

# On the Complexity and Volume of Hyperbolic 3-Manifolds.

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## Abstract

We compare the volume of a hyperbolic 3-manifold  $M$  of finite volume and the complexity of its fundamental group. <sup>1</sup>

## 1 Introduction.

**Complexity of 3-manifolds and groups.** One of the most striking corollaries of the recent solution of the geometrization conjecture for 3-manifolds is the fact that every aspherical 3-manifold is uniquely determined by its fundamental group. It seems to be natural to think that a topological/geometrical description of a 3-manifold  $M$  produces the simplest way to describe its fundamental group  $\pi_1(M)$ ; on the other hand, the simplest way to define the group  $\pi_1(M)$  gives rise to the most efficient way to describe  $M$ . More precisely, we want to compare the complexity of 3-manifolds and their fundamental groups.

The study of the complexity of 3-manifolds goes back to the classical work of H. Kneser [K]. Recall that the Kneser complexity invariant  $k(M)$  is defined to be the minimal number of simplices of a triangulation of the manifold  $M$ . The main result of Kneser is that this complexity serves as a bound of the number of embedded incompressible 2-spheres in  $M$ , and bounds the numbers of factors in a decomposition of  $M$  as a connected sum. A version of this complexity was used by W. Haken to prove the existence of hierarchies for a large class of compact 3-manifolds (called since then Haken manifolds). Another measure of the complexity  $c(M)$  for the 3-manifold  $M$  is due to S. Matveev. It is the minimal number of vertices of a special spine of  $M$  [Ma]. It is shown that in many important cases (e.g. if  $M$  is a non-compact hyperbolic 3-manifold of finite volume) one has  $k(M) = c(M)$  [Ma].

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The rank (minimal number of generators) is also a measure of complexity of a finitely generated group. According to the classical theorem of I. Grushko [Gr], the rank of a free product of groups is the sum of their ranks. This immediately implies that every finitely generated group is a free product of finitely many freely indecomposable factors, which is an algebraic analogue of Kneser theorem.

For a finitely presented group  $G$  a measure of complexity of  $G$  was defined in [De]. Here is its definition :

**Definition 1.1.** *Let  $G$  be a finitely presented group. We say that  $T(G) \leq t$  if there exists a simply-connected 2-dimensional complex  $P$  such that  $G$  acts freely and simplicially on  $P$  and the number of 2-faces of the quotient  $\Pi = P/G$  is less than  $t$ .*

*If the group  $G$  is defined by a presentation  $\langle a_1, \dots, a_r; R_1, \dots, R_n \rangle$  the sum  $\sum(|R_i| - 2)$  serves as a natural bound for  $T(G)$ .*

Note that an inequality between Kneser complexity and this invariant is obvious. Indeed, by contracting a maximal subtree of the 2-dimensional skeleton of a triangulation of  $M$  one obtains a triangular presentation of the group  $\pi_1(M)$ . Since every 3-simplex has four 2-faces it follows

$$T(\pi_1(M)) \leq 4k(M).$$

In order to compare the complexity of a manifold and that of its fundamental group, it is enough to find a function  $\theta$  such that  $\theta(\pi_1(M)) \leq T(\pi_1(M))$ . Note that the existence of such a function follows from G. Perelman's solution of the geometrization conjecture [Pe 1-3]. Indeed there could exist at most finitely many different 3-manifolds having the fundamental groups isomorphic to the same group  $G$  (for irreducible 3-manifolds with boundary this was shown much earlier in [Swa]). The question which still remains open is to describe the asymptotic behavior of the function  $\theta$ .

Note that for certain lens spaces the following inequality is proven in [PP]:

$$c(L_{n,1}) \leq \ln n \approx \text{const} \cdot T(\mathbb{Z}/n\mathbb{Z}).$$

However, the above problem remains widely open for irreducible 3-manifolds with infinite fundamental group. If  $M$  is a compact hyperbolic 3-manifold, D. Cooper showed [C]:

$$\text{Vol}M \leq \pi \cdot T(\pi_1(M)) \tag{C}.$$

where  $\text{Vol}M$  is the hyperbolic volume of  $M$ . Note that the converse inequality in dimension 3 is not true: there exists infinite sequences of different hyperbolic 3-manifolds  $M_n$  obtained by Dehn filling on a fixed finite volume hyperbolic manifold  $M$  with cusps such that  $\text{Vol}M_n < \text{Vol}M$  [Th]. The ranks of the groups  $\pi_1(M_n)$  are all bounded by  $\text{rank}(\pi_1(M))$  and since  $\pi_1(M_n)$  are not isomorphic, we must have  $T(\pi_1(M_n)) \rightarrow \infty$ . So the invariant  $T(\pi_1(M))$  is not comparable

with the volume of hyperbolic 3-manifolds. This difficulty can be overcome using the following relative version of the invariant  $T$  introduced in [De]:

**Definition 1.2.** *Let  $G$  be a finitely presented group, and  $\mathcal{E}$  be a family of subgroups. We say that  $T(G, \mathcal{E}) \leq t$  if there exists a simply-connected 2-dimensional complex  $P$  such that  $G$  acts simplicially on  $P$ , the number of 2-faces of the quotient (an orbihedron)  $\Pi = P/G$  is less than  $t$ , and the stabilizers of vertices of  $P$  are elements of  $\mathcal{E}$ .*

The main goal of the present paper is to obtain uniform constants comparing the volume of a hyperbolic 3-manifold  $M$  of finite volume and the relative invariant  $T(\pi_1(M), E)$  where  $E$  is the family of its elementary subgroups.

To finish our historical discussion let us point out that the relative invariant  $T(G, E)$  allows one to prove the accessibility of a finitely presented group  $G$  without 2-torsion over elementary subgroups [DePo1]. Using these methods it was shown recently that for hyperbolic groups without 2-torsion any canonical hierarchy over finite subgroups and one-ended subgroups is finite [Va]. The relative invariant  $T$  and the hierarchical accessibility was used in [DePo2] to give a criterion of the co-Hopf property for geometrically finite discrete subgroups of  $\text{Isom}(\mathbb{H}^n)$ .

**Main Results.** Let  $M$  be a hyperbolic 3-manifold of finite volume. We consider the family  $E_\mu$  of all elementary subgroups of  $\pi_1(M)$  having translation length less than the Margulis constant  $\mu = \mu(3)$ . The family  $E_\mu$  includes all parabolic subgroups of  $G$  as well as cyclic loxodromic ones representing geodesics in  $M$  of length less than  $\mu$  (see also the next Section).

The first result of the paper is the following:

**Theorem A.** *There exists a constant  $C$  such that for every hyperbolic 3-manifold  $M$  of finite volume the following inequality holds:*

$$C^{-1}T(G, E_\mu) \leq \text{Vol}(M) \leq CT(G, E_\mu) \quad (*)$$

■

The following are corollaries of Theorem A.

**Corollary 1.3.** *Suppose  $M_n \xrightarrow{f_n} M$  is a sequence of finite coverings over a finite volume 3-manifold  $M$  such that  $\text{deg} f_n \rightarrow +\infty$ . Then  $T(\pi_1(M_n), E_n) \rightarrow +\infty$ , where  $E_n$  is the above system of elementary subgroups of  $\pi_1(M_n)$  whose translation length is less than  $\mu$ .* ■

*Proof:* The statement follows immediately from the right-hand side of (\*) since  $\text{Vol}(M_n) \rightarrow \infty$ . QED.

**Corollary 1.4.** *Let  $M_n$  be a sequence of different hyperbolic 3-manifolds obtained by Dehn surgery on a cusped hyperbolic 3-manifold of finite volume  $M$ . Then*

$$T(\pi_1(M_n), E_n) \leq C \cdot \text{Vol}(M) < +\infty.$$

■

*Proof:* The left-hand side of (\*) gives

$$T(\pi_1(M_n), E_n) \leq C \cdot \text{Vol}(M_n),$$

and by [Th] one has  $\text{Vol}(M_n) < \text{Vol}(M)$ . *QED.*

As it is pointed out in Corollary 1.3 above we must have  $T(\pi_1(M_n)) \rightarrow +\infty$  for the absolute invariant. Our next result is the following :

**Theorem B.** *(Generalized Cooper inequality) Let  $E$  be the family of elementary subgroups of  $G$ , then one has*

$$\text{Vol}(M) \leq \pi \cdot T(\pi_1(M), E) \tag{**}$$

■

Note that Theorem B gives a generalization of the Cooper inequality (C) for the relative invariant  $T(G, E)$ . Furthermore, if one puts  $E = E_\mu$ , then Theorem B implies the right-hand side of (\*) in Theorem A. Theorems A and B together have several immediate consequences:

**Corollary 1.5.** *For the constant  $C$  from Theorem A the following statements hold:*

- i) *Let  $M$  be a finite volume hyperbolic 3-manifold and  $E_\mu$  and  $E$  be the above families of elementary subgroups of  $\pi_1(M)$ . Then*

$$T(\pi_1(M), E_\mu) \leq C \cdot \pi \cdot T(\pi_1(M), E).$$

- ii) *Let  $M$  be a hyperbolic 3-manifold such that  $M = M_{\mu\text{thick}}$ , i.e. every loop in  $M$  of length less than  $\mu$  is homotopically trivial. Then*

$$T(\pi_1(M)) \leq C \cdot \pi \cdot T(\pi_1(M), E).$$

*Proof:* i) By Theorems A and B we have

$$T(\pi_1(M), E_\mu) \leq C \text{Vol}(M) \leq C \cdot \pi \cdot T(\pi_1(M), E). \quad QED.$$

ii) Since  $E_\mu = \emptyset$  the result follows from i). *QED.*

Let us now briefly describe the content of the paper. In Section 2 we provide some preliminary results needed in the future. The proof of Theorem B is given in Section 3, it provides a "simplicial blow-up" procedure for an orbihedron. In Section 4 we prove the left-hand side of the inequality (\*) using some standard techniques and the results of Section 2. In the last Section 5 we discuss some open questions related to the present paper.

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## 2 Preliminary results.

Let us recall few standard definitions which we will use in the future. We say that  $G$  *splits* as a graph of groups  $X_* = (X, (C_e)_{e \in X^1}, (G_v)_{v \in X^0})$  (where  $C_e$  and  $G_v$  denote respectively edge and vertex groups of the graph  $X$ ) if  $G$  is isomorphic to the fundamental group  $\pi_1(X_*)$  in the sense of Serre [Se]. The Bass-Serre tree  $T$  is the universal cover of the graph  $X = T/G$ . When  $X$  has only one edge, we will say that  $G$  splits as an amalgamated free product (resp. an HNN-extension) if  $X$  has two vertices (resp. one vertex).

**Definition 2.1.** *Let  $G$  be a group acting on a tree  $T$ . A subset  $H$  of  $G$  is elliptic (resp. hyperbolic) in  $T$  (and in the graph  $T/G$ ) if  $H$  fixes a point in  $T$  (resp. does not fix a point in  $T$ ). If  $T$  is the Bass-Serre tree of a splitting of  $G$  as a graph of groups,  $H$  is elliptic if and only if it is conjugate into a vertex group of this graph.*

*We say that  $G$  splits relatively to a family of subgroups  $(E_1, \dots, E_n)$ , or that the pair  $(G, (E_i)_{1 \leq i \leq n})$  splits as a graph of groups, if  $G$  splits as a graph of groups such that all the groups  $E_i$  are elliptic in this splitting. A  $(G, (E_i)_{1 \leq i \leq n})$ -tree is a  $G$ -tree in which  $E_i$  are elliptic for all  $i$ . ■*

**Definition 2.2.** *Suppose  $G$  splits as a graph of groups*

$$G = \pi_1(X, C_e, G_v) \tag{1}$$

*relatively to a family of subgroups  $E_i$   $\{i = 1, \dots, n\}$ .*

The decomposition (1) such that all edge groups are non-trivial is called **reduced** if every vertex group  $G_v$  cannot be decomposed relatively to the subgroups  $E_i \in G_v$  as a graph of groups having one of the subgroups  $C_e$  as a vertex group.

The decomposition (1) is called **rigid** if whenever one has a  $(G, (E_i)_{i \in \{1, \dots, n\}})$ -tree  $T^*$  such that the subgroup  $C_e$  contains a non-trivial edge stabilizer then  $C_e$  acts elliptically on  $T^*$ . ■

It was shown in [De] that the sum of relative  $T$ -invariants of the vertex groups of a reduced splitting is less than or equal to the absolute invariant of  $G$ .

Recall that the Margulis constant  $\mu = \mu(n)$  is a number for which any  $n$ -dimensional hyperbolic manifold  $M$  can be decomposed into thick and thin parts :  $M = M_{\mu\text{thick}} \sqcup M_{\mu\text{thin}}$  such that the injectivity radius at each point of  $M_{\mu\text{thin}}$  is less than  $\mu/2$ , and  $M_{\mu\text{thick}} = M \setminus M_{\mu\text{thin}}$ . By the Margulis Lemma the components of  $M_{\mu\text{thin}}$  are either parabolic cusps or regular neighborhoods (tubes) of closed geodesics of  $M$  of length less than  $\mu$ . We will denote by  $E = E(\pi_1(M))$  (respectively  $E_\mu = E_\mu(\pi_1(M))$ ) the system of elementary subgroups of  $\pi_1(M)$  (respectively the systems of subgroups of  $\pi_1 M_{\mu\text{thick}}$ ). We will need the following:

**Lemma 2.3.** *Let  $H$  be a group admitting the following splitting as a graph of groups:*

$$H = \pi_1(X, C_e, G_v), \tag{2}$$

where each vertex group  $G_v$  is a lattice in  $\text{Isom}(\mathbb{H}^n)$  ( $n > 2$ ) and  $C_e \in E(G_v)$  ( $n > 2$ ).

Then (2) is a reduced and rigid splitting of the couple  $(H, \mathcal{E})$  where  $\mathcal{E} = \bigcup_v E(G_v)$ .

**Remark 2.4.** *The above Lemma will be further used in a very particular geometric situation when the group  $H$  is the fundamental group of the double of the thick part  $M_{\mu\text{thick}}$  of  $M$  along its boundary.* ■

*Proof:* We first claim that it is enough to prove that every vertex group  $G_v$  of the graph  $X$  cannot split non-trivially over an elementary subgroup. Indeed, if it is the case then obviously (2) is reduced. If it is not rigid, then the couple  $(H, \mathcal{E})$  acts on a simplicial tree  $T^*$  such that one of the groups  $C_e$  contains an edge stabiliser  $C_e^*$  of  $T^*$  and therefore acts hyperbolically on  $T^*$ . It follows that the vertex group  $G_v$  containing  $C_e$  also acts hyperbolically on  $T^*$  and so is decomposable over elementary subgroups.

Let us now fix a vertex  $v$  and set  $G = G_v$ . The Lemma now follows from the following statement:

**Sublemma 2.5.** *[Be] Let  $G$  be the fundamental group of a Riemannian manifold  $M$  of finite volume of dimension  $n > 2$  with pinched sectional curvature within  $[a, b]$  for  $a \leq b < 0$ . Then  $G$  does not split over a virtually nilpotent group.*

*Proof:* We provide below a direct proof of this Sublemma in the case of the constant curvature. Suppose, on the contrary, that

$$G = A *_C B \quad \text{or} \quad G = A *_C, \quad (3)$$

where  $C$  is an elementary subgroup. Let  $\tilde{C}$  be the maximal elementary subgroup containing  $C$ . The group  $\tilde{C}$  is virtually abelian and contains a maximal abelian subgroup  $\tilde{C}_0$  of finite index. We have the following

**Claim 2.6.** *The group  $\tilde{C}_0$  is separable in  $G$ .*

*Proof:*<sup>2</sup> Recall that the subgroup  $\tilde{C}_0$  is said *separable* if  $\forall g \in G \setminus \tilde{C}_0$  there exists a subgroup of finite index  $G_0 < G$  such that  $\tilde{C}_0 < G_0$  and  $g \notin G_0$ . Since  $\tilde{C}_0$  is a maximal abelian subgroup of  $G$ , and  $g \notin \tilde{C}_0$ , it follows that there exists  $h \in \tilde{C}_0$  such that  $\gamma = gh_0g^{-1}h_0^{-1} \neq 1$ . The group  $G$  is residually finite, so there exists an epimorphism  $\tau : G \rightarrow K$  to a finite group  $K$  such that  $\tau(\gamma) \neq 1$ . Since  $\tau(\tilde{C}_0)$  is abelian,  $\tau(\gamma) \notin \tau(\tilde{C}_0)$  and the subgroup  $G_0 = \tau^{-1}(\tau(\tilde{C}_0))$  satisfies our Claim. *QED.*

Denote  $C_0 = C \cap \tilde{C}_0$  (the maximal abelian subgroup of  $C$ ). We have  $\tilde{C} = \bigcup_{i=1}^m c_i C_0 \cup C_0$ . So by the Claim we can find a subgroup of finite index  $G_0$  of  $G$  containing  $C_0$  such that  $c_i \notin G_0$  ( $i = 1, \dots, m$ ). Then  $G_0 \cap \tilde{C} = C_0$  is abelian group and by the Subgroup Theorem [SW] we have that  $G_0$  splits as :

$$G_0 = A_0 *_C B_0 \quad \text{or} \quad G_0 = A_0 *_C, \quad (3')$$

where  $C_0 < C_0$  is also abelian. Suppose first that  $G_0 = A_0 *_C B_0$ , since  $G_0$  is not elementary group, one of the vertex subgroups of this splitting, say  $A_0$  is not elementary too. Then the map  $\varphi : G_0 \rightarrow (cA_0c^{-1}) *_C B_0$ ,  $c \in C_0$ , such that  $\varphi|_{A_0} = cA_0c^{-1}$  and  $\varphi|_{B_0} = \text{id}$  is an exterior automorphism (as  $c$  commutes with every element of  $C_0$ ) of infinite order. So the group of the exterior automorphisms  $\text{Out}(G_0)$  is infinite. This contradicts to the Mostow rigidity as  $G_0$  is still a lattice. In the case of HNN-extension  $G_0 = A_0 *_C \langle A_0, t \mid tC_0t^{-1} = \psi(C_0) \rangle$  suppose first that  $t$  does not belong to the centralizer  $Z(C_0)$  of  $C_0$  in  $G_0$ . Then we put  $\varphi|_{A_0} = cA_0c^{-1}$  for some  $c \in C_0$  such that  $[c, t] \neq 1$  and  $\varphi(t) = t$ . Since  $t \notin Z(C_0)$  we obtain again that  $\varphi$  is an infinite order exterior automorphism which is impossible. If, finally,  $t \in Z(C_0)$  then put  $\varphi|_{A_0} = \text{id}$  and  $\varphi(t) = t^2$  and it is easy to see that  $G'_0 = \varphi(G_0)$  is a subgroup of index 2 of  $G_0$  isomorphic to  $G_0$ . Then  $\text{Vol}(\mathbb{H}^n/\varphi(G_0)) < +\infty$  and again by Mostow rigidity we must have  $\text{Vol}(\mathbb{H}^n/G_0) = \text{Vol}(\mathbb{H}^n/\varphi(G_0))$ , and so  $\varphi : G_0 \rightarrow G_0$  should be surjective. A contradiction. The Sublemma 2.5 and Lemma 2.3 follow. *QED.*

<sup>2</sup>The argument is due to M. Kapovich and one of the authors is thankful for sharing it with him (about 20 years ago).

### 3 Proof of the generalized Cooper inequality.

The aim of this Section is to prove Theorem B stated in the Introduction:

**Theorem B.** *Let  $E$  be an arbitrary family of elementary subgroups of  $G$ , then*

$$\text{Vol}(M) \leq \pi \cdot T(\pi_1(M), E) \tag{1}$$

■

*Proof:* If  $E = \emptyset$ , then  $\text{Vol}(M) < \pi \cdot (L - 2n)$ , where  $L$  is the sum of the word-lengths of the relations of  $\pi_1(M)$  and  $n$  is the number of relations [C]. Let  $D$  be a disk representing a relation in the presentation complex  $R$  of  $\pi_1(M)$ . Then, triangulating  $D$  by triangles having vertices on  $\partial D$ , we obtain  $|D| - 2$  triangles. So  $L - 2n$  represents the total number of triangles in  $R$ . Thus Cooper's result implies  $\text{Vol}(M) \leq \pi \cdot T(\pi_1(M))$ .

Suppose now that  $M = \mathbb{H}^3/G$  where  $G < \text{Isom}(\mathbb{H}^3)$  is a lattice (uniform or not) and let  $E$  be a family of elementary subgroups of  $G$ . Let  $P$  be a simply-connected 2-dimensional polyhedron admitting a simplicial action of  $G$  such that the vertex stabilizers are elements of the system  $E$ . Let us also assume that the quotient  $\Pi = P/G$  is a finite orbihedron. We will need the following:

**Lemma 3.1.** *There exists a  $G$ -equivariant simplicial continuous map  $f : P \rightarrow \mathbb{H}^3 \cup \partial\mathbb{H}^3$  such that the images of the 2-simplices of  $P$  are geodesic triangles or ideal triangles of  $\mathbb{H}^3$ .*

*Proof:* Let us first construct a  $G$ -equivariant continuous map  $f : P \rightarrow \overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \partial\mathbb{H}^3$  such that the image of the fixed points for the action  $G$  on  $P$  belong to  $\partial\mathbb{H}^3$ . To do it we apply the construction from [DePo, Lemma 1.6] where instead of a tree as the goal space we will use the hyperbolic space  $\mathbb{H}^3$ . Let us first construct a map  $\rho : E \rightarrow \mathbb{H}^3$  as follows. Since the group  $G$  is torsion-free we can assume that all non-trivial groups in  $E$  are infinite. Then for every elementary group  $E_0 \in E$  we put  $\rho(E_0) = x \in \partial\mathbb{H}^3$  to be one of the fixed points for the action of  $E_0$  on  $\partial\mathbb{H}^3$  (by fixing a point  $O \in \partial\mathbb{H}^3$  for the image of the trivial group  $\rho(id)$ ). The map  $\rho$  has the following obvious properties :

- a)  $\forall E_1, E_2 \in E$  if  $E_1 \cap E_2 \neq \emptyset$  then  $\rho(E_1) = \rho(E_2)$ ;
- b) if  $\tilde{E}_0$  is a maximal elementary subgroup then  $\rho(E_0) = \rho(\tilde{E}_0)$  and  $\rho(g\tilde{E}_0g^{-1}) = g\rho(\tilde{E}_0)$  ( $g \in G$ ).

We now choose the set of  $G$ -non-equivalent vertices  $\{p_1, \dots, p_l\} \subset P$  representing all vertices of  $\Pi = P/G$ . We first construct a map  $f$  on zero-skeleton  $P^{(0)}$  of the complex  $P$  by putting  $f(p_i) = \rho(E_i)$  and then extend it equivariantly  $f(gp_i) = gf(p_i)$  ( $g \in G$ ).



Suppose now  $y = (q_1, q_2)$  ( $q_1, q_2 \in P^{(0)}$ ) is an edge of  $P$ . To define  $f$  on  $y$  we distinguish two cases: 1)  $H = \text{Stab}(y) \neq 1$  and 2)  $H = 1$ .

In the first case we have necessarily that  $E_{g_1} \cap E_{g_2} = H_0$  is an infinite elementary group where  $E_{q_i}$  is the stabilizer of  $q_i$ . Then there exist  $g_i \in G$  such that  $q_i = g_i(p_{k_i})$  ( $i = 1, 2$ ). So  $E_{q_i} = g_i E_{p_{k_i}} g_i^{-1}$  and  $g_1 E_{p_1} g_1^{-1} \cap g_2 E_{p_2} g_2^{-1} = H_0$ . It follows that  $E_{p_1} \cap g_1^{-1} g_2 E_{p_1} g_2^{-1} g_1$  is an infinite group and, therefore  $f(p_1) = g_1^{-1} g_2 (f(p_2))$  implying that

$$f(q_1) = f(g_1 p_1) = f(g_2 p_2) = f(q_2).$$

In the case 2) the stabilizer of the infinite geodesic  $l = ]f(q_1), f(q_2)[ \subset \mathcal{P}$  is trivial so we extend  $f : y \rightarrow l$  by a piecewise-linear homeomorphism. Having defined the map  $f$  as above on the maximal set of non-equivalent edges of  $P^{(1)}$  under  $G$ , we extend it equivariantly to the 1-skeleton  $P^{(1)}$  by putting  $f(gy) = gf(y)$  ( $g \in G$ ). Finally we extend  $f$  piecewise linearly to the 2-skeleton  $P^{(2)}$ .

We obtain a  $G$ -equivariant continuous map  $f : P \rightarrow \overline{\mathbb{H}^3}$  such that the all 2-faces of the simplicial complex  $f(P) \cap \mathbb{H}^3$  are ideal geodesic triangles. The Lemma is proved. *QED.*

**Remarks 3.2.** 1. *Note that the above Lemma is true in any dimension. We restricted our consideration to dimension 3 since the further argument will only concern this case.*

2. *If the system  $E$  contains only parabolic subgroups one can claim that the action of  $G$  on  $f(P) \cap \mathbb{H}^3$  is in addition proper. Indeed, using the convex hull  $\mathcal{P} \subset \mathbb{H}^3$  of the maximal family of non-equivalent parabolic points constructed in [EP] the above argument gives the map  $f : P \rightarrow \overline{\mathcal{P}} \subset \overline{\mathbb{H}^3}$ . By [EP, Proposition 3.5] the set of faces of  $\mathcal{P}$  is locally finite in  $\mathbb{H}^3$ . Since the boundary of each face of the 2-orbihedron  $f(P)$  constructed above belongs to  $\partial\mathcal{P}$ , we obtain that the set of 2-faces of  $f(P) \subset \mathbb{H}^3$  is locally finite in this case.*

■

If now  $W$  is the set of the fixed points for the action of  $G$  on  $P$ , we put  $P' = P \setminus W$  and  $Q' = f(P') = f(P) \cap \mathbb{H}^3$ . Let also  $\nu : P \rightarrow \Pi$  and  $\pi : \mathbb{H}^3 \rightarrow M = \mathbb{H}^3/G$  denote the natural projections. Then by Lemma 3.1 the map  $f$  projects to a simplicial map  $F : (\Pi' = P'/G) \rightarrow Q'/G \subset M$  such that the following diagram is commutative:

$$\begin{array}{ccc} P' & \xrightarrow{f|_{P'}} & Q' \subset \mathbb{H}^3 \\ \nu \downarrow & & \pi \downarrow \\ \Pi' & \xrightarrow{F} & Q'/G \subset M \end{array}$$

Note that, if  $\Pi$  is a simplicial polyhedron, it is proved in [C] that the hyperbolic area of  $F(\Pi)$  bounds the volume of the manifold  $M$ . This argument does not work if  $\Pi$  is an orbihedron but

not a polyhedron. Indeed the complex  $Q'$  above is not necessarily simply connected. So the group  $G$  is not isomorphic to  $\pi_1(Q'/G)$  but is a non-trivial quotient of it. Our goal now is to construct a new simplicial polyhedron  $\Sigma$  with the fundamental group  $G$  whose image into  $M$  has area arbitrarily close to that of  $F(\Pi')$ . So the main step in the proof of Theorem B is the following :

**Proposition 3.3.** *(simplicial blow-up procedure). For every  $\varepsilon > 0$  there exists a 2-dimensional complex  $\Sigma_\varepsilon$  and a simplicial map  $\varphi_\varepsilon : \Sigma_\varepsilon \rightarrow M$  such that*

1) *The induced map  $\varphi_\varepsilon : \pi_1 \Sigma_\varepsilon \rightarrow M$  is an isomorphism.*

*and*

2) *For the hyperbolic area one has:*

$$|\text{Area}(\varphi_\varepsilon(\Sigma_\varepsilon)) - \text{Area}(F(\Pi'))| < \varepsilon.$$

*Proof of the Proposition:* Let  $\Pi$  be a finite orbihedron with elementary vertex groups and such that  $\pi_1^{\text{orb}}(\Pi) \cong G$ . Let us fix a vertex  $\sigma$  of  $\Pi$  and let  $\tilde{\sigma} \in \nu^{-1}(\sigma)$  be its lift in  $P$ . We denote by  $G_\sigma$  the group of the vertex  $\sigma$  in  $G$ . By Lemma 3.1 the point  $f(\tilde{\sigma}) \in \partial\mathbb{H}^3$  is fixed by the elementary group  $G_\sigma$ . We will distinguish between the two cases when the group  $G_\sigma$  is loxodromic cyclic or parabolic subgroup of rank 2.

**Case 1. The group  $G_\sigma$  is loxodromic.**

Let  $V \subset \Pi$  be a regular neighborhood of the vertex  $\sigma$ . Then the punctured neighborhood  $V \setminus \sigma$  is homotopically equivalent to the one-skeleton  $L^{(1)}$  of the link  $L$  of  $\sigma$ .

We will call *realization* of  $L$  a graph  $\Lambda \subset V \setminus \sigma$  such that the canonical map  $L \rightarrow \Lambda$  is a homeomorphism. Let us fix a maximal tree  $T$  in  $\Lambda$ , and let  $y_i$  be the edges from  $\Lambda \setminus T$  which generate the group  $\pi_1(L)$  ( $i = 1, \dots, k$ ).

By its very definition, the  $G$ -equivariant map  $f : P \rightarrow \mathbb{H}^3$  sends the edges of  $P$  to geodesics of  $\mathbb{H}^3$ . So let  $G_\sigma = \langle g \rangle$  and let  $\gamma \subset M$  be the corresponding closed geodesic in  $M$ . We denote by  $A_g \subset \mathbb{H}^3$  the axis of the element  $g$  and by  $g^+, g^-$  its fixed points on  $\partial\mathbb{H}^3$ . Let us assume that  $f(\tilde{\sigma}) = g^+$ . For  $X \subset M$  we denote by  $\text{diam}(X)$  the diameter of  $X$  in the hyperbolic metric of  $M$ .

Recall that the map  $f : P \rightarrow \mathbb{H}^3 \cup \partial\mathbb{H}^3$  constructed in Lemma 3.1 induces the map  $F : \Pi' \rightarrow M$ . We start with the following:

**Step 1.** *For every  $\eta > 0$  there exists a realization  $\Lambda$  of  $L$  in  $\Pi$  such that for the maximal tree  $T$  of  $\Lambda$  one has*

$$\text{diam}(F(T)) < \eta,$$

Furthermore, for every edge  $y_i \in \Lambda \setminus T$  its image  $F(y_i)$  is contained in a  $\eta$ -neighborhood  $N_\eta(\gamma) \subset M$  of the geodesic  $\gamma$  ( $i=1, \dots, k$ ).

*Proof:* We fix a sufficiently small neighborhood  $V$  of a vertex  $\sigma$  in  $\Pi$  (the "smallness" will be specified later on). Let  $\tilde{\sigma} \in \nu^{-1}(\sigma)$  be its lift to  $P$  and let  $\tilde{\Lambda}$  and  $\tilde{T}$  be the lifts of  $\Lambda$  and  $T$  to a neighborhood  $\tilde{V} \subset \nu^{-1}(V)$  of  $\tilde{\sigma}$ . We are going first to show that, up to decreasing  $V$ , the image  $f(\tilde{T})$  belongs to a sufficiently small horosphere in  $\mathbb{H}^3$  centered at the point  $g^+$ .

Let  $\alpha$  be an edge of  $\Pi$  having  $\sigma$  as a vertex and  $\tilde{\alpha}$  be its lift starting at a point  $\tilde{\sigma}$ . Then  $a = f(\tilde{\alpha}) \subset \mathbb{H}^3$  is the geodesic ray ending at the point  $g^+$ , let  $a(t)$  be its parametrization. For a given  $t_0$  we fix a horosphere  $S_{t_0}$  based at  $g^+$  and passing through the point  $a(t_0)$ . Suppose there is a simplex in  $P$  having two edges  $\tilde{\alpha} = [\tilde{\sigma}, s]$ ,  $\tilde{\alpha}_1 = [\tilde{\sigma}, s_1]$  at the vertex  $\tilde{\sigma}$  and an edge  $[s, s_1]$  in  $\Lambda$ . The horosphere  $S_{t_0}$  is the level set of the Busemann function  $\beta_{g^+}$  based at the point  $g^+$ . So for the geodesic rays  $a = f(\tilde{\alpha})$  and  $a_1 = f(\tilde{\alpha}_1)$  issuing from the point  $g^+$  we have that the points  $f(s) = a(t_0)$  and  $f(s_1) = a_1(t_0)$  belong to the horosphere  $S_{t_0}$ . Proceeding in this way for all simplices whose edges share the vertex  $\sigma$ , we obtain that  $f(\tilde{T}^{(0)}) \subset S_{t_0} \subset \mathbb{H}^3$ . Since  $\Lambda$  is finite, so is the tree  $\tilde{T}$ . By choosing  $t_0$  sufficiently large ( $t_0 > \Delta$ ) we may assume that  $d(\alpha_i(t_0), \alpha_j(t_0)) < \eta$  and  $d(\alpha_i(t_0), A_g) < \eta$  ( $i, j = 1, \dots, k$ ). We now connect all the vertices of  $f(\tilde{T})$  by geodesic segments  $b_i \subset \mathbb{H}^3$ . By convexity, and up to increasing the parameter  $t_0$ , we also have  $d(b_i, A_g) < \eta$ .

By Lemma 3.1 the map  $f$  sends the lifts  $\tilde{y}_i \in \tilde{T}$  of the edges  $y_i \in \Lambda \setminus T$  simplicially to  $b_i$  ( $i = 1, \dots, k$ ); and  $f$  maps  $G_\sigma$ -equivariantly the preimage  $\tilde{\Lambda} = \nu^{-1}(\Lambda)$  to  $\mathbb{H}^3$ . Hence the map  $f$  projects to the map  $F : \Lambda \rightarrow M$  satisfying the claim of Step 1.  $\blacksquare$

## Step 2. Definition of the polyhedron $\Pi^\sim$

Using the initial orbihedron  $\Pi$  we will construct a new polyhedron  $\Pi^\sim$  having the following properties :

- a)  $\Pi^{(0)} = \Pi^{\sim(0)}$  and  $\Pi = \Pi^\sim$  outside of  $V$ ;
- b)  $\pi_1(L^*) = G_\sigma$ , where  $L^*$  is the link of  $\sigma$  in  $\Pi^\sim$ ;
- c)  $\pi_1(\Pi^\sim) \cong G$ .

The graph  $\Lambda$  realizes the link of the vertex  $\sigma$  so there exists an epimorphism  $\pi_1(\Lambda) \rightarrow \langle g \rangle$ . Every edge  $y_i \in \Lambda \setminus T$  which is a generator of the group  $\pi_1 \Lambda$  is mapped onto  $g^{n_{y_i}}$  in  $G_\sigma$  ( $i = 1, \dots, k$ ). We now subdivide each edge  $y_i$  by edges  $y_{ij}$  ( $i = 1, \dots, k, j = 1, \dots, n_{y_i}$ ), and denote by  $\Lambda'$  the obtained graph. Let  $S$  be a circle considered as a graph with one edge  $e$  and one vertex  $u$ . Then there exists a simplicial map from  $\Lambda'$  to  $S$  mapping simplicially each edge  $y_{ij}$  onto  $S$ .

To construct polyhedron  $\Pi^\sim$ , we replace the neighborhood  $V$  by the cone of the above map. Namely, we first delete the vertex  $\sigma$  from  $\Pi$  as well as all edges connecting  $\sigma$  with  $L$ . Then we connect the vertices of the edge  $y_{ij}$  with the vertex  $u \in S$  by edges which we call *vertical*

$(i = 1, \dots, k, j = 1, \dots, n_{y_i})$ . So  $\Pi^\sim$  is the union of  $\Pi \setminus V$  and the rectangles  $R_{ij}$ , which are bounded by  $y_{ij}$ , two vertical edges and the loop  $S$ . The set of rectangles  $\{R_{ij} \mid i = 1, \dots, k, j = 1, \dots, n_{y_i}\}$  realizes the epimorphism  $\pi_1(L) \rightarrow G_\sigma$ . By Van-Kampen theorem we have  $\pi_1(\Pi^\sim) \cong G$ , and the conditions a)-c) follow.  $\blacksquare$

**Step 3.** *There exists a constant  $c$  (depending only on the topology of  $\Pi$ ) such that for all  $\eta > 0$ , there exists a map  $F^\sim : \Pi^\sim \rightarrow M$  such that*

1)  $F^\sim$  induces an isomorphism on the fundamental groups,

2)  $F^\sim|_{\Pi \setminus V} = F$ ,

3)  $\sum_{ij} \text{Area}(F^\sim(R_{ij})) < c \cdot \eta$ . (2)

*Proof:* We choose a neighborhood  $V$  of the singular point  $\sigma$  and put  $F^\sim = F|_{\Pi \setminus V}$ . Using Step 2 we transform the orbihedron  $\Pi$  to  $\Pi^\sim$  in the neighborhood  $V$  and let  $P^\sim$  be the universal covering of  $\Pi^\sim$ . Note that, by construction,  $P^\sim$  is obtained by adding the  $G$ -orbit of the rectangles  $R_{ij}$  to the preimage  $\tilde{\Lambda}' = \nu^{-1}(\Lambda')$  of the graph  $\Lambda'$  ( $i = 1, \dots, k, j = 1, \dots, n_{y_i}$ ).

We will now extend the map  $f$  defined on  $P \setminus V$  to the polyhedron  $P^\sim \setminus P$  as follows. We first subdivide every segment  $b_i$  in  $n_{y_i}$  geodesic subsegments  $b_{ij} \subset b_i$  corresponding to the edges  $y_{ij}$ . We now project orthogonally each  $b_{ij}$  to  $A_g$  and let  $\tilde{\gamma} \subset A_g$  denote its image. Let  $\tau_{ij} \subset \mathbb{H}^3$  be the rectangle formed by  $b_{ij}$ ,  $\tilde{\gamma}$  and these two orthogonal segments from  $b_{ij}$  to  $A_g$  whose lengths are by Step 1 less than  $\eta$ . We extend the map  $f$  simplicially to a map  $f^\sim$  sending the rectangle  $\nu^{-1}(R_{ij})$  to the rectangle  $\tau_{ij}$  ( $i = 1, \dots, k, j = 1, \dots, n_{y_i}$ ). Note that by construction the lift  $\tilde{S}$  of the circle  $S$  is mapped on  $\tilde{\gamma}$ . The map  $f^\sim$  descends to a map  $F^\sim : \Pi_* \setminus \Pi \rightarrow N_\eta(\gamma)$ . It induces the epimorphism  $\pi_1 \Pi^\sim \rightarrow G$ .

Let us now make the area estimates for the added rectangles  $\tau_{ij}$ . Each rectangle  $\tau = \tau_{ij}$  has four vertices  $A, B, C, D$  in  $\mathbb{H}^3$  where  $B = gA, D = g(C)$  and the segment  $[A, B] \subset A_g$  is the orthogonal projection of  $[C, D]$  on  $A_g$ . The rectangle  $\tau$  is bounded by these two segments and two perpendicular segments  $l_1 = [A, C]$  and  $l_2 = [B, D]$  to the geodesic  $A_g$  ( $l_2 = g(l_1)$ ). We have  $\tau \subset ABC'D$  where  $\angle BDC' = \frac{\pi}{2}$  and  $\beta = \angle BC'D < \frac{\pi}{2}$ . Then by [Be, Theorem 7.17.1] one has  $\cos(\beta) \leq \sinh(d(B, D)) \cdot \sinh l(\gamma)$ . Therefore  $\text{Area}(\tau) < \frac{\pi}{2} - \beta$ , and  $\sin(\text{Area}(\tau)) \leq \sinh \eta \cdot \sinh l(\gamma)$ . Summing up over all segments  $b_{ij}$  we arrive to the formula (2). This proves Case 1.  $\blacksquare$

**Case 2. The group  $G_\sigma$  is parabolic.**

The proof is similar and even simpler in this case. Let again  $T$  be the maximal tree of the graph  $\Lambda$  realizing the link  $L$  of the vertex  $\sigma$ . We start by embedding a lift  $\tilde{T}^{(0)}$  of the zero-skeleton

of  $T^0$  into a horosphere  $S_{t_0} \subset \mathbb{H}^3$  based at the parabolic fixed point  $p \in \partial\mathbb{H}^3$  of the group  $G_\sigma = \langle g_1, g_2 \rangle \cong \mathbb{Z} + \mathbb{Z}$ . Then, using Lemma 3.1, we construct an embedding  $f : \tilde{\Lambda}^{(0)} \rightarrow S_{t_0}$  of the zero-skeleton of the graph  $\tilde{\Lambda} = \nu^{-1}(\Lambda)$  into the same horosphere  $S_{t_0}$  invariant under  $G_\sigma$  (which was not so in the previous case). Since the number of vertices of  $\tilde{T}$  is finite, for any  $\eta > 0$  we can choose a horosphere  $S_{t_0}$  ( $t_0 > \Delta$ ) such that  $\text{diam}\tilde{T} < \eta$ . Fixing a point  $O \in S_{t_0}$ , we can also assume that  $d(O, \tilde{T}^{(0)}) < \eta$ .

Now, let us modify the orbihedron  $\Pi$  in the neighborhood  $V$  of  $\sigma$ . First we delete the vertex  $\sigma$  from  $\Pi$  and all edges connecting  $\sigma$  with the graph  $\Lambda$ . We then add to the obtained orbihedron a torus  $\mathcal{T}$  with two intersecting loops  $C_1$  and  $C_2$  representing the generators of  $\pi_1(T, u)$  where  $u \in C_1 \cap C_2$ . To realize the epimorphism  $\pi_1\Lambda \rightarrow G_\sigma$  in  $M$  we proceed as before. For any edge  $y \in \Lambda \setminus T$  corresponding to the element  $g = ng_1 + mg_2$  in  $G_\sigma$  we add a rectangle  $R$  bounded by  $y$ , two edges connecting the end points of  $y$  with  $u$  and a loop  $C \subset \mathcal{T}$  representing the element  $g$  in  $\pi_1(T, u)$ . Let  $\tilde{\Pi}$  denote the obtained orbihedron.

Coming back to  $\mathbb{H}^3$ , let us assume for simplicity that  $p = \infty$  and the horosphere  $S_{t_0}$  is a Euclidean plane. By Lemma 3.1 the map  $f$  sends the edges  $\tilde{y}_i \in \tilde{\Lambda} \setminus \tilde{T}$  to the geodesic edges  $b_i$  connecting the vertices of  $f(\tilde{T})$ .

We now construct the rectangles  $\tau_i$  by projecting the end points of the edges  $b_i$  to the corresponding vertices of the Euclidean lattice given by the orbit  $G_\sigma O$ . Let us briefly describe this procedure in case of one rectangle  $\tau$ . Suppose that the edge  $y \in \Lambda \setminus T$  represents the element  $g = ng_1 + mg_2 \in G_\sigma$ . Let  $A$  and  $gA$  be vertices of  $f(\tilde{T})$  belonging to  $S_{t_0}$  connected by a geodesic segment  $b$  corresponding to  $y$ . Let  $\tau \subset \mathbb{H}^3$  be the geodesic bounded by the edges  $b, l = [O, A], gl, gb$ . We extend the map  $f : \tilde{R} \rightarrow \tau$  where  $\tilde{R}$  is a lift of the corresponding rectangle  $R$  added to  $\Pi$ . The map  $f$  descends now to a simplicial map  $F : \tilde{\Pi} \rightarrow M$  sending the torus  $\mathcal{T}$  into a cusp neighborhood of the manifold  $M$ . Since the rectangle  $\tau$  belongs to  $\eta$ -neighborhood of the horosphere  $S_{t_0}$ , its area, being close to the Euclidean one, is bounded by  $c \cdot \eta^2$  for some constant  $c > 0$ . Summing up over all edges  $y_i$  we obtain that the area of added rectangles does not exceed  $k \cdot c \cdot \eta^2$ . This proves Case 2.  $\blacksquare$

To finish the proof of Proposition 3.3, we note that the initial orbihedron  $\Pi$  is finite, so it has a finite number of vertices  $v_1, \dots, v_l$  whose vertex groups are either loxodromic or parabolic. So for a fixed  $\varepsilon > 0$ , we apply the above simplicial "blow-up" procedure in a neighborhood of each vertex  $v_i$  ( $i = 1, \dots, l$ ). Finally, we obtain a 2-complex  $\Sigma_\varepsilon$ ; and the simplicial map  $\phi_\varepsilon : \Sigma_\varepsilon \rightarrow M$  which induces an isomorphism on the fundamental groups and such that  $|\text{Area}(\varphi_\varepsilon(\Sigma_\varepsilon)) - \text{Area}(f(\Pi'))| < \psi(\eta)$ , where  $\psi$  is a continuous function such that  $\lim_{\eta \rightarrow 0} \psi(\eta) = 0$ . So for  $\eta$  sufficiently small we have  $\psi(\eta) < \varepsilon$  which proves the Proposition. *QED.*

*Proof of Theorem B.* Let  $G$  be the fundamental group of a hyperbolic 3-manifold  $M$  of finite volume. Let  $\Pi = P/G$  be a finite orbihedron realizing the invariant  $T(G, E)$ , i.e.  $\pi_1^{\text{orb}}(\Pi) \cong G$ , all vertex groups of  $\Pi$  are elementary and  $|\Pi^{(2)}| = T(G, E)$ . Hence  $\text{Area}(F(\Pi')) = \pi \cdot T(G, E)$ .

Then by Proposition 3.3 for any  $\varepsilon > 0$  there exists a 2-polyhedron  $\Sigma_\varepsilon$  and a map  $\psi_\varepsilon : \Sigma_\varepsilon \rightarrow M$  which induces an isomorphism on the fundamental groups and such that

$$\text{Area}(\psi_\varepsilon(\Sigma_\varepsilon)) < \pi T(G, E) + \varepsilon$$

By [C] we have  $\text{Vol}M < \text{Area}(\psi_\varepsilon(\Sigma_\varepsilon)) < \pi T(G, E) + \varepsilon$  ( $\forall \varepsilon > 0$ ). It follows  $\text{Vol}M \leq \pi T(G, E)$ . Theorem B is proved. *QED.*

## 4 Proof of Theorem A.

In this Section we finish the proof of

**Theorem A.** *There exists a constant  $C$  such that for every hyperbolic 3-manifold  $M$  of finite volume the following inequality holds:*

$$C^{-1}T(G, E_\mu) \leq \text{Vol}(M) \leq CT(G, E_\mu) \quad (*)$$

■

The right-hand side of the inequality (\*) follows from Theorem B if one puts  $E = E_\mu$ . So we only need to prove the left-hand side of (\*). We start with the following Lemma dealing with  $n$ -dimensional hyperbolic manifolds :

**Lemma 4.1.** *Let  $M$  be a  $n$ -dimensional hyperbolic manifold of finite volume. Then there exists a 2-dimensional triangular complex  $W \subset M_{\mu\text{thick}}$  such that  $\pi_1(W) \hookrightarrow \pi_1 M_{\mu\text{thick}}$  is an isomorphism and*

$$|W^2| \leq \sigma \cdot \text{Vol}(M),$$

where  $|W^2|$  is the number of 2-simplices of  $W$  and  $\sigma = \sigma(\mu)$  is a constant depending only on  $\mu$ .

*Proof:* The Lemma is a quite standard fact, proved for  $n = 3$  in [Th] and more generally in [G], [BGLM], [Ge]. We provide a short proof of it for the sake of completeness. Consider a maximal set of points  $\mathcal{A} = \{a_i \mid a_i \in M_{\mu\text{thick}}, d(a_i, a_j) > \mu/4\}$  where  $d(\cdot, \cdot)$  is the hyperbolic distance of  $M$  restricted to  $M_{\mu\text{thick}}$ . By the triangle inequality we obtain

$$B(a_i, \mu/8) \cap B(a_j, \mu/8) = \emptyset \quad \text{if } i \neq j,$$

where  $B(a_i, \mu)$  is an embedded ball in  $M$  (isometric to a ball in  $\mathbb{H}^n$ ) centered at  $a_i$  of radius  $\mu$ . By the maximality of  $\mathcal{A}$  we have  $M_{\mu\text{thick}} \subset \mathcal{U} = \bigcup_i B(a_i, \mu/4)$ . Recall that the nerve  $N\mathcal{U}$  of the covering  $\mathcal{U}$  is constructed as follows. Let  $N\mathcal{U}^0 = \mathcal{A}$  be the vertex set. The vertices  $a_{i_1}, \dots, a_{i_{k+1}}$

span a  $k$ -simplex if for the corresponding balls we have  $\bigcap_{j=1}^{k+1} B(a_{i_j}, \mu/4) \neq \emptyset$ . Since the covering  $\mathcal{U}$  is given by balls embedded into  $M$ , the nerve  $N\mathcal{U}$  is homotopy equivalent to  $\mathcal{U}$  [Hat, Corollary 4G.3].

Note that  $M_{\mu\text{thick}} \hookrightarrow \mathcal{U} \hookrightarrow M_{\frac{\mu}{2}\text{thick}}$ . Indeed if  $x \in \partial B(a_i, \mu/4)$  then by the triangle inequality we have  $B(x, \mu/4) \subset B(a_i, \mu/2)$ , and so both are embedded in  $M$ . Then  $x \in M_{\frac{\mu}{2}\text{thick}}$ . By the Margulis lemma, as the corresponding components of their thin parts are homeomorphic, the embedding  $M_{\mu\text{thick}} \hookrightarrow M_{\frac{\mu}{2}\text{thick}}$  is a homotopy equivalence. It implies that the complex  $N\mathcal{U}$  is homotopy equivalent to  $M_{\mu\text{thick}}$ . Let  $W$  denote the 2-skeleton of  $N\mathcal{U}$ . Then it is a standard topology fact that  $W$  carries the fundamental group of  $N\mathcal{U}$  [Hat]. Therefore,  $\pi_1 W \cong \pi_1 M_{\mu\text{thick}}$ .

It remains to count the number of 2-faces of  $W$ . We have for the cardinality  $|\mathcal{A}|$  of the set  $\mathcal{A}$ :

$$|\mathcal{A}| \leq \frac{\text{Vol}(M_{\mu\text{thick}})}{\text{Vol}(B(\mu/8))} \leq \frac{\text{Vol}(M)}{\text{Vol}(B(\mu/8))},$$

where  $B(\mu)$  denotes a ball of radius  $\mu$  in the hyperbolic space  $\mathbb{H}^n$ . The number of faces of  $W$  containing a point of  $\mathcal{A}$  as a vertex is at most  $m = \frac{\text{Vol}(B(\mu/2))}{\text{Vol}(B(\mu/8))}$ . Then

$$|W^{(2)}| \leq C_m^2 \frac{\text{Vol}(M)}{\text{Vol}(B(\mu/8))} = \sigma \cdot \text{Vol}(M),$$

where  $\sigma = \sigma(\mu) = \frac{C_m^2}{\text{Vol}(B(\mu/8))}$ . This completes the proof of the Lemma. ■

Suppose now that  $M$  is a hyperbolic 3-manifold of finite volume and let  $\mu = \mu(3)$  be the 3-dimensional Margulis constant. We are going to use a result of [De] which we need to adapt to our Definition 1.2 of the invariant  $T$ . So we start with the following:

**Remark 4.2.** *In the definition of the invariant  $T$  in [De] there is one more additional condition compared to our Definition 1.2. Namely, it requires that every element of a system  $E$  fixes a vertex of  $P$ . To be able to use the results of [De] we will denote by  $T_0(G, E)$  the invariant defined in [De] and keep the notation  $T(G, E)$  for that of our Definition 1.2. Notice that nothing changes for the absolute invariant  $T(G)$ .*

Let  $l_1, \dots, l_k$  be the set of closed geodesics in  $M$  of length less than  $\mu$ . Then by [Ko] the manifold  $M' = M \setminus \bigcup_i l_i$  is a complete hyperbolic manifold of finite volume and  $\pi_1 M_{\mu\text{thick}} \cong \pi_1 (M)'$ .

Let  $\mathcal{E}_\mu$  denote the system  $\pi_1(\partial M_{\mu\text{thick}})$  of fundamental groups of the boundary components of the thick part  $M_{\mu\text{thick}}$ . We have the following :

**Lemma 4.3.**

$$T_0(\pi_1(M), \mathcal{E}_\mu) \leq T_0(\pi_1(M'), \pi_1(\partial M')) \leq T_0(\pi_1(M), \mathcal{E}_\mu) + 2k. \quad (5)$$

*Proof:* 1) Consider first the left-hand side. Let  $G = \pi_1(M)$  and  $G' = \pi_1(M')$ . Let  $\mathcal{E}'_\mu = \{E_{k+1}, \dots, E_n\}$  be the set of fundamental groups of cusps of  $M_{\mu\text{thin}}$ . Let us fix a two-dimensional  $(G', \mathcal{E}'_\mu)$ -orbihedron  $P'$  containing  $T_0(G', \mathcal{E}'_\mu)$  triangular 2-faces. The pair  $(G', \mathcal{E}'_\mu)$  acts on its orbihedral universal cover  $P'$  [H]. Let  $N(l_i)$  be a regular neighborhood of the geodesic  $l_i \in M$  ( $i = 1, \dots, k$ ) and  $H_i = \langle \alpha_i, \beta_i \rangle$  be the fundamental group of the torus  $T_i = \partial N(l_i)$  where  $\alpha_i$  is freely homotopic to  $l_i$  in  $N(l_i)$ . The group  $H_i$  fixes a point  $x_i \in P'$ . We will now construct a 2-orbihedron  $P$  for the couple  $(G, \mathcal{E}_\mu)$  as follows. The group  $G$  is the quotient of  $G'$  by adding the relation  $\beta_i = 1$  ( $i = 1, \dots, k$ ). We identify the vertices of  $P'$  equivalent under the groups generated by  $\beta_i$  ( $i = 1, \dots, k$ ). The natural projection map  $P' \rightarrow P$  consists of contracting each edge of  $P'$  of the type  $(y, \beta_i(y))$  ( $y \in P'^{(0)}$ ) to a point. The projection has connected fibres so the 2-orbihedron  $P$  is simply connected and the pair  $(G, \mathcal{E}_\mu)$  acts on it. The procedure did not increase the number of 2-faces, and we have:  $|\Pi^{(2)} = P/G| \leq |\Pi^{(2)} = P'/G'|$ . Thus  $T_0(\pi_1(M), \mathcal{E}_\mu) \leq T_0(\pi_1(M'), \pi_1(\partial M') = \mathcal{E}'_\mu)$ .

2) Let  $\Pi$  be the 2-orbihedron which realizes  $T_0(\pi_1(M), \mathcal{E}_\mu)$ , and let  $P$  be its universal cover. To obtain a  $(\pi_1(M)', \mathcal{E}'_\mu)$ -orbihedron we modify  $P$  as follows. Let  $H_i = \langle h_i \rangle$  be the loxodromic subgroup corresponding to the geodesic  $l_i \subset M$  of length less than  $\mu$  ( $i = 1, \dots, k$ ). Let  $x_i \in P$  be a vertex fixed by the subgroup  $H_i$ . Notice that the group  $G'$  is generated by  $G$  and elements  $\beta_i$  such that  $[h_i, \beta_i] = 1$  ( $i = 1, \dots, k$ ). So we add to  $\Pi$  a new loop  $\beta_i$  (by identifying it with the corresponding element in  $G$ ) and glue a disk whose boundary is the loop corresponding to  $[h_i, \beta_i]$ . By triangulating each such a disk we add  $2k$  new triangles to  $\Pi^{(2)}$ . Thus the universal cover  $P'$  is obtained by adding to  $P$  a vertex  $y_i$  and its orbit  $\{Gy_i\}$ , so that the points  $\beta_i h_i g y_i$  are identified with  $h_i \beta_i g y_i$ . We further add the rectangle  $gD_i$  ( $g \in G$ ) whose vertices are  $h_i g y_i, \beta_i h_i g y_i, \beta_i g y_i, g y_i$  and subdivide it by one of the diagonal edges, say  $(h_i g y_i, \beta_i g y_i)$  ( $i = 1, \dots, k$ ). The construction gives a new 2-complex  $P'$  on which the pair  $(G', \mathcal{E}'_\mu)$  acts simplicially. We claim that  $P'$  is simply connected. Indeed if  $\alpha$  is a loop on it, since  $P$  is simply connected,  $\alpha$  is homotopic to a product of loops belonging to the disks  $gD_i$  so  $\alpha$  is a trivial loop. Since the 2-orbihedron  $\Pi' = P'/G'$  contains  $|\Pi^{(2)}| + 2k$  faces, we obtain  $T_0(\pi_1(M)', \pi_1(\partial M')) \leq T_0(\pi_1(M), \mathcal{E}_\mu) + 2k$  which was promised. *QED.*

**Remark 4.4.** *It is worth pointing out that in the context of volumes of hyperbolic 3-manifolds the following inequality (similar to (5)) is known:*

$$\text{Vol}(M) < \text{Vol}(M') < k \cdot (C_1(R) \cdot \text{Vol}(M) + C_2(R)), \quad (\dagger)$$

where  $R$  is the maximum of radii of the embedded tubes around the short geodesics  $l_i$  ( $i = 1, \dots, k$ ) and  $C_i(R)$  are functions of  $R$  ( $i = 1, 2$ ). The left-hand side of  $(\dagger)$  is classical and due to W. Thurston [Th], the right-hand side is proved recently by I. Agol, P. A. Storm, and W. Thurston [AST] ■



*Proof of the left-hand side of the inequality (\*):* By Lemma 4.1 the thick part  $M_{\mu\text{thick}}$  of  $M$  contains a 2-dimensional complex  $W$  such that  $\pi_1 W \hookrightarrow \pi_1 M_{\mu\text{thick}}$  is an isomorphism and  $|W^{(2)}| < \sigma \cdot \text{Vol}(M)$  for some uniform constant  $\sigma$ . Consider now the double  $N = DM_{\mu\text{thick}}$  of the manifold  $M_{\mu\text{thick}}$  along the boundary  $\partial M_{\mu\text{thick}}$ . By repeating the argument of Lemma 4.1 to each half of  $N$  we obtain two complexes  $W$  and  $\tau(W)$  embedded in  $N$  where  $\tau : N \rightarrow N$  is the involution such that  $M_{\mu\text{thick}} = N/\tau$ . By Van-Kampen theorem the fundamental group of the complex  $V = W \cup \tau(W)$  is generated by  $\pi_1 W$  and  $\pi_1(\tau(W))$  and is isomorphic to  $\pi_1(N)$ . Furthermore, for the number of two-dimensional faces in  $V$  we have  $|N^{(2)}| = 2|W^{(2)}|$ . So by Lemma 4.1  $T(\pi_1 N) \leq |V^{(2)}| < 2\sigma \cdot \text{Vol}(M)$ . The group  $\pi_1 N$  splits as the graph of groups whose two vertex groups are  $\pi_1 M_{\mu\text{thick}}$ . The edge groups of the graph of groups are given by the system  $\mathcal{E}_\mu$ . As  $\pi_1 M_{\mu\text{thick}} \cong \pi_1(M)'$  and  $M'$  is a complete hyperbolic 3-manifold of finite volume it follows from Lemma 2.3 that the above splitting is reduced and rigid. So by [De] we have:

$$T(\pi_1 N) \geq 2T_0(\pi_1 M_{\mu\text{thick}}, \mathcal{E}_\mu). \quad (6)$$

Then by Lemma 4.3  $T_0(\pi_1 M_{\mu\text{thick}}, \mathcal{E}_\mu) \geq T_0(\pi_1(M), \mathcal{E}_\mu)$ , and therefore

$$\sigma^{-1} \cdot T_0(\pi_1(M), \mathcal{E}_\mu) < \text{Vol}(M).$$

Recall that the initial system  $E_\mu$  of elementary subgroups includes all elementary subgroups of  $\pi_1(M)$  whose translation length is less than  $\mu$ . So  $\mathcal{E}_\mu \subset E_\mu$  implying that  $T(\pi_1(M), E_\mu) \leq T_0(\pi_1(M), \mathcal{E}_\mu)$ . We finally obtain

$$C^{-1} \cdot T(\pi_1(M), E_\mu) < \text{Vol}(M),$$

where  $C = \sigma$ . The left-hand side of (\*) is now proved. Theorem A follows. ■

## 5 Concluding remarks and questions.

The finiteness theorem of Wang affirms that there are only finitely many hyperbolic manifolds of dimension greater than 3 having the volume bounded by a fixed constant [W]. So it is natural to compare the volume of a hyperbolic manifold  $M = \mathbb{H}^n/\Gamma$  with the absolute invariant  $T(\Gamma)$ . In the case  $n > 3$  the inequality

$$\text{const} \cdot T(\Gamma) \leq \text{Vol}(M)$$

follows from [Ge, Thm 1.7] (see also Section 2 above, where instead of  $T(\pi_1(M), E)$  one needs to consider  $T(\pi_1(M))$  and use the fact that  $\pi_1 M_{\mu\text{thick}} \cong \pi_1(M)$ ). However, the result [C] is not known in higher dimensions. Thus we have the following :

**Question 5.1.** *Is there a constant  $C_n$  such that for every lattice  $\Gamma$  in  $\text{Isom}(\mathbb{H}^n)$  one has*

$$\text{Vol}(\Gamma) \leq C_n \cdot T(\Gamma) ?$$

**Remark 5.2.** (*M. Gromov*) The answer is positive if  $M$  is a compact hyperbolic manifold of dimension 4. Indeed in this case by the Gauss-Bonnet formula one has  $\text{Vol}(M) = \frac{\Omega_4}{2} \cdot \chi(M)$ , where  $\Omega_4$  is the volume of the standard unit 4-sphere. Hence  $\text{Vol}(M) < \frac{\Omega_4}{2} \cdot (2 - 2b_1 + b_2)$  where  $b_i = \text{rank}(H_i(M, \mathbb{Z}))$  is the  $i$ -th Betti number of  $M$  ( $i = 1, 2$ ). Since  $b_2 < T(\pi_1(M))$ , one has  $\text{Vol}(M) < \frac{\Omega_4}{2} \cdot (2 + b_2) < \Omega_4 \cdot T(\pi_1(M))$  (as  $T(\pi_1(M)) > 1$ ).

Recently it was shown by D. Gabai, R. Meyerhoff, and P. Milley that the Matveev-Weeks 3-manifold  $M_0$  is the unique closed 3-manifold of the smallest volume [GMM]. Furthermore, C. Cao and R. Meyerhoff found cusped 3-manifolds  $m003$  and  $m004$  of the smallest volume [CM], [GMM]. In this context we have the following :

**Question 5.3.** *Is the invariant  $T(\pi_1(M), E_\mu)$  on the set of compact hyperbolic 3-manifolds attained on the manifold  $M_0$  ? Is the minimal relative invariant  $T(\pi_1(M), E_\mu)$  on the set of cusped finite volume 3-manifolds attained on the manifolds  $m003$  and  $m004$  ?*

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