Topological Hochschild cohomology I

A.Lazarev

1 Introduction

A new definition of topological Hochschild cohomology of an R-algebra in the sense of [EKMM] is given. This definition is based on the notion of a derivation of an R-algebra. We also introduce the notion of a singular extension of R-algebras and relate those to topological Hochschild cohomology. We use these tools to introduce multiplicative structure into Postnikov towers of R-algebras.[1] We compute the topological Hochschild cohomology of the Eilenberg-MacLane spectrum $K(Z/pZ, 0)$ as an MU-algebra. This computation is an essential ingredient in our study of deformations of MU-algebras which will be the subject of the subsequent paper.

2 Derived module of differentials

Let R be a commutative S-algebra (in older terminilogy, an E_∞ ring spectrum), A a (not necessarily commutative) R-algebra. Throughout the paper the notation $\wedge$ will mean $\wedge_R$.

Definition 2.1 *The (derived) module of differentials $\Omega_{A/R}$ or just Ω_A if the "ground ring" R is clear from the context is, by definition, the homotopy fibre of the multiplication map:*

$$\Omega_{A/R} \to A \wedge A \to A \tag{1}$$

[1] A similar theory was independently worked out by J.Smith and S.Schwede [Sch] in the context of symmetric spectra. Our approaches have some substantial differences, however, particularly in the definition of a singular extension.

Notice that $\Omega_{A/R}$ is an A-bimodule. The exact triangle (1) is split by the map

$$A \stackrel{id \wedge 1}{\to} A \wedge A$$

in the homotopy category of left A-modules, therefore Ω_A is weakly equivalent as a left A-module to $A \wedge A/R$. Symmetrically, Ω_A is weakly equivalent to $A/R \wedge A$.

Now let M be an A-bimodule. Recall that M is then a left $A \wedge A^{op}$ module where A^{op} is an R-algebra which as an R-module coincides with A but with opposite multiplication. Define the topological Hochschild homology spectrum $\mathbf{THH_\bullet}(A, M)$ as

$$\mathbf{THH_\bullet}(A, M) = \Omega_A \wedge_{A \wedge A^{op}} M$$

and the topological Hochschild cohomology spectrum $\mathbf{THH^\bullet}(A, M)$ as

$$\mathbf{THH^\bullet}(A, M) = F_{A \wedge A^{op}}(\Omega_A, M)$$

Recall that the standard definition of Hochschild homology and cohomology (cf. [EKMM])is respectively

$$\overline{\mathbf{THH_\bullet}}(A, M) = A \wedge_{A \wedge A^{op}} M;$$

$$\overline{\mathbf{THH^\bullet}}(A, M) = F_{A \wedge A^{op}}(A, M)$$

Now from (1) we see that the two definitions are related via cofibre sequences of A-bimodules:

$$\mathbf{THH_\bullet}(A, M) \to M \to \overline{\mathbf{THH_\bullet}}(A, M);$$

$$\overline{\mathbf{THH^\bullet}}(A, M) \to M \to \mathbf{THH^\bullet}(A, M)$$

Now define the derived module of derivations of A with values in M as

$$Der(A, M) = [A, A \vee M]_{R-alg/A} \tag{2}$$

The right hand side of the last formula means homotopy classes of maps in the category of R-algebras over A, that is, R-algebras B supplied with a map $B \to A$. This category inherits the structure of a topological model category from the category of R-algebras and therefore it is legitimate to consider homotopy classes of maps in $R - alg/A$. Further, A is an R-algebra over

A in the obvious way and $A \vee M$ is given the structure of an R-algebra as follows:

$$(A \vee M) \wedge (A \vee M) = A \wedge A \vee A \wedge M \vee M \wedge A \vee M \wedge M \to A \vee M,$$

where the last map is induced by the multiplication on the first summand, by the A-bimodule structure on M on the second and third summands and zero on the last summand. The structure map $A \vee M \to A$ is the usual projection.

We will see shortly that $Der(A, M)$ is an abelian group and

$$Der(A, M) = [\Omega_A, M]_{A-bimod}$$

The right hand side of the last equality is the homotopy classes of maps from Ω_A to M in the category of A-bimodules (or, equivalently, in the category of left $A \wedge A_{op}$-modules). This gives, therefore, another definition of topological Hochschild cohomology, namely

$$THH^n(A, M) = Der(A, \Sigma^n M)$$

Denote the category of R-algebras over A as $\mathcal{C}_{R/A}$ and $ho\mathcal{C}_{R/A}$ - its homotopy category. Define $\Omega^B{}_A$ from the following homotopy fibre sequence of A-bimodules:

$$\Omega^B{}_A \to A \wedge A \to A \wedge_B A \qquad (3)$$

where $B \in \mathcal{C}_{R/A}$. Smashing the cofibre sequence

$$\Omega_B \to B \wedge B \to B$$

on both sides with A over B we see that

$$\Omega^B{}_A = A \wedge_B \Omega_B \wedge_B A$$

Theorem 2.2 *There is a natural equivalence*

$$[B, A \vee M]_{R-alg/A} = [\Omega^B{}_A, M]_{A-bimod}$$

In other words the functors $B \to \Omega^B{}_A : ho\mathcal{C}_{R/A} \to ho(A - bimod)$ and $M \to A \vee M : ho(A - bimod) - ho\mathcal{C}_{R/A}$ are adjoint.

Proof. As a first step we construct the "universal map" of A- bimodules

$$\Omega_A^{A\vee M} \to M.$$

Since $\Omega_A^{A\vee M} = A \wedge_{A\vee M} \Omega_{A\vee M} \wedge_{A\vee M} A$ it is enough to construct a map of $A \vee M$-bimodules $\Omega_{A\vee M} \to M$. Consider the following diagram

$$\begin{array}{ccccc} \Omega_{A\vee M} & \to & A\wedge A\vee A\wedge M\vee M\wedge A\vee M\wedge M & \to & A\vee M \\ & & \downarrow & & \downarrow \\ M & \to & A\vee M\vee M & \to & A\vee M \end{array}$$

Here the rows are homotopy cofibre sequences, the middle vertical map is determined by the algebra structure on A and an A-bimodule structure on M on the first three summands, and zero on the last summand.

Since the lower row is obviously split in the homotopy category of $A\vee M$-bimodules, we conclude that there is a map

$$\Omega_{A\vee M} \to M$$

making the above diagram commutative and it is unique.

So we constructed a map $\Omega_A^{A\vee M} \to M$. Now for $f : B \to A \vee M$ we have a map

$$\Omega_A^B \to \Omega_A^{A\vee M} \to M$$

and therefore a correspondence

$$D : [B, A\vee M]_{R-alg/A} \to [\Omega_A^B, M]_{A-bimod}.$$

We need a special case of the Theorem 2.2 to prove the general case:

Lemma 2.3 *For*

$$B = TV = \{R \vee V \vee V^{\wedge 2} \vee \ldots \xrightarrow{\mu} A\} \in \mathcal{C}_{R/A},$$

the free associative algebra on an R-module V over A this correspondence establishes an isomorphism

$$[TV, A\vee M]_{R-alg/A} \to [\Omega_A^{TV}, M]_{A-bimod} \tag{4}$$

Proof. First compute both sides of (4) separately. Obviously. the left hand side is equal to

$$[V, A \vee M]_{R-mod/A} = [V, M]_{R-mod}.$$

We have the following homotopy fibre sequence of TV-bimodules:

$$(TV) \wedge V \wedge TV \xrightarrow{f} TV \wedge TV \xrightarrow{m} TV \tag{5}$$

where $f = f_1 - f_2$ and

$$f_1 : TV \wedge V \wedge TV \xrightarrow{m \wedge id} TV \wedge TV$$

$$f_2 : TV \wedge V \wedge TV \xrightarrow{id \wedge m} TV \wedge TV;$$

m is the multiplication map. Smashing (5) with A on both sides over TV we get

$$A \wedge V \wedge A \to A \wedge A \to A \wedge_{TV} A.$$

Therefore $\Omega_A^{TV} = A \wedge V \wedge A$ and

$$[\Omega^{T(V)}{}_A, M]_{A-bimod} = [V, M]_{R-mod} = [T(V), A \vee M]_{ho\mathcal{C}_{R/A}}.$$

The second equality uses the (obvious) fact that the functor

$$M \to A \vee M : ho(R-mod) \to ho(R-mod/A)$$

is right adjoint to the forgetful functor

$$ho(R-mod/A) \to ho(R-mod).$$

So we identified each side of (4) with $[V, M]_{R-mod}$. To prove our lemma we have to show that D respects this identification. In other words for an R-module map $f : V \to M$ the composite

$$\Omega_{TV} \to \Omega_{A \vee M} \to M$$

should correspond under this identification to the map f.

Consider now the diagram

$$\begin{array}{ccccc}
TV \wedge V \wedge TV & \longrightarrow & \Omega_{A \vee M} & \longrightarrow & M \\
\downarrow & & \downarrow & & \downarrow \\
TV \wedge TV & \longrightarrow & A \wedge A \vee A \wedge M \vee M \wedge A \vee M \wedge M & \longrightarrow & A \vee M \vee M \\
\downarrow & & \downarrow & & \downarrow \\
TV & \longrightarrow & A \wedge M & = & A \wedge M
\end{array}$$

Here all three columns are homotopy cofibre sequences. Also recall that the upper left vertical map is $m \wedge id - id \wedge m$. Clearly the composite

$$V \to TV \wedge V \wedge TV \to TV \wedge TV \to$$

$$A \wedge A \vee A \wedge M \vee M \wedge A \vee M \wedge \to A \vee M \vee M \to M \vee M$$

coincides with the composite

$$V \to V \vee V \to A \wedge M \vee M \wedge A \to M \vee M$$

where the first map is $id \vee 0 - 0 \vee id$, the second map is $\mu \wedge f - f \wedge \mu$ and the last map is induced by the A-bimodule structure on M. Therefore the composite map $V \to M \vee M$ is $f \vee 0 - 0 \vee f$. Taking into account that the exact triangle

$$M \to M \vee M \to M$$

splits we see that there is a unique lifting

$$V \to TV \wedge V \wedge TV \to M$$

and it coincides with f. The lemma is proved.

Return now to the general case. The set $[B, A \vee M]_{R-alg/A}$ can be computed by resolving B by the free associative simplicial algebra (e.g. the monadic bar construction). That is,

$$[B, A \vee M]_{R-alg/A} = [|\mathbf{B}_\star|, A \vee M]_{R-alg/A}$$

where $\mathbf{B}_\star = TV_\star$ is a simplicial associative algebra whose components are free R-algebras on R-modules V_i equivalent to B and $|\mathbf{B}_\star|$ is its geometric realization (which by [EKMM], Proposition VII 3.3 is the same in the category of R-algebras as in the underlying category of R-modules).

Denote $F_{\mathcal{C}_{A/R}}(-,-)$ the topological space of maps between two objects in $\mathcal{C}_{A/R}$ (not homotopy classes of maps). We have:

$$F_{\mathcal{C}_{A/R}}(|\mathbf{B}_\star|, A \wedge M) \cong F_{\mathcal{C}_{A/R}}(\mathbf{B}_\star, A \wedge M)$$

where $F_{\mathcal{C}_{A/R}}(\mathbf{B}_\star, A \wedge M)$ is the topological space of maps between two simplicial R-algebras $\mathbf{B}_\star$ and $A \wedge M$, the latter regarded as a constant simplicial algebra.

Next, $F_{\mathcal{C}_{A/R}}(\mathbf{B}_\star, A \vee M)$ is the total space of the cosimplicial topological space

$$\{F_{\mathcal{C}_{A/R}}(\mathbf{B}_i, A \wedge M)\}.$$

Likewise

$$F_{A-bimod}(\Omega_A^B, M) = F_{B-bimod}(\Omega_B, M)$$

is the total space of the cosimplicial space

$$\{F_{B_i-bimod}(\Omega_{B_i-bimod}, M).$$

There is a map between these two cosimplicial spaces which, according to the previous lemma induces an equivalence on the ith simplices for all i. Therefore this map is a weak equivalence and our theorem is proved.

Remark 2.4 *It will be useful to generalize the notion of the module of differentials a little. Let R' be a (not necessarily commutative) R-algebra and consider the category of R'-algebras which means just R-algebras A equipped with a map $R' \to A$. For an R'-algebra A, we can define $\Omega_{A/R'}$ (replacing $\wedge$ by $\wedge_{R'}$. The analogue of Theorem (2.2) reads as follows: for an R'-algebra B over an R'-algebra A one has a natural equivalence*

$$[B, A \vee M]_{ho\mathcal{C}_{R'/A}} = [\Omega^B{}_{A/R'}, M]_{A-bimod}$$

where $\Omega^B{}_{A/R'}$ is defined from the fibre sequence

$$\Omega^B{}_{A/R'} \to A \wedge_{R'} A \to A \wedge_B A.$$

The proof is the same, one only has to notice that there still are adjunctions

$$R' - algebras \overset{\leftarrow}{\to} R' - bimodules;$$

$$A - bimodules \overset{\leftarrow}{\to} R' - bimodules$$

where the "free" R'-algebra functor is given for an R'-bimodule V as

$$TV = R' \vee V \vee V \wedge_{R'} V \ldots$$

and the "free" A-bimodule on a R'-bimodule V is $A \wedge_{R'} V \wedge_{R'} A$. The R'-bimodule $F_{A-bimod}(\Omega_{A/R'}, M)$ will be called R' relative Hochschild cohomology of A with coefficients in M and denoted as $THH(A;, R', M)$. Notice, that even if R' happens to be commutative $THH(A; R', M)$ is not the same as $THH_{R'}(A, M)$ since the latter is relevant to the category of R'-relative A-bimodules, that is, those A-bimodules whose induced R'-bimodule structure is symmetric.

Corollary 2.5 *The functor*

$$B \to {\Omega^B}_A : \mathcal{C}_{\mathcal{R}/\mathcal{A}} \to A - bimod$$

respects homotopy colimits.

Proposition 2.6 *(Transitivity exact sequence) Let $A \to B \to C$ be morphisms of R-algebras. Then one has the following homotopy cofibre sequence of C-bimodules:*

$$C \wedge_B \Omega_{B/A} \wedge_B C \to \Omega_{C/A} \to \Omega_{C/B}$$

Proof. We have the following homotopy cofibre sequence of B-bimodules:

$$\Omega_{B/A} \to B \wedge_A B \to B.$$

Smashing it with C over B from both sides we get the homotopy cofibre sequence of C-bimodules:

$$C \wedge_B \Omega_{B/A} \wedge_B C \to C \wedge_A C \to C \wedge_B C.$$

Consider the following commutative diagram:

$$\begin{array}{ccccc} C \wedge_B \Omega_{B/A} \wedge_B C & \longrightarrow & C \wedge_A C & \longrightarrow & C \wedge_B C \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C & \longrightarrow & C \end{array}$$

Here the right and the middle vertical arrows are induced by multiplication on C. Taking the homotopy fibres of the vertical arrows we get the desired cofibre sequence.

Corollary 2.7 *For a C-bimodule M we have the following homotopy cofibre sequence*

$$THH(C; B, M) \to THH(C; A, M) \to THH(B; A, M)$$

Proof. Apply the functor $F_{C-bimod}(\bullet, M)$ to the transitivity exact sequence.

3 Extensions of R-algebras

Definition 3.1 *An extension of an R-algebra B by an R-module I is the following homotopy fibre sequence of R-modules:*

$$I \to A \to B. \tag{6}$$

where A and B are algebras, I is an algebra without unit and both arrows are algebra maps.

Next denote I/I^2 the homotopy cofibre

$$I \wedge_A I \to I \to I/I^2.$$

(notice that I is naturally an A-module. Then we have the following

Lemma 3.2 *(Conormal exact sequence) The R-module I/I^2 has a structure of a B-bimodule and there is a homotopy cofibre sequence of B-bimodules*

$$I/I^2 \to B \wedge_A \Omega_A \wedge_A B \to \Omega_B \tag{7}$$

Proof. We have the following transitivity exact sequence (corresponding to algebra maps $R \to A \to B$):

$$B \wedge_B \Omega_B \wedge_B B \to \Omega_B \to \Omega_{B/A}.$$

So it suffices to show that $\Sigma(I/I^2) = \Omega_{B/A}$. Consider the following diagram:

$$\begin{array}{ccccc}
I \wedge_A I & \longrightarrow & I & \longrightarrow & I/I^2 \\
\downarrow & & \downarrow & & \downarrow \\
I & \longrightarrow & A & \longrightarrow & B \\
\downarrow & & \downarrow & & \downarrow \\
I \wedge_A B & \longrightarrow & B & \longrightarrow & B \wedge_A B
\end{array}$$

Here the arrows going to the right lower corner are both equal to $1 \wedge id$, the arrows going out of the left upper corner are multiplication maps on I and the map from I to $I \wedge_A B$ is $id \wedge 1$. This diagram is commutative in the homotopy category of R-modules and all rows and columns are homotopy cofibre sequences of R-modules. Next we have the following commutative diagram in the homotopy category of R- modules (again, all rows and columns are homotopy cofibre sequences):

$$\begin{array}{ccccc} I/I^2 & \longrightarrow & 0 & \longrightarrow & \Omega_{B/A} \\ \downarrow & & \downarrow & & \downarrow \\ I/I^2 & \longrightarrow & B & \longrightarrow & B\wedge_A B \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B & \longrightarrow & B \end{array}$$

From the upper row we see that there is a weak equivalence of R-modules $\Omega_{B/A}$ and $\Sigma I/I^2$ and the lemma is proved.

Definition 3.3 *Suppose, that in (6) the module I has a structure of a B-bimodule compatible with its A-bimodule structure and, in addition, the B-bimodule map $I \to I/I^2$ is an B-bimodule retraction, i.e. it admits a left inverse in the category of B-bimodules. Then we call (6) a singular extension of B by I. A morphism between two singular extensions (I, A, B) and (I', A', B') is the following strictly commutative diagram*

$$\begin{array}{ccccc} I & \to & B & \to & A \\ \downarrow & & \downarrow & & \downarrow \\ I' & \to & B' & \to & A' \end{array}$$

such that the vertical arrows are algebra maps. We also require that a morphism preserve all the structure in sight. If, in addition, the vertical maps in the above diagram are homotopy equivalences, we say that the extensions (I, A, B) and (I', A', B') are equivalent. Denote the set of equivalence classes of singular extensions of B by I as $Ext(B, I)$.

The next proposition describes more concretely the boundary map

$$\Omega_B \to \Sigma I/I^2$$

in the case of a singular extension.

Proposition 3.4 *let (6) be a singular extension, so $\Sigma I/I^2 \simeq \Sigma I \vee \Sigma I \wedge_A \Sigma I$. Denote by*

$$d : B \to \Sigma I$$

the boundary map in (6). Recall that there is an R-module weak equivalence $\Omega_B \cong B \wedge B/R$. Then for the boundary map in (7)

$$\partial : B \wedge B/R \to \Sigma I \vee \Sigma I \wedge_A \Sigma I$$

we have $\partial = \partial_1 + \partial_2$ where

$$\partial_1 : B \wedge B/R \stackrel{1\wedge d}{\to} B \wedge \Sigma I \to B \wedge_B \Sigma I = \Sigma I;$$

and ∂_2 is the composite map

$$B \wedge B/R \stackrel{d\wedge d}{\to} \Sigma I \wedge \Sigma I \to \Sigma I \wedge_A \Sigma I$$

Proof. Consider the following homotopy commutative diagram of R-modules:

$$\begin{tikzcd}
B \arrow[r, "id\wedge 1"] & B \wedge_A B \arrow[r, "id\wedge d"] & B \wedge_A \Sigma I \\
B \arrow[u, equal] \arrow[r, "id\wedge 1"] & B \wedge B \arrow[u] \arrow[r] & B \wedge B/R \arrow[u, "id \wedge d"']
\end{tikzcd}$$

Here both rows are split homotopy cofibre sequences , the R-module $B \wedge_A \Sigma I$ is isomorphic to $\Sigma I/I^2$ as we saw in the proof of the lemma 3.2 and the right vertical map is the map ∂ that we are interested in. Now the proposition follows from the following diagram , where again both rows are split cofibre sequences of R-modules:

$$\begin{tikzcd}
\Sigma I \arrow[r, "1\wedge d"] & B \wedge_A \Sigma I \arrow[r] & \Sigma I \wedge_A \Sigma I \\
A \wedge B/R \arrow[u] \arrow[r] & B \wedge B/R \arrow[u, "id \wedge d"'] \arrow[r] & \Sigma I \wedge B/R \arrow[u, "id \wedge d"']
\end{tikzcd}$$

Remark 3.5 *There is a slight catch in the above proposition. We considered the map*

$$1 \wedge d : B \wedge B/R \to B \wedge \Sigma I$$

denoting the extension of the map $B \to \Sigma I$ to B/R by the same letter d. This extension is not unique, but the indeterminacy vanishes upon smashing with B.

Theorem 3.6 *There is a one-to one correspondence between the set of equivalence classes of singular extensions of an R-algebra B by an B-bimodule M and the abelian group $THH^1(B, M) = Der(B, \Sigma M)$.*

Proof. Let $I \to A \to B$ be a singular extension. By the previous lemma we have a cofibre sequence of B-bimodules:

$$I/I^2 \to B \wedge_A \Omega_A \wedge_A B \to \Omega_B \to \Sigma I/I^2.$$

Taking the composition of the last arrow with the projection $\Sigma I/I^2 \to \Sigma I$ we get an element of

$$[\Omega_B, \Sigma I]_{B-bimod} = THH^1(B, I).$$

The map in the opposite direction is constructed as follows: let $d : B \to B \vee \Sigma I$ be a derivation. Consider the diagram

$$\begin{array}{ccc} I & = & I \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \\ \downarrow & & \downarrow \\ B & \xrightarrow{d} & B \vee \Sigma I \end{array} \tag{8}$$

Here the lower vertical map is the inclusion of a direct summand; the lower square is a homotopy pullback in the category of R-algebras (which is the same as the pullback of underlying R-modules), therefore the left column is a homotopy cofibre sequence of R-modules.

Claim: the sequence $I \to X \to B$ is a singular extension. First, I is a B-bimodule by its very construction. Next, we have the following commutative diagram:

$$\begin{array}{ccccc} I \wedge_B I & \to & I & \to & \Omega_{(B \vee \Sigma I)/B} \\ \uparrow & & \| & & \uparrow \\ I \wedge_X I & \to & I & \to & \Omega_{B/X} \end{array}$$

Since the upper row is clearly a singular extension we can construct a retraction $\Omega_{B/X} \to I$ as a composition of the map

$$\Omega_{B/X} \to \Omega_{(B \vee \Sigma I)/B}$$

and the retraction $\Omega_{(B \vee \Sigma I)/B} \to I$.

So we constructed two maps

$$f : Ext(B, I) \to THH^1(B, I);$$

$$Ext(B, I) \to THH^1(B, I) : g$$

Let us show that they are indeed inverse to each other.

Take a derivation $d : B \to B \wedge \Sigma I$. As before, we have the diagram (8)

Consider the following commutative (up to homotopy diagram of B-modules:

$$\begin{array}{ccc} \Omega_B & \to & \Omega_{B\wedge\Sigma I} \\ \downarrow & & \downarrow \\ \Omega_{B/X} & = & \Sigma I \vee \Sigma^2 I \wedge_X I \end{array}$$

Here the columns are fragments of the conormal exact sequences corresponding to the columns of (8). To prove that $g \circ f = id$ we have to show that the composite map

$$\Omega_B \to \Omega_{B\wedge\Sigma I} \to \Sigma I$$

corresponds to our original derivation $d : B \to B \wedge \Sigma I$. But this is exactly what we did in the course of the proof of Theorem 2.2.

Conversely, suppose we have a singular extension

$$I \to X \to B. \tag{9}$$

Consider a fragment of the corresponding conormal sequence

$$B \wedge_X \Omega_X \wedge_X B \to \Omega_B \to \Sigma I \vee \Sigma^2 I \wedge_X I. \tag{10}$$

To (9) there corresponds a derivation $d : \Omega_B \to \Sigma I$ induced by the last map in (10). The restriction of d to X is trivial as it corresponds to the composite of two consecutive maps in the cofibre sequence (10). Therefore the following diagram is commutative in the homotopy category of R-algebras:

$$\begin{array}{ccc} X & \to & B \\ \downarrow & & \downarrow \\ B & \to & B \vee \Sigma I \end{array}$$

Here the lower map corresponds to the derivation d and the right vertical map is the inclusion of a direct summand. There is a map of this diagram to the pullback diagram

$$\begin{array}{ccc} X' & \to & B \\ \downarrow & & \downarrow \\ B & \to & B \vee \Sigma I \end{array}$$

and it is clear that $X \to X'$ is a weak equivalence. This shows that $f \circ g = id$. Our theorem is proved.

4 The universal derivation

In this section we will compute the homotopy type of the "universal derivation", that is, of the composite map

$$A \to A \vee \Omega_A \to \Omega_A$$

where the first map is the algebra map adjoint to the identity map $\Omega_A \to \Omega_A$ and the second map is the projection.

Theorem 4.1 *Let A be an R-algebra. Recall that the R-module Ω_A is weakly equivalent to $A \wedge A/R$.*

Upon this identification the universal derivation $A \to \Omega_A$ is the same as the map

$$A \xrightarrow{d \wedge 1} A \wedge A/R.$$

where d is defined by the homotopy cofibre sequence

$$R \to A \xrightarrow{d} A/R.$$

More symmetrically: the universal derivation is the unique (in the homotopy category of R-modules) lifting $A \to \Omega_A$ in the diagram

$$\begin{array}{ccc} & & \Omega_A \\ & & \downarrow \\ A & \xrightarrow{f} & A \wedge A \end{array}$$

where $f = 1 \wedge id - id \wedge 1$

Proof. Consider the following extension

$$I \to R \to A$$

where the second map is just the unit map and I is the kernel of this map. From the conormal sequence of this extension we see that there is a map

$$\Omega_A \to \Sigma I/I^2 \tag{11}$$

and, consequently, a singular extension with the kernel I/I^2:

$$I/I^2 \to X \to I. \tag{12}$$

We have the following homotopy commutative diagram of R-modules:

$$\begin{array}{ccc} \Omega_A & \xrightarrow{g} & \Sigma I/I^2 \vee \Sigma^2 I/I^2 \wedge_X I/I^2 \\ \uparrow & \nearrow & \\ A & & \end{array}$$

Here the horizontal row is a fragment of the conormal sequence corresponding to (11), the vertical arrow is the universal derivation and the slanted arrow is the boundary map in (11).

We claim that the map g is the canonical inclusion of a direct summand:

$$f : \Omega_A \cong \Sigma I/I^2 \to \Sigma I/I^2 \vee \Sigma^2 I/I^2 \wedge_X I/I^2.$$

To see that observe that there is a map of extensions

$$\begin{array}{ccccc} I & \to & R & \to & A \\ \downarrow & & \downarrow & & \downarrow \\ I/I^2 & \to & X & \to & A \end{array}$$

and therefore the following homotopy commutative diagram of R-modules:

$$\begin{array}{ccccc} \Omega_A = I/I^2 & \to & A & \stackrel{1\wedge id}{\to} & A \wedge A \\ & & \downarrow & & \downarrow \\ \Omega_{A/X} = I/I^2 \vee \Sigma^2 I/I^2 \wedge_X I/I^2 & \to & A & \stackrel{1\wedge id}{\to} & A \wedge_X A \end{array}$$

Notice that since both rows split there exist only one up to homotopy map

$$\Omega_A \to \Omega_{A/X}$$

making the diagram commute and this map is obviously the inclusion of a direct summand. Our claim is proved. Therefore, our universal derivation is just the boundary map (12):

$$A \to \Sigma I/I^2.$$

Next from the diagram

$$\begin{array}{ccccccccc} I & \to & R & \to & A & \to & \Sigma I & = & A/R \\ \downarrow & & \downarrow & & \| & & \downarrow & & \\ I/I^2 & \to & R/I^2 & \to & A & \to & \Sigma I/I^2 & = & A \wedge A/R \end{array}$$

we see that the boundary map $A \to \Sigma I/I^2$ is the composition

$$A \to A/R = \Sigma I \to \Sigma I/I^2$$

and the diagram

$$\begin{array}{ccccccc} I & \to & R & \to & A & \to & A/R \\ \downarrow & & \downarrow & & 1 \wedge id \downarrow & & 1 \wedge id \downarrow \\ I/I^2 & \to & A & \stackrel{1 \wedge id}{\to} & A \wedge A & \to & A \wedge A/R \end{array}$$

tells us that the map $\Sigma I \to \Sigma I/I^2$ is the same as

$$A \stackrel{id \wedge d}{\to} A \wedge A/R$$

and the first part of our theorem is proved.

Consider now the following homotopy commutative diagram

$$\begin{array}{ccccc} \Omega_A & \to & A \wedge A & \to & A \wedge A/R \\ & & f \uparrow & \nearrow & \\ & & A & & \end{array}$$

Here the slanted arrow is $id \wedge d$ and the composite of the two horizontal arrows is an isomorphism identifying Ω_A with $A \wedge A/R$. Clearly the map $A \to \Omega_A$ fitting in the above diagram exists and is unique. With this our theorem is proved.

Remark 4.2 . *The universal derivation allows us to define a useful forgetful map*

$$l : THH(A, M) \to F(A, M)$$

as the composite

$$THH(A, M) = F_{A-bimod}(\Omega_A, M) \to F(\Omega_A, M) \to F(A, M)$$

where the last map is induced by d.

At the level of homotopy groups the map l has the following simple description: let $d : A \to A \vee M$ be a derivation of A with coefficients in M. Then $l(d)$ is the homotopy class of the following composite map

$$A \stackrel{d}{\to} A \vee M \to M$$

where the second map is just the projection onto the direct summand. Let $M = A$. Then from the fact that d is a map of algebras it follows that the following diagram is homotopy commutative:

$$\begin{array}{ccc} A \wedge A & \longrightarrow & A \\ \downarrow & & \downarrow \\ A \wedge A \vee A \wedge A & \to & A \end{array}$$

Here the upper arrow is the multiplication map m, the lower arrow is $m \vee m$, the right vertical arrow is d and the left vertical arrow is $d \wedge 1 \vee 1 \wedge d$. We see, therefore, that the image of l is contained in the subspace of primitive cohomology operations from A into itself.

5 Pullbacks and pushouts of singular extensions

Here we will describe some basic constructions with singular extensions. Let

$$E : I \to A \to B \tag{13}$$

be a singular extension and

$$f : C \to B$$

a map of algebras. Consider the diagram

$$\begin{array}{ccccc} I & \to & X & \to & C \\ \downarrow & & \downarrow & & \downarrow \\ I & \to & A & \to & B \end{array} \tag{14}$$

where the right vertical map is F and the right square is a homotopy pullback of R-algebras. It is easy to see that the upper row in (14) is a singular extension. We will denote it $f^\star(E)$.

We have the following simple interpretation of $f^\star(E)$ in terms of Hochschild cohomology. Let

$$\Omega_B \to \Sigma I$$

be the derivation corresponding to the singular extension E. Then the composite map

$$\Omega_C \to \Omega_B \to \Sigma I$$

represents the element in $THH^1(C, I)$ that corresponds to $f^\star(E)$. It follows immediately from the commutativity of the following diagram:

$$\begin{array}{ccccc}
\Omega_B & \to & \Omega_{B/A} & = & \Sigma I \vee \Sigma^2 I \wedge_A I \\
\uparrow & & \uparrow & & \\
\Omega_C & \to & \Omega_{C/X} & &
\end{array}$$

Next consider again an extension (13) and a map of B-modules

$$g : I \to M$$

The map
$$d : \Omega_B \to \Sigma I \to \Sigma M$$
gives rise to an element in $THH^1(B, M)$ and, therefore, to a singular extension of B by M. We will call this extension the pushout of E by g and denote it $g_\star(E)$.

Proposition 5.1 *There is a map from E to $g_\star(E)$ which is identical on B.*

Proof. Consider the diagram

$$\begin{array}{ccccc}
I & & & & \\
\downarrow & & & & \\
A & \to & B & \cong & B \\
\downarrow & & \downarrow & & \downarrow \\
B & \stackrel{d}{\to} & B \vee \Sigma I & \to & B \vee \Sigma M
\end{array}$$

Here the left square is a homotopy pullback, and from the commutativity of the outer square it follows that there is a map from it to the pullback diagram
$$\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
B & \to & B \vee \Sigma I
\end{array}$$
and therefore from E to $g_\star(E)$ and the proposition is proved.

Remark 5.2 *Consider the diagram*

$$\begin{array}{ccccc}
I & \to & A & \to & B \\
\downarrow & & \downarrow & & \downarrow \\
M & \to & Y & \to & B
\end{array}$$

where the upper row is E and the lower row is $g_\star(E)$. Since both rows are homotopy cofibre sequences of R-modules, Y is homotopically the same as $A \vee_I M$ but it is not clear apriori how to introduce strict multiplication on $A \vee_I M$

Proposition 5.3 *Let*

$$\begin{array}{cccccc} E_1: & I & \longrightarrow & A & \longrightarrow & B \\ & g\big\downarrow & & \big\downarrow & & \big\downarrow f \\ E_2: & M & \longrightarrow & C & \longrightarrow & D \end{array}$$

be a map of singular extensions $E_1 \to E_2$. Then

$$g_\star E_1 \cong f^\star E_2$$

Proof. We have the following commutative diagram

$$\begin{array}{ccccc} \Omega_B & \to & \Omega_{B/A} & = & \Sigma I \vee \Sigma^2 I \wedge_A I \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_D & \to & \Omega_{D/C} & = & \Sigma M \vee \Sigma^2 M \wedge_D M \end{array} \tag{15}$$

Now the singular extension $g_\star(E_1)$ corresponds to the composite map

$$\Omega_B \to \Sigma I \to \Sigma M$$

and $f^\star(E_2)$ corresponds to the composite map

$$\Omega_B \to \Omega_D \to \Sigma M$$

but (15) tells us that that these maps are equal and we are done.

Application: obstruction theory.

Proposition 5.4 *For a singular extension (13) and a map of algebras $f : C \to B$ the diagram*

$$\begin{array}{ccc} & & A \\ & & \downarrow \\ C & \to & B \end{array}$$

admits a lifting $C \to A$ if and only if $f^\star(E)$ is a trivial singular extension. Therefore the obstruction to lifting the map f is a certain class in

$THH^1(C, I)$. Furthermore, if this obstruction vanishes, then homotopy classes of liftings are in $1-1$ correspondence with derivations

$$C \to C \vee I,$$

that is with elements in $THH^0(C, I)$.

The proof should be clear.

Consider the dual problem of extending a map. It is not enirely obvious how to dualize correctly, but once the correct formulation is found, the proof is almost automatic.

Again take a singular extension (13) and a map of B-bimodules

$$f : I \to M.$$

We will call an extension of the map f to A a B-derivation d of A with values in M compatible with f, that is we require the following composite map $I \to M$ coincide with f:

$$I \to A \xrightarrow{d} B \wedge M \to M.$$

Proposition 5.5 *For a singular extension (13)and a map $f : I \to M$ admits an extension in the sense explained above if and only is the singular extension $f_\star(E)$ is trivial. The obstruction to extending f, therefore, lies in the group $THH^1(B, M)$. Moreover, the set of equivalence classes of possible extensions (provided one exists) is in $1-1$ correspondence with the group $THH^0(B, M)$.*

6 Postnikov towers of R-algebras

In this section we prove, as an application of the techniques developed before, that a Postnikov tower of an R-algebra is a tower of R-algebras. The analogous statement for commutative R-algebras was proved in [Kri] and [Bas]. The proof presented here is somewhat sketchy since it differs little from the one found in the above mentioned references.

Theorem 6.1 *Let A be a connective R-algebra and here we assume that R is connective also (these assumptions are needed in order to be able to kill higher homotopy groups). Then there exists a tower of R-algebras*

$$\ldots \leftarrow A_i \leftarrow A_{i+1} \leftarrow \ldots$$

and maps of R-algebras

$$f_i : A \to A_i$$

compatible with the maps in the tower such that

1)$\pi_k(A_i) = 0$ *for* $k > i$ *and*

2)f_i *induces an isomorphism* $\pi_k(A) \to \pi_k(A_i)$ *for* $k \leq i$. *Moreover, the homotopy fibre sequence*

$$K(\pi_{i+1}(A), i+1) \to A_{i+1} \to A_i$$

is a singular extension and the whole construction is functorial in the homotopy category of R-algebras

Proof. Take A_0 to be equal to $K(\pi_0(A,)0)$. This is an R-algebra and it is well-known that there exists an R-algebra map

$$A \to A_0$$

realizing the isomorphism on π_0.

Suppose by induction that the map $A \to A_i$ has already been constructed. Consider the following diagram:

$$\begin{array}{ccccc} A & \to & A_i & \xrightarrow{k} & \Sigma K(\pi_{i+1}(A), i+1) \\ & & \downarrow & \nearrow & \\ & & \Omega_{A_i/A} & & \end{array}$$

Here the vertical map is the canonical derivation, k is the ith k- invariant. The existence of the diagonal arrow can be seen as follows. The first nontrivial homotopy group of $\Omega_{A_i/A}$ is the same as the only nontrivial homotopy group of $\Sigma K(\pi_{i+1}(A), i+1)$, so we could construct a map

$$\Omega_{A_i/A} \to \Sigma K(\pi_{i+1}(A), i+1)$$

by attaching cells in the category of A_i-bimodules. This gives a derivation

$$A_i \to A_i \vee \Sigma K(\pi_{i+1}(A), i+1)$$

and the corresponding singular extension

$$K(\pi_{i+1}(A), i+1) \to A_{i+1} \to A_i$$

which is the $(i+1)$th stage of the Postnikov tower and the theorem is proved.

7 The spectral sequence for topological Hochschild cohomology

In this section we discuss the spectral sequence that relates topological Hochschild cohomology and ordinary Hochschild cohomology of graded algebras. Since we know of no published source which describes algebraic Hochschild cohomology the way we need it we begin with an outline of the classical Hochschild cohomology.

Definition 7.1 . *Let $R^\bullet$ be a commutative graded ring, $A^\bullet$ is an algebra over $R^\bullet$, $M^\bullet$ is a graded module over $A^\bullet$. Then Hochschild cohomology of $A^\bullet$ with coefficients in $M^\bullet$ is the module*

$$\overline{HH}^{\bullet\bullet}_{R^\bullet}(A^\bullet, M^\bullet) = Ext_{A^\bullet \otimes^L_{R^\bullet} A^{\bullet op}}(A^\bullet, M^\bullet)$$

where $\otimes^L_{R^\bullet}$ denotes derived tensor product over $R^\bullet$

From now on we will write $\otimes$ instead of $\otimes_{R_\bullet}$ Notice that if $A^\bullet$ is flat over $R^\bullet$ then this definition coincides with the usual one. There is also a generalization of the standard complex which computes Hochschild cohomology; let $\tilde{A}^\bullet$ be a differential graded $R^\bullet$-algebra which is quasiisomorphic to $A^\bullet$ and $R^\bullet$-flat. Then $\overline{HH}^{\bullet\bullet}_{R^\bullet}(A^\bullet, M^\bullet)$ is the cohomology of the bicomplex

$$\overline{C}^{ij}(A^\bullet, M^\bullet) = Hom^i_{R^\bullet}(\tilde{A}^{\otimes j}, M^\bullet).$$

The differentials in $\overline{C}^{\bullet\bullet}$ are the internal one induced by the differential in $\tilde{A}^\bullet$ and the standard bar-differential.

We will be interested mostly in the truncated version of Hochschild cohomology which will be denoted as $HH^\bullet(A^\bullet, M^\bullet)$. To define it let us introduce the module of differentials $\Omega^\bullet_{A^\bullet/R^\bullet}$ from the folowing short exact sequence of complexes:

$$\Omega^\bullet_{A^\bullet/R^\bullet} \to \tilde{A}^\bullet \otimes \tilde{A}^\bullet \overset{m}{\to} \tilde{A}^\bullet,$$

the second arrow being the multiplication map.

Now define truncated Hochschild cohomology of $A^\bullet$ with coefficients in $M^\bullet$ as

$$HH^\bullet A^\bullet, M^\bullet = Ext_{R^\bullet}(\Omega^\bullet_{A^\bullet/R^\bullet}, M^\bullet).$$

As in the topological case there is a "universal derivation"

$$d : \tilde{A}^\bullet \to \Omega^\bullet_{A^\bullet/R^\bullet}$$

whose composition with the inclusion

$$\Omega^{\bullet}_{A^{\bullet}/R^{\bullet}} \hookrightarrow \tilde{A}^{\bullet} \otimes_{R^{\bullet}} \tilde{A}^{\bullet}$$

is the map $1 \wedge id - id \wedge 1$. It gives, therefore, the forgetful map

$$l : HH^{\bullet}(A^{\bullet}, M^{\bullet}) \to Ext^{\bullet}_{R^{\bullet}}(A^{\bullet}, M^{\bullet})$$

Lemma 7.2 *Consider the truncated Hochschild complex*

$$C^{ij}(A^{\bullet}, M^{\bullet}) = Hom^{i}_{R^{\bullet}}(\tilde{A}^{\otimes j-1}, M^{\bullet})$$

with the same differentials as in $\overline{C}^{\bullet\bullet}(A^{\bullet}, M^{\bullet})$*. Then the cohomology of* $C^{\bullet\bullet}(A^{\bullet}, M^{\bullet})$ *coincides with* $HH^{\bullet}(A^{\bullet}, M^{\bullet})$ *and the forgetful map* l *is induced by the projection*

$$C^{\bullet\bullet}(A^{\bullet}, M^{\bullet} \to C^{\bullet 0}(A^{\bullet}, M^{\bullet}) = Hom^{\bullet}_{R^{\bullet}}(\tilde{A}^{\bullet}, M^{\bullet})$$

times (-1).

Proof. Considering the standard two-sided bar-resolution of $\tilde{A}^{\bullet}$:

$$\tilde{A}^{\bullet} \xleftarrow{m} \tilde{A}^{\bullet} \otimes_{R^{\bullet}} \tilde{A}^{\bullet} \leftarrow \tilde{A}^{\bullet \otimes_{R^{\bullet}} 3} \leftarrow \ldots$$

we see that the complex $\tilde{A}^{\bullet \otimes_{R^{\bullet}} 3} \leftarrow \tilde{A}^{\bullet \otimes_{R^{\bullet}} 4} \leftarrow \ldots$ maps quasiisomorphically onto the complex $\Omega^{\bullet}_{A^{\bullet}/R^{\bullet}} = Ker(m)$. The universal derivation

$$d : \tilde{A}^{\bullet} \to \Omega^{\bullet}_{A^{\bullet}/R^{\bullet}}$$

lifts to the map

$$\tilde{d} : \tilde{A}^{\bullet} \to \tilde{A}^{\bullet \otimes_{R^{\bullet}} 3};$$
$$\tilde{d}(a) = -1 \otimes a \otimes 1.$$

Now the statement of the lemma becomes clear after we apply the functor $Hom_{\tilde{A}^{\bullet} \otimes \tilde{A}^{\bullet}}(-, M^{\bullet})$ to the complex $\tilde{A}^{\bullet \otimes_{R^{\bullet}} 3} \leftarrow \tilde{A}^{\bullet \otimes_{R^{\bullet}} 4} \leftarrow \ldots$.

We now return to the topological situation; as before A is an R-algebra, M is an A-bimodule; denote $\pi_{\bullet}(A)$ as $A_{\bullet}$, $\pi_{\bullet}(R)$ as $R_{\bullet}$ and $\pi_{\bullet}(M)$ as $M_{\bullet}$.

Proposition 7.3 *Suppose that*
1)the Kunneth spectral sequence for $\pi_{\bullet}(A, A^{op})$ *collapses so we have*

$$\pi_{\bullet}(A, A^{op}) = Tor^{\bullet}_{R_{\bullet}}(A^{\bullet}, A^{\bullet});$$

2)the algebras $A_\bullet \otimes^L_{R_\bullet} A_\bullet$ and $Tor^\bullet_{R_\bullet}(A^\bullet, A^\bullet)$ are quasiisomorphic.
Then there are the following spectral sequences:

$$^1E_2^{ij} = HH^{ij}_{R^\bullet}(A^\bullet, M^\bullet) \Longrightarrow THH^{i+j}(A, M);$$

$$^2E_2^{ij} = Ext_{R^\bullet}(A^\bullet, M^\bullet) \Longrightarrow F(A, M)$$

and a map

$$^1E_2^{ij} \to \ ^2E_2^{ij}$$

which at the level of E_∞ terms gives the forgetful map

$$l : grTHH^\bullet(A, M) \to gr[A, M]^\bullet$$

Proof. There is a spectral sequence (cf. [EKMM], Thm 4.1)

$$E_2^{\bullet,\bullet} = Ext_{\pi_\bullet(A\wedge A^{op})}(\pi_\bullet\Omega_A, M^\bullet) \Longrightarrow F_{A\wedge A^{op}}(\Omega_A, M) = THH(A, M)$$

Our assumptions 1) and 2) ensure that

$$\pi_\bullet(A \wedge A^{op}) = A_\bullet \otimes^L_{R_\bullet} A_\bullet$$

and

$$\pi_\bullet(\Omega_A) = \Omega_{A^\bullet/R^\bullet},$$

so we identify the E_2-term of our spectral sequence as $HH_{R^\bullet}(A^\bullet, M^\bullet)$.

The second spectral sequence is just the hypercohomology spectral sequence for calculating $F(A, M)$. Next from the naturality of the hypercohomology spectral sequences it follows that we have the following commutative diagram of spectral sequences:

$$\begin{array}{ccccc} Ext^\bullet_{\pi_\bullet(A\wedge A^{op})}(\pi_\bullet\Omega_A, M^\bullet) & \to & Ext^\bullet_{R^\bullet}(\pi_\bullet\Omega_A, M^\bullet) & \to & Ext^\bullet_{R^\bullet}(A^\bullet, M^\bullet) \\ \| & & \| & & \| \\ HH^\bullet_{R^\bullet}(A^\bullet, M^\bullet) & \to & Ext^\bullet_{R^\bullet}(\Omega_{A^\bullet/R^\bullet}, M^\bullet) & \to & Ext^\bullet_{R^\bullet}(A^\bullet, M^\bullet) \end{array} \tag{16}$$

and since the composite map

$$A^\bullet \to \pi_\bullet\Omega_A \simeq \Omega_{A^\bullet/R^\bullet}$$

is just the universal derivation of the algebra $A^\bullet$, we conclude that the lower row in (16) gives us the forgetful map

$$l : HH^\bullet_{R^\bullet}(A^\bullet, M^\bullet) \to Ext^\bullet_{R^\bullet}(A^\bullet, M^\bullet).$$

Our proposition is proved.

8 Calculations

In this section we calculate the topological Hochschild cohomology of the Eilenberg-MacLane spectrum $K(Z/pZ, 0)$ which will be denoted in the sequel by K considered as an algebra over the complex cobordism spectrum MU,

Theorem 8.1
(i)

$$\overline{THH}^{\bullet}_{MU}(K,K) = Z/pZ[y_2, y_4, \ldots, y_{2n}\ldots];$$
$$THH_{MU}(K,K)^{\bullet-1} = Z/pZ[y_2, y_4, \ldots]/(Z/pZ).$$

(ii)

$$[K,K]^{\bullet}_{MU} = \Lambda_{Z/pZ}(z_1, z_3, \ldots, z_{2n+1}\ldots)$$

and the map

$$THH_{MU}(K,K)^{\bullet} \to [K,K]^{\bullet}_{MU}$$

sends the elements $y_{2i} \in THH_{MU}(K,K)^{2i-1}$ to $z_2 i - 1 \in [K,K]_{MU}$.
(iii) In the forgetful map

$$[K,K]^{\bullet}_{MU} \to [K,K]^{\bullet}_{S} = \mathcal{A}_p \tag{17}$$

the elements z_{2p^i-1} get mapped to the elements Q_i which form the set of primitive elements in the Steenrod algebra, and all other z's get mapped to zero.

Proof. First compute $\pi_\bullet(K \wedge_{MU} K)$. We have the (strongly converging) Kunneth spectral sequence

$$Tor_{MU_\bullet}(Z/pZ, Z/pZ) = \Lambda_{Z/pZ}(x_1, x_3, \ldots x_{2n+1}, \ldots) \Longrightarrow \pi_\bullet(K \wedge_{MU} K).$$

This is a spectral sequence of Hopf algebras, so all the higher differentials applied to the primitive elements x_i's should be zero and we conclude that this spectral sequence collapses and

$$\pi_\bullet(K \wedge_{MU} K) = \Lambda_{Z/pZ}(x_1, x_3, \ldots x_{2n+1}, \ldots).$$

Next consider the spectral sequence

$$Ext_{\pi_\bullet(K\wedge_{MU}K)}(Z/pZ, Z/pZ) \Longrightarrow \overline{THH}_{MU}(K,K).$$

Its E_2-term is easily computed and equal to

$$Ext_{\Lambda_{Z/pZ}(x_1,x_3,\ldots x_{2n+1},\ldots)}(Z/pZ, Z/pZ) = Z/pZ[y_2, y_4, \ldots, y_{2n}\ldots].$$

Again, all higher differentials are zero this time because the spectral sequence is even. Now we know that our spectral sequence lies in the right half plane and converges conditionally. Since all higher differentials are zero Boardman's theorem (cf. [GM], Theorem B.6) tells us that it converges strongly. All extensions problems are clearly trivial and we conclude that

$$\overline{THH}^{\bullet}_{MU}(K,K) = Z/pZ[y_2, y_4, \ldots, y_{2n}\ldots].$$

The formula for $THH^{\bullet}_{MU}(K,K)$ follows from the homotopy cofibre sequence

$$\overline{THH}^{\bullet}_{MU}(K,K) \to K \to THH^{\bullet}_{MU}(K,K)$$

and part (i) of our theorem is proved.

For part (ii) consider the Kunneth spectral sequence

$$Ext_{MU_\bullet}(Z/pZ, Z/pZ) = \Lambda_{Z/pZ}(x_1, x_3, \ldots x_{2n+1}, \ldots) \Longrightarrow [K,K]_{MU}.$$

This is a sequence of Hopf algebras and the considerations similar to the foregoing show that it collapses, with no additive or multiplicative extension problems.

To compute the image of the map

$$THH_{MU}(K,K)^{\bullet} \to [K,K]^{\bullet}_{MU} \tag{18}$$

we first look at the map

$$HH_{MU_\bullet}(Z/pZ, Z/pZ) \to Ext_{MU_\bullet}(Z/pZ, Z/pZ)$$

and then apply Proposition 7.3.

It is an easy exercize in homological algebra to show the elements

$$y_{2i} \in \overline{HH}^{1,2i-1}_{MU_\bullet}(Z/pZ, Z/pZ) = Z/pZ[y_2, y_4\ldots]$$

corresponds to the elements

$$z_{2i+1} \in Ext^{1,2i}_{Z[x_2,x_4\ldots]}(Z/pZ, Z/pZ),$$

so Proposition 7.3 tells us that the map (18) is as stated at the level of E_2-terms. However, the image of(18) should be contained in the subgroup of primitive cohomology operations, which is spanned by the elements $z_{2i+1} \in [K,K]_{MU}$ and therefore no elements of lower bar filtration might appear in this image. Part *(ii)* is proved.

To prove part *(iii)* notice that the map (17) is a map of Hopf algebras, and therefore the primitive elements z_{2i+1} should map to the primitive elements in the Steenrod algebra. Hence only the elements z_{2p^i-1} could have a nonzero image (namely, Q_i). We will show now that this is indeed the case. Consider the spectrum $k(n)$, the nth connective Morava K-theory. Recall, that $k(n)$ is constructed out of MU by killing off all polynomial generators x_{2i} of $MU_\bullet$ except x_{2p^n-2}. Since the procedure of killing a regular ideal in the coefficient ring is legitimate in the category of MU-modules, that shows that $k(n)$ is an MU-module. But then the Postnikov tower of $k(n)$ can also be constructed in the category of MU-modules, and in particular the first nontrivial k-invariant of $k(n)$ (which is just Q_n) is given as a map of MU-modules $K \to K$. In other words the cohomological operation Q_n can be lifted as a map of MU-modules and part *(iii)* is proved.

Remark 8.2 *Recall that for the ordinary topological Hochschild cohomology of K we have (cf., for example [FLS])*

$$\overline{THH}^\bullet(K,K) = Z/pZ[x]$$

where x is a 2-dimensional homotopy class. It follows that

$$THH^{\bullet-1}(K,K) = Z/pZ[x]/(Z/pZ).$$

Then under the map

$$l : THH^\bullet(K,K) \to [K,K]_S^\bullet = \mathcal{A}_p \tag{19}$$

the elements x^{p^n} correspond to $Q_n \in \mathcal{A}_p$ and the rest gets mapped to zero. Indeed, the Hochschild cohomology classes should give rise to primitive cohomology operations, therefore only elements x^{p^n} may have a nonzero image. But we know that the elements Q_i's do lift to Hochschild cohomology classes (even in the category of MU-modules). Therefore the map (19) is as described.

References

[Bas] M.Basterra. *Andre-Quillen cohomology of commutative S-algebras.* Preprint.

[EKMM] A.D.Elmendorf, I.Kriz, M.A.Mandell, J.P.May. *Rings, Modules and Algebras in Stable Homotopy Theory.* Mathematical Surveys and Monographs. Vol. 47. AMS,1996

[FLS] V.Franjou, J.Lannes, L.Schwartz. Autour de la cohomologie de MacLane des corps finis. Invent.Math.,115, no.3, 513-538.

[GM] J.P.C.Greenlees, J.P.May.*Generalized Tate Cohomology.* Memoirs of the AMS,Vol.113, 1995

[Kri] I.Kriz. *Towers of E_∞ spectra with an application to BP.* Preprint

[Sch] S.Schwede. *Private communication*