inf}}})$ with f_{n+1} lifting $\underline{p}^{\alpha_{n+1}-\alpha_n}\cdot f_n$. This is possible because $\underline{p}^{\alpha_{n+1}-\alpha_n}$ annihilates the obstruction to lift f_n. Then the $p^{\alpha-\alpha_n}\cdot f_n$ form a compatible system of lifts of $\underline{p}^\alpha\cdot f_1=p^\alpha\cdot f_1$. Thus finally

$S_{\inf} = \Gamma(V_{\inf}, \mathcal{O}_{V_{\inf}})$ is an $A_{\inf}(R)$ -algebra with $S_{\inf}/\xi \cdot S_{\inf} = S_\infty$. Furthermore Frobenius induces an almost automorphism of $S_{\inf}$.

Next consider the cokernel M of the map

$$S_{\inf} \otimes_{A_{\inf}(R)} S_{\inf} \to \Gamma(V_{\inf}, \mathcal{O}_{V_{\inf}} \otimes_{A_{\inf}(R)} \mathcal{O}_{V_{\inf}}):$$

On it Frobenius is an almost isomorphism. On the other hand $M/\xi \cdot M$ (almost) injects into the cokernel of $S_\infty \otimes_{R_\infty} S_\infty \to \Gamma(V_\infty, \mathcal{O}_{V_\infty} \otimes_{R_\infty} \mathcal{O}_{V_\infty})$, that is $H^1_{\mathfrak{m}}(S_\infty \otimes_{R_\infty} S_\infty)$.

By the same reasoning as before $H^1_{\mathfrak{m}}(S_\infty \otimes_{R_\infty} S_\infty)$ is for each $\alpha > 0$ p^α -isomorphic to a finitely presented torsion-module, thus $\lambda(p^\alpha \cdot M/\xi M) < \infty$ (all $\alpha > 0$) and $M/\xi M \approx 0$. Finally the canonical idempotent defining the diagonal defines an almost section of $\mathcal{O}_{V_{\inf}} \otimes_{A_{\inf}(R)} \mathcal{O}_{V_{\inf}}$ lifting the corresponding idempotent e_{S_∞/R_∞}. It is thus at least modulo ξ in the image of $S_{\inf} \otimes_{A_{\inf}(R)} S_{\inf}$, and e_{S_∞/R_∞} lies almost in the image of $S_\infty \otimes_{R_\infty} S_\infty$.

Appendix 1. Some toroidal geometry. Assume $Spec(R) \to \bar{T}_\lambda$ is formally étale in characteristic p. We assume that R is local with maximal ideal containing π. We want to study how much of this parametrisation is canonical.

Obviously we can replace σ by its face parametrising the stratum of $\bar{T}$ which contains the image of the closed point of $Spec(R)$. Let $L' \subseteq L$ denote the lattice spanned by σ. Then we have a formally smooth map

$$Spec(R) \to \bar{T}'_\lambda \subseteq \bar{T}',$$

into the torus-embedding defined by (L', σ). Thus change notation and assume $L = L'$, but the map only formally smooth (i.e. flat with geometrically regular fibres).

4. Proposition. R is normal. The complement of the open stratum in $Spec(R)$ is the union of irreducible divisors E. They correspond one to one to the extremal rays of σ. If $\rho \in L^\vee \cap \sigma^\vee$ is the indivisible generator of such a ray, then for any $\mu \in L$ the function $m \in R$ has valuation $\mu(\rho)$ in the generic point of the correspodning divisor.

Furthermore $L^\vee \cap \sigma^\vee =$ invertible ideals $I \subset R$ with support in the complement of the open stratum, via $\mu \mapsto R \cdot \mu$.

Proof. Choose a $\rho \in L$ which lies in the interior of σ. Then define a decreasing filtration on $V[L^\vee \cap \sigma^\vee]$ by $F^b(V[L^\vee \cap \sigma^\vee]) = \oplus_{\mu(\varphi) \geq n} V \cdot \mu$. It induces a quotient filtration on $V[L^\vee \cap \sigma^\vee]/(\pi - \lambda)$ with $gr^\circ_{F_\rho}(V[L^\vee \cap \sigma^\vee]/(\pi - \lambda)) \cong k[L^\vee \cap \sigma^\vee]$, and by pushout a filtration F°_ρ on R with $gr_{F_\rho}(R) \cong \kappa[L^\vee \cap \sigma^\vee], \kappa =$ residue-field of R. Furthermore this filtration defines the $\mathfrak{m}$ -adic topology on R. As the associated graded is normal, so is R.

For each extremal ρ we obtain a divisor in $\bar{T}$ isomorphic to the torus-embedding defined by $(L/\langle\rho\rangle$, projection of $\sigma)$, and the valuation of $\mu \in L^\vee$ along this divisor is $\mu(\rho)$. The intersection with $\bar{T}_\lambda$ is as follows: If $\lambda(\rho) = 0$, it is $\bar{T}'_\lambda$ with T' corresponding to $L/\langle\rho\rangle$. Its preimage in $Spec(R)$ is normal and thus irreducible. If $\lambda(\rho) = n > 0$, then the intersection does not meet the generic fibre of $\bar{T}_\lambda$. In the special fibre $\bar{T}_\lambda \otimes_V k = Spec(k[L^\vee \cap \sigma^\vee](\lambda))$ the intersection is the boundary component of $Spec(k[L^\vee \cap \sigma^\vee])$ defined by ρ. Its preimage in $Spec(R)$ is irreducible for the same reasons as before, and also the valuation of μ is as required (the intersection with $\bar{T}_\lambda$ is transverse). As the complement of the open stratum in $Spec(R)$ is the union of the divisors, we have shown the second assertion.

Finally assume $I = R \cdot f \subseteq R$ is an invertible ideal. For any $\rho \in L^\vee \cap \sigma^\vee$ we can as before define filtrations F_ρ on $V[L^\vee \cap \sigma^\vee], V[L^\vee \cap \sigma^\vee]/(\pi - \lambda)$ and R. Now $gr^0_{F_\rho}(R)$ is formally smooth over $gr^0_{F_\rho}(V[L^\vee \cap \sigma^\vee]/(\pi - \lambda)) = k[L^\vee \cap \sigma^\vee \cap \rho^\perp]$, and $gr_{F_\rho}(R)$ is an integral domain (as this holds for $\bar{T}_\lambda$).

Now the conditions on I mean that f divides some $\mu_0 \in L^\vee \cap \sigma_j^\vee$, $\mu_0 = f \cdot g$. First choose $\rho \in \overset{\circ}{\sigma}$. Then if a and b are maximal with $f \in F^a_\rho(R), g \in F^b_\rho(R)$, it follows by compution in $gr_{F_\rho}(R) = \kappa[L^\vee \cap \sigma^\vee]$ that $a + b = \mu_0(\rho)$, and after multiplication by units that there is a decomposition $\mu_0 = \mu - \nu$ with $f - \mu \in F^{a+1}_\rho(R), g - \nu \in F^{b+1}_\rho(R)$. μ and ν do not change if we vary slightly the ray $\mathbb{R}_+ \cdot \rho$. As σ^0 is convex they are thus independant of ρ.

Next choose ρ generating an extremal ray of σ. It corresponds to an irreducible divisor E. We show that for the corresponding valuation $v_E(f) \leq v_E(\mu) = \mu(\rho)$. If this holds for all E then f divides μ (by normality) and g divides ν (by symmetry). But then $I = R \cdot \mu$. Next assume that $v_E(f) > \mu(\rho) = n$. Thus $f \in F^{n+1}_\rho(R)$, which is generated by finitely many $\chi \in L^\vee \cap \sigma^\vee$ with $\chi(\rho) \geq n + 1$. Choose a ray in the interior $\overset{\circ}{\sigma}$ sufficiently close to ρ. It is generated by a $\tilde{\rho}$ such that $\frac{n \cdot \tilde{\rho}}{\mu(\tilde{\rho})}$ is very close to ρ. Especially we can achieve that each of the κ's has values $> n$ on it. Thus if $\tilde{n} = \mu(\tilde{\rho})$, then $f \in F^{\tilde{n}+1}_{\tilde{\rho}}$, which cannot happen. Hence we have shown surjectivity of the map.

Injectivity follows because we already know how to recover μ from I_μ. Conversely once we have identified $\sigma^\vee$ with the semigroup of effective Cartier-divisors supported in the boundary, we obtain $L^\vee$ as the group of all such divisors, and then also L and σ. Also λ corresponds to π. Now extend λ to a basis of $L^\vee$, and choose equations for the remaining basis-elements. This defines a map $Spec(R) \to \bar{R}_\lambda$ which differs from the given one by the action of T via a point z of $T(R)$. Passing to the $\mathfrak{m}$ -adic completion $\hat{R}$ the two maps differ by an automorphism of $\hat{R}$, which is the identity on $\kappa = R/\mathfrak{m}$ and multiplies $\mu \in L^\vee \cap \sigma^\vee$ by $\mu(z)$.

All in all (L, σ) is essentially unique up to automorphism once given $Spec(R)$ and its open stratum.

5. Remark. One might ask whether the fibre $R \otimes_V k$ is reduced. This happens if and only if this holds for $\bar{T}_\lambda$, i.e. if $k[L^\vee \cap \sigma^\vee]/(\lambda)$ is reduced, that is if $\lambda(\rho) \in \{0, 1\}$ for all extremal rays ρ of σ. We can always achieve this by replacing L by $\tilde{L} = \lambda^{-1}(e\mathbb{Z})$ with e sufficiently divisible, and λ by λ/e. Choose e such that $\lambda(\rho)$ divides e if $\lambda(\rho) \neq 0$. Then ρ changes to $\tilde{\rho} = \frac{e}{\lambda(p)} \cdot \rho$, and $\lambda/e(\tilde{\rho}) = 1$. This operation amounts to adjoining $\sqrt[e]{\pi}$ to V. We shall usually assume that this has been done. Then $R \otimes_V A$ is normal for any flat normal V-algebra A. Especially $R \otimes_V \bar{V}$ is normal.

There is also an intimate relation to log -structures as defined by Fontaine-Illusie-Kato: Namely on $X = Spec(R)^{et}$ consider the sheaf of monoids $P = j_*(\mathcal{O}^*_{X^0}) \cap \mathcal{O}_X$ ($j : X^0 \hookrightarrow X$ inclusion). Its fibre at $x \in \tau-$ stratum is an extension $0 \to \mathcal{O}^*_{x,x} \to P_x \to \tau^\vee \to 0$, with $\mathcal{O}^*_{x,x}$ acting freely on P_x. Thus P defines a fine log -structure, log-smooth over $Spec(V)$ with the log-structure defined by $V - \{0\}$. Conversely if $X^0 \subseteq X$ is such that $P = j_*(\mathcal{O}^*_{x^0}) \cap \mathcal{O}_x$ defines a fine log-structure log-smooth $Spec(V)$ with $V - \{0\}$, then for each $x \in X$ $P_x((\mathcal{O}^{sh}_{x,x})^*$ injects into the group of Weil-divisors supported on the complement of the open stratum in $Spec(\mathcal{O}^{sh}_{x,x})$. If

$L_x^\vee = \langle P_x/(\mathcal{O}_{x,x}^{sh})^*\rangle^{\text{group}}$, $L = (L^\vee)^\vee$, then the valuations at irreducible components of the boundary induce maps $L^\vee \to \mathbb{Z}$ and thus elements $\rho \in L$, which span a polyhedral cone $\sigma \subseteq L_\mathbb{R}$ with $P_x/(\mathcal{O}_{x,x}^{sh})^* = L^\vee \cap \sigma^\vee$. Furthermore elements of P_x lifting a basis of L contained in $\sigma^\vee$ define a formally smooth map $Spec(\mathcal{O}_{x,x}^{sh}) \to \bar{T}_\lambda$, where $\lambda \in L^\vee$ corresponds to π. $\lambda \in L^\vee$ may not be indivisible (it is only indivisible by p), but assume that it is. Hence our toroidal description can be translated into log -structures. However as all the important tools we need from the theory have been formulated and proved in toroidal geometry long before the advent of log-structures, we use the toroidal language.

Appendix 2: Length computations. Let us start with the torus-embedding

$$T = \mathbb{G}_m^{d+1} \subseteq X = Spec(V[L^\vee \cap \sigma^\vee]) = Spec(R).$$

For a sublattice $L' \subseteq L$ let $X' = Spec(R') = Spec(V[L'^\vee \cap \sigma^\vee])$. As an R-module

$$R' = \bigoplus_{\mu \in L'^\vee/L^\vee} V[(\mu + L^\vee) \cap \sigma^\vee] = \bigoplus_\mu R'_\mu,$$

where the isomorphism-class of R'_μ as R-module depends only on the integral parts $[\mu(l)], l \in L$ the generator of an extremal ray of σ. Namely for $\kappa \in L^v$ $(\mu + \kappa)$ lies in $\sigma^\vee$ if any only if $\kappa(l) \geq -[\mu(l)]$ for all such l. As μ can be chosen in a compact set (representatives for $L_\mathbb{R}^\vee/L^\vee$) the numbers $[\mu(l)]$ are bounded. One derives that there are onyl finitely many isomorphism classes of R'_μ's, independant of the choice of L'. Furthermore if for some integer N $L' \subseteq N \cdot L$, then the multiplicity of a given isomorphism-class of R_μ's is $[L : L'] \cdot (\text{constant } + O(\frac{1}{N}))$. Also the constant is strictly positive for the multiplicity of the trivial R-module $R_\mu \cong R$. Now firstly if $\lambda \in L^\vee$ is such that $\sigma \cap \ker(\lambda)$ is a simplex spanned by a partial basis of L, then R_λ is regular and R'_λ is flat over R_λ (being finite and Cohen-Macaulay). Thus if $P. \to P'$ is a resolution of R by finitely generated projective R-modules there exists on N such that λ^N is nullhomotopic on P_1, i.e. $\lambda^N \cdot id = d \circ s + s \circ d$ for suitable $s : P_1 \to P_2, P_0 \to P_1$.

Thus λ^N annihilates $Tor_1^R(R', M)$ for all R-modules M, and also all higher Tor's. It now follows that N can be chosen independantly from L'. The same holds after flat pushout. This already shows one of our claims: If we let $L' = L_n$ (with L_n as before) and if $\sigma \cap \ker(\lambda)$ is a simplex, then for a suitable choice we obtain first on the level of torus-embeddings and then for $R \subseteq R_n \subseteq R_\infty$, that $p^\alpha \cdot Tor_1^{R_n}(R_\infty,$ any R_n -module$) = 0$ for $n >> 0$.

Now for the definition of $\lambda()$: We assume that for a cofinal sequence of L_n's $L_n = N \cdot L$ with $N \to \infty$. Let $d_n = [L : L_n]$ denote its index. Then after pushout we obtain that for all n (in our cofinal sequence)

$$R_{n+m} = \bigoplus_{\mu \in L_{n+m}/L_n^\vee} R_{n+m,\mu}$$

where only finitely many isomorphism types of direct summands occur, and the relative frequency of each summand converges (for $m \to \infty$) to a constant, which is non-zero for the type $R_{n+m,\mu} \cong R$. Moreover this summand occurs with multiplicity $\geq c \cdot (d_{n+m}/d_m), c > 0$ a constant independant of n and m.

Now assume M_∞ is a finitely presented R_∞ -module which is $\mathfrak{m}$ -torsion. ($\mathfrak{m}$ = maximal ideal of R). Then $M_\infty = M_n \otimes_{R_n} R_\infty$ for a finitely presented R_n -module M_n which is also $\mathfrak{m}$ -torsion and thus of finite length. Define

$$\lambda(M_\infty) = \lim_{m\to\infty} \frac{d_n}{d_{n+m}} length_R(M_n \otimes_{R_n} R_{n+m})$$

The limit exists because the relative multiplicities of the $R_{n+m,\mu}$ converge. Furthermore $length_R(M_n \otimes_{R_n} R_{n+m}) \geq c \cdot d_{n+m} \cdot \lambda(M_\infty)$ for all m. Especially M_∞ vanishes if $\lambda(M_\infty) = 0$. In more generality we have: Let $M_n \twoheadrightarrow \tilde{M}_n$ denote a surjection, inducing $M_\infty \twoheadrightarrow \tilde{M}_\infty$. Then for all m

$$length(kernel(M_n \otimes_{R_n} R_{n+m} \to \tilde{M}_n \otimes_{R_n} R_{n+m}) \geq c \cdot d_{n+m} \cdot (\lambda(M_\infty) - \lambda(\tilde{M}_\infty))$$

Next assume M_∞ is only finitely generated but still M torsion. Choose a finite set of generators and denote by M_n the R_n -submodule generated by them. Then M_{n+m} is a quotient of $M_n \otimes_{R_n} R_{n+m}$, and the sequence $\lambda(M_n \otimes_{R_n} R_\infty)$ is decreasing.

6. Claim.

$$\lim_{n\to\infty} \frac{1}{d_n} length_R(M_n) = \lim_{n\to\infty} \lambda(M_n \otimes_{R_n} R_\infty)$$

Define this limit as $\lambda(M_\infty)$.

Proof of claim. For $\varepsilon > 0$ choose n such that $\lambda(M_n \otimes_{R_n} R_\infty) \leq \lambda(M_\infty) + \varepsilon$. Then if $N_{n+m} = \ker(M_n \otimes_{R_n} R_{n+m} \to M_{n+m})$, we have that for $l \geq 0$

$$length_R \ker(M_n \otimes_{R_n} R_{n+m+l} \to M_{n+m} \otimes_{R_{n+m}} R_{n+m+l}) \geq c \cdot (d_{n+m+l}/d_{n+m}) \cdot length_R(N_{n+m})$$

Passing to the limit $l \to \infty$ we obtain

$$\begin{aligned} length_R(N_{n+m}) &\leq \frac{d_{n+m}}{c}(\lambda(M_n \otimes_{R_n} R_\infty) - \lambda(M_{n+m} \otimes_{R_{n+m}} R_\infty)) \\ &\leq \frac{d_{n+m} \cdot \varepsilon}{c} \end{aligned}$$

Thus $0 \leq \frac{1}{d_{n+m}}(length_R(M_n \otimes_{R_n} R_{n+m}) - length_R(M_{n+m})) \leq \frac{\varepsilon}{c}$, which implies the claim.

Next for an arbitrary $\mathfrak{m}$ -torsion R_∞ -module M define

$$\lambda(M) = \sup\{\lambda(M') \mid M' \subseteq M \text{ finitely generated submodule}\},$$

so $0 \leq \lambda(M) \leq \infty$.

7. Lemma (additivity). Suppose $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $\mathfrak{m}$ -torsion R_∞ -modules. Then

$$\lambda(M) = \lambda(M') + \lambda(M'')$$

Proof. We easily reduce to M and M'' finitely generated. For a finite system of generators of M let M_n denote the R_n -module generated by them, M''_n its image in M'', and M'_n its intersection with M'. Then we have

$$\lim_{n\to\infty}\frac{1}{d_n}length_R(M'_n) = \lambda(M) - \lambda(M'').$$

For any finite set of elements of M' the R_n -module $\tilde{M}'_n$ generated by them is for big n contained in M'_n, and has $length \leq length_R(M'_n)$. It follows that

$$\lambda(M') \leq \lambda(M) - \lambda(M'').$$

On the other hand for $\varepsilon > 0$ choose n such that

$$\lambda(M'') \geq \lambda(M''_n \otimes_{R_n} R_\infty) - \varepsilon.$$

Then for $m \geq 0$ the kernel of $M''_n \otimes_{R_n} R_{n+m} \to M''_m$ has length bounded by $\frac{\varepsilon}{n} \cdot d_{n+m}$, by the previous arguments. From the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & M'_n \otimes_{R_n} R_{n+m} & \longrightarrow & M_n \otimes_{R_n} R_{n+m} & \longrightarrow & M''_n \otimes_{R_n} R_{n+m} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M'_{n+m} & \longrightarrow & M_{n+m} & \longrightarrow & M''_{n+m} & \longrightarrow & 0
\end{array}$$

we obtain a surjection

$$\ker(M''_n \otimes_{R_n} R_{n+m} \to M''_{n+m}) \twoheadrightarrow M'_{n+m}/R_{n+m} \cdot M'_n$$

Thus $length_R(M'_{n+m}) \leq length_R(R_{n+m} \cdot M'_n) + \frac{\varepsilon}{c} \cdot d_{n+m}$ and in the limit $m \to \infty$

$$\lambda(M) - \lambda(M'') \leq \lambda(R_\infty \cdot M'_n) + \frac{\varepsilon}{c} \leq \lambda(M') + \frac{\varepsilon}{c} \leq \lambda(M') + \frac{\varepsilon}{c}.$$

As ε was arbitrary we derive the lemma. Finally if M is finitely presented and annihilated by $\pi^{1/p}$, then $\lambda(M \otimes_{R_\infty \varphi} R_\pi R_\infty) = p^{d+1} \cdot \lambda(M)$: This follows because first on the level of torus-embeddings pushout be Frobenius induces an isomorphism

$$k[L_n^\vee \cap \sigma^\vee]/(\lambda^{1/p}) \xrightarrow{\sim} k[pL_n^\vee \cap \sigma^\vee]/(\lambda)$$

Thus for modules annihilated by $\pi^{1/p}$ or $\lambda^{1/p}$ pushout by Frobenius replaces L_n by $\frac{1}{p}L_n$, hence d_n by $p^{-(d+1)}d_n$, and the assertion follows. It then also holds for finitely generated M by passage to the limit from $M_n \otimes_{R_n} R_\infty$.

Finally if M' is submodule of a finitely presented M_1 and $\lambda(M') = 0$, then $M'' = M/M'$ is also finitely presented, with $\lambda(M) = \lambda(M'')$. Then $length_R(\ker(M_n \to M''_n)) \leq \frac{1}{c}(\lambda(M) - \lambda(M'')) = 0$, hence $M' = (0)$.

Remark. There exist finitely generated $M \neq (0)$ with $\lambda(M) = 0$, for example the residue field of R_∞.

Suppose G is a finite group, A a commutative ring (with unit), B an A-algebra with G-action. Assume furthermore that for $b \in B$ $tr(b) = \sum_{g \in G} g(b)$ lies in the image of A in B. If M is a B-module with semilinear G-action, we may compute the cohomology $H^*(G, M)$ using a G-acyclic resolution by such $B - G$ -modules. If $b \in B$ has image $a = tr(b) \in A$, then a annihilates all higher cohomology:

For $m \in M$ we have $tr(b \cdot m) = \sum_{g \in G} g(b \cdot m) \in M^G$, and if $m \in M^G$ then $tr(b \cdot m) = a \cdot m$. Applying this to a resolution of M the claim follows.

If we also assume that there exist elements $b_i, c_i \in B$ $(1 \leq i \leq r)$ such that for any $b \in B$

$$a = \sum_{i=1}^{r} b_i \cdot c_i$$

$$0 = \sum_{i=1}^{r} b_i \cdot g(c_i)$$

for $g \in G, g \neq 1$ we can define a map

$$M \to B \otimes_A M^G$$

$$m \longmapsto \sum_{i=1}^{r} b_i \otimes_A tr(c_i \cdot m)$$

The composition either way of this map with the natural

$$B \otimes_A M^G \to M$$

will be a. For example if $A \subseteq B$ is an almost étale Galois-covering with group G, then we apply this to $a = \pi^\alpha$ (any $\alpha > 0$), $\sum_{i=1}^{r} b_i \otimes c_i = \pi^\alpha \cdot e_{B/A}$, and obtain that

$$H^i(G, M) \approx 0 \quad (i > 0)$$

$$M \approx B \otimes_A M^G$$

Passing to the limit this will also hold if B is an inductive limit of almost étale coverings and G the corresponding profinite group, i.e. for the extensions $R_\infty \subseteq \bar{R}$. Here $\bar{R}$ denotes the normalisation of R in the maximal étale covering of the preimage of $T \subseteq \bar{T}$ in $Spec(R)$, that is the union of all finite extensions $R \subseteq S$ with S normal and $S[\frac{1}{\pi}]/R[\frac{1}{\pi}]$ ramified only along the divisor "at infinity". For simplicity we assume that R is an integral domain (otherwise we would take everywhere products of integral domains). If $\bar{V}$ denotes the integral closure of V in $\bar{K}, \bar{R}$ contains as subring the normalisation of the image of $R \otimes_V \bar{V} \to \bar{R}$. If $\lambda = n \cdot \lambda_0$ with λ_0 indivisible in L then firstly R contains the subring $V' = V[\sqrt[n]{\pi}]$. Replacing V by V' we then may assume that $\lambda = \lambda_0$. In this case the normalisation of V in R is unramified over V. Thus if the residue field k is algebraically closed it coincides with V_j and $R \otimes_V \bar{V}$ is an integral domain. In any case we denote by Δ the Galois-group of $\bar{R}$ over the normalisation of the image of $R \otimes_V \bar{V}$. If $R \otimes_V \bar{V}$ is an integral domain this is $\Delta = \pi_1(Spec(R \otimes_V \bar{K})^0)$. The coverings defined by the R_n give a map

$$\Delta \to L \otimes \hat{\mathbb{Z}}(1) = \hat{L}(1)(\text{profinite completion}).$$

Let Δ_∞ denote its image and $\Delta_0 \subseteq \Delta$ its kernel. As R is formally *é*tale over $\bar{T}_\lambda$ and the R_n are totally ramified over R at all generic characteristic p-points if $[L : L_n]$ is a p-power, there is a surjection $\Delta_\infty \twoheadrightarrow L \otimes \mathbb{Z}_p(1)$, with kernel of order prime to p. For most arguments we can in fact replace Δ_∞ by $L \otimes \mathbb{Z}_p(1)$.

Now suppose $\bar{M}$ is a p-torsion $\bar{R}$ -module with a semilinear continuous Δ -action. To study the cohomology $H^*(\Delta, M)$ we use the spectral sequence

$$E_2^{a,b} = H^a(\Delta_\infty, H^b(\Delta_0, M)) \Rightarrow H^{a+b}(\Delta, M).$$

The extension governed by Δ_0 is a composition of almost *é*tale extensions. Thus all higher Δ_0 -cohomology almost vanishes,

$$M \approx (M^{\Delta_0}) \otimes \bar{R},$$

and $H^*(\Delta, M) \approx H^*(\Delta_\infty, M^{\Delta_0})$. As Δ_∞ has cohomological dimension d this vanishes in degree $> d$. Also after choice of topological generators $\delta_1, \dots, \delta_d$ of Δ_∞ the cohomology is represented by the Koszul-complex of $\delta_1 - 1, \dots, \delta_d - 1$ acting on M^{Δ_0}. This commutes with formally *é*tale base change $R \to R'$.

For example we can study the cohomology of quotients $\bar{R}/p^s \cdot \bar{R}$. By the above we may (up to almost isomorphism) replace R by the affine ring of the torus-embedding $\bar{T}_\lambda$, and have to study the cohomology of $\mathbb{Z}_p(1)^d$ acting on

$$V[\sigma^\vee \cap L^\vee[\frac{1}{p}]]/(\lambda - \pi) \otimes_V \bar{V}/p^s\bar{V}$$

This module is free over $\bar{V}/p^s\bar{V}$, where a basis can be obtained as follows:

λ acts freely on the monoid $\sigma^\vee \cap L^\vee[\frac{1}{p}]$. For each orbit $\{\mu, \mu + \lambda, \mu + 2\lambda \dots\}$ choose the minimal representative μ. Then these μ form a basis.

All the basis-elements μ are eigenvectors for $\mathbb{Z}_p(1)^d$, with character $e^\mu : \mathbb{Z}_p(1)^d \to \mu_{p^\infty}$. If this character has order p^t the corresponding cohomology is annihilated by $\zeta_{p^t} - 1, \zeta_{p^t}$ a primitive $p^t - th$ root of unity. Thus up to terms annihilated by $\zeta_p - 1$ only the eigenspaces with trivial e^μ contribute to cohomology. e^μ is trivial if and only if $\mu \mid \ker(\lambda)$ is in $\ker(\lambda)^\vee$. Then there exists a unique $\alpha \in \mathbb{Z}[\frac{1}{p}], 0 \le \alpha < 1$, with $\mu - \alpha\lambda \in L^\vee$ integral. Let $\mathcal{R} = \bigoplus_{\alpha,\mu} V\pi^\alpha \cdot \mu$ (sum over such α, μ). Then $\mathcal{R} = R_\infty \cap R \otimes_V K_\infty$ is the normalisation of $R \otimes_{\mathbf{V}} V_\infty$, and $\pi\mathcal{R} \subseteq R \otimes_V \bar{V}$. Furthermore (as $H^*(\mathbb{Z}_p(1)^d, V) = \Lambda^{\cdot}(V(-1)^d)$)

$$H^*(\Delta, \bar{R}/p^s\bar{R}) \approx \mathcal{R} \otimes \Lambda^{\cdot}(Hom(\Delta_\infty, \bar{V}/p^s\bar{V})) \oplus \quad (\text{terms annihilated by} \quad \zeta_p - 1).$$

Especially we get compatible almost maps

$$tr : H^d(\Delta, \bar{R}/p^s\bar{R}) \to \mathcal{R} \otimes_{V_\infty} \bar{V}/p^s\bar{V}(-d).$$

(Compatibility forces them to vanish on the second direct summand). The target can be more canonically identified with $\Omega^{d,\log}_{R/V} \otimes_R \mathcal{R} \otimes_{V_\infty} \bar{V}/p^s\bar{V}(-d)$. Here $\Omega^{\log}_{R/V}$ denotes the logarithmic module of differentials, for the log -structures on $\bar{T}_\lambda$ and

$Spec(V)$ given by $\sigma^\vee \cap L^\vee$ and $\mathbb{N}$ $\quad(\mu \in \sigma^\vee \cap L^\vee$ maps to μ, and $1 \in \mathbb{N}$ to $\pi)$. By etaleness

$$\Omega^{\log}_{R/V} \otimes V/p^sV \cong \ker(\lambda)^\vee \otimes R$$

Of course $\Omega^{d,\log}_{R/V}$ is the $d'th$ exterior power. At least modulo p^s it is free of rank one. Checking all the identification we get almost maps

$$tr : H^d(\Delta, \bar{R}/p^s\bar{R}) \to \Omega^{d,\log}_{R/V} \otimes_R \mathcal{R} \otimes_{V_\infty} \bar{V}/p^s\bar{V}(-d)$$

which are independant of the choice of generators $\delta_1, \dots, \delta_d$. In fact they only depend on R (with its log-structure) and not on the formally étale map $Spec(R) \to \bar{T}_\lambda$. This follows (as in [F2], Sect. I, 4) from the following facts, which can be checked by direct calculation for $\bar{T}_\lambda$ and then also hold for R by formal étaleness and almost étaleness:

i) The sequence

$$0 \to \Omega^{\log}_{\bar{V}/V} \otimes_{\bar{V}} \bar{R} \to \Omega^{\log}_{\bar{R}/R} \to \Omega^{\log}_{\bar{R}/R\otimes\bar{V}} \to 0$$

is almost exact

ii)

$$\Omega^{\log}_{\bar{V}/V} \cong \bar{V}[\frac{1}{p}]/\rho^{-1}\bar{V}(1)$$

for an element $\rho \in \bar{V}, \rho \neq 0$. The isomorphism is induced from $d\log : \mu_{p^\infty} \to \Omega^{\log}_{\bar{V}/V}$.

iii) Consider the exact sequence

$$\Omega^{\log}_{R/V} \otimes_R \bar{R} \to \Omega^{\log}_{\bar{R}/\bar{V}} \to \Omega^{\log}_{\bar{R}/R\otimes_V\bar{V}} \to 0$$

Then the kernel of the first map is almost contained in $\bigcap_{s\geq 0} p^s \cdot (\Omega^{\log}_{R/V} \otimes_R \bar{R})$. Furthermore the induced map

$$Hom(\frac{1}{p^s}\mathbb{Z}/\mathbb{Z}, \Omega^{\log}_{\bar{R}/R\otimes_V\bar{V}}) \to \Omega^{\log}_{R/V} \otimes_R \bar{R}/p^s\bar{R}$$

is an almost isomorphism.

iv) Applying $Hom(\frac{1}{p^s}\mathbb{Z}/\mathbb{Z},)$ to the exact sequence in i) gives an almost exact sequence

$$0 \to \rho^{-1}(\bar{R}/p^s\bar{R}(1)) \to \text{middle term} \to \Omega^{\log}_{R/V} \otimes_R \bar{R}/p^s\bar{R} \to 0,$$

thus a map

$$\Omega^{\log}_{R/V} \to H^1(\Delta, (\bar{R}/p^s\bar{R})(1))$$

By skew-commutativity of the cup-product we obtain

$$\Omega^{d,\log}_{R/V} \otimes_V \bar{V}/p^s\bar{V}(-d) \to H^d(\Delta, \bar{R}/p^s\bar{R})$$

which is $\rho^d \cdot d!$ -times the map constructed before.

v) As $\rho^d \cdot d!$ -times our maps are canonical, compatibility for varying s implies the assertion.

Remark. So from now on we assume that $\mathcal{R} = R \otimes_V \bar{K}$, equivalent to the cone-condition above $((\mathbb{Q}^+ \cdot \lambda + L^\vee) \cap \sigma^\vee = \mathbb{Q}^+ \cdot \lambda + (L^\vee \cap \sigma^\vee))$. Other equivalences: if $\mu + \alpha\lambda (\mu \in L^\vee, \alpha \in \mathbb{Q}, 0 \leq \alpha < 1)$ lies in $\sigma^\vee$, then so does μ.
Also: The special fibre $\bar{T}_\lambda \otimes_V k$ is reduced.

Before we start to treat duality we have to introduce coefficients. Suppose $\mathbb{L}$ is a finite $\mathbb{Z}_p$ -module with a continuous action of Δ. We denote by $\mathbb{L}^t = Hom(\mathbb{L}, \mathbb{Q}_p/\mathbb{Z}_p)$ its dual. Next recall that $\sigma \cap \ker(\lambda)$ was supposed to be a simplex spanned by a partial basis $\rho_1, \ldots, \rho_r$ of L. On the generic fibre of $\bar{T}_\lambda$ these define normal crossings divisors $D_1, \ldots, D_r$, which are cut out by a (partial) dual basis of $L^\vee$. Choose a subset I of $\{1, \ldots, r\}$ and denote by D_I the union of the $\{D_i \mid i \in I\}$. Then for each n the preimage of D_I in $\bar{T}_{\lambda,n} \otimes_V K$ is a divisor. Let $J_{I,n} \subseteq V[L_n^\vee \cap \sigma^\vee]/(\lambda - \pi)$ denote the ideal of its reduced closure. That is $J_{I,n}$ is spanned by all $\mu \in L_n^\vee \cap \sigma^\vee$ with $\mu(\rho_i) > 0$ for $i \in I$. Finally $J_{I,\infty} \subseteq \Gamma(\bar{T}_{\lambda,\infty}, \mathcal{O})$ denotes the union of the $J_{I,n}$. One checks that for any $\mu \in L^\vee \cap \sigma^\vee$ with $\mu(\rho_i) > 0$ for $i \in I, \mu(\rho_i) = 0$ for $i \notin I, J_{I,\infty}$ is almost generated by the fractional powers μ^{1/p^n}. (Use that any μ' vanishing on $\sigma \cap \ker(\lambda)$ divides a power of π). Thus $J_{I,\infty}$ is almost a union of principal ideals and almost flat.

Finally we apply pushout to R. If we assume π is contained in the Jacobsen radical of R (for example if R is π -adically complete) then R is normal, and $R[\frac{1}{\pi}]$ is regular with the pullback of the D_i defining a normal crossings divisor. Then one checks that the ideal $J_{I,\infty} \otimes \bar{R}[\frac{1}{\pi}]$ defines the reduced pullback-divisor and thus only depends on the pullback of D_I to $Spec(R[\frac{1}{\pi}])$, not on the particular representation $Spec(R) \to \bar{T}_\lambda$. Finally be almost étaleness of $\bar{R}$ over $\Gamma(\bar{T}_{\lambda,\infty}, \mathcal{O})$ the ideal $J_{I,\infty} \cdot \bar{R}$ is almost equal to the ideal defining the closure, thus also as canonical as the divisor D_I. Call it $\bar{J}_I$.

Now our aim is to study the cohomology of Δ with coefficients $\mathbb{L} \otimes \bar{J}_I$. This depends only on the image of $\bar{J}_I$ in the π -adic completion of R, so that everything is canonical even if π should not lie in the Jacobsen radical of R. Next comes finiteness. As $R \otimes_V \bar{V}$ is the inductive limit of noetherian rings ($R \otimes_V V', V'$ a finite extension of V) with flat transition-maps, it is coherent. That is any finitely presented $R \otimes_V \bar{V}$ -module has a projective resolution by finitely generated free $R \otimes_V \bar{V}$ -modules, and the finitely presented $R \otimes_V \bar{V}$ -modules form an abelian subcategory of all $R \otimes_V \bar{V}$ -modules.

We say that an $R \otimes_V \bar{V}$ -module M is almost finitely presented if for each $\alpha > 0$ there exists a finitely presented M_α and a π^α -isomorphism $f : M_\alpha \to M$ i.e. kernel and cokernel of f are annihilated by π^α. The almost finitely presented $R \otimes_V \bar{V}$ -modules also form an abelian subcategory. Now suppose $\mathbb{L}$ is a finite $\mathbb{Z}_p$ -module with a continuous Δ -action.

8. Proposition. The cohomology-groups $H^i(\Delta, \mathbb{L} \otimes \bar{J}_I)$ are almost finitely presented over $R \otimes_V \bar{V}$.

Proof. By devissage we may assume that $p \cdot \mathbb{L} = (0)$. We also may replace R by its p-adic completion, so $Spec(R)$ is formally étale over $\bar{T}_\lambda$. Especially we have that $R \otimes_V K$ is regular, and the complement of $T \subseteq \bar{T}$ defines a normal crossing divisor. Replacing L by a sublattice $L' \subseteq L$ we may assume that $\mathbb{L}$ is unramified over $Spec(R \otimes_V \bar{K})$. (If Δ' is the group corresponding to L', use the Hochschild spectral sequence for $0 \to \Delta' \to \Delta \to \Delta/\Delta' \to 0$). Next by almost faithfully falt descent (for $\bar{R}/R_\infty$) $(\mathbb{L} \otimes \bar{R})^{\Delta_\infty}$ is almost finitely presented. Thus for each

$\alpha > 0$ there exists an integer n, a finitely presented R_n -module M_α action, and a π^α-isomorphism

$$M_\alpha \otimes_{R_n} R_\infty \to (\mathbb{L} \otimes \bar{R})^{Gal(\bar{R}/R_\infty \otimes_V \bar{K})}$$

By increasing n we make this $Gal(R_\infty/R_n \otimes_V \bar{V})$ -linear (trivial operation on M_α. It suffices if the images of generators of M_α are fixed.

Again by the Hochschild spectral sequence we may replace R by R_n, i.e. study $H^*(\Delta_\infty, M \otimes_R J_{\infty,I})$. Now for each n $J_{\infty,I}$ is the direct sum of a finitely presented $R_n \otimes_V \bar{V}$ -module and modules on which a generator δ of Δ_∞ acts like ζ_{p^n}, ζ_{p^n} a primitive $p^n - th$ root of unity. (There are generated by $\mu's, \mu \mid \ker(\lambda)$ not integral on $p^{n-1}L$.) For these the cohomology is annihilated by $\zeta_{p^n} - 1$, while the first direct summand has finitely presented cohomology.

Finally we come to duality. Recall ([H], Ch. V, § 2) that a dualising complex D^* over a noetherian ring A is a complex with coherent cohomology which has an injective resolution in which each indecomposable injective appears precisely once. Equivalently for each prime $\mathfrak{p} \in Spec(A)$ there exists an integer $d(\mathfrak{p})$ (related to the codimension) with $Ext^i_{A_\mathfrak{p}}(K(\mathfrak{p}), D^*_\mathfrak{p}) = \kappa(\mathfrak{p})$ for $i = d(\mathfrak{p})$, and 0 else. Then $K^* \to \mathbb{R}Hom(K^*, D^*)$ defines a perfect duality on the derived category of finite complexes with finitely generated cohomology. If A is Cohen-Macaulay D^* has only one non-trivial cohomology group, the dualising mdoule ω_A. For example for the torus-embedding $\bar{T}$ a dualising module is given by $\omega_{\bar{T}} = J \otimes \Omega^{d+1,\log}$, where J is the ideal generated by all $\mu \in L^\vee$ which are strictly positive on $\sigma - \{0\}$. This follows because on the smooth locus the dualising module coincides with the differentials, and in general it is the reflexive extension of its restriction to the smooth locus. Also for $\bar{T}_\lambda$ the dualising module is $Ext^1_{O_{\bar{T}}}(O_{\bar{T}_\lambda}, \omega_{\bar{T}}) \cong J \otimes O_{\bar{T}_\lambda}$.

It is the ideal in $\mathcal{O}_{\bar{T}_\lambda}$ (or better in $\Omega^{d,\log}_{\bar{T}_\lambda}$) generated by all $\mu \in L^\vee$ which are strictly positive on $\sigma - \{0\}$. Finally if π is contained in the Jacobson-radical of R the map $\mathcal{O}_{\bar{T}_\lambda} \to R$ is flat with Gorenstein-fibres. Thus $\omega_{\bar{T}_\lambda} \otimes_{\bar{T}_\lambda} R = \omega_R$ is a dualising module on R. It is isomorphic to J. Similarly the dualising modules on R_n are isomorphic to J_n. By the general theory $\omega_{R_n} \cong Hom_R(R_n, \omega_R)$. This isomorphism is determined by the evaluation at $1 \in R_n$, which defines a trace-map

$$tr : \omega_{R_n} \to \omega_R.$$

It is obtained by pushout from the trace-map on $O_{\bar{T}}$ or $O_{\bar{T}_\lambda}$. The latter sends to character $\mu \in L_n^\vee$ to itself if $\mu \in L^\vee$, and to zero else. Similarly we can define a dualising module $\omega_{R \otimes_V \bar{V}} = \omega_R \otimes_V \bar{V}$.

By passing to the limit over finite extensions of V,

$$K^* \to D(K^*) = \mathbb{R}Hom_{R \otimes_V \bar{V}}(K^*, \omega_R \otimes_V \bar{V})$$

still defines a perfect duality for finite complexes with finitely presented (= coherent) cohomology, or an almost perfect duality for finite complexes with almost finitely presented cohomology. Again $\omega_{R \otimes_V \bar{V}} \subseteq R \otimes_V \bar{V}$ is the ideal J generated by all $\mu \in L^\vee$ which are strictly positive on $\sigma - \{0\}$. Similarly $\omega_{R_n \otimes_{V_n} \bar{V}} \cong J_n \subseteq R_n \otimes_{V_n} \bar{V}$ is the ideal generated by $\mu \in L_n^\vee$ strictly positive on $\sigma - \{0\}$, and we have the union $J_\infty \subseteq R_\infty \otimes_{V_\infty} \bar{V}$. It coincides almost with the ideal generated by $\mu \in \sigma^\vee$ which are strictly positive on $\sigma \cap \ker(\lambda)) - \{0\}$:

For such a μ $\mu + \alpha \cdot \lambda$ is strictly positive on $\sigma - \{0\}$, for each $\alpha > 0$. That is J_∞ almost coincides with the ideal previously named by the same symbol. Finally the trace-map becomes more complicated:

As $V_n = V[\sqrt[e]{\pi}] = V[\pi_n]$ (for some e)

$$\frac{1}{e} \cdot tr_{V_n/V} : \frac{\pi_n}{\pi} \cdot V_n \to V$$

defines an isomorphism

$$\frac{\pi_n}{\pi} V_n \cong Hom_V(V_n, V).$$

Thus our old trace defines a new trace from J_n to $J \otimes_V (\frac{\pi_n}{\pi} V_n)$ and by $\otimes_{V_n} \bar{V}$ a trace-map

$$tr : J_n \otimes_{V_n} \bar{V} \to J \otimes_V (\frac{\pi_n}{\pi} \bar{V}) = \frac{\pi_n}{\pi} \otimes (J \otimes_V \bar{V})$$

It sends a $\mu \in L_n^\vee$ strictly positive on $\sigma - \{0\}$ to itself or zero, depending on whether $\mu \in \ker(\lambda)^\vee + \mathbb{Q} \cdot \lambda$ or not. Obviously these combine to define

$$tr : R_\infty \otimes_{V_\infty} \bar{V} \to \frac{1}{\pi}(J \otimes_V \bar{V})$$

As on finite levels we obtain isomorphisms

$$J_n \otimes_V \bar{V} \cong Hom_{R \otimes_V \bar{V}}(R_n \otimes_V \bar{V}, \frac{\pi_n}{\pi} \cdot (J \otimes_V \bar{V})),$$

we have in the limit for each $\mu \in \ker(\lambda)^\vee \otimes \mathbb{Q}/\mathbb{Z}$ an almost isomorphism of μ -components

$$(J_\infty \otimes_V \bar{V})_\mu \approx Hom_{R \otimes_V \bar{V}}(R_\infty \otimes_V \bar{V}, \frac{1}{\pi}(J \otimes_V \bar{V}))_\mu$$

Of course the map

$$J_\infty \otimes_V \bar{V} \to Hom_{R \otimes_V \bar{V}}(R_\infty \otimes_V \bar{V}, \frac{1}{\pi} J \otimes_V \bar{V})$$

cannot be an almost-isomorphism because of the difference between direct sum and product (over all μ -components).

However this difficulty disappears if we take Δ_∞ -cohomology: If μ has denominator p^n the cohomology of the μ -component is annihilated by $\zeta_{p^n} - 1$. It thus follows that we obtain almost-isomorphisms on cohomology. The same holds after tensoring with $(\mathbb{L} \otimes \bar{R})^{Gal(\bar{R}/R_\infty \otimes_V \bar{V})}$ (π^α -isomorphic to $M_\alpha \otimes_{R_n} R_\infty$ as before). Thus (using almost étaleness).

9. Proposition. tr induces almost isomorphisms

$$H^i(\Delta, \mathbb{L} \otimes \bar{R}) \approx H^i(\Delta, \mathbb{L} \otimes Hom_{\bar{R} \otimes_V \bar{V}}(\bar{R}, \frac{1}{\pi} J \otimes_V \bar{V}))$$

Finally cohomology can be (almost) computed by taking invariants under $Gal(\bar{R}/R_\infty \otimes_V \bar{V})$ (an almost exact operation) and then applying the Koszul-complex with $(\delta_1 - 1), \dots, (\delta_d - 1)$, δ_i topological generators of $\Delta_\infty \cong \mathbb{Z}_p(1)^d$. However the dual of the Koszul-complex is again a Koszul-complex shifted by d. We thus obtain:

10. Theorem. Assume $\mathbb{L}$ is annihilated by p^s, and let $\mathbb{L}^t = Hom(\mathbb{L}, \mathbb{Z}/p^s\mathbb{Z})$ denote the dual. Then

i) $\mathbb{R}\Gamma(\Delta, \mathbb{L} \otimes \bar{R})$ is almost isomorphic to a finite complex of $R \otimes_V \bar{V}$ -modules concentrated in degrees $[0, d]$ with almost finitely presented cohomology ($\mathbb{R}\Gamma(\Delta, \cdot)$ denotes one of the canonical complexes computing group-cohomology)
ii) The previously defined trace-maps $tr : H^d(\Delta, \bar{R}/p^s\bar{R}) \to \mathcal{R}/p^s\mathcal{R}$ induce maps $tr : H^d(\Delta, \bar{J}/p^s\bar{J}) \to \frac{1}{\pi}\omega_R \otimes \bar{V}/p^s\bar{V}$ ($\mathcal{R}$ is contained in $\frac{1}{\pi}R \otimes_V \bar{V}$)
iii) tr defines an almost isomorphism

$$\mathbb{R}\Gamma(\Delta, \mathbb{L} \otimes \bar{J})(d)[d] \approx \mathbb{R}Hom_{R\otimes_V \bar{V}/p^s\bar{V}}(\mathbb{R}\Gamma(\Delta, \mathbb{L}^\vee \otimes \bar{R}), \frac{1}{\pi}\omega_R \otimes \bar{V}/p^s\bar{V})$$

Proof. Only some minor details need to be checked: Firstly our trace map (from duality) $J_\infty/(\delta_i - 1) \to J \otimes_V \frac{1}{\pi}\bar{V}$ induces on cohomology the previously defined trace, (obviously direct checking), and secondly we may replace R by its π -adic completion so that the duality-theory works.

Variant. We may also use coefficients $\bar{J}_I; I \subseteq \{1, \dots, r\}$ a subset. The ideals $J_I \subseteq R$ and $J_{I,n} \subseteq R_n$ are Cohen-Macaulay:

Reduce to the torus-embedding $\bar{T}$. If (say) $\delta_1 \in I$, let $\tilde{L}$ denote the torus-embedding defined by $\tilde{L} = L/\mathbb{Z} \cdot \delta_1$ and the image $\tilde{\sigma}$ of σ in $\tilde{L}_\mathbb{R}$. Then $\tilde{\sigma}$ has extremal rays generated by $\delta_2, \dots, \delta_s$. Let $\tilde{I} = I \cap \{\delta_2, \dots, \delta_r\}$. Then there is an exact sequence

$$0 \to J_I \to J_{I-\{1\}} \to \tilde{J}_{\tilde{I}} \to 0.$$

If the other two terms are Cohen-Macaulay (of relative dimension $d + 1$ respectively d), then J_I also is. Now duality holds between $J_{I,n}$ and $Hom_R(J_{I,n}, \omega_R) = Hom_{R_n}(J_{I,n}, \omega_{R_n})$, and similarly between $J_{I,n} \otimes_{V_n} \bar{V}$ and $Hom_{R\otimes_V\bar{V}}(J_{I,n} \otimes_V \bar{V}, \omega_R \otimes_V \bar{V})$. However one checks that the union

$$\bigcup_n Hom_{R\otimes_V\bar{V}}(J_{I,n} \otimes_V \bar{V}, \omega_R \otimes_V \bar{V}) \approx J_{I^c,\infty}$$

is almost $J^c_{I,\infty}, I^c \subseteq \{1, \dots, r\}$ the complement of I:

The product $J_{I,\infty} \times J_{I^c,\infty} \to J_\infty$ shows that the union contains $J_{I^c,\infty}$. On the other hand it is contained in R_∞, and any character μ in it must be strictly positive on $\delta_i \in I^c$ (choose $\nu \in L^\vee \cap \sigma^\vee$ which is strictly positive on I but vanishes on δ_i. Then $\nu \in J_I$, and $\mu + \nu$ must be strictly positive on δ_i too). However these μ's are almost in $J_{I^c,\infty}$. With the same proofs as before we then have

10'. Theorem.

$$\mathbb{R}\Gamma(\Delta, \mathbb{L} \otimes \bar{J}_{I^c})(d)[d] \approx \mathbb{R}Hom_{R\otimes_V \bar{V}/p^s\bar{V}}(\mathbb{R}\Gamma(\Delta, \mathbb{L}^\vee \otimes \bar{R}), \frac{1}{\pi}\omega_R \otimes \bar{V}/p^s\bar{V})$$

Last we need a local Künneth-formula. Suppose we have given two lattices L', L'' with cones $\sigma' \subseteq L'_\mathbb{R}, \sigma'' \subseteq L''_\mathbb{R}$ and elements $\lambda' \in L'^\vee, \lambda'' \in L''^\vee$. Then in general the fibered product $\bar{T}_{\lambda'} \otimes_V \bar{T}_{\lambda''}$ will not be normal. However we can achieve this by adjoining roots of π. Namely pass to $V[\sqrt[e]{\pi}]$ (replacing L' by $\lambda'^{-1}(e\mathbb{Z}), L''$ by $\lambda''^{-1}(e\mathbb{Z})$) such that $\mathcal{R}' = R' \otimes_V \bar{V}, \mathcal{R}'' = R'' \otimes_V \bar{V}$. That means for any linear combination $\mu' + \alpha\lambda' \in \sigma'^{\vee}(\mu' \in L', 0 \leq \alpha < 1)$ we have $\mu' \in \sigma'^{\vee}$, and similarly for $\mu'' + \alpha\lambda''$.

11. Lemma. Under the conditions $\bar{T}_{\lambda'} \otimes_V \bar{T}''_{\lambda''}$ is normal, and equal to $\bar{T}_\lambda$ where the torus-embedding $\bar{T}$ is defined by $L = \ker(\lambda', -\lambda'') \subseteq L' \times L'', \sigma = L_{\mathbb{R}} \cap \sigma' \times \sigma'', \lambda = (\lambda', 0) \mid L = (0, \lambda'') \mid L$

Proof. There is a map

$$\bar{T}_\lambda \to \bar{T}'_{\lambda'} \otimes_V \bar{T}''_{\lambda''}$$

induced from the tensor-product

$$V[\sigma' \cap L'^{\vee}] \otimes_V V[\sigma'' \cap L''^{\vee}] \to V[\sigma \cap L^{\vee}],$$

by dividing by $(\pi - \lambda', \pi - \lambda'')$. The rest follows from remark 5.

Now assume that this condition holds. Suppose R is formally *é*tale over $R'_1 \otimes_V R''$. Furthermore assume given representations $\mathbb{L}'$ of Δ', L'' of Δ'', and let $\mathbb{L} = \mathbb{L}' \otimes \mathbb{L}''$. We assume that $\mathbb{L}'$ and $\mathbb{L}''$ are free $\mathbb{Z}/p^s\mathbb{Z}$ -modules (to avoid using derived tensor-products). Furthermore choose finite subsets $I' \subseteq \{1, \ldots, r\}, I'' \subseteq \{1, \ldots, r''\}$. As $\sigma \cap \ker(\lambda) = (\sigma' \cap \ker(\lambda') \times (\sigma'' \cap \ker(\lambda'')), I = I' \times I''$ identifies with a subset of the vertices of $\sigma \cap \ker(\lambda)$. Then pullback and cup-product define a map

$$\mathbb{R}\Gamma(\Delta', \bar{J}_{I'} \otimes \mathbb{L}') \otimes^{\mathbb{L}}_{\bar{V}/p^s\bar{V}} \mathbb{R}\Gamma(\Delta'', \bar{J}_{I''} \otimes \mathbb{L}'') \otimes^{\mathbb{L}}_{R_1 \otimes_V R_2} R \to \mathbb{R}\Gamma(\Delta, \bar{J}_I \otimes \mathbb{L})$$

12. Theorem. This induces almost isomorphisms on cohomology.

Proof. Reduce to $\Delta'_\infty, \Delta''_\infty, \Delta_\infty = \Delta'_\infty \times \Delta''_\infty$ acting on $R'_\infty, R''_\infty, R_\infty$, and apply the obvious isomorphism on the level of Koszul-complexes.

3. Global Cohomology

Assume $X \to Spec(V)$ is proper and flat. Then there exists a dualising complex D_X on X, such that $D_X(K) = \mathbb{R}Hom_{\mathcal{O}_X}(M, D_X)$ defines a perfect duality on bounded complexes with coherent cohomology. Furthermore (if X has say pure relative dimension d) $\mathbb{R}\Gamma(X, D_X(M)) \cong \mathbb{R}Hom_V(\mathbb{R}\Gamma(M,R),V)[d]$, induced from a trace-map

$$tr : \mathbb{R}\Gamma(X, D_X) \to V[-d]$$

(see [H], Ch. VII, Th. 3.3). By base-extension $\otimes_V \bar{V}$ this extends to $X \otimes_V \bar{V}, D_X \otimes_V \bar{V}$, and bounded complexes with coherent cohomology. ($\mathcal{O}_{X\otimes_V \bar{V}}$ is still coherent). Only slightly less trivial is the extension to complexes $M^\cdot$ such that for any $\alpha > 0$ there exists a bounded complex $M^\cdot_\alpha$ with coherent cohomology, and maps $M^\cdot_\alpha \xrightarrow{\beta_\alpha} M^\cdot \xrightarrow{\lambda_\alpha} M^\cdot_\alpha$ with $\beta_\alpha \circ \lambda_\alpha = \pi^\alpha \cdot id, \lambda_\alpha \circ \beta_\alpha = \pi^\alpha \cdot id$. There duality holds up to almost quasi-isomorphisms (the induced maps on cohomology are almost isomorphic).

1. Lemma. A complex $M^\cdot$ with bounded cohomology satisfies these conditions (i.e. $\underline{M}^\cdot$ is π^α -isomorphic to a coherent $M^\cdot_\alpha$, for each $\alpha > 0$) if and only if the direct sum $\oplus \underline{H}^i(M^\cdot)$ is almost quasicoherent and locally (over each affine $Spec(R)$) almost finitely presented.

Proof. The non-trivial direction is local $\Rightarrow$ global. Assume first that $M^\cdot$ is given by an almost quasicoherent sheaf M. Replacing M by $\mathfrak{m}M/\mathfrak{m}$ -torsion we can assume that M is an honest quasicoherent sheaf. Then M is the filtering inductive limit of coherent sheaves $M_i, M = \varinjlim M_i$. For given $\alpha > 0$ and i big enough the cokernel of $M_i \to M$ is annihilated by π^α. We thus may replace M by the image of M_i, as this also satisfies the local condition (the coherent $R \otimes_V \bar{V}$ -modules form an abelian category, and so do the almost finitely presented modules). Also the kernel N of $M_i \to M$ satisfies the local condition. Applying the same argument we may assume that it is also annihilated by π^α. Then $M_i \to M$ is an $\pi^{2\alpha}$ -isomorphism.

For general M choose i minimal with $H^i(M) \neq (0)$. Then M is an extension (i.e. one has an exact triangle)

$$0 \to H^i(M)[-i] \to M \to M' \to 0,$$

which up to isomorphism is determined by a class $c \in H^1(\mathbb{R}Hom(M', H^i(M))[-i])$. By pullback with β'_α we get a class $c_\alpha \in H^1(\mathbb{R}Hom(M'_\alpha, N))(N = H^i(M)[-i])$, defining $0 \to N \to M_\alpha \to M'_\alpha \to 0$ and a map $M_\alpha \to M$. Further pullback (via λ'_α) to $H^1(\mathbb{R}Hom(M', N))$ gives $\pi^\alpha \cdot c$, which also can be obtained from c via pushforward on N by $\pi^\alpha \cdot id$. We thus obtain a map $M \to M_\alpha$ inducing λ'_α on M' and $\pi^\alpha \cdot id$ on N. The two compositions of $M \to M_\alpha \to M$ differ from $\pi^\alpha \cdot id$ by maps $M' \to N$ or $M'_\alpha \to N$ (respectively), which are nullhomotopic for degree-reasons. Thus M'_α approximates M up to π^α. With the same type of argument we can replace N by an approximation N_α.

We want to apply this theory to complexes made up from $\mathbb{R}\Gamma(\Delta, \mathbb{L} \otimes \bar{R})'s$. For this we first have to define an appropriate topos.

For a scheme X we denote by X^{Zar} resp. X^{et} the Zariski respectively *étale* topos. Furthermore $X^{et,lc}$ denotes the topos defined by finite *étale* covers of X. If X is connected and x a geometric point, $X^{et,lc}$ is equivalent to the topos of sets with continuous action of $\pi_1(X,x)$ (SGAI). In any case for noetherian X $X^{et,lc}$

is isomorphic to the topos of étale sheaves which are inductive limits of locally constant sheaves. Also there is a continuous $\rho_X : X^{et} \to X^{et,lc}$ with $\rho_{X,*}(\mathcal{F}) =$ restriction of $\mathcal{F}$ to finite étale covers, $\rho_X^*(\mathbb{L}) = \mathbb{L}$, for $\mathbb{L}$ locally constant on X.

We say X is a $K(\pi,1)$ if for each $\mathbb{L}$ $H^*(X^{et,lc},\mathbb{L}) \cong H^*(X^{et},\rho_X^*\mathbb{L})$.

Next assume given $X \to Spec(V)$, and an open subset $X^0 \subseteq X \otimes_V K$. Define a situs whose objects consists of pairs (U,V), where $U \to X$ is étale and $V \to (U \cap X^0) \otimes_K \bar{K}$ is a finite étale covering. As maps we use compatible pairs of maps, and coverings are just pairs of surjective maps. Let $\mathcal{X}^0_{\bar{K}}$ denote the corresponding topos. Its sheaves consist of ind-locally constant sheaves $\mathbb{L}_U$ on $(U \cap X^0) \otimes_K \bar{K}$, for each U as above, together with transition maps (for $U' \to U$) satisfying the usual cocycle condition. Furthermore for surjective $f : U' \twoheadrightarrow U$ $\mathbb{L}_U$ has to be the universal ind-locally constant sheaf mapping to the equaliser of $f_*\mathbb{L}_U \rightrightarrows f_*(\mathbb{L}U' \times_U U'))$. Equivalently sections $\mathbb{L}_U(V)$ can be glued for étale coverings of V induced from U. There exist a continuous

$$u_X : (X^0 \otimes_V \bar{K})^{et} \to \mathcal{X}^0_{\bar{K}},$$

with $u_{X,*}(\mathcal{F})_U = \rho_{U,*}(\mathcal{F} \mid (U \cap X^0) \otimes_V \bar{K})$.

Finally assume that $X \otimes_V K = \coprod X_\alpha$ is stratified, locally like by a "simple" normal crossings-divisor. Especially the closure X_α of each stratum is smooth over K. Define an order on indices by

$$\bar{X}_\alpha - X_\alpha = \coprod_{\beta > \alpha} X_\beta.$$

If $j_\alpha : X_\alpha \to X$ denotes the inclusion, it is known that for $\mathbb{L}_\alpha$ locally constant on X_α and $\beta > \alpha$, the restrictions $j_\beta^*\mathbb{R}^\nu j_{\alpha,*}\mathbb{L}_\alpha$ are locally constant on X_β. This follows from Abhyankhar's lemma. Now define a topos $\mathcal{X}_{\bar{K}}$ as sheaves on the following site:

Objects are triples (U,α,V), where $U \to X$ is étale $V \to U_\alpha \otimes_K \bar{K}$ is finite étale ($U_\alpha = (U \otimes_V K) \times_X X_\alpha$). Maps $(U',\beta,V') \to (U,\alpha,V)$ exist only of $\beta \geq \alpha$. Then they consist of pairs of compatible maps $U' \to U, V' \to$ normalisation of V in $U' \times_U \bar{U}_\alpha$. (It also follows from Abhyankhar's lemma that the restriction of this normalisation to U'_β, with its reduced structure, defines a finite étale cover). Coverings are pairs of surjective maps with the same α. Thus sheaves in $\mathcal{X}_{\bar{K}}$ associate to each U locally constant sheaves $\mathbb{L}_{U,\alpha}$ on $U_\alpha \otimes_K \bar{K}$, with pullback-maps (for $U' \to U$) as before and maps $\mathbb{L}_{U,\beta} \to j_\beta^* j_{\alpha,*}\mathbb{L}_{U,\alpha}$ for $\beta > \alpha$ satisfying transitivity for $\gamma > \beta > \alpha$ ($j_\lambda^* \circ j_{\alpha,*} = j_\lambda^* \circ j_{\beta,*} \circ j_\beta^* \circ j_{\alpha,*}$, by local calculation) and compatibility with pullbacks. Finally for fixed α the $\mathbb{L}_{U,\alpha}$ have to satisfy the previous gluing-condition for coverings $U' \twoheadrightarrow U$. There is a continuous $u_X : (X \otimes_V \bar{K})^{et} \to \mathcal{X}_{\bar{K}}$, with direct image $u_{X,*}(\mathcal{F})$ defined as follows: For minimal α, $u_{X,*}(\mathcal{F})_{U,\alpha}$ is the restriction of $\mathcal{F}$ to finite étale covers of $U_\alpha \otimes_K \bar{K}$. If $u_{X,*}(\mathcal{F})_{U,\alpha}$ is already defined for all $\alpha < \beta$, then $u_{X,*}(\mathcal{F})_{U,\beta}$ is the universal (inductive limit of) locally constant sheaves fitting into a commutative diagram

$$\begin{array}{ccc} u_{X,*}(\mathcal{F})_{U,\beta} & \longrightarrow & \mathcal{F} \mid U_\beta \otimes_V \bar{K} \\ \downarrow & & \downarrow \\ \prod'_{\alpha<\beta} j_\beta^* j_{\alpha,*}(u_{X,*}(\mathcal{F})_{U,\alpha}) & \longrightarrow & \prod_{\alpha<\beta} j_\beta^* j_{\alpha,*}(\mathcal{F} \mid U_\alpha \otimes_V \bar{K}) \end{array}$$

Here in the lower-keft corner $\prod'_{\alpha<\beta}$ denotes the subset of the product consisting of compatible families (for $\tilde{\alpha} < \alpha < \beta$ the α -components maps to the $\tilde{\alpha}$ -component via

$$j_\beta^* j_{\alpha,*}(u_{X,*}(\mathcal{F})_{U,\alpha}) \to j_\beta^* j_{\alpha,*} j_\alpha^* j_{\tilde{\alpha},*}(u_{X,*}(\mathcal{F})_{U,\tilde{\alpha}}) = j_\beta^* j_{\tilde{\alpha},*}(u_{X,*}(\mathcal{F})_{U,\tilde{\alpha}})$$

One checks without much difficulty that this defines indeed a map of topoi. Also if $\mathcal{F}$ is such that all restrictions $\mathcal{F}_\alpha = \mathcal{F} \mid X_\alpha \otimes_K \bar{K}$ are locally constant, then $u_{X,*}(\mathcal{F})_{U,\alpha} = \mathcal{F} \mid U_\alpha \otimes_K \bar{K}$. Finally there is an obvious commutative diagram of maps

$$\begin{tikzcd} (X \otimes_V \bar{K})^{et} \arrow[r] \arrow[dr] & \mathcal{X}_{\bar{K}} \arrow[d, "v_X"] \\ & X^{et} \end{tikzcd}$$

The direct image for the vertical map sends a collection $(\mathbb{L}_{U,\alpha})$ to the sheaf which associates to an étale $U \to X$ compatible families of sections in $\prod_\alpha \Gamma(U_\alpha, \mathbb{L}_{U,\alpha})$. We shall need another model for extensions by zero. For $\alpha < \beta$ define a functor $\Psi_{\alpha,\beta}$ on sheaves on $\mathcal{X}_{\bar{K}}^0$ by

$$\Psi_{\alpha,\beta}(\mathbb{L})_U(V) = \Gamma(\bar{V}_\beta, j_\beta^* j_{\alpha,*}\mathbb{L}_U \mid V)$$

Here $\bar{V}$ denotes the normalisation of $U \otimes_V \bar{K}$ in $V \to U_\alpha \otimes_V \bar{K}, \bar{V}_\beta$ its β -stratum (the preimage of U_β) and $\mathbb{L}_U \mid V$ the pullback of $\mathbb{L}_U$ to V). One checks without difficulty that this satisfies the sheaf property with respect to étale coverings of U. Also for $\alpha > \beta > \gamma$ there are transition maps

$$\Psi_{\alpha,\beta}(\mathbb{L}) \to \Psi_{\alpha,\gamma}(\mathbb{L})$$

satisfying the obvious commutativity for $\alpha < \beta < \gamma_1, \gamma_2 < \delta$. To proceed further we need a number of local assumptions.

Local Conditions LC: Let $R = \mathcal{O}_{X,x}^{sh}$ denote the strict henselisation of X in a point x. We assume:

a) The closures of the strata on $Spec(R \otimes_V \bar{K})$ are normal subvarieties of $Spec(R \otimes_V \bar{V})$, defining a stratification (again indexed by $\alpha, \beta, \dots$)

b) For each α, the inclusion of the α -stratum admits a retraction

$$r_\alpha : Spec(R \otimes_V \bar{V}) \to \overline{Spec(R \otimes_V \bar{V})_\alpha},$$

sending any β -stratum with $\beta < \alpha$ into the (open) α -stratum

c) There exists $l_1, \dots, l_r \in R$ such that the stratification on $Spec(R \otimes_V \bar{K})$ is defined by the ideals generated by subsets of $\{l_1, \dots, l_r\}$

2. Lemma. Suppose LC hold. Then

i) The $\Psi_{\alpha,\beta}$ are exact

ii) If $\mathbb{L}$ is an injective sheaf on $\mathcal{X}_{\alpha,\bar{K}}^0$, then all $\Psi_{\alpha,\beta}(\mathbb{L})$ are acyclic for the direct image v_X (to X^{et}). Also all $j_\beta^* j_{\alpha,*}(\mathbb{L})$ are acyclic for $\mathcal{X}_\beta^0 \otimes_V \bar{K} \to X$, and the direct images $R^\nu j_{\alpha,*}\mathbb{L}_\alpha (j_\alpha : \mathcal{X}_\alpha^0 \otimes_V \bar{K} \to \mathcal{X} \otimes_V \bar{K})$ are defined by the functors $j_\beta^* R^\nu j_{\alpha,*}$.

Proof. The assertions are local in X^{et}. By an inductive limit argument we may replace $\mathcal{X}$ by $Spec(R), R = O^{sh}_{X,x}$ as before. Then $R \otimes_V \bar{V}$ is a finite product of strictly henselian local rings. We thus may assume that it is local (eventually replace K by a finite extension). Then by b) $Spec(R \otimes_V \bar{V})$ is stratified with all closed strata normal and thus irreducible. Let Δ_α denote the fundamental group of the α -stratum. For each $\beta > \alpha$ we obtain subgroups $I_\beta \subseteq D_\beta \subseteq \Delta_\alpha$, well-determined up to conjugation, namely the inertia and decomposition-group of the β -stratum. If this is defined by (say) $l_1 = \ldots = l_t = 0$, then $I_\beta \cong \hat{\mathbb{Z}}(1)^t$, via the coverings obtained by adjoining roots of l_i. Furthermore there is a canonical map $\Delta_\beta \to D_\beta / I_\beta$ (Δ_β - fundamental group of β -stratum) which is an isomorphism by b). Now any locally constant sheaf $\mathbb{L}$ on the generic stratum corresponds to a continuous representation of Δ. Then one checks that $\Psi_{0,\beta}(\mathbb{L})$ is defined by $Ind^{\Delta}_{D_\beta}(\mathbb{L})$ = continuous D_β - linear maps $\Delta \to \mathbb{L}$. This is an exact functor and implies the first assertion (over proper subsets $U \subsetneqq Spec(R)$ one has to localize further).

For ii) we need to compute direct images for $v_{X_\alpha} : \mathcal{X}_\alpha \otimes_V \bar{K} \to X^{et}$. Again the fibre in $x \in X$ is the cohomology of $Spec(R \otimes_V \bar{V})$, that is the cohomology of the topos of locally constant sheaves on the α -stratum. (This topos is a retract of the full topos and has the same cohomology). This coincides with the group-cohomology $H^*(\Delta, \ldots)$. However (using inclusions $\Delta_\alpha \supseteq D_{\alpha,\beta} \supseteq I_{\alpha,\beta} \cong \hat{\mathbb{Z}}(1)^t$) $H^*(\Delta_\alpha, Ind^{\Delta_\alpha}_{D_{\alpha,\beta}} L) = H^*(D_{\alpha,\beta}, \mathbb{L})$ vanishes in positive degrees for injective $L's$. Similar for the cohomology of $j^*_\beta j_{\alpha,*}\mathbb{L}$, which corresponds to L^{I_β}.

Now we come to one of the main results. For a sheaf $\mathbb{L}$ on $\mathcal{X}^0_{\bar{K}}$ define a complex $\Psi(\mathbb{L})$

$$\Psi(\mathbb{L}) : 0 \to \mathbb{L} \to \bigoplus_{codim(X_\beta)=1} \Psi_{0,\beta}(\mathbb{L}) \to \bigoplus_{codim(X_\beta)=2} \Psi_{0,\beta}(\mathbb{L}) \to \ldots$$

the maps being sums of restrictions with $\pm$ signs. On the other hand we have a sheaf $j_{0,!}\mathbb{L}$ on $\mathcal{X}_{\bar{K}}$, which is $\mathbb{L}$ on $\mathcal{X}^0_{\bar{K}}$ and trivial on all other strata.

3. Proposition. $\mathbb{R}v_{X,*}(\Psi(\mathbb{L})) \cong \mathbb{R}v_{X,*}(j_{0,!}\mathbb{L})$ canonically in the category of étale sheaves on X.

Proof. Choose an injective resolution $\mathbb{L} \to I^\cdot$ in $\mathcal{X}^0_{\bar{K}}$. Then $\Psi(I^\cdot)$ is $v_{X,*}$ -acyclic, thus $\mathbb{R}v_{X,*}(\Psi(\mathbb{L}))$ is represented by $v_{X,*}(\Psi(I^\cdot))$. This associated to $U \to X$ étale the complex of global sections $\Gamma((U \cap X_0) \otimes_V \bar{K}, \Psi(I^\cdot))$, i.e. the double complex made up from

$$\bigoplus_{codim(\beta)=a} \Gamma((U \cap X_0) \otimes_V \bar{K}, \Psi_{0,\beta}(I^b)) = \bigoplus_{codim(\beta)=a} \Gamma((U \cap X_\beta) \otimes_V \bar{K}, j^*_\beta j_{0,*}(I^b))$$

This is the direct image of the total complex associated to a double complex on $\mathcal{X} \otimes_V \bar{K}$ with components

$$\bigoplus_{codim(\beta)=a} j_{\beta,*} j^*_\beta j_{0,*}(I^b),$$

which is thus acyclic for $v_{X,*}$. On the α -stratum its components are

$$\bigoplus_{\substack{\beta\leq\alpha\\ codim(\beta)=a}} j_\alpha^* j_{\beta,*} j_\beta^* j_{0,x}(I^b) = \bigoplus_{\substack{\beta\leq\alpha\\ codim(\beta)=a}} j_\alpha^* j_{0,*}(I^b)$$

By one of the standard calculations in homological algebra it then follows that for $codim(\alpha) > 0$ this forms an acyclic complex. As one the 0-stratum our complex gives $\mathbb{L}$ it represents $j_{0,!}(l)$ and its direct image $v_{X,*}(j_{0,!}(\mathbb{L}))$. As the resolution $I^\cdot$ is unique up to homotopy, this construction is also canonical (in the derived category) and functorial.

We leave it to the reader to formulate a variant where one extends $\mathbb{L}$ by zero along some of the codimension-one strata and by $\mathbb{R}_{j_0,*}(\mathbb{L})$ along the others. Of course one then uses only $\Psi'_{0,\beta}s$ with the β -stratum an intersection of the codimension-one strata of the first type.

Finally we apply our theory to the previous case of schemes formally étale over the torus-embedding $\bar{T}_\lambda \subseteq \bar{T}$. We have to check the local conditions (LC): By assumption $\sigma \cap \ker(\lambda)$ is spanned by a partial basis $\{\rho_1, \dots, \rho_t\}$ of the lattice L. The strata are indexed by subsets $I \subseteq \{1, \dots, t\}$, and the ideal of the I-stratum is spanned by those $\mu \in L^\vee \cap \sigma^\vee$ which are strictly positive on some $\rho_i, i \in I$. If L_I denotes the quotient-lattice $L_I = l/\langle \rho_i \mid i \in I\rangle$ and $\sigma_I \subseteq L_{I,\mathbb{R}}$ the image of σ, then σ_I defines a torus-embedding $T_I \subseteq \bar{T}_I$. Furthermore there is a projection $\bar{T} \to \bar{T}_I$ which has partial inverse given by an isomorphism $\bar{T}_I \xrightarrow{\sim} I$-stratum $\subseteq \bar{T}$. The latter map is given on affine rings by sending $\mu \in L^\vee \cap \sigma^\vee$ to either $\mu \in L_I^\vee \cap \sigma^\vee$ or to zero (if $\mu \notin L_I^\vee$, that is $\mu(\rho_i) > 0$ for some $i \in I$). This picture persists after base change to $\bar{T}_\lambda$ (dividing rings by the ideal $(\pi - \lambda)$), and also to the strictly henselian local ring R. Furthermore by ajoining a suitable root $\pi^{1/e}$ we can achieve that all strata in $X \otimes_V \bar{V}$ are normal. Thus we have conditions a) and b).

Next we define a sheaf $\bar{\mathcal{O}}$ on $\mathcal{X}^0 \otimes_V \bar{K}$ as follows: To U and $V \to U_{\bar{K}}$ finite étale associate the global sections of the normalisation $\mathcal{O}_V^{norm}$ of $\mathcal{O}_U$ in V. We also want to define a sheaf $A_{\inf}(\bar{\mathcal{O}}_V)$, but for this we need that the Frobenius is surjective on $\bar{\mathcal{O}}/p \cdot \bar{\mathcal{O}}$. In fact we have a slightly stronger statement.

4. Lemma. Suppose $U = Spec(R) \to \bar{T}_\lambda$ is étale over $\bar{T}_\lambda$. Then Frobenius is surjective on $\bar{R}/p \cdot \bar{R}$.

Proof. Frobenius is surjective on the affine ring of the curve $\bar{T}_{\lambda,\infty}$, and thus also on $R_\infty/p \cdot R_\infty$ by étaleness. Then for any almost étale cover S of R_∞ it is almost surjective on S/pS (by Frobenius-invariance of almost étale coverings) and thus also almost surjective on $\bar{R}/p \cdot \bar{R}$. Thus if $x \in \bar{R}$ there is a y with $p^{\frac{1}{2}} \cdot x \equiv y^p mod\, p$. Then y is divisible by $p^{\frac{1}{2p}}$ and x is a $p - th$ power $mod\, p^{1/2}$. Repeating we see that it is a $p - th$ power $mod\, p$.

A little bit more involved is

5. Lemma. Suppose $a \in \bar{V}$ divides p^α for $\alpha < 1$. Then the map $X \mapsto X^p - aX$ is surjective on $\bar{\mathcal{O}}/p \cdot \bar{\mathcal{O}}$.

Proof. It suffices to check local rings. So let $R = \mathcal{O}_{X,x}^{sh} \subseteq \bar{R}$. Obviously we may assume that p lies in the maximal ideal of R and thus also in the Jacobson-radical of $\bar{R}$. For $y \in \bar{R}$ consider $S = \bar{R}[X]/(X^p - aX - y)$. It suffices to show that $S[\frac{1}{p}]$ is étale over $\bar{R}[\frac{1}{p}]$, or that $X^p - aX - y$ and $pX^{p-1} - a$ have no common zero in any extension-field L of $\bar{R}[\frac{1}{p}]$. But if not, then $X =^{p-1} \sqrt{a/p}, y =^{p-1} \sqrt{a/p}(\frac{a}{p} - a)$. Thus

$$y^{p-1}(\frac{p}{a})^p = (1-pa)^p$$
$$0 = (1-pa)^p - y^{p-1}(\frac{p}{a})^p$$

However the right hand side lies in $\bar{R}$ and is $\equiv 1 \, mod \, p^\alpha$ (some $\alpha > 0$), thus is a unit.

We also need to consider the kernel of the endomorphism $X^p - aX$ of $\bar{R}/p\bar{R}$. Assume now that $a = b^{p-1}$ with b^p dividing $p^\alpha, \alpha < 1$. Then $X^p - b^{p-1}X \in p\bar{R}$ if and only if

$$(\frac{X}{b})^p - (\frac{X}{b}) \in \frac{p}{b^p} \cdot \bar{R}$$

This happens if and only if locally in the Zariski - or $\acute{e}$tale topology

$$X \in \mathbb{Z}_p \cdot b + b \cdot \frac{p}{b^p} \cdot \bar{r}$$

Thus $Ker(X^p - b^{p-1}X) = b \cdot \mathbb{F}_p \cdot b \oplus (\frac{p}{b^{p-1}})\bar{R}/p\bar{R}$ (as sheaves).

We also need cohomology with values in the projective limit $\mathcal{R}(\bar{\mathcal{O}}) = \varprojlim \bar{\mathcal{O}}/p\bar{\mathcal{O}}$ (transition maps are Frobenius). This is a "topological" sheaf. However we do not need continuous cohomology. Projective system $\mathcal{F} = \{\mathcal{F}_0 \leftarrow \mathcal{F}_1 \leftarrow \mathcal{F}_2 \leftarrow \dots\}$ indexed by integers form a new topos. The functor of global sections maps this to a projective system of abelian groups, whose projective limit is $H^0(\mathcal{F})$. Deriving this functor gives higher cohomology. The derived functor of (projective systems $\mathcal{F}_n$) $\to$ (projective systems of abelian groups) can be computed termwise. Finally the derived functors of $\varprojlim\limits_{\mathbb{N}}$ are wellknown: There is only a non-zero $\varprojlim\limits_{\mathbb{N}}{}^{(1)}$, which vanishes if the projective system satisfies a Mittag-Leffler-condition. By Leray we have short exact sequences

$$0 \to \varprojlim^{(1)} H^{i-1}(\mathcal{F}_n) \to H^i(\mathcal{F}) \to \varprojlim H^i(\mathcal{F}_n) \to 0.$$

Thus we have defined the cohomology of $\mathcal{R}(\bar{\mathcal{O}}) = \varprojlim \bar{\mathcal{O}}/p \cdot \bar{\mathcal{O}}$ as well as direct images $R^a v_{X,*}(\mathcal{R}(\bar{\mathcal{O}}))$ (which are projective systems on X^{et}). Also choose any $a \in \mathcal{R}(\bar{V}), a \neq 0$. Then a can be represented by a system of elements $a_n \in \hat{\bar{V}}, a_n = a_{n+1}^p$. Also $a = b^{p-1}$ for some b. Then for n big enough the $a_n' s$ and $b_n' s$ satisfy the condition in Lemma. Thus the sequence

$$0 \to \mathbb{F}_p \oplus (\frac{p}{b_n^{p-1}}\bar{\mathcal{O}})/p\bar{\mathcal{O}} \to \bar{\mathcal{O}}/p\bar{\mathcal{O}} \xrightarrow{\Phi - a} \bar{\mathcal{O}}/p - \bar{\mathcal{O}} \to 0$$

(Φ = Frobenius) is exact.

Passing to the limit we obtain a sequence

$$0 \to \mathbb{F}_p \to \mathcal{R}(\bar{\mathcal{O}}) \xrightarrow{\Phi - a} \mathcal{R}(\bar{\mathcal{O}}) \to 0$$

whose cohomology has trivial transition-maps for $n >> 0$, and thus will disappear if we take cohomology. In the same vein we can pass to $A_{\inf}(\bar{\mathcal{O}}) = W(\mathcal{R}(\bar{\mathcal{O}}))$, which a priori is a double projective limit but can be written as a single projective limit.

However we really only need the $A_{\inf}(\bar{\mathcal{O}})/p^s A_{\inf}(\bar{\mathcal{O}}) = W_s(\mathcal{R}(\bar{\mathcal{O}}))$. They fit into a sequence

$$0 \to \mathbb{Z}/p^s\mathbb{Z} \to W_s(\mathcal{R}(\bar{\mathcal{O}})) \xrightarrow{\Phi-[a]} W_s(\mathcal{R}(\bar{\mathcal{O}})) \to 0$$

with again essentially trivial cohomology. Finally we tensor with $u_{X,*}(\mathbb{L}), \mathbb{L}$ a locally constant $\mathbb{Z}/p^s\mathbb{Z}$ -sheaf on $(X^0 \otimes_V \bar{K})^{et}$, to obtain

$$0 \to u_{X,*}(\mathbb{L}) \to u_{X,*}(\mathbb{L}) \otimes A_{\inf}(\bar{\mathcal{O}}) \xrightarrow{\Phi-[a]} u_{X,*}(\mathbb{L}) \otimes A_{\inf}(\bar{\mathcal{O}}) \to 0.$$

As a consequence we obtain exact sequences

$$\begin{aligned} 0 \to Coker(\Phi - [a] \mid H^{i-1}(\mathcal{X}^0 \otimes_V \bar{K}, u_X^*(\mathbb{L}) \otimes A_{\inf}(\bar{\mathcal{O}})) \to H^i(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L}) \\ \to \ker(\Phi - [a] \mid H^i(\mathcal{X}^0 \otimes_V \bar{K}, u_{X,*}(\mathbb{L}) \otimes A_{\inf}(\bar{\mathcal{O}}))) \to 0 \end{aligned}$$

We also need a variant for cohomology with compact support. For each stratum α we have sheaves $\bar{\mathcal{O}}_\alpha$ on $\mathcal{X}^0_\alpha \otimes_V \bar{K}$ as before. For $\alpha < \beta$ we define a sheaf $\tilde{\psi}_{\alpha,\beta}(\bar{\mathcal{O}}_\alpha)$ on $\mathcal{X}^0_\alpha \otimes_V \bar{K}$ as follows:

Suppose we have $U \to X$ étale and $V \to U_\alpha \otimes_V \bar{K}$ finite étale, with normalisation $V^{norm} \to \bar{U}_\alpha \otimes_V \bar{K}$. It is stratified with reduced strata obtained by taking closures of generic fibres. We define $\tilde{\psi}_{\alpha,\beta}(\bar{\mathcal{O}}_\alpha)$ as the normalisation of $\mathcal{O}_U$ in V^{norm}_β. If $U = Spec(R)$ is affine, $S =$ normalisation of R in V, then there are finitely many primes $\mathfrak{p}_\nu \subseteq S$ corresponding to the irreducible components of V^{norm}_β. $\tilde{\psi}_{\alpha,\beta}(\bar{\mathcal{O}}_\alpha)$ is then the direct sum of the normalisations of $(S/\mathfrak{p}_\nu)$. $(S/\mathfrak{p}_\nu[\frac{1}{\pi}]$ is already normal). Also for $\beta < \gamma$ there are morphisms

$$\tilde{\psi}_{\alpha,\beta}(\bar{\mathcal{O}}_\alpha) \to \tilde{\psi}_{\alpha,\gamma}(\bar{\mathcal{O}}_\alpha)$$

satisfying some transitivity for $\alpha < \beta < \gamma_1, \gamma_2 < \delta$.

6. Lemma.

i) Frobenius Φ is surjective on $\tilde{\psi}_{\alpha,\beta}(\bar{\mathcal{O}}_\alpha)/p \cdot \tilde{\psi}_{\alpha,\beta}(\bar{\mathcal{O}}_\alpha)$

ii) For $b \in \bar{V}$ with b^p dividing $p^\alpha, \alpha < 1, \Phi - b$ is surjective on $\tilde{\psi}_{\alpha,\beta}(\bar{\mathcal{O}}_\alpha)/p \cdot \tilde{\psi}_{\alpha,\beta}(\bar{\mathcal{O}}_\alpha)$.

Proof. It suffices to check these over $Spec(R), R = \mathcal{O}^{sh}_{X,x}$. We may assume $\alpha = 0$. Then $\tilde{\psi}_{\alpha,\beta}(\bar{R})$ the subset of the product $\prod(\bar{R}/\mathfrak{p}_\nu)^{norm}$ where $\Delta = \pi_1(Spec(R \otimes_V \bar{K}))$ acts continuously. The product is over all minimal primes of the β -stratum. Equivalently it is the induced module (via $D_\beta \subseteq \Delta$) from one $(\bar{R}/\mathfrak{p}_\nu)^{norm}$, which by LC coincides with the version of $\bar{R}$ for the β -stratum. There assertions i) and ii) are already known. In fact for i) we can prove surjectivity of Frobenius on global section over sufficiently small affines, reducing to R_∞ and checking there directly. We now have the obvious complexes

$$\tilde{\psi}(\bar{\mathcal{O}}_\alpha) : \bar{\mathcal{O}}_\alpha \to \oplus_{codim \beta=1} \tilde{\psi}_{\alpha,\beta}(\bar{\mathcal{O}}_\alpha) \to \oplus_{codim(\beta)=2} \tilde{\psi}_{\alpha,\beta}\ (\bar{\mathcal{O}}_\alpha) \to \dots$$

$\mathcal{R}(\tilde{\psi}(\bar{\mathcal{O}}_\alpha))$ and $A_{\inf}(\bar{\psi}(\bar{\mathcal{O}}_\alpha))$. For any $\mathbb{L}$ locally constant $\mathbb{L}$ on $(X^0 \otimes_V \bar{K})^{et}$, annihilated by p^s, we obtain a complex of projective systems

$$0 \to \psi(\mathbb{L}) \to u_{X,*}(\mathbb{L}) \otimes A_{\inf}(\tilde{\psi}(\bar{\mathcal{O}})) \xrightarrow{\Phi-[a]} u_{X,*}(\mathbb{L}) \otimes A_{\inf}(\tilde{\psi}(\bar{\mathcal{O}}))) \to 0$$

with irrelevant cohomology, and thus short exact sequences

$$0 \to Coker(\Phi - [a] \mid H^{i-1}(\mathcal{X}^0 \otimes_V \bar{K}, u_{X,*}(\mathbb{L}) \otimes A_{\inf}(\tilde{\psi}(\bar{\mathcal{O}}))) \to$$
$$\to H^i_!(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L}) \to \ker(\Phi - [a] \mid H^i(\mathcal{X}^0 \otimes_V \bar{K}, u_{X,*}(\mathbb{L}) \otimes A_{\inf}(\tilde{\psi}(\bar{\mathcal{O}}))) \to 0$$

There are variants with partial compact support at infinity, with obvious formulations and proofs.

Finally we get back to "almost mathematics". There is also a subsheaf $\bar{\mathcal{J}} \subseteq \bar{\mathcal{O}}, \bar{\mathcal{J}} = kernel(\bar{\mathcal{O}} \to \bigoplus_{codim\beta=1} \tilde{\psi}_{0,\beta}\ (\bar{\mathcal{O}}))$.

Lemma.

$$\bar{\mathcal{J}} \approx \tilde{\psi}(\bar{\mathcal{O}})$$

is an almost isomorphism.

Proof. Again we may work with $R = O^{sh}_{X,x}, Spec(R) \to \bar{T}_\lambda$. The assertion holds for $J_\infty \subseteq R_\infty$: Here each stratum is irreducible, with ideal generated by the $\mu \in \bigcup_n L_n^\vee$ with $\mu(\rho_i) > 0$ for some $i \in I =$ index-set of the stratum. By direct calculation we have an isomorphism. After that $\bar{R} \supseteq R_\infty$ is the limit of almost *é*tale coverings. Assume we have such a covering $S \supseteq R_\infty$ with group G. Then if $\mathfrak{q}_\alpha \subseteq S$ is the prime ideal giving an irreducible component of the α-stratum, $D_\alpha \subseteq G$ its decomposition-group, and $\mathfrak{p}_\alpha = \mathfrak{q}_\alpha \cap R_\infty \subseteq R_\infty$, we have a map $R_\infty/\mathfrak{p}_\alpha \otimes_{R_\infty} S \to Ind_D^G((S/\mathfrak{q}_\alpha)^{norm})$, which is an isomorphism after inverting π. As the left side is almost *é*tale over $R_\infty/\mathfrak{p}_\alpha$ and thus has non-degenerate trace-form, the map must be an almost isomorphism. Passing to the limit over all S we obtain that $\tilde{\psi}(\bar{R})$ is almost isomorphic to $\bar{J}_\infty \otimes_{R_\infty} \bar{R} \approx \bar{J}$.

If we define $\mathcal{R}(\bar{\mathcal{J}}) = \varprojlim \bar{\mathcal{J}}/p\bar{\mathcal{J}} = \ker(\mathcal{R}(\bar{\mathcal{O}}) \to \mathcal{R}(\bar{\mathcal{O}}/\bar{\mathcal{J}}))$ and similarly $A_{\inf}(\bar{\mathcal{J}}) = W(\mathcal{R}(\bar{\mathcal{J}}))$, we get almost isomorphisms

$$H^*(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes A_{\inf}(\bar{\mathcal{J}})) \approx H^*(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes A_{\inf}(\tilde{\psi}(\bar{\mathcal{O}}))).$$

Next consider the direct images $R^\nu q_{X,*}(\bar{\mathcal{J}}/p^s\bar{\mathcal{J}})$ which are sheaves on X^{et}. By Th. 2.10.i) there are almost zero unless $0 \leq \nu \leq d$, are almost quasi-isomorphic to bounded complexes of quasicoherent sheaves, and there is a canonical almost map

$$R^d q_{X,*}(\bar{\mathcal{J}}/p^s\bar{\mathcal{J}}) \to \omega_X \otimes \bar{V}/p^s\bar{V},$$

inducing almost quasi-isomorphisms

$$\mathbb{R}q_{X,*}(\mathbb{L} \otimes \bar{\mathcal{J}}) \approx \mathbb{R}Hom_{O_X \otimes \bar{V}/p^s\bar{V}}(\mathbb{R}q_{X,*}(\mathbb{L}^\vee \otimes \bar{\mathcal{O}}), \omega_X \otimes \bar{V}/p^s\bar{V})[-d]$$

for $\mathbb{L}$ locally constant annihilated by p^s. Furthermore there is a canonical trace-map

$$H^d(X^{et}, \omega_X) = H^d(X, \omega_X) \to V$$

inducing global duality for coherent sheaves on X. As the direct iamges $\mathbb{R}q_{X,*}(\mathbb{L}\otimes\bar{\mathcal{J}})$ etc. are almost coherent these combine to produce almost quasi-isomorphisms

$$\mathbb{R}\Gamma(\mathcal{X}^0\otimes_V\bar{K},\mathbb{L}\otimes\bar{\mathcal{J}})\approx\mathbb{R}Hom_{\bar{V}/p^s\bar{V}}(\mathbb{R}\Gamma(\mathcal{X}^0\otimes_V\bar{K},\mathbb{L}\otimes\bar{\mathcal{O}}),\bar{V}/p^s\bar{V})[-2d]$$

Remark. Previously we always worked with $\frac{1}{\pi}\omega_X$ instead of ω_X. This is because in the torus-embedding we identified the dualising complex for the quotient $V[L^\vee\cap\sigma^\vee]/(\lambda-\pi)$ as the tensor-product of that for $V[L^\vee\cap\sigma^\vee]$ with the quotient, instead of the Ext^1. Then we used the trace-map for $V[L^\vee\cap\sigma^\vee]$. This differs from the trace-map used in relative duality by a factor π. That this is so can be seen easily for the affine torus-embedding $V[\lambda]$, and follows in the general case by functoriality $V[\lambda]\subseteq V[L^\vee\cap\sigma^\vee]$. It is also related to the fact that we have tacitly identified $\Lambda^{d+1}L^\vee$ and $\Lambda^d\ker(\lambda)^\vee$.

The structure of $H^i(\mathcal{X}^0\otimes_V\bar{K},\mathbb{L}\otimes A_{\mathrm{inf}}(\bar{\mathcal{O}}))$ is surprisingly simple. Assume for the moment that $\mathbb{L}$ is annihilated by p. Then this cohomology is a module over $\mathcal{R}(\bar{V})$.

8. Theorem. For each i $H^i(\mathcal{X}^0\otimes_V,\bar{K},\mathbb{L}\otimes\mathcal{R}(\bar{J}_I))$ is almost isomorphic to a finitely generated free $\mathcal{R}(\bar{V})$- module, namely to $H^i_{(!)}(\mathcal{X}^0\otimes_V\bar{K},\mathbb{L})\otimes\mathcal{R}(\bar{V})$ (support-condition given by I).

Proof. Let $M=H^i(\mathcal{X}^0\otimes_V\bar{K},\mathbb{L}\otimes\mathcal{R}(\bar{J}_I))$. Then for $\xi=\underline{p}\in\mathcal{R}(\bar{V})$, $M/\xi M\subseteq H^i(\mathcal{X}^0\otimes_V\bar{K},\mathbb{L}\otimes\bar{J}_I)$, which is almost finitely presented over $\mathcal{R}(\bar{V})/\xi\cdot\mathcal{R}(\bar{V})=\bar{V}/p\bar{V}$. Also Frobenius Φ induces an isomorphism on M. Now for torsion $\bar{V}$ -modules N we can define a normalised length $\lambda(N)$ similarly as before: Write $V=\cup V_\alpha$ with V_α finite over V. Then for $N=\bar{V}\otimes_{V_\alpha}N_\alpha$

$$\lambda(N)=\frac{1}{[V_\alpha:V]}length_V(N_\alpha),$$

which defines $\lambda(N)$ for N finitely presented. For N only finitely generated, $N=\bar{V}\cdot N_\alpha$, we have

$$\lambda(N)=\lim_{\beta\to\infty}\frac{1}{[V_\beta:V]}length_V(V_\beta\cdot N_\alpha),$$

and for arbitrary N

$$\lambda(N)=\sup\{\lambda(N'))N'\subseteq N\quad\text{finitely generated}\}.$$

Also $\lambda(\bar{V}/\mathfrak{a})=\inf\{v(x)\mid x\in\mathfrak{a}\}$, where v is the valuation normalised such that $v(\pi)=1$. Finally for a submodule N of a finitely presented $\bar{V}$ -module, $\lambda(N)=0$ implies $N=(0)$.

By additivity we extend λ to ξ -torsion $\mathcal{R}(\bar{V})$ -modules. Pushout by Frobenius multiplies λ by p. Next for each $\alpha>0$

$$\xi^\alpha\cdot H^i(\mathcal{X}^0\otimes_V\bar{K},\mathbb{L}\otimes\bar{\mathcal{J}}_I)$$

has finite λ -invariant, and by devissage this holds for $\xi^\alpha\cdot H^i(\mathcal{X}^0\otimes_V\bar{K},\mathbb{L}\otimes\mathcal{R}(\bar{\mathcal{J}}_I)/\xi^n)$, all n. Applying Frobenius-pushout to this module multiplies λ by p, and produces

$$\xi^{p\alpha} H^i(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes \mathcal{R}(\bar{\mathcal{J}}_I)/\xi^{np}).$$

On the other hand filter $\mathbb{L} \otimes \mathcal{R}(\bar{\mathcal{J}}_I)/\xi^{np}$ by submodules $\xi^{in}(\mathbb{L} \otimes \mathcal{R}(\bar{J}_I))/\xi^{np}$, apply the exact cohomology-sequences and use that for an exact sequence

$$N_1 \to N_2 \to N_3$$

we have

$$\lambda(\xi^{\alpha+\beta} N_2) \leq \lambda(\xi^\alpha N_1) + \lambda(\xi^\beta N_3),$$

with equality for all $\alpha, \beta > 0$ only possible if

$$\lambda(\xi^\alpha \cdot \ker(N_1 \to N_2)) = \lambda(\xi^\beta \cdot Coker(N_2 \to N_3)) = 0, \quad \text{all} \quad \alpha, \beta > 0.$$

It then follows that for $m \geq 0$ the exact sequences

$$\begin{aligned} H^i(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes \mathcal{R}(\bar{\mathcal{J}}_I)/\xi^m) &\to H^i(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes \mathcal{R}(\bar{\mathcal{J}}_I)/\xi^{m+1}) \\ &\to H^i(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes \mathcal{R}(\bar{\mathcal{J}}_I)/\xi^m) \end{aligned}$$

satisfy this condition. If (say) N denotes the kernel of the first map, for each $\alpha, \beta > 0$ multiplication by ξ^β on $\xi^\alpha N$ factors as $\xi^\alpha N \twoheadrightarrow N_1 \to \xi^\alpha N$ with N_1 submodule of a finitely presented $\bar{V}/p\bar{V}$ -module. As $\lambda(N_1) \leq \lambda(\xi^\alpha \cdot N) = 0$, we have $N_1 = (0)$, and $N \approx 0$. Quite similarly (or from the long exact sequence in cohomology) the cokernel is almost zero. Especially the projective system of $H^i(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes \mathcal{R}(\bar{\mathcal{J}}_I)/\xi^m)$ has almost surjective transition-maps, and $\lim^{(1)}$ of it is almost zero. Also if $M^i = H^i(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes \mathcal{R}(\bar{\mathcal{J}}_I))$, then $H^i(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes \mathcal{R}(\bar{\mathcal{J}}_I)/\xi^m) \approx M^i/\xi^m \cdot M^i$, and $M = \varprojlim_m M/\xi^m M$: If $z \in H^i(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes \mathcal{R}(\bar{\mathcal{J}}_I)/\xi^m)$, we show that for each $\alpha > 0$ $\xi^\alpha \cdot z$ lifts to M. For this choose a strictly decreasing sequence $\{\alpha_n, n \geq m\}, 0 < \alpha_{n+1} < \alpha_n < \alpha = \alpha_m$. Then choose $z_n \in H^i(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes \mathcal{R}(\bar{J}_I)(\xi^n)$ with $z_m = z, z_{n+1}$ lifting $\xi^{\alpha_n - \alpha_{n+1}} \cdot z_n$. Then $\xi^\alpha \cdot z$ is lifted by $\varprojlim(\xi^{\alpha_n} \cdot z_n)$.

Also it follows that multiplication by ξ defines an almost isomorphism $\xi^n M^i/\xi^{n+1} M^i \approx \xi^{n+1} M^i/\xi^{n+2} M^i$, so ξ is almost injective on M^i. Next if $M^i/\xi M^i$ is not almost zero, choose an element $m \in M^i$. m is not almost zero in $M^i/\xi M^i$. Multiplication by m defines a map

$$\xi \cdot \mathcal{R}(\bar{V}) \cong \mathcal{R}(\bar{V}) \to M^i,$$

injective as m is not ξ^l -torsion for any l. Consider $\alpha, 0 \leq \alpha \leq 1$, such that this map extends to $\xi^\alpha \cdot \mathcal{R}(\bar{V})$ (this is not possible for $\alpha < 0$, as $m \notin \xi \cdot M^i$). Such an extension is almost unique. Thus if $\underline{\alpha}$ is the infimum of these α's, the map extends uniquely to the union

$$\mathfrak{a}(\underline{\alpha}) = \bigcup_{\alpha < \underline{\alpha}} \xi^\alpha \cdot \mathcal{R}(\bar{V}).$$

Hence we get an injection $\mathfrak{a}(\underline{\alpha}) \subseteq M^i$, with quotient $\tilde{M}^i = M^i/\mathfrak{a}(\underline{\alpha})$ again almost without ξ -torsion. If $\tilde{M}^i$ is not almost zero we can repeat the construction to obtain an increasing sequences of submodules of M^i, with subquotients $\mathfrak{a}(\underline{\alpha}_\nu)$, and almost ξ -torsion-free quotients. Especially

$$\lambda(\xi \cdot M/\xi^2 \cdot M) \geq \sum_\nu \lambda(\xi \cdot \mathfrak{a}(\underline{\alpha}_\nu)\xi^2 \cdot \mathfrak{a}(\underline{\alpha}_\nu))$$

$=$ number of $\underline{\alpha}_\nu$'s,

so the process must stop. Hence we have found a finite filtration

$$(0) = M_0 \subset M_1^{\subseteq} \ldots \subset M_r = M = M^i,$$

with $M_n/M_{n-1} \approx \mathfrak{a}(\underline{\alpha}_n)$.

It then follows easily that for each $\alpha > 0$ there exists a free submodule $\mathcal{R}(\bar{V})^r \subseteq M$ with quotient annihilated by ξ^α. Especially M is almost projective of finite rank r over $\mathcal{R}(\bar{V})$. Next define an $\mathcal{R}(\bar{V})$ -algebra

$$\mathcal{A} = S[M]/(\Phi(m) - m^p)$$

(symmetric algebra divided by the relations displayed). Obviously $\mathcal{A}[\xi^{-1}]$ is finite flat over $\mathcal{R}(\bar{V})[\xi^{-1}]$ of rank p^r, generated by monomials with exponents $< p$ in a basis of $M[\xi^{-1}] = \mathcal{R}(\bar{V})[\xi^{-1}]^r$. Also as the free submodules $N \cong \mathcal{R}(\bar{V})^r \subseteq M$ form a filtering inductive system with union almost M, we obtain (taking such monomials in a basis of N) a filtering inductive system of free submodules of $\mathcal{A}$ (of rank p^r) with union almost $\mathcal{A}$. Thus $\mathcal{A}$ is almost projective of finite rank p^r over $\mathcal{R}(\bar{V})$, and almost finitely generated. Also assume given such a free submodule with basis $\{m_1, \ldots, m_r\}$, quotient annihilated by ξ^α, and let $\xi^\alpha \cdot \Phi(m_i) = \sum_j a_{ij} \cdot m_j$ with $a_{ij} \in \mathcal{R}(\bar{V})$. Then the matrix (a_{ij}) has an inverse up to ξ^β for any $\beta > (p+1)\alpha$, as $\xi^\beta \cdot m_i \in \xi^\alpha \cdot \Phi(\xi^\alpha \cdot M)$. Also all monomials with exponents $< p$ in the $\xi^{\alpha/p-1} \cdot m_i = T_i$ form an subalgebra $\mathcal{A}_\alpha$ isomorphic to

$$\mathcal{R}(\bar{V})[T_1, \ldots, T_r]/(T_i^p - \sum_j a_{ij} \cdot T_j).$$

¿From ramification-theory it follows that the trace-form identifies the dual of $\mathcal{A}_\alpha$ with $\det(a_{ij})^{-1} \cdot \mathcal{A}_\alpha$. Especially $\mathcal{A}[1/\xi] = \mathcal{A}_\alpha[1/\xi]$ is *é*tale over $\mathcal{R}(\bar{V})[1/\xi]$, and the corresponding canonical idempotent $e \in \mathcal{A}_\alpha \otimes_{\mathcal{R}(\bar{V})} \mathcal{A}_\alpha[1/\xi]$ has denominator $\det(a_{ij})$ which divides $\xi^{r\beta}$ for any $\beta > (r+1)\alpha$. Letting $\alpha \to 0$ we see that the canonical idempotent in $\mathcal{A} \otimes_{\mathcal{R}(\bar{V})} \mathcal{A}[1/\xi]$ is almost integral, thus $\mathcal{A}$ is almost *é*tale over $\mathcal{R}(\bar{V})$.

Hence we can almost lift $\mathcal{A}$ to an almost *é*tale cover of $\mathcal{A}_{\inf}(\bar{V})$ and its quotient $\hat{\bar{V}}$. However $\hat{\bar{V}}[\frac{1}{p}]$ is algebraically closed, so this cover is almost trivial. It follows that $\mathcal{A} \approx \mathcal{R}(\bar{V})^{p^r}$. We thus obtain p^r different (almost) homomorphisms of $\mathcal{A}$ into $\mathcal{R}(\bar{V})$, corresponding to (almost) maps $M \to \mathcal{R}(\bar{V})$ which are Frobenius linear. These maps almost span the (module) dual of $\mathcal{A}$ and thus also of M (check with free submodules). Now the $\mathbb{F}_p$ -vectorspace $Hom^{almost}_{\Phi,\mathcal{R}(\bar{V})}(M, \mathcal{R}(\bar{V}))$ has p^r elements and thus dimension r. A basis of it defines an almost map $M \to \mathcal{R}(\bar{V})^r$ inducing an almost surjection on duals, and thus an almost isomorphism. Also the canonical $\mathbb{F}_p$-structure on $\mathcal{R}(\bar{V})^r$ can be recovered as the space of Φ -invariants.

Next replace M by $H^i(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes \tilde{\psi}(\mathcal{R}(\bar{\mathcal{O}}))) = \tilde{M}^i$, which is almost the same. Then we have the exact sequences ($b \in \mathcal{R}(\bar{V})$ as before)

$$0 \to Coker(\Phi - b^{p-1} \mid \tilde{M}^{i-1}) \to H^i_{(!)}(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L}) \to Ker(\Phi - b^{p-1} \mid \tilde{M}_i) \to 0.$$

But $\tilde{M}^i \approx \mathcal{R}(\bar{V})^{r_i}$ (Φ -linear). Thus $\Phi - b^{p-1}$ is almost surjective on $\tilde{M}^{i-1}$. Now the above sequence comes (with tensor-products etc.) from the first line in

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{F}_p & \xrightarrow{b\cdot} & \mathcal{R}(\bar{\mathcal{O}}) & \xrightarrow{\Phi - b^{p-1}} & \mathcal{R}(\bar{\mathcal{O}}) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle b'^p} & & \\
0 & \longrightarrow & \mathbb{F}_p & \xrightarrow{b\cdot b'} & \mathcal{R}(\bar{\mathcal{O}}) & \xrightarrow{\Phi - (bb')^{p-1}} & \mathcal{R}(\bar{\mathcal{O}}) & \longrightarrow & 0
\end{array}$$

Extend this as shown with $b' \in \mathcal{R}(\bar{V}), b' \neq 0$, unit, to get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & coker(\Phi - b^{p-1})/\tilde{M}^{i-1}) & \longrightarrow & H^i_{(!)}(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L}) & \longrightarrow & \ker(\Phi - b^{p-1}/\tilde{M}^i) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle b'^p\cdot} & & \| & & \downarrow{\scriptstyle b'} & & \\
0 & \longrightarrow & Coker(\Phi - (bb')^{p-1}/\tilde{M}^{i-1}) & \longrightarrow & H^i_{(!)}(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L}) & \longrightarrow & \ker(\phi - (bb')^{p-1}/\tilde{M}^i) & \longrightarrow & 0
\end{array}$$

The first vertical arrow is zero, thus $H^i_{(!)}(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L})$ injects into $\tilde{M}_i$ for all b (we may vary b, b'). Also multiplication by b respectively b' applied to $\tilde{M} \approx \mathcal{R}(\bar{V})^{r_i}$ induces maps
$\mathbb{F}_p^{r_i} = \Phi$ -invariants in $\mathcal{R}(\bar{V})^{r_i}$

$$\begin{aligned}
&\to \ker(\Phi - b^{p-1}/\tilde{M}^i) \\
&\to \ker(\Phi - (bb')^{p-1}/\mathcal{R}(\bar{V})^{r_i}) = b \cdot \mathbb{F}_p^{r_i},
\end{aligned}$$

and one derives that indeed

$$\tilde{M}^i \approx H^i_{(!)}(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L}) \otimes \mathcal{R}(\bar{V}).$$

Corollary 1. For any $\mathbb{L}$ (only annihilated by p^s)

$$\begin{aligned}
&\mathbb{R}\Gamma(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes A_{\inf}(\bar{\mathcal{J}}_I)) \approx \mathbb{R}\Gamma_{(!)}(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L}) \otimes A_{\inf}(\bar{V}) \\
&\mathbb{R}\Gamma(\mathcal{X}^0 \otimes \bar{K}, \mathbb{L} \otimes \bar{\mathcal{J}}_I) \approx \mathbb{R}\Gamma_{(!)}(\mathcal{X}^0 \otimes_V \bar{K}, L) \otimes \bar{V}
\end{aligned}$$

Proof. Devissage in $\mathbb{L}$

Corollary 2. $\mathbb{R}\Gamma_{(!)}(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L})$ satisfies Künneth.

Proof. Descente from $\mathbb{R}\Gamma(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes \bar{\mathcal{J}}_I)$ (for $\bar{V} \otimes$ (a $\mathbb{Z}/p^s\mathbb{Z}$ -module), "almost zero" implies zero).

Also the map $u_X : (X^0 \otimes_V \bar{K})^{et} \to \mathcal{X}^0 \otimes_V \bar{K}$ and its variants induce a map

$$\mathbb{R}\Gamma_{(!)}(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L}) \to \mathbb{R}\Gamma_{(!)}((X^0 \otimes_V \bar{K})^{et}, \mathbb{L}).$$

Especially the trace-form on $H^{2d}_!((X^0 \otimes_V \bar{K})^{et}, \mathbb{Z}/p^s\mathbb{Z}(d))$ (if X has pure relative dimension d) induces a trace-form tr^{et} on $H^{2d}(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{Z}/p^s\mathbb{Z}(d))$. We shall show later that it coincides with the restriction of the trace-form on $H^{2d}(\mathcal{X}^0 \otimes_V \bar{K}, \bar{\mathcal{J}}/p^s\bar{\mathcal{J}}(d))$. Hence it induces by descente a perfect duality.

Corollary 3. The étale trace induces a perfect duality

$$\mathbb{R}\Gamma_{(!)}(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L}) \otimes^{\mathbb{L}} \mathbb{R}\Gamma_{(!^c)}(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L}^\vee)(d)[2d] \to \mathbb{Z}/p^s\mathbb{Z}.$$

4. Chern-classes and Gysin-maps

Suppose $\mathcal{L}$ is a line-bundle on X. We define its first Chern-Class $c_1(\mathcal{L}) \in H^2(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{Z}/p^s\mathbb{Z}(1))$ as follows: $\mathcal{L}$ is given by a class in $H^1(X, O_X^*) = H^1(X^{et}, \mathcal{O}_X^*)$. Pullback via $v_X^* \mathcal{O}_X^* \to \bar{\mathcal{O}}^*$ gives a class in $H^1(\mathcal{X} \otimes_V \bar{K}, \bar{\mathcal{O}}^*)$, and via the Kummer-sequence $0 \to \mu_{p^s} \to \bar{\mathcal{O}}^* \to \bar{\mathcal{O}}^* \to 0$ we obtain a class in $H^2(\mathcal{X} \otimes_V \bar{K}, \mu_{p^s})$. It is clear that pullback to $H^2(X \otimes_V \bar{K})^{et}, \mu_{p^s})$ maps it to the étale first Chern-class of $\mathcal{L} \mid X \otimes_V \bar{K}$.

Next we consider a closed immersion $i : Y \to X$ such that locally in the étale topology $X \cong X_1 \times \mathbb{A}^1, Y \cong X_1 \times \{0\}$, with X_1 having toroidal singularities, stratifications etc. as before. We intend to give a geometric description of the Gysin-map i_*. We define a new stratification on X by adding Y as well as its intersections with X_α's to the strata, and call the new topos $\tilde{X}$. Thus $\tilde{X}^0 = X^0 - Y^0$. Next there are maps of topoi

$$\begin{aligned}
&j : (\mathcal{X} \setminus \mathcal{Y})^0 \otimes_V \bar{K} \to \mathcal{X}^0 \otimes_V \bar{K} \\
&w : \tilde{\mathcal{X}} \otimes_V \bar{K} \to \mathcal{X} \otimes_V \bar{K} \\
&i : \mathcal{Y} \otimes_V \bar{K} \to \tilde{\mathcal{X}} \otimes_V \bar{K}
\end{aligned}$$

Now a sheaf $\mathcal{F}$ on $(X \otimes_V \bar{K})^{et}$ which is locally constant on all strata induces $\mathbb{L} = u_{X,*}(\mathcal{F})$ on $\mathcal{X} \otimes_V \bar{K}$ and $\tilde{\mathbb{L}} = w^*(\mathbb{L})$ on $\tilde{\mathcal{X}} \otimes_V \bar{K}$. Furthermore on $(X \otimes_V \bar{K})^{et}$ there is an exact sequence (or better an exact triangle)

$$0 \to \mathcal{F} \to \mathbb{R}j_* j^* \mathcal{F} \to i_*(\mathcal{F}/Y)(-1)[-1] \to 0,$$

whose $\mathbb{R}u_{X,*}$ receives a map from

$$0 \to \tilde{\mathbb{L}} \to \mathbb{R}j_* j^* \tilde{\mathbb{L}} \to i_*(\tilde{\mathbb{L}}/Y)(-1)[-1] \to 0.$$

It is wellknown that the Gysin-homomorphism in étale topology is defined by the connecting map

$$\mathbb{R}\Gamma((Y^0 \otimes_V \bar{K})^{et}, \mathcal{F}(1))[-2] \to \mathbb{R}\Gamma((X^0 \otimes_V \bar{K})^{et}, \mathcal{F})$$

from the first triangle, and similar for variants with (partial) compact support.

Sketch. Taking cup-product with classes in $\mathbb{R}\Gamma_!((X^0 \otimes_V \bar{K})^{et}, \mathcal{F}^\vee)$ we reduce to constant coefficients and cohomology in highest degrees. Then the assertion follows from the explciit description of the cohomology-class of a point in Y^0.

Hence the second triangle defines the Gysin-homomorphism for $i : \mathcal{Y} \otimes_V \bar{K} \to \tilde{\mathcal{X}} \otimes_V \bar{K}$.

Finally we come to coefficients $\mathbb{L} \otimes \bar{\mathcal{O}}$.

1. Proposition. There is an almost exact sequence (better triangle)

$$0 \to \mathbb{L} \otimes \bar{\mathcal{O}}_X \to \mathbb{R}j_*(\mathbb{L} \otimes \bar{\mathcal{O}}_{\tilde{X}}) \to i_*(\mathbb{L} \otimes \bar{\mathcal{O}}_Y)(-1)[-1] \to 0$$

The maps can be described as follows:

i) The first is induced from

$$\bar{\mathcal{O}}_X/p^s \cdot \bar{\mathcal{O}}_X \to j_* \bar{\mathcal{O}}_{\tilde{X}}/p^s \bar{\mathcal{O}}_{\tilde{X}}.$$

We claim that this is an almost isomorphism. Applying this to all strata and tensoring with $\mathbb{L}$ then gives an almost isomorphism of H^0's.

ii) Let ι denote the stratum Y in $\tilde{\mathcal{X}}$. Then there is a map $\mathbb{R}j_*(\mathbb{L}\otimes\bar{\mathcal{O}}_{\tilde{X}}) \to \mathbb{R}^1 j_*(\mathbb{L}\otimes\tilde{\psi}_{0\iota}(\bar{\mathcal{O}}_{\tilde{X}}))$ Furthermore we claim that

$$j_*\tilde{\psi}_{0,\iota}(\bar{\mathcal{O}}_{\tilde{X}}) = i_*(\bar{\mathcal{O}}_Y)$$

and that the cup-product

$$\mathbb{R}^1 j_*\mathbb{Z}/p^s\mathbb{Z} \times j_*(\tilde{\psi}_{0,\iota}(\bar{\mathcal{O}}_{\tilde{X}}) \to R^1 j^*(\bar{\mathcal{O}}_Y/p^s\bar{\mathcal{O}}_Y)$$

is an almost isomorphism. Again this persists after tensoring with $\mathbb{L}$

iii) The higher direct images $\mathbb{R}^\nu j_*\bar{\mathcal{O}}_{\tilde{X}}/p^s\mathcal{O}_{\tilde{X}} \approx 0$ for $\nu \geq 2$, and also after tensoring with $\mathbb{L}$.

The assertions in i), ii), iii) imply almost exactness of the sequences as well as that the corresponding sequence with constant coefficients maps to it. If we show

iv) The connecting homomorphism in our exact sequence computes i_*,

we also get the required compatability for i_*.

Now to the required assertions. For i), ii), iii) we have already reduced to constant coefficients. That $j_*(\tilde{\psi}_{0,\iota}(\bar{\mathcal{O}}_{\tilde{X}})) = i_*(\mathcal{O}_{\bar{Y}})$ follows from the definitions:

To evaluate the left hand side on a finite *é*tale covering $V \to \mathcal{U}^0 \otimes_V \bar{K}$ $(\mathcal{U} \subseteq X$ *é*tale) we first restrict to the preimage of $X \setminus Y$, then extend by normalisation to recover V, next restrict to the preimage of $Y \otimes_V \bar{K}$, and then form global sections of the normalisation. But this gives the right hand side.

For the remaining assertions it suffices to check them on stalks $Spec(R), R = \mathcal{O}^{sh}_{X,x}$. Let $T \in R$ define the stratum Y. Extending V we may assume that $R \otimes_V \bar{V}$ is local and normal. Consider the normalisations $\bar{R}$ and $\bar{\bar{R}}$ of R in the maximal *é*tale cover of $Spec(R)^0 \otimes_V \bar{K}$ respectively of the complement of Y in it. By the theory of almost *é*tale extension $\bar{R}$ and $\bar{\bar{R}}$ are limits of almost *é*tale covers of R_∞ respectively $\tilde{R}_\infty$, which in turn are induced by toroidal coverings. As locally in the *é*tale topology $X \cong \mathbb{A}^1 \times X_1$ we may use for $\tilde{R}_\infty$ the toroidal covering induced from toroidal coverings of X_1 and from adjoining roots of T, and for R_∞ we use the toroidal covering induced from X_1 as well as from p-power roots of a suitable unit, say $1+T$. Especially $\bar{\bar{R}}$ is almost *é*tale over the subsextension obtained by adjoining to $\bar{R}$ all roots of T (and normalising the result). Now it Δ and $\tilde{\Delta}$ denote the relevant fundamental groups, and $\underset{\sim}{\Delta} \subseteq \tilde{\Delta}$ the kernel of the surjection to Δ, we have to compute the cohomology $H^*(\underset{\sim}{\Delta}, \bar{\bar{R}}/p^s\bar{\bar{R}})$: We need that

i) $H^0(\underset{\sim}{\Delta}, \bar{\bar{R}}/p^s\bar{\bar{R}}) \approx \bar{R}/p^s\bar{R}$

ii) $H^1(\underset{\sim}{\Delta}, \bar{\bar{R}}/p^s\bar{\bar{R}}) \approx (\bar{R}/T\bar{R})^{norm} \otimes \mathbb{Z}/p^s\mathbb{Z}(-1)$

iii) $H^\nu(\underset{\sim}{\Delta}, \bar{\bar{R}}/p^s\bar{\bar{R}}) \approx (0)$ for $\nu \geq 2$.

By the theory of almost *é*tale extensions we may first replace $\bar{R}$ by R_∞ and $\bar{\bar{R}}$ by the subextension corresponding to the quotient $\underset{\sim}{\Delta} \to \hat{\mathbb{Z}}(1)$, i.e. where we adjoin roots of T. Furthermore as groups of order prime to p have trivial cohomology, we reduce to the quotient $\underset{\sim}{\Delta} \to \mathbb{Z}_p(1)$. Let $S_\infty \subseteq \bar{\bar{R}}$ denote the corresponding normalisation,

$$S_\infty/\bar{\bar{J}}_\infty \approx (R_\infty/T\cdot R_\infty)^{norm} \cong Hom_{cont}(\mathbb{Z}_p(1),\bar{V})$$

The assertions we have to show are

i) $H^0(\hat{\mathbb{Z}}(1), S_\infty/p^s\cdot S_\infty) \approx R_\infty/p^s\cdot R_\infty$

ii) $H^1(\hat{\mathbb{Z}}(1), S_\infty/p^s\cdot S_\infty) \approx (R_\infty/TR_\infty)^{norm}\otimes \mathbb{Z}/p^s\mathbb{Z}(-1)$

iii) $H^\nu(\hat{\mathbb{Z}}(1), S_\infty/p^s\cdot S_\infty) = 0$ for $\nu > 1$.

Here iii) is clear because if cohomological dimension one. Also we may replace $\hat{\mathbb{Z}}(1)$ by the quotient $\mathbb{Z}_p(1)$, as the kernel $\prod_{l\neq p}\mathbb{Z}_l(1)$ has trivial cohomology. This amounts to changing S_∞ and $\tilde{R}_\infty$ by adjoining only p-power roots of T. We do this, but retain the notation for simplicity. Now write $S_\infty = \bigcup_n S_n$, with

$$S_n = R_\infty[T^{1/p^n}]^{norm}$$

Note that $S_n[\frac{1}{T}]$ is almost étale over $R_\infty[\frac{1}{T}]$, by the purity theorem. Also R_∞ is the inductive limit of regular rings $R_m = V_m[(1+T)^{p^{-n}}]^{sh}$, and thus $S_n = \bigcup_m S_{n,m}$ with $S_{n,m} = R_m[T^{p^{-n}}]^{norm}$ a free R_n -module of rank $q = p^n$. The group $\Delta_n = \mathbb{Z}/p^n\mathbb{Z}(1)$ operates on $S_{n,m}$. We have $H^0(\Delta_n, S_{n,m}) = R_m$. For the higher cohomology consider the infinite complex

$$K^\cdot_{n,m} :\to S_{n,m} \xrightarrow{(\sigma-1)} S_{n,m} \xrightarrow{tr} S_{n,m} \xrightarrow{(\sigma-1)} S_{n,m} \xrightarrow{tr} S_{n,m}$$

where $tr = \sum_{i=0}^{q-1}\sigma^i$.

Its cohomology $H^i(K_{n,m}) = \tilde{H}^i(\Delta_n, S_{n,m})$ is called the Tate-cohomology ($i\in\mathbb{Z}$). It coincides for $i > 0$ with $H^i(\Delta_n, S_{n,m})$, while $\tilde{H}^0(\Delta_n, S_{n,m}) = R_m/tr(S_{n,m})$. Obviously $\tilde{H}^i$ is periodic with period two. If $S^\vee_{n,m} = Hom_{R_m}(S_{n,m}, R_m)$, we can similarly compute $\tilde{H}^i(\Delta_n, S^\vee_{n,m})$ from the version of $K_{n,m}$ where $S_{n,m}$ is replaced by duals. However this is isomorphic to $(K^\cdot_{n,m})^\vee[1]$. As R_m has finite cohomological dimension two we thus obtain a spectral sequence

$$E_2^{a,b} = Ext^a_{R_m}(\tilde{H}^{-b}(\Delta_n, S^\vee_{n,m}), R_m) \Rightarrow \tilde{H}^{a+b+1}(\Delta_n, S_{n,m})$$

with $E_2^{a,b} = 0$ unless $0\leq a\leq 2$.

Now we pass to the limit $m\to\infty$. The trace-form defines embeddings $S_{n,m}\hookrightarrow S^\vee_{n,m}$. If we localise at height one primes $\mathfrak{p}$ of R_m these are determined by the different of the extension. It is a unit except if $\mathfrak{p}$ divides p or T. However at the prime-divisors of P the p-valuation of the different converges to zero because $S_n[\frac{1}{T}]$ is almost étale over $R_\infty[1/T]$. For $\mathfrak{p}$ dividing T the different is generated by $T^{\frac{q-1}{q}}$. It thus follows that

$$T^{\frac{q-1}{q}}\cdot S_{n,m}\subset S^\vee_{n,m},$$

with the quotient annihilated by p^{α_m}, $\lim\limits_{m\to\infty}\alpha_m = 0$. It also follows that p^{α_m} annihilates $S_n/S_{n,m}\otimes_{R_m} R_\infty$. Hence we obtain:

a) S_n is almost projective finitely generated $/R_\infty$

b) The trace map factors

$$tr: J_n = T^{\frac{1}{q}}\cdot S_n \to T\cdot R_\infty,$$

and $\frac{1}{T} \cdot tr$ induces an almost isomorphism

$$J_n \approx Hom_{R_\infty}(S_n, R_\infty)$$

Especially $tr(J_n) \approx T \cdot R_\infty$.

c) The $\tilde{H}^i(\Delta_n, S_n)$ are almost finitely presented. There is a spectral sequence (up to almost zero modules)

$$E_2^{a,b} = Ext^a_{R_\infty}(\tilde{H}^{-b}(\Delta_n, J_n), R_\infty) \Rightarrow \tilde{H}^{a+b+1}(\Delta_n, S_n)$$

with $E_2^{a.b} \approx 0$ unless $0 \leq a \leq 2$.

((c) follows because $S_{n,m} \otimes_{R_m} R_\infty \to S_n$ is a p^{α_n} -isomorphismus)

Now $\tilde{H}^*(\Delta_n, J_n)$ is annihilated by p^n as well as by $tr(S_n)$ which almost contains T. As (p^n, T) form a regular sequence in R_∞, $Ext^a_{R_\infty}(M, R_\infty) = (0)$ for $a \leq 1$ if M is annihilated by p^n and T. Thus the spectral sequence almost degenerates to give (using period two)

$$\tilde{H}^{a+1}(\Delta_n, S_n) \approx Ext^2_{R_\infty}(\tilde{H}^a(\Delta_n, J_n), R_\infty)$$

However $\tilde{H}^0(\Delta_n, J_n) = (J_n \cap R_\infty)/tr(J_n) \approx T \cdot R_\infty / T \cdot R_\infty \approx 0$, and thus $\tilde{H}^{even}(\Delta_n, J_n) \approx \tilde{H}^{odd}(\Delta_n, S_n) \approx (0)$. Passing to the limit $n \to \infty$ implies

$$\begin{aligned}
&H^i(\Delta_\infty, J_\infty) = (0), i > 0\,\text{even}\\
&H^i(\Delta_\infty, S_\infty) = (0), i > 0\,\text{odd}\\
&H^0(\Delta_\infty, J_\infty) \approx T \cdot R_\infty\\
&H^0(\Delta_\infty, S_\infty) \approx R_\infty
\end{aligned}$$

Also all cohomology vanishes in degree > 1 for p-torsion modules).

Finally we compute

$$H^*(\Delta_\infty, S_\infty/J_\infty) \approx H^*(\mathbb{Z}_p(1), (R_\infty/TR_\infty)^{norm}).$$

Here the action of $\mathbb{Z}_p(1)$ on $(R_\infty/TR_\infty)^{norm}$ is trivial. Thus the module coincides with H^0. Also H^1 are the continuous homomorphisms $\mathbb{Z}_p(1) \to (R_\infty/T \cdot R_\infty)^{norm}$. As the target has no torsion these vanish. Finally H^2 can be computed by tensoring with $0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$. The result:

$$\begin{aligned}
&H^0(\Delta_\infty, (R_\infty/T \cdot R_\infty)^{norm}) = (R_\infty/T \cdot R_\infty)^{norm}\\
&H^1(\Delta_\infty, (R_\infty/T \cdot R_\infty)^{norm}) = (0)\\
&H^2(\Delta_\infty, (R_\infty/T \cdot R_\infty)^{norm}) = (R_\infty/T \cdot R_\infty)^{norm} \otimes \mathbb{Q}_p/\mathbb{Z}_p(-1)
\end{aligned}$$

Next using $0 \to J_\infty \to S_\infty \to S_\infty/J_\infty \to 0$ we compute

$$\begin{aligned}
&H^0(\Delta_\infty, S_\infty) = R_\infty\\
&H^1(\Delta_\infty, S_\infty) \approx (0)\\
&H^2(\Delta_\infty, S_\infty) \approx (R_\infty/T \cdot R_\infty)^{norm} \otimes \mathbb{Q}_p/\mathbb{Z}_p(-1)
\end{aligned}$$

Finally $0 \to p^s \cdot S_\infty \to S_\infty \to S_\infty/p^s \cdot S_\infty \to 0$ gives

$$H^0(\Delta_\infty, S_\infty/p^s \cdot S_\infty) \approx R_\infty/p^s \cdot R_\infty$$
$$H^1(\Delta_\infty, S_\infty/p^s \cdot S_\infty) \approx (R_\infty/T \cdot R_\infty)^{norm} \otimes \mathbb{Z}/p^s\mathbb{Z}(-1)$$

Thus we have shown our claim (the maps are easily seen to be a described there). Apply $\mathbb{R}q_{X,*}$ to the almost exact sequence

$$0 \to \bar{\mathcal{O}}_X/p^s \cdot \bar{\mathcal{O}}_X \to \mathbb{R}j_*\bar{\mathcal{O}}_{\tilde{X}}/p^s \cdot \bar{\mathcal{O}}_{\tilde{X}} \to i_*(\bar{\mathcal{O}}_Y/p^s \cdot \bar{\mathcal{O}}_Y)[-1](-1) \to 0$$

to obtain in highest degree

$$\mathbb{R}^d q_{X,*}(\bar{\mathcal{O}}_X/p^s\bar{\mathcal{O}}_X) \to \mathbb{R}^d q_{\tilde{X},*}(\bar{\mathcal{O}}_{\tilde{X}}/p^s\bar{\mathcal{O}}_{\tilde{X}}) \to \mathbb{R}^{d-1}q_{Y,*}(\bar{\mathcal{O}}_Y/p^s \cdot \bar{\mathcal{O}}_Y)(-1) \to 0$$

on X^{et}. These map to the exact sequence

$$0 \to \omega_X(-d) \to \omega_X(Y)(-d) = \omega_{\tilde{X}}(-d) \to i_*\omega_Y(-d) \to 0$$

with the resulting two squares commutative:

This is an easy local calculation. The first square commutes on $X \setminus Y$ which is dense. For the second choose a local equation T for Y and extend to local parameter $X_1 = T, X_2, \dots, X_d$. Then we obtain a $\mathbb{Z}_p(1)^d = \Delta$ -extension of $\mathcal{O}_{\tilde{X}}$ by adjoining p-power roots of the X_i. Then the trace-map sends the generator of $H^d(\Delta, \mathbb{Z}_p(d)) \cong \mathbb{Z}_p$ to $d\log X_1 \wedge \dots \wedge d\log X_d$. When mapping to the "$Y$-terms" the first goes to the canonical generator of $H^{d-1}(\Delta_Y, \mathbb{Z}_p(d-1)) = \mathbb{Z}_p$, where $\Delta_Y \cong \mathbb{Z}_p(1)^{d-1}$ corresponds to adjoining p-power roots of $X_2, \dots, X_d$. In turn this goes to $d\log X_2 \wedge \dots \wedge d\log X_d$, which is the residue of $d\log X_1 \wedge \dots \wedge d\log X_d$. However the direct image in coherent cohomology is defined by the exact sequence of ω's.

By taking cup-products with classes of complementary degree it follows that the Gysin-map

$$i_* : \mathbb{R}\Gamma_{(!)}(Y, \mathbb{L} \otimes \bar{\mathcal{O}}_Y)(-1)[-2] \to \mathbb{R}\Gamma_!(X, \mathbb{L} \otimes \bar{\mathcal{O}}_X)$$

is the connecting map from the exact sequence (triangle)

$$0 \to \mathbb{L} \otimes \bar{\mathcal{O}}_X \to \mathbb{R}j_*(\mathbb{L} \otimes \bar{\mathcal{O}}_{\tilde{X}}) \to i_*(\mathbb{L} \otimes \bar{\mathcal{O}}_Y)(-1)[-1] \to 0$$

Also by direct computation at stalks there is an exact triangle

$$0 \to \mathbb{L} \to \mathbb{R}j_*(\mathbb{L}) \to i_*(\mathbb{L})(-1)[-1] \to 0$$

mapping to the previous one (onto the almost invariants of Frobenius). It then induces an "adjoint"

$$\mathbb{R}\Gamma_{(!)}(Y, \mathbb{L})(-1)[-2] \to \mathbb{R}\Gamma_{(!)}(X, \mathbb{L})$$

for the trace-maps induced from cohomology with values in $\bar{\mathcal{O}}$. If we know that these trace-maps take values in $\mathbb{Z}/p^s\mathbb{Z}$ we could use them to define a Poincaré-duality and i_* would be the adjoint to i^*. In any case u_X^* maps our exact sequence to its pendant in $(X \otimes_V \bar{K})^{et}$, which defines the direct image in *étale* cohomology. Thus one checks that our triangle defines an adjoint i_* for the inner product defined by $tr^{\acute{e}tale}$.

The upshot of this discussion:

There exists an adjoint i_* for the $tr^{\acute{e}tale}$ -theory inducing the adjoint in $\bar{\mathcal{O}}$ -cohomology.

Next let us compute in some examples that the comparison-maps $H^i(\mathcal{X} \otimes_V \bar{K}, \mathbb{Z}/p^s\mathbb{Z}) \to H^i((X \otimes_V \bar{K})^{et}, \mathbb{Z}/p^s\mathbb{Z})$ tend to be isomorphisms:

Example 1. Let $X = \mathbb{P}^n, X^0 = \mathbb{G}_m^d = \mathbb{P}^d \setminus \cup_{i=0}^d H_i$, the standard-hyperplanes $\{T_i = 0\}$. Then X can be covered by standard affines $\mathbb{A}^n$ with fundamental group $\pi_1(\mathbb{G}_m^d \otimes_V \bar{K}) = \hat{\mathbb{Z}}$ corresponding to n-power coverings $\mathbb{P}^d \to \mathbb{P}^d$. Hence

$$H^*(\mathcal{X}^0 \otimes_V \bar{K}, \bar{\mathcal{O}}/p^s\bar{\mathcal{O}} = \mathbb{R}\Gamma(\hat{\mathbb{Z}}(+1)^d, \mathbb{R}\Gamma(\mathbb{P}^d, \mathcal{O}/p^s\mathcal{O}) \otimes_V \bar{V})$$

$= (\text{exterior algebra in } \mathbb{Z}_p(-1)^d) \otimes \bar{V}/p^s\bar{V}$.

Taking Frobenius-invariants

$$\begin{aligned} H^*(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{Z}/p^s\mathbb{Z}) &= \text{exterior algebra in} \quad \mathbb{Z}/p^s\mathbb{Z}(-1)^d \\ &= H^*((X^0 \otimes_V \bar{K})^{et}, \mathbb{Z}/p^s\mathbb{Z}) \end{aligned}$$

By using the fundamental triangle for the embeddings $H_i \hookrightarrow \mathbb{P}^d \hookleftarrow \mathbb{P}^d - H_i$ one obtains by induction that

$$H^*(\mathcal{X} \otimes_V \bar{K}, \mathbb{Z}/p^s\mathbb{Z}) = H^*((X \otimes_V \bar{K})^{et}, \mathbb{Z}/p^s\mathbb{Z}) = \bigoplus_{i=0}^d \mathbb{Z}/p^s\mathbb{Z} \cdot \xi^i, \xi = c_1(\mathcal{O}(1)).$$

Example 1.* Let $\mathcal{E}$ denote a vectorbundle on X of rank $r+1$, $Y = \mathbb{P}_X(\mathcal{E}) \xrightarrow{pr} X$. Then $H^*(\mathcal{Y}^0 \otimes_V \bar{K}, pr^*(\mathbb{L}) \otimes \bar{\mathcal{O}}_Y) = \bigoplus_{i=0}^r H^{*-2i}(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes \bar{\mathcal{O}}_X) \cdot \xi^i$ with $\xi = c_1(O(1)), pr^*(\mathcal{E}) \to O(1)$ the universal quotient. Also $H^*(\mathcal{Y}^0 \otimes_V \bar{K}, pr^*\mathbb{L}) = \bigoplus_{i=0}^r H^{*-2i}(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L}) \circ \xi^i$.

Proof. Both assertions are equivalent. We prove the corresponding equality for the derived image $\mathbb{R}pr_* \circ \mathbb{R}q_{X,*}$, that is *étale* locally in X^{et}. But over some $Spec(O_{X,x}^{sh})$ Y is a product $X \times \mathbb{P}^r$, and the result follows from Künneth.

In example 1 the $tr : H^{2d}(\mathbb{P}^d, \bar{\mathcal{O}}/p^s\bar{\mathcal{O}}(d)) \to \bar{V}/p^s\bar{V}$ sends $c_1(\mathcal{O}(1))^d$ to 1: This follows by induction from (i = inclusion $H_0 \hookrightarrow \mathbb{P}^d$)

$$i_*c_1(\mathcal{O}(1))^{d-1} = c_1(\mathcal{O}(1))^d.$$

The formula holds in *étale* cohomology, and we already know that both versions of i_* are compatible.

By checking fibrewise we derive that in example 1* $pr_*(\xi^r) = 1$. Also one defines the Chern-classes $c_i(\mathcal{E}) \in H^{2i}(\mathcal{X} \otimes_V \bar{K}, \mathbb{Z}/p^s\mathbb{Z}(i))$ by the usual formula $\sum_{i=0}^{r+1}(-1)^i c_i(\mathcal{E}) \cdot \xi^{r-i} = 0$. They map to the corresponding Chern-classes in *étale* cohomology.

Example 2. Let $\tilde{X} \to \mathbb{P}^d = X$ denote the blow-up in the point $Y = (1 : 0 \dots)$. Then $\tilde{X}$ is a $\mathbb{P}^1$ -bundle over $\mathbb{P}^{d-1}$, with exceptional divisor $E \cong \mathbb{P}^{d-1}$. Using example 1* one checks that for each i the sequence

$$0 \to H^i_{(!)}(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{Z}/p^s\mathbb{Z}) \to H^i_{(!)}(\tilde{\mathcal{X}}^0 \otimes_V \bar{K}, \mathbb{Z}/p^s\mathbb{Z}) \oplus H^i_{()}(\mathcal{Y}^0 \otimes_V \bar{K}, \mathbb{Z}/p^s\mathbb{Z})$$
$$\to H^i_{(!)}(\mathcal{E}^0 \otimes_V \bar{K}, \mathbb{Z}/p^s\mathbb{Z}) \to 0$$

is exact.

Example 2.* Let $Y \subseteq X$ denote a closed subvariety such that locally in the étale topology $(X, Y) \cong X_1 \times (\mathbb{A}^t, 0)$, with X_1 toroidal etc. Let $\tilde{X}$ denote the blow-up of X in $Y, E \subseteq \tilde{X}$ the exceptional divisor, $E \cong \mathbb{P}_Y(J_Y/J_Y^2)$. Then for each i the sequence

$$0 \to H^i_{(!)}(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L}) \to H^i_{(!)}(\tilde{\mathcal{X}}^0 \otimes_V \bar{K}, \mathbb{L}) \oplus H^i_{(!)}(\mathcal{Y}^0 \otimes_V \bar{K}, \mathbb{L}) \to H^i_{(!)}(\mathcal{E}^0 \otimes_V \bar{K}, \mathbb{L}) \to 0$$

is exact.

Proof. As in example 1* one checks this by a localisation in X^{et}. This would give a priori a long exact sequence in cohomology, but for example the pullback $p_X^* : H^i_{(!)}(X^0 \otimes_V \bar{K}, \mathbb{L}) \to H^i_{(!)}(\tilde{\mathcal{X}}^0 \otimes_V \bar{K}, \mathbb{L})$ is injective, with left-inverse $p_{X,*}$.

Thus consider a product $X_1 \times \mathbb{A}^t = X$ and the blow-up $\tilde{X}$ in $X_1 \times \{0\}$. $\mathbb{A}^t$ as well as its blow-up in zero obtain the structure of a torus-embedding from $\mathbb{G}_m^t \subseteq \mathbb{A}^t$. On $\mathbb{A}^t$ the codimension one strata are the coordinate hyperplanes, while on the blow-up we have their proper transforms as well as the exceptional divisor ($\cong \mathbb{P}^{t-1}$).

Now the blow-up induces an almost isomorphism on the cohomology of the stratum $X_1 \times \mathbb{G}_m^t$, using that we retain the same fundamental group and the theory of toroidal embeddings. Also the other strata can be treated by induction.

Remark. For constant coefficients we also could have used local Künneth. The above procedure also works if $\mathbb{L}$ is not induced from the factor X_1 in $X_1 \times \mathbb{A}^t$.

Finally we need the self-intersection formula:

2. Lemma. For the inclusion $i : Y \hookrightarrow X$ of a divisor $i^* i_*(z) = -z \cup c_1(J_Y/J_Y^2) = z \cup c_1(\mathcal{N}_Y)$.

Proof. This amounts to computing the extension class of the pullback by i^* of the exact sequence

$$0 \to \mathbb{Z}/p^s\mathbb{Z} \to \mathbb{R}j_*\mathbb{Z}/p^s\mathbb{Z} \to i_*\mathbb{Z}/p^s\mathbb{Z}(-1)[-1] \to 0$$

Cover X by affines U_ν on which Y has a local equation f_ν. Then the $p^s - th$ roots of f_ν form a μ_{p^s} -torsor on $\mathcal{U}_\nu \cap (X \setminus Y)$, hence give a class in $\Gamma(\mathcal{U}_\nu \otimes_V \bar{K}, \mathbb{R}^1 j_* \mu_{p^s})$ which maps to $1 \in i_*(\mathbb{Z}/p^s\mathbb{Z})$. Thus the restriction to $\mathcal{U}_\nu \cap Y$ trivializes the extension there. The trivialisation is not global. Namely one intersections $\mathcal{U}_\nu \cap \mathcal{U}_\mu$ we get a difference given by the μ_{p^s}-torsor of roots of $u_{\nu\mu} = (f_\nu/f_\mu) \mid Y$. However this defines $c_1(J_Y/J_Y^2)$, and the negative sign arises from sign conventions.

Finally we treat immersions of higher codimensions. Let $Y \subseteq X$ be a closed immersion such that locally in the étale topology $(X, Y) \cong X_1 \times (\mathbb{A}^t, 0)$. Let $\tilde{X} =$ blow-up of X in $Y, E =$ exceptional divisor $= \mathbb{P}_Y(J_Y/J_Y^2)$.

$$\begin{array}{ccc} E & \stackrel{\tilde{i}}{\hookrightarrow} & \tilde{X} \\ \downarrow p_Y & & \downarrow p_X \\ Y & \stackrel{i}{\hookrightarrow} & X \end{array}$$

On E we have the vectorbundle $\mathcal{F} = pr_Y^*(J_Y/J_Y^2)^*/\mathcal{O}(-1)$. $\mathcal{F}$ has rank $t-1$, and from the usual formulas one computes that $c_{t-1}(\mathcal{F}) = \sum_{i=0}^{t-1} c_{r-1}(pr_Y^*(J_Y/J_Y^2)^*) \cup \xi^i$. Especially $p_{Y,*}(c_{t-1}(\mathcal{F})) = 1$ (in all cohomology-theories).

3. Proposition. $p_X^* \circ i_*(z) = \tilde{i}_*(c_{t-1}(\mathcal{F}) \cup p_Y^*(z))$ for $z \in H^*(\mathcal{Y} \otimes_V \bar{K}, \mathbb{L})$.

Proof. The right-hand side lies in the image of pr_X^*: To check this it suffices to check the restriction to E. By the self-intersection formula this is $c_1(\mathcal{O}(-1)) \cup c_{t-1}(\mathcal{F}) \cup p_Y^*(z) = pr_Y^*(c_t(J_Y/J_Y^2) \cup z)$ and indeed induced from Y. It now suffices to apply $p_{X,*}$:

$$\begin{aligned} &p_{X,*}(\tilde{i}_*(c_{t-1}(\mathcal{F}) \cup p_Y^*(z)) \\ &= i_* \circ p_{Y,*}(c_{t-1}(\mathcal{F}) \cup p_Y^*(z)) \\ &= i_*(z). \end{aligned}$$

Remark. The proof actually shows the formula in the derived category.

4. Corollary. The transformation

$$\mathbb{R}\Gamma(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L}) \to \mathbb{R}\Gamma((X^0 \otimes_V \bar{K})^{et}, \mathbb{L})$$

commutes with i_*.

Proof. It commutes with everything else in the formula.

5. Corollary. The two trace-maps on $H_!^{2d}(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{Z}/p^s\mathbb{Z}(d))$ coincide.

Proof. We may assume that the generic fibre $X \otimes_V \bar{K}$ is irreducible (extend K if necessary). As the special fibre $X \otimes_V k$ is generically reduced, there exists (after perhaps again extending K) a section $i : Y = Spec(V) \hookrightarrow X$ with image in the smooth locus. Then both traces are characterised by the fact that they take the value 1 on $i_*(1)$.

We intend to apply this theory to the diagonal embedding

$$\delta : X \hookrightarrow X \times_V X.$$

As this does not satisfy the conditions on the nose, we shall modify it to

$$\tilde{\delta} : \tilde{X} \to (X \times_V X)^\sim.$$

This modification is done using toroidal geometry. We assume given a finite number of convex polyhedral cones $\{\sigma_\nu\}, \sigma_\nu \subseteq L_\mathbb{R}, L = \mathbb{Z}^{d+1}$, and also an indivisible $\lambda \in L^\vee$. We assume that each face of a σ_ν also appears among them, and that $\sigma_\nu \cap \ker(\lambda)$ is a simplex spanned by a partial basis of L.

Furthermore we assume given an étale covering of X by $X(\sigma_\nu)$'s, and étale maps $X(\sigma_\nu) \to \bar{T}_{\lambda,\sigma_\nu}$ (the torus-embedding defined by σ_ν). Also for τ a face of $\sigma, X(\tau)$ should be equal to the preimage of the open $\bar{T}_{\lambda,\{\tau\}} \subseteq \bar{T}_{\lambda,\{\sigma_\nu\}}$, with the induced map.

Each $X(\sigma)$ is stratified, with strata indexed by faces τ of σ. If $x_1 \in X(\sigma_1)$ and $x_2 \in X(\sigma_2)$ lie over the same point $x \in X$, and in the closed σ_1 -stratum respectively σ_2 -stratum, then there exists an automorphism ($\langle\sigma_i\rangle \subseteq L$, the sublattice spanned by σ_i) $(\langle\sigma_1\rangle, \sigma_1, \lambda) \cong (\langle\sigma_2\rangle, \sigma_2, \lambda)$, such that the two induced isomorphisms between $\mathcal{O}^{sh}_{X,x} = \mathcal{O}^{sh}_{X(\sigma_1),x_1} = \mathcal{O}^{sh}_{X(\sigma_2),x_2}$ and the strict henselisations of $\bar{T}_{\lambda,\{\sigma_1\}}$ respectively $\bar{T}_{\lambda,\{\sigma_2\}}$ differ by the induced automorphism and the action of an element of T.

We construct for each σ_ν a finite collection $M(\sigma_\nu)$ of T-invariant ideals $I \subseteq V[L^\vee \cap \sigma_\nu^\vee]$, as follows: First, for any integer n, consider the set of elements $\mu \in L^\vee \cap \sigma_\nu^\vee$ such that

i) $\mu(\rho) \leq n$ for each generator ρ of an extremal ray of σ_ν

ii) If $\lambda(\rho) = 0$ for such a ρ, then $\mu(\rho) = 0$ except for at most one such ρ, for which $\mu(\rho) = 1$.

Choose n so big that the elements generate the dual of the subgroup $\langle\sigma_\nu\rangle \subseteq L$ generated by σ_ν, for all ν. Then define $M_0(\sigma_\nu)$ as the (finite) set of principal ideals $V[L^\vee \cap \sigma_\nu^\vee] \cdot \mu$ generated by such μ. If τ is a face of σ then each ideal in $M_0(\sigma)$ generates an ideal in $M_0(\tau)$ in $V[L^\vee \cap \tau^\vee] \supseteq V[L^\vee \cap \sigma^\vee]$. Also the set of ideals $M_0(\sigma)$ is stable under isomorphisms $(\langle\sigma_1\rangle, \sigma_1, \lambda) \cong (\langle\sigma_2\rangle, \sigma_2, \lambda)$. We define $M(\sigma)$ as the set of all ideals of the form

$$\ker : V[L^\vee \cap \sigma^\vee] \to V[L^\vee \cap \tau^\vee]/I,$$

with $I \in M_0(\tau), \tau$ a face of σ. Then:

i) $M(\sigma)$ is stable under isomorphisms of $(\langle\sigma\rangle, \sigma, \lambda)$.

ii) For τ a face of σ, the map $I \mapsto I \cdot V[L^\vee \cap \tau^\vee]$ induces a surjection $M(\sigma) \twoheadrightarrow M(\tau)$.

iii) If $\sigma \subseteq \ker(\lambda)$, then $M(\sigma)$ consists of the unit ideals and the ideals defining the irreducible components of the boundary-divisor

iv) $M(\sigma)$ contains principal ideals generated by μ, for μ a system of generators of $\langle\sigma\rangle^\vee$.

Now define new torus-embeddings $T \subseteq \tilde{\tilde{\bar{T}}}, \tilde{\tilde{\bar{T}}}_\lambda$ by normalised blow-up of all ideals $I_1 + I_2$, for $I_1, I_2 \in M(\sigma)$. By ii) the commutes with open immersions defined by faces $\tau \subseteq \sigma$, and by i) the induced modifications of $X(\sigma)$ descend to a proper $f : \tilde{X} \to X$ inducing $\tilde{X}^0 \xrightarrow{\sim} X^0$. By iii) on the generic fibre $\tilde{X} \otimes_V K \to X \otimes_V K$ is common blow-up of all intersections of pairs of boundary-divisors. Similarly on $X \times_V X$ we blow-up (and normalise) all ideals $pr_1^*(I_1) + pr_1^*(I_2), pr_1^*(I_1) + pr_2^*(I_2), pr_2^*(I_1) + pr_2^*(I_2)$ with $I_1 \in M(\sigma_1), I_2 \in M(\sigma_2)$ (over $X(\sigma_1) \times X(\sigma_2)$). Then the diagonal extends to $\tilde{\Delta} : \tilde{X} \to \widetilde{X \times_V X}$. Note that normalised blow-up has the following toroidal description: Each $I \in M(\sigma)$ defines a convex piecewise linear function ψ_I on σ by $\psi_I(\rho) = Min\{\mu(\rho) \mid \mu \in I \cap \sigma^\vee\}$. Then the normalised blow-up amounts to subdividing σ (in a minimal way) such that ψ_I becomes linear on each stratum.

Especially on $X \times_V X$ (defined locally by $L^{(2)} = \ker(\lambda, -\lambda) \subseteq L \times L, \sigma^{(2)} = \sigma \times \sigma \cap L^{(2)}_{\mathbb{R}}$), the functions $Min(\mu(x), \mu(y))$ are linear on the strata of the subdivision, for μ in a system of generators of $\langle\sigma\rangle^\vee$ (condition iv)). It follows that on each open stratum meeting the diagonal we have $\mu(x) = \mu(y)$, and thus this stratum is totally contained in the diagonal. Also on the diagonal we obtain the subdivision defining $\tilde{X}$. Thus the diagonal embedding $\bar{T} \hookrightarrow \bar{T} \times_V \bar{T}$ extends to an open immersion

$(T \times 1) \times \tilde{\tilde{T}} \subseteq (\bar{T} \times_V \bar{T})^{\sim}$. Hence locally in the *é*tale cohomology $(\widetilde{X \times_V X}, \tilde{X}) \cong X_1 \times (\mathbb{A}^d, 0)$.

Next consider support-conditions at infinity for a toroidally defined $\tilde{X} \to X$. If $J \subseteq \mathcal{O}_X$ partially defines the boundary, there are two canonical choices for an ideal $\tilde{J} \subseteq \mathcal{O}_{\tilde{X}}, \tilde{J} \supseteq f^*(J)$. Either $\tilde{J}$ defines the total preimage of $V(J)$ (the minimal choice), or the union of boundary divisors $\tilde{D}$ not projecting onto irreducible components outside $V(J)$.

6. Lemma.

i) $\tilde{X} \otimes_V K$ has normal crossings singularities
ii) For either choice of $\tilde{J}$ $\mathbb{R}f_* \tilde{J} = J$.

Proof.

i) On the generic fibre we have to treat the torus-embedding defined by $\sigma \cap \ker(\lambda)$, which up to a torus-factor is $\mathbb{G}_m^d \subseteq \mathbb{A}^d$. Then $\tilde{X}$ is the union of $d!$ embeddings $\mathbb{G}_m^d \subseteq \mathbb{A}^d$. One of them maps to X via $(X_1, X_1 \cdot X_2, \dots, X_1 \cdot \dots \cdot X_d)$, the others by permuting variables.
ii) If $\rho_1, \dots, \rho_r$ are the extremal rays for σ, those for the cones of $\tilde{\sigma}$ are sums $\rho_s = \sum_{i \in S} \rho_i$ for non-empty subsets $S \subseteq \{1, \dots, d\}$. Those lie in a common simplex if and only if the S's form a chain. Assume J is defined by $\{1, \dots, a\}$. Then the bigger transform $\tilde{J}$ corresponds to $S \subseteq \{1, \dots, a\}$, while the total transform to $S \cap \{1, \dots a\} \neq \phi$.

By [KKMS], §, 3, Th. 12 for any $\mu \in L^\vee$ the μ -component in $H^i(\tilde{\tilde{T}}; \tilde{J})$ is $H^i_A(\sigma, k)$ for a certain closed subset $A \subseteq \mid \tau \mid$, namely the union of all faces τ with $\mu \in \Gamma$ (τ -open, $\tilde{J}$). This is also equal to the relative cohomology $H^i(\sigma, \sigma - A; k)$. First choose $\tilde{J}$ minimal. Now $\mu \notin \Gamma(\tau$ -open, $\tilde{J})$ if either not $\mu \in \tau^\vee$, or $\mu \in \tau^\vee$ and $\mu(\rho_s) = 0$

for some ρ_s belonging to $\tilde{J}$ with $\rho_s \in \tau$, i.e. $S \cap \{1, \dots, a\} \neq \phi$. Choose a $\delta\mu \in L^\vee$ with $\delta\mu(\rho_i) < 0$ for $i = 1, \dots, a, \delta_\mu(\rho_j) = 0$ for $j = a+1, \dots, r, \delta\mu(\rho) \geq 0$ for any extremal ρ of a face τ with $\lambda(\rho) > 0$. Then for $N >> 0$ $\mu \in \Gamma(\tau$ -open, $\tilde{J}) \Leftrightarrow N\mu + \delta\mu \in \tau^\vee$, thus the μ -component becomes the $(N\mu + \delta\mu)$ component in $H^i(\tilde{\tilde{T}}, \mathcal{O})$ which is known to be 0 unless $i = 0$ and $N\mu + \delta\mu \in \sigma^\vee$, which means $\mu \in J$. For the maximal J choose $\delta\mu$ as before except that $\delta\mu(\rho_j) > -a \cdot \min\{\partial\mu(\rho_i) \mid 1 \leq i \leq 0\}$ for $a+1 \leq j \leq r$. This proves the assertion ii) for $\tilde{\tilde{T}} \to \bar{T}$, and by basechange also for $\tilde{X} \to X$.

As an application let as before R_∞ denote the normalisation of $R \otimes_V \bar{V}$ in the extension defined by adjoining all p-power roots of $\mu \in L^\vee, \tilde{X}_\infty \to \tilde{X}$ the corresponding normalised pullback to $f : \tilde{X} \to X = Spec(R)$, with Galois-group $\Delta_\infty = \mathbb{Z}_p(1)^d$. If M_∞ denotes an R_∞ -module with continuous semilinear Δ_∞ -action, and $J_\infty \subset R_\infty$ a support-ideal extending (in two possible ways) to $\tilde{J}_\infty \subset \mathcal{O}_{\tilde{X}_\infty}$, then $\mathbb{R}f_{\infty,*}(M_\infty \otimes^{\mathbb{L}} \tilde{J}_\infty) = M_\infty \otimes^{\mathbb{L}} J_\infty$ (by basechange), and this equality persists after applying $\mathbb{R}\Gamma(\Delta_\infty, \quad)$. However by the almost-*é*tale theory these are cohomologies of $\mathbb{L} \otimes \bar{\mathcal{O}}$, if $M = (\bar{R} \otimes \mathbb{L})^{Gal(\bar{R}/R_\infty)}$ for a locally constant p-torsion sheaf $\mathbb{L}$ on $Spec(R)^0 \otimes_V \bar{K}$. That is we know that for the map $q_X : \mathcal{X}^0 \otimes_V \bar{K} \to X^{et}$ $Rq_{X,*}(\mathbb{L} \otimes \bar{J})$ is almost isomorphic to a complex of quasicoherent sheaves, and the above assertion means that the map

$$\mathbb{R}q_{X,*}(\mathbb{L}\otimes\bar{J}) \to \mathbb{R}f_* \circ \mathbb{R}q_{\tilde{X},*}(\mathbb{L}\otimes\tilde{\tilde{\bar{J}}})$$

is an almost isomorphism. Thus:

7. Corollary. $\mathbb{R}\Gamma(\mathcal{X}^0\otimes_V \bar{K}, \mathbb{L}\otimes\bar{J}) \approx \mathbb{R}\Gamma(\tilde{\mathcal{X}}^0\otimes_V \bar{K}, \mathbb{L}\otimes\tilde{\tilde{\bar{J}}})$

$$\mathbb{R}\Gamma_{(!)}(\mathcal{X}^0\otimes_V \bar{K}, \mathbb{L}) = \mathbb{R}\Gamma_{(!)}(\tilde{\mathcal{X}}^0\otimes_V \bar{K}, \mathbb{L})$$

In the second equation (!) always denotes the relevant support-condition at infinity. The second assertion follows from the first by descente. Also the corresponding *é*tale analogue holds:

8. Lemma. $\mathbb{R}\Gamma_{(!)}((X^0\otimes_V \bar{K})^{et}, \mathbb{L}) = \mathbb{R}\Gamma_{(!)}((\tilde{X}^0\otimes_V \bar{K})^{et}, \mathbb{L})$

Proof. Let $X - X^0 = A\cup B$ denote the decomposition of the boundary in union of irreducible components, such that we have compact support along B. Similarly $\tilde{X} - \tilde{X}^0 = \tilde{A}\cup\tilde{B}$, with either $\tilde{A} = f^{-1}(A)$ or $\tilde{B} = f^{-1}(B)$. As Poincar*é*-duality exchanges A and B (and $\tilde{A}, \tilde{B}$) we may assume that $\tilde{A} = f^{-1}(A)$. Now with $j_A : X - A \hookrightarrow X$ denoting the inclusion, we have $\mathbb{R}f_*\circ\mathbb{R}j_{\tilde{A}} = \mathbb{R}j_A\circ\mathbb{R}f_*$. Thus it suffices if

$$\mathbb{R}f_*\circ\mathbb{R}j_{\tilde{B},!}(\mathbb{L}) = \mathbb{R}j_{B,!}\circ\mathbb{R}f_*(\mathbb{L}) \quad \text{on} \quad X - A.$$

This assertion is local in $X - A$. We thus reduce to $X^0 = \mathbb{G}_m^d \subset X = \mathbb{A}^d, A = \phi, B = X - X^0$ and thus $\tilde{B} = \tilde{X} - \tilde{X}^0$. As f is proper $\mathbb{R}f_*$ commutes with duality which exchanges $j_{B,!}$ and $\mathbb{R}j_{B,*}$, and similar for $\tilde{B}$. This reduces us again to $A = X - X^0, B = \phi$, and then to $A = B = \phi$.

Now the main application: The diagonal-embedding

$$\delta : X \hookrightarrow X\times X$$

induces

$$\tilde{\delta} : \tilde{X} \hookrightarrow (\widetilde{X\times X}).$$

This embedding is such that locally in the *é*tale topology $((\widetilde{X\times_V X}), \tilde{X}) \cong X_1 \times (\mathbb{A}^d, 0)$. Now choose a decomposition of the boundary

$$X - X^0 = A\cup B,$$

and extend to $\tilde{X} - \tilde{X}^0 = \tilde{A}\cup\tilde{B}$ with $\tilde{A}$ = full preimage of A. Then define $\mathbb{R}\Gamma_{A,B}((X^0\otimes_V \bar{K})^{et}, \mathbb{L})$ as cohomology with compact support along B, and similarly for the other cohomology-theories or for $\tilde{X}$. On $X\times_V X$ we consider the pair

$$(A^{(2)}, B^{(2)}) = (A\times_V X\cup X\times_V B, B\times_V X\cup X\times_V A),$$

and similarly $\tilde{A}^{(2)}$ = full preimage of $A^{(2)}$ in $(\widetilde{X\times_V X})$, etc. Assume first that $\mathbb{L}$ is annihilated by p. We then have a commutative diagram

$$\begin{array}{ccc}
\mathbb{R}\Gamma((\tilde{X}^0 \otimes_V \bar{K})^{et}, End(\mathbb{L})) & \xrightarrow{\tilde{\delta}_*} & \mathbb{R}\Gamma_{\tilde{A}^{(2)},\tilde{B}^{(2)}}(((\widetilde{X\times_V X})^0 \otimes_V \bar{K})^{et}, \mathbb{L}\otimes\mathbb{L}^\vee)(-d)[-2d] \\
\uparrow & & \uparrow \\
\mathbb{R}\Gamma(\tilde{\mathcal{X}}^0 \otimes_V \bar{K}, End(\mathbb{L})) & \xrightarrow{\tilde{\delta}_*} & \mathbb{R}\Gamma_{\tilde{A}^{(2)},\tilde{B}^{(2)}}((\widetilde{\mathcal{X}\times_V\mathcal{X}})^0 \otimes_V \bar{K}, \mathbb{L}\otimes\mathbb{L}^\vee)(-d)[-2d],
\end{array}$$

which can be identified with

$$\begin{array}{ccc}
\mathbb{R}\Gamma((X^0 \otimes_V \bar{K})^{et}, End(\mathbb{L})) & \xrightarrow{\delta_*} & \mathbb{R}\Gamma_{A^{(2)},B^{(2)}}((X\times_V X)^0 \otimes_V \bar{K})^{et}, \mathbb{L}\otimes\mathbb{L}^V)(-d)[-2d] \\
\uparrow & & \uparrow \\
\mathbb{R}\Gamma(\mathcal{X}^0 \otimes_V \bar{K}, End(\mathbb{L})) & \xrightarrow{\delta_*} & \mathbb{R}\Gamma_{A^{(2)},B^{(2)}}((\mathcal{X}\times_V\mathcal{X}_0)^0 \otimes_V \bar{K}, \mathbb{L}\otimes\mathbb{L}^V)(-d)[-2d]
\end{array}$$

Apply this to the canonical class $1 \in H^0(X^0 \otimes_V \bar{K})^{et}, End(\mathbb{L}))$ or its variants. Its image in

$$\begin{aligned}
&\mathbb{R}\Gamma_{A^{(2)},B^{(2)}}(((X\times_V X)^0 \otimes_V \bar{K})^{et}, \mathbb{L}\otimes\mathbb{L}^\vee)(-d)[-2d] \\
\cong\; &\mathbb{R}\Gamma_{A,B}((X\otimes_V \bar{K})^{et}, \mathbb{L}) \otimes \mathbb{R}\Gamma_{B,A}((X\otimes_V \bar{K})^{et}, \mathbb{L}^\vee)(-d)[-2d]
\end{aligned}$$

corresponds via Poincaré-duality to the identity, and similarly for $\mathbb{R}\Gamma(\mathcal{X}^0\otimes_V\bar{K},\dots)$. As the comparison-map $\mathbb{R}\Gamma_{A,B}(\mathcal{X}^0\otimes_V\bar{K},\mathbb{L}) \to \mathbb{R}\Gamma_{A,B}((\mathcal{X}^0\otimes_V\bar{K})^{et},\mathbb{L})$ respects the trace and cup-product it induces an isometry on cohomology, and thus an injection. That it also preserves $\delta_*(1)$ means that its adjoint also preserves products, thus is also an injection. Hence the comparison-map is a quasi-isomorphism. By devissage this also holds for general p-torsion $\mathbb{L}$'s.

9. Theorem. $H^i_{A,B}(\mathcal{X}^0\otimes_V,\mathbb{L}) = H^i_{A,B}((X^0\otimes_V \bar{K})^{et},\mathbb{L})$.

Remark. We have given a "global" proof. In many cases one can also proceed "locally": Namely if for each $x \in X$ all strata in $Spec(\mathcal{O}^{sh}_{X,x}\otimes_V \bar{K})$ are $K(\pi,1)$'s in the étale topology, then the stalks of $\mathbb{R}q_{X,*}\mathbb{L}$ coincide for $q_X : \mathcal{X}^0\otimes_V \bar{K} \to X^{et}$ respectively $(X^0\otimes_V \bar{K})^{et} \to X^{et}$.

5. Crystalline Cohomology

Let $X \to Spec(V)$ etc. be as before. Choose a uniformiser π of V. Let $V_0 = W(k) \subseteq V$ denote the maximal unramified subring of Witt-vectors over k. Then

$$V = V_0[\pi] = V_0[t]/(f(t)),$$

where $f(t) = t^e + a_{e-1} \cdot t^{e-1} + \ldots + a_0$ is an Eisenstein polynomial: All a_i are divisible by p, and a_0/p is a unit in V_0. $D_{(f(t))}(V_0[t])$ denotes the divided power-hull of $(f(t))$, obtained by adjoining divided powers $f(t)^n/n!$ or $t^{en}/n!$, and R_V its p-adic completion. $R_V \subset K_0[[t]]$ consists of powerseries $\sum_{n=0}^{\infty} a_n \cdot t^n$ with

$$[\frac{n}{e}]! \cdot a_n \in V_0 \quad (\text{all } n), \quad \lim_{n\to\infty}([\frac{n}{e}]! \cdot a_n) \to 0$$

It has the PD-filtration $F^n(R_V) =$ ideal generated by divided powers $\frac{f^m}{m!}, m \geq n$. $R_V/F^1(R_V) = V$. We want to study the logarithmic crystalline topics of X/R_V. For lack of suitable reference we first tensor with a fixed $\mathbb{Z}/p^s\mathbb{Z}$. So let $X_s = X \otimes_V V/p^sV$. Then the logarithmic crystalline site ([B], [K]) $(X_s/R_{V,s})_{cris}$ consists of PD-embeddings

$$U \hookrightarrow V$$

where U, V are schemes/ $R_{V,s}, U \to X_s$ is étale and V has a "logarithmic structure" which should mean the following:

The covering by $X(\sigma) \to X$ induces an étale covering by $U(\sigma) \to U$, with $U(\sigma) \to \bar{T}_{\lambda,\sigma} \subseteq \bar{T}$. $\bar{T}$ is a scheme over $V_0[t]$ with t corresponding to λ. As the ideal of U in V is nilpotent $U(\sigma)$ extends canonically to $V(\sigma) \to V$ étale, with $U(\sigma) \hookrightarrow V(\sigma)$. Then the "logarithmic structure" on V is an étale local extension of $U(\sigma) \to \bar{T}$ to $V(\sigma) \to \bar{T}$ (as schemes over $V_0[t]$), unique up to homomorphisms $V(\sigma) \to T_\lambda = \ker(\lambda : T \to \mathbb{G}_m)$. That is the extensions exist on étale cover on $V'(\sigma) \twoheadrightarrow V(\sigma)$ and the two pullbacks to $V''(\sigma) = V'(\sigma) \times_{V(\sigma)} V'(\sigma)$ differ by a map to $V''(\sigma) \to T_\lambda$ satisfying the cocycle condition. It then defines a class in $H^1(V(\sigma), T_\lambda) = (Pic\, V(\sigma))^d$. Hence we only need a Zariski-covering of $V(\sigma)$.

Maps $f : (U_1 \hookrightarrow V_1) \to (U_2 \to V_2)$ are pairs of maps $f_U : U_1 \to U_2, f_V : V_1 \to V_2$ with the usual commutation rule, and $\rho : V_1(\sigma) \to T_\lambda$ such that

$$\begin{tikzcd} V_1(\sigma) \arrow[r] \arrow[dr] & V_2(\sigma) \arrow[d] \\ & \bar{T} \end{tikzcd}$$

commutes up to the action of ρ. (Composition is obvious) Coverings in the site are induced by étale covers of U. A crystal (of quasicoherent sheaves) on $(X_s/R_{V,s})_{crys}$ is a sheaf $\mathcal{E}$ such that for $U \hookrightarrow V, \mathcal{E}_V$ is a quasicoherent sheaf on V. Also for any map $(U_1 \hookrightarrow V_1) \to (U_2 \hookrightarrow V_2), \mathcal{E}_{V_1}$ should be isomorphic to the pullback of $\mathcal{E}_{V_1}$, via the pullback map. A filtered crystal is an $\mathcal{E}$ as above together with a decreasing sequence of subsheaves $F^n(\mathcal{E}) \subseteq \mathcal{E}$ (indexed by $n \in \mathbb{Z}$) such that on each $U \hookrightarrow V$, the $F^n(\mathcal{E})_V$ are quasicoherent subsheaves with $F^{[a]}(\mathcal{O}_V) \cdot F^b(\mathcal{E})_V \subseteq F^{a+b}(\mathcal{E})_V$ ($F^{[a]}(\mathcal{O}_V) : PD$ -filtration on the ideal of $U \hookrightarrow V$), and for $f : (U_1, V_1) \to (U_2, V_2)$ $F^n(\mathcal{E})_{V_1} \subseteq \mathcal{E}_{V_1}$ is the subsheaf generated $\sum_{a+b\geq n} F^{[a]}(\mathcal{O}_{V_1}) \cdot f_V^*(F^b(\mathcal{E})_{V_2})$. Obviously any (unfiltered) $\mathcal{E}$ can be filtered by the $F^{[n]}(J_{U/V}) \cdot \mathcal{E}_V$. An example if a filtered crystal is $\mathcal{O}_X^{crys}\{a\}$, with underlying sheaf $\mathcal{E}_V = \mathcal{O}_V$ and $F^n(\mathcal{E})_V = F^{a+m}(\mathcal{O}_V)$. A filtered crystal of vectorbundles $(\mathcal{E}, F^{\cdot})$ is called locally filtered free if for any $U \hookrightarrow V$ $(\mathcal{E}_V, F^{\cdot}(\mathcal{E}_V))$ is locally isomorphic to a direct sum of $\mathcal{O}_{crys}\{a\}$'s (no compatability with pullbacks required). Now choose an *é*tale covering of X by $X(\sigma)$'s, with *é*tale maps $X(\sigma) \to \bar{T}_\lambda$. These lift to *é*tale maps of formal schemes (with adic topology defined by $\pi - \lambda$) $\widehat{X(\sigma)} \to \overset{\wedge}{\bar{T}} =$ formal completion of $\bar{T}$ along $\bar{T}_\lambda$. Thus we can find an *é*tale covering $Y \twoheadrightarrow X$ and a lift to an *é*tale scheme $\hat{Y}$ over $\overset{\wedge}{\bar{T}}$, namely for example $Y = \amalg X(\sigma)$. Fix such a pair $(\hat{U}, U)$. We need the logarithmic products $\hat{Y}^{n+1,\log}$. These will be formal schemes with underlying topological space $Y \times_X Y \times \ldots \times_X Y$ $\quad (n+1)$ -times), defined by *é*tale glueing from the following: For $\bar{T}$ the $(n+1)$ -fold log -product is the formal completion of $\bar{T} \times_{V_0} T_\lambda^n$ along $(\bar{T} \times \{1, \ldots, 1\})$. Its projections to $\bar{T}$ are given by $pr_0(z, g_1, \ldots, g_n) = z, pr_i(z, g_1, \ldots, g_n) = g_i \cdot z$. Now locally in the *é*tale topology $Y = X = \bar{T}_\lambda, \hat{Y} = \overset{\wedge}{\bar{T}}$, and we use pullback. It follows that via any of the projections pr_i $\hat{Y}^{n+1,\log}$ is isomorphic to a product of formal schemes $\hat{Y}^{n+1,\log} \cong \hat{Y} \times \hat{T}_\lambda^n (\hat{T}_\lambda :$ formal completion along origine). This log -product corresponds to the "exactified" product in the Fontaine-Illusie-Kato theory. The $\hat{Y}^{n+1,\log}$ form a simplicial formal scheme. Next if we reduce modulo some fixed p^s we can form the PD-hulls of the embeddings $Y \times_X Y \ldots \times_X Y \hookrightarrow \hat{Y}^{n+1,\log}$ and obtain schemes (not just formal) schemes over $R_V/p^s R_V$, or (again) formal simplicial schemes over R_V, with the p-adic topology:

$$Y_X^{\cdot} \hookrightarrow D(\hat{Y}^{\cdot,\log}).$$

Any crystal $\mathcal{E}\, mod\, p^s$ can be evaluated on $D(\hat{Y}^{\cdot;\log}) \otimes_{R_V} R_V/p^s R_V$, so gives $\mathcal{E}_n$ on each $D(\hat{Y}^{\cdot;\log})_n \otimes_{R_V} R_V/p^s R_V$ with pullback-maps which are all isomorphisms. As usual this is equivalent to giving $\mathcal{E}_0$ on $D(\hat{U}^{\cdot;\log})_0$ together with an isomorphism of the two pullbacks to $D(\hat{Y}^{\cdot;\log})_1$, satisfying a cocycle condition, or to give $\mathcal{E}_0$ with an integrable log-connection $\bigtriangledown : \mathcal{E}_0 \to \mathcal{E}_0 \otimes \Omega^1_{\hat{Y},\log}$, where $\Omega^1_{\hat{Y},\log}$ are the logarithmic differential forms, dual to the Lie-algebra of T_λ. The relation is given by Taylor's formula. With this connection we define the de Rham complex

$$DR(\mathcal{E}) = (\mathcal{E} \otimes \Omega^{\cdot}_{\hat{Y},\log}, \bigtriangledown + d)$$

on $D(\hat{Y}^{\cdot,\log})$.. On $D(\hat{Y}^{\cdot,\log})_n$ its underlying sheaves are $pr_0^*(\mathcal{E}) \otimes \Lambda \oplus_{i=0}^n pr_i^*((\Omega^1 1_{\hat{Y},\log})$, with

$$(\bigtriangledown + d)(pr_0^*(e) \otimes \alpha) = pr_0^*(\bigtriangledown e) \cdot \alpha + e \otimes d\alpha.$$

For any PD-thickening $U \hookrightarrow V$ of $U \to X$ *é*tale we find *é*tale locally a map $(U, V) \xrightarrow{f} (Y, \hat{Y})$. Thus $\mathcal{E}_V$ is determined by pullback. In fact one can form the log-product $(V \times \hat{Y}.)^{\log}$, a formal simplicial scheme with same underlying topological space as the graph of f_V. Etale locally it is isomorphic to $V \times \bar{T}_\lambda$. As usual modulo p^s the divided power hull of this is a scheme, and we obtain a diagram

$$(U, V) \xleftarrow{pr_1} D((V \times \hat{Y}^{\cdot})^{\log}) \xrightarrow{pr_2} D(\hat{Y}^{\cdot})$$

Then it is known that

$$\mathbb{R}pr_{1,*}pr_2^*(DR(\mathcal{E}))$$

globalises (using stratifications) and forms an acyclic resolution of $\mathcal{E}$. Thus $\mathbb{R}u_{X,*}(\mathcal{E})$ on X^{et} pulls back on $(U_X^{\cdot})$. to $DR(\mathcal{E})$. By cohomological descente $\mathbb{R}u_{X,*}(\mathcal{E})$ is represented by $\mathbb{R}pr_*(DR(\mathcal{E}).)$, $pr : U_X \to X$ the projection. The same holds in the filtered context, that is for $\mathcal{E}$ filtered the de Rham complexes give filtered resolutions. In local coordinates $D((Y \times \hat{Y}_0)^{\log})_n$ corresponds to a free PD-algebra $\mathcal{O}_V\{t_\nu\}$ in $(n+1)\cdot d$ variables t_ν (= local coordinates near 1 in T_λ^{n+1}), and the de Rham complex is the tensorproduct of $\mathcal{E}_V$ and the de Rham complex in the $\{t_\nu\}$, with its product filtration.

Basically we are interested in the cohomology of crystals of vectorbundles. However we also need boundary conditions. For this we assume (for simplicity) that the special fibre $X \otimes_V k$ is reduced, or equivalently that $X \otimes_V \bar{V}$ is normal. We can always achieve this by adjoining a root $\pi^{1/e}$. Then locally the extremal rays of σ are generated by ρ_i with $\lambda(\rho_i) \in \{0,1\}$. Any subset I of them defines a reduced divisor with support in the boundary, and thus an ideal $J_I \subseteq \mathcal{O}_X$. For $I = \{\rho \mid \lambda(\rho) = 1\}$ we have $J_I = \pi \cdot \mathcal{O}_X$. Now restrict to subsets I with $\lambda(\rho) = 0$ for $\rho \in I$, and let I^c denote the complement of I in the set of all such ρ's, which defines the ideal $J \subseteq \mathcal{O}_X$. Then $J \cdot \Omega^{d,\log}_{X/Y}$ is the relative dualising complex, $J_{I_c} = Hom_{\mathcal{O}_X}(J_I, J)$. Globally we assume given an ideal J_I which locally is of the type above. For example this is always possible for $\mathcal{O}_X$ or J. Also if J_I is defined so is J_{I^c}.

The ideals J_I extend naturally to $\bar{T}$ and thus to any object (U,V) of the log-crystalline topos, and form crystals. We are interested in the cohomologies $\mathbb{R}\Gamma(X^{crys}/R_V, \mathcal{E} \otimes J_I)$, $\mathcal{E}$ a crystal of vectorbundles on X^{crys}. As it stands this is defined for crystals modulo a fixed power p^s, and then can be represented by the hypercohomology of $\mathcal{E} \otimes J_I$ on $D(\hat{Y}^{\cdot,\log})$. If we assume that Y is affine (which we may do) then so is each $Y_n = Spec(S_n)$, $D(\hat{Y}_n) = Spec(D(S_n))$, and the de Rham complexes $(\mathcal{E} \otimes J_I)(D(\hat{Y}_n) \otimes \Omega^{\cdot,\log}_{S_n}$ form (modulo p^s) a double complex whose associated total complex represents the crystalline cohomology. It is in fact easily seen (comparing two coverings) that this is (up to canonical isomorphism in the filtered derived category) independent of choices. Also if we have a compatible system $\mathcal{E}$ of crystals of vectorbundles $\mathcal{E}_s$ modulo each p^s, we may form the projective limit (for $s \to \infty$) of the corresponding complexes to get the cohomology of $\mathcal{E}$ (in the same spirit as earlier the cohomology of $A_{\inf}(\bar{\mathcal{O}})$). As the sheaves J_I are modulo each p^s flat over R_V/p^sR_V, the proof in [F7] applies to show that $\mathbb{R}\Gamma(X^{crys}/R_V, \mathcal{E} \otimes J_I)$ can be represented by a finite complex of filtered free R_V -modules if $\mathcal{E}$ is locally filtered free. It is also shown there (the proof extends) that Berthelot's construction from [B] gives a trace-form

$$H^{2d}(X^{crys}/R_V, J) \to R_V\{-d\},$$

which induces a perfect duality (of complexes of filtered free R_V -modules)

$$\mathbb{R}\Gamma(X^{crys}/R_V, \mathcal{E}^\vee \otimes J_{I^c}) \xrightarrow{\approx} \mathbb{R}Hom_{R_V}(\mathbb{R}\Gamma(X^{crys}/R_V, \mathcal{E} \otimes J_I), R_V\{-d\}[-2d])$$

Also there is a Künneth-formula. In short all the usual formalism is again available.

For a closed immersion of a divisor $i : Y \hookrightarrow X$ transversal to all strata (i.e. $\acute{e}$tale locally $(X,Y) = X_1 \times (\mathbb{A}^1, 0)$) there is an exact sequence (triangle) with $\tilde{X}$ corresponding to the new toroidal structure with open stratum $X^0 - Y^0$

$$0 \to \mathbb{R}\Gamma(X^{crys}/R_V, \mathcal{E}\otimes J_I) \to \mathbb{R}\Gamma(\tilde{X}^{crys}/R_V, \mathcal{E}\otimes J_I) \to \mathbb{R}\Gamma(Y^{crys}/R_V, \mathcal{E}\otimes J_I)\{-1\}[-1] \to 0$$

(see [F3]) and the connecting map represents i_*:

It suffices that the trace on $H^{2d-2}(Y^{crys}/R_V, J)$ is equal to the composite of the connecting map with the trace on $H^{2d}(X^{crys}/R_V, J)$. But the trace is characterised by its values on local cohomology

$$H^{2d-2}_{\{y\}}(Y^{crys}/R_V, J) \quad (y \in Y \otimes_V k \quad \text{a closed point}).$$

This can be computed by locally lifting to

$$Y = X_1 \times \{0\} \hookrightarrow X = X_1 \times \mathbb{A}^1,$$

with X_1 étale over some torus-embedding $\bar{T}_1$. The exact sequence reduces to the tensor-product of $J \circ \Omega^{\cdot,\log}_{X_1}$ with

$$0 \to \Omega^{\cdot}_{\mathbb{A}^1} \to \Omega^{\cdot}_{\mathbb{A}^1}(d\log 0) \stackrel{\text{Residue}}{\longrightarrow} \mathcal{O}_{\{0\}}\{-1\}[-1] \to 0,$$

and the assertion is wellknown.

Finally for a line-bundle $\mathcal{L}$ on X the first Chern-class $c_1(\mathcal{L}) \in H^2(X^{crys}/R_V, F^1(\mathcal{O}))$ is defined via the connecting map

$$Pic(X) = H^1(X^{crys}/R_V, (\mathcal{O}/F^1(\mathcal{O}))^*) \to$$
$$H^2(X^{crys}/R_V, (1+F^1(\mathcal{O}), \cdot) \stackrel{\log}{\longrightarrow} H^2(X^{crys}/R_V, F^1(\mathcal{O})).$$

As usual this defines higher Chern classes, the selfintersection formula holds, and also (for immersions $Y \hookrightarrow X$ with locally $(X,Y) \cong X_1 \times (\mathbb{A}^t, 0)$) the formula $pr_X^* \circ i_* = \tilde{i}_*(c_{t-1}(\mathcal{F}) \cup pr_Y^*(\,))$

$$\begin{tikzcd} E \arrow[rr] \arrow[d] & & \tilde{X} \arrow[d] \\ Y \arrow[r, hook] & & X \end{tikzcd}$$

So far the review.

Next relations to étale cohomology. For a ring R with $Spec(R)$ étale over $\bar{T}_\lambda$ we have defined $A_{\inf}(\bar{R})$, with an isomorphism $A_{\inf}(\bar{R})/\xi \cdot A_{\inf}(\bar{R}) \xrightarrow{\sim} \overset{\wedge}{\bar{R}}$ (p-adic completion). We define $B^+_{cr}(\bar{R})$ as the p-adically completed divided power hull of $\xi \cdot A_{\inf}(\bar{R})$. For varying R the $B^+_{cr}(\bar{R})/p^s \cdot B^+_{cr}(\bar{R})$ form an almost sheaf of rings on $\mathcal{X}^0 \otimes_V \bar{K}$. Thus we may form the cohomology $\mathbb{R}\Gamma(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes B^+_{cr}(\bar{\mathcal{O}}))$, which however turns out to be almost equal to $\mathbb{R}\Gamma((X^0 \otimes_V \bar{K})^{et}, \mathbb{L}) \otimes^{\mathbb{L}} B^+_{cr}(\bar{V})$, as

$$B^+_{cr}(\bar{R})/p^s \cdot B^+_{cr}(\bar{R}) \cong A_{\inf}(\bar{R})/p^s \cdot A_{\inf}(\bar{R}) \otimes_{A_{\inf}(\bar{V})} B^+_{cr}(\bar{V}).$$

Similarly let $B^+_{cr}(\bar{J}_I) = \ker(B^+_{cr}(\bar{R}) \to B^+_{cr}(\bar{R}/\bar{J}_I)) = p$-adic completion of $A_{\inf}(\bar{J}_I) \otimes_{A_{\inf}(\bar{V})} B^+_{cr}(\bar{V})$, and then

$$\mathbb{R}\Gamma_{(!)}((X^0 \otimes_V \bar{K})^{et}, \mathbb{L}) \otimes^{\mathbb{L}} B^+_{cr}(\bar{V}) \approx \mathbb{R}\Gamma(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes B^+_{cr}(\bar{J}_I)).$$

Also $A_{\inf}(\bar{R})$ and $B^+_{cr}(\bar{R})$ get toroidal structures by choosing for each μ in a basis of $L^\vee$ a compatible system of p-power roots of μ, defining $\underline{\mu} = [\mu] \in A_{\inf}(\bar{R})$. Next the preimage of R in $B^+_{cr}(R)$ defines a PD-thickening. Thus if $\mathcal{E}$ is a crystal modulo p^s on X^{crys}/R_V, we can evaluate it on this preimage to get a module with an action of $Gal(\bar{R}/R \otimes_V \bar{V})$. This is functorial in R, thus defines a presheaf (in fact almost a sheaf for small enough $Spec(R)$) on $\mathcal{X}^0 \otimes_V \bar{K}$.

Now assume given a locally constant sheaf $\mathbb{L}$ (annihilated by p^s) on $(X^0 \otimes_V \bar{K})^{et}$, and a functorial map

$$\mathcal{E}\,(\text{preimage of } R) \quad \to \mathbb{L} \otimes B^+_{cr}(\bar{R}),$$

linear over the preimage of R in $B^+_{cr}(\bar{R})$ and commuting with the Galois-action. Then any resolution of $\mathcal{E}$ on the crystalline topos (like the de Rham resolution) defines by evaluation a resolution of the presheaf $\mathcal{E}$ on $\mathcal{X}^0 \otimes_V \bar{K}$, which maps to an injective resolution of $\mathbb{L} \otimes B^+_{cr}(\bar{R})$. Hence we obtain a transformation

$$v_X : \mathbb{R}\Gamma(X^{crys}/R_V, \mathcal{E}) \to \mathbb{R}\Gamma(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes B^+_{cr}(\bar{\mathcal{O}})) \approx \mathbb{R}\Gamma((X^0 \otimes_V \bar{K})^{et}, \mathbb{L}) \otimes B^+_{cr}(\bar{V}).$$

If $\mathcal{E}$ is filtered and the basic maps to $\mathbb{L} \otimes B^+_{cr}(\bar{R})$ preserve filtrations, we obtain a transformation in the filtered derived category. An example is $\mathcal{E} = \mathcal{O}_X$ and $\mathbb{L} = \mathbb{Z}/p^s\mathbb{Z}$. Also there is the obvious variant

$$v_X : \mathbb{R}\Gamma(X^{crys}/R_V, \mathcal{E} \otimes J_I) \to \mathbb{R}\Gamma(\mathcal{X}^0 \otimes_V \bar{K}, \mathbb{L} \otimes B^+_{cr}(\bar{J}_I)) \approx \mathbb{R}\Gamma_{(!)}((X^0) \otimes_V \bar{K})^{et}, \mathbb{L}) \otimes B^+_{cr}(\bar{V}).$$

Next we check how much the map commutes with Chern-classes and direct images. Recall that there is a canonical map

$$\beta : \mathbb{Z}_p(1) \to F^1(B^+_{crys}(V)).$$

If $t \in \mathbb{Z}_p(1)$ is a generator, corresponding to a compatible sequence of primitive roots $\zeta_n \in \mu_{p^n}(\bar{V}), \underline{1} = \varprojlim \zeta_n$ defines $[\underline{1}] \in A(\underline{V})$. Then $\beta(t) = \log \underline{1}$.

1. Theorem.

i) For a vectorbundle $\mathcal{E}$ on X, $c_i^{cr}(\mathcal{E}) \in F^i H^{2i}(X^{crys}/R_V, \mathcal{O}_X)$ maps to

$$v_X(c_i^{cr}(\mathcal{E})) = \beta^{\otimes i} \cdot c_i^{et}(\mathcal{E}) \in H^{2i}((X^0 \otimes_V \bar{K})^{et}, \mathbb{Z}/p^s\mathbb{Z}(i)) \otimes F^i B^+_{cr}(\bar{V}).$$

ii) Assume $i : Y \hookrightarrow X$ is a closed immersion, such that locally in the étale topology $(X, Y) = X_1 \times (\mathbb{A}^t, \mathcal{O})$. Then $v_X \circ i_*^{cr} = \beta^{\otimes t} \cdot i_*^{et} \circ v_Y$, as transformations

$$\mathbb{R}\Gamma(Y^{cr}, \mathcal{E} \otimes J) \to \mathbb{R}\Gamma(X^{cr}, \mathbb{L} \otimes B^+_{cr}(\bar{J}))$$

Proof. For i) it suffices to treat $c_1(\mathcal{L})$ for line-bundles $\mathcal{L}$. Both Chern-classes are boundaries of the class of $\mathcal{L}$ in $H^1(X, \mathcal{O}_X^*)$, with respect to either

$$0 \to F^1(\mathcal{O}_X^{crys}) \xrightarrow{\exp} \mathcal{O}_X^{crys,*} \to \mathcal{O}_X^* \to 0$$

respectively

$$0 \to \mathbb{Z}_p(1) \to \mathcal{R}(\bar{\mathcal{O}})^* \to \hat{\bar{\mathcal{O}}}^* \to 0$$

However evaluated on $B_{cr}^+(\bar{\mathcal{O}})$ the second sequence maps to the first via β respectively the map

$$\mathcal{R}(\bar{O})^* \to A_{\inf}(\bar{\mathcal{O}})^* \to B_{crys}^+(\bar{\mathcal{O}})^*,$$

hence the assertion.

Similarly by the fundamental formula i) is reduced to the case of a divisor, that is $t = 1$. As before we denote by $\tilde{X}$ X with the modified toroidal structure where Y is added to the boundary. There exists an étale covering of X by affines $Spec(R)$ such that either $Spec(R)$ lies over $X \setminus Y$ and has an étale toric parametrisation $Spec(R) \to \bar{T}_\lambda$, or $Spec(R) \to \bar{T}_{1,\lambda} \times \mathbb{A}^1$ is an étale parametrisation for the two possible toric structures (depending on whether $\{0\} \subseteq \mathbb{A}^1$ is added to the boundary). In the second case we also assume that R has enough units to define a good R_∞. For example $Spec(R)$ could lie above $\bar{T}_{1,\lambda} \times (\mathbb{A}^1 - \{1\})$. Let $X_0' = \tilde{X}_0' = \amalg Spec(R)$ denote the corresponding affine cover of X, $\tilde{X}_n' \to X_n'$ the $(n+1)$ -fold logarithmic fibre-products (in the two different log -structures),

$$\hat{\tilde{X}}_n' = Spf(\tilde{R}_n'), \hat{X}_n' = Spf(R_n')$$

affine formal lifts locally étale over $\hat{\bar{T}}^{n+1,\log}$ (respectively $(\bar{T}_1 \times \mathbb{A}^1)^{\wedge_{n+1,\log}}$, and derived power hulls $Spf(D(\tilde{R}_n')), Spf(D(R_n'))$. The crystalline cohomology $\mathbb{R}\Gamma(X^{crys}/R_V, \mathcal{E} \otimes J)$ can be represented by the total complex associated to the de Rham complexes $(\mathcal{E} \otimes J)(D(R_n')) \otimes \Omega_{R_n'}^{\cdot,\log}$, and similarly for $\tilde{X}$. However the $\hat{\tilde{X}}'$ are also log -smooth in the log -structure of X, i.e. without adding Y to the boundary (the logarithmic product $(\mathbb{A}^1, \{0\}^{n+1,\log} = \mathbb{A}^1 \times \mathbb{G}_m^n$ is smooth). Thus they may also be used to compute the crystalline cohomology of X. That is there are the subcomplexes

$$(\mathcal{E} \otimes J)(D(\tilde{R}_n')) \otimes \Omega_{\underset{\sim}{\tilde{R}_n'}}^{\cdot,\log} \subseteq (\mathcal{E} \otimes J)(D(\tilde{R}_n')) \otimes_{\tilde{R}_n'} \Omega_{\tilde{R}_n'}^{\cdot,\log}$$

whose associated total complex represents

$$\mathbb{R}\Gamma(X^{crys}/R_V, \mathcal{E} \otimes J).$$

The quotient represents $\mathbb{R}\Gamma(Y^{crys}/R_V, \mathcal{E} \otimes J)\{-1\}[-1]$, with connecting map i_*^{crys}.

Also the local parametrisations give an equation $f_0 \in R_0'$ for the Y-stratum. By definition of the logarithmic product $f_1 = pr_0^*(f_0)/pr_1^*(f_1) \in \tilde{R}_1^{\prime,*}$ is a unit. $d\log f_0$ is a 1-cochain with residue 1 in the de Rham complex $D(\tilde{R}_0') \otimes \Omega_{\tilde{R}_0'}^{\cdot,\log}$. Also the de Rham complex for Y can be identified with the quotient of the $(\mathcal{E} \otimes J)(D(\tilde{R}_n')) \otimes \Omega_{\underset{\sim}{\tilde{R}_n'}}^{\cdot,\log}$ obtained by setting $f_0 = df_0 = 0$. Thus to compute $i_*^{crys}(z)$ for a class

$z \in H^m(Y^{crys}/R_V, \mathcal{E} \otimes J)$, represent z by an m-cocycle $\bar{\lambda}$ in this quotient complex, lift to an m-cochaine λ in the complex for X, form the product $(d\log f_0, 0) \cup \lambda$ and apply to it the differential to get a class in $H^{m+2}(X^{crys}/R_V, \mathcal{E} \otimes J)$ representing $i_*^{crys}(z)$.

Now let us map to *é*tale cohomology. There is a diagram of topoi

$$\tilde{\mathcal{X}} \otimes_V \bar{K} \overset{j}{\hookrightarrow} \mathcal{X} \otimes_V \bar{K} \overset{i}{\longleftarrow} \mathcal{Y} \otimes_V \bar{K}.$$

We know that

$$j_*(\mathbb{L} \otimes B_{cr}^+(\tilde{\bar{\tilde{J}}})) \approx \mathbb{L} \otimes B_{cr}^+(\bar{J}))$$

$$\mathbb{R}^1 j_*(\mathbb{L} \otimes B_{cr}^+(\tilde{\bar{\tilde{J}}})) \approx i_*(\mathbb{L} \otimes B_{cr}^+(\bar{J}_Y))(-1),$$

and higher direct images

$$\mathbb{R}^\nu j_*(\mathbb{L} \otimes B_{cr}^+(\tilde{\bar{\tilde{J}}})) \quad (\nu \geq 2) \quad \text{or} \quad \mathbb{R}^\nu i_*(\mathbb{L} \otimes B_{cr}^+(\bar{J}_Y)) \quad (\nu \geq 1)$$

are almost zero.

Next our de Rham resolutions define complexes L_X on $\mathcal{X}^0 \otimes_V \bar{K}$, $L_{\tilde{X}}$ on $\tilde{\mathcal{X}}^0 \otimes_V \bar{K}$ and L_Y on $\mathcal{Y}^0 \otimes_V \bar{K}$, acyclic in degrees $\neq 0$, with an injection $L_X \hookrightarrow j_* L_{\tilde{X}}$. We also get a map $i_* L_Y \leftarrow (j_* L_{\tilde{X}}/L_X)[-1]$. Locally these can be described as follows: Assume that an *é*tale $Spec(R) \to X$ lifts to $Spec(R) \to X'$. Then L_X evaluated on $\bar{R}$ is made up from the de Rham complexes

$$\mathcal{E} \otimes J(D(B_{cr}^{+,0}(\bar{R}) \otimes^{\log} \tilde{R}_n') \otimes \Omega^{\cdot,\log}_{\underset{\sim}{\tilde{R}_n}}$$

where $B_{cr}^{+,0}(\bar{R})$ = preimage of $R \subseteq \overset{\wedge}{\bar{R}}$ in $B_{cr}^+(\bar{R})$, and "D" denotes PD-hull along the graph of $Spec(R) \to X_n'$. Similarly $\tilde{L}_X$ on $\tilde{\bar{\tilde{R}}}$ is represented by

$$\mathcal{E} \otimes J(D(B_{cr}^{+,0}(\tilde{\bar{\tilde{R}}}) \tilde{\otimes}^{\log} \tilde{R}_n') \otimes \Omega^{\cdot,\log}_{\tilde{R}_n}$$

with $\tilde{\otimes}^{\log}$ denoting log-product in the $\tilde{X}-\log$ -structure. Over $\bar{R}$ $\quad j_*\tilde{L}_X$ is locally given by the invariants under $\underset{\sim}{\Delta} = \ker(\tilde{\Delta} \to \Delta)$.

The map $j_*\tilde{L}_X \to i_* L_Y[1]$ is defined by choosing an extension $\tilde{\bar{\tilde{R}}} \to \bar{R}_Y$ of $R \to R_Y$, which is possible. Then the induced $B_{cr}^+(\tilde{\bar{\tilde{R}}}) \to B_{cr}^+(\bar{R}_Y)$ and the residue give a map. (One can check easily that things are independent of choices, as evaluation of crystals on PD-thickenings). With this definitions obviously the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & (\text{DR-complex for } X) \arrow[r] \arrow[d] & (\text{DR-complex for } \tilde{X}) \arrow[r, "Res"] \arrow[d] & (\text{DR-complex for } Y)[-1] \arrow[r] & 0 \\
0 \arrow[r] & \Gamma(\mathcal{X}^0 \otimes_V \bar{K}, L_X) \arrow[r] & \Gamma(\tilde{\mathcal{X}}^0 \otimes_V \bar{K}, L_{\tilde{X}}) \arrow[r] & \Gamma(\mathcal{Y}^0 \otimes_V \bar{K}, L_Y)[-1] \arrow[r] & 0
\end{tikzcd}$$

strictly commutes.

Next first assume $\mathbb{L} = \mathbb{Z}/p^s\mathbb{Z}, \mathcal{E} = \mathcal{O}_X/p^s\mathcal{O}_X, J = \mathcal{O}_X$. Then the canonical element $1 \in H^0(Y^{crys}/R_V, \mathcal{O}/p^s\mathcal{O})$ maps to an element of $\Gamma(\tilde{\mathcal{X}}^0 \otimes_V \bar{K}, L^1_{\tilde{X}})/\Gamma(\mathcal{X}^0 \otimes_V \bar{K}, L^1_X)$, and thus to a global section of

$$\Gamma(\mathcal{X}^0 \otimes_V \bar{K}, j_* L^1_{\tilde{X}}/L^1_X),$$

and a global section of

$$\underline{H}^1(j_* L^1_{\tilde{X}}/L^1_X) = \underline{H}^1(j_* L^1_{\tilde{X}}).$$

Mapping to an injective resolution of $B^+_{cr}(\bar{\mathcal{O}}_{\tilde{X}})/p^s B^+_{cr}(\bar{\mathcal{O}}_{\tilde{X}})$ we get a global section of

$$\mathbb{R}^1 j_* B^+_{cr}(\bar{\mathcal{O}}_{\tilde{X}})/p^s \cdot B^+_{cr}(\bar{\mathcal{O}}_{\tilde{X}}) \approx i_*(B^+_{cr}(\bar{\mathcal{O}}_Y)/p^s \cdot B^+_{cr}(\bar{\mathcal{O}}_Y))(-1).$$

We claim that this is given by $\beta \in B^+_{cr}(\bar{V})(-1)$:

This is an entirely local assertion, thus can be checked over strict henselisations $R = \mathcal{O}^{sh}_{X,x}$. Especially we have a lift to $Spec(R) \to X'$. Let $f_0 \in \hat{R}'_0$ denote the global equation for Y on $\hat{X}'_0$, $f_1 = pr_0^* f_0 / pr_1^* f_0 \in \hat{R}'^*_1$. Then $(\frac{df_0}{f_0}, 0)$ defines a 1-cochain in the de Rham complex for $\tilde{X}$, with boundary $(0, \frac{df_1}{f_1}, 0)$ in the de Rham complex for X. It thus defines a cocycle in the quotient, with residue one. Hence we have to find its image in $B^+_{cr}(\bar{R}_Y)/p^s \cdot B^+_{cr}(\bar{R}_Y)[-1]$. Thus consider the section $(d\log(1 \otimes f_0), 0)$ of $L^1_{\tilde{X}}$, where

$$1 \otimes f_0 \in D(B^+_{cr}(\tilde{\bar{R}}) \otimes \hat{\tilde{R}}_0)$$

We remark that $1 \otimes f_1$ is a 1-unit in $D(B^+_{cr}(\bar{R}) \otimes \hat{\tilde{R}}'_1)$, thus its logarithm g_1 defines a section of L^1_X. We thus also may consider $(d\log(1 \otimes f_0), g_1)$ which is a 1-cocycle for $L_{\tilde{X}}$, and $\tilde{\Delta}$ -invariant. We want to compute its class in cohomology, at least after mapping to $j_* \tilde{I}^{\cdot}$.

For this we not that also $(1 \otimes f_0)/([f_0] \otimes 1) = \exp(g_0)$ is a 1-unit, where $[f_0] \in A_{crys}(\tilde{\bar{R}})$ is defined by an element of $\mathcal{R}(\tilde{\bar{R}})$ made up from p-power roots of f_0. Moreover the differential of g_0 is $(d\log(1 \otimes f_0), g_1)$. However g_0 is not $\tilde{\Delta}$ -invariant: Namely any $\sigma \in \tilde{\Delta}$ multiplies $[f_0]$ by $\exp(\beta(\bar{\sigma}))$, with $\bar{\sigma} \in \mathbb{Z}_p(1)$ describing the action of σ on p-power roots of f_0. Thus $\sigma(g_0) = g_0 - \beta(\bar{\sigma}) \otimes 1$, and the obstruction to make g $\tilde{\Delta}$ -invariant is given by the 1-cocycle $\sigma \mapsto \beta(\bar{\sigma})$. Restriction to inertia $(\cong \mathbb{Z}_p(1))$ at Y then implies the assertion.

For general $\mathcal{E}$ and J we need cup-products. The de Rham and $\check{C}$ech-complexes admit strictly associative differential products (in the $\check{C}$ech-setting, $(f \cup g)_{0\ldots a+b} = f_{0\ldots a} \cup g_{a\ldots b}$ for f in degree a, g in degree b). This defines such products on $L_X(\mathcal{O}), L_{\tilde{X}}(\mathcal{O}), L_Y(\mathcal{O})$, and makes $L_X(\mathcal{E} \otimes J)$ etc. to differentially graded modules over them. Furthermore the elements $pr_0^* f_0 \in \hat{\tilde{R}}'_n$ generate an invertible ideal $\langle f_0 \rangle$ invariant under all simplicial maps, and thus a differential ideal $\langle f_0, df_0 \rangle \subseteq L_X(\mathcal{X})$. The quotient $L_X(\mathcal{O})/\langle f_0, df_0 \rangle$ naturally maps to $i_* L_Y(\mathcal{O})$ induced from $B^+_{cr}(\bar{\mathcal{O}}_X) \to i_* B^+_{cr}(\bar{\mathcal{O}}_Y)$. Next we have defined a class $\psi \in \Gamma(\mathcal{X}^0 \otimes_V \bar{K}, j_* L^1(\mathcal{O}_{\tilde{X}}]/L^1(\mathcal{O}_X))$, the image of $\frac{df_0}{f_0}$. Under the multiplication of $L^{\cdot}(\mathcal{O}_X)$ on $j_* L(\mathcal{O}_{\tilde{X}})/L(\mathcal{O}_X)$ this class is annihilated by $\langle f_0, df_0 \rangle$. Thus cup-product defines a map

$$\psi\cup : L_X(\mathcal{E}\otimes J)/\langle f_0, df_0\rangle \to j_* L_{\tilde{X}}(\mathcal{X}\otimes J)/L_X(\mathcal{E}\otimes J)[-1]$$

Furthermore in the derived category we had defined a canonical map

$$j_* L_{\tilde{X}}(\mathcal{E}\otimes J)/L_X(\mathcal{E}\otimes J)[-1] \to \mathbb{R}i_*(\mathbb{L}\otimes B^+_{cr}(\bar{J})),$$

which is compatible with the action of cup-products from $L_X/\langle f_0, df_0\rangle$, acting on the right hand side via

$$L_X(\mathcal{E}\otimes J)/\langle f_0, df_0\rangle \to i_* L_Y(\mathcal{E}\otimes J) \to \mathbb{R}i_*(\mathbb{L}\otimes B^+_{cr}(\bar{\mathcal{O}}_Y))$$

Remark. The maps $L_X(\mathcal{E}\otimes J) \to \mathbb{L}\otimes B^+_{cr}(\bar{J}_X)$ are a priori only defined in the derived category. However we may replace $B^+_{cr}(\bar{\mathcal{O}}_X)$ by $B^+_{cr}(\bar{\mathcal{O}}_X)\otimes_{L^0_X(\mathcal{O})} L_X(\mathcal{O})$ (which is quasi-isomorphic) and $\mathbb{L}\otimes B^+_{cr}(\bar{J}_X)$ by $(\mathbb{L}\otimes B^+_{cr}(\bar{J}_X))\otimes_{L^0_X(\mathcal{O})} L_X(\mathcal{O})$ to represent those by maps of DG-algebras respectively DG-modules. Then compatability with cup-products follows automatically, for example by computing cohomology via hypercoverings.

As ψ maps to β we see that

$$\begin{array}{ccc}
L_X(\mathcal{E}\otimes J)/\langle f_0, df_0\rangle & \xrightarrow{\psi\cup} & j_* L_{\tilde{X}}(\mathcal{E}\otimes J)/L_X(\mathcal{E}\otimes J)[+1] \\
\downarrow & & \downarrow \\
i_* L_Y(\mathcal{E}\otimes J) & & \mathbb{R}j_*(\mathbb{L}\otimes B^+_{cr}(J_{\tilde{X}}))/\mathbb{L}\otimes B^+_{cr}(\bar{J}_X)[+1] \\
\downarrow & & \downarrow \\
i_*(\mathbb{L}\otimes B^+_{cr}(\bar{J}_Y)) & \xrightarrow{\beta\circ} & i_*(\mathbb{L}\otimes B^+_{cr}(\bar{J}_Y)(-1)
\end{array}$$

commutes up to homotopy. Now i_*^{crys} is defined by the connecting map for

$$0 \to DR_X(\mathcal{E}\otimes J) \to DR_{\tilde{X}}(\mathcal{E}\otimes J) \to DR_Y(\mathcal{E}\otimes J)[-1] \to 0.$$

This maps to global sections in the exact sequence

$$0 \to L_X(\mathcal{E}\otimes J) \to L_{\tilde{X}}(\mathcal{E}\otimes J) \to j_* L_{\tilde{X}}(\mathcal{E}\otimes J)/L_X(\mathcal{E}\otimes J) \to 0$$

or also

$$0 \to L_X(\mathcal{E}\otimes J) \to L^*_{\tilde{X}}(\mathcal{E}\otimes J) \to L_X(\mathcal{E}\otimes J)/\langle f_0, df_0\rangle[-1] \to 0,$$

the pullback via

$$\psi\cup : L_X(\mathcal{E}\otimes J)/\langle f_0, df_0\rangle[-1] \to j_* L_{\tilde{X}}(\mathcal{E}\otimes J)/L_X(\mathcal{E}\otimes J),$$

which in turn maps to

$$0 \to \mathbb{L}\otimes B^+_{cr}(\bar{J}_X) \to \mathbb{R}j_*(\mathbb{L}\otimes B^+_{cr}(\bar{J}_{\tilde{X}})) \to i_*(\mathbb{L}\otimes B^+_{cr}(\bar{J}_Y))[-1](-1) \to 0,$$

where the maps on the first two terms are the canonical comparison maps, while on the third we have the comparison-map multiplied by β. However as this last sequence defines i_*^{et}, we get indeed $v_Y \circ i_*^{crys} = \beta \cdot i_*^{et} \circ v_X$. Thus the theorem is shown.

2. Corollary. For the trace-forms on $H^{2d}(X/R_V, J_X)$ respectively $H_!^{2d}((X^0 \otimes_V \bar{K})^{et}, \mathbb{Z}_p)$ we have

$$tr^{et} \circ v_X = \beta^d \cdot tr^{crys}$$

Proof. Check on the class of a smooth point. Also toric blow-ups $\tilde{X} \to X$ induce isomorphisms on crystalline cohomology. This is true on de Rham cohomology, thus modulo $F^1(R_V) \subseteq R_V$, and follows in general because $F^1(R_V)$ consists of topologically nilpotent elements. Hence:

3. Corollary. Assume we are given functorial transformations

$$\begin{aligned}
&v_{\mathcal{E}} : \mathcal{E}(B_{cris}^{+,0}(R)) \to \mathbb{L} \otimes B_{cris}^{+}(R)\\
&v_{\mathcal{E}^\vee} : \mathcal{E}^\vee(B_{crys}^{+,0}(R)) \to \mathbb{L}^\vee \otimes B_{crys}^{+}(R)\\
&(\mathcal{E}^\vee = Hom(\mathcal{E}, \mathcal{O}_X/p^s\mathcal{O}_X), \mathbb{L}^\vee = Hom(\mathbb{L}, \mathbb{Z}/p^s\mathbb{Z})),
\end{aligned}$$

with $\langle v_{\mathcal{E}}(e), v\mathcal{E}^\vee(e^\vee)\rangle = \beta^{\otimes m} \cdot v\langle e, e^\vee\rangle$ for some fixed m.

Then

$$v_X : \mathbb{R}\Gamma(X^{crys}/R_V, \mathcal{E} \otimes J) \otimes_{R_V} B_{crys}^{+}(V) \to \mathbb{R}\Gamma_{(!)}((X^0 \otimes_V \bar{K})^{et}, \mathbb{L}) \otimes B_{cris}^{+}(V)$$

has an inverse up to $\beta^{\otimes(d+m)}$, that is composition either way is multiplication by $\beta^{\otimes(d+m)}$

Proof. The "inverse" is the adjoint of the map v_X for $\mathcal{E}^\vee$. One has to compare the classes of the diagonals, passing to $\tilde{\Delta} : \tilde{X} \to (\widetilde{X \times X})$. (The class of the diagonal $c_\delta \in H_{(!)}^{2d}(\widetilde{X \times X}, \mathbb{R}\underline{Hom}(pr_2^*\mathcal{E}, pr_1^*\mathcal{E}))$ is the image of a class in $H_{(!)}^{2d}(\widetilde{X \times X}, \underline{Hom}(pr_2^*\mathcal{E}, pr_1^*\mathcal{E}))$, the direct image of the identity map on $\mathcal{E}$).

Remarks.

1) If $\mathcal{E}$ is in fact a crystal in the crystalline topos of X relative V_0 (that is one has integrable connections $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_{R'/V_0}^{1,\log} = \mathcal{E} \otimes Lie(T)^\vee$), the crystalline cohomology $\mathbb{R}\Gamma(X^{crys}/R_V, \mathcal{E})$ inherits a logarithmic connection (as a perfect complex). Especially all cohomology-groups have such a connection. Also the Galois-group $Gal(\bar{R}/R)$ acts on $\mathbb{R}\Gamma(X^{crys}/R_V, \mathcal{E} \otimes J) \otimes_{R_V} B_{crys}^{+}$, although the map $R_V \to B_{crys}^{+}$ is not Galois-linear. It suffices that this holds modulo $F^1(B_{crys}^{+})$. Finally if $\mathbb{L}$ is defined over $K, Gal(\bar{R}/R)$ also operates on $\mathbb{R}\Gamma_{(!)}((X^0 \otimes_V \bar{R})^{et}, \mathbb{L})$. And if the local comparison-maps

$$\mathcal{E}(B_{crys}^{+,0}(\bar{R})) \to \mathbb{L} \otimes B_{crys}^{+}(\bar{R})$$

also respect the action of $Gal(\bar{R}/R)$ the comparison-map v_X is $Gal(\bar{K}/K)$ -linear.

2) Everything works in the filtered context: If $\mathcal{E}$ is a filtered crystal, and the comparison

$$\mathcal{E}(B_{cr}^{+,0}(R)) \to \mathbb{L} \otimes B_{cr}^{+}(R)$$

preserves filtrations, that is it sends $F^i(\mathcal{E}(B_{cr}^{+,0}(R)))$ to $\mathbb{L} \otimes F^i(B_{cr}^{+}(R))$, the comparison-map

$$\mathbb{R}\Gamma(X^{crys}/R_V, \mathcal{E} \otimes J) \to \mathbb{R}\Gamma_{(!)}((X^0 \otimes_K \bar{V})^{et}, \mathbb{L}) \otimes B_{cr}^{+}(V)$$

exists in the filtered derived catgeory, and similarly for inverese "up to β^{d+e}".

3) If $\mathcal{E}$ is a Frobenius-cyrstal (in the log-sense) then $\mathcal{E}(B_{cr}^{+,0}(R)) \otimes_{B_{cr}^{+,0}(R)} B_{cr}^{+}(R)$ has a Frobenius-action ϕ, as B_{cr}^{+} has a canonical Frobenius-lift (which is a logarithmic: $\phi([\underline{\mu}]) = [\underline{\mu}]^p$, for $\mu \in \sigma^\vee \cap L^\vee$). It then makes sense to require that the comparison-map to $\mathbb{L} \otimes B_{cr}^{+}(R)$ is Frobenius-linear. In this case

$$v_X : \mathbb{R}\Gamma(X^{crys}/R_V, \mathcal{E} \otimes J) \to \mathbb{R}\Gamma_{(!)}((X^0 \otimes_K \bar{V})^{et}, \mathbb{L}) \otimes B_{cr}^{+}(V)$$

will also preserve Frobenius (note that $Frob^*(J) \subseteq J$, as J is locally generated by elements $\mu \in \sigma^\vee \cap L^\vee$ with $Frob(\mu) = \mu^p \cdot$ unit). Here the "Frobenius" on R_V is Frobenius on V_0, and sends t to t^p.

4) In many cases (for example constant coefficients) one can pass to $s \to \infty$ to obtain a $\mathbb{Z}_p$ -theory and then also a $\mathbb{Q}_p$ -theory. In more generality this is done in the context of convergent isocrystals (see [Bo], [O]):

Our base will be the "open unit disk": That is for each $\varepsilon = \frac{1}{n}$ consider the algebra $R_{V,\varepsilon} = V_0[t, t^n/p]^\wedge$. ($\wedge = p$ -adic completion). For $n >> 0\, R_{V,\varepsilon} \subseteq R_V, R_{V,\varepsilon}$ has a toroidal or log -structure generated by t. An affine logarithmic enlargement (we do not need others) is given by an $R_{V,\varepsilon}$ -algebra A (some $\varepsilon > 0$) with a log-structure (étale locally $Spec(A) \to \bar{T}, R_\varepsilon$ -linear), and ideal $I \subset A$ such that A is I-adically complete and such that some power of I is contained in pA, and an R_ε -linear log-map $Spec(R/I) \to X$. The definition of maps is the obvious one. A convergent (logarithmic) isocrystal associates to any enlargement A a finitely generated projective $A[\frac{1}{p}]$ -module $\mathcal{E}(A)$, with $\mathcal{E}(B) = B \otimes_A \mathcal{E}(A)$ for maps $A \to B$. Similarly filtered convergent isocrystals associate filtered modules, $A[\frac{1}{p}]$ itself being filtered by powers of I.

In the usual way (compare [Fa3], IV.e.) one defines the cohomology $\mathbb{R}\Gamma(X^{crys}/R_\varepsilon, \mathcal{E} \otimes J)$. I do not know whether this is in general representable by a perfect complex over $R_\varepsilon[\frac{1}{p}]$, satisfying Poincaré-duality. However for Frobenius-crystals ($\Phi^*\mathcal{E} \cong \mathcal{E}$) one can at least show that Frobenius induces a quasi-isomorphism on cohomology, that the special fibre at $t = 0$ is representable by a perfect complex $\mathbb{K}_0$ (i.e. has finite-dimensional cohomology), and that there are Φ -linear and parallel comparsion-maps

$$(H^i(X^{crys} \otimes_V k/V_0, \mathcal{E} \otimes J) \otimes_{K_0} R_\varepsilon[\frac{1}{p}], d + N) \to (H^i(X^{crys}/R_\varepsilon, \mathcal{E} \otimes J), \bigtriangledown),$$

where $N = Res(t\frac{d}{dt})$ is the endomorphism defined by the logarithmic connection, satisfying $\Phi \circ N = pN \circ \Phi$ (and thus N is nilpotent): For the finite-dimensionality one constructs integral lattices and proceeds as in [Fa7]. The comparison-map is constructed as in [Fa4]: Start with any lift of the identity $mod\, t$. Then transform it by powers of Frobenius and pass to the limit.

I expect the comparison-map to be isomorphisms. This holds if $H^i(X^{crys}/R_\varepsilon, \mathcal{E} \otimes J)$ is finitely generated over $R_\varepsilon[\frac{1}{p}]$, and for example if $\mathcal{E}$ comes from a usual crystal (for example if $\mathcal{E}$ = constant). Then we also have Poincaré-duality (check in the fibre at $t = \pi$), and the usual construction of the diagonal-class. In any case the appendix contains more details on what can be shown and what not.

Define $B^+_{crys,\varepsilon}(\bar{R}) = p$ -adic completion of $A_{inf}(\bar{R})[\frac{\xi^n}{p}]$ (as before, $\varepsilon = \frac{1}{n}$). This is an enlargement of R, thus $\mathcal{E}(B^+_{crys,\varepsilon}(\bar{R}))$ is defined. If we are given a smooth $\mathbb{Q}_p$ -adic sheaf $\mathbb{L}$ on $(X^0 \otimes_V \bar{K})^{et}$ and functorial comparison-maps

$$\mathcal{E}(B^+_{crys,\varepsilon}(\bar{R})) \to \mathbb{L} \otimes B^+_{crys,\varepsilon}(\bar{R})$$

respecting Galois-actions and compatible for varying ε's) we obtain a comparison-map

$$v_X : \mathbb{R}\Gamma(X^{crys}/R_\varepsilon, \mathcal{E} \otimes J) \otimes B^+_{crys,\varepsilon}(\bar{V}) \to \mathbb{R}\Gamma_{(!)}((X^0 \otimes_V \bar{K})^{et}; \mathbb{L}) \otimes B^+_{crys,\varepsilon}(\bar{V}),$$

with the usual properties concerning duality, characteristic classes, action of $Gal(\bar{K}/K)$ and of Frobenius, filtrations. Also for a Frobenius-crystal we can compose with the comparison-maps to obtain

$$v_X : \mathbb{R}\Gamma(X^{crys} \otimes_V k/V_0, \mathcal{E} \otimes J) \otimes_{K_0} B^+_{crys,\varepsilon}(\bar{V}) \to \mathbb{R}\Gamma_{(!)}((X^0 \otimes_V \bar{K})^{et}, \mathbb{L}) \otimes B^+_{crys,\varepsilon}(\bar{V})$$

This will respect Galois-actions if the other maps do, where however on the left hand side the Galois-action is not just via the second factor but has to be twisted, because of the connection $d + N$. There is a 1-cocycle $Gal(\bar{K}/K) \to \mathbb{Z}_p(1)$

$$\sigma \mapsto \bar{\sigma}$$

describing the action on all p-power roots of π, or equivalently on $[\underline{\pi}] \in A_{\mathrm{inf}}(\bar{V})$. Then the Galois-action of σ on

$$H^i(X^{crys} \otimes_V k/V_0, \mathcal{E} \otimes J) \otimes B^+_{crys,\varepsilon}(\bar{V})$$

is defined as

$$\sigma \mapsto \exp(\beta(\bar{\sigma})N) \otimes \sigma$$

This corresponds to Fontaine's description: If $B^+_{st} = B^+_{cr}[u]$ with $Nu = 1(u =$ " $\log([\underline{\pi}]/\pi)$") then $(E_0 = H^i(X^{crys} \otimes_V k/V_0, \mathcal{E} \otimes J))$

$$E_0 \otimes B^+_{cr} \cong \ker(N \otimes 1 + 1 \otimes N \mid E_0 \otimes B_{st})$$

via $\exp(-N \otimes u)$, and as $\sigma(u) = u + \beta(\bar{\sigma})$ this intertwines Galois-actions.

5*. Appendix: Cohomology of Frobenius-isocrystals

As before X is *é*tale locally *é*tale over some $\bar{T}_\lambda$. The cohomology of convergent isocrystals can be defined via de Rham complexes on smooth local lifts, as it is done in crystalline cohomology. However one has to replace PD-hulls by open " $> \varepsilon$" -neighbourhoods, as follows:

Let $R_{>\varepsilon}$ respectively $R_{\geq\varepsilon}$ denote the rings of formal powerseries convergent in the open (respectivly closed) disk of radius p^ε, that is formal powerseries $f = \sum_{n\geq 0} a_n t^n$ with ($v_p = p$ -adic valuation).

$$a_n \in K_0, \liminf_{n\to\infty}(v_p(a_n)n) \geq -\varepsilon$$

(respectively $a_n \in K_0, \liminf_{n\to\infty}(v_p(a_n) + n\varepsilon) = \infty$). Obviously $R_{<\varepsilon} = \bigcap_{\delta>\varepsilon} R_{\geq\delta}$. Also if $\varepsilon = \frac{a}{b}$ is rational (which we assume from now on)

$$R_{\geq\varepsilon} = K_0 \otimes_{V_0} (V_0[t]/(t^b - p^a))^\wedge.$$

This is not true for $R_{>\varepsilon}$, but unfortunately we shall need this ring because it satisfies the Poincar*é*-lemma:

The de Rham complex

$$R_{>\varepsilon} \xrightarrow{d} R_{>\varepsilon} \cdot dt$$

is quasi-isomorphic to K_0.

Next to define and compute cohomology choose ε small enough such that π has valuation $> \varepsilon$. Then $X \otimes_V K$ is a scheme over $R_{>\varepsilon}$ and $R_{\geq\varepsilon}$, and in fact X itself is a scheme over $R^0_{\geq\varepsilon} = V_0[t]/(t^b - p^a)(\varepsilon = \frac{a}{b})$. If X embeds into a p-adic affine formal scheme Y over $\bar{\bar{T}} \otimes_{V_0} R^0_{\geq\varepsilon}$, respecting toroidal structures, let $J_X \subseteq \mathcal{O}_Y$ denote the ideal of X. For $\delta = \frac{c}{d}$, $0 < \delta < \varepsilon$, define the closed δ -tube of X in Y as the affine Y-scheme with algebra $\mathcal{O}_Y[J^d_X/p^c]$. Call it $D_{X,\geq\delta}(Y)$. Define the open δ -tube as the *ind*-scheme

$$"\varinjlim_{\delta'>\delta}" D_{X_1\geq\delta'}(Y) = D_{X,>\delta}(Y).$$

As each $D_{X_1\geq\delta'}(Y)$ is an enlargement of $X/R_{\geq\varepsilon}$ we can evaluate $\mathcal{E}$ on it, and form the de Rham complex $\mathcal{E}(D_{X_1\geq\delta'}(Y)) \otimes \Omega^{\cdot,\log}_Y$. Then define $\mathbb{R}\Gamma(X/R_{\geq\varepsilon}, \mathcal{E})$ as represented by the projective limit ($\delta' \to \delta+$) of the de Rham complexes. Similarly we define $\mathbb{R}\Gamma(X/R_{\geq\varepsilon}, \mathcal{E} \otimes J)$, and if X cannot be embedded globally we use local embeddings any hypercoverings. One checks that the result is independant of all choices, the key being the Poincar*é*-lemma which allows to compare two local smooth embeddings (and there we unfortunately need open disks).

Next we first treat the case that $X = Spec(R)$ is affine, and lifts to an *é*tale formal scheme over $\hat{\bar{T}}$. For each $\varepsilon > 0$ we then obtain by pushout formal p-adic schemes $Spf(S_{\geq\varepsilon})$ over $R^0_{\geq\varepsilon}$, which are enlargements of X. $Spf(S_\varepsilon)$ is formally *é*tale over $\bar{T} \otimes_{V_0[t]} R^0_{\geq\varepsilon}$. The de Rham complex computing the cohomology of X is $\mathcal{E}(S_\varepsilon)\otimes$ (pullback of $J \otimes \Omega^{\cdot,\log}_{\bar{T}_\lambda}$). Also $S_{\geq\varepsilon}$ has a canonical Frobenius-lift ϕ induced from $\bar{T}$ (via $\mu \mapsto \mu^p$ for $\mu \in L$). Thus a Frobenius Φ on cohomology, induced from Frobenius Φ^* on $\mathcal{E}(S_\varepsilon)$ as well as $d\phi$ on $\Omega^{\cdot,\log}_{\bar{T}_\lambda}$.

4. Lemma. Φ^* induces an isomorphism on $H^*(X/R_\varepsilon, \mathcal{E}\otimes J)$.

Proof (following Mazur, Ogus). Let $p^{-\cdot}\cdot\Omega^{\cdot,\log}_{\bar{T}_\lambda}$ denote the de Rham complex with $\Omega^{q,\log}$ replaced by $p^{-q}\cdot\Omega^{q,\log}$. Then ϕ^* induces a quasi-isomorphism

$$p^{-\cdot}\Omega^{\cdot,\log}_{\bar{T}_\lambda}\otimes J(S_\varepsilon)\otimes_\phi R^0_\varepsilon \to \Omega^{\cdot,\log}_{\bar{T}_\lambda}\otimes J(S_\varepsilon),$$

even modulo p^s for all $s\geq 0$. It suffices to prove this modulo p, and for $\bar{T}$ itself (étale localisation). Then the left hand side is the direct sum, over all $\mu\in L^\vee\cap\sigma^\vee$ such that $\mu-\lambda\notin\sigma^\vee$, and $\mu(\rho_i)>0$ for certain extremal ρ_i (with $\lambda(\rho_i)=0$), of $R^0_\varepsilon/pR^0_\varepsilon\otimes\Lambda^\cdot\ker(\lambda)^*$, with trivial differential. On the right we have the same modules but with differential equal to exterior multiplication by μ. The map between both sides is induced from $\mu\mapsto p\mu$. Now on the right the μ -eigenspace has trivial cohomology unless $\mu\in p\cdot\ker(\lambda)^*, \mu=p\cdot\mu_1$. Then $\mu(\rho_i)>0\Leftrightarrow\mu_1(\rho_i)>0$, and $\mu\in\sigma^\vee\Leftrightarrow\mu_1\in\sigma^\vee$. Also $\mu-\lambda\notin\sigma^\vee$ implies $\mu_1-\lambda\notin\sigma^\vee$. If conversely $\mu-\lambda\in\sigma^\vee$, then $\mu_1-\frac{\lambda}{p}=(\mu_1-\lambda)+\frac{p-1}{p}\lambda\in\sigma^\vee$. By the condition on σ (i.e. $X\otimes_V k$ is geometrically reduced) this implies $\mu_1-\lambda\in\sigma^\vee$. Hence the assertion.

Now choose a system of generators e_α for $\mathcal{E}(S_\varepsilon)$. Then

$$\nabla(e_\alpha)=\sum_\beta e_\beta\otimes w_{\alpha\beta}$$

with certain $w_{\alpha\beta}\in S_\varepsilon[\frac{1}{p}]\otimes\Omega^{1,\log}_{\bar{T}_\lambda}$. Replacing e_α by $\Phi^N(e_\alpha)$ with $N>>0$ we may assume that all $w_{\alpha\beta}\in p\cdot S_\varepsilon\otimes\Omega^{1,\log}_{\bar{T}_\lambda}$. Then the e_α span an S_ε -submodule $\mathcal{E}^0(S_\varepsilon)$, and there is a welldefined map

$$\Phi^*:\mathcal{E}^0(S_\varepsilon)\otimes p^{-\cdot}\cdot\Omega^{\cdot,\log}_{\bar{T}_\lambda}\otimes J(S_\varepsilon)\to\mathcal{E}^0(S_\varepsilon)\otimes\Omega^{\cdot,\log}_{\bar{T}_\lambda}\otimes J(S_\varepsilon)$$

which is a quasi-isomorphism modulo p and thus in general. Composing with Frobenius $\Phi^*(\mathcal{E}_0)\to\mathcal{E}_0$ and inverting p proves the lemma.

Remarks. The role of the base $R_{\geq\varepsilon}$ is rather auxiliary. One can improve the result by noting that Frobenius induces a finite flat map from $R_{\geq\varepsilon}$ to $R_{\geq\varepsilon/p}$ (which has basis $\{1,t,\ldots,t^{p-1}\}$ over $\rho(R_{\geq\varepsilon})$). Then also we have an isomorphism

$$\Phi^*:H^i(X/R_{\geq\varepsilon},\mathcal{E}\otimes J)\otimes_{R_{\geq\varepsilon}\phi}R_{\geq\varepsilon/p}\xrightarrow{\sim}H^i(X/R_{\geq\varepsilon/p},\mathcal{E}\otimes J).$$

Also we might use the base R_V, if $\mathcal{E}$ is actually a crystal (in the usual crystalline topos).

5. Corollaries.

a) For any X (not necessarily affine or lifting to $\hat{\bar{T}}$) Frobenius induces isomorphisms

$$\Phi^*:H^i(X/R_{\geq\varepsilon},\mathcal{E}\otimes J)\otimes_{R_{\geq\varepsilon}\phi}R_{\geq\varepsilon}\xrightarrow{\sim}H^i(X/R_{\geq\varepsilon},\mathcal{E}\otimes J)$$
$$H^i(X/R_{\geq\varepsilon},\mathcal{E}\otimes J)\otimes_{R_{\geq\varepsilon}\phi}R_{\geq\varepsilon/p}\xrightarrow{\sim}H^i(X/R_{\geq\varepsilon/p},\mathcal{E}\otimes J)$$

b) Suppose $\mathcal{E}$ is given by an actual crystal. Then

$$\Phi^* : H^i(X/R_V, \mathcal{E} \otimes J) \otimes^{\mathbb{L}}_{R_V \phi} R_V \to H^i(X/R_V, \mathcal{E} \otimes J)$$

is an isogeny, that is kernel and cokernel are annihilated by some fixed power p^N.

These follow by cohomological descent, using an *é*tale covering of X by good affines.

Next we show that $\mathcal{E}$ always satisfies the assumptions in b), that is it can be defined by a crystal $\mathcal{E}^0$. For this choose an affine *é*tale covering $X' = Spec(R') \to X$ with $X' \to \bar{T}_\lambda$ *é*tale and lift to $\hat{X}' \to \hat{\bar{T}}$. Let $\hat{R}'$ define the p-adic completion of $\hat{X}' \otimes_{V_0[t]} R_V, \hat{R}''$ = completed PD-hull of $X' \times_X X' \hookrightarrow Spf(\hat{R}' \hat{\otimes}_{R_V} \hat{R}'), \hat{R}'''$ etc. as usual. The $\hat{R}^{(m)}$ form a cosimplicial ring with Frobenius ϕ, and all transition-maps flat (modulo any fixed power p^s, that is the sense of formal p-adic schemes). They also define affine enlargements of X. We can choose a finitely generated $\hat{R}'$ -submodule $\mathcal{E}_0^* \in \mathcal{E}(\hat{R}')$ spanning $\mathcal{E}(R')$, and stable under $p^N \cdot \Phi$ (some fixed N) and $\bigtriangledown$. Then $pr_0^* \mathcal{E}_0^* \subseteq p^{-N} \cdot pr_0^* \mathcal{E}_0^*$ in $\mathcal{E}(\hat{R}'')$, after (if necessary) enlarging N. Then let

$$\begin{aligned} \mathcal{E}^0(\hat{R}') &= \{e \in \mathcal{E}_0^* : pr_1^*(e)) \in pr_0^*(\mathcal{E}_0^*)\} \\ &= \{e \in \mathcal{E}(\hat{R}') : pr_1^*(e) \in pr_0^*(\mathcal{E}_0^*)\} \end{aligned}$$

(use the section $\hat{R}'' \to \hat{R}'$). Then $p^N \cdot \mathcal{E}_0^* \subseteq \mathcal{E}_0^0(\hat{R}') \subseteq \mathcal{E}_0^*$ (agian enlarging N), $\mathcal{E}_0(\hat{R}')$ is stable under $p^N \cdot \Phi$ and $\bigtriangledown$, and

$$pr_1^*(\mathcal{E}_0(\hat{R}')) = pr_0^*(\mathcal{E}_0(\hat{R}')) \subseteq \mathcal{E}(\hat{R}'')$$

(direct check via simplicial identities, using flatness modulo p^N to compute in $\mathcal{E}(\hat{R}''')$). Thus $\mathcal{E}_0(\hat{R}')$ defines a crystal.

We use this as follows: For each $s > 0$ $\mathbb{R}\Gamma(X^{crys}/R_V, \mathcal{E}^0/p^s\mathcal{E}^0 \otimes J)$ is representable by a bounded complex over R_V/p^sR_V, and

$$p^N \cdot \Phi^* : \mathbb{R}\Gamma(X^{crys}/R_V, \mathcal{E}^0/p^s\mathcal{E}^0 \otimes J) \otimes^{\mathbb{L}}_{\phi} R_V \to \mathbb{R}\Gamma(X^{crys}/R_V, \mathcal{E}^0/p^s\mathcal{E}^0 \otimes J)$$

induces on cohomology maps with kernel and cokernel annihilated by p^{2N}, N sufficiently big but independant of s.

Next we consider the special fibre, that is the pushout via $R_V \to V_0$ (sending t to 0). This computes the cohomology $\mathbb{R}\Gamma((X \times_V k)^{crys}/V_0, \mathcal{E} \otimes J)$. The restriction of $\mathcal{E}^0$ defines a crystal on the special fibre, whose evaluation on $\hat{R}' \otimes_{V_0[t]} V_0 = \hat{R}'/t\hat{R}'$ is the tensor-product (over R_V) with R_0. If we divide by p-torsion we obtain a new crystal $\bar{\mathcal{E}}^0$ which is flat over V_0. Furthermore

$$\bar{\mathcal{E}}^0(\hat{R}'/t\hat{R}') \subseteq \mathcal{E}(\hat{R}'/t\hat{R}')$$

is contained in a finitely generated $\hat{R}'/t\hat{R}'$ -submodule (induced from the $\tilde{\mathcal{E}}^0$ used in the construction of $\mathcal{E}^0$), thus itself finitely generated. Hence its cohomology modulo t (given by the de Rham complex on $X \otimes_V k$ is finite over k. By the methods of

[Fa7] it follows that $\mathbb{R}\Gamma((X \otimes_V k)^{crys}/V_0, \bar{\mathcal{E}}^0 \otimes J)$ is representable by a perfect complex over k. Furthermore we claim that the kernel of $\mathcal{E}^0 \otimes_{R_V} V_0 \twoheadrightarrow \bar{\mathcal{E}}^0$ as well as all higher $Tor_i^{R_V}(\mathcal{E}^0, V_0)$ are annihilated by a fixed power p^N: It suffices to treat evaluations on $\hat{R}'/t\hat{R}'$. We know that $\mathcal{E}^0(\hat{R}')[\frac{1}{p}]$ is finitely generated projective, so there exist maps

$$\mathcal{E}^0(\hat{R}')[\frac{1}{p}] \underset{\beta}{\overset{\alpha}{\rightleftarrows}} (\hat{R}'[\frac{1}{p}])^n$$

with $\beta \circ \alpha = id$. Also $\mathcal{E}^0(\hat{R}')$ is contained in a finitely generated $\hat{R}'$ -submodule. This multiplying α and β by suitable p-powers we obtain

$$\mathcal{E}^0(\hat{R}') \underset{\beta}{\overset{\alpha}{\rightleftarrows}} (\hat{R}')^n$$

with $\beta \circ \alpha = p^N \cdot id$. This p^N does the job.

As a corollary we see that

$$H^i(X^{crys}/R_V, \mathcal{E}^0 \otimes J) \otimes^{\mathbb{L}}_{R_V} V_0 \to H^i((X \otimes_V k)^{crys}/V_0, \bar{\mathcal{E}}^0 \otimes J)$$

has kernel and cokernel annihilated by some fixed p^N. Let L^* denote a topologically flat model of the total de Rham complex representing $\mathbb{R}\Gamma(X^{crys}/R_V, \mathcal{E}^0 \otimes J)$. Then L^* consists of R_V -modules, has a Frobenius $p^N \cdot \Phi$ and is p-adically complete. Furthermore if $I \subset R_V$ denotes the kernel of $R_V \twoheadrightarrow V_0$, then $L^* \otimes_{R_V} R_V/I$ represents the cohomology of the special fibre. Note that I is generated by all fractional powers $t^m/m!$. Applying Frobenius i-times to such an element gives

$$\phi^i(t^m/m!) = t^{p^i m}/m! = (\frac{(p^i m)!}{m!} \cdot t^{p^i m}/(p^i m)!$$

which is divisible by $p^{(m+pm+\ldots+p^{i-1}m)}$. Thus $\phi^i(I) \subseteq p^{p^{i-1}} \cdot I$. We have seen that for each i

$$H^i(L^*/I \cdot L^*) \cong M_0^i \oplus (p^N - \text{torsion}),$$

with M_0^i a finitely generated free V_0-module. Replacing Φ by $p^{2N} \cdot \Phi$ we may assume that Φ annihilates the second summands, Φ is an isogeny on the first (i.e. an isomorphism on $M_0^i[\frac{1}{p}]$). We use this to lift M_0^i to a submodule of $H^i(L^*[\frac{1}{p}])$, as follows: Choose a map $\tilde{\alpha} : M_0^i \to L^i$ lifting a basis of M_0^i. Then

$$\Phi \circ \tilde{\alpha} - \tilde{\alpha} \circ \Phi = \beta + d \circ \gamma$$

with $\beta : M_0^i \to I \cdot L^i$

$$\gamma : M_0^i \to L^{i-1}$$

Define $\alpha = \tilde{\alpha} + \sum_{m=0}^{\infty} \Phi^m \circ \beta \circ \Phi^{-(m+1)}$. This sum converges in $Hom(M_0^i, L^i[\frac{1}{p}])$ as $\Phi^m \circ \beta$ is divisible by $p^{p^{m-1}}$, while the p-power in the denominator of $\Phi^{-(m+1)}$ grows linearly in m. Then

$$\Phi \circ \alpha - \alpha \circ \Phi = d \circ \gamma,$$

$$\begin{aligned} d \circ \alpha &= d \circ \tilde{\alpha} + \sum_{m=0}^{\infty} \Phi^m \circ d \circ \beta \circ \Phi^{-(m+1)} \\ &= d \circ \tilde{\alpha} + \sum_{m=0}^{\infty} \Phi^m \circ (d \circ \Phi \circ \tilde{\alpha} - d \circ \tilde{\alpha} \circ \Phi) \circ \Phi^{-(m+1)} \\ &= d \circ \tilde{\alpha} + \lim_{m \to \infty} (\Phi^{m+1} \circ d \circ \tilde{\alpha} \circ \Phi^{-(m+1)} - d \circ \tilde{\alpha}) \\ &= \lim_{m \to \infty} \Phi^{m+1} \circ d \circ \tilde{\alpha} \circ \Phi^{(-m+1)} = 0, \end{aligned}$$

as $d \circ \tilde{\alpha}(M_0^i) \subseteq I \cdot L^i$. Hence if M_0^* denotes the complex made up from the M_0^i, with trivial differential and with Frobenius $\Phi, M^* = M_0^* \otimes_{V_0} R_V$, we have for N sufficiently big a map of complexes

$$\alpha : M^* \to p^{-N} \cdot L^*$$

with $\Phi \circ \alpha - \alpha \circ \Phi$ homotopic to zero, and the cone $C(\alpha)$ has the property that the cohomology of $C(\alpha)/I \cdot C(\alpha)$ is annihilated by p^{2N}. We also may extend Φ to a Frobenius on $C(\alpha)$. Then p^{2N} sends $H^i(C(\alpha))$ to the image of $H^i(I \cdot C(\alpha))$, thus $p^{2N} \cdot \Phi^m$ is divisible by $p^{p^{m-1}}$ on $H^i(C(\alpha))$. On the other hand we know that the cohomology of M^* as well as L^* some Frobenius divides some power of p. This is then also true on $C(\alpha)$. It follows that for some $m >> 0, p^m$ divides p^{m+1} on this cohomology. Then $H^i(C(\alpha)) = (p^m$ -torsion$)$ +(p-divisible submodule). Unfortunately I cannot prove in general that the second summand vanishes. What is however true in general is that the $H^i(X \otimes_V k/V_0, \mathcal{E} \otimes J)$ still satisfy Poincaré-duality. More precisely $(X \otimes_V k/V_0, \bar{\mathcal{E}}_0 \otimes J_I)$ is paired with

$$\mathbb{R}\Gamma(X \otimes_V k/V_0, \mathbb{R}Hom(\bar{\mathcal{E}_0} \otimes J_I, J))[-2d]$$

The pairing exists because there is a map $H^{2d}(X \otimes_V k/V_0, J) \to V_0$. It is non-degenerate because this is true modulo t (both objects are representable by perfect modules over V_0, see [Fa7]). Finally $\mathbb{R}Hom(\bar{\mathcal{E}_0} \otimes J_I, J)$ is up to p-torsion isomorphic to $\bar{\mathcal{E}}_0^\vee \otimes J_{I^c}$, as $\bar{\mathcal{E}}_0$ is up to p^N a direct summand in a projective module. Since there is also a Poincaré-pairing on $\mathbb{R}\Gamma(X/R_V, \mathcal{E} \otimes J)$ (the trace-form exists) we see that $H^i(X \otimes_V k_0/V_0, \mathcal{E} \otimes J) \otimes_{V_0} R_V$ injects into $H^i(X/R_V, \mathcal{E} \otimes J)$. Finally by pushout via powers of Frobenius (which map R_V to $R^0_{\geq \varepsilon}$) we obtain the same for cohomology over $R_{\geq \varepsilon}$, that is $H^i(X \otimes_V k_0/V_0, \mathcal{E} \otimes J) \otimes_{K_0} R_{\geq \varepsilon} \subseteq H^i(X/R_{\geq \varepsilon}, \mathcal{E} \otimes J)$. Finally mapping $R_{\geq \varepsilon} \to K$ (or $R_V \to V$) we know that the pushout is de Rham cohomology of $\mathcal{E}(\hat{X})$ over V. This also satisfies Poincaré-duality. By the usual calculation with characteristic of diagonals the induced maps

$$H^i(X \otimes_V k_0/V_0, \mathcal{E} \otimes J) \otimes_{K_0} K \to H^i_{DR}(X, \mathcal{E} \otimes J)$$

are isomorphisms (the Hyodo-Kato isomorphisms).

Remarks. Pragmatically one could replace $H^i(X/R_{\geq \varepsilon}, \mathcal{E} \otimes J)$ by $H^i(X \otimes_V k_0/V_0, \mathcal{E} \otimes J) \otimes_{K_0} R_{\geq \varepsilon}$ and then have all the desired properties. However philosophically this looks a little bit like cheating.

6. Complements: Relative case, Fontaine-Lafaille theory

As in [Fa3] one can generalise to the case of proper maps $f : X \to Y$ with $f(X^0) \subseteq Y^0$. For a reasonable theory one needs some local conditions on f. We assume that étale locally f is isomorphic to a map of torus-embeddings $\bar{S}_\lambda \to \bar{T}_\lambda$, where $\bar{S}_\lambda \subseteq \bar{S}, \bar{T}_\lambda \subseteq \bar{T}, \bar{S}$ corresponds to a lattice L and a cone $\sigma \subseteq L_{\mathbb{R}}, \bar{T}$ to a lattice M and a cone $\tau \subseteq M_{\mathbb{R}}$, and the map f to a homomorphism $L \to M$ which sends σ into τ such that each face of σ maps onto a face of τ. We assume that the cokernel of $L \to M$ is torsion free. Also $\lambda \in M^\vee$ maps to the corresponding element $\lambda \in L^\vee$. Finally σ and τ should satisfy our usual assumptions, namely that $\sigma \cap \ker(\lambda)$ and $\tau \cap \ker(\lambda)$ are spanned by partial basis of L, respectively M. This implies that $f_K : X \otimes_V K \to Y \otimes_V K$ sends each open stratum of X smoothly onto an open stratum of Y. Also for each locally constant torsion-sheaf $\mathbb{L}$ on $X^0 \otimes_V \bar{K}$, the direct images $\mathbb{R}^\nu f_* \mathbb{L}$ on $Y^0 \otimes_V \bar{K}$ are again locally constant. Furthermore f induces a map of topoi $f : \mathcal{X}^0 \otimes_V \bar{K} \to \mathcal{Y}^0 \otimes_V \bar{K}$:

For $U \to Y$ étale, $V \to U^0 \otimes_V \bar{K}$ an étale covering, the pullbacks to X are in the situs defining $\mathcal{X}^0 \otimes_V \bar{K}$. This defines f^* on sites, etc. We intend to sketch the proof of the following

6. Theorem. Suppose $\mathbb{L}$ is locally constant torsion on $(X^0 \otimes_V \bar{K})^{et}$, defining such a sheaf $\mathbb{L}$ on $\mathcal{X}^0 \otimes_V \bar{K}$. Then the direct images $\mathbb{R}^\nu f_* \mathbb{L}$ under $f : (X^0 \otimes \bar{K})^{et} \to (Y^0 \otimes_V \bar{K})^{et}$ and $f : \mathcal{X}^0 \otimes_V \bar{K} \to \mathcal{Y}^0 \otimes_V \bar{K}$ correspond. Furthermore $R^\nu f_*(\mathbb{L} \otimes \bar{\mathcal{O}}_X) \approx R^\nu f_* \mathbb{L} \otimes \bar{\mathcal{O}}_Y$, and similarly for $A_{\inf}(\bar{\mathcal{O}})$, or direct images with compact support (along boundary strata of X mapping to Y^0).

Proof. This is done as in the absolute case, replacing $\bar{V}$ by $\bar{R}$, for $Spec(R) \to Y$ étale and also $Spec(R)$ étale over $\bar{S}_\lambda, R$ strictly henselian. As usual we can for many purposes replace $\bar{R}$ by R_∞. First we have to study Poincaré-duality for the direct images $\mathbb{R} f_*(\mathbb{L} \otimes \bar{\mathcal{O}}_X)$. There is an (almost defined) trace-map

$$\mathbb{R}^{2\delta} f_*(\bar{J}_X / p^s \bar{J}_X)(\delta) \to \bar{J}_Y / p^s \bar{J}_Y$$

($\delta =$ relative dimension) inducing an almost quasi-isomorphism

$$\mathbb{R} f_*(\mathbb{L} \otimes \bar{\mathcal{O}}_X)[2\delta](\delta) \approx \mathbb{R} Hom_{\bar{R}}(\mathbb{R} f_*(\mathbb{L}^\vee \otimes \bar{J}_X), \bar{J}_Y p^s \bar{J}_Y)$$

(and similarly for cohomology with compact support at infinity). The proofs are easy generalisations of the previous ones.

Secondly $\mathbb{R} f_*(\mathbb{L})$ is given by the "almost Frobenius-invariants" on $\mathbb{R} f_*(\mathbb{L} \otimes A_{\inf}(\bar{\mathcal{O}}_X))$. We claim that (if R is strictly henselian)

$$R^\nu f_*(\mathbb{L} \otimes \bar{\mathcal{O}}_X) \approx R^\nu f_*(\mathbb{L}) \otimes \bar{R}$$

(and similarly with coefficients $A_{\inf}(\bar{R})$):

Reduce to $\mathbb{L}$ annihilated by $p, s = 1$. Then choose the highest index ν for which $\mathcal{M} = R^\nu f_*(\mathbb{L} \otimes \mathcal{R}(\bar{\mathcal{O}}_X))$ is not almost isomorphic to (almost Φ -invariants in $M) \otimes \mathcal{R}(\bar{R})$. Then $M = \mathcal{M}/\xi\mathcal{M}$ is almost finitely presented over $\bar{R}/p\bar{R}$, and Frobenius is an almost isomorphism on $\mathcal{M}$. Equivalently $\mathcal{M} = \varprojlim(M, \Phi)$, and Frobenius induces an isomorphism $M/p^{1/p} \cdot M \approx M$. Thus everything can be formulated in terms of M. Furthermore we can replace $\bar{R}$ by R_∞ and M by $M_\infty = Gal(\bar{R}/R_\infty)$ -invariants, as $M \approx M_\infty \otimes_{R_\infty} \bar{R}$. M_∞ is an almost finitely presented R_∞ / pR_∞ -module with Frobenius $\Phi : M_\infty / p^{1/p} M_\infty \approx M_\infty$. In the appendix it is shown that for some almost $\acute{e}$tale covering $\bar{A}$ of R_∞/pR_∞ $M_\infty \otimes_{R_\infty} \bar{A} \approx$ (almost Φ -invariants) $\otimes A$. Thus be descending induction over ν we can find such an A almost $\acute{e}$tale over $\bar{R}/p\bar{R}$ with $\mathbb{R}^\nu f_*(\mathbb{L} \otimes \bar{\mathcal{O}}_X) \otimes_{\bar{R}} \bar{A} \approx M_0^\nu \otimes_{\mathbb{F}_p} \bar{A}$ for certain $\mathbb{F}_p$ -vectorspaces M_0^ν. $\bar{A}$ lifts to an almost $\acute{e}$tale covering $\hat{A}$ of $\hat{\bar{R}}$, the p-adic completion of $\bar{R}$. Temporarily working over $\hat{A}$ we see that the M_0^ν are the direct images of $\mathbb{L}$ on the topos $\mathcal{X}^0 \otimes_{\mathcal{Y}} \hat{A}$, the analogue of the topos $\mathcal{X}^0$ with the base $\bar{V}$ replaced by $\hat{A}$. These map to the $\acute{e}$tale images, and these maps are isomorphisms (by duality, or basechange to discrete valuation rings). If we replace $\hat{A}$ by a Galois-covering the Galois-group $Gal(\hat{A}/\hat{\bar{R}})$ then operates trivially on the M_0^ν, as it operates trivially on $\acute{e}$tale direct images. Hence by taking invariants

$$R^\nu f_*(\mathbb{L} \otimes \bar{\mathcal{O}}) \cong M_0^\nu \otimes \bar{R},$$

and comparison is shown.

Obviously there are consequences for somparison to crystalline direct images. Finally the Fontaine-Lafaille theory from [Fa3] carries over without much change: Consider a filtered Frobenius-crystal $(\mathcal{E}, F^i(\mathcal{E}))$ which locally is the direct sum of shifted filtered modules $\mathcal{O}_X/p^s\mathcal{O}_X\{e_i\}$, with shifts $0 \leq e_i \leq p-2$. Assume that Frobenius Φ on F^i is divisible by p^i, that is there exist (assuming we evaluate on a PD-thickening with Frobenius-lift) $\Phi_i : F^i(\mathcal{E}) \otimes_\rho \mathcal{O}_X \to \mathcal{E}$ with $\Phi_{i-1} \mid F^i = p \cdot \Phi^i$. Furthermore assume that for a local filtered basis m_i, of degree e_i, the $\Phi^{e_i}(m_i)$ also form a basis. Then $\mathbb{L} = Hom_{F,\Phi}(\mathcal{E} \otimes B^+_{cr}(R)), B^+_{cr}(R) \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ defines a locally constant $\acute{e}$tale sheaf on $X \otimes_V \bar{K}$. This is shown as in [Fa3] (and uses in fact little of the structure of $\bar{R}$). Finally sometimes such crystals can be constructed as crystalline direct images under $f : X \to Y$. However for this one at least needs more stringent restrictions on f ("of relative Cartier-type" in Kato's terminology, see also [Fa3]), and also criteria under which the derived direct image splits into the direct sum of its cohomology groups.

6^*. Appendix: Some more almost mathematics

The ring $R_\infty = \lim\limits_{n\to\infty} R_n$ is the increasing union of noetherian rings R_n, such that for a sequence $\varepsilon_n \to 0$ we have

$$p^{\varepsilon_n} \cdot Tor^{R_n}, (R_{n+m}, \quad \text{any} \quad R_n\text{-module}) \quad = 0.$$

If M is any almost finitely presented R_∞ -module, then for each $\alpha > 0, M$ is p^α -isomorphic to $M_n \otimes_{R_n} R_\infty$ for some finitely presented R_n -module. Resolve M_n by finitely generated free R_n -module, to get by base extension a complex of finitely generated free R_∞-modules which up to $p^{\alpha+\varepsilon_n}$ resolves M. Similarly for a map $M \to N$ of such modules. With these methods one sees easily that the almost finitely presented R_∞ -modules form an abelian subcategory of all R_∞ -modules, which is closed under forming $Tor_i^{R_\infty}(M, N)$ or $Ext_i^{R_\infty}(M, N)$. The same holds for R_∞/pR_∞ (and by the way all this remains true for $\bar{R}$). Form now on assume that R is strictly henselian.

We now show that for an almost finitely presented R_∞/pR_∞ -module M with a Frobenius-linear almost isomorphism $\Phi : M/p^{1/p}M \approx M$, for an almost faithfully flat almost étale covering A of $R_\infty/pR_\infty, M \otimes_{R_\infty} A \approx M_0 \otimes_{\mathbb{F}_p} A$ with $M_0 = Hom(\mathcal{M}; M)^\Phi$ the almost invariants under Frobenius.

We proceed by induction over $\dim(R)$ and may assume that the assertion already holds for the strict localisation in any prime $\mathfrak{p} \subseteq R, \mathfrak{p} \neq \mathcal{M}$. By descent it follows that the localisations $M_\mathfrak{p}$ are almost projective on $(R_\infty/pR_\infty)_\mathfrak{p}$. Also the assertion has already been shown for $d = 0(R$ a discrete valuation ring), so we may assume that $\dim(R) \geq 2$. Replace M by $Hom(\mathcal{M}, M)$. Then $M_0 = M^\Phi$, frobenius is surjective on M, and M has no $\mathcal{M}$-torsion. If $\mathcal{M} = \varprojlim(M, \Phi)$, then $\mathcal{M}$ has no ξ -torsion, $\mathcal{M}/\xi\mathcal{M} \cong M$, and Frobenius is bijective in the $\mathcal{R}_\infty$ -module $\mathcal{M}$. Next consider $\mathcal{N} = Ext^1_{R_\infty}(\mathcal{M}, R_\infty)$. Then frobenius is also bijective on $\mathcal{N}$, and $\mathcal{N}/\xi\mathcal{N}$ injects into $Ext^1_{R_\infty}(\mathcal{M}, R_\infty/pR_\infty) = Ext^1_{R_\infty}/pR_\infty(M, R_\infty/pR_\infty)$, which is almost finitely presented and almost $\mathcal{M}$ -torsion. By a previously used argument (using $d > 0$ and $\lambda(\xi^\alpha\mathcal{N}/\xi\mathcal{N}))$ $\mathcal{N}/\xi\mathcal{N} \approx 0$, and it then follows that the maps any map $\mathcal{M} \to \mathcal{R}_\infty/\xi^n \cdot \mathcal{R}_\infty$ almost lifts to $\mathcal{R}_\infty/\xi^{n+1} \cdot \mathcal{R}_\infty$, and thus in the limit to $\mathcal{R}_\infty$.

Next consider the cokernel $\mathcal{N}$ of

$$\mathcal{M} \otimes_{\mathcal{R}_\infty} Hom_{\mathcal{R}_\infty}(\mathcal{M}, \mathcal{R}_\infty) \to End_{R_\infty}(\mathcal{M}).$$

Then Frobenius is almost isomorphic on $\mathcal{N}$. Also $\mathcal{N}/\xi \cdot \mathcal{N}$ almost injects into the cokernel of

$$M \otimes_{R_\infty} Hom_{R_\infty}(M, R_\infty/pR_\infty) \to End_{R_\infty}(M).$$

This cokernel is almost finitely presented and almost $\mathcal{M}$-torsion. By the usual argument $\mathcal{N}/\xi \cdot \mathcal{N}$ almost vanishes, and so does $\mathcal{N}$ (lifting modulo ξ^n step by step). Hence for each $\alpha > 0, \xi^\alpha \cdot id_\mathcal{N}$ lies in the image, thus is of the form $\sum x_i \otimes y_i$ with $x_i \in \mathcal{M}, y_i \in Hom_{\mathcal{R}_\infty}(\mathcal{M}, \mathcal{R}_\infty)$. Thus $\xi^d \cdot z = \sum y_i(z) \cdot x_i$ for each $z \in \mathcal{M}$, and $\mathcal{M}$ is almost projective.

Each localisation $M_\mathfrak{p}$ has finite rank, for $\mathfrak{p} \neq \mathfrak{m}$. If r denotes the maximum of these ranks for $\mathfrak{p}$ a minimal prime divisor of pR, then M and $\mathfrak{m}$ have rank $\leq r$. For $n > r(p-1)$ consider the cokernel $\mathcal{N}$ of the map

$$\mathfrak{m} \otimes S^{n-p}(\mathfrak{n}) \to S^n(\mathfrak{n})$$
$$m \otimes m' \to m^p \cdot m!$$

Then $\mathcal{N}/\xi\mathcal{N}$ injects into the corresponding cokernel for M, which is $\mathcal{M}$ -torsion and almost finitely presented. Also Frobenius is an isomorphism on $\mathcal{N}$, hence as usual $\mathcal{N}/\xi N \approx 0$, and the map above is almost surjective ($\mathcal{N} \approx 0$). Thus

$$\mathcal{A} = S^{\cdot}(\mathcal{N})/\langle m^p - \Phi(m)\rangle$$

is almost generated by symmetric powers $S^n(\mathcal{N})$ ($n \leq r(p-1)$), hence almost finitely presented. Also Frobenius is almost isomorphic on $\mathcal{A}$, thus $\mathcal{A}$ itself is almost projective (same argument as for $\mathcal{N}$). Finally $\mathcal{A}$ defines an almost *é*tale covering of $A_{\text{inf}}(R_\infty)$ which is almost faithfully flat ($A_{\text{inf}}(R_\infty)$ is a direct summand). The universal map $\mathcal{N} \to \mathcal{A}$ is Φ invariant. Passing to a Galois-hull of $\mathcal{A}$ we obtain the desired A.

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