Compactifications of contractible affine 3-folds into smooth Fano 3-folds with $B_2 = 2$

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Contents

After the contributions of Furushima, Nakayama, Peternell and Schneider, in 1993, Furushima [Fur93] finally succeeded in the classification of the compactifications of the affine 3-space $\mathbf{A}^3$ into smooth Fano 3-folds with $B_2 = 1$. In this paper, we consider the compactifications of the contractible affine 3-folds X (e.g. $X = \mathbf{A}^3$) into smooth Fano 3-folds V with $B_2 = 2$. Consequently, we classify all such compactifications $X \hookrightarrow (V, D_1 \cup D_2)$ in the case where $K_V + D_1 + D_2$ is not nef. Furthermore, we know that infinitely many mutually non-isomorphic exotic $\mathbf{A}^3$'s can be compactified into Fano 3-folds with $B_2 = 2$. This phenomenon never occurs when $B_2 = 1$.

1 Introduction

Throughout the present article we work over the field of complex numbers $\mathbf{C}$. Concerning the compactifications of the affine 3-space $\mathbf{A}^3$ into smooth Fano 3-folds with the second Betti number $B_2 = 1$, after the contributions of Furushima, Nakayama, Peternell, Schneider (cf. [Fur86, Fur90, Fur92, F-N89a, F-N89b, Pe89, P-S88]), finally in [Fur93], Furushima completed the classification. Let $\mathbf{A}^3 \hookrightarrow (V, D)$ be the compactification of the affine 3-space $\mathbf{A}^3$, where V is a smooth Fano 3-fold with $B_2 = 1$ of index $r \in \{1, 2, 3, 4\}$ and D is the boundary divisor. Note that since $B_2 = 1$ and the coordinate ring

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of $\mathbf{A}^3$ is factorial, D is irreducible and is an ample generator of $\mathrm{Pic}(V) \cong \mathbf{Z}$. At first, in [Fur86] and [Fur90], Furushima classified such compactifications when $r \geq 2$ and D is normal. As the next step, in [F-N89a], Furushima and Nakayama investigated the compactifications when index $r = 2$ and D is possibly non-normal. As a result, they obtained the following:

Theorem 1.1 (cf. [Fur86, F-N89a]) *Let (V, D) be a compactification of $\mathbf{A}^3$ into a smooth Fano 3-fold V with the second Betti number $B_2 = 1$ and of index $r \geq 2$. Then:*

(1) $r = 4 \Rightarrow (V, D) \cong (\mathbf{P}^3, \mathbf{P}^2)$;

(2) $r = 3 \Rightarrow (V, D) \cong (\mathbf{Q}^3, \mathbf{Q}^2_0)$;

(3) $r = 2$ *and* D *is normal* $\Rightarrow (V, D) \cong (V_5, H^0_5)$;

(4) $r = 2$ *and* D *is non-normal* $\Rightarrow (V, D) \cong (V_5, H^\infty_5)$,

where V_5 is a linear section $\mathrm{Gr}(2,5) \cap \mathbf{P}^6$ of the Grassmann variety $\mathrm{Gr}(2,5) \hookrightarrow \mathbf{P}^9$ by three hyperplanes in $\mathbf{P}^9$. H^0_5 is a normal hyperplane section on V_5 which is a normal del Pezzo surface of degree $(-K_{H^0_5})^2 = 5$ with exactly one A_4-type Du Val singularity. H^∞_5 is a non-normal hyperplane section on V_5 which is a non-normal del Pezzo surface of degree $(-K_{H^\infty_5})^2 = 5$ whose singular locus is a line Σ with a normal bundle $\mathcal{N}_{\Sigma/V_5} \cong \mathcal{O}_\Sigma(-1) \oplus \mathcal{O}_\Sigma(1)$.

In the case where the index r is equal to 1, the situation is more subtle. But, in [Fur93], [Pe89] and [P-S88], they have shown the following:

Theorem 1.2 (cf. [Fur93, Pe89, P-S88]) *Let (V, D) be a compactification of $\mathbf{A}^3$ into a smooth Fano 3-fold V with the second Betti number $B_2 = 1$ and of index $r = 1$. Then:*

(1) $(V, D) \cong (V_{22}, H^0_{22})$ *or* (V_{22}, H^∞_{22}), *where $V_{22} \hookrightarrow \mathbf{P}^{13}$ is a smooth Fano 3-fold of index $r = 1$ with genus $g := \frac{1}{2}(-K_V)^3 + 1 = 12$, of degree 22 by the anti-canonical embedding, and H^0_{22} and H^∞_{22} are non-normal hyperplane sections on V_{22};*

(2) *Let Σ be a non-normal locus of H^0_{22} (resp. H^∞_{22}). Then Σ is a line on V_{22} with the normal bundle $\mathcal{N}_{\Sigma/V_{22}} \cong \mathcal{O}_\Sigma(-2) \oplus \mathcal{O}_\Sigma(1)$;*

(3) $\mathrm{mult}_\Sigma H^0_{22} = 2$ *and* $\mathrm{mult}_\Sigma H^\infty_{22} = 3$, *in particular, H^∞_{22} is a ruled surface swept out by conics on V_{22} intersecting with the line Σ.*

Note that the arguments in [Fur86, Fur90, Fur92, F-N89a, F-N89b, Fur93, Pe89, P-S88] can be applied without any modification even in the case of the compactifications of contractible affine 3-folds into a smooth Fano 3-folds with $B_2 = 1$. Indeed, their arguments made use of only the contractibility of the affine 3-space $\mathbf{A}^3$. Thus we have the following result as a consequence:

Corollary 1.1 (cf. [Fur86, Fur90, Fur92, Fur93, F-N89a, F-N89b, Pe89, P-S88]) *Let X be a contractible smooth affine 3-fold. Assume that X is embedded into a smooth Fano 3-fold V with the second Betti number $B_2 = 1$ and D the boundary divisor. Then X is isomorphic to the affine 3-space $\mathbf{A}^3$ and $(V, D) \cong (\mathbf{P}^3, \mathbf{P}^2), (\mathbf{Q}^3, \mathbf{Q}^2_0), (V_5, H^0_5), (V_5, H^\infty_5), (V_{22}, H^0_{22})$ or (V_{22}, H^∞_{22}).*

Now we hope to investigate whether or not the similar statements hold for the case $B_2(V) = 2$. Namely, our main interest is stated as follows:

Problem 1.1 Let X be a contractible affine 3-fold which is embedded into a smooth Fano 3-folds V with the second Betti number $B_2 = 2$, and let $V - X = D_1 \cup D_2$ be the boundary components. Then, classify such an X and a compactification $X \hookrightarrow (V, D_1 \cup D_2)$.

Note that the contractibility of X implies that the coordinate ring $A := \Gamma(\mathcal{O}_X)$ is factorial and the multiplicative group A^* consisting of all invertible elements of A coincides with non-zero constants $A^* = \mathbf{C}^*$ (cf. [Fuj82]). Hence Pic (V) is free generated by irreducible components in $V - X$, so the boundary consists of two components, say D_1 and D_2 such that Pic $(V) = \mathbf{Z}[D_1] \oplus \mathbf{Z}[D_2]$ (cf. Lemma 2.1).

As mentioned above, in the case where the second Betti number equals $B_2 = 1$, the first progress towards the classification of compactifications lies in the case of Fano index $r \geq 2$, in other words, the case where the line bundle $K_V + D$ is not nef. In consideration of this fact, as the first step to Problem 1.1 above, we shall investigate compactifications $X \hookrightarrow (V, D_1 \cup D_2)$ satisfying the following condition (†):

(†) : The line bundle $K_V + D_1 + D_2$ is not nef.

In this article, we shall classify the contractible affine 3-folds X and their compactifications $X \hookrightarrow (V, D_1 \cup D_2)$ into smooth Fano 3-folds V with $B_2(V) = 2$ satisfying (†). In order to state our main result, we prepare the notation. Since V is a Fano 3-fold with $B_2 = 2$, $\overline{\mathrm{NE}}(V)$ is spanned by two extremal rays, say R_1 and R_2. Let $\varphi_i : V \to W_i$ be the contraction of R_i for $i = 1, 2$. Let L_i be an ample generator of $\mathrm{Pic}(W_i) \cong \mathbf{Z}$ and we put $H_i := \varphi_i^*(L_i)$ for $i = 1, 2$. In **Table 1**, (No.♠) of each V means that V is a smooth Fano 3-fold with the indicated number in [M-M81, Table 2]. When we write $[\star - \star\star]$ in **Table 1**, it means that φ_1 is of $\star$-type and φ_2 is of $\star\star$-type. With these notations, we obtain the following:

Theorem 1.3 *Let X be a contractible affine algebraic 3-fold. Assume that X is embedded into a smooth Fano 3-fold V with the second Betti number $B_2(V) = 2$ and with $V - X = D_1 \cup D_2$ the boundary components satisfying the condition (†). Then the complete list of the pair $(V, D_1 \cup D_2)$ up to the permutation of D_1 and D_2, and the inside affine algebraic 3-fold $X = V - (D_1 \cup D_2)$ is given as in* **Table 1***, where affine 3-folds X marked as $(*)$ (Types XII, XIII and XXII) is described as follows:*

$(*)$: In Types XII, XIII (resp. Type XXII), we put $Z := \mathbf{Q}^3 - \overline{D_1}$ (resp. $Z := \mathbf{P}^3 - \overline{D_1}$). Then X is obtained from Z via the affine modification with locus $(p \in S)$. (See Definition 2.1 for the definition on affine modifications.)

Remark 1.1 It is clear that $Z \cong \mathbf{A}^3$ in the cases of Types XII and XXII. On the other hand, in the case of Type XIII, Z is isomorphic to $\mathrm{SL}(2; \mathbf{C})$.

Corollary 1.2 *Infinitely many exotic $\mathbf{A}^3$'s can be embedded into smooth Fano 3-folds with $B_2 = 2$.*

Proof. For two distinct pairs (a, b) and (a', b') of positive integers satisfying $a > b \geq 2$, $a' > b' \geq 2$ and $\gcd(a, b) = 1$, $\gcd(a', b') = 1$, the affine 3-folds $S_{a,b} \times \mathbf{A}^1$ and $S_{a',b'} \times \mathbf{A}^1$ are mutually non-isomorphic exotic $\mathbf{A}^3$ and they can be embedded into a smooth Fano 3-folds of type XII or XXII in **Table 1**. (See Remark 1.3 for the notation.) Thus we complete the proof. Note that the phenomenon that exotic $\mathbf{A}^3$'s can be compactified never occurs when $B_2 = 1$. (Compare this corollary with Corollary 1.1.) □

Remark 1.2 In [Mu90], he classified the compactifications of the affine 3-space $\mathbf{A}^3$ into smooth projective 3-folds V with $B_2(V) = 2$ in the case where the boundary two components D_1 and D_2 are normal and intersect each other along a smooth curve ideal-theoretically. Note that in our classification (cf. **Table 1**), we do not assume the normalities and normal crossings of D_1 and D_2. In fact, when the normality condition is excluded we can construct the compactifications of exotic $\mathbf{A}^3$'s in Types XII and XXII as seen in Corollary 1.2.

Remark 1.3 (1) X in Types XII, XIII and XXII in **Table 1** are not necessarily isomorphic to the affine 3-space $\mathbf{A}^3$. In fact, infinitely many exotic $\mathbf{A}^3$'s can be choosen as X. For example, in Types XII or XXII, we choose the following affine surface $S_{a,b}$ as S:

$$S := S_{a,b} := \left\{ f_{a,b}(x, y, z) := \frac{(xz+1)^a - (yz+1)^b}{z} = 0 \right\} \subset \mathbf{A}^3{}_{(x,y,z)},$$

where $a > b \geq 2$ is positive integers with $\gcd(a, b) = 1$. It is known that $S_{a,b}$ is smooth and contractible with log Kodaira dimension $\overline{\kappa}(S_{a,b}) = 1$ (cf. [Miy01]). Then X is isomorphic to an affine 3-fold $X_{a,b}$ which is obtained from $\mathbf{A}^3$ via the affine modification with a locus $(0 \in S_{a,b})$. (See Section 5 and we use the terminology in [K-Z99] and Definition 2.1.) $X_{a,b}$ is expressed as the hypersurface in $\mathbf{A}^4{}_{(x,y,z,u)}$ defined by $\left\{f_{a,b}(xu, yu, zu) - u\right\}/u = 0$. We can see that $X_{a,b}$'s are pairwise non-isomorphic contractible smooth affine 3-folds.

(2) An acyclic affine surface S of Types XII and XXII in **Table 1** need not to be smooth. For instance, let S be the following non-normal acyclic affine surface:

$$S := \left\{(x+1)^a - (y+1)^b = 0\right\} \subset \mathbf{A}^3{}_{(x,y,z)},$$

where $a > b \geq 2$ are positive integers with $\gcd(a, b) = 1$. Then X, which is obtained from $\mathbf{A}^3$ via the affine modification with a locus $(0 \in S)$, is isomorphic to $S_{a,b} \times \mathbf{A}^1$, where $S_{a,b}$ is an affine surface in (1) above. It is well known that such an X is an exotic $\mathbf{A}^3$ (cf. [Za97]).

We shall state the scheme of this article. Let $X \hookrightarrow (V, D_1 \cup D_2)$ be an embedding of a contractible affine algebraic 3-fold X into a smooth Fano 3-fold V with $B_2(V) = 2$ satisfying (†). We divide the situation into the three cases according as the reduced boundary divisor $D_1 + D_2$ is nef or not. Since V is a Fano 3-fold with $B_2(V) = 2$, the cone $\overline{\mathrm{NE}}(V)$ of effective 1-cycles on V is spanned by two extremal rays, say $R_1 = \mathbf{R}_+[l_1]$ and $R_2 = \mathbf{R}_+[l_2]$. Let $\varphi_i := \mathrm{cont}_{R_i} : V \to W_i$ be the contraction of R_i for $i = 1, 2$. We shall make use of the classification of smooth Fano 3-folds with $B_2 = 2$ due to Mori and Mukai in our proof frequently (cf. [M-M81, M-M83, M-M85]).

In Section 2, we shall summarize preliminary results and recall the definition on affine modifications briefly for the subsequent arguments in order to prove Theorem 1.3.

In Section 3, we consider, at first, the case where $D_1 + D_2$ is not nef. Then it is not difficult to see that we may assume that one of contractions φ_1 and φ_2 is divisorial of $E5$-type with one boundary component as an exceptional divisor. Hereafter we shall determine the structure of the embeddings $X \hookrightarrow (V, D_1 \cup D_2)$. For this purpose, we need the result due to Abe and Furushima concerning the classification on non-normal del Pezzo surfaces (cf. [A-F]).

In Section 4, we consider the case where $D_1 + D_2$ is nef but not ample. Then at least one of contractions φ_1 and φ_2 is divisorial with one boundary component as an exceptional divisor. We consider, separately, according as type of this contraction $(E1, E2, E3, E4, E5)$ in the subsequent. We need to make use of the result of Furushima and Nakayama (cf. [Fur86, F-N89a, F-N89b] and Theorem 1.1) concerning the compactifications of the affine 3-space $\mathbf{A}^3$ into smooth Fano 3-folds with $B_2 = 1$ of index $r \geq 2$. As in the previous section, we also need the result of [A-F] in order to exclude the cases of $E3$ and $E4$-types.

In Section 5, we finally consider the case where $D_1 + D_2$ is ample. By noting that $K_V + D_1 + D_2$ is not nef, we can see that one contraction of φ_1 and φ_2 is either of type $E2$, $D2$, $D3$ or $C2$. In the subsequent of this section, we consider according as the type of this contraction, separately. Note that, in the case where $E2$-type occurs, X can be an exotic $\mathbf{A}^3$. In order to see this, we need the theory of affine modifications (cf. [K-Z99]).

Notation.

- K_V : canonical divisor of a variety V;
- $F|_W$: scheme-theoretic restriction of a Cartier divisor F on V to a subvariety $W \subset V$;
- $F|_F$: self-intersection of an irreducible Cartier divisor F on V;
- $B_i(V)$: the i-th Betti number of V as a real topological manifold;
- $\chi_{\mathrm{top}}(V)$: the topological Euler number of a topological manifold V;
- $F \cdot C$: the intersection number of a ($\mathbf{Q}$-)Cartier divisor F and a curve C on a threefold;

- $(C_1 \cdot C_2)_Y$: the intersection number of curves C_1 and C_2 on a surface Y;
- $p_a(C)$: the arithmetic genus of a curve C;
- $F_1 \sim F_2$: F_1 is linearly equivalent to F_2;
- $F_1 \equiv F_2$: F_1 is numerically equivalent to F_2;
- $\mathbf{F}_a$: Hirzebruch surface of degree a, $(a \geq 0)$;
- M_a : the minimal section on $\mathbf{F}_a$;
- L_a : a fiber on $\mathbf{F}_a$;
- $\mathbf{A}^n$: n-dimensional affine space;
- $\mathbf{P}^n$: n-dimensional projective space;
- $\mathbf{Q}_0^2$: a quadric cone in $\mathbf{P}^3$;
- $\mathbf{Q}^3$: a smooth quadric hypersurface in $\mathbf{P}^4$;
- $\mathbf{C}^{*k}$: the affine line with k points punctured;
- V_5 : a linear section $\mathrm{Gr}(2,5) \cap \mathbf{P}^6$ of the Grassmann variety $\mathrm{Gr}(2,5) \hookrightarrow \mathbf{P}^9$ by three hyperplanes in $\mathbf{P}^9$, which is a Fano threefold of Fano index 2 and degree $(-\frac{1}{2}K_{V_5})^3 = 5$ in $\mathbf{P}^6$;
- H_5^0 : a normal hyperplane section of V_5 with exactly one A_4-type Du Val singularity;
- H_5^∞ : a non-normal hyperplane section of V_5 whose singular locus is a line $\Sigma \cong \mathbf{P}^1$ with the normal bundle $\mathcal{N}_{\Sigma/V_5} \cong \mathcal{O}_\Sigma(-1) \oplus \mathcal{O}_\Sigma(1)$;
- $\overline{\mathrm{NE}}(V)$: the closure of the cone of effective 1-cycles on V.

An **extremal ray** R of $\overline{\mathrm{NE}}(V)$ is a one-dimensional edge of $\overline{\mathrm{NE}}(V)$ which is K_V-negative. For the types of contractions of extremal rays on smooth projective 3-folds ($E1, E2, E3, E4, E5, C1, C2, D1, D2, D3$), see [Mo82, Theorems (3.3) and (3.5)]. Concerning divisorial contractions, $E1, E2, E3, E4$ and $E5$-type corresponds to (3.3.1), (3.3.2), (3.3.3), (3.3.4) and (3.3.5) in [Mo82, Theorem (3.3)].

2 Preliminaries

In this section, we shall summarize the preliminary results in order to prove Theorem 1.3. Furthermore, we recall the definition on affine modifications (cf. [K-Z99]) for the convenience of readers. Let X be a smooth contractible affine 3-fold. Assume that X is embedded into a smooth Fano 3-fold V with $B_2(V) = 2$.

Lemma 2.1 *The boundary $V - X$ consists of two irreducible components, say D_1 and D_2. Furthermore, the Picard group $\mathrm{Pic}(V)$ of V is free generated by the classes of D_1 and D_2.*

Proof. Note that the coordinate ring $A := \Gamma(\mathcal{O}_X)$ is factorial and $A^* = \mathbf{C}^*$ (cf. [Fuj82]). Hence the assertion is easy to see. □

We shall follow the following convention which are used consistently in the subsequent of this article.

Convention. Since V is a smooth Fano 3-fold with $B_2 = 2$, the cone $\overline{\mathrm{NE}}(V)$ of effective 1-cycles on V is spanned by two extremal rays $R_1 = \mathbf{R}_+[l_1]$ and $R_2 = \mathbf{R}_+[l_2]$. We put $\mu_i := \min\{(-K_V \cdot l) \in \mathbf{N}; l \subset V \text{ and } [l] \in R_i\}$ for $i = 1, 2$. We may assume that $(-K_V \cdot l_i) = \mu_i$. Let $\varphi_i := \mathrm{cont}_{R_i} : V \to W_i$ be the contraction of R_i. Since $B_2(V) = 2$, the Picard number of W_i is equal to 1. Let L_i be an ample generator of $\mathrm{Pic}(W_i) \cong \mathbf{Z}$ and put $H_i := \varphi^*(L_i)$ for $i = 1, 2$.

With these notations the following result holds:

Proposition 2.1 (cf. [M-M83, Proposition 5.1])

(1) $\mathrm{Pic}(V) \cong \mathbf{Z}[H_1] \oplus \mathbf{Z}[H_2]$,

(2) $-K_V \sim \mu_2 H_1 + \mu_1 H_2$,

(3) $(H_i \cdot l_j) = \delta_{ij}$ *for* $i, j = 1, 2$, *where* δ_{ij} *is the Kroneker's delta.*

In the subsequent sections of this article, we consider according as the numerical conditions of the reduced boundary divisor $D_1 + D_2$. For this purpose, we shall prove the following:

Lemma 2.2 *Assume that the reduced boundary divisor* $D_1 + D_2$ *is not nef (resp. nef but not ample). Then we may assume that* φ_1 *is a divisorial contraction of* $E5$*-type (resp. of type either* $E1, E2, E3, E4$ *or* $E5$*) with one boundary component* D_1 *as an exceptional divisor. The intersection* $(D_1 + D_2) \cdot l_2$ *of* $D_1 + D_2$ *with the other extremal ray* $R_2 = \mathbf{R}_+[l_2]$ *is positive.*

Proof. Assume that D is not nef, i.e., there exists an irreducible curve $C \subset V$ such that $(D_1 + D_2) \cdot C < 0$. Since $\overline{\mathrm{NE}}(V) = R_1 + R_2$, at least one of intersections $(D_1 + D_2) \cdot l_1$ and $(D_1 + D_2) \cdot l_2$ must be negative, say $(D_1 + D_2) \cdot l_1 < 0$. Then φ_1 is divisorial with the exceptional divisor contained in $D_1 \cup D_2$. Hence we may assume that $\mathrm{Exc}(\varphi_1) = D_1$. Since $D_1 + D_2$ is connected and $D_1 \cdot l_1 + D_2 \cdot l_1 < 0$, the possibility is one of the following two cases:

(a) $\varphi_1 : V \to W_1$ is of $E1$-type and D_2 meets D_1 along the rulings of $\varphi_1|_{D_1} : D_1 \to C_1 := \varphi_1(D_1) \subset W_1$,

(b) $\varphi_1 : V \to W_1$ is of $E5$-type and D_2 meets D_1 along a line on $D_1 \cong \mathbf{P}^2$.

Assume that (a) occurs. Then, via the contraction φ_1, $(W_1, \varphi_1(D_2) \cup C_1)$ is also a compactification of X. But the boundary $\varphi_1(D_2) \cup C_1$ is not of pure codimension 1, which is a contradiction as X is affine. Hence (a) does not occur and (b) occurs. If $(D_1 + D_2) \cdot l_2 \leq 0$ for the remaining extremal ray R_2, then $(D_1 + D_2) \cdot B$ for every irreducible curve $B \subset V$. Since X contains no complete curves, this implies that $B \subset D_1 \cup D_2$, which is obviously a contradiction. Assume that $D_1 + D_2$ is nef but not ample. Then there exists an irreducible curve $C \subset V$ such that $(D_1 + D_2) \cdot C = 0$. Hence we may assume that $(D_1 + D_2) \cdot l_1 = 0$. Since X contains no complete curves, this means that $l_1 \subset D_1 \cup D_2$. So, we may assume that φ_1 is divisorial and $D_1 = \mathrm{Exc}(\varphi_1)$. The same argument as above deduces that the remaining extremal ray $R_2 = \mathbf{R}_+[l_2]$ satisfies $(D_1 + D_2) \cdot l_2 > 0$. □

In Section 4, we need the information of some kinds of curves on H_5^0 ($\subset V_5 \hookrightarrow \mathbf{P}^6$) (see **Notation**). For this purpose we prove the following:

Lemma 2.3 *Let* $S := H_5^0$*, i.e.,* S *is a normal del Pezzo surface of degree* $(-K_S{}^2) = 5$ *of rank* $B_2(S) = 1$ *with only one singular point* p *which is Du Val of* A_4*-type. Let* $\mu : \widehat{S} \to S$ *be the minimal resolution of* S*. For each of the following (i)*$\sim$*(iii) there exists a smooth rational curve* C *on* S *with this numerical property:*

(i) $(K_S \cdot C)_S = (K_{\widehat{S}} \cdot \widehat{C})_{\widehat{S}} = -3$ *and* $(\widehat{C}^2)_{\widehat{S}} = 1$,

(ii) $(K_S \cdot C)_S = (K_{\widehat{S}} \cdot \widehat{C})_{\widehat{S}} = -2$ *and* $(\widehat{C}^2)_{\widehat{S}} = 0$,

(iii) $(K_S \cdot C)_S = (K_{\widehat{S}} \cdot \widehat{C})_{\widehat{S}} = -1$ *and* $(\widehat{C}^2)_{\widehat{S}} = -1$,

where we denote by $\widehat{C}$ *the proper transform by* μ *of* C.

Proof. The surface $\widehat{S}$, which is the minimal resolution of $p \in S := H_5^0$, is obtained from the projective plane $\mathbf{P}^2$ via the 4-points blow-up admitting infinitely near points lying on the smooth cubic curve $\Gamma_0 \subset S_0 := \mathbf{P}^2$. Let $\{p_0, p_1, p_2, p_3\}$ be these points and let $\nu_j : S_j \to S_{j-1}$ be the blow-up at the point p_{j-1} with the exceptional curve l_j for $1 \leq j \leq 4$, where p_0 is one of the inflection points of Γ_0. Let

$\nu := \nu_1 \circ \nu_2 \circ \nu_3 \circ \nu_4 : \widehat{S} = S_4 \to \mathbf{P}^2$ be the composite and let $\widehat{l_j}$ be the proper transform of l_j on $\widehat{S}$. Then we have the configuration of curves on $\widehat{S}$ as in Figure 1, where $\widehat{l_0}$ is the proper transform of the line l_0 on $\mathbf{P}^2$ meeting Γ_0 at p_0 with order 3. Note that $\mu : \widehat{S} \to S$ is the contraction of (-2)-curves $l_0 \cup l_1 \cup l_2 \cup l_3$ to a point p. The curve C in (iii) is obtained as the proper image on S of $\widehat{l_4}$. Let $\widehat{C}$ be the proper transform on $\widehat{S}$ of a line on $\mathbf{P}^2$ which passes through (resp. does not pass through) p_0. Then the proper image $C = \mu(\widehat{C})$ on S satisfies the mentioned property (ii) (resp. (i)). $\square$

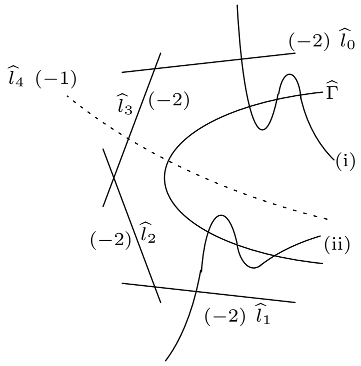

Figure 1.

We shall recall the definition on affine modifications which are used in Lemmas 5.4 and 5.5. For more detailed treatment and relevant results on these topics, see [K-Z99] and [Za97].

Definition 2.1 Let R be an affine domain and let $Z := \mathrm{Spec}(R)$. Let $I \subset R$ be an ideal and let f be an element of I. We denote by $D := \mathrm{V}_Z(f)$ and $C := \mathrm{V}_Z(I)$ the subvarieties in Z defined by f and I, respectively. Then the **affine modification of Z with a locus** $(C \subset D)$ is the affine variety Z' with a coordinate ring $R' := \Gamma(\mathcal{O}_{Z'}) = R[I/f]$. It is clear that $R \subset R' \subset Q(R)$, where $Q(R)$ is the quotient field of R. The canonical inclusion $R \subset R'$ induces a birational morphism $\sigma : Z' \to Z$. We often call this morphism σ itself as an affine modification with a locus $(C \subset D)$. If f is not a unit in R', it defines a divisor $E := \mathrm{div}(f)_{Z'}$ on Z' and we have an isomorphism $\sigma|_{Z'-E} : Z' - E \xrightarrow{\sim} Z - D$. Then E (resp. D) is said to be an **exceptional divisor** (resp. **divisor**) of the modification σ.

3 The case $D_1 + D_2$ is not nef

In this section, we consider the case where the reduced boundary divisor $D_1 + D_2$ is not nef. By Lemma 2.2, we may assume that the contraction $\varphi_1 : V \to W_1$ of the extremal ray R_1 is divisorial of type $E5$ with $D_1 = \mathrm{Exc}(\varphi_1) \cong \mathbf{P}^2$, $\mathcal{O}_{D_1}(D_1) \cong \mathcal{O}_{\mathbf{P}^2}(-2)$ as an exceptional divisor and D_2 meets D_1 along a line on $D_1 \cong \mathbf{P}^2$. In the list [M-M81, Table 2], smooth Fano 3-folds with $B_2 = 2$ having extremal rays of $E5$-type are No. 28 and No. 36. In fact, we shall prove the following:

Lemma 3.1 *The case of No. 28 does not occur.*

Proof. Our proof consists of three steps.

Step 1. Assume that V is No. 28 in [M-M81, Table2]. Namely, V is obtained from $W_2 = \mathbf{P}^3$ via the blow-up $\varphi_2 : V \to W_2$ along the plane cubic $C_2 \subset W_2$ with an exceptional divisor $E_2 := \mathrm{Exc}(\varphi_2)$ and $\varphi_1 : V \to W_1$ is the contraction of D_1, which is the proper transform on V of the hyperplane in W_2 containing C_2, to an index 2 point $p_1 := \varphi_1(D_1)$. We put $\overline{D_2} := \varphi_1(D_2) \subset W_1$.

Claim. $D_1 \sim H_1 - 2H_2$ *and* $E_2 \sim -H_1 + 3H_2$. *Furthermore,* $-K_{W_1} \equiv \frac{3}{2}L_1$

Proof of Claim. By Proposition 2.1 (2), we have $H_1 + H_2 \sim -K_V = \varphi_2^*(-K_{W_2}) - E_2 \sim 4H_2 - E_2$. Thus $E_2 \sim -H_1 + 3H_2$. Since D_1 is the proper transform by φ_2 of the hyperplane containing C_2, we have $D_1 \sim H_2 - E_1 \sim H_1 - 2H_2$. Since $-K_V \equiv \varphi_1^*(-K_{W_1}) - \frac{1}{2}D_1$, the remaining assertion is easy to see. $\square$

By noting that $V - D_1 \cong W_1 - \{p_1\}$, we write the following exact sequences:

$$\begin{array}{ccccccccccc} 0 \to H_3(V) & \longrightarrow & H_3(V, D_1) & \longrightarrow & H_2(D_1) & \longrightarrow & H_2(V) & \longrightarrow & H_2(V, D_1) \to 0 \\ \downarrow & & \downarrow\cong & & \downarrow & & \downarrow & & \downarrow\cong \\ 0 \to H_3(W_1) & \to & H_3(W_1, \{p_1\}) & \to & H_2(\{p_1\}) & \to & H_2(W_1) & \to & H_2(W_1, \{p_1\}) \to 0. \end{array}$$

Since $B_2(V) = B_3(V) = 2$ and $B_2(W_1) = 1$, we have $B_3(W_1) = 2$ by these sequences. On the other hand, we write the exact sequence associated to the pair $(W_1, \overline{D_2})$:

$$H_4(W_1, \overline{D_2}) \to H_3(\overline{D_2}) \to H_3(W_1) \to H_3(W_1, \overline{D_2}) \to H_2(\overline{D_2}) \to H_2(W_1) \to H_2(W_1, \overline{D_2}).$$

Since $H_j(W_1, \overline{D_2}) \cong H^{6-j}(X) = 0$ for $j = 2, 3, 4$ by the contractibility of X, we have $B_2(\overline{D_2}) = 1$ and $B_3(\overline{D_2}) = 2$.

Step 2. Note that since $p_1 \in W_1$ is an index 2 terminal singular point, for every irreducible Weil divisor F on W_1, $2F$ is Cartier (cf. [Ka88]). Since $\mathrm{Pic}(W_1) = \mathbf{Z}[L_1]$, we can write $\overline{D_2} \equiv \alpha L_1$ with $\alpha \in \frac{1}{2}\mathbf{N}$. We claim the following:

Claim. $\alpha = \frac{1}{2}$ *or* 1.

Proof of Claim. Assume to the contrary, namely, $\alpha = \frac{\beta}{2}$ with $\beta \geq 3$. Assume that β is odd. It is obvious that $L_1 \neq \overline{D_2}$. Put $L_1^\circ := L_1 \cap X$, where we note $X \cong W_1 - \overline{D_2}$. Since $A := \Gamma(\mathcal{O}_X)$ is factorial, there exists a prime element $g \in A$ such that $\mathrm{div}(g)_X = L_1^\circ$. When we consider g as an element of the quotient field $Q(A) = \mathbf{C}(W_1)$, it follows $\mathrm{div}(g)_{W_1} = L_1 - m\overline{D_2}$ with $m \in 2\mathbf{N}$. Hence $\overline{D_2} \equiv \frac{1}{m}L_1 \equiv \frac{\beta}{2}L_1$, which is a contradiction. By the similar argument, we can obtain a contradiction in the case where β is even also. □

We also claim the following:

Claim. $\overline{D_2}$ *is a Gorenstein (possibly non-normal) del Pezzo surface.*

Proof of Claim. It is clear that D_2 is Gorenstein. Note that D_2 meets the exceptional divisor $D_1 \cong \mathbf{P}^2$ of φ_1 along the line, so $\varphi_1|_{D_2} : D_2 \to \overline{D_2}$ is the contraction of a (-2)-curve. Thus $\overline{D_2}$ is Gorenstein. By the adjunction, we have $-K_{\overline{D_2}} \equiv (\frac{3}{2} - \alpha)L_1|_{\overline{D_2}}$. Since α is equal to either $\frac{1}{2}$ or 1, $-K_{\overline{D_2}}$ is ample. □

Step 3. Note that every normal del Pezzo surface is rational (cf. [De80], [H-W81]). If $\overline{D_2}$ is normal, then the rationality of $\overline{D_2}$ implies that $B_3(\overline{D_2}) = 0$. This is a contradiction because $B_3(\overline{D_2}) = 2$ as seen in Step 1. Thus $\overline{D_2}$ is a non-normal del Pezzo surface. Let $\sigma : \widetilde{D_2} \to \overline{D_2}$ be the normalization. Let $\mathcal{J} \subset \mathcal{O}_{\overline{D_2}}$ be the conductor sheaf of σ, and let $\overline{\Sigma} := \mathrm{V}_{\overline{D_2}}(\mathcal{J})$ and $\widetilde{\Sigma} := \mathrm{V}_{\widetilde{D_2}}(\mathcal{J})$. Then we have the following exact sequence associated with σ (cf. [B-K82]):

$$\cdots \to H^1(\widetilde{\Sigma}) \to H^2(\overline{D_2}) \to H^2(\widetilde{D_2}) \oplus H^2(\overline{\Sigma}) \to H^2(\widetilde{\Sigma}) \to H^3(\overline{D_2}) \to H^3(\widetilde{D_2}) \oplus H^3(\overline{\Sigma}).$$

Here we shall make use of the result of Abe and Furushima [A-F, Theorem 1.5] concerning the classification on non-normal del Pezzo surfaces. According to this, it follows that $H^1(\widetilde{\Sigma}) = 0$, $H^2(\overline{\Sigma}) \cong \mathbf{Z}$, $H^2(\widetilde{\Sigma}) \cong \mathbf{Z}$ or $\mathbf{Z}^{\oplus 2}$ and $H^3(\widetilde{D_2}) \oplus H^3(\overline{\Sigma}) = 0$. Since $B_2(\overline{D_2}) = 1$ and $B_3(\overline{D_2}) = 2$, this exact sequence is:

$$0 \to \mathbf{Z} \to H^2(\widetilde{D_2}) \oplus \mathbf{Z} \to H^2(\widetilde{\Sigma}) \cong \mathbf{Z} \text{ or } \mathbf{Z}^{\oplus 2} \to \mathbf{Z}^{\oplus 2} \to 0.$$

Hence $H^2(\widetilde{\Sigma}) \cong \mathbf{Z}^{\oplus 2}$ and $H^2(\widetilde{D_2}) = 0$, which is a contradiction because $B_2(\widetilde{D_2}) > 0$. Thus we complete the proof. □

By Lemma 3.1, we have only to consider the case of No. 36. In fact, we have the following:

Lemma 3.2 *The embedding* $X \hookrightarrow (V, D_1 \cup D_2)$ *is described as follow:* $X \cong \mathbf{A}^3$ *and* V *is a Fano 3-fold of No. 36 in* [M-M81, Table 2]*, i.e.,* $V \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(2))$*.* D_1 *is a section corresponding to the surjection* $\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(2) \to \mathcal{O}_{\mathbf{P}^2} \to 0$ *and* $D_2 \sim H_2$*.*

Proof. Let $V = \mathbf{P}(\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(2))$ be the Fano 3-fold of No. 36 in [M-M81, Table 2]. Then $\varphi_2 : V \to W_2 = \mathbf{P}^2$ is the $\mathbf{P}^1$-bundle structure associated with the rank 2 vector bundle $\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(2)$ and the exceptional divisor D_1 of $\varphi_1 : V \to W_1$ is a section of φ_2 corresponding to the surjection $\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(2) \to \mathcal{O}_{\mathbf{P}^2} \to 0$. Associated with φ_2 we have $\mathrm{Pic}(V) \cong \mathbf{Z}[D_1] \oplus \mathbf{Z}[H_2]$. On the other hand, it follows that $\mathrm{Pic}(V) \cong \mathbf{Z}[D_1] \oplus \mathbf{Z}[D_2]$ by Lemma 2.1. Therefore, when we write $D_2 \sim aD_1 + bH_2$ with $a, b \in \mathbf{Z}$ the matrix $\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}$ have to be unimodular. Hence $b = 1$ or -1. On the other hand, since D_2 meets $D_1 \cong \mathbf{P}^2$ with a normal bundle $\mathcal{O}_{D_1}(D_1) \cong \mathcal{O}_{\mathbf{P}^2}(-2)$ along the line on D_1, we have $-2a + b = 1$. Hence either $(a, b) = (0, 1)$ or $(-1, -1)$ holds. If $(a, b) = (-1, -1)$, then $(D_2 \cdot l_2) = (-D_1 - H_2 \cdot l_2) = -1$, which is a contradiction. So, $(a, b) = (0, 1)$, i.e., $D_2 \sim H_2$. Hence the restriction $\varphi_2|_X : X = V - (D_1 \cup D_2) \to W_2 - \varphi_2(D_2)$ gives rise to an $\mathbf{A}^1$-bundle structure over the affine plane $W_2 - \varphi_2(D_2) \cong \mathbf{A}^2$. Since $\mathrm{Pic}\,(\mathbf{A}^2) = 0$, this is a trivial $\mathbf{A}^1$-bundle structure over $\mathbf{A}^2$. Thus we have $X \cong \mathbf{A}^2 \times \mathbf{A}^1 \cong \mathbf{A}^3$. □

The embedding $\mathbf{A}^3 \cong X \hookrightarrow (V, D_1 \cup D_2)$ in Lemma 3.2 is of Type XXIV in **Table 1**, where we shall interchange the roles of φ_1 and φ_2.

4 The case $D_1 + D_2$ is nef but not ample

In this section we consider the case where $D_1 + D_2$ is nef but not ample. By Lemma 2.2, we may assume that the contraction $\varphi_1 : V \to W_1$ of the extremal ray R_1 is divisorial with $D_1 = \mathrm{Exc}(\varphi_1)$ as an exceptional divisor. We consider according as the type of the divisorial contraction φ_1 $(E1, E2, E3, E4, E5)$.

4.1 φ_1 is of $E1$-type

First of all we consider the case where $\varphi_1 : V \to W_1$ is of $E1$-type, i.e., it contracts D_1 onto a curve, say $C_1 := \varphi_1(D_1)$, in a smooth projective 3-fold W_1. Let $\overline{D_2} := \varphi_1(D_2)$. Then:

Lemma 4.1 *$X \cong \mathbf{A}^3$ and $(W_1, \overline{D_2}) \cong (\mathbf{P}^3, \mathbf{P}^2)$, $(\mathbf{Q}^3, \mathbf{Q}^2_0)$, (V_5, H^0_5) or (V_5, H^∞_5).*

Proof. Since the exceptional divisor D_1 of φ_1 is contained in the boundary $D_1 \cup D_2$, it is clear that $X \cong W_1 - \overline{D_2}$. Note that W_1 is a smooth Fano 3-fold with $B_2(W_1) = 1$ of index $r \geq 2$ (cf. [M-M83, Proposition 4.10]). Therefore we have the assertion by Theorems 1.1, 1.2 and Corollary 1.1. □

In consideration of Lemma 4.1 and [M-M81, Table 2], the possibilities of Fano 3-folds V are No. 4, No. 7, No. 9, No. 12, No. 13, No. 14, No. 15, No. 17, No. 19, No. 20, No. 21, No. 22, No. 23, No. 25, No. 26, No. 27, No. 28, No. 29, No. 30, No. 31 and No. 33 in [M-M81, Table 2]. We can easily exclude some cases among these:

Lemma 4.2 *The cases No. 4, No. 7, No. 9, No. 12, No. 13, No. 15, No. 19, No. 21, No. 25, No. 27 does not occur as compactifications of X.*

Proof. Note that the center C_1 of the blow-up φ_1 is contained in $\overline{D_2}$. Namely, if $(W_1, \overline{D_2}) \cong (\mathbf{P}^3, \mathbf{P}^2)$ (resp. $(\mathbf{Q}^3, \mathbf{Q}^2_0)$, (V_5, H^0_5) or (V_5, H^∞_5)), then C_1 is contained in $\overline{D_2} \cong \mathbf{P}^2$ (resp. $\mathbf{Q}^2_0$, H^0_5 or H^∞_5). (Note that No. 19 is obtained from $W_1 = \mathbf{P}^3$ via the blow-up $\varphi_1 : V \to W_1$ along the curve of degree 5 and genus 2.) But the center $C_1 \subset W_1$ of the blow-up φ_1 cannot be contained in $\overline{D_2}$ when the indicated cases in the statement occur. This is a contradiction. □

We consider the remaining cases No. 14, No. 17, No. 20, No. 22, No. 23, No. 26, No. 28, No. 29, No. 30, No. 31 and No. 33, separately,

Lemma 4.3 *Assume that V is of No. 14 in* [M-M81, Table 2]. *Then $X \cong \mathbf{A}^3$. D_1 is the exceptional divisor of the blow-up $\varphi_1 : V \to W_1 := V_5 \hookrightarrow \mathbf{P}^6$ and D_2 is the proper transform by φ_1 of $\overline{D_2} := H^0_5 \in |-\frac{1}{2}K_{V_5}|$ containing the center C_1 of φ_1 which does not pass through the singular point of $\overline{D_2}$.*

Proof. Let V be a Fano 3-fold of No. 14, i.e., it is obtained from $W_1 = V_5 \hookrightarrow \mathbf{P}^6$ via the blow-up φ_1 along the curve C_1 which is an intersection of two hyperplane sections $\in |-\frac{1}{2}K_{V_5}|$. Of course, we can take $\overline{D_2}$ as one of them. By Theorem 1.1, $\overline{D_2}$ is either H_5^0 or H_5^∞. Assume that $\overline{D_2} = H_5^\infty$. Since C_1 is Cartier on H_5^∞ and smooth, it is disjoint from the singular locus $\Sigma := \mathrm{Sing}\, H_5^\infty$. But since $H_5^\infty - \Sigma \cong \mathbf{A}^2$ (cf. [A-F, F-T92]), C_1 have to meet Σ, a contradiction. Thus $\overline{D_2} = H_5^0$ and we complete the proof. □

Thus we obtain Type I in **Table 1**.

Lemma 4.4 *The case No. 17 in* [M-M81, Table 2] *does not occur.*

Proof. Assume that V is of No. 17. Then there are two possibilities with respect to the contraction $\varphi_1 : (D_1 \subset V) \to (C_1 \subset W_1)$. Namely, the one is $W_1 = \mathbf{P}^3$, C_1 is of degree 5 of genus 1 and the other is $W_1 = \mathbf{Q}^3 \hookrightarrow \mathbf{P}^4$, C_1 is of degree 5 of genus 1. The former is obviously absurd because such a curve C_1 is not contained in a hyperplane $\overline{D_2} \cong \mathbf{P}^2$. We consider the latter case and assume that C_1 is contained in a hyperplane section $\overline{D_2} \cong \mathbf{Q}_0^2$. Since $\deg(C_1) = (H_1 \cdot C_1) = 5$ we have $C_1 \equiv 5l$ on $\overline{D_2}$, where l is a generating line on $\overline{D_2} \cong \mathbf{Q}_0^2$. Then $(K_{\overline{D_2}} \cdot C_1)_{\overline{D_2}} = (K_{W_1} + \overline{D_2} \cdot C_1) = (-2\overline{D_2} \cdot C_1) = -10$ and $(C_1^2)_{\overline{D_2}} = 25(l^2)_{\overline{D_2}} = \frac{25}{2}$. Let $\mu : \widehat{D_2} \to \overline{D_2}$ be the resolution of singularity and let $\widehat{e}$ be the exceptional curve which is a (-2)-curve. We write $\mu^*(C_1) \equiv \widehat{C_1} + a\widehat{e}$ with $a \geq 0$. Then $(\widehat{C_1}^2)_{\widehat{D_2}} = ({C_1}^2)_{\overline{D_2}} - 2a^2 = \frac{25}{2} - 2a^2$. Since $\widehat{C_1}$ is an elliptic curve and $(K_{\widehat{D_2}} \cdot \widehat{C_1})_{\widehat{D_2}} = (K_{\overline{D_2}} \cdot C_1)_{\overline{D_2}} = -10$, we have $a = \frac{\sqrt{5}}{2}$, a contradiction. □

Lemma 4.5 *Assume that V is of No. 20 in* [M-M81, Table 2]*. Then $X \cong \mathbf{A}^3$. D_1 is the exceptional divisor of the blow-up $\varphi_1 : V \to W_1 := V_5 \hookrightarrow \mathbf{P}^6$ along the twisted cubic C_1 and D_2 is the proper transform by φ_1 of $\overline{D_2} := H_5^0 \in |-\frac{1}{2}K_{V_5}|$ containing the center C_1 passing through the singular point of $\overline{D_2}$.*

Proof. Let V be a Fano 3-fold of No. 20, i.e., V is obtained from $W_1 := V_5 \hookrightarrow \mathbf{P}^6$ via the blow-up along the twisted cubic C_1 on W_1. By Lemma 4.1, the boundary $\overline{D_2}$ is either H_5^0 or H_5^∞. We claim the following:

Claim. The case $\overline{D_2} = H_5^\infty$ does not occur.

Proof. Assume $\overline{D_2} = H_5^\infty$. Let $\sigma : \widetilde{D_2} \to \overline{D_2}$ be the normalization and let $\mathcal{J} \subset \mathcal{O}_{\overline{D_2}}$ be the conductor sheaf with respect to σ. Let $\overline{\Sigma} := \mathrm{V}_{\overline{D_2}}(\mathcal{J})$ and let $\widetilde{\Sigma} := \mathrm{V}_{\widetilde{D_2}}(\mathcal{J})$. Then $\widetilde{D_2} \cong \mathbf{F}_3$, $\widetilde{\Sigma} = M_3 + L_3$ and $\overline{\Sigma}$ is a line on V_5 with the normal bundle $\mathcal{N}_{\overline{\Sigma}/V_5} \cong \mathcal{O}_{\overline{\Sigma}}(-1) \oplus \mathcal{O}_{\overline{\Sigma}}(1)$ (cf. [F-T92, A-F]). It is clear that C_1 is distinct from $\overline{\Sigma}$. Let $\widetilde{C_1}$ be the proper transform on $\widetilde{D_2}$ of $C_1 \subset \overline{D_2}$. We write $\widetilde{C_1} \equiv \alpha M_3 + \beta L_3$, where $(\alpha, \beta) = (0, 1)$ or $\alpha \geq 3\beta$ with $\beta > 0$. Note that $K_{\widetilde{D_2}} \equiv \sigma^*(K_{\overline{D_2}}) - \widetilde{\Sigma}$ (cf. [Mo82]). Hence $(K_{\widetilde{D_2}} \cdot \widetilde{C_1})_{\widetilde{D_2}} = (K_{\overline{D_2}} \cdot C_1)_{\overline{D_2}} - (M_3 + L_3 \cdot \widetilde{C_1})_{\widetilde{D_2}} = 2\alpha - \beta - 3$. Thus we have $(\widetilde{C_1} \cdot \widetilde{C_1} + K_{\widetilde{D_2}})_{\widetilde{D_2}} = -(3\alpha^2 - 2\alpha + 3) + (2\alpha - 1)\beta$. When $(\alpha, \beta) = (0, 1)$, it follows that $(\widetilde{C_1} \cdot \widetilde{C_1} + K_{\widetilde{D_2}})_{\widetilde{D_2}} = -4$, a contradiction. When $\alpha \geq 3\beta$ with $\beta > 0$ we have $(\widetilde{C_1} \cdot \widetilde{C_1} + K_{\widetilde{D_2}})_{\widetilde{D_2}} \geq -(3\alpha^2 - 2\alpha + 3) + 3\alpha(2\alpha - 1) = 3\alpha^2 - \alpha - 3 \geq -1$, which is also a contradiction. □

Thus we may assume that $\overline{D_2} = H_5^0$. Let $\mu : \widehat{D_2} \to \overline{D_2}$ be the minimal resolution. Let $\widehat{C} \subset \widehat{D_2}$ be the curve obtained as in Lemma 2.3 (i). Then the proper image $\mu(\widehat{C}) \subset \overline{D_2} \subset V_5$ is a twisted cubic. Therefore, we can take $\mu(\widehat{C})$ as the center C_1 of the blow-up $\varphi_1 : V \to W_1$. Thus we obtain the assertion. □

Thus we have Type II in **Table 1**.

The similar argument can be applied to the case of No. 22.

Lemma 4.6 *Assume that V is of No. 22 in* [M-M81, Table 2]*. Then $X \cong \mathbf{A}^3$. D_1 is the exceptional divisor of the blow-up $\varphi_1 : V \to W_1 := V_5 \hookrightarrow \mathbf{P}^6$ along the conic C_1 and D_2 is the proper transform by φ of $\overline{D_2} := H_5^0 \in |-\frac{1}{2}K_{V_5}|$ containing the center C_1 passing through the singular point of $\overline{D_2}$.*

Proof. Assume that V is of No. 22. Then there are two possibilities with respect to the blow-up $\varphi_1 : (D_1 \subset V) \to (C_1 \subset W_1)$, namely, the one is $W_1 = \mathbf{P}^3$, C_1 is $\deg(C_1) = 4$ of genus $g(C_1) = 0$ and the other is $W_1 = V_5 \hookrightarrow \mathbf{P}^6$, C_1 is a conic on V_5. The former is obviously absurd because such a curve C_1 cannot be contained in a hyperplane. Hence we have only to consider the latter case. Hence $\overline{D_2}$ is either H_5^0 or H_5^∞. The similar argument as in the proof of Lemma 4.5 with an adequate modification shows that the case $\overline{D_2} = H_5^\infty$ does not occur. So, let $\overline{D_2} = H_5^0$. Let $\mu : \widehat{D_2} \to \overline{D_2}$ be the minimal resolution. Let $\widehat{C}$ be the curve obtained as in Lemma 2.3 (ii). Then the proper image $\mu(\widehat{C}) \subset \overline{D_2} \subset V_5$ is a smooth conic. Hence we can take $\mu(\widehat{C})$ as the center C_1 of the blow-up φ_1. Thus we obtain the assertion. □

The embedding in Lemma 4.6 is of Type III in **Table 1**.

Lemma 4.7 *Assume that V is one of No. 23, No. 28, No. 29, No. 30 and No. 31 in* [M-M81, Table 2]*. Then $X \cong \mathbf{A}^3$. D_1 is the exceptional divisor of the blow-up $\varphi_1 : V \to W_1$ and D_2 is the proper transform by φ_1 of the hyperplane (section) $\overline{D_2} \in |L_1|$ containing the center C_1 of φ_1.*

Proof. Let V be one of smooth Fano 3-folds mentioned in the statement. Then V has exactly one extremal ray of $E1$-type and W_1 is either $\mathbf{P}^3$ or $\mathbf{Q}^3 \hookrightarrow \mathbf{P}^4$. In any case it is easy to see that the center C_1 of the blow-up $\varphi_1 : V \to W_1$ can be contained in $\overline{D_2}$. Thus we have the assertion. □

Thus we obtain the embeddings of Types IV, VIII, IX, X and XIV in **Table 1**.

Lemma 4.8 *Assume that V is of No. 26 in* [M-M81, Table 2]*. Then $X \cong \mathbf{A}^3$. D_1 is the exceptional divisor of the blow-up $\varphi_1 : V \to W_1 := V_5 \hookrightarrow \mathbf{P}^6$ along the line C_1 and D_2 is the proper transform by φ_1 of $\overline{D_2} \in |-\frac{1}{2}K_{V_5}|$, which is either H_5^0 or H_5^∞, containing the center C_1 of φ_1. Furthermore, in the case $\overline{D_2} = H_5^\infty$, C_1 is distinct from the singular locus of $\overline{D_2}$.*

Proof. Assume that V is of No. 26. Then there are two possibilities concerning $\varphi_1 : (D_1 \subset V) \to (C_1 \subset W_1)$. The one is $W_1 = \mathbf{Q}^3 \hookrightarrow \mathbf{P}^4$, C_1 is a twisted cubic, and the other is $W_1 = V_5 \hookrightarrow \mathbf{P}^6$, C_1 is a line on V_5. The former is obviously impossible because a twisted cubic C_1 cannot contained in a hyperplane section $\overline{D_2} \cong \mathbf{Q}_0^2$. Hence we have only to consider the latter case. By Lemma 4.1, the boundary $\overline{D_2}$ is either H_5^0 or H_5^∞. If $\overline{D_2} = H_5^0$, then it is known that there exists exactly one line contained in $\overline{D_2}$ (cf. [Fur86, Proposition 15]). Indeed, let $\mu : \widehat{D_2} \to \overline{D_2}$ be the minimal resolution and let $\widehat{C}$ be the curve on $\widehat{D_2}$ obtained as in Lemma 2.3 (iii). Then the proper image $\mu(C)$ is a such a line. Hence we have the assertion. On the other hand, if $\overline{D_2} = H_5^\infty$, it is swept out by lines on V_5 intersecting the singular locus $\overline{\Sigma}$ of $\overline{D_2}$. Note that if we choose $\overline{\Sigma}$ as the center C_1 of the blow-up φ_1, then $D_1 + D_2 = \varphi_1^*(\overline{D_2}) - D_1$ and $(D_1 + D_2 \cdot l_1) = (-D_1 \cdot l_1) = 1$, which is a contradiction. Hence we have to choose a line on $\overline{D_2}$ distinct from $\overline{\Sigma}$ as C_1. Thus we complete the assertion. □

Hence we have the embedding of Type VI in **Table 1**. The remaining case in this subsection is of No. 33.

Lemma 4.9 *Assume that V is of No. 33 in* [M-M81, Table 2]*. Then $X \cong \mathbf{A}^3$. D_1 is the exceptional divisor of the blow-up $\varphi_1 : V \to W_1 := \mathbf{P}^3$ along the line C_1 and D_2 is the proper transform by φ_1 of $\overline{D_2} \in |\mathcal{O}_{\mathbf{P}^3}(1)|$ containing the center C_1 of φ_1.*

Proof. Assume that V is of No. 33, i.e., $V \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbf{P}^1}(1))$ and $\varphi_1 : V \to W_1 := \mathbf{P}^3$ is the blow-up along the line C_1. Hence the assertion is easy to see. □

Thus we have the embedding of Type XVII with $x = -1$ in **Table 1**, where we shall interchange the roles of φ_1 and φ_2.

4.2 φ_1 is of $E2$-type

In this subsection we consider the case where $\varphi_1 : V \to W_1$ is of $E2$-type, i.e., it contracts $D_1 \cong \mathbf{P}^2$ with a normal bundle $\mathcal{O}_{D_1}(D_1) \cong \mathcal{O}_{\mathbf{P}^2}(-1)$ to a smooth point, say $p_1 := \varphi_1(D_1)$, in a smooth projective

3-fold W_1. Let $\overline{D_2} := \varphi_1(D_2)$. Since the exceptional divisor D_1 of φ_1 is contained in the boundary, we have $X = V - (D_1 \cup D_2) \cong W_1 - \overline{D_2}$. In [M-M81, Table 2], there exist two Fano 3-folds with $B_2 = 2$ having extremal ray of $E2$-type, i.e., No. 30 and No. 35.

Lemma 4.10 *Assume that V is of No. 30 in* [M-M81, Table 2]. *Then $X \cong \mathbf{A}^3$. D_1 is the exceptional divisor of the blow-up $\varphi_1 : V \to W_1 := \mathbf{Q}^3 \hookrightarrow \mathbf{P}^4$ and D_2 is the proper transform by φ_1 of the hyperplane section $\mathbf{Q}_0^2 \cong \overline{D_2} \in |\mathcal{O}_{\mathbf{Q}^3}(1)|$ which contains the center p_1 of φ_1 as a smooth point.*

Proof. Assume that V is of No. 30. Then $\varphi_1 : (D_1 \subset V) \to (p_1 \in W_1 := \mathbf{Q}^3)$ is the blow-up at the point p_1. Since $(\mathbf{Q}^3, \overline{D_2})$ is the compactification of the contractible affine 3-fold X, it follows that $X \cong \mathbf{A}^3$ and $\overline{D_2} \cong \mathbf{Q}_0^2$. If p_1 is a vertex of $\overline{D_2}$, then we have $D_1 + D_2 = \varphi_1^*(\overline{D_2}) - D_1$ and $(D_1 + D_2 \cdot l_1) = (-D_1 \cdot l_1) = 1$, which is a contradiction. Hence p_1 is a smooth point on $\overline{D_2}$. Thus we have the assertion. □

Hence we have the embedding of Type XI in **Table 1**.

Lemma 4.11 *Assume that V is of No. 35 in* [M-M81, Table 2]. *Then $X \cong \mathbf{A}^3$. D_1 is the exceptional divisor of the blow-up $\varphi_1 : V \to W_1 := \mathbf{P}^3$ and D_2 is the proper transform by φ_1 of the hyperplane $\overline{D_2} \in |\mathcal{O}_{\mathbf{P}^3}(1)|$ containing the center p_1 of φ_1.*

Proof. Since the asssertion is easy to see, we shall omit the detail. □

The embedding in Lemma 4.11 is a special case of Type XXIII in **Table 1**, where we interchange the roles of φ_1 and φ_2.

4.3 φ_1 is of $E3$-type

In this subsection we consider the case where $\varphi_1 : V \to W_1$ is of $E3$-type, i.e., it contracts $D_1 \cong \mathbf{P}^1 \times \mathbf{P}^1$ with a normal bundle $\mathcal{O}_{D_1}(D_1) \cong \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(-1, -1)$ to a Gorenstein terminal singularity, say $p_1 := \varphi_1(D_1)$, in W_1. Let $\overline{D_2} := \varphi_1(D_2)$. Since the exceptional divisor D_1 of φ_1 is contained in the boundary, we have $X = V - (D_1 \cup D_2) \cong W_1 - \overline{D_2}$. Since $(D_1 + D_2 \cdot l_1) = -1 + (D_2 \cdot l_1) = 0$, D_2 meets D_1 in such a way that $D_2|_{D_1} \in |\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1, 1)|$. First of all we prove the following:

Lemma 4.12 *With the notations and assumptions as above. Then:*

(1) $B_3(W_1) = B_3(V) + 1$,

(2) $B_3(\overline{D_2}) = B_3(V) + 1$,

(3) $B_2(\overline{D_2}) = 1$.

Proof. By noting that $V - D_1 \cong W_1 - \{p_1\}$, we consider the following exact sequences associated with the pairs (V, D_1) and $(W_1, \{p_1\})$:

$$\begin{array}{ccccccccccc} 0 \to H_3(V) & \longrightarrow & H_3(V, D_1) & \longrightarrow & H_2(D_1) & \longrightarrow & H_2(V) & \longrightarrow & H_2(V, D_1) \to 0 \\ \downarrow & & \downarrow\cong & & \downarrow & & \downarrow & & \downarrow\cong \\ 0 \to H_3(W_1) & \to & H_3(W_1, \{p_1\}) & \longrightarrow & 0 & \longrightarrow & H_2(W_1) & \to & H_2(W_1, \{p_1\}) \to 0. \end{array}$$

Since $B_2(D_1) = B_2(V) = 2$ and $B_2(W_1) = 1$, these exact sequences imply that $B_3(W_1) = B_3(V) + 1$. Thus we obtain (1). On the other hand, by writing the exact sequence of the pair $(W_1, \overline{D_2})$, we have $B_2(\overline{D_2}) = B_2(W_1)$ and $B_3(\overline{D_2}) = B_3(W_1)$ in consideration of the contractibility of $X \cong W_1 - \overline{D_2}$. Thus we complete the proof. □

In [M-M81, Table 2] there exist three smooth Fano 3-folds No. 8 (a), No. 15 (a), No. 23 (a) having an extremal ray of $E3$-type. We prove the following:

Lemma 4.13 (1) *$\overline{D_2}$ is an ample generator of* $\mathrm{Pic}(W_1)$,

(2) $-K_{W_1} \equiv 2\overline{D_2}$,

(3) $D_1 \sim H_1 - H_2$,

(4) $D_2 \sim H_2$.

Proof. Since $p_1 = \varphi_1(D_1) \in W_1$ is a Gorenstein terminal singularity, every Weil divisor on W_1 is Cartier (cf. [Ka88]). In particular, $\overline{D_2}$ is Cartier. Since the coordinate ring of $X \cong W_1 - \overline{D_2}$ is factorial, $\overline{D_2}$ is an ample generator of $\mathrm{Pic}(W_1)$. When we write $-K_{W_1} \sim \alpha\overline{D_2}$ with $\alpha \in \mathbf{N}$ we have $H_1+H_2 \sim -K_V = \varphi_1^*(-K_{W_1})-D_1 \sim \alpha H_1 - D_1$ by Proposition 2.1. Hence $D_1 \sim (\alpha-1)H_1-H_2$. On the other hand, by [M-M83, Theorem 5.1], it follows that $1 = (D_1 \cdot l_2) = ((\alpha-1)H_1 - H_2 \cdot l_2) = \alpha - 1$. Hence $\alpha = 2$, i.e., $-K_{W_1} \sim 2\overline{D_2}$ and $D_1 \sim H_1 - H_2$. Since $\overline{D_2}$ is Cartier, we can write $H_1 = \varphi_1^*(\overline{D_2}) = D_2 + \beta D_1$ for some $\beta \in \mathbf{N}$. Hence $D_2 \sim (1-\beta)H_1 + \beta H_2$. Note that $0 \leq (D_2 \cdot l_2)$. In fact, if V is No. 8 (a), then φ_2 is of $C1$-type so it is clear that $(D_2 \cdot l_2) \geq 0$. Meanwhile, if V is either of No. 15 (a) or No. 23 (a), then D_2 is distinct from the exceptional divisor of $E1$-type divisorial contraction $\varphi_2 : V \to W_2$ as D_2 meets $D_1 \cong \mathbf{P}^1 \times \mathbf{P}^1$ in such a way that $D_2|_{D_1} \in |\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,1)|$ (see the descriptions in No. 15 (a) and No. 23 (a) in [M-M81, Table 2]). Hence $(D_2 \cdot l_2) \geq 0$ in this case also. Therefore $(D_2 \cdot l_2) = ((1-\beta)H_1 + \beta H_2 \cdot l_2) = 1 - \beta \geq 0$, i.e., $\beta \leq 1$. Thus we have $\beta = 1$ and $D_2 \sim H_2$ as desired. □

In fact we shall prove the following:

Lemma 4.14 *These cases No. 8 (a), No. 15 (a) and No. 23 (a) do not occur.*

Proof. Since $-K_{W_1} \sim 2\overline{D_2}$ and $\overline{D_2}$ is Gorenstein, $\overline{D_2}$ is a (possibly non-normal) del Pezzo surface. If $\overline{D_2}$ is a normal del Pezzo surface, then $B_3(\overline{D_2}) = 0$ because of the rationality of normal del Pezzo surfaces (cf. [De80], [H-W81]). But by Lemma 4.12, we have $B_3(\overline{D_2}) = B_3(V) + 1 = 19$ (resp. $= 9$, $= 3$) when V is of No. 8 (a) (resp. No. 15 (a), No. 23 (a)). Hence this is a contradiction. Thus we may assume that $\overline{D_2}$ is a non-normal del Pezzo surface. Let $\sigma : \widetilde{D_2} \to \overline{D_2}$ be the normalization. Let $\mathcal{J} \subset \mathcal{O}_{\overline{D_2}}$ be the conductor sheaf of σ and let $\overline{\Sigma} := \mathrm{V}_{\overline{D_2}}(\mathcal{J})$, $\widetilde{\Sigma} := \mathrm{V}_{\widetilde{D_2}}(\mathcal{J})$ be subvarieties in $\overline{D_2}$ and $\widetilde{D_2}$ defined by $\mathcal{J} \subset \mathcal{O}_{\overline{D_2}}$ and $\mathcal{J} \subset \mathcal{O}_{\widetilde{D_2}}$, respectively. Then we have the exact sequence associated with σ as follows (cf. [B-K82]):

$$\cdots \to H^1(\widetilde{\Sigma}) \to H^2(\overline{D_2}) \to H^2(\widetilde{D_2}) \oplus H^2(\overline{\Sigma}) \to H^2(\widetilde{\Sigma}) \to H^3(\overline{D_2}) \to H^3(\widetilde{D_2}) \oplus H^3(\overline{\Sigma}).$$

By [A-F, Theorem 1.5], it follows that $H^2(\widetilde{\Sigma}) \cong \mathbf{Z}$ or $\mathbf{Z}^{\oplus 2}$ and $H^3(\widetilde{D_2}) \oplus H^3(\overline{\Sigma}) = 0$. Hence we have the surjection $H^2(\widetilde{\Sigma}) \cong \mathbf{Z}$ or $\mathbf{Z}^{\oplus 2} \to H^3(\overline{D_2})$. Since $B_3(\overline{D_2}) = B_3(V) + 1 = 19$ (resp. $= 9$, $= 3$) when V is of No. 8 (a) (resp. No. 15 (a), No. 23 (a)), this is a contradiction. □

4.4 φ_1 is of $E4$-type

In this present subsection, we consider the case where $\varphi_1 : V \to W_1$ is of $E4$-type, i.e., it contracts $D_1 \cong \mathbf{Q}_0^2 \hookrightarrow \mathbf{P}^3$ with a normal bundle $\mathcal{O}_{D_1}(D_1) \cong \mathcal{O}_{\mathbf{P}^3}(-1) \otimes \mathcal{O}_{\mathbf{Q}_0^2}$ to a Gorenstein terminal singularity, say $p_1 := \varphi_1(D_1)$, in W_1. Let $\overline{D_2} := \varphi_1(D_2)$. Since the exceptional divisor D_1 of φ_1 is contained in the boundary, we have $X = V - (D_1 \cup D_2) \cong W_1 - \overline{D_2}$. Since $(D_1 + D_2 \cdot l_1) = -1 + (D_2 \cdot l_1) = 0$, D_2 meets D_1 in such a way that $D_2|_{D_1} \in |\mathcal{O}_{\mathbf{P}^3}(1) \otimes \mathcal{O}_{\mathbf{Q}_0^2}|$. In [M-M81, Table 2], there exist three Fano 3-folds with $B_2 = 2$ having an extremal ray of $E4$-type, i.e., No. 8 (b), No. 15 (b) and No. 23 (b). We shall prove that, in fact, these cases cannot occur. The argument for the proof is similar to that in the previous subsection, so we shall sketch the proof.

Lemma 4.15 *The cases No. 8 (b), No. 15 (b) and No. 23 (b) do not occur.*

Proof. By considering the exact sequences as in the proof of Lemma 4.12 we have $B_3(W_1) = B_3(V)$, where we note that $B_2(D_1) = 1$ as $D_1 \cong \mathbf{Q}_0^2$. Since $X \cong W_1 - \overline{D_2}$ is contractible, we have $B_3(\overline{D_2}) =$

$B_3(\overline{W_1})$ and $B_2(\overline{D_2}) = B_2(W_1) = 1$. The same argument as in Lemma 4.13 implies that $\mathrm{Pic}(W_1) = \mathbf{Z}[\overline{D_2}]$ and $-K_{W_1} \sim 2\overline{D_2}$. Hence $\overline{D_2}$ is a (possibly non-normal) del Pezzo surfaces. When V is of No. 8 (b) (resp. No. 15 (b), No. 23 (b)) we have $B_3(\overline{D_2}) = B_3(V) = 18$ (resp. $= 8$, $= 2$), so $\overline{D_2}$ cannot be a normal del Pezzo surface. Thus we may assume that $\overline{D_2}$ is a non-normal del Pezzo surface. Let $\sigma : \widetilde{D_2} \to \overline{D_2}$ be the normalization. Let $\mathcal{J} \subset \mathcal{O}_{\overline{D_2}}$ be the conductor sheaf of σ and let $\overline{\Sigma} := \mathrm{V}_{\overline{D_2}}(\mathcal{J})$, $\widetilde{\Sigma} := \mathrm{V}_{\widetilde{D_2}}(\mathcal{J})$. Then we have the exact sequence associated with σ (cf. [B-K82]):

$$\cdots \to H^1(\widetilde{\Sigma}) \to H^2(\overline{D_2}) \to H^2(\widetilde{D_2}) \oplus H^2(\overline{\Sigma}) \to H^2(\widetilde{\Sigma}) \to H^3(\overline{D_2}) \to H^3(\widetilde{D_2}) \oplus H^3(\overline{\Sigma}).$$

By [A-F, Theorem 1.5], it follows that $H^1(\widetilde{\Sigma}) = H^3(\widetilde{D_2}) \oplus H^3(\overline{\Sigma}) = 0$, $H^2(\overline{\Sigma}) = \mathbf{Z}$ and $H^2(\widetilde{\Sigma}) = \mathbf{Z}$ or $= \mathbf{Z}^{\oplus 2}$. Hence the above sequence is:

$$0 \to \mathbf{Z} \to H^2(\widetilde{D_2}) \oplus \mathbf{Z} \to \mathbf{Z} \text{ or } \mathbf{Z}^{\oplus 2} \to H^3(\overline{D_2}) \to 0,$$

so $B_3(\overline{D_2}) \leq 2$. Hence the only the possibility is No. 23 (b) and $B_3(\overline{D_2}) = 2$. But then this sequence means that $B_2(\widetilde{D_2}) = 0$, which is a contradiction. $\square$

4.5 φ_1 is of $E5$-type

We consider the case where $\varphi_1 : V \to W_1$ is of $E5$-type, i.e., it contracts $D_1 \cong \mathbf{P}^2$ with a normal bundle $\mathcal{O}_{D_1}(D_1) \cong \mathcal{O}_{\mathbf{P}^2}(-2)$ to an index 2 point $p_1 := \varphi_1(D_1)$ in W_1. Let $\overline{D_2} := \varphi_1(D_2)$. Since the exceptional divisor D_1 of φ_1 is contained in the boundary, we have $X = V - (D_1 \cup D_2) \cong W_1 - \overline{D_2}$. Since $(D_1 + D_2 \cdot l_1) = -2 + (D_2 \cdot l_1) = 0$, D_2 meets D_1 in such a way that $D_2|_{D_1} \in |\mathcal{O}_{\mathbf{P}^2}(2)|$. In [M-M81, Table 2] there exist two Fano 3-folds with $B_2 = 2$ having an extremal ray of $E5$-type, i.e., No. 28 and No. 36.

Lemma 4.16 *The case of No. 28 does not occur.*

Proof. Our proof is similar to that in Lemma 3.1. Assume that V is No. 28 in [M-M81, Table 2], i.e., V is obtained from $W_2 = \mathbf{P}^3$ via the blow-up $\varphi_2 : V \to W_2$ along the plane cubic C_2 with an exceptional divisor E_2 and $\varphi_1 : V \to W_1$ is the contraction of D_1, whicn is the proper transform on V of the hyperplane containing C_2, to an index 2 point $p_1 := \varphi_1(D_1) \in W_1$. Let $\overline{D_2} := \varphi_1(D_2) \subset W_1$. The same argument as in Lemma 3.1 implies that $-K_{W_1} \equiv \frac{3}{2}L_1$, $D_1 \sim H_1 - 2H_2$ and $E_2 \sim -H_1 + 3H_2$. Furthermore, we have $B_2(\overline{D_2}) = 1$, $B_3(\overline{D_2}) = 2$ and $\overline{D_2} \equiv \alpha L_1$ with $\alpha = \frac{1}{2}$ or 1. If $\alpha = \frac{1}{2}$, then $\overline{D_2}$ is a Gorenstein non-normal del Pezzo surface and we obtain a contradiction by the same argument in Lemma 3.1. Thus we may assume that $\alpha = 1$, i.e., $\overline{D_2}$ is an ample generator of $\mathrm{Pic}(W_1)$. Since $\overline{D_2}$ is Cartier, we can write $H_1 \sim \varphi_1^*(\overline{D_2}) = D_2 + mD_1$, i.e., $D_2 \sim (1-m)H_1 + 2mH_2$, where m is a positive integer. Since D_1 and D_2 generate $\mathrm{Pic}(V) \cong \mathbf{Z}[H_1] \oplus \mathbf{Z}[H_2]$ by Lemma 2.1, the matrix $M := \begin{pmatrix} 1 & -2 \\ 1-m & 2m \end{pmatrix}$ have to be unimodular. But $\det(M) = 2$, which is a contradiction. $\square$

Lemma 4.17 *The case of No. 36 does not occur.*

Proof. Assume that V is of No. 36 in [M-M81, Table 2], i.e., $V = \mathbf{P}(\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(2))$. $\varphi_1 : V \to W_1$ is the contraction of the negative section $D_1 \cong \mathbf{P}^2$ of the $\mathbf{P}^1$-bundle structure $\varphi_2 : V \to W_2 \cong \mathbf{P}^2$ with a normal bundle $\mathcal{O}_{D_1}(D_1) \cong \mathcal{O}_{\mathbf{P}^2}(-2)$ to an index 2 point $p_1 := \varphi_1(D_1)$. Associated with φ_2, we have $\mathrm{Pic}(V) \cong \mathbf{Z}[D_1] \oplus \mathbf{Z}[H_2]$. We write $D_2 \sim \alpha D_1 + \beta H_2$ with $\alpha, \beta \in \mathbf{Z}$. Since D_1 and D_2 generate $\mathrm{Pic}(V)$ (cf. Lemma 2.1), we have $\beta \pm 1$. On the other hand, since D_2 meets D_1 in such a way that $D_2|_{D_1} \in |\mathcal{O}_{\mathbf{P}^2}(2)|$, we have $D_2|_{D_1} \sim (\alpha D_1 + \beta H_2)|_{D_1} \sim \mathcal{O}_{\mathbf{P}^2}(-2\alpha + \beta) = \mathcal{O}_{\mathbf{P}^2}(2)$. Hence $-2\alpha + \beta = 2$, i.e., $\alpha = \frac{\beta}{2} - 1 \notin \mathbf{Z}$, which is a contradiction. $\square$

5 The case $D_1 + D_2$ is ample

In this section, we finally consider the case where $D_1 + D_2$ is ample. By noting that $K_V + D_1 + D_2$ is not nef, we have the following:

Lemma 5.1 *At least one of the extremal rays R_1 and R_2 is $(K_V + D_1 + D_2)$-negative. Let R_1 be such an extremal ray. Then the contraction $\varphi_1 : V \to W_1$ of R_1 is of type either $E2$, $C2$, $D2$ or $D3$.*

Proof. We put $t := \sup\{\mu \in \mathbf{Q}\ ; D_1 + D_2 + \mu K_V \text{ is nef}\}$, which is greater than 0 as $D_1 + D_2$ is ample. Furthermore, t is less than 1 as $K_V + D_1 + D_2$ is not nef. Then there exists an extremal ray contained in $\overline{\mathrm{NE}}(V) \cap \{D_1 + D_2 + tK_V\}^{\perp}$. We may assume that $R_1 = \mathbf{R}_+[l_1]$ is such an extremal ray. Then we have $(D_1 + D_2) \cdot l_1 = t(-K_V \cdot l_1) < (-K_V \cdot l_1)$. Since $D_1 + D_2$ is ample, $(D_1 + D_2) \cdot l_1$ is a positive integer. Hence $(-K_V \cdot l_1) \geq 2$. By the classification of extremal rays on smooth projective 3-folds (cf. [Mo82]), we know that φ_1 is of type either $E2$, $C2$, $D2$ or $D3$. □

In the subsequent of this section, we consider according as the type of $\varphi_1 : V \to W_1$.

5.1 φ_1 is of $E2$-type

First of all, we consider the case where $\varphi_1 : V \to W_1$ is of type $E2$. As seen below, some kinds of exotic $\mathbf{A}^3$'s can be compactified in this situation (cf. Theorem 1.3, Types XII, XIII and XXII in **Table 1** and Remark 1.3). Let E_1 be the exceptional divisor of φ_1 with a normal bundle $\mathcal{O}_{E_1}(E_1) \cong \mathcal{O}_{\mathbf{P}^2}(-1)$ which is contracted to a smooth point $p_1 := \varphi_1(E_1) \in W_1$. In [M-M81, Table 2], there are two smooth Fano 3-folds with $B_2 = 2$ having an extremal ray of type $E2$, namely, No. 30 and No. 35. We prove the following:

Lemma 5.2 (1) *E_1 is not contained in the boundary $D_1 \cup D_2$,*

(2) *One of the boundary components, say D_1, meets E_1 in such a way that $D_1|_{E_1} \in |\mathcal{O}_{\mathbf{P}^2}(1)|$, and the other component D_2 is disjoint from E_1.*

Proof. Note that when V is of No. 30 (resp. No. 35), W_1 is isomorphic to $\mathbf{Q}^3 \hookrightarrow \mathbf{P}^4$ (resp. $\mathbf{P}^3$). Assume that E_1 is contained in the boundary $D_1 \cup D_2$, say $E_1 = D_1$. Let $\overline{D_2} := \varphi_1(D_2)$ be the proper image on W_1. Then $(W_1, \overline{D_2})$ is a smooth compactification of a contractible affine 3-fold X, so $X \cong \mathbf{A}^3$ and $\overline{D_2}$ is an ample generator of $\mathrm{Pic}(W_1) \cong \mathbf{Z}$. If W_1 is isomorphic to $\mathbf{Q}^3 \hookrightarrow \mathbf{P}^4$ (resp. $\mathbf{P}^3$), then $\overline{D_2} \in |\mathcal{O}_{W_1}(1)|$ is a quadric cone $\mathbf{Q}^2_0$ (resp. a hyperplane $\mathbf{P}^2$). If p_1 is a smooth point on $\overline{D_2}$, then we have $\varphi_1^*(\overline{D_2}) = D_1 + D_2$. Hence $(D_1 + D_2) \cdot l_1 = 0$, which is a contradiction. Thus p_1 is a singular point on $\overline{D_2}$, so the possibility is No. 30: $W_1 \cong \mathbf{Q}^3$, $\overline{D_2} \cong \mathbf{Q}^2_0$ and p_1 is a vertex of $\overline{D_2}$. Then $D_1 + D_2 = \varphi_1^*(\overline{D_2}) - D_1 \sim H_1 - D_1$. We have $-K_V = \varphi_1^*(-K_{W_1}) - 2D_1 \sim \varphi_1^*(3\overline{D_2}) - 2D_1 \sim 3H_1 - 2D_1$. Since $-K_V \sim H_1 + 2H_2$ by Proposition 2.1, we have $D_1 \sim H_1 - H_2$. Thus $D_1 + D_2 = \varphi_1^*(\overline{D_2}) - D_1 \sim H_1 - (H_1 - H_2) = H_2$, which is not ample. This is a contradiction. Hence E_1 is not contained in $D_1 \cup D_2$. Since $D_1 + D_2$ is ample and $(D_1 + D_2) \cdot l_1 = t(-K_V \cdot l_1) = 2t$ with $0 < t < 1$, we have $t = \frac{1}{2}$ and $(D_1 + D_2) \cdot l_1 = 1$. This means the assertion (2). □

We put $\overline{D_i} := \varphi_1(D_i) \subset W_1$ for $i = 1, 2$. By Lemma 5.2, both of $\overline{D_1}$ and $\overline{D_2}$ are divisors on W_1, and $p_1 := \varphi_1(E_1)$ is contained in $\overline{D_1}$ as a smooth point but not contained in $\overline{D_2}$.

Lemma 5.3 *$\overline{D_2}$ is an ample generator of* $\mathrm{Pic}(W_1)$*, i.e.,* $\mathrm{Pic}(W_1) \cong \mathbf{Z}[\overline{D_2}]$.

Proof. Since $\mathrm{Pic}(V) = \mathbf{Z}[D_1] \oplus \mathbf{Z}[D_2]$ (cf. Lemma 2.1), we can write $E_1 \sim a_1 D_1 + a_2 D_2$ with $a_1, a_2 \in \mathbf{Z}$. By noting that $-1 = E_1 \cdot l_1 = (a_1 D_1 + a_2 D_2) \cdot l_1$, D_1 meets E_1 in such a way that $D_1|_{E_1} \in |\mathcal{O}_{\mathbf{P}^2}(1)|$ and D_2 is disjoint from E_1, we have $a_1 = -1$. Hence $\mathrm{Pic}(W_1) \cong \mathrm{Pic}(V)/ < E_1 >\cong \mathbf{Z}[D_1] \oplus \mathbf{Z}[D_2]/ < -D_1 + a_2 D_2 >\cong \mathbf{Z}[\overline{D_2}]$. Thus $\overline{D_2}$ is an ample generator of $\mathrm{Pic}(W_1)$. □

Lemma 5.4 *Assume that V is No. 35 in* [M-M81, Table 2]*, i.e., $V \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(1))$. Then we have:*

(1) *The complement* $Z := W_1 - \overline{D_2}$ *is isomorphic to the affine 3-space* $\mathbf{A}^3$,

(2) X *is obtained as the affine modification of* Z *with a locus* $(p_1 \in D_1^\circ)$, *where* $D_1^\circ := \overline{D_1} \cap Z$*: see Definition 2.1 for the definition on affine modifications,*

(3) D_1° *is an acyclic hypersurface.*

Proof. The first assertion (1) is obvious because $W_1 \cong \mathbf{P}^3$ and $\overline{D_2}$ is an ample generator of $\mathrm{Pic}(W_1)$ (cf. Lemma 5.3). Note that $X = V - (D_1 \cup D_2)$ is obtained from Z via the blow-up at the point p_1 and deleting the proper transform of D_1°. Hence X is just the affine modification of Z with a locus $(p_1 \in D_1^\circ)$. Let $\sigma : X \to Z$ be this affine modification. Note that $E_1^\circ := E_1 \cap X \cong \mathbf{A}^2$ is an exceptional divisor and D_1° is a divisor of σ in the terminology of Definition 2.1 (see also [K-Z99]). Since σ preserves the contractibility, $\sigma|_{E_1^\circ}$ induces isomorphims of homology groups $(\sigma|_{E_1^\circ})_* : H_*(E_1^\circ; \mathbf{Z}) \xrightarrow{\sim} H_*(D_1^\circ; \mathbf{Z})$ by [K-Z99, Theorem 3.1]. Thus we complete the proof. □

The embedding stated in Lemma 5.4 is of Type XXII in **Table 1**.

Lemma 5.5 *Assume that* V *is No. 30. Then we have:*

(1) $\overline{D_2}$ *is isomorphic to a smooth quadric* $\mathbf{P}^1 \times \mathbf{P}^1$ *or a quadric cone* $\mathbf{Q}_0^2$,

(2) *If* $\overline{D_2} \cong \mathbf{P}^1 \times \mathbf{P}^1$ *(resp.* $\cong \mathbf{Q}_0^2$*), then the complement* $Z := W_1 - \overline{D_2}$ *is isomorphic to* $\mathrm{SL}(2; \mathbf{C})$ *(resp.* $\mathbf{A}^3$*),*

(3) X *is obtained as the affine modification of* Z *with a locus* $(p_1 \in D_1^\circ)$, *where* $D_1^\circ := \overline{D_1} \cap Z$,

(4) *If* $\overline{D_2} \cong \mathbf{P}^1 \times \mathbf{P}^1$, *then* D_1° *satisfies* $H_0(D_1^\circ; \mathbf{Z}) \cong H_1(D_1^\circ; \mathbf{Z}) \cong \mathbf{Z}$ *and* $H_i(D_1^\circ; \mathbf{Z}) = 0$ *for* $i > 1$. *On the other hand, if* $\overline{D_2} \cong \mathbf{Q}_0^2$, *then* D_1° *is an acyclic surface.*

Proof. The first assertion (1) is obvious as $W_1 \cong \mathbf{Q}^3$ and $\overline{D_2}$ is an ample generator of $\mathrm{Pic}(W_1)$. The remaining assertions (1)∼(4) for the case $\overline{D_2} \cong \mathbf{Q}_0^2$ can be obtained by the same argument as in the proof of Lemma 5.4. Hence we have only to consider the case where $\overline{D_2} \in |\mathcal{O}_{W_1}(1)|$ is a smooth quadric. Then it is not difficult to see that $Z := \mathbf{Q}^3 - \overline{D_2}$ is isomorphic to the hypersurface in $\mathbf{A}^4{}_{(x,y,z,u)}$ defined by $xy - zu - 1 = 0$, i.e., $Z \cong \mathrm{SL}(2; \mathbf{C})$. We can see that X is obtained from Z via the affine modification with a locus $(p_1 \in D_1^\circ)$, where $D_1^\circ := \overline{D_1} \cap Z$. Let $\sigma : X \to Z$ be this affine modification. Note that $E_1^\circ := E_1 \cap X \cong \mathbf{A}^2$ is an exceptional divisor and D_1° is a divisor of σ (cf. Definition 2.1). We put $X^* := X - E_1^\circ$ and $Z^* := Z - D_1^\circ$. Associated with σ, we write the exact sequences:

$$\begin{array}{ccccccccccc}
\cdots \to & H_{i+1}(X, X^*) & \longrightarrow & H_i(X^*) & \longrightarrow & H_i(X) & \longrightarrow & H_i(X, X^*) & \longrightarrow & H_{i-1}(X^*) & \to \cdots \\
 & & & \downarrow\cong & & \downarrow \sigma_* & & & & \downarrow\cong & \\
\cdots \to & H_{i+1}(Z, Z^*) & \longrightarrow & H_i(Z^*) & \longrightarrow & H_i(Z) & \longrightarrow & H_i(Z, Z^*) & \longrightarrow & H_{i-1}(Z^*) & \to \cdots
\end{array}$$

Note that $H_i(X, X^*) \cong H_{i-2}(E_1^\circ)$ and $H_i(Z, Z^*) \cong H_{i-2}(D_1^\circ)$ by Thom isomorphism. We shall investigate what topological data on D_1° are needed in order that X is contractible. Since p_1 is a smooth point of D_1°, we have $\sigma^*(D_1^\circ) = E_1^\circ$. Hence σ induces an isomorphism of the fundamental groups $\sigma_* : \pi_1(X) \xrightarrow{\sim} \pi_1(Z) = \{1\}$ (cf. [K-Z99, Proposition 3.1]). Since $Z \cong \mathrm{SL}(2; \mathbf{C})$, Z is simply connected and satisfies $H_0(Z) = H_3(Z) = \mathbf{Z}$ and $H_i(Z) = 0$ for $i \neq 0, 3$. Hence X is acyclic if and only if D_1° satisfies $H_0(D_1^\circ) = H_1(D_1^\circ) = \mathbf{Z}$ and $H_i(D_1^\circ) = 0$ for $i > 1$ by the above exact sequences. Thus X is contractible if and only if D_1° satisfies $H_0(D_1^\circ) = H_1(D_1^\circ) = \mathbf{Z}$ and $H_i(D_1^\circ) = 0$ for $i > 1$ by a theorem of J.H.C. Whitehead. □

Thus we have the embeddings of Types XII and XIII in **Table 1**.

Example 5.1 Let $\overline{D_2} \in |\mathcal{O}_{\mathbf{Q}^3}(1)|$ be isomorphic to a smooth quadric $\mathbf{P}^1 \times \mathbf{P}^1$. Then the complement $Z = \mathbf{Q}^3 - \overline{D_2}$ is isomorphic to $\mathrm{SL}(2; \mathbf{C})$. We may assume that $Z = \{x^2 + y^2 + zu - 1 = 0\} \subset \mathbf{A}^4{}_{(x,y,z,u)}$. We shall choose, for example, $\{u = 0\} \cap Z$ as D_1°. Note that since $D_1^\circ \cong \mathbf{C}^* \times \mathbf{A}^1$, the conditions mentioned in Lemma 5.5 (4) are satisfied. Let $p_1 := (1, 0, 0, 0)$. Then the affine 3-fold X, which is obtained from Z via the affine modification with a locus $(p_1 \in D_1^\circ)$, is isomorphic to the hypersurface in $\mathbf{A}^4{}_{(x,y,z,u)}$ defined by $x^2u + 2x + y^2u + zu = 0$. Hence $X \cong \mathbf{A}^3$.

5.2 φ_1 is of $C2$-type

In this subsection, we consider the case where $\varphi_1 : V \to W_1$ is of $C2$-type. In the list [M-M81, Table 2], there are seven deformation types of smooth Fano 3-folds having extremal rays of $C2$-type, namely, No. 24, No. 27, No. 31, No. 32, No. 34, No. 35 and No. 36. Note that the base surface W_1 is isomorphic to the projective plane $\mathbf{P}^2$ (cf. [M-M83]). We have the following:

Lemma 5.6 (1) *$t = \frac{1}{2}$ and $(D_1 + D_2) \cdot l_1 = 1$,*

(2) *We may assume that D_1 satisfies $D_1 \cdot l_1 = 1$ and $D_2 \sim H_1 = \varphi_1^* \mathcal{O}_{\mathbf{P}^2}(1)$. In particular, D_1 is a birational section of φ_1,*

(3) *The restriction $\varphi_1|_X$ of φ_1 onto $X = V - (D_1 \cup D_2)$ yields a trivial $\mathbf{A}^1$-bundle structure over the affine plane $\mathbf{A}^2$, i.e., $X \cong \mathbf{A}^2 \times \mathbf{A}^1 \cong \mathbf{A}^3$.*

Proof. By the choice of the extremal ray $R_1 = \mathbf{R}_+[l_1]$, we have $(D_1 + D_2) \cdot l_1 = t(-K_V \cdot l_1) = 2t$. Since $D_1 + D_2$ is ample and t is less than 1, we have $t = \frac{1}{2}$ and $(D_1 + D_2) \cdot l_1 = 1$. Thus we may assume that $D_1 \cdot l_1 = 1$ and $D_2 \cdot l_1 = 0$. By Proposition 2.1, we can write $D_1 \sim aH_1 + H_2$ and $D_2 \sim bH_1$ with $a, b \in \mathbf{Z}$. Since $\mathrm{Pic}(V) = \mathbf{Z}[D_1] \oplus \mathbf{Z}[D_2]$ (cf. Lemma 2.1), the matrix $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ is unimodular, so we have $b = \pm 1$. Since the case $b = -1$, i.e., $D_2 \sim -H_1$ is obviously impossible, it follows that $b = 1$, i.e., $D_2 \sim H_1$. Thus the restriction $\varphi_1|_X$ yields an $\mathbf{A}^1$-fibration over the affine plane $W_1 - \varphi_1(D_2) \cong \mathbf{P}^2 - L_1 \cong \mathbf{A}^2$. Since X is a smooth contractible affine 3-fold, this $\mathbf{A}^1$-fibration is surjective (cf. [Ka02], [K-S02]). In fact, all fibers of $\varphi_1|_X$ are isomorphic to the affine line $\mathbf{A}^1$ as $D_1 \cdot l_1 = 1$. Since $\mathrm{Pic}(\mathbf{A}^2) = 0$, we know that $\varphi_1|_X : X \to \mathbf{A}^2$ is a trivial $\mathbf{A}^1$-bundle structure. Hence $X \cong \mathbf{A}^2 \times \mathbf{A}^1 \cong \mathbf{A}^3$. □

Conversely, we shall investigate the configuration of two components D_1 and D_2 on a smooth Fano 3-fold V of No. 24, No. 27, No. 31, No. 32, No. 34, No. 35 or No. 36 in [M-M81, Table 2] in order to see when the complement $X = V - (D_1 \cup D_2)$ is isomorphic to the affine 3-space $\mathbf{A}^3$. In consideration of Lemma 5.6 (2), we may assume that D_1 is a birational section of $\varphi_1 : V \to W_1 \cong \mathbf{P}^2$ and $D_2 \sim H_1$.

Lemma 5.7 *Let V be a smooth Fano 3-fold of No. 24, No. 27, No. 31, No. 32, No. 34, No. 35 or No. 36 in* [M-M81, Table 2]. *Let D_1 and D_2 be two irreducible divisors on V such that $D_1 + D_2$ is ample. Assume that D_1 is a birational section of $\varphi_1 : V \to W_1 \cong \mathbf{P}^2$ and $D_2 \sim H_1 := \varphi_1^* \mathcal{O}_{\mathbf{P}^2}(1)$, where $\varphi_1 : V \to W_1$ is a Mori fiber space of $C2$-type. Then the complement $X = V - (D_1 \cup D_2)$ is isomorphic to the affine 3-space $\mathbf{A}^3$ if and only if $H^2(D_1 \cup D_2; \mathbf{Z}) \cong \mathbf{Z}^{\oplus 2}$.*

Proof. Only if part is easy to see, so we have only to show the another implication, i.e., the condition $H^2(D_1 \cup D_2; \mathbf{Z}) \cong \mathbf{Z}^{\oplus 2}$ implies $X \cong \mathbf{A}^3$. By the assumption, we can write $D_1 \sim mH_1 + H_2$ and $D_2 \sim H_1$ for some $m \in \mathbf{Z}$. Since $D_1 + D_2$ is ample, the complement $X = V - (D_1 \cup D_2)$ is affine.

Claim. X is factorial, i.e., the coordinate ring $\Gamma(\mathcal{O}_X)$ of X is factorial.

Proof of Claim. Since the matrix $\begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix}$ is unimodular, we have $\mathrm{Pic}(V) \cong \mathbf{Z}[D_1] \oplus \mathbf{Z}[D_2]$. Let G_0 be an irreducible divisor on X and let G be the closure of G_0 in V. Then we can write $G = a_1 D_1 + a_2 D_2 + (g)$ for some a_1, $a_2 \in \mathbf{Z}$ and $g \in \mathbf{C}(V) = Q(A)$, where $A := \Gamma(\mathcal{O}_X)$. It is clear that $g \in A$ because g is regular on X. Then $\mathrm{div}(g)_X = G_0$. Hence A is factorial as desired. □

We consider the exact sequence with respect to the pair $(V, D_1 \cup D_2)$:

$$\cdots \to H^2(V, D_1 \cup D_2) \to H^2(V) \to H^2(D_1 \cup D_2) \to H^3(V, D_1 \cup D_2) \to H^3(V) \to \cdots,$$

where the cohomology groups have the ring of integers $\mathbf{Z}$ as the coefficient group. Note that $H^2(V, D_1 \cup D_2) \cong H_4(X) = 0$ (cf. [Mil63, Theorem 7.1]) and $H^3(V) = 0$ (cf. [M-M81]). Since $H^2(D_1 \cup D_2) \cong \mathbf{Z}^{\oplus 2}$, we have $H_3(X) = 0$ by the above sequence. On the other hand, the restriction $\varphi_1|_X$ of φ_1 onto $X = V - (D_1 \cup D_2)$ yields an $\mathbf{A}^1$-fibration over the affine plane $W_1 - \varphi_1(D_2) \cong \mathbf{A}^2$ as D_1 is a birational section of φ_1. Since X is factorial and $H_3(X) = 0$, this $\mathbf{A}^1$-fibration is surjective (cf. [Ka02, Prop. 3.2]). This implies that, for every point $y \in W_1 - \varphi_1(D_2)$, the fiber $l_y := \varphi_1^{-1}(y) \cong \mathbf{P}^1$ is not contained in D_1.

Since $D_1 \cdot l_y = 1$, it follows that l_y intersects D_1 in a single point transversally. Thus all the fibers of $\varphi_1|_X$ are isomorphic to the affine line $\mathbf{A}^1$. Since $\mathrm{Pic}(\mathbf{A}^2) = 0$, we know that $\varphi_1|_X$ is a trivial $\mathbf{A}^1$-bundle structure over $\mathbf{A}^2$, i.e., $X \cong \mathbf{A}^2 \times \mathbf{A}^1 \cong \mathbf{A}^3$. □

It is not difficult to find the components D_1 and D_2 on V satisfying the condition mentioned in Lemma 5.7 actually when V is of No. 32, No. 34, No. 35 and No. 36. Even if V is either of No. 24, No. 27 or No. 31, we can also find such two components D_1 and D_2. In the following, we shall demonstrate how to find them in the case where V is of No. 27.

Lemma 5.8 *Let V be a smooth Fano 3-fold of No. 27 in* [M-M81, Table 2]*, i.e., V is obtained via the blow-up $\varphi_2 : V \to \mathbf{P}^3$ along a twisted cubic on $\mathbf{P}^3$. Then we can find a normal member $D_1 \in |H_2|$ and $D_2 \in |H_1|$ in such a way that $H^2(D_1 \cup D_2) \cong \mathbf{Z}^{\oplus 2}$, and hence the complement $X = V - (D_1 \cup D_2)$ is isomorphic to the affine 3-space $\mathbf{A}^3$.*

Proof. Let $C \subset \mathbf{P}^3$ be a twisted cubic which is the center of φ_2. Then the restriction homomorphism $H^0(\mathcal{O}_{\mathbf{P}^3}(1)) \to H^0(C(1))$ is an isomorphism as C is not contained in any hyperplane. We fix a point p on C. Then we can take a hyperplane $L_2 \in |\mathcal{O}_{\mathbf{P}^3}(1)|$ in such a way that $L_2|_C = 3p$. Let $D_1 := \varphi_2^*(L_2) \in |H_2|$. We claim the following:

Claim. D_1 is a normal singular del Pezzo surface of degree $(-K_{D_1})^2 = 6$.

Proof of Claim. Note that $-K_V \sim H_1 + 2H_2$ by Proposition 2.1. Hence $-K_{D_1} = (H_1 + H_2)|_{D_1}$. In particular, D_1 is a (possibly singular) del Pezzo surface of degree $(-K_{D_1})^2_{D_1} = (H_1 + H_2)^2 \cdot H_2 = 6$. Assume that D_1 is smooth. Then D_1 is obtained from $\mathbf{P}^2$ by the blow-ups of 3-points in a general position. On the other hand, there exists only one exceptional curve of $\varphi_2|_{D_1} : D_1 \to L_2 \cong \mathbf{P}^2$ by the choice of the hyperplane L_2. This is a contradiction. Hence D_1 is singular. Suppose that D_1 is non-normal. Let $\sigma : \overline{D_1} \to D_1$ be the normalization. By virtue of the classification of non-normal del Pezzo surfaces (cf. [A-F]), there are three possibilities concerning $\overline{D_1}$, namely, $\overline{D_1}$ is isomorphic to $\mathbf{F}_4$, $\mathbf{F}_2$ or S_6, where $S_6 \subset \mathbf{P}^7$ is a cone over a smooth rational curve of degree 6 in $\mathbf{P}^6$. This is a contradiction because the composite $(\varphi_1|_{D_1}) \circ \sigma : \overline{D_1} \to D_1 \to \mathbf{P}^2$ yields a birational morphism. Therefore D_1 is a normal singular del Pezzo surface. □

Let $\mu : \widetilde{D_1} \to D_1$ be the minimal resolution. Then $\widetilde{D_1}$ is obtained from $\mathbf{P}^2$ by 3-points blow-ups in almost general position, say $\nu : \widetilde{D_1} \to \mathbf{P}^2$ (cf. [H-W81]). These 3-points lie on a suitable line on $\mathbf{P}^2$, say L_1. By the construction, ν is a composite of μ and $\varphi_1|_{D_1}$. We put $D_2 := \varphi_1^*(L_1) \sim H_1$. Then it is easy to see that $H^2(D_1 \cup D_2) \cong \mathbf{Z}^{\oplus 2}$ and, as a consequence, $X \cong \mathbf{A}^3$ by Lemma 5.7. □

We can construct factorial smooth affine 3-folds X with non-surjective $\mathbf{A}^1$-fibrations over the affine plane $\mathbf{A}^2$ as follows. Indeed, the homology groups $H_3(X)$ of X in Example 5.2 below are non-trivial.

Example 5.2 (1) Let V be of No. 32, i.e., $V = \mathbf{P}(\Theta_{\mathbf{P}^2})$. Then both extremal contractions $\varphi_1 : V \to W_1$ and $\varphi_2 : V \to W_2$ are of $C2$-type. For instance, we choose $H_2 = \varphi_2^*\mathcal{O}_{W_2}(1)$ as D_1. Then $D_1 \cong \mathbf{F}_1$ and $\varphi_1|_{D_1} : D_1 \to W_1 \cong \mathbf{P}^2$ is a contraction of the minimal section. Hence there exists a unique point $y \in W_1$ such that the fiber $l_y := \varphi_1^{-1}(y)$ is contained in D_1 as the minimal section. Let L_1 be a line on W_1 which does not pass through y and let $D_2 := \varphi_1^*(L_1) \sim H_1$. Then $H^2(D_1 \cup D_2) \cong \mathbf{Z}^{\oplus 3}$ and the complement $X = V - (D_1 \cup D_2)$ is a factorial smooth affine 3-fold with $H_3(X) \cong \mathbf{Z}$. By the construction, the restriction $\varphi_1|_X$ yields an $\mathbf{A}^1$-fibration over the affine plane $W_1 - L_1 \cong \mathbf{A}^2$ and the fiber of it over the point y is empty, i.e., $(\varphi_1|_X)^{-1}(y) = \emptyset$. Thus $\varphi_1|_X$ gives rise to an $\mathbf{A}^1$-bundle over the punctured affine plane.

(2) More generally, we can construct a factorial smooth affine 3-fold X with an arbitrary large homology group $H_3(X)$. With the notation as in (1), let D_1 be a smooth member of $|mH_1 + H_2|$, where $m \geq 0$. Then $(-K_{D_1})_{D_1} = ((2-m)H_1 + H_2)^2 \cdot (mH_1 + H_2) = -m^2 - m + 8$. Hence the restriction $\varphi_1|_{D_1} : D_1 \to \mathbf{P}^2$ is a composite of $(m^2 + m + 1)$-points blow-ups. More precisely, there are $(m^2 + m + 1)$-fibers of φ_1 contained in D_1, and each of them is a (-1)-curve on D_1. Let L_1 be a line on $\mathbf{P}^2$ which does not pass through any of the centers of the blow-ups $\varphi_1|_{D_1}$, and let $D_2 := \varphi_1^*(L_1) \sim H_1$. Then $H^2(D_1 \cup D_2) \cong \mathbf{Z}^{\oplus(m^2+m+3)}$ and $H_3(X) \cong \mathbf{Z}^{\oplus(m^2+m+1)}$. By the construction, the restriction $\varphi_1|_X$

gives rise to an $\mathbf{A}^1$-fibration over $\mathbf{P}^2 - L_1 \cong \mathbf{A}^2$ and there are exactly (m^2+m+1)-points on $\mathbf{A}^2$ over which the fibers of $\varphi_1|_X$ are empty.

Summarizing the argument in this subsection we have the embeddings of Types V, VII, XV, XVI, XIX, XXIII and XXIV in **Table 1**.

5.3 φ_1 is of $D2$-type

In this subsection, we consider the case where $\varphi_1 : V \to W_1$ is of $D2$-type. We shall prove that, in fact, this case does not occur. At first, we prove the following:

Lemma 5.9 (1) *$t = \frac{1}{2}$ and $(D_1 + D_2) \cdot l_1 = 1$,*

(2) *We may assume that D_1 satisfies $D_1 \cdot l_1 = 1$ and $D_2 \sim H_1 = \varphi_1^* \mathcal{O}_{\mathbf{P}^1}(1)$,*

(3) *The restriction $\varphi_1|_X$ of φ_1 onto $X = V - (D_1 \cup D_2)$ yields a fibration onto the affine line $\mathbf{A}^1 = W_1 - \{\infty\}$, where $\infty := \varphi_1(D_2)$,*

(4) *Let S be the general fiber of $\varphi_1|_X$. Then S is an affine surface obtained by deleting a diagonal from $\mathbf{P}^1 \times \mathbf{P}^1$.*

Proof. It is easy to see that the base curve W_1 is isomorphic to $\mathbf{P}^1$. The assertions (1), (2) and (3) can be obtained by the similar argument as in the proof of Lemma 5.6. Since $D_1 \cdot l_1 = 1$, it follows that $D_1|_{F_y} \sim \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1,1)$, where $F_y := \varphi_1^{-1}(y) \cong \mathbf{P}^1 \times \mathbf{P}^1$ $(y \in W_1)$ is a general fiber of φ_1. Since $\varphi_1|_{D_1} : D_1 \to W_1$ gives rise to a $\mathbf{P}^1$-fibration, we know that $D_1|_{F_y}$ is a diagonal on $F_y \cong \mathbf{P}^1 \times \mathbf{P}^1$ for a general $y \in W_1$. This means the assertion (4). □

We shall investigate fibers of $\varphi_1|_X$ which are not isomorphic to S (cf. Lemma 5.9). We denote by F_y (resp. F_y°) the fiber of φ_1 (resp. of $\varphi_1|_X$) over the point $y \in W_1$ (resp. $y \in W_1 - \{\infty\}$). Note that, with this notation, we have $F_\infty = D_2$. Let $\Sigma := \{y \in W_1\ ; F_y \cong \mathbf{Q}_0^2\}$ and let $\Delta := \{y \in W_1 - \Sigma\ ;\ D_1|_{F_y}$ is a union of two distinct rulings on $F_y \cong \mathbf{P}^1 \times \mathbf{P}^1\}$.

Lemma 5.10 *We have the following with respect to the fibers of $\varphi_1|_X$:*

(1) *For $y \in W_1 - (\Sigma \cup \Delta \cup \{\infty\})$, we have $F_y^\circ \cong S$,*

(2) *For $y \in \Delta - \{\infty\}$ (unless $\Delta - \{\infty\} = \emptyset$), we have $F_y^\circ \cong \mathbf{A}^2$,*

(3) *For $y \in \Sigma - \{\infty\}$ (unless $\Sigma - \{\infty\} = \emptyset$), there are two possibilities (a) and (b) concerning the fiber F_y°:*

   (a) *F_y° is isomorphic to the affine surface which is obtained as the complement of a hyperplane section on $F_y \cong \mathbf{Q}_0^2 \hookrightarrow \mathbf{P}^3$ not passing through the vertex of F_y;*

   (b) *$F_y^\circ \cong \mathbf{A}^1 \times \mathbf{C}^*$.*

Proof. The assertions (1) and (2) are not difficult to see. We shall prove (3). Since the fibers of $\varphi_1|_{D_1} : D_1 \to W_1$ are of type $(1,1)$ on the smooth fibers $\cong \mathbf{P}^1 \times \mathbf{P}^1$ of φ_1 and hence D_1 is locally a graph everywhere (cf. [Mu90]). Thus D_1 is smooth. Since $\varphi_1|_{D_1}$ yields a $\mathbf{P}^1$-fibration, there is no multiple fiber of it. On the other hand, for $y \in \Sigma$, the intersection $D_1|_{F_y}$ is linearly equivalent to $\mathcal{O}_{\mathbf{P}^3}(1) \otimes \mathcal{O}_{\mathbf{Q}_0^2}$. Hence the possibility of configuration of $D_1|_{F_y}$ is a smooth hyperplane section, a union of two distinct generating lines or a twofold generating line on $F_y \cong \mathbf{Q}_0^2 \hookrightarrow \mathbf{P}^3$. But the last possibility is excluded because of the absence of multiple fibers of $\varphi_1|_{D_1}$. Thus we obtain the assertion (3). □

In [M-M81, Table 2], there are three smooth Fano 3-folds with $B_2 = 2$ having an extremal ray of type $D2$, namely, No. 18, No. 25 and No. 29. We see how many singular fibers of φ_1 exist by calculating topological Euler number χ_{top}.

Lemma 5.11 *We have $\sharp\Sigma = 6$ (resp. $= 4$ or $= 2$) if V is of No. 18 (resp. No. 25 or No. 29) in* [M-M81, Table 2].

Proof. Since $B_3(V) = 4$ (resp. $= 2$ or $= 0$) when V is of No. 18 (resp. No. 25 or No. 29), it is easy to see $\chi_{\rm top}(V) = 2$ (resp. $= 4$ or $= 6$). On the other hand, we calculate $\chi_{\rm top}(V)$ associated with the Mori fiber space $\varphi_1 : V \to W_1 \cong \mathbf{P}^1$, namely, we have:

$$\chi_{\rm top}(V) = \chi_{\rm top}(\mathbf{P}^1 - \Sigma)\chi_{\rm top}(\mathbf{P}^1 \times \mathbf{P}^1) + \chi_{\rm top}(\Sigma)\chi_{\rm top}(\mathbf{Q}_0^2) = 8 - \sharp\Sigma.$$

Hence we have the assertion. □

By using the previous lemmas, we have the following:

Lemma 5.12 *The cases No. 18 and No. 25 do not occur.*

Proof. Assume that V is of No. 18. Then there are 6 singular fibers $\cong \mathbf{Q}_0^2$ of φ_1 by Lemma 5.11. Let $\Sigma = \{y_1, \cdots, y_6\}$. Note that $\chi_{\rm top}(F_i^\circ)$ is equal to 1 or 0 according as F_i° is of type (a) or (b) in Lemma 5.10 (3) unless $y_i = \infty$. We calculate $\chi_{\rm top}(X)$ associated with the fibration $\varphi_1|_X : X \to \mathbf{A}^1 \cong W_1 - \{\infty\}$. For this purpose, we need to consider according as the configurations of the fiber $D_2 = F_\infty$ and the intersection $D_1|_{D_2}$, separately. There are following 3 possibilities:

(i) $D_2 \cong \mathbf{P}^1 \times \mathbf{P}^1$ and $D_1|_{D_2}$ is a diagonal on D_2, i.e., $\infty \in W_1 - (\Sigma \cup \Delta)$,

(ii) $D_2 \cong \mathbf{P}^1 \times \mathbf{P}^1$ and $D_1|_{D_2}$ is a union of two rulings in distinct directions on D_2, i.e., $\infty \in \Delta$,

(iii) $D_2 \cong \mathbf{Q}_0^2$, i.e., $\infty \in \Sigma$.

In the case (i) we have the following equality:

$$1 = \chi_{\rm top}(X) = \chi_{\rm top}(\mathbf{P}^1 - \Sigma \cup \Delta \cup \{\infty\})\chi_{\rm top}(S) + \chi_{\rm top}(\Delta)\chi_{\rm top}(\mathbf{A}^2) + \sum_{i=1}^{6} \chi_{\rm top}(F_{y_i}^\circ).$$

By noting that $\chi_{\rm top}(F_i^\circ) = 0$ or $= 1$ for $i \le i \le 6$, we have:

$$1 = \chi_{\rm top}(X) \le 2(2 - 6 - \sharp\Delta - 1) + \sharp\Delta + 6 = -\sharp\Delta - 4 \le -4,$$

which is a contradiction. In the case (ii) we have:

$$1 = \chi_{\rm top}(X) = \chi_{\rm top}(\mathbf{P}^1 - \Sigma \cup \Delta)\chi_{\rm top}(S) + \chi_{\rm top}(\Delta - \{\infty\})\chi_{\rm top}(\mathbf{A}^2) + \sum_{i=1}^{6} \chi_{\rm top}(F_{y_i}^\circ)$$

$$\le 2(2 - 6 - \sharp\Delta) + \sharp\Delta - 1 + 6 = -\sharp\Delta - 3 \le -4,$$

which is a contradiction. In the remaining case (iii), we may assume $y_6 = \infty$ and obtain the inequality:

$$1 = \chi_{\rm top}(X) = \chi_{\rm top}(\mathbf{P}^1 - \Sigma \cup \Delta)\chi_{\rm top}(S) + \chi_{\rm top}(\Delta)\chi_{\rm top}(\mathbf{A}^2) + \sum_{i=1}^{5} \chi_{\rm top}(F_{y_i}^\circ)$$

$$\le 2(2 - 6 - \sharp\Delta) + \sharp\Delta + 5 = -\sharp\Delta - 3 \le -3,$$

which is also a contradiction. Thus we know that the case of No. 18 does not occur. By the similar argument as above, we can see that the case of No. 25 also does not occur. □

Lemma 5.13 *The case No. 29 does not occur.*

Proof. Our proof consists of three steps.

Step 1. Assume that V is of No. 29, namely, V is obtained from $W_2 := \mathbf{Q}^3 \hookrightarrow \mathbf{P}^4$ via the blow-up $\varphi_2 : V \to W_2$ along a conic $C_2 \subset W_2$ with an exceptional divisor $E_2 \cong \mathbf{P}^1 \times \mathbf{P}^1$. By Lemma 5.11, $\varphi_1 : V \to W_1 \cong \mathbf{P}^1$ has 2 singular fibers $\cong \mathbf{Q}_0^2$. Let $\Sigma = \{y_1, y_2\}$. Although there are three possibilities (i), (ii) and (iii) concerning the fiber $D_2 = F_\infty$ and the intersection $D_1|_{D_2}$ as in the proof of Lemma 5.12, the same argument there shows that the cases (i) and (ii) are impossible. Hence we may assume that (iii) occurs and $y_2 = \infty$. We have the following equality associated with the fibration $\varphi_1|_X$:

$$1 = \chi_{\mathrm{top}}(X) = \chi_{\mathrm{top}}(\mathbf{P}^1 - \Sigma \cup \Delta)\chi_{\mathrm{top}}(S) + \chi_{\mathrm{top}}(\Delta)\chi_{\mathrm{top}}(\mathbf{A}^2) + \chi_{\mathrm{top}}(F_{y_1}^\circ)$$

$$= 2(2 - 2 - \sharp\Delta) + \sharp\Delta + \chi_{\mathrm{top}}(F_{y_1}^\circ) = -\sharp\Delta + \chi_{\mathrm{top}}(F_{y_1}^\circ).$$

Hence $\Delta = \emptyset$ and $\chi_{\mathrm{top}}(F_{y_1}^\circ) = 1$.

Step 2. By Proposition 2.1, we have $H_1 + 2H_2 \sim -K_V = \varphi_2^*(-K_{W_2}) - E_2 \sim 3H_2 - E_2$. Hence $E_2 \sim -H_1 + H_2$. Note that D_1 is distinct from E_2. Indeed, suppose that $D_1 = E_2$. Then, via the contraction φ_2, we have $(W_2, \overline{D_2} := \varphi_2(D_2)) \cong (\mathbf{Q}^3, \mathbf{Q}_0^2)$. We have $D_1 + D_2 = \varphi_2^*(\overline{D_2})$, which is a contradiction as $D_1 + D_2$ is ample. Let $\overline{D_i} := \varphi_2(D_i)$ for $i = 1, 2$. We write $\overline{D_1} \sim nL_2$ with $n \in \mathbf{N}$ and put $m := \mathrm{mult}_{C_2}\overline{D_1} \geq 0$. Then we have $D_1 = \varphi_2^*(\overline{D_1}) - mE_2 \sim nH_2 - m(-H_1 + H_2) = mH_1 + (n-m)H_2$. On the other hand, since $\mathrm{Pic}(V) = \mathbf{Z}[D_1] \oplus \mathbf{Z}[D_2]$ by Lemma 2.1, the matrix $\begin{pmatrix} m & n-m \\ 1 & 0 \end{pmatrix}$ have to be unimodular, so $n - m \pm 1$. But if $n - m = -1$, then $D_1 \cdot l_1 = -1$, which is obviously contradiction. Hence $n - m = 1$, i.e., $D_1 \sim mH_1 + H_2$. Then $(-K_{D_1})^2_{D_1} = (-K_V - D_1)^2 \cdot D_1 = ((1-m)H_1 + H_2)^2 \cdot (mH_1 + H_2) = m(1-m)^2(H_1^3) + (1+m)(1-m)(H_1^2 \cdot H_2) + (2-m)(H_1 \cdot H_2^2) + (H_2^3)$. Note that $(H_1^3) = (H_1^2 \cdot H_2) = 0$ and $(H_1 \cdot H_2^2) = (H_2^3) = 2$. Hence $(-K_{D_1})^2_{D_1} = 2(3-m)$.

Step 3. D_1 is smooth by the same reason as in the proof of Lemma 5.10. Since $\Delta = \emptyset$, a fiber of the $\mathbf{P}^1$-fibration $\varphi_1|_{D_1} : D_1 \to W_1 \cong \mathbf{P}^1$ over any point except for $\Sigma = \{y_1, y_2 = \infty\}$ is smooth. On the other hand, since $\chi_{\mathrm{top}}(F_{y_1}^\circ) = 1$, the intersection $D_1|_{F_{y_1}}$ is a hyperplane section not passing through the vertex of $F_{y_1} \cong \mathbf{Q}_0^2 \hookrightarrow \mathbf{P}^3$ (cf. Lemma 5.10). Thus $\varphi_1|_{D_1}$ is either a $\mathbf{P}^1$-bundle structure or has a unique singular fiber $D_2|_{D_1}$ consisting of two components. Hence $(-K_{D_1})^2_{D_1} = 7$ or 8, which is a contradiction as $(-K_{D_1})^2_{D_1} = 2(3-m)$ with $m \geq 0$. □

5.4 φ_1 is of $D3$-type

In this subsection, we consider the case where $\varphi_1 : V \to W_1 \cong \mathbf{P}^1$ is of $D3$-type, i.e., all the fibers of φ_1 are isomorphic to $\mathbf{P}^2$. We shall prove the following:

Lemma 5.14 *We may assume that one of the following (i) and (ii) occurs:*

(i) *D_1 meets each fiber $F \cong \mathbf{P}^2$ of φ_1 in such a way that $D_1|_F \in |\mathcal{O}_{\mathbf{P}^2}(1)|$, and D_2 is a fiber of φ_1,*

(ii) *D_1 and D_2 meet each fiber $F \cong \mathbf{P}^2$ of φ_1 in such a way that $D_i|_F \in |\mathcal{O}_{\mathbf{P}^2}(1)|$ for $i = 1, 2$.*

Proof. Since $(D_1 + D_2) \cdot l_1 = t(-K_V \cdot l_1)$ with $0 < t < 1$ and $D_1 + D_2$ is ample, we have either $t = \frac{1}{3}$, $(D_1 + D_2) \cdot l_1 = 1$ or $t = \frac{2}{3}$, $(D_1 + D_2) \cdot l_1 = 2$. In the former case, we have (i) in the statement. Suppose that the latter case occurs. Since $(D_1 + D_2) \cdot l_1 = 2$, we may assume that one of the following (a) and (b) occurs:

(a) $D_1 \cdot l_1 = 2$ and $D_2 \cdot l_1 = 0$,

(b) $D_1 \cdot l_1 = D_2 \cdot l_1 = 1$.

In the case (a), the Picard group $\mathrm{Pic}(X)$ contains 2-torsion elements. This is a contradiction because the coordinate ring of X is factorial. Hence we may assume that (b) occurs. This completes the proof. □

In [M-M81, Table 2], there are two smooth Fano 3-folds with $B_2 = 2$ having extremal rays of $D3$-type, i.e., No. 33: $V = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbf{P}^1}(1))$ and No. 34: $V = \mathbf{P}^1 \times \mathbf{P}^2$.

First of all, we consider the case (i) in Lemma 5.14.

Lemma 5.15 *Suppose that the case (i) in Lemma 5.14 occurs. Then* $X \cong \mathbf{A}^3$.

Proof. We consider the restriction $\varphi_1|_X$ of φ_1 onto $X = V - (D_1 \cup D_2)$. Since D_2 is a fiber of φ_1 and D_1 meets every fiber $F \cong \mathbf{P}^2$ of φ_1 in such a way that $D_1|_F \in |\mathcal{O}_{\mathbf{P}^2}(1)|$, $\varphi_1|_X$ is an $\mathbf{A}^2$-bundle structure over the affine line $W_1 - \varphi_1(D_2) \cong \mathbf{A}^1$. Hence $X \cong \mathbf{A}^1 \times \mathbf{A}^2 \cong \mathbf{A}^3$. □

The embeddings in the previous lemma correspond to Type XVII with $x \geq 1$ and Type XX in **Table 1**.

Next we consider the case (ii) in Lemma 5.14, i.e., D_1 and D_2 meet each fiber $F \cong \mathbf{P}^2$ of φ_1 in such a way that $D_i|_F \in |\mathcal{O}_{\mathbf{P}^2}(1)|$ for $i = 1, 2$. Hence a general fiber of the restriction $\varphi_1|_X : X \to W_1 \cong \mathbf{P}^1$ of φ_1 onto $X = V - (D_1 \cup D_2)$ is isomorphic to $\mathbf{C}^* \times \mathbf{A}^1$. Put $\Delta := \{y \in W_1 \ ; \ D_1 \cap F_y = D_2 \cap F_y\}$, where $F_y := \varphi_1^{-1}(y) \cong \mathbf{P}^2$ for $y \in W_1 \cong \mathbf{P}^1$. Note that, for a point y in Δ, the fiber $F_y^\circ := (\varphi_1|_X)^{-1}(y)$ is isomorphic to the affine plane $\mathbf{A}^2$. We prove the following:

Lemma 5.16 (1) $\sharp\Delta = 1$,

(2) $\varphi_1|_{D_i} : D_i \to W_1 \cong \mathbf{P}^1$ *gives rise to a* $\mathbf{P}^1$*-bundle structure for* $i = 1, 2$,

(3) $D_1 \cap D_2$ *consists of two curves. The one is a cross-section of* $\varphi_1|_{D_i} : D_i \to W_1$ *and the other is a fiber of* $\varphi_1|_{D_i}$ *over the point in* Δ,

(4) $X \cong \mathbf{A}^3$.

Proof. Note that $F_y^\circ := F_y \cap X$ is isomorphic to $\mathbf{A}^2$ for $y \in \Delta$. We shall calculate $\chi_{\mathrm{top}}(X)$ associated with the fibration $\varphi_1|_X : X \to W_1 \cong \mathbf{P}^1$, namely, we have:

$$1 = \chi_{\mathrm{top}}(X) = \chi_{\mathrm{top}}(\mathbf{P}^1 - \Delta)\chi_{\mathrm{top}}(\mathbf{A}^1 \times \mathbf{C}^*) + \chi_{\mathrm{top}}(\Delta)\chi_{\mathrm{top}}(\mathbf{A}^2) = \sharp\Delta.$$

Hence $\sharp\Delta = 1$. This proves (1). The assertion (2) is clear. Since $\sharp\Delta = 1$, the assertion (3) is also easy to see. We shall verify (4) in the subsequent. We may assume that $\Delta = \{\infty\}$ and $F_\infty^\circ \cong \mathbf{A}^2$. Put $X' := X - F_\infty^\circ$. Then $\varphi_1|_{X'} : X' \to \mathbf{A}^1$ yields a trivial $(\mathbf{A}^1 \times \mathbf{C}^*)$-bundle structure over $\mathbf{A}^1$, so $X' \cong \mathbf{C}^* \times \mathbf{A}^2$. Since the coordinate ring $A := \Gamma(\mathcal{O}_X)$ is factorial, there exists a prime element $a \in A$ such that $\mathrm{div}(a)_X = F_\infty^\circ$, hence $A[a^{-1}] = \mathbf{C}[x, y, t, t^{-1}]$. Since A is a finitely generated $\mathbf{C}$-algebra, the locally nilpotent derivations $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ on $A[a^{-1}]$ can be extended to those $\delta_x := a^N(\frac{\partial}{\partial x})$ and $\delta_y := a^N(\frac{\partial}{\partial y})$ on A by multiplying a^N for a sufficiently large $N >> 0$. Put $A_0 := \mathrm{Ker}(\delta_x) \cap \mathrm{Ker}(\delta_y)$, which is a finitely generated $\mathbf{C}$-algebra of dimension one (cf. [Miy78, Chap. I]). Since A is factorial with $A^* = \mathbf{C}^*$ and A_0 is an inert sub-algebra of A, it follows that A_0 is also factorial with $A_0^* = \mathbf{C}^*$. Hence $A_0 = \mathbf{C}^{[1]}$, a polynomial ring in one variable over $\mathbf{C}$. Let $f : X \to Z := \mathrm{Spec}\,(A_0) \cong \mathbf{A}^1$ be a morphism associated with the canonical inclusion $A_0 \hookrightarrow A$. By the construction, the restriction $f|_{X'}$ of f onto $X' = X - F_\infty^\circ$ coincides with the canonical projection $p : X' \cong \mathbf{C}^* \times \mathbf{A}^2 \to Z' := \mathbf{C}^*$. The fiber $f^{-1}(z)$ over the point $z \in Z'$ coincides with $p^{-1}(z) \cong \mathbf{A}^2$. On the other hand, the fiber of f over the point $Z - Z'$ is equal to $F_\infty^\circ \cong \mathbf{A}^2$. Thus all the fibers of f are isomorphic to the affine plane $\mathbf{A}^2$. Since $\mathrm{Pic}(Z) = 0$, we know that f is a trivial $\mathbf{A}^2$-bundle structure over $Z \cong \mathbf{A}^1$. Therefore we have $X \cong Z \times \mathbf{A}^2 \cong \mathbf{A}^3$ as desired. □

As seen in the previous lemma, the intersection $D_1 \cap D_2$ of boundary components consists of two curves, say C_1 and C_2. We may assume that C_1 (resp. C_2) is a section (resp. a fiber) of the $\mathbf{P}^1$-bundle structure $\varphi_1|_{D_i} : D_i \to W_1 \cong \mathbf{P}^1$ $(i = 1, 2)$. Since D_1 and D_2 satisfy $D_1 \cdot l_1 = D_2 \cdot l_1 = 1$ and $\mathrm{Pic}(V) \cong \mathbf{Z}[D_1] \oplus \mathbf{Z}[D_2]$ (cf. Lemma 2.1), we may write $D_1 \sim H_2 + xH_1$ and $D_2 \sim H_2 + yH_1$ with x, $y \in \mathbf{Z}$ such that $x - y = 1$. We shall see, conversely, that if D_1 and D_2 on V satisfy these conditions, then the complement $X = V - (D_1 \cup D_2)$ is isomorphic to the affine 3-space $\mathbf{A}^3$.

Lemma 5.17 *Assume that two irreducible components D_1 and D_2 on V of No. 33 or No. 34 satisfy the following two conditions:*

(i) $D_1 \sim H_2 + xH_1$ *and* $D_2 \sim H_2 + yH_1$ *with* x, $y \in \mathbf{Z}$ *such that* $x - y = 1$;

(ii) *The intersection $D_1 \cap D_2$ consists of two curves, say C_1 and C_2, and C_1 (resp. C_2) is a section (resp. a fiber) of the $\mathbf{P}^1$-bundle structure $\varphi_1|_{D_i} : D_i \to W_1 \cong \mathbf{P}^1$ $(i = 1, 2)$.*

Then the complement $X = V - (D_1 \cup D_2)$ is isomorphic to the affine 3-space $\mathbf{A}^3$.

Proof. It is easy to see that $D_1 + D_2$ is ample and the coordinate ring $A := \Gamma(\mathcal{O}_X)$ is factorial by (i). The restriction $\varphi_1|_X : X \to W_1 \cong \mathbf{P}^1$ yields a $(\mathbf{A}^1 \times \mathbf{C}^*)$-fibration. Exactly one fiber, say F_∞°, of $\varphi_1|_X$ is isomorphic to $\mathbf{A}^2$ in consideration of (ii). Let $a \in A$ be a prime element defining $F_\infty^\circ \subset X$. (Note that A is factorial.) Since $X - F_\infty^\circ \cong \mathbf{C}^* \times \mathbf{A}^2$, we have $A[a^{-1}] = \mathbf{C}[x, y, t, t^{-1}]$. Then, by the same argument as in the proof of Lemma 5.16, we can deduce that $X \cong \mathbf{A}^3$. □

The embeddings in Lemma 5.17 is of Types XVIII and XXI in **Table 1**. Summarizing the arguments we prove Theorem 1.3.

Type	V	$D_1 \cup D_2$	X
I:($No.14$) $[E1{-}D1]$	V : a blow-up of V_5 along an elliptic curve which is an intersection of two members of $\lvert -\frac{1}{2}K_{V_5}\rvert$	$D_1 = \mathrm{Exc}(\varphi_1)$, D_2 : a proper transform of H_5^0 containing a center of φ_1	$\mathbf{A}^3$
II:($No.20$) $[E1{-}C1]$	V : a blow-up of V_5 along a twisted cubic	$D_1 = \mathrm{Exc}(\varphi_1)$, D_2 : a proper transform of H_5^0 containg a center φ_1	$\mathbf{A}^3$
III:($No.22$) $[E1{-}E1]$	V : a blow-up of V_5 along a conic	$D_1 = \mathrm{Exc}(\varphi_1)$, D_2 : a proper transform of H_5^0 containg a center of φ_1	$\mathbf{A}^3$
IV:($No.23(b)$) $[E1{-}E4]$	V : a blow-up of $\mathbf{Q}^3$ along a curve which is an intersection of $\mathbf{Q}_0^2 \cong \overline{D_2} \in \lvert\mathcal{O}_{\mathbf{Q}}(1)\rvert$ and a member of $\lvert\mathcal{O}_{\mathbf{Q}}(2)\rvert$	$D_1 = \mathrm{Exc}(\varphi_1)$, D_2 : a proper transform of $\overline{D_2}$	$\mathbf{A}^3$
V:($No.24$) $[C2{-}C1]$	V : a divisor on $\mathbf{P}^2 \times \mathbf{P}^2$ of bi-degree $(2,1)$	D_1 : a birational section of φ_1, $D_2 \sim H_1$, s.t. $H^2(D_1 \cup D_2) \cong \mathbf{Z}^{\oplus 2}$	$\mathbf{A}^3$
VI:($No.26$) $[E1{-}E1]$	V : a blow-up of V_5 along a line	$D_1 = \mathrm{Exc}(\varphi_1)$, D_2 : a proper transform of H_5^0 or H_5^∞ containing a center of φ_1	$\mathbf{A}^3$
VII:($No.27$) $[C2{-}E1]$	V : a blow-up of $\mathbf{P}^3$ along a twisted cubic	D_1 : a birational section of φ_1, $D_2 \sim H_1$, s.t. $H^2(D_1 \cup D_2) \cong \mathbf{Z}^{\oplus 2}$	$\mathbf{A}^3$
$VIII$:($No.28$) $[E1{-}E5]$	V : a blow-up of $\mathbf{P}^3$ along a plane cubic	$D_1 = \mathrm{Exc}(\varphi_1)$, D_2 : a proper transform of a hyperplane containing a center of φ_1	$\mathbf{A}^3$
IX:($No.29$) $[E1{-}D2]$	V : a blow-up of $\mathbf{Q}^3$ along a conic	$D_1 = \mathrm{Exc}(\varphi_1)$, D_2 : a proper transform of $\mathbf{Q}_0^2 \in \lvert\mathcal{O}_{\mathbf{Q}}(1)\rvert$ containing a center of φ_1	$\mathbf{A}^3$
X:($No.30$) $[E1{-}E2]$	V : a blow-up of $\mathbf{P}^3$ along a conic	$D_1 = \mathrm{Exc}(\varphi_1)$, D_2 : a proper transform of a hyperplane containing a center of φ_1	$\mathbf{A}^3$
XI:($No.30$) $[E2{-}E1]$	V : a blow-up of $\mathbf{Q}^3$ at a point p	$D_1 = \mathrm{Exc}(\varphi_1)$, D_2 : a proper transform of $\mathbf{Q}_0^2 \in \lvert\mathcal{O}_{\mathbf{Q}}(1)\rvert$ containing p as a smooth point	$\mathbf{A}^3$
XII:($No.30$) $[E2{-}E1]$	V : a blow-up of $\mathbf{Q}^3$ at a point p	D_1 : a proper transform of a hyperplane section $\mathbf{Q}_0^2 \cong \overline{D_1}$ not containing p, D_2 : a proper transform of $\overline{D_2} \subset \mathbf{Q}^3$ containing p as a smooth point s.t. $S := \overline{D_2} \cap (\mathbf{Q}^3 - \overline{D_1})$ is acyclic	$(*)$
$XIII$:($No.30$) $[E2{-}E1]$	V : a blow-up of $\mathbf{Q}^3$ at a point p	D_1 : a proper transform of a hyperplane section $\mathbf{P}^1 \times \mathbf{P}^1 \cong \overline{D_1}$ not containing p, D_2 : a proper transform of $\overline{D_2} \subset \mathbf{Q}^3$ containing p as a smooth point s.t. $S := \overline{D_2} \cap (\mathbf{Q}^3 - \overline{D_1})$ satisfies $H_0(S) = H_1(S) \cong \mathbf{Z}$ and $H_i(S) = 0$ $(\forall i > 1)$	$(*)$
XIV:($No.31$) $[E1{-}C2]$	V : a blow-up of $\mathbf{Q}^3$ along a line	$D_1 = \mathrm{Exc}(\varphi_1)$, D_2 : a proper transform of $\mathbf{Q}_0^2 \in \lvert\mathcal{O}_{\mathbf{Q}}(1)\rvert$ containing a center of φ_1	$\mathbf{A}^3$
XV:($No.31$) $[C2{-}E1]$	V : a blow-up of $\mathbf{Q}^3$ along a line	D_1 : a birational section of φ_1, $D_2 \sim H_1$, s.t. $H^2(D_1 \cup D_2) \cong \mathbf{Z}^{\oplus 2}$	$\mathbf{A}^3$

Type	V	$D_1 \cup D_2$	X
XVI:$(No.32)$ $[C2-C2]$	$V = \mathbf{P}(\Theta_{\mathbf{P}^2})$: a divisor on $\mathbf{P}^2 \times \mathbf{P}^2$ of bi-degree $(1,1)$	D_1 : a birational section of φ_1, $D_2 \sim H_1$, s.t. $H^2(D_1 \cup D_2) \cong \mathbf{Z}^{\oplus 2}$	$\mathbf{A}^3$
$XVII$:$(No.33)$ $[D3-E1]$	$V = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbf{P}^1}(1))$	$D_1 \sim H_2 + xH_1$ $(x = -1,\ x \geq 1)$, $D_2 \sim H_1$	$\mathbf{A}^3$
$XVIII$:$(No.33)$ $[D3-E1]$	$V = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbf{P}^1}(1))$	D_1 and D_2 satisfy the conditions (i), (ii) in Lemma 5.17	$\mathbf{A}^3$
XIX:$(No.34)$ $[C2-D3]$	$V = \mathbf{P}^1 \times \mathbf{P}^2$	D_1 : a birational section of φ_1, $D_2 \sim H_1$, s.t. $H^2(D_1 \cup D_2) \cong \mathbf{Z}^{\oplus 2}$	$\mathbf{A}^3$
XX:$(No.34)$ $[D3-C2]$	$V = \mathbf{P}^1 \times \mathbf{P}^2$	$D_1 \sim H_2 + xH_1$ $(x \geq 0)$, $D_2 \sim H_1$	$\mathbf{A}^3$
XXI:$(No.34)$ $[D3-C2]$	$V = \mathbf{P}^1 \times \mathbf{P}^2$	D_1 and D_2 satisfy the conditions (i), (ii) in Lemma 5.17	$\mathbf{A}^3$
$XXII$:$(No.35)$ $[E2-C2]$	V : a blow-up of $\mathbf{P}^3$ at a point p	D_1 : a proper transform of a hyperplane $\overline{D_1} \subset \mathbf{P}^3$ not containing p, D_2 : a proper transform of a hypersurface $\overline{D_2} \subset \mathbf{P}^3$ containing p as a smooth point s.t. $S := \overline{D_2} \cap (\mathbf{P}^3 - \overline{D_1})$ is acyclic	$(*)$
$XXIII$:$(No.35)$ $[C2-E2]$	$V = \mathbf{P}(\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(1))$	D_1 : a birational section of φ_1, $D_2 \sim H_1$, s.t. $H^2(D_1 \cup D_2) \cong \mathbf{Z}^{\oplus 2}$	$\mathbf{A}^3$
$XXIV$:$(No.36)$ $[C2-E5]$	$V = \mathbf{P}(\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(2))$	D_1 : a birational section of φ_1, $D_2 \sim H_1$, s.t. $H^2(D_1 \cup D_2) \cong \mathbf{Z}^{\oplus 2}$	$\mathbf{A}^3$

Table 1.

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