TOPOLOGICAL EQUIVALENCE OF LINEAR REPRESENTATIONS FOR CYCLIC GROUPS

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June 17, 1997

Abstract. In this paper we prove that the Reidemeister torsion invariants determine topological equivalence of G–representations, for G a finite cyclic group.

1. Introduction

Let G be a finite group and V, V' finite dimensional real orthogonal representations of G. Then V is said to be *topologically equivalent* to V' (denoted $V \sim_t V'$) if there exists a homeomorphism $h: V \to V'$ which is G-equivariant. If V, V' are topologically equivalent, but not linearly isomorphic, then such a homeomorphism is called a non-linear similarity. These notions were introduced and studied by de Rham [33], [34], and developed extensively in [3], [4], [23], [24], and [7]. In this paper, we complete de Rham's program by showing that Reidemeister torsion invariants and number theory determine non-linear similarity for finite cyclic groups (see Section 2 for the detailed statement of results in Theorems A–D).

In [3], Cappell and Shaneson showed that non-linear similarities exist for cyclic groups $G = C(4q)$ of every order $4q \geqq 8$. On the other hand, if $G = C(q)$ or $G = C(2q)$, for q odd, Hsiang–Pardon [23] and Madsen–Rothenberg [24] proved that topological equivalence of G–representations implies linear equivalence (the case $G = C(4)$ is trivial). Since linear G-equivalence for general finite groups G is detected by restriction to cyclic subgroups, it is reasonable to study this case first. For the rest of the paper, unless otherwise mentioned, G denotes a finite cyclic group.

Further positive results can be obtained by imposing assumptions on the isotropy subgroups allowed in V_1 and V_2. For example, if the V_i are free G-representions, then de Rham [33] proved in 1935 that piecewise linear similarity implies linear equivalence, by using Reidemeister torsion and the Franz Independence Lemma. Topological invariance of Whitehead torsion shows that his method also rules out non–linear similarity in this case. In [19,Thm.A] we studied "first–time" similarities, where $\operatorname{Res}_K V_1 \cong \operatorname{Res}_K V_2$ for all proper subgroups $K \subsetneqq G$, and showed that

Partially supported by NSERC grant A4000 and NSF grant DMS 9104026. The authors also wish to thank the Max Planck Institut für Mathematik, Bonn, for its hospitality and support.

topological equivalence implies linear equivalence if V_1, V_2 have no isotropy subgroup of index 2. This result is an application of bounded surgery theory (see [18], [19,§4]), and these techniques are extended here to provide a necessary and sufficient condition for non-linear similarity in terms of the vanishing of a bounded transfer map (see Theorem 3.5). This provides a new approach to de Rham's problem. The main work of the present paper is to establish methods for effective calculation of the bounded transfer.

In [4], Cappell and Shaneson initiated the study of *stable* topological equvalence for G–representations. We say that V_1 and V_2 are stably topologically similar ($V_1 \approx_t V_2$) if there exists a G–representation W such that $V_1 \oplus W \sim_t V_2 \oplus W$. Let $R_{\mathrm{Top}}(G) = R(G)/R_t(G)$ denote the quotient group of the real representation ring of G by the subgroup $R_t(G) = \{[V_1]-[V_2] \mid V_1 \approx_t V_2\}$. In [4], $R_{\mathrm{Top}}(G) \otimes \mathbf{Z}[1/2]$ was computed, and the torsion subgroup was shown to be 2–primary. As an application of our general results, we determine the structure of the torsion in $R_{\mathrm{Top}}(G)$, for G cyclic (see Section 20).

One interesting question in non–linear similarity concerns the minimum possible dimension for examples. It is easy to see that the existence of a non–linear similarity $V_1 \sim_t V_2$ implies $\dim V_i \geqq 5$. Cappell, Shaneson, Steinberger and West [7], proved that 6-dimensional similarities exist for $G = C(2^r)$, $r \geqq 4$, and asserted that 5–dimensional similarities do not exist for any finite group. On the other hand, the reference cited [5] has not yet appeared. In Theorem A we give a reduction of this question to an explicit number theory criterion. Using this criterion, we show in Corollary 19.2 that cyclic 2–groups do not have any 5–dimensional non–linear similarities. We have not been able to supply either a complete proof or a counterexample in the general case.

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2. Statement of Results

We first introduce some notation, and then give the main results. Let $G = C(4q)$, where $q > 1$, and let $H = C(2q)$ denote the subgroup of index 2 in G. The maximal odd order subgroup of G is denoted G_{odd}. We fix a generator $G = \langle t\rangle$ and a primitive $4q^{th}$-root of unity $\zeta = \exp 2\pi i/4q$. The group G has both a trivial 1-dimensional real representation, denoted $\mathbf{R}_+$, and a non-trivial 1-dimensional real representation, denoted $\mathbf{R}_-$.

A free G–representation is a sum of faithful 1-dimensional complex representations. Let t^a, $a \in \mathbf{Z}$, denote the complex numbers $\mathbf{C}$ with action $t \cdot z = \zeta^a z$ for all $z \in \mathbf{C}$. This representation is free if and only if $(a, 4q) = 1$, and the coefficient a is well–defined only modulo $4q$. Since $t^a \cong t^{-a}$ as real G–representations, we can always choose the weights $a \equiv 1 \pmod 4$. This will be assumed unless otherwise mentioned.

Now suppose that $V_1 = t^{a_1} + \cdots + t^{a_k}$ is a free G–representation. The Reidemeister torsion invariant of V_1 is defined as

$$\Delta(V_1) = \prod_{i=1}^{k}(t^{a_i} - 1) \in \mathbf{Z}[t]/\{\pm t^m\} \ .$$

If $V_2 = t^{b_1} + \cdots + t^{b_k}$ is another free representation, such that $S(V_1)$ and $S(V_2)$ are G-homotopy equivalent, then

$$\Delta(V_1)/\Delta(V_2) = \frac{\prod(t^{a_i} - 1)}{\prod(t^{b_i} - 1)} \in \mathrm{Wh}(\mathbf{Z}G)$$

is the Whitehead torsion of any G-homotopy equivalence $f\colon S(V_2) \to S(V_1)$ [33]. When there exists a G-homotopy equivalence $f\colon S(V_2) \to S(V_1)$ which is normally cobordant to the identity map on $S(V_1)$, we say that $S(V_1)$ and $S(V_2)$ are *normally cobordant*. More generally, we say that $S(V_1)$ and $S(V_2)$ are *s–normally cobordant* if $S(V_1 \oplus U)$ and $S(V_2 \oplus U)$ are normally cobordant for all free G–representations U. This is a necessary condition for non-linear similarity, which can be decided by explicit congruences in the weights.

This quantity, $\Delta(V_1)/\Delta(V_2)$ is the basic invariant determining non-linear similarity. To state concrete results we need to evaluate this invariant modulo suitable indeterminacy.

The involution $t \mapsto t^{-1}$ induces the identity on $\mathrm{Wh}(\mathbf{Z}G)$, so we get an element

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^0(\mathrm{Wh}(\mathbf{Z}G))$$

where we use $H^i(A)$ to denote the Tate cohomology $H^i(\mathbf{Z}/2; A)$ of $\mathbf{Z}/2$ with coefficients in A.

Let $\mathrm{Wh}(\mathbf{Z}G^-)$ denote the Whitehead group $\mathrm{Wh}(\mathbf{Z}G)$ together with the involution induced by $t \mapsto -t^{-1}$. Then for $\tau(t) = \frac{\prod(t^{a_i}-1)}{\prod(t^{b_i}-1)}$, we compute

$$\tau(t)\tau(-t) = \frac{\prod(t^{a_i} - 1)\prod((-t)^{a_i} - 1)}{\prod(t^{b_i} - 1)\prod((-t)^{b_i} - 1)} = \prod \frac{(t^2)^{a_i} - 1}{((t^2)^{b_i} - 1)}$$

which is clearly induced from $\mathrm{Wh}(\mathbf{Z}H)$. Hence we also get a well defined element

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\,\mathrm{Wh}(\mathbf{Z}H)) \ .$$

Consider the exact sequence of $\mathbf{Z}/2$ modules:

$$K_1(\mathbf{Z}H) \to K_1(\mathbf{Z}G) \to K_1(\mathbf{Z}H \to \mathbf{Z}G) \to \widetilde{K}_0(\mathbf{Z}H) \to \widetilde{K}_0(\mathbf{Z}G) \tag{2.1}$$

and let $\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G) = K_1(\mathbf{Z}H \to \mathbf{Z}G)/\{\pm G\}$. We then have a short exact sequence

$$0 \to \mathrm{Wh}(\mathbf{Z}G)/\,\mathrm{Wh}(\mathbf{Z}H) \to \mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G) \to \mathbf{k} \to 0$$

where $\mathbf{k} = \ker(\widetilde{K}_0(\mathbf{Z}H) \to \widetilde{K}_0(\mathbf{Z}G))$. Such an exact sequence of $\mathbf{Z}/2$-modules induces a long exact sequence in Tate cohomology. In particular, we have a coboundary map

$$\delta\colon H^0(\mathbf{k}) \to H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\,\mathrm{Wh}(\mathbf{Z}H)) \ .$$

Theorem A. *Let $V_1 = t^{a_1} + \cdots + t^{a_k}$ and $V_2 = t^{b_1} + \cdots + t^{b_k}$ be free G–representations, with $a_i \equiv b_i \equiv 1 \pmod 4$. There exists a topological similarity $V_1 \oplus \mathbf{R}_- \sim_t V_2 \oplus \mathbf{R}_-$ if and only if*

(i) $\prod a_i \equiv \prod b_i \pmod{4q}$,
(ii) $\mathrm{Res}_H V_1 \cong \mathrm{Res}_H V_2$, *and*
(iii) *the element $\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\,\mathrm{Wh}(\mathbf{Z}H))$ is in the image of the coboundary $\delta\colon H^0(\mathbf{k}) \to H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\,\mathrm{Wh}(\mathbf{Z}H))$.*

Remark 2.2. Theorem A should be compared with [3,Cor.1], where more explicit conditions are given for similarities of this kind under the assumption that q is odd, or a 2-power, or $4q$ is a "tempered" number. See also Theorem 15.2 for a more general result concerning similarities without $\mathbf{R}_+$ summands. The case $\dim V_1 = \dim V_2 = 4$ gives a reduction to number theory for the existence of 5-dimensional similarities (see Remark 12.2). □

Our next result uses a more elaborate setting for the invariant. Let

$$\Phi = \begin{pmatrix} \mathbf{Z}H & \to & \widehat{\mathbf{Z}}_2H \\ \downarrow & & \downarrow \\ \mathbf{Z}G & \to & \widehat{\mathbf{Z}}_2G \end{pmatrix}$$

and consider the exact sequence

$$0 \to K_1(\mathbf{Z}H \to \mathbf{Z}G) \to K_1(\widehat{\mathbf{Z}}_2H \to \widehat{\mathbf{Z}}_2G) \to K_1(\Phi) \to \widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G) \to 0 \ . \tag{2.3}$$

Again we can define the Whitehead group versions by dividing out trivial units $\{\pm G\}$, and get a double coboundary

$$\delta^2\colon H^1(\widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G^-)) \to H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-)) \ .$$

There is a natural map $H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\,\mathrm{Wh}(\mathbf{Z}H)) \to H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$.

Theorem B. *Let $V_1 = t^{a_1} + \cdots + t^{a_k}$ and $V_2 = t^{b_1} + \cdots + t^{b_k}$ be free G–representations. There exists a topological similarity $V_1 \oplus \mathbf{R}_- \oplus \mathbf{R}_+ \sim_t V_2 \oplus \mathbf{R}_- \oplus \mathbf{R}_+$ if and only if*

(i) $\prod a_i \equiv \prod b_i \pmod{4q}$,
(ii) $\mathrm{Res}_H V_1 \cong \mathrm{Res}_H V_2$, *and*
(iii) *the element $\{\Delta(V_1)/\Delta(V_2)\}$ is in the image of the double coboundary*

$$\delta^2 \colon H^1(\widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G^-)) \to H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-)) \ .$$

This result can be applied to 6-dimensional similarities.

Corollary 2.4. *Let $G = C(4q)$, with q odd, and suppose that the fields $\mathbf{Q}(\zeta_d)$ have odd class number for all $d \mid 4q$. Then G has no 6-dimensional non-linear similarities.*

Remark 2.5. This corrects [7,Thm.1(i)], and invalidates the computations of $R_{\mathrm{Top}}(G)$ given in [7,Thm.2]. □

Our final computation of the bounded transfer is suitable for determining stable non–linear similarities inductively. To state the conditions, we must again generalize the indeterminacy for the Reidemeister torsion invariant to include bounded K–groups (see Section 6). Since $K_i(\mathbf{Z}H \to \mathbf{Z}G) = K_i(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$, we can consider the analogous double coboundary

$$\delta^2 \colon H^1(\widetilde{K}_0(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))) \to H^1(\mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})))$$

and note that there is a map $\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}) \to \mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ induced by the inclusion on the control spaces, so our invariant $\{\Delta(V_1)/\Delta(V_2)\}$ maps forward into this new domain.

Theorem C. *Let $V_1 = t^{a_1} + \cdots + t^{a_k}$ and $V_2 = t^{b_1} + \cdots + t^{b_k}$ be free G–representations. Let W be a complex G–representation with no $\mathbf{R}_+$ summands. Then there exists a topological similarity $V_1 \oplus W \oplus \mathbf{R}_- \oplus \mathbf{R}_+ \sim_t V_2 \oplus W \oplus \mathbf{R}_- \oplus \mathbf{R}_+$ if and only if*

(i) *$S(V_1)$ is s–normally cobordant to $S(V_2)$,*
(ii) $\mathrm{Res}_H(V_1 \oplus W) \oplus \mathbf{R}_+ \sim_t \mathrm{Res}_H(V_2 \oplus W) \oplus \mathbf{R}_+$, *and*
(iii) *the element $\{\Delta(V_1)/\Delta(V_2)\}$ is in the image of the double coboundary*

$$\delta^2 \colon H^1(\widetilde{K}_0(\mathcal{C}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z}))) \to H^1(\mathrm{Wh}(\mathcal{C}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z}))) \ ,$$

where $0 \subseteq W_{max} \subseteq W$ is a 2–dimensional complex subrepresentation with maximal isotropy group among the isotropy groups of W with 2-power index.

Remark 2.6. Note that $W_{max} = 0$ in condition (iii) if W has no isotropy subgroups of 2-power index. Theorem C suffices to handle stable topological similarities, but leaves out cases where W has an odd number of $\mathbf{R}_-$ summands (see Section 16). Simpler conditions can be given when $G = C(2^r)$ (see Section 19).

The double coboundary in (iii) can also be expressed in more "classical" terms by using the short exact sequence

$$(2.7) \quad 0 \to \mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to \mathrm{Wh}(\mathcal{C}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z})) \to K_1(\mathcal{C}^{>\mathbf{R}_-}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z})) \to 0$$

derived in Corollary 8.9. We have $K_1(\mathcal{C}^{>\mathbf{R}_-}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z})) = K_{-1}(\mathbf{Z}K)$, where K is the isotropy group of W_{max}, and $\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) = \mathrm{Wh}(\mathbf{Z}H\to\mathbf{Z}G)$. The indeterminacy in Theorem C is then generated by the double coboundary

$$\delta^2\colon H^1(\widetilde{K}_0(\mathbf{Z}H\to\mathbf{Z}G^-)) \to H^1(\mathrm{Wh}(\mathbf{Z}H\to\mathbf{Z}G^-))$$

used in Theorem B and the coboundary

$$\delta\colon H^0(K_{-1}(\mathbf{Z}K)) \to H^1(\mathrm{Wh}(\mathbf{Z}H\to\mathbf{Z}G^-))$$

from the Tate cohomology sequence of (2.7). $\square$

Finally, we will apply these results to $R_{\mathrm{Top}}(G)$. In Section 4, we will define a filtration

$$(2.8) \qquad R_t(G) \subseteq R_n(G) \subseteq R_h(G) \subseteq R(G)$$

on the real representation ring $R(G)$, inducing a filtration on

$$R_{\mathrm{Top}}(G) = R(G)/R_t(G)\ .$$

Here $R_h(G)$ consists of those virtual elements with no homotopy obstruction to similarity, and $R_n(G)$ the virtual elements with no normal invariant obstruction to similarity (see Section 4 for more precise definitions). Note that $R(G)$ has the nice basis $\{t^i, \delta, \epsilon \mid 1 \leqq i \leqq 2q-1\}$, where $\delta = [\mathbf{R}_-]$ and $\epsilon = [\mathbf{R}_+]$.

Let $R^{free}(G) = \{t^a \mid (a, 4q) = 1\} \subset R(G)$ be the subgroup generated by the free representations. To complete the definition, we let $R^{free}(C(2)) = \{\mathbf{R}_-\}$ and $R^{free}(e) = \{\mathbf{R}_+\}$. Then

$$R(G) = \bigoplus_{K\subseteq G} R^{free}(G/K)$$

and this direct sum splitting can be intersected with the filtration above to define $R_h^{free}(G)$, $R_n^{free}(G)$ and $R_t^{free}(G)$. In addition, we can divide out $R_t^{free}(G)$ and obtain subgroups $R^{free}_{h,\mathrm{Top}}(G)$ and $R^{free}_{n,\mathrm{Top}}(G)$ of $R^{free}_{\mathrm{Top}}(G) = R^{free}(G)/R_t^{free}(G)$. By induction on the order of G, we see that it suffices to study the summand $R^{free}_{\mathrm{Top}}(G)$.

Let $\widetilde{R}^{free}(G) = \ker(\mathrm{Res}\colon R^{free}(G) \to R^{free}(G_{odd}))$, and then project into $R_{\mathrm{Top}}(G)$ to define

$$\widetilde{R}^{free}_{\mathrm{Top}}(G) = \widetilde{R}^{free}(G)/R_t^{free}(G)\ .$$

In Section 5 we prove that $\widetilde{R}^{free}_{\mathrm{Top}}(G)$ is precisely the torsion subgroup of $R^{free}_{\mathrm{Top}}(G)$, and the structure of this subgroup is determined in Section 20. Here is a sample computation.

Theorem D. *Let $G = C(4q)$, with q odd, and suppose that the fields $\mathbf{Q}(\zeta_d)$ have odd class number for all $d \mid 4q$. Then $\widetilde{R}^{free}_{\mathrm{Top}}(G) = \mathbf{Z}/4$ generated by $(t - t^{1+2q})$.*

For any cyclic group G, both $R^{free}(G)/R_h^{free}(G)$ and $R_h^{free}(G)/R_n^{free}(G)$ are torsion groups which can be explicitly determined by congruences in the weights [3] (but see [40] for a correction to [6,Condition A]). In Section 20 we show that the remaining subquotient $\widetilde{R}^{free}_{n,\mathrm{Top}}(G) = \widetilde{R}^{free}_n(G)/R_t^{free}(G)$ always has exponent two.

3. A criterion for non–linear similarity

Our approach to the non-linear similarity problem is through bounded surgery theory (see [18], [19]). First an elementary observation about topological equivalences for cyclic groups.

Lemma 3.1. *If $V_1 \oplus W \sim_t V_2 \oplus W$, where V_1, V_2 are free G–representations, and W and W' have no free summands, then there is a G-homeomorphism $h\colon V_1 \oplus W \to V_2 \oplus W$ such that*

$$h| \bigcup_{1 \neq H \leq G} W^H$$

is the identity.

Proof. Let h be the homeomorphism. We will successively change h, stratum by stratum. For every subgroup K of G, consider the homeomorphism of K-fixed sets

$$f^K\colon W^K \to W'^K.$$

This is a homeomorphism of G/K, hence of G–representations. As G–representations we can split

$$V_2 \oplus W' = U \oplus W'^K \sim_t U \oplus W^K = V_2 \oplus W''$$

where the similarity uses the product of the identity and $(f^K)^{-1}$. Rename W'' and repeat this successively for all subgroups. We end up with $W = W'$ and a homeomorphism as claimed. $\square$

One consequence is

Lemma 3.2. *If $V_1 \oplus W \sim_t V_2 \oplus W$ then there exists a G-homotopy equivalence $S(V_2) \to S(V_1)$.*

Proof. If we 1-point compactify h we obtain a G-homeomorphism

$$h^+\colon S(V_1 \oplus W \oplus \mathbf{R}) \to S(V_2 \oplus W \oplus \mathbf{R}).$$

After an isotopy, the image of the free G-sphere $S(V_1)$ may be assumed to lie in the complement $S(V_2 \oplus W \oplus \mathbf{R}) - S(W \oplus \mathbf{R})$ of $S(W \oplus \mathbf{R})$ which is G-homotopy equivalent to $S(V_2)$. $\square$

Any homotopy equivalence $f\colon S(V_2)/G \to S(V_1)/G$ defines an element $[f]$ in the structure set $\mathcal{S}^h(S(V_1)/G)$. We may assume that $\dim V_i \geqq 4$. This element must be non-trivial: otherwise $S(V_2)/G$ would be topologically h-cobordant to $S(V_1)/G$, and Stallings infinite repetition of h-cobordisms trick would produce a homeomorphism $V_1 \to V_2$ contradicting [1], since V_1 and V_2 are free representations. More precisely, we use Wall's extension of the Atiyah–Singer equivariant index formula to the topological locally linear case [38]. If $\dim V_i = 4$, we can cross with $\mathbf{CP}^2$ to avoid low–dimensional difficulties. Crossing with W and parameterising by projection on W defines a map from the classical surgery sequence to the bounded surgery exact sequence

$$
(3.3)\qquad
\begin{array}{ccccc}
L^h_n(\mathbf{Z}G) & \longrightarrow & \mathcal{S}^h(S(V_1)/G) & \longrightarrow & [S(V_1)/G, \mathrm{F}/\mathrm{Top}] \\
\downarrow & & \downarrow & & \downarrow \\
L^h_{n+k}(\mathcal{C}_{W,G}(\mathbf{Z})) & \longrightarrow & \mathcal{S}^h_b\begin{pmatrix} S(V_1)\times W/G \\ \downarrow \\ W/G \end{pmatrix} & \longrightarrow & [S(V_1)\times_G W, \mathrm{F}/\mathrm{Top}]
\end{array}
$$

The L–groups in the upper row are the ordinary surgery obstruction groups for oriented manifolds and surgery up to homotopy equivalence. In the lower row, we have bounded L–groups (see Section 6) corresponding to an orthogonal action $\rho_W\colon G \to O(W)$, with orientation character given by $\det(\rho_W)$. Our main criterion for non–linear similarities is:

Theorem 3.4. *Let V_1 and V_2 be free G–representations with $\dim V_i \geqq 2$, and suppose that $f\colon S(V_2) \to S(V_1)$ is a G–homotopy equivalence. Then, there is a topological equivalence $V_1 \oplus W \sim_t V_2 \oplus W$ if and only if the element $[f] \in \mathcal{S}^h(S(V_1)/G)$ is in the kernel of the bounded transfer map*

$$
trf_W\colon \mathcal{S}^h(S(V_1)/G) \to \mathcal{S}^h_b\begin{pmatrix} S(V_1)\times_G W \\ \downarrow \\ W/G \end{pmatrix}.
$$

Proof. For necessity, we refer the reader to [19] where this is proved using a version of equivariant engulfing. For sufficiency, we notice that crossing with $\mathbf{R}$ gives an isomorphism of the bounded surgery exact sequences parameterized by W to simple bounded surgery exact sequence parameterized by $W \times \mathbf{R}$. By the bounded s-cobordism theorem, this means that the vanishing of the bounded transfer implies that

$$
\begin{array}{ccc}
S(V_2)\times W\times\mathbf{R} & \xrightarrow{f\times 1} & S(V_1)\times W\times\mathbf{R} \\
 & & \downarrow \\
 & & W\times\mathbf{R}
\end{array}
$$

is within a bounded distance of an equivariant homeomorphism h, where distances are measured in $W \times \mathbf{R}$. We can obviously complete $f \times 1$ to the map

$$
f * 1\colon S(V_2) * S(W\times\mathbf{R}) \to S(V_1) * S(W\times\mathbf{R})
$$

and since bounded in $W \times \mathbf{R}$ means small near the subset

$$
S(W\times\mathbf{R}) \subset S(V_i) * S(W\times\mathbf{R}) = S(V_i\oplus W\oplus\mathbf{R}),
$$

we can complete h by the identity to get a homeomorphism

$$
S(V_2\oplus W\oplus\mathbf{R}) \to S(V_1\oplus W\oplus\mathbf{R})
$$

and taking a point out we have a homeomorphisms $V_2 \times W \to V_2 \times W$ $\square$

By comparing the ordinary and bounded surgery exact sequences (3.3), and noting that the bounded transfer induces the identity on the normal invariant term, we see that a necessary condition for the existence of any stable similarity $f\colon V_2 \approx_t V_1$ is that $f\colon S(V_2) \to S(V_1)$ has s–normal invariant zero. Assuming this, under the natural map

$$L_n^h(\mathbf{Z}G) \to \mathcal{S}^h(S(V_1)/G),$$

where $n = \dim V_1$, the element $[f]$ is the image of $\sigma(f) \in L_n^h(\mathbf{Z}G)$, obtained as the surgery obstruction (relative to the boundary) of a normal cobordism from f to the identity. The element $\sigma(f)$ is well-defined in $\tilde{L}_n^h(\mathbf{Z}G) = \operatorname{Coker}(L_n^h(\mathbf{Z}) \to L_n^h(\mathbf{Z}G))$.

Theorem 3.5. *Let V_1 and V_2 be free G–representations with $\dim V_i \geqq 2$, and suppose that $f\colon S(V_2) \to S(V_1)$ is a G–homotopy equivalence which is G–normally cobordant to the identity. Then, there is a topological equivalence $V_1 \oplus W \sim_t V_2 \oplus W$ if and only if $trf_W(\sigma(f)) = 0$, where $trf_W\colon L_n^h(\mathbf{Z}G) \to L_n^h(\mathcal{C}_{W,G}(\mathbf{Z}))$ is the bounded transfer.*

The rest of the paper is about the computation of these bounded transfer homomorphisms in L–theory. We will need the following result (proved for K_0 in [19,6.3]).

Theorem 3.6. *Let W be a G–representation with $W^G = 0$. For all $i \in \mathbf{Z}$, the image of the bounded transfer $trf_W\colon K_i(\mathbf{Z}G) \to K_i(\mathcal{C}_{W,G}(\mathbf{Z}))$ is contained in the image of the cone point inclusion $c_*\colon K_i(\mathbf{Z}G) = K_i(\mathcal{C}_{pt,G}(\mathbf{Z})) \to K_i(\mathcal{C}_{W,G}(\mathbf{Z}))$.*

Proof. Let G be a finite group and V a representation. Crossing with V defines a transfer map in K–theory $K_i(RG) \to K_i(\mathcal{C}_{V,G}(R))$ for $i \leq 1$, where R is any ring with unit. This map is actually defined for all i, and turns out to be equal to the map $K_i(\mathcal{C}_{0,G}(R)) \to K_i(\mathcal{C}_{V,G}(R))$ induced by the inclusion $0 \subset V$. To see this we need to choose models for K–theory. For RG we choose the category of finitely generated free RG modules, but we think of it as a category with cofibrations and weak equivalences with weak equivalences isomorphisms and cofibrations split inclusions. For $\mathcal{C}_{V,G}(R)$ we use the category of finite length chain complexes, with weak equivalences chain homotopy equivalences and cofibrations sequences that are split short exact at each level. The K–theory of this category is the same as the K-theory of $\mathcal{C}_{V,G}(R)$. For an argument working in this generality see [8]. Tensoring with the chain complex of (V, G) induces a map of categories with cofibrations and weak equivalences hence a map on K–theory. It is elementary to see this agrees with the geometric definition in low dimensions, since the identification of the K–theory of chain complexes of an additive category with the K–theory of the additive category is an Euler characteristic (see e. g. [8]). By abuse of notation we denote the category of finite chain complexes in $\mathcal{C}_{V,G}(R)$ simply by $\mathcal{C}_{V,G}(R)$. We need to study various related categories. First there is $\mathcal{C}_{V,G}^{iso}(R)$ where we have replaced the weak equivalences by isomorphisms. Obviously the transfer map, tensoring with the chains of (V, G) factors through this category. Also the transfer factors through the category $\mathcal{D}_{V,G}^{iso}(R)$ with the same objects, and isomorphisms as weak equivalences but the control condition is 0-control instead of bounded control. The category

$\mathcal{D}^{iso}_{V,G}(R)$ is the product of the full subcategories on objects with support at 0 and the full subcategory on objects with support on $V-0$, $\mathcal{D}^{iso}_{0,G}(R)\times\mathcal{D}^{iso}_{V-0,G}(R)$, and the transfer factors through chain complexes concentrated in degree 0 in $\mathcal{D}^{iso}_{0,G}(R)$ crossed with chain complexes in the other factor. But the subcategory of chain complexes concentrated in degree zero of $\mathcal{D}^{iso}_{0,G}(R)$ is precisely the same as $\mathcal{C}_{0,G}(R)$ and the map to $\mathcal{C}_{V,G}(R)$ is induced by inclusion. So to finish the proof we have to show that the other factor $\mathcal{D}^{iso}_{V-0,G}(R)$ maps to zero. For this we construct an intermediate category $\mathcal{E}^{iso}_{V-0,G}(R)$ with the same objects, but where the morphisms are bounded radially and 0-controlled otherwise (i. e. a nontrivial map between objects at different points is only allowed if the points are on the same radial line, and there is a bound on the distance independent of the points). This category has trivial K-theory since we can make a radial Eilenberg swindle toward infinity. Since the other factor $\mathcal{D}^{iso}_{V-0,G}(R)$ maps through this category, we find that the transfer maps through the corner inclusion as claimed.

Remark 3.7. It is an easy consequence of the filtering arguments based on [18,Thm. 3.12] that the bounded L–groups are finitely generated abelian groups with 2–primary torsion subgroups. We will therefore localize all the L–groups by tensoring with $\mathbf{Z}_{(2)}$ (without changing the notation), and this loses no information for computing bounded transfers. One concrete advantage of working with the 2–local L–groups is that we can use the idempotent decomposition [15,§6] and the direct sum splitting $L^h_n(\mathcal{C}_{W,G}(\mathbf{Z})) = \oplus_{d|q} L^h_n(\mathcal{C}_{W,G}(\mathbf{Z}))(d)$. Since the "top component" $L^h_n(\mathcal{C}_{W,G}(\mathbf{Z}))(q)$ is just the kernel of the restriction map to all odd index subgroups of G, the use of components is well-adapted to inductive calculations. □

A first application of these techniques was given in [19,5.1].

Theorem 3.8. *For any G–representation W, let $W = W_1\oplus W_2$ where W_1 is the direct sum of the irreducible summands of W with isotropy subgroups of 2–power index. If $W^G=0$, then*

(i) *the inclusion $L^h_n(\mathcal{C}_{W_1,G}(\mathbf{Z}))(q)\to L^h_n(\mathcal{C}_{W,G}(\mathbf{Z}))(q)$ is an isomorphism on the top component,*

(ii) *the bounded transfer*
$$trf_{W_2}\colon L^h_n(\mathcal{C}_{W_1,G}(\mathbf{Z}))(q)\to L^h_n(\mathcal{C}_{W,G}(\mathbf{Z}))(q)$$
is an injection on the top component, and

(iii) $\ker(trf_W)=\ker(trf_{W_1})\subseteq L^h_n(\mathbf{Z}G)(q)$.

Proof. In [19] we localized at an odd prime $p\nmid |G|$ in order to use the Burnside idempotents for all cyclic subgroups of G. The same proof works for the L–groups localized at 2, to show that trf_{W_2} is injective on the top component. □

Lemma 3.9. *For any choice of normal cobordism between f and the identity, the surgery obstruction $\sigma(f)$ is a nonzero element of infinite order in $\tilde{L}^h_n(\mathbf{Z}G)$.*

Proof. See [19,4.5] □

The following result (combined with Theorem 3.5) shows that there are no non-linear similarities between semi-free G–representations, since $L^h_{n+1}(\mathcal{C}_{\mathbf{R},G}(\mathbf{Z})) = L^p_n(\mathbf{Z}G)$ and the natural map $L^h_n(\mathbf{Z}G)\to L^p_n(\mathbf{Z}G)$ may be identified with the bounded transfer $trf_{\mathbf{R}}\colon L^h_n(\mathbf{Z}G)\to L^h_{n+1}(\mathcal{C}_{\mathbf{R},G}(\mathbf{Z}))$ [31,§15].

Corollary 3.10. *Under the natural map $L_n^h(\mathbf{Z}G) \to L_n^p(\mathbf{Z}G)$, the image of $\sigma(f)$ is nonzero.*

Proof. The kernel of the map $L_n^h(\mathbf{Z}G) \to L_n^p(\mathbf{Z}G)$ is the image of $H^n(\widetilde{K}_0(\mathbf{Z}G))$ which is a torsion group. □

4. A splitting of $R_{\mathrm{Top}}(G)$

In this section we point out an elementary splitting of $R_{\mathrm{Top}}(G)$, and some useful filtrations.

For G any finite group, we denote by $R(G)$ the *real* representation ring of G. Elements in $R(G)$ can be given as formal differences $(V_1 - V_2)$ of G–representations, and $(V_1 - V_2) \sim 0$ if and only if there exists a representation W such that $V_1 \oplus W \cong V_2 \oplus W$.

Notice that for K any normal subgroup of G, taking fixed sets gives a retraction of the inflation map

$$\mathrm{Inf}_K \colon R(G/K) \to R(G)$$

defined by pulling back a G/K representation using the composition with the quotient map $G \to G/K$. More explicitly,

$$\mathrm{Fix}_K \colon R(G) \to R(G/K)$$

is defined by $\mathrm{Fix}_K(V_1 - V_2) = (V_1^K - V_2^K)$ for each normal subgroup $K \triangleleft G$. Then $\mathrm{Fix}_K \circ \mathrm{Inf}_K = id \colon R(G/K) \to R(G/K)$.

Definition 4.1. A G–representation V is *free* if $V^K = \{0\}$ for all non–trivial normal subgroups $1 \neq K \triangleleft G$.

This is the same as the usual definition (no non–identity element of G fixes any non–zero vector) for cyclic groups. We let

$$R^{free}(G) = \{(V_1 - V_2) \mid V_1, V_2 \text{ free representations}\} \tag{4.2}$$

denote the subgroup of $R(G)$ such that there is a representative of the stable equivalence class $(V_1 - V_2)$ so that V_1, V_2 are free.

Proposition 4.3. *There is a direct sum splitting*

$$R(G) = \bigoplus_{K \triangleleft G} R^{free}(G/K)$$

indexed by the normal subgroups K in G.

Inside $R(G)$ have the subgroup of stably topologically similar representations

$$R_t(G) = \{(V_1 - V_2) \mid V_1 \oplus W \sim_t V_2 \oplus W \text{ for some } W\} \tag{4.4}$$

and the quotient group is $R_{\mathrm{Top}}(G)$ by definition. Since this subgroup is preserved by inflation and taking fixed sets, we obtain

Corollary 4.5. *There is a direct sum decomposition*

$$R_{\mathrm{Top}}(G) = \bigoplus_{K \lhd G} {R^{free}}_{\mathrm{Top}}(G/K)$$

where the summand are the quotients $R^{free}(G/K)/R_t^{free}(G/K)$.

We will also need a certain filtration of $R(G)$. First we define

$$R_h(G) = \{(V_1 - V_2) \mid S(V_1 \oplus W) \simeq_G S(V_2 \oplus W) \text{ for some } W\} \tag{4.6}$$

where $\simeq_G$ denotes G–homotopy equivalence. This is a subgroup of $R(G)$, in fact a sub-Mackey functor since it has induction and restriction for subgroups of G. If there exists a G–homotopy equivalence $f\colon S(V_1 \oplus W) \to S(V_2 \oplus W)$ such that

$$S(V_1 \oplus W \oplus U) = S(V_1 \oplus W) * S(U) \xrightarrow{f*1} S(V_1 \oplus W) * S(U) = S(V_1 \oplus W \oplus U)$$

which is G–normally cobordant to the identity for all G–representations U, then we write $S(V_1 \oplus W) \doteq_G S(V_2 \oplus W)$. Define

$$R_n(G) = \{(V_1 - V_2) \mid S(V_1 \oplus W) \doteq_G S(V_2 \oplus W) \text{ for some } W\} \tag{4.7}$$

and note that $R_n(G)$ is a subgroup of $R(G)$. Indeed, if $(V_1 - V_2)$ and $(V_1' - V_2')$ are in $R_n(G)$, there exist G-homotopy equivalences $f\colon S(V_1 \oplus W) \to S(V_2 \oplus W)$ and $f'\colon S(V_1' \oplus W') \to S(V_2' \oplus W')$ with $f*1$ and $f'*1$ normally cobordant to the identity under any stabilization. But $f*f' \simeq_G (1*f') \circ (f*1)$, so we just glue together the normal cobordisms for $f*1$ (after stabilizing by $U = V_1' \oplus W'$) and for $1*f'$ (after stabilizing by $U = V_2 \oplus W$) along the common boundary $id\colon S(V_2 \oplus W \oplus V_1' \oplus W')$.

Since $R_t(G) \subseteq R_n(G)$, we have defined a filtration

$$R_t(G) \subseteq R_n(G) \subseteq R_h(G) \subseteq R(G) \tag{4.8}$$

of $R(G)$, natural with respect to induction and restriction of representations. Now we intersect this filtration with the splitting described above. Define a subgroup of $R(G)$ and subgroups for each decoration $(?) = h, n, t$ by

$$R^{free}_{(?)}(G) = \{(V_1 - V_2) \mid V_1, V_2 \text{ free representations}\} \tag{4.9}$$

or, in other words, we assume that there is a representative of the stable equivalence class $(V_1 - V_2)$ so that V_1, V_2 are free.

Lemma 4.10. *The subfiltration $R^{free}_{(?)}(G)$ is well–defined.*

Proof. Any V has a unique direct sum splitting $V = V_1 \oplus W$ with V_1 free of maximal dimension ($V_1 = 0$ is possible). Any G-homotopy equivalence $\phi\colon S(V_1 \oplus W \to S(V_2 \oplus W)$ is stably G-homotopic to one of the form $f*1$, where $f\colon S(V_2) \to S(V_1)$ is a G-homotopy equivalence on the free parts [11]. Since homotopy equivalences of free G–spheres are detected by the first k–invariant (which is multiplicative under joins [38,14E]), we note that $S(V_1 \oplus U) \simeq_G S(V_2 \oplus U)$ if and only if $S(V_1) \simeq_G S(V_2)$ when V_1, V_2 and U are free representations. It follows that $R_h^{free}G$ is just the subgroup of virtual elements $(V_1 - V_2)$ with V_1, V_2 free, and $S(V_1) \simeq_G S(V_2)$. Similarly, $(V_1 - V_2) \in R_n^{free}G$ if and only if there exists a free G-homotopy equivalence $f\colon S(V_2) \to S(V_1)$ such that $f*1\colon S(V_1 \oplus U) \to S(V_2 \oplus U)$ is freely G-normally cobordant to the identity for all free representations U. $\square$

Using the groups $R^{free}_{(?)}(G/K)$ we can refine our previous splitting.

Lemma 4.11. *For $R(G)$ and $(?) = h, n, t$ there is a direct sum splitting*

$$R_{(?)}(G) = \bigoplus_{K \triangleleft G} R^{free}_{(?)}(G/K)$$

so that the filtration (4.8) is the direct sum of the filtrations on the free parts.

Proof. Just checking through the definitions. □

We conclude this list of definitions by pointing out an even finer direct sum splitting of $R(G)$ for G cyclic, after localizing at 2, using Mackey functor structure. The components are indexed by the family $\mathcal{O}(G)$ of subgroups $C \subset G$ of odd order [15,§6], [25,11b]. Let $q = q(G)$ denote the odd part of the order of G, then for $(?) = h, n, t$ we have

$$R_{(?)}(G) \otimes \mathbf{Z}_{(2)} = \bigoplus_{d|q} R_{(?)}(G)(d) \tag{4.12}$$

We call this splitting the *Burnside* splitting (or the 2–local Burnside splitting if we need to distinguish it from the corresponding splittings at other primes). It is induced by the action of idempotents e_C in the 2–local Burnside ring.

Proposition 4.13. *Let G be a cyclic group. The Burnside splitting of $R(G)$ induces a splitting*

$$R^{free}_{(?)}(G/K) \otimes \mathbf{Z}_{(2)} = \bigoplus_{d|q(G/K)} R^{free}_{(?)}(G/K)(d)$$

for $(?) = h, n, t$ compatible with the splitting of (4.8) into free summands.

Proof. We must compare the operation of the idempotents e_C with the splitting maps $\mathrm{Inf}_K: R(G/K) \to R(G)$ and $\mathrm{Fix}_K: R(G) \to R(G/K)$. The cyclic subgroups of G/K are just the cyclic subgroups of G containing K. Therefore the maps e_C commute with Inf_K and Fix_K whenever C is a cyclic subgroup of G containing K. □

5. A rational computation

In this section we use Theorem 3.5 and the splitting of the last section to describe the torsion subgroup of $R^{free}_{\mathrm{Top}}(G)$. We also give a new proof of Cappell and Shaneson's result computing $R_{\mathrm{Top}}(G) \otimes \mathbf{Q}$ for all finite groups G.

First we consider cyclic groups. Let $G = C(2^r q)$ be a cyclic group, where $q \geqq 1$ is odd. Notice that the Odd Order Theorem [23], [24] gives $R(G) = R_{\mathrm{Top}}(G)$ if $r \leqq 1$. Recall that

$$\widetilde{R}^{free}_{\mathrm{Top}}(G) = \widetilde{R}^{free}(G)/R^{free}_t(G)$$

where $\widetilde{R}^{free}(G) = \ker\{\mathrm{Res}: R^{free}(G) \to R^{free}(G_{odd})\}$. Notice that $\widetilde{R}^{free}(C(2q)) = \widetilde{R}^{free}(C(q)) = 0$, for $q \geqq 1$ odd.

Theorem 5.1. *For G cyclic, the kernel and cokernel of*

$$\mathrm{Res}\colon R^{free}_{\mathrm{Top}}(G) \to R^{free}_{\mathrm{Top}}(G_{odd}) = R^{free}(G_{odd})$$

are 2–primary torsion groups.

Corollary 5.2. *Let $G = C(2^r q)$, q odd, be a finite cyclic group.*

(i) *The torsion subgroup of $R^{free}_{\mathrm{Top}}(G)$ is $\widetilde{R}^{free}_{\mathrm{Top}}(G)$.*
(ii) *The rank of $R^{free}_{\mathrm{Top}}(G)$ is $\varphi(q)/2$ (resp. 1) if $q > 1$ (resp. $q = 1$).*
(iii) *We have the formula*

$$rank(R_{\mathrm{Top}}(G)) = \begin{cases} (r+1)\{\sum_{d|q}(\varphi(d)/2+1)\}, & for\ q > 1 \\ (r+1) & for\ q = 1. \end{cases}$$

where $\varphi(d)$ is the Euler function.

Proof. The first part follows directly from Theorem 5.1. For parts (ii) and (iii) count the free representations of G_{odd}. □

Now let F be the composite of all subfields of $\mathbf{R}$ of the form $\mathbf{Q}(\zeta + \zeta^{-1})$ where $\zeta \in \mathbf{C}$ is an odd root of unity.

Lemma 5.3. *For G cyclic, the composition of the natural map $R_F(G) \to R(G)$ and $\mathrm{Res}\colon R(G) \to R(G_{odd})$ induces a p–local isomorphism $R^{free}_F(G) \to R^{free}(G_{odd})$, for any odd prime p.*

Proof. According to a result of Brauer [35,Thm.24] $R_F(G_{odd}) = R(G_{odd})$, and any representation of G can be realized over the field $\mathbf{Q}(\zeta_{|G|})$. In addition, the restriction map

$$\mathrm{Res}\colon R_F(G) \to R_F(G_{odd})$$

is a p–local surjection (since $\mathrm{Res}_{G_{odd}} \circ \mathrm{Ind}_{G_{odd}}$ is just multiplication by 2^r). But the rank of $R_F(G)$ given in [35,12.4] equals the rank of $R(G_{odd})$, so we are done. □

Corollary 5.4. *For G cyclic and any odd prime p, the natural map $R_F(G) \to R_{\mathrm{Top}}(G)$ induces a p–local isomorphism.*

Proof. This follows from the Lemma 5.3 and Theorem 5.1. □

Proof of Theorem 5.1. Since a F–representation is free if and only if it is the sum of Galois conjugates of free G–representations, we can decompose $R_F(G)$ as in Section 4, and conclude that $R^{free}_F(G_{odd}) = R^{free}(G_{odd})$. It remains to show that $\widetilde{R}^{free}_{\mathrm{Top}}(G)$ is a torsion group with 2–primary exponent. For this we use the filtration of §4, and claim that when intersected with $\widetilde{R}^{free}_{\mathrm{Top}}(G)$ each successive quotient is 2–primary torsion.

For $\widetilde{R}^{free}(G)/\widetilde{R}^{free}_h(G)$ this is easy since the k–invariant gives a homomorphism (via joins of free G–spheres) to $(\mathbf{Z}/2^r)^\times$ and this is a 2–group. The next quotient is also 2–primary torsion, by results of [40]: a sufficiently large 2–power join of a G–homotopy equivalence between two free G–spheres, which are linearly equivalent over G_{odd}, becomes s–normally cobordant to the identity. The point is that the normal invariant is detected by a finite number of 2–power congruences conditions among the Hirzebruch L–classes of the tangent bundles of the lens spaces, and this can be satisfied after sufficiently many joins.

Finally, the last quotient $\widetilde{R}_n^{free}(G)/\widetilde{R}_t^{free}(G)$ is shown to be 2–primary torsion in the next proposition. $\square$

Proposition 5.5. *Let $G = C(2^r q)$, q odd, and assume*

$$\sigma \in \ker(\mathrm{Res}\colon L^h_{2k}(\mathbf{Z}G) \to L^h_{2k}(\mathbf{Z}G_{odd}).$$

Then there exists a complex representation W with $W^G = 0$ such that $trf_W(2^r\sigma) = 0$.

Proof. We will take $W = \mathbf{R}_- \oplus \mathbf{R}_- \oplus W_0$, where W_0 is the sum of all the irreducible 2–dimensional representations of G with isotropy of 2–power index. Note that the $\mathbf{R}_-$-transfer is just the compact I_- transfer of one-sided codimension 1 surgery followed by adding rays to infinity, so whenever the I_-transfer is 0, the $\mathbf{R}_-$-transfer will have to be 0. This will be discussed in more detail in Section 7.

Step 1: If G has odd order, there is nothing to prove. Otherwise, let $H \subset G$ be of index 2. If $\mathrm{Res}_H(\sigma) = 0$, then $trf_{\mathbf{R}_-}(\sigma) \in \mathrm{Im}(L^h_1(\mathbf{Z}G^-) \to L^h_1(\mathbf{Z}H \to \mathbf{Z}G^-)$. But $L^h_1(\mathbf{Z}G^-)$ has exponent 2 [20], so $trf_{\mathbf{R}_-}(2\sigma) = 0$. Then take $W = \mathbf{R}_- \oplus \mathbf{R}_-$. Note that this case applies to $G = C(2q)$, so we can always get started.

Step 2: We may assume that $r \geqq 2$. If $\mathrm{Res}_H(\sigma) \neq 0$ note that

$$\mathrm{Res}_H(2\sigma - \mathrm{Ind}_H \mathrm{Res}_H(\sigma)) = 2\,\mathrm{Res}_H(\sigma) - 2\,\mathrm{Res}_H(\sigma) = 0.$$

By induction $\mathrm{Res}_H W_0$ works for $\mathrm{Res}_H(\sigma)$: say

$$trf_{\mathrm{Res}_H W_0}(2^{r-1}\,\mathrm{Res}_H(\sigma)) = 0$$

and $W_0^H = 0$. Let $\dim W_0 = m$ and consider the commutative diagram

$$\begin{tikzcd}
L_{2k}(\mathbf{Z}G) \arrow[r, "\mathrm{Res}"] \arrow[d, "trf_{W_0}"'] & L_{2k}(\mathbf{Z}H) \arrow[r, "\mathrm{Ind}"] \arrow[d, "trf_{\mathrm{Res}_H W_0}"'] & L_{2k}(\mathbf{Z}G) \arrow[d, "trf_{W_0}"'] \\
L_{2k+m}(\mathcal{C}_{W_0,G}(\mathbf{Z})) \arrow[r, "\mathrm{Res}"] & L_{2k+m}(\mathcal{C}_{\mathrm{Res}\,W_0,H}(\mathbf{Z})) \arrow[r, "\mathrm{Ind}"] & L_{2k+m}(\mathcal{C}_{W_0,G}(\mathbf{Z})).
\end{tikzcd}$$

From this we get

$$2^{r-1} \cdot trf_{W_0}(\mathrm{Ind}_H \mathrm{Res}_H(\sigma)) = 0\ .$$

The first step implies that $trf_{\mathbf{R}^2_-}(2\sigma_1) = 0$, where $\sigma_1 = 2\sigma - \mathrm{Ind}_H \mathrm{Res}_H(\sigma)$. Let $W = \mathbf{R}_- \oplus \mathbf{R}_- \oplus W_0$, so that we have W complex and $W^G = 0$. Note that $trf_W = trf_{\mathbf{R}^2_-} \circ trf_{W_0} = trf_{W_0} \circ trf_{\mathbf{R}^2_-}$. But

$$2^r \cdot trf_W(\sigma) = 2^{r-1}\, trf_W(2\sigma - \mathrm{Ind}_H \mathrm{Res}_H(\sigma)) + 2^{r-1}\, trf_W(\mathrm{Ind}_H \mathrm{Res}_H(\sigma))$$

and both terms vanish (because $r \geqq 2$ and by the property of W_0 respectively). $\square$

A similar argument to that in Step 2 above gives:

Proposition 5.6. *If* $\mathrm{Res}_H(\mathit{trf}_W(x)) = 0$ *for* $x \in L_0^h(\mathbf{Z}G)$, *then* $4 \cdot \mathit{trf}_{W\times\mathbf{R}_-}(x) = 0$.

Proof. Since

$$4\cdot \mathit{trf}_{W\times\mathbf{R}_-}(\sigma) = 2\,\mathit{trf}_W\,\mathit{trf}_{\mathbf{R}_-}(2\sigma - \mathrm{Ind}_H\,\mathrm{Res}_H(\sigma)) + 2\,\mathit{trf}_{\mathbf{R}_-}\,\mathit{trf}_W(\mathrm{Ind}_H\,\mathrm{Res}_H(\sigma))$$

we conclude as above that both terms vanish. □

In [4], Cappell and Shaneson obtained the following result by a different argument.

Theorem 5.7. *Let F be the composite of all subfields of $\mathbf{C}$ of the form $\mathbf{Q}(\zeta+\zeta^{-1})$ where $\zeta \in \mathbf{C}$ is an odd root of unity. Then the natural map $R_F(G) \to R(G)$ induces an isomorphism $R_F(G)\otimes\mathbf{Z}_{(p)} \cong R_{\mathrm{Top}}(G)\otimes\mathbf{Z}_{(p)}$ for any odd prime p.*

Proof. The result holds for cyclic groups G by Corollary 5.4, and we apply induction theory to handle general finite groups.

First we observe that the Mackey functors $R(G)\otimes\mathbf{Z}_{(p)}$ and $R_F(G)\otimes\mathbf{Z}_{(p)}$ are generated by induction from p–elementary subgroups (see [35,Thm.27] and note that $\Gamma_F = \{\pm 1\}$). Therefore, by [25,11.2] they are p–elementary computable in the sense of Dress induction theory [12]. We may therefore assume that G is p–elementary.

Now if G is p–elementary, it is a product of a p–group and a cyclic group prime to p. Any irreducible complex representation of G is then induced from a linear representation on a subgroup [35,Thm.16].

It follows that $R_{\mathrm{Top}}(G)$ is generated by generalized induction (i.e. inflation followed by induction) from cyclic subquotients. Consider the following commutative diagram

$$\begin{tikzcd}
R_F(G) \arrow[r] \arrow[d, shift left, "\mathrm{Res}"] & R(G) \arrow[r] \arrow[d, shift left] & R_{\mathrm{Top}}(G) \arrow[d, shift left] \\
\bigoplus_C R_F(C) \arrow[r] \arrow[u, shift left, "\mathrm{Ind}"] & \bigoplus_C R(C) \arrow[r] \arrow[u, shift left] & \bigoplus_C R_{\mathrm{Top}}(C) \arrow[u, shift left, two heads]
\end{tikzcd}$$

It follows from Corollary 5.4 that the top composite map $R_F(G)\otimes\mathbf{Z}_{(p)} \to R_{\mathrm{Top}}(G))\otimes\mathbf{Z}_{(p)}$ is surjective.

The sum of the (ordinary) restriction maps to cyclic subgroups induces a rational injection on $R_F(G)$ (see [37,2.5,2.10]). Since $R_F(G)$ is torsion–free, it follows again from Corollary 5.4 that the map $R_F(G)\otimes\mathbf{Z}_{(p)} \to R_{\mathrm{Top}}(G))\otimes\mathbf{Z}_{(p)}$ is injective. □

6. Excision in bounded surgery theory

A small additive category with involution $\mathcal{A}$ is a small additive category together with a contravariant endofunctor $*$ such that $*^2 = 1_{\mathcal{A}}$. Ranicki defines algebraic L–theory $L_*^h(\mathcal{A})$ for such categories and corresponding spectra $\mathbb{L}^h(\mathcal{A})$ with $L_*^h(\mathcal{A}) = \pi_*(\mathbb{L}^h(\mathcal{A}))$ [29]. The obstruction groups for bounded surgery are obtained this way for appropriately chosen additive categories. We shall also need a simple version of such groups. For this, the additive category must come equipped with a system of stable isomorphisms and a subgroup $s \subset K_1(\mathcal{A})$, such that any composition resulting in an automorphism defines an element in s. The point here is that whenever two objects are stably isomorphic, there is a canonically chosen stable isomorphism, canonical up to automorphisms defining elements of a. In this situation Ranicki refines the definition of $\mathbb{L}^h(\mathcal{A})$ to give the simple L–theory spectrum $\mathbb{L}^s(\mathcal{A})$, by requiring appropriate isomorphisms to give elements of $K_1(\mathcal{A})$ belonging to the subgroup s. More generally, We also get the $\mathbb{L}^{\overline{s}}(\mathcal{A})$-spectra for any involution invariant subgroup a with $s \subset a \subset K_1(\mathcal{A})$, coinciding with $\mathbb{L}^h(\mathcal{A})$ when $a = K_1(\mathcal{A})$.

Example 6.1. Let $\mathcal{A}$ be the category of free $\mathbf{Z}G$-modules with a G-invariant $\mathbf{Z}$-basis, and $\mathbf{Z}G$-module morphisms. Two objects are stably isomorphic if and only if they have the same rank. The preferred isomorphisms are chosen to be the ones sending a $\mathbf{Z}$-basis to a $\mathbf{Z}$-basis, so automorphisms define elements of $\{\pm G\} \subset K_1(\mathbf{Z}G)$. In this situation one obtains Wall's L^s-groups. □

The theory of projective L–groups fits into the scheme as follows: one defines $\mathbb{L}^p(\mathcal{A}) = \mathbb{L}^h(\mathcal{A}^\wedge)$, where $\mathcal{A}^\wedge$ is the idempotent completion of $\mathcal{A}$. The objects of $\mathcal{A}^\wedge$ are pairs (A, p) with A an object of $\mathcal{A}$ and $p^2 = p$. The morphisms $\phi: (A,p) \to (B,q)$ are the $\mathcal{A}$-morphisms $\phi: A \to B$ with $q\phi p = \phi$. Again it is possible to "partially" complete $\mathcal{A}$. If $K_0(\mathcal{A}) \subset k \subset K_0(\mathcal{A}^\wedge)$ is an involution invariant subgroup , we define $\mathcal{A}^{\wedge k}$ to be the full subcategory of $\mathcal{A}^\wedge$ with objects defining elements of $k \subset K_0(\mathcal{A}^\wedge)$. This way we may define $\mathbb{L}^k(\mathcal{A}) = \mathbb{L}(\mathcal{A}^{\wedge k})$. Similarly to the above, for $k = K_0(\mathcal{A}) \subset K_0(\mathcal{A}^\wedge)$, $L^k_*(\mathcal{A})$ is naturally isomorphic to $L^h_*(\mathcal{A})$. The quotient $\widetilde{K}_0(\mathcal{A}) = K_0(\mathcal{A}^\wedge)/K_0(\mathcal{A})$ is called the reduced projective class group of $\mathcal{A}$.

Example 6.2. If $\mathcal{A}$ is the category of free $\mathbf{Z}G$-modules then $\mathcal{A}^\wedge$ is isomorphic to the category of projective $\mathbf{Z}G$-modules and $L^p_*(\mathcal{A})$ are Novikov's original L^p-groups. □

Suppose M is a metric space with the finite group G acting by isometries, R a ring with involution. In [18] we defined an additive category $\mathcal{C}_{M,G}(R)$ with involution as follows:

Definition 6.3. An object A is a free left $R(G)$-module together with a map $f: A \longrightarrow F(M)$, where $F(M)$ is the set of finite subsets of M, satisfying

(1) f is G-equivariant.
(2) $A_x = \{a \in A \mid f(a) \subseteq \{x\}\}$ is a finitely generated free sub R-module
(3) As an R-module $A = \bigoplus_{x\in M} A_x$
(4) $f(a+b) \subseteq f(a) \cup f(b)$
(5) For each ball $B \subset M$, $\{x \in B \mid A_x \neq 0\}$ is finite.

A morphism $\phi: A \longrightarrow B$ is a morphism of RG-modules, satisfying the following condition: there exists k so that the components $\phi^m_n: A_m \longrightarrow B_n$ (which are R-module morphisms) are zero when $d(m,n) > k$. $\mathcal{C}_{M,G}(R)$ is an additive category in an obvious way.

Given an object A, an R-module homomorphism $\phi: A \longrightarrow R$ is said to be locally finite if the set of $x \in M$ for which $\phi(A_x) \neq 0$ is finite. We define $A^* = \mathrm{Hom}^{lf}_R(A, R)$, the set of locally finite R-homomorphisms. We want to make $*$ a functor from $\mathcal{C}_{M,G}(R)$ to itself to make $\mathcal{C}_{M,G}(R)$ a category with involution. We define $f^*: A^* \longrightarrow FM$ by $f^*(\phi) = \{x \mid \phi(A_x) \neq 0\}$ which is finite by assumption. A^* has an obvious right action of G turning it into a right RG module given by $\phi g(a) = \phi(ga)$, and f^* is equivariant with respect to the right action on M given by $xg = g^{-1}x$. To make $*$ an endofunctor of $\mathcal{C}_{M,G}(R)$ we need to replace the right action by a left action. We do this by the standard way in surgery theory by letting g act on the left by letting g^{-1} act on the right. In the unoriented case, given a homomorphism $w\colon G \longrightarrow \{\pm 1\}$, we let g act on the left of A^* by $w(g) \cdot g^{-1}$ on the right.

Example 6.4. Let $\rho_W\colon G \to O(W)$ be an orthogonal action of G on a finite dimensional real vector space W. We take $M = W$ with the action through ρ_W, and orientation character $\det(\rho_W)$. This will be called the *standard orientation* on $\mathcal{C}_{W,G}(\mathbf{Z})$. $\square$

Remark 6.5. We will need to find a system of stable isomorphisms for the category $\mathcal{C}_{M,G}(R)$ to be able to do simple L–theory. To do this we choose a point x in each G-orbit, and an RG_x-basis for A_x, where G_x is the isotropy subgroup of x. We then extend that by equivariance to an R-basis of the module. Having a basis allows defining an isomorphism in the usual fashion. In each case we need to describe the indeterminacy in the choices coming from the choice of R-basis and points in the orbit. For our particular choices of M it will be easy to determine the subgroup s, so we will not formulate a general statement. $\square$

We will study the L–theory of the categories $\mathcal{C}_{M,G}(R)$ using excision. Let N be a G-invariant metric subspace of M. Denoting $\mathcal{C}_{M,G}(R)$ by $\mathcal{U}$, let $\mathcal{A}$ be the full subcategory on modules A so that $A_x = 0$ except for x in some bounded neighborhood of N. The category $\mathcal{A}$ is isomorphic to $\mathcal{C}_{N,G}(R)$ in an obvious fashion. The quotient category $\mathcal{U}/\mathcal{A}$, which we shall denote by $\mathcal{C}^{>N}_{M,G}(R)$, has the same objects as $\mathcal{U}$ but two morphisms are identified if the difference factors through $\mathcal{A}$, or in other words, if they differ in a bounded neighborhood of N. This is a typical example of an additive category $\mathcal{U}$ which is $\mathcal{A}$-filtered in the sense of Karoubi. We recall the definition.

Definition 6.6. Let $\mathcal{A}$ be a full subcategory of an additive category $\mathcal{U}$. Denote objects of $\mathcal{A}$ by the letters A through F and objects of $\mathcal{U}$ by the letters U through W. We say that $\mathcal{U}$ is $\mathcal{A}$-filtered, if every object U has a family of decompositions $\{U = E_\alpha \oplus U_\alpha\}$, so that

(1) For each U, the decomposition form a filtered poset under the partial order that $E_\alpha \oplus U_\alpha \leq E_\beta \oplus U_\beta$, whenever $U_\beta \subseteq U_\alpha$ and $E_\alpha \subseteq E_\beta$.
(2) Every map $A \to U$, factors $A \to E_\alpha \to E_\alpha \oplus U_\alpha = U$ for some α.
(3) Every map $U \to A$ factors $U = E_\alpha \oplus U_\alpha \to E_\alpha \to A$ for some α.
(4) For each U, V the filtration on $U \oplus V$ is equivalent to the sum of filtrations $\{U = E_\alpha \oplus U_\alpha\}$ and $\{V = F_\beta \oplus V_\beta\}$ i.e. to $U \oplus V = (E_\alpha \oplus F_\beta) \oplus (U_\alpha \oplus V_\beta)$

The main excision results were proved in [26], [8], [9], [30]. We give a slight generalization of the L–theory results. Let $\mathbb{K}$ denote the Quillen K–theory spectrum, and $\mathbb{K}^{-\infty}$ its non-connective delooping (with the K_{-i}-groups as homotopy groups).

Theorem 6.7. *Let $\mathcal{U}$ be an $\mathcal{A}$-filtered additive category with involution. Consider the map $i\colon K_0(\mathcal{A}^\wedge) \to K_0(\mathcal{U}^\wedge)$ induced by inclusion, and let $k = i^{-1}(K_0(\mathcal{U}))$. There* *are fibrations of spectra*

$$\mathbb{K}(\mathcal{A}^{\wedge k}) \to \mathbb{K}(\mathcal{U}) \to \mathbb{K}(\mathcal{U}/\mathcal{A})$$

and

$$\mathbb{K}^{-\infty}(\mathcal{A}) \to \mathbb{K}^{-\infty}(\mathcal{U}) \to \mathbb{K}^{-\infty}(\mathcal{U}/\mathcal{A})$$

If $\mathcal{U}$ and $\mathcal{A}$ admit compatible involutions there is a fibration of spectra

$$\mathbb{L}^k(\mathcal{A}) \to \mathbb{L}^h(\mathcal{U}) \to \mathbb{L}^h(\mathcal{U}/\mathcal{A}) \ .$$

More generally, if

(i) *$a \subset K_i(\mathcal{A})$,$b \subset K_i(\mathcal{U}^\wedge)$, and $c \subset K_i((\mathcal{U}/\mathcal{A})^\wedge)$, for $i \leqq 1$,*
(ii) *$a = i^{-1}(b)$ and $b \to c$ is onto,*
(iii) *a, b, and c contain $K_0(\mathcal{A})$, $K_0(\mathcal{U})$ and $K_0(\mathcal{U}/C A)$ respectively, if $i = 0$, and*
(iv) *a, b and c contain the indeterminacy subgroup given by the system of stable isomorphisms in the case $i = 1$,*

then we have a fibration of spectra

$$\mathbb{L}^a(\mathcal{A}) \to \mathbb{L}^b(\mathcal{U}) \to \mathbb{L}^c(\mathcal{U}/\mathcal{A})$$

Proof. The K–theory statements are implicitly contained in [26]. A simpler, more modern proof and explicit statements are given in [8]. The first L–theory statement was proved in [9], and the other L–theory statements follow by the following argument: we have an exact sequence

$$K_0(\mathcal{A}^{\wedge k}) \to K_0(\mathcal{U}) \to K_0(\mathcal{U}/\mathcal{A}) \to 0,$$

where the map from $K_0(\mathcal{U}) \to K_0(\mathcal{U}/\mathcal{A})$ is onto because the categories have the same objects. Letting I denote the image of $K_1(\mathcal{U}/\mathcal{A})$ in $K_0(\mathcal{A}^{\wedge k})$, we consider the diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, equal] & K_0(\mathcal{A}^{\wedge k}) \arrow[r] \arrow[d] & K_0(\mathcal{U}) \arrow[r] \arrow[d] & K_0(\mathcal{U}/\mathcal{A}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[d, equal] & a \arrow[r] \arrow[d, equal] & b' \arrow[r] \arrow[d] & K_0(\mathcal{U}/\mathcal{A}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[d, equal] & a \arrow[r] \arrow[d] & b \arrow[r] \arrow[d] & c \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I \arrow[r] & K_0(\mathcal{A}^\wedge) \arrow[r] & K_0(\mathcal{U}^\wedge) \arrow[r] & K_0((\mathcal{U}/\mathcal{A})^\wedge) &
\end{tikzcd}$$

The vertical arrows are either equalities or inclusions. We define b' simultaneously as the pullback of $a/I \to b \to c$ and as the pushout of $0 \to K_0(\mathcal{A}^{\wedge k})/I \to K_0(\mathcal{U}) \to K_0(\mathcal{U}/\mathcal{A}) \to 0$. We have

$$a/K_0(\mathcal{A}^{\wedge k}) \cong b'/K_0(\mathcal{U}),$$

so using the Ranicki–Rothenberg fibrations of spectra [30]

$$\begin{tikzcd}
\mathbb{L}^k(\mathcal{A}) \arrow[r] \arrow[d] & \mathbb{L}^a(\mathcal{A}) \arrow[r] \arrow[d] & \mathbb{H}(a/K_0(\mathcal{A}^{\wedge k})) \arrow[d] \\
\mathbb{L}^h(\mathcal{U}) \arrow[r] & \mathbb{L}^{b'}(\mathcal{U}) \arrow[r] & \mathbb{H}(b'/K_0(\mathcal{U}))
\end{tikzcd}$$

we get a fibration

$$\mathbb{L}^a(\mathcal{A}) \to \mathbb{L}^{b'}(\mathcal{U}) \to \mathbb{L}^h(\mathcal{U}/\mathcal{A}).$$

We now repeat this argument using the isomorphisms $b/b' \cong c/K_0(\mathcal{U}/\mathcal{A})$ to obtain the desired fibration of spectra. Since L^h-groups may be understood as simple L-groups with all of K_1 as allowed torsions, the above bootstrapping argument extends to fibrations of the L-spectra stated, using the isomorphism

$$K_1(\mathcal{U}/\mathcal{A})/\ker(\partial) \cong \text{image}(\partial)$$

where ∂ is the boundary map $\delta\colon K_1(\mathcal{U}/\mathcal{A}) \to K_0(\mathcal{A}^\wedge)$. $\square$

7. Bounded $\mathbf{R}_-$ transfers

Let G denote a finite group of even order, with a subgroup $H < G$ of index 2. We first describe the connection between the bounded $\mathbf{R}_-$ transfer and the compact line bundle transfer of [38,12C] by means of the following diagram:

$$\begin{tikzcd}
L_n^h(\mathbf{Z}G, w) \arrow[r, "trf_{I_-}"] \arrow[d, "trf_{\mathbf{R}_-}"'] & L_{n+1}^h(\mathbf{Z}H \to \mathbf{Z}G, w\phi) \arrow[d, "j_*"] \\
L_{n+1}^h(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w\phi) & L_{n+1}^{k,h}(\mathbf{Z}H \to \mathbf{Z}G, w\phi) \arrow[l, "r_*"]
\end{tikzcd}$$

where $w\colon G \to \{\pm 1\}$ is the orientation character for G and $\phi\colon G \to \{\pm 1\}$ has kernel H. On $\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})$ we use the standard orientation defined in Example 6.4. Note that the orientation induced on $\mathcal{C}_{pt}(\mathbf{Z}G)$ via the cone point inclusion $c\colon \mathcal{C}_{pt}(\mathbf{Z}G) \to \mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})$ is *non–trivial*. The homomorphism

$$r_*\colon L_{n+1}^{k,h}(\mathbf{Z}H \to \mathbf{Z}G, w\phi) \to L_{n+1}^h(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w\phi)$$

is obtained by adding a ray $[1, \infty)$ to each point of the boundary double cover in domain and range of a surgery problem. Here $\mathbf{k}$ in the decoration means that we are allowing projective $\mathbf{Z}H$-modules that become free when induced up to $\mathbf{Z}G$

Theorem 7.1. *The map $r_*\colon L_{n+1}^{k,h}(\mathbf{Z}H \to \mathbf{Z}G, w\phi) \to L_{n+1}^h(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w\phi)$ is an isomorphism, and under this identification, the bounded $\mathbf{R}_-$ transfer corresponds to the line bundle transfer, followed by the relaxation of projectivity map j_* given by $\mathbf{k}$.*

Proof. Let $\mathcal{A}$ be the full subcategory of $\mathcal{U} = \mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})$ with objects that are only nontrivial in a bounded neighborhood of 0. Then $\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})$ is $\mathcal{A}$-filtered. The category $\mathcal{A}$ is equivalent to the category of free $\mathbf{Z}G$-modules (with the non-orientable involution). The quotient category $\mathcal{U}/\mathcal{A}$ is equivalent to $\mathcal{C}^{>0}_{[0,\infty),H}(\mathbf{Z})$, which has the same L-theory as $\mathcal{C}_{\mathbf{R}}(\mathbf{Z}H)$, so we get a fibration of spectra

$$\mathbb{L}^k(\mathbf{Z}H) \to \mathbb{L}^h(\mathbf{Z}G) \to \mathbb{L}^h(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \ .$$

This shows that

$$\mathbb{L}^h(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \simeq \mathbb{L}^{k,h}(\mathbf{Z}H \to \mathbf{Z}G) \quad \square$$

The line bundle transfer can be studied by the long exact sequence

$$(7.2) \quad \begin{aligned} \cdots \to LN_n(\mathbf{Z}H\to\mathbf{Z}G, w\phi) \to L^h_n(\mathbf{Z}G, w) \to L^h_{n+1}(\mathbf{Z}H\to\mathbf{Z}G, w\phi) \\ \to LN_{n-1}(\mathbf{Z}H\to\mathbf{Z}G, w\phi) \to L^h_{n-1}(\mathbf{Z}G, w) \to \dots \end{aligned}$$

given in [38,11.6]. The obstruction groups $LN_n(\mathbf{Z}H\to\mathbf{Z}G, w\phi)$ for codimension 1 surgery have an algebraic description

$$(7.3) \qquad LN_n(\mathbf{Z}H\to\mathbf{Z}G, w\phi) \cong L^h_n(\mathbf{Z}H, \alpha, u)$$

given by [38,12.9]. The groups on the right–hand side are the algebraic L–groups of the "twisted" anti–structure defined by choosing some element $t \in G - H$ and then setting $\alpha(x) = w(x)t^{-1}x^{-1}t$ for all $x \in H$, and $u = w(t)t^{-2}$. Another choice of $t \in G - H$ gives a scale equivalent anti–structure on $\mathbf{Z}H$. The same formulas also give a "twisted" anti–structure $(\mathbf{Z}G, \alpha, u)$ on $\mathbf{Z}G$, but since the conjugation by t is now an inner automorphism of G this is scale equivalent to the standard structure $(\mathbf{Z}G, w)$. We can therefore define the twisted induction map

$$\tilde{i}_*: L^h_n(\mathbf{Z}H, \alpha, u) \to L^h_n(\mathbf{Z}G, w)$$

and the twisted restriction map

$$\tilde{\gamma}_*: L^h_n(\mathbf{Z}G, w) \to L^h_n(\mathbf{Z}H, \alpha, u)$$

as the composites of the ordinary induction or restriction maps (induced by the inclusion $(\mathbf{Z}H, w) \to (\mathbf{Z}G, w)$) with the scale isomorphism.

The twisted anti–structure on $\mathbf{Z}H$ is an example of a "geometric anti–structure" [21,p.110]:

$$\alpha(g) = w(g)\theta(g^{-1}), \qquad\qquad u = \pm b$$

where $\theta: G \to G$ is a group automorphism with $\theta^2(g) = bgb^{-1}$, $w \circ \theta = w$, $w(b) = 1$ and $\theta(b) = b$.

Example 7.4. For G cyclic, the orientation character restricted to H is trivial, $\theta(g) = tgt^{-1} = g$ and $u = w(t)t^2$. Choosing $t \in G$ a generator we get $b = t^2$, which is a generator for H. $\square$

There is an identification [14,Thm.3], [22,50-53] of the exact sequence (7.2) for the line bundle transfer, extending the scaling isomorphism $L^h_n(\mathbf{Z}G, w) \cong L^h_n(\mathbf{Z}G, \alpha, u)$ and (7.3), with the long exact sequence of the "twisted" inclusion

$$\dots L^h_n(\mathbf{Z}H, \alpha, u) \xrightarrow{\tilde{i}_*} L^h_n(\mathbf{Z}G, \alpha, u) \to L^h_n(\mathbf{Z}H\to\mathbf{Z}G, \alpha, u) \to L^h_{n-1}(\mathbf{Z}H, \alpha, u) \to \dots$$

These identifications can then be substituted into the following "twisting diagram" in order to compute the various maps (see [22,Appendix 2] for a complete tabulation in the case of finite 2–groups).

$$
\begin{tikzcd}[column sep=small]
LN_n(\mathbf{Z}H \to \mathbf{Z}G, w\phi) \arrow[dr] \arrow[rr, bend left, "\tilde{i}_*"] & & L_n(\mathbf{Z}G, w) \arrow[dr] \arrow[rr, bend left, "\gamma_*"] & & L_n(\mathbf{Z}H, w) \\
& L_{n+1}(\gamma_*) \arrow[ur] \arrow[dr] & & L_{n+1}(\mathbf{Z}H \to \mathbf{Z}G, w\phi) \arrow[ur] \arrow[dr] & \\
L_{n+1}(\mathbf{Z}H, w) \arrow[ur] \arrow[rr, bend right, "i_*"'] & & L_{n+1}(\mathbf{Z}G, w\phi) \arrow[ur] \arrow[rr, bend right, "\tilde{\gamma}_*"'] & & LN_{n-1}(\mathbf{Z}H \to \mathbf{Z}G, w\phi)
\end{tikzcd}
\tag{7.5}
$$

The existence of the diagram depends on the identifications $L_{n+1}(\gamma_*) \cong L_n(\tilde{\gamma}_*)$ and $L_{n+1}(i_*) \cong L_n(\tilde{i}_*)$ obtained geometrically in [14] and algebraically in [28].

In Section 16 we need to use bounded surgery groups with geometric anti–structure (see [18]). The new ingredient is a counterpart to the automorphism $\theta: H \to H$ at the metric space level. The necessary definitions for $(\mathcal{C}_{M,H}(R), \alpha, u)$, for M a metric space on which H acts by isometries and R a ring, were given in Section 6.

Let $\theta_H: H \longrightarrow H$ be a group automorphism so that the data (H, θ_H, w, b) gives a geometric anti–structure on RH. Let $\theta_M: M \longrightarrow M$ be an isometry with the properties $\theta_M(g \cdot m) = \theta_H(g) \cdot \theta_M(m)$, $\theta_M^2(m) = bm$, and $\theta_H^2(g) = bgb^{-1}$.

Given an object $A \in \mathcal{G}_{M,H}(R)$, we have the functor $*$ from $\mathcal{G}_{M,H}(R)$ to itself so that $\mathcal{G}_{M,H}(R)$ is a category with involution. We may then twist the involution $*$ by composing with the functor sending (A, f) to (A^θ, f^θ) where A^θ is the same R-module, but g acts on the left by multiplication by $\theta(g)$ and $f^\theta = \theta_M^{-1} \cdot f$. This defines the bounded anti–structure on $\mathcal{G}_{M,H}(R)$ and on the subcategory $\mathcal{C}_{M,H}(R)$ of free RH modules.

Example 7.6. Bounded geometric anti–structures arise geometrically as above. The bounded $\mathbf{R}_-$ transfer sits in the long exact sequence

$$
\begin{aligned}
LN_n(\mathcal{C}_{W \times \mathbf{R}_-, G}(\mathbf{Z}), w\phi) &\to L^h_n(\mathcal{C}_{W,G}(\mathbf{Z}), w) \to L^h_{n+1}(\mathcal{C}_{W \times \mathbf{R}_-, G}(\mathbf{Z}), w\phi) \\
&\to LN_{n-1}(\mathcal{C}_{W \times \mathbf{R}_-, G}(\mathbf{Z}), w\phi) \to L^h_{n-1}(\mathcal{C}_{W,G}(\mathbf{Z}), w) \to \dots
\end{aligned}
$$

where $w = \det(\rho_W)$ is the standard orientation (see Example 6.4). The bounded LN–group

$$LN_n(\mathcal{C}_{W \times \mathbf{R}_-, G}(\mathbf{Z}), w\phi) \cong L_n(\mathcal{C}_{W,H}(\mathbf{Z}), \alpha, u)$$

where $\theta_W(x) = t \cdot x$ and $\theta_H(h) = tht^{-1}$ for a fixed $t \in G - H$.

Conversely, given a bounded geometric antistructure $(\theta_H, \theta_M, b, w)$, we can define $G = \langle H, t \mid t^{-1}ht = \theta_H(h), t^2 = b \rangle$ and $t \cdot m = \theta_M(m)$. Then $\mathcal{C}_{M,G}(R)$ induces

$(\theta_H, \theta_M, b, w)$ as above, showing that all geometric antistructures arise by twisting and restricting to an index two subgroup. $\square$

The L-theory of these bounded geometric anti–structures also has a useful vanishing property which we now wish to formulate. We first give a basic construction.

Definition 7.7. If $\mathcal{A}$ is an additive category, then the *opposite category* $\mathcal{A}^{op}$ is the category with the same objects as $\mathcal{A}$ but $\hom_{\mathcal{A}^{op}}(A,B) = \hom_{\mathcal{A}}(B,A)$. The product category $\mathcal{A} \times \mathcal{A}^{op}$ is an additive category with involution given by $*\colon (A,B) = (B,A)$ on objects and $*\colon (\alpha,\beta) = (\beta,\alpha)$ on morphisms.

Clearly $K_i(\mathcal{A}^{op}) = K_i(\mathcal{A})$, so we can identify $K_i(\mathcal{A} \times \mathcal{A}^{op}) = K_i(\mathcal{A}) \times K_i(\mathcal{A})$.

Lemma 7.8. *Let $b \subseteq K_i(\mathcal{A})$ for some $i \leqq 1$, and $q = b \times b \subseteq K_i(\mathcal{A} \times \mathcal{A}^{op})$. Then $L^q_n(\mathcal{A} \times \mathcal{A}^{op}) = 0$ for all n.*

Proof. Let $\mathcal{P}(\mathcal{A})$ denote the category with the same objects as $\mathcal{A}$, but with morphisms given by $\mathcal{A}$–isomorphisms. Then it suffices to prove that the quadratic category $\mathcal{Q}(\mathcal{A} \times \mathcal{A}^{op}) \simeq \mathcal{P}(\mathcal{A})$ via the hyperbolic map (see [39,p.122]). This shows that $L^h_*(\mathcal{A} \times \mathcal{A}^{op}) = 0$ and other decorations follow trivially from the Ranicki Rothenberg exact sequences (note that the Tate cohomology $H^*(q) = 0$). The result for lower L–groups follows by replacing $\mathcal{A}$ by $\mathcal{C}_{\mathbf{R}}(\mathcal{A})$.

Suppose $(\nu_1, \nu_2)\colon (A,B) \to (B,A)$ is a non–singular quadratic form representing an element in $\mathcal{Q}(\mathcal{A} \times \mathcal{A}^{op})$. This means that the bilinearization $\nu_1 + \nu_2$ is an isomorphism, and we are allowed to change (ν_1, ν_2) by terms of the form $(\alpha,\beta) - (\beta,\alpha)$. We have

$$(\nu_1, \nu_2) + (\nu_2, 0) - (0, \nu_2) = (\nu_1 + \nu_2, 0)$$

and the right hand side is a hyperbolic form. $\square$

We encounter the $\mathcal{A} \times \mathcal{A}^{op}$ situation in the following setting:

Example 7.9. Let $M = M_1 \cup M_2$ be a metric space given as the union of two sub–metric spaces M_1 and M_2, where we denote $M_1 \cap M_2$ by N. Suppose that

(i) G acts by isometries on M, such that each $g \in G$ preserves or switches M_1 and M_2 in this decomposition,
(ii) $H = \{g \in G \mid g(M_1) = M_1\}$ is an index two subgroup of G,
(iii) for every $k > 0$ there exists an $l > 0$ such that, if $x \in M_1$ (resp. $x \in M_2$) with $d(x,N) > l$, then $d(x, M_2) > k$ (resp. $d(x, M_1) > k$).

The category $\mathcal{C}_{M,H}(R)$ has a bounded geometric antistructure (α, u) given by $(\theta_H, \theta_M, b, w)$ as in Example 7.6, with $\theta_M(m) = t \cdot m$ and $\theta_H(h) = tht^{-1}$ for a fixed $t \in G - H$. Next, observe that the category

$$\mathcal{C}^{>N}_{M,H}(R) = \mathcal{C}^{>N}_{M_1,H}(R) \times \mathcal{C}^{>N}_{M_2,H}(R)$$

because of our separation condition (ii). Moreover, the functor $T\colon \mathcal{C}^{>N}_{M_1,H}(R) \to \mathcal{C}^{>N}_{M_2,H}(R)^{op}$ defined by $T(A,f) = (A^*, \theta_M^{-1} \circ f^*)$ on objects and $T(\phi) = \phi^*$ on morphisms is an equivalence of categories. We are thus in the $\mathcal{A} \times \mathcal{A}^{op}$ situation described above and $L^h_*(\mathcal{C}^{>N}_{M,H}(R), \alpha, u) = 0$ by Lemma 7.8. $\square$

For any bounded geometric antistructure, notice that the action of θ_M on M takes H–orbits to H–orbits since $\theta_M(g \cdot m) = \theta_H(g) \cdot \theta_M(m)$. Let $M_{(H,\theta)}$ denote the subset consisting of H–orbits in M which are *fixed* by the θ–action. Then $M_{(H,\theta)} = \{m \in M \mid \theta_M(m) \in H \cdot m\}$. Note that $M_{(H,\theta)}$ is a H–invariant subspace of M.

Theorem 7.10. *Suppose that $(\mathcal{C}_{M,H}(R), \alpha, u)$ has a bounded geometric antistructure (α, u) given by $(\theta_H, \theta_M, b, w)$, such that:*

(i) *$M = O(K)$, where K is a finite H–CW complex and M has the cone of the given H–action on K,*
(ii) *θ_M is induced by a simplicial map on K,*
(iii) *$M_{(H,\theta)} \subseteq O(L) := N$ for some H–invariant subcomplex $L \subset K$, and*
(iv) *for some $i \leqq 1$, $I \subset K_i(\mathcal{C}^{>N}_{M,H}(R))$ is a subgroup with $H^*(I) = 0$.*

Then $L^I_n(\mathcal{C}^{>N}_{M,H}(R), \alpha, u) = 0$ for all n.

Corollary 7.11. *Let $G = C(2^r q)$, q odd, be a cyclic group and $H \subset G$ the subgroup of index 2. Let W be a G–representation, and $N = \bigcup_K \{W^K \mid [G : K] \text{ is odd}\}$. Then $L^I_n(\mathcal{C}^{>N}_{W,H}(\mathbf{Z}), \alpha, u)(q) = 0$ on the top component, where (α, u) is the antistructure given above.*

Proof of Theorem 7.10. We extend the given H–action on M to a simplicial action of $G = \langle H, t \mid t^{-1}ht = \theta_H(h), t^2 = b\rangle$ as described above. The proof is by induction on cells, so suppose that K is obtained from L by attaching exactly one G–equivariant k–cell $D^k \times G/G_0$. Since $M_{(H,\theta)} \subset O(L) = N$, it follows that $G_0 \subset H$ and we may write $G/G_0 = H/G_0 \sqcup tH/G_0$. Now we define $M_1 = O(L \cup (D^k \times H/G_0))$ and consider the category $\mathcal{A} = \mathcal{C}^{>O(L)}_{M_1,H}(R)$. By construction, we have

$$\mathcal{C}^{>N}_{M,H}(R) = \mathcal{A} \times \mathcal{A}^{op}$$

which has trivial L–theory by Lemma 7.8. Since the Tate cohomology of the K_1 decoration I vanishes, we get $L^I_n(\mathcal{C}^{>N}_{M,H}(R), \alpha, u) = 0$. $\square$

8. Calculations in bounded K-theory

We begin to compute the bounded transfers trf_W by considering the bounded K–theory analogue. In this section, $G = C(2^r q)$ is cyclic of order $2^r q$, with $r \geqq 2$ and $q \geqq 1$ odd. By Theorem 3.8 we can restrict our attention to those W where the isotropy subgroups have 2–power index. Let $G_i \subseteq G$ denote the subgroup of index $[G : G_i] = 2^i$ for $i = 0, 1, \dots, r$. As above, we reserve the notation $H < G$ for the subgroup of index 2.

Any real, orthogonal G–representation W can be decomposed uniquely into isotypical direct summands indexed by the subgroups $K \subseteq G$, where in each summand G operates with isotropy group K away from the origin. Since we assume that W has isotropy of 2–power index, we can write

$$W = W[0] \oplus W[1] \oplus \cdots \oplus W[r]$$

where $W[i]$ is isotypic with isotropy group G_i. Thus $W[0] = \mathbf{R}^k$ is a trivial G–representation, and $W[1]$ is a sum of $\mathbf{R}_-$ factors. We note that W is *complex* if $\dim W[1]$ is even. If W is complex, then $W_{max} \subseteq W$ denotes a 2-dimensional complex sub-representation with maximal proper isotropy subgroup. Then W_{max} is either irreducible or $W_{max} = \mathbf{R}_- \oplus \mathbf{R}_-$.

We study bounded K–theory by means of equivariant filtrations of the control space. The basic sequence is (see[18]):

$$\cdots \to K_{i+1}(\mathcal{C}^{>U}_{V,G}(\mathbf{Z})) \xrightarrow{\partial_{i+1}} K_i(\mathcal{C}_{U,G}(\mathbf{Z})) \to K_i(\mathcal{C}_{V,G}(\mathbf{Z})) \to K_i(\mathcal{C}^{>U}_{V,G}(\mathbf{Z})) \to \dots$$

valid for $U \subseteq V$ a closed G–invariant subspace. If W_1 is a complex representation with $\dim W_1 = 2$ and isotropy group $K \neq G$, let $U = \bigcup \ell_\alpha$ be the union of $[G:K]$ rays from the origin in W_1, which are freely permuted by G/K. Then $W_1 \setminus \bigcup \ell_\alpha$ is a disjoint union of open fundamental domains for the free G/K–action.

If $W = W_1 \oplus W_2$, we call $W_2 \subset \bigcup \ell_\alpha \times W_2 \subset W$ the *orbit type* filtration of W.

Lemma 8.1. *Let $W = W_1 \oplus W_2$, where W_1 is a complex 2–dimensional sub-representation of W with minimal isotropy subgroup $K \neq G$. Then*

$$K_{i+1}(\mathcal{C}^{>\bigcup \ell_\alpha \times W_2}_{W,G}(\mathbf{Z})) \cong K_{i-1-k}(\mathbf{Z}K)$$
$$K_i(\mathcal{C}^{>W_2}_{\bigcup \ell_\alpha \times W_2, G}(\mathbf{Z})) \cong K_{i-1-k}(\mathbf{Z}K),$$

where $k = \dim W_2$. The boundary map $\partial_{i+1} = 1 - t_*$ in the long exact sequence of the orbit type filtration for W.

Proof. The bounded category $\mathcal{C}^{>W_2}_{\bigcup \ell_\alpha \times W_2, G}(\mathbf{Z})$ of germs away from W_2 has effective fundamental group K, as defined in [18,3.13]. It therefore has the same K–theory as $\mathcal{C}_{\mathbf{R}^{k+1}}(\mathcal{C}_{pt}(\mathbf{Z}K))$. The other case is similar. □

Since $K_{-j}(\mathbf{Z}K) = 0$ for $j \geqq 2$ by [10], this Lemma gives vanishing results for bounded K–theory as well.

Lemma 8.2. *Suppose that W is complex, and $W^G = 0$. Then the inclusion map induces an isomorphism*

$$K_i(\mathcal{C}_{W_{max} \times \mathbf{R}_-, G}(\mathbf{Z})) \to K_i(\mathcal{C}_{W \times \mathbf{R}_-, G}(\mathbf{Z}))$$

for $i \leqq 1$.

Proof. This is an argument using the orbit type filtration. Let $W = W_{max} \oplus W_2$, and suppose that $W_2 \neq 0$ or equivalently $\dim W_2 \geqq 2$, since W is complex. Write $W = W' \oplus W''$ with $W' \subseteq W_2$, $\dim W' = 2$ and $\operatorname{Iso}(W') = K$ minimal. We choose W'' containing W_{max}, and by induction we assume the result holds for W''.

Then applying Lemma 8.1, we get the calculations

$$K_i(\mathcal{C}^{>W'' \times \mathbf{R}_-}_{\bigcup \ell_\alpha \times W'' \times \mathbf{R}_-, G}(\mathbf{Z})) = K_{i-2-|W''|}(\mathbf{Z}K) \tag{8.3}$$

and

$$K_i(\mathcal{C}^{>\bigcup \ell_\alpha \times W'' \times \mathbf{R}_-}_{W \times \mathbf{R}_-, G}(\mathbf{Z})) = K_{i-3-|W''|}(\mathbf{Z}K)\ . \tag{8.4}$$

Since $\dim W'' \geqq 2$, we get $K_i(\mathcal{C}^{>W'' \times \mathbf{R}_-}_{\bigcup \ell_\alpha \times W'' \times \mathbf{R}_-, G}(\mathbf{Z})) = 0$ for $i \leqq 2$ by [10], and $K_i(\mathcal{C}^{>\bigcup \ell_\alpha \times W'' \times \mathbf{R}_-}_{W \times \mathbf{R}_-, G}(\mathbf{Z})) = 0$ for $i \leqq 3$. From the exact sequence of the filtration, it follows that $K_i(\mathcal{C}^{>W'' \times \mathbf{R}_-}_{W \times \mathbf{R}_-, G}(\mathbf{Z})) = 0$ for $i \leqq 2$, and therefore

$$K_i(\mathcal{C}_{W'' \times \mathbf{R}_-, G}(\mathbf{Z})) \xrightarrow{\approx} K_i(\mathcal{C}_{W \times \mathbf{R}_-, G}(\mathbf{Z}))$$

for $i \leqq 1$. We are done, by induction. □

Corollary 8.5. *$K_i(\mathcal{C}^{>W_{max}\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))=0$ for $i\leqq 2$.*

Proof. We continue the notation from above, and look at part of the filtration sequence

$$K_i(\mathcal{C}^{>W_{max}\times\mathbf{R}_-}_{W''\times\mathbf{R}_-,G}(\mathbf{Z}))\to K_i(\mathcal{C}^{>W_{max}\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))\to K_i(\mathcal{C}^{>W_{max}\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))\ .$$

The first term is zero for $i\leqq 2$ by induction on dimension, and the third term is zero for $i\leqq 2$ as above. □

We can obtain a little sharper result with some additional work. First a useful observation:

Lemma 8.6. *Let $\mathcal{A}$ be an additive category (with involution). Then the map $\mathbf{R}\to\mathbf{R}$ sending x to $-x$ induces minus the identity on K-theory (and L-theory) of $\mathcal{C}_{\mathbf{R}}(\mathcal{A})$.*

Proof. The category $\mathcal{C}_{\mathbf{R}}(\mathcal{A})$ is filtered by the full subcategory whose objects have support in a bounded neighborhood of 0. This subcategory is equivalent to $\mathcal{A}$ and the quotient category my be identified with $\mathcal{C}^{>0}_{[0,\infty)}(\mathcal{A})\times\mathcal{C}^{>0}_{(-\infty,0]}(\mathcal{A})$ via the projection maps in an obvious way.

Consider the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d, equal] & \mathcal{C}_{\mathbf{R}}(\mathcal{A}) \arrow[r] \arrow[d] & \mathcal{C}^{>0}_{[0,\infty)}(\mathcal{A})\times\mathcal{C}^{>0}_{(-\infty,0]}(\mathcal{A}) \arrow[d] \\
\mathcal{A} \arrow[r] & \mathcal{C}_{[0,\infty)}(\mathcal{A}) \arrow[r] & \mathcal{C}^{>0}_{[0,\infty)}
\end{tikzcd}$$

where the vertical map is induced by $x\mapsto|x|$. In the lower horizontal row, K and L–theory of the middle term is trivial, so the boundary map will be an isomorphism. The lower row splits off the upper row in two different ways, one induced by including $[0,\infty)\subset\mathbf{R}$ and the other by sending $x\in[0,\infty)$ to $-x\in\mathbf{R}$. Under these two splittings we may identify K or L–theory of the quotient $\mathcal{C}^{>0}_{[0,\infty)}(\mathcal{A})\times\mathcal{C}^{>0}_{(-\infty,0]}(\mathcal{A})$ with $\mathcal{C}^{>0}_{[0,\infty)}\times\mathcal{C}^{>0}_{[0,\infty)}$ and under this identification, the flip map of $\mathbf{R}$ corresponds to interchanging the two factors. On K–theory (or L–theory) we conclude that the exact sequence is of the form

$$0\to A_*\to A_*\times A_*\xrightarrow{+}A_*\to 0\ .$$

The flip action on the last term is trivial, and on the middle term it interchanges the two factors, so the inclusion must send a to $(a,-a)$. Hence the flip action on the first term must be $a\mapsto -a$. □

Let t_* denote the induced action of a generator $t\in G$ on K–theory, and $\varepsilon\colon G\to\{\pm1\}$ the action of G on $\mathbf{R}_-$.

Proposition 8.7. *Let W be a complex 2-dimensional G–representation with $W^G=0$, and isotropy subgroup K. Then under the isomorphisms of (8.3) and (8.4) the complex*

$$K_{i+1}(\mathcal{C}^{>\bigcup\ell_\alpha\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))\xrightarrow{\partial_a}K_i(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup\ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z}))\xrightarrow{\partial_b}K_{i-1}(\mathcal{C}^{>0}_{\mathbf{R}_-,G}(\mathbf{Z}))$$

with $\partial_b \circ \partial_a = 0$ *is isomorphic to*

$$K_{i-2}(\mathbf{Z}K) \xrightarrow{\partial'_a} K_{i-2}(\mathbf{Z}K) \xrightarrow{\partial'_b} K_{i-2}(\mathbf{Z}H)$$

where $\partial'_a = 1 - \varepsilon(t)t_* = 2$ *and* $\partial'_b = \mathrm{Ind}_H \circ (1 - t_*) = 0$.

Proof. The orbit type filtration is based on a G–equivariant simplicial model for W, where G acts through the projection to G/K. The third term in the complex is $K_{i-1}(\mathcal{C}^{>0}_{\mathbf{R}_-,G}(\mathbf{Z})) = K_{i-2}(\mathbf{Z}H)$ and the identification of the boundary maps follows from the definition of the germ categories.

To compute ∂_a, we use the isomorphisms between the domain of ∂_a and $K_{i-2}(\mathbf{Z}K)$ obtained by noticing that every element is induced from an element of

$$K_{i+1}(\mathcal{C}^{>\partial C\times\mathbf{R}}_{C\times\mathbf{R},K}(\mathbf{Z})),$$

where C is the region between two adjacent half–lines of $\bigcup \ell_\alpha$. This follows since the regions in the complement of $\bigcup \ell_\alpha \times \mathbf{R}_-$ are disjoint and the boundedness condition ensures there is no interference. Similarly the isomorphism of the range of ∂_a with $K_{i-2}(\mathbf{Z})$ is obtained by noticing that every element is induced from $\mathcal{C}^{>\mathbf{R}}_{h\times\mathbf{R},K}$ where h is just one half–line in $\bigcup \ell_\alpha$, and we can think of $\partial C = h \cup th$. To compute the boundary we first take the standard boundary to $\mathcal{C}_{\partial C\times\mathbf{R},K}(\mathbf{Z})$ which is an isomorphism, and then map away from $0 \times \mathbf{R}$. It follows from the proof of Lemma 8.6 above that this map is of the form $a \mapsto (a, -a)$ in K or L–theory. In this picture, the support of one of the boundary components is along h and the other along th. We need to use the group action to associate both elements to the same ray. Since t flips the $\mathbf{R}$–factor, we get a change of sign before adding, so ∂_a sends a to $2a$.

To compute ∂_b, we start with an element in the source of ∂_b which as above is identified via induction with $K_i(\mathcal{C}^{>\mathbf{R}}_{h\times\mathbf{R},K,G}(\mathbf{Z}))$. The boundary first sends this isomorphically to $\mathcal{C}_{\mathbf{R},K}(\mathbf{Z})$, then by induction to $\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})$, and then via the natural map to the range of ∂_b, which is $\mathcal{C}^{>0}_{\mathbf{R}_-,G}$. We have a commutative square

$$\begin{array}{ccc}
K_1(\mathcal{C}^{>\mathbf{R}}_{h\times\mathbf{R},K}(\mathbf{Z})) & \xrightarrow{\mathrm{Ind}} & K_1(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z})) \\
\downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial_b} \\
K_0(\mathcal{C}^{>0}_{\mathbf{R},K}(\mathbf{Z})) & \xrightarrow{\mathrm{Ind}} & K_0(\mathcal{C}^{>0}_{\mathbf{R}_-,G}(\mathbf{Z}))
\end{array}$$

where $\mathcal{C}^{>0}_{\mathbf{R},K}(\mathbf{Z}) = \mathcal{C}_{\mathbf{R},K}(\mathbf{Z}) \times \mathcal{C}_{\mathbf{R},K}(\mathbf{Z})$. Under this identification, the natural map $K_0(\mathcal{C}_{\mathbf{R},K}(\mathbf{Z})) \to K_0(\mathcal{C}^{>0}_{\mathbf{R},K}(\mathbf{Z}))$ is just $a \mapsto (a, -a)$, and the induction map

$$\mathrm{Ind}\colon K_0(\mathcal{C}^{>0}_{\mathbf{R},K}(\mathbf{Z})) \to K_0(\mathcal{C}^{>0}_{\mathbf{R}_-,G}(\mathbf{Z}))$$

is given by $(a, b) \mapsto \mathrm{Ind}_H(a + t_* b)$. In this case the action of t on $\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})$ is the identity since any element is invariant under the action of G, hence under the action of t. It follows that $\mathrm{Ind}(a, -a) = \mathrm{Ind}_H(a - t_* a) = 0$ as required. $\square$

Lemma 8.8. *Let W be a complex 2-dimensional G–representation with proper isotropy group K. The composition*

$$K_{i+1}(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z})) \to K_{i+1}(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to K_i(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$$

is zero for $i \leqq 1$.

Proof. By Proposition 8.7 the given composite map factors through the cone point inclusion $K_i(\mathbf{Z}G) \to K_i(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$. For $i = 1$, the result follows from the fact that $K_2(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z}) = K_0(\mathbf{Z}K)$ is torsion, except for the factor $K_0(\mathbf{Z}) = \mathbf{Z}$, and $\mathrm{Wh}(\mathbf{Z}G)$ is torsion free. For $i = 0$, we use the surjection $K_0(\widehat{\mathbf{Q}}K) \to K_{-1}(\mathbf{Z}K)$, and compute with $\widehat{\mathbf{Q}}$ coefficients and $i = 1$. Note that the given composite is just the boundary map $\partial\colon K_{i+1}(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z})) \to K_i(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$.

We will list the steps, and leave the details to the reader. First, compute that $K_1(\mathcal{C}_{\mathbf{R}_-,G}(\widehat{\mathbf{Q}})) = K_1(\widehat{\mathbf{Q}}G)/K_1(\widehat{\mathbf{Q}}H)$ surjects onto $K_1(\mathcal{C}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\widehat{\mathbf{Q}}))$, by means of a braid containing the cone point inclusions into $\mathcal{C}_{\mathbf{R}_-,G}(\widehat{\mathbf{Q}})$ and $\mathcal{C}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\widehat{\mathbf{Q}})$. Second, prove that $K_1(\mathcal{C}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\widehat{\mathbf{Q}}))$ fits into a short exact sequence

$$0 \to K_1(\mathcal{C}_{\bigcup \ell_\alpha,H}(\widehat{\mathbf{Q}})) \to K_1(\mathcal{C}_{\bigcup \ell_\alpha,G}(\widehat{\mathbf{Q}})) \to K_1(\mathcal{C}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\widehat{\mathbf{Q}})) \to 0$$

by means of a braid containing the inclusion $\bigcup \ell_\alpha \subseteq \bigcup \ell_\alpha \times \mathbf{R}_-$. Finally, compute $K_1(\mathcal{C}_{\bigcup \ell_\alpha,G}(\widehat{\mathbf{Q}})) = K_1(\widehat{\mathbf{Q}}G)/K_1(\widehat{\mathbf{Q}}K)$, and $K_1(\mathcal{C}_{\bigcup \ell_\alpha,H}(\widehat{\mathbf{Q}})) = K_1(\widehat{\mathbf{Q}}H)/K_1(\widehat{\mathbf{Q}}K)$ by comparing the groups under the inclusion $H < G$. We conclude that

$$K_1(\mathcal{C}_{\mathbf{R}_-,G}(\widehat{\mathbf{Q}})) \cong K_1(\mathcal{C}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\widehat{\mathbf{Q}}))$$

under the inclusion map, and hence $\partial = 0$. □

Corollary 8.9. *Let W be a complex G–representation with $W^G = 0$. Then the inclusion induces an isomorphism $K_i(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to K_i(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ for $i \leqq 0$, and an injection for $i = 1$. If $K_{-1}(\mathbf{Z}K) = 0$ for the maximal proper isotropy groups K of W, then $K_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \cong K_1(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$.*

Proof. We may assume that $\dim W = 2$, and apply the filtering argument again. By (8.3) and (8.4) we get $K_i(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) = 0$ for $i \leqq 0$, and $K_1(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ is a quotient of $K_1(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z})) = K_{-1}(\mathbf{Z}K)$. The boundary map

$$\partial\colon K_1(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to K_0(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$$

composed with the natural map $K_0(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to K_0(\mathcal{C}^{>0}_{\mathbf{R}_-,G}(\mathbf{Z}))$ is just the map ∂_b of Proposition 8.7. Since $\partial_b = 0$ on K–theory, the image $\operatorname{Im}\partial$ factors through the inclusion $K_0(\mathbf{Z}H) \to K_0(\mathbf{Z}G)$ (because $K \subseteq H$), hence vanishes in $K_0(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ by Lemma 8.8.

Similarly, $K_2(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ is a quotient of $K_2(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z})) = K_0(\mathbf{Z}K)$, because the boundary map $K_2(\mathcal{C}^{>\bigcup \ell_\alpha\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) = K_{-1}(\mathbf{Z}K)$ to $K_1(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z})) = K_{-1}(\mathbf{Z}K)$ is multiplication by 2, and hence injective. Then we make the same argument, based on Lemma 8.8.

If we also assume $K_{-1}(\mathbf{Z}K) = 0$, then $K_1(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) = 0$ so we get the isomorphism $K_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \xrightarrow{\approx} K_1(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$. □

9. The double coboundary

The composite Δ of maps from the $L^h - L^p$ and $L^s - L^h$ Rothenberg sequences

$$\begin{tikzcd} H^i(\widetilde{K}_0(\mathbf{Z}G)) \arrow[r] \arrow[dr, "\Delta"] & L_i^h(\mathbf{Z}G) \arrow[d] \\ & H^i(\mathrm{Wh}(\mathbf{Z}G)) \end{tikzcd}$$

(see [20]) has an algebraic description by means of a "double coboundary" homomorphism

$$\delta^2 \colon H^i(\widetilde{K}_0(\mathbf{Z}G)) \to H^i(\mathrm{Wh}(\mathbf{Z}G))$$

In this section, we will give a brief description due to Ranicki [32] of this homomorphism (see also [27,§6.2] for related material on "interlocking" exact sequences in K and L–theory).

Let X be a space with a $\mathbf{Z}/2$ action $T\colon X \to X$, and define homomorphisms

$$\Delta\colon H^i(\pi_n(X)) \to H^i(\pi_{n+1}(X))$$

by sending $g\colon S^n \to X$ to

$$h \cup (-1)^i Th\colon S^{n+1} = D_+^{n+1} \cup_{S^n} D_-^{n+1} \to X$$

for any null-homotopy $h\colon D^{n+1} \to X$ of the map $g + (-1)^{i+1}Tg\colon S^n \to X$.

The maps Δ lead to a universal description of double coboundary maps, as follows. Let $f\colon X \to Y$ be a $\mathbf{Z}/2$–equivariant map of spaces with $\mathbf{Z}/2$ action, and consider the long exact sequence

$$\cdots \to \pi_n(X) \xrightarrow{f} \pi_n(Y) \to \pi_n(f) \to \pi_{n-1}(X) \to \pi_{n-1}(Y) \to \ldots$$

We define $I_n = \ker(f\colon \pi_n(X) \to \pi_n(Y))$ and $J_n = \mathrm{Im}(\pi_n(Y) \to \pi_n(f))$, and get an exact sequence

$$0 \to \pi_n(X)/I_n \to \pi_n(Y) \to \pi_n(f) \to I_{n-1} \to 0$$

which can be spliced together from the short exact sequences

$$\begin{aligned} &0 \to \pi_n(X)/I_n \to \pi_n(Y) \to J_n \to 0 \\ &0 \to J_n \to \pi_n(f) \to I_{n-1} \to 0 \ . \end{aligned} \tag{9.1}$$

Then it follows directly from the definitions that the double coboundary

$$\delta^2\colon H^i(I_{n-1}) \xrightarrow{\delta} H^{i+1}(J_n) \xrightarrow{\delta} H^i(\pi_n(X)/I_n)$$

from the Tate cohomology sequences induced by (9.1) is given by the composite

$$\delta^2\colon H^i(I_{n-1}) \xrightarrow{inc_*} H^i(\pi_{n-1}(X)) \xrightarrow{\Delta} H^i(\pi_n(X)) \xrightarrow{proj_*} H^i(\pi_n(X)/I_n) \ .$$

If we can pick the map $f\colon X \to Y$ appropriately, say with $I_n = 0$ and $I_{n-1} = \pi_{n-1}(X)$, this gives an algebraic description of Δ. We give some examples, with G denoting a finite cyclic group as usual.

Example 9.2. There is an exact sequence [20]

$$0 \to \mathrm{Wh}(\mathbf{Z}G) \to \mathrm{Wh}(\widehat{\mathbf{Z}}_2 G) \to \mathrm{Wh}(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2 G) \to \widetilde{K}_0(\mathbf{Z}G) \to 0$$

of $\mathbf{Z}/2$ modules and the associated double coboundary in Tate cohomology equals

$$\Delta\colon H^i(\widetilde{K}_0(\mathbf{Z}G)) \to H^i(\mathrm{Wh}(\mathbf{Z}G)) .$$

The point here is that $\ker(\mathrm{Wh}(\mathbf{Z}G) \to \mathrm{Wh}(\widehat{\mathbf{Z}}_2 G)) = 0$ [25], and the map $\widetilde{K}_0(\mathbf{Z}G) \to \widetilde{K}_0(\widehat{\mathbf{Z}}_2 G)$ is zero by a result of Swan [36]. We could also use the exact sequence

$$0 \to \mathrm{Wh}(\mathbf{Z}G) \to \mathrm{Wh}(\widehat{\mathbf{Z}}G) \oplus \mathrm{Wh}(\mathbf{Q}G) \to \mathrm{Wh}(\widehat{\mathbf{Q}}G)) \to \widetilde{K}_0(\mathbf{Z}G) \to 0$$

to compute the same map Δ. □

Example 9.3. There is an exact sequence

$$\begin{aligned} 0 \to \mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to \mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-}(\widehat{\mathbf{Z}} \oplus \mathbf{Q})) &\to \\ \mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-}(\widehat{\mathbf{Q}})) &\to \widetilde{K}_0(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to 0 \end{aligned}$$

where $\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) = \mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-)$, $\widetilde{K}_0(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) = \widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G^-)$, and $\mathbf{Z}G^-$ denotes the group ring $\mathbf{Z}G$ with the non–oriented involution induced by $t \to -t^{-1}$. The point here is that $K_2(\widehat{\mathbf{Q}}G) = K_2(\mathcal{C}_{\mathbf{R}_-}(\widehat{\mathbf{Q}}))$ maps trivially through $K_1(\mathbf{Z}G)$ into $\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ because $SK_1(\mathbf{Z}G) = 0$ [25]. We therefore get an algebraic description of $\delta^2\colon H^i(\widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G^-)) \to H^i(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$ as used in the statement of Theorem B. □

Example 9.4. There is an exact sequence

$$\begin{aligned} 0 \to \mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to \mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-}(\widehat{\mathbf{Z}} \oplus \mathbf{Q})) &\to \\ \mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-}(\widehat{\mathbf{Q}})) &\to \widetilde{K}_0(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to 0 \end{aligned}$$

for any complex G–representation W with $W^G = 0$. We therefore get an algebraic description of $\delta^2\colon H^i(\widetilde{K}_0(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))) \to H^i(\mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})))$ as used in the statement of Theorem C. □

Lemma 9.5. *For complex G–representations $W_1 \subseteq W$ with $W^G = 0$, there is a commutative diagram*

$$\begin{array}{ccc} H^i(\widetilde{K}_0(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))) & \xrightarrow{\delta^2} & H^i(\mathrm{Wh}(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))) \\ \Big\downarrow{\scriptstyle c_*} & & \Big\downarrow{\scriptstyle c_*} \\ H^i(\widetilde{K}_0(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))) & \xrightarrow{\delta^2} & H^i(\mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))) \end{array}$$

where the vertical maps are induced by the inclusion $W_1 \subseteq W$.

For our applications, the main point of the double coboundary description is that it permits these maps induced by cone point inclusions to be computed using bounded K–theory, instead of bounded L–theory.

The double coboundary maps also commute with restriction to subgroups of G.

Proposition 9.6. *Let $G_1 < G$ be a subgroup of odd index, and $H_1 < G_1$ have index 2, then there are twisted restriction maps $H^i(\widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G^-)) \to H^i(K_1(\mathbf{Z}H_1 \to \mathbf{Z}G_1^-))$ such that the diagram*

$$\begin{tikzcd} H^i(\widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G^-)) \arrow[r, "\delta^2"] \arrow[d, "\mathrm{Res}"'] & H^i(K_1(\mathbf{Z}H \to \mathbf{Z}G^-)) \arrow[d, "\mathrm{Res}"] \\ H^i(\widetilde{K}_0(\mathbf{Z}H_1 \to \mathbf{Z}G_1^-)) \arrow[r, "\delta_1^2"] & H^i(K_1(\mathbf{Z}H_1 \to \mathbf{Z}G_1^-)) \end{tikzcd}$$

commutes.

Proof. The vertical maps are twisted restriction maps given by the composite of the twisting isomorphisms $H^i(\widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G^-)) \cong H^{i+1}(\widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G))$ and $H^i(K_1((\mathbf{Z}H \to \mathbf{Z}G^-)) \cong H^{i+1}(K_1((\mathbf{Z}H \to \mathbf{Z}G))$ with the restriction maps induced by the inclusion $\mathbf{Z}G_1 \to \mathbf{Z}G$ of rings with involution. Since $G_1 < G$ has odd index, $H_1 = H \cap G_1$ and the composition $\mathrm{Res}_{G_1} \circ \mathrm{Ind}_H$ lands in the image of Ind_{H_1} by the double coset formula. □

This can be generalized to the double coboundary maps used in the statement of Theorem C, under certain conditions.

Proposition 9.7. *Let $G_1 < G$ be an odd index subgroup, and $H_1 < G_1$ have index 2. Suppose that W only has proper isotropy subgroups of 2–power index. Then there are a twisted restriction maps $H^i(\widetilde{K}_0(\mathcal{C}_{W \times \mathbf{R}_-, G}(\mathbf{Z}))) \to H^i(\widetilde{K}_0(\mathcal{C}_{\mathrm{Res}\, W \times \mathbf{R}_-, G_1}(\mathbf{Z})))$ and $H^i(\mathrm{Wh}(\mathcal{C}_{W \times \mathbf{R}_-, G}(\mathbf{Z}))) \to H^i(\mathrm{Wh}(\mathcal{C}_{\mathrm{Res}\, W \times \mathbf{R}_-, G_1}(\mathbf{Z})))$ which commute with the corresponding double coboundary maps δ^2_W and $\delta^2_{\mathrm{Res}\, W}$.*

10. Some basic facts in K- and L-theory

In this section we record various calculational facts from the literature about K–and L–theory of cyclic groups. A general reference for the K–theory is [25], and for L–theory computations is [20]. Recall that $\widetilde{K}_0(\mathcal{A}) = K_0(\mathcal{A}^\wedge)/K_0(\mathcal{A})$ for any additive category $\mathcal{A}$, and $\mathrm{Wh}(\mathcal{A})$ is the quotient of $K_1(\mathcal{A})$ by the subgroup defined by the system of stable isomorphisms.

Theorem 10.1. *Let G be a cyclic group, K a subgroup. We then have*

(1) *$K_1(\mathbf{Z}G) = (\mathbf{Z}G)^* \subset K_1(\mathbf{Q}G)$ Here $(\mathbf{Z}G)^*$ denotes the units of $\mathbf{Z}G$.*
(2) *The torsion in $K_1(\mathbf{Z}G))$ is precisely $\{\pm G\}$, so $\mathrm{Wh}(\mathbf{Z}G)$ is torsion free.*
(3) *The maps $K_1(\mathbf{Z}K) \to K_1(\mathbf{Z}G)$ and*
$$\mathrm{Wh}(\mathbf{Z}G)/\mathrm{Wh}(\mathbf{Z}K) \to \mathrm{Wh}(\mathbf{Q}G)/\mathrm{Wh}(\mathbf{Q}K)$$
are injective.
(4) *$\widetilde{K}_0(\mathbf{Z}G)$ a torsion group and the map $\widetilde{K}_0(\mathbf{Z}G) \to \widetilde{K}_0(\mathbf{Z}_{(p)}G$ is the zero map for all primes p.*
(5) *$K_{-1}(\mathbf{Z}G)$ is torsion free, and sits in an exact sequence*
$$0 \to K_0(\mathbf{Z}) \to K_0(\widehat{\mathbf{Z}}G) \oplus K_0(\mathbf{Q}G) \to K_0(\widehat{\mathbf{Q}}G) \to K_{-1}(\mathbf{Z}G) \to 0\ .$$
(6) *$K_{-1}(\mathbf{Z}K) \to K_{-1}(\mathbf{Z}G)$ is an injection.*
(7) *$K_{-j}(\mathbf{Z}G) = 0$ for $j \geqq 2$.*

Proof. The proof mainly consists of references. See [25,pp.6,14] for the first two parts. Part (3) follows from (1) and the relation $(\mathbf{Z}G)^* \cap (\mathbf{Q}K)^* = (\mathbf{Z}K)^*$. Part (4) is due to Swan [36], and part (7) is a result of Carter [10]. Part (5) gives the arithmetic sequence for computing $K_{-1}(\mathbf{Z}G)$, and the assertion that $K_{-1}(\mathbf{Z}G)$ is torsion free is easy to deduce (see also [10]). Since $\mathrm{Res}_K \circ \mathrm{Ind}_K$ is multiplication by the index $[G:K]$, part (6) follows from (5). □

Tate cohomology of K_i-groups play an important role. The involution on K–theory is induced by duality on modules. It is conventionally chosen to have the boundary map

$$K_1(\widehat{\mathbf{Q}}(G) \to \tilde{K}_0(\mathbf{Z}G)$$

preserve the involution, so to make this happen we choose to have the involution on K_0 be given by sending $[P]$ to $-[P^*]$, and the involution on K_1 is given by sending τ to τ^*. This causes a shift in dimension in Ranicki-Rothenberg exact sequences

$$\ldots \to H^0(\widetilde{K}_0(\mathcal{A})) \to L^h_{2k}(\mathcal{A}) \to L^p_{2k}(\mathcal{A}) \to \ldots$$

compared to

$$\ldots \to H^1(\mathrm{Wh}(\mathcal{A})) \to L^s_{2k}(\mathcal{A}) \to L^h_{2k}(\mathcal{A}) \to \ldots$$

Theorem 10.2. *Let G be a cyclic group, K a subgroup.*

1. $L^s_{2k}(\mathbf{Z}G)$, $L^p_{2k}(\mathbf{Z}G)$, *and* $L^{\langle -1\rangle}_{2k}(\mathbf{Z}G)$ *are torsion–free when k is even, and when k is odd the only torsion is a $\mathbf{Z}/2$-summand generated by the Arf invariant element.*
2. *The groups* $L^h_{2k+1}(\mathbf{Z}G) = L^s_{2k+1}(\mathbf{Z}G) = L^p_{2k+1}(\mathbf{Z}G)$ *are zero (k even), or $\mathbf{Z}/2$ (k odd).*
3. $L^{\langle -1\rangle}_{2k+1}(\mathbf{Z}G) = H^1(K_{-1}(\mathbf{Z}G))$ *(k even), or* $\mathbf{Z}/2 \oplus H^1(K_{-1}(\mathbf{Z}G))$ *(k odd).*
4. *The Ranicki-Rothenberg exact sequence gives*

$$0 \to H^0(\widetilde{K}_0(\mathbf{Z}G)) \to L^h_{2k}(\mathbf{Z}G) \to L^p_{2k}(\mathbf{Z}G) \to H^1(\widetilde{K}_0(\mathbf{Z}G)) \to 0$$

   so $L^h_{2k}(\mathbf{Z}G)$ *has the torsion subgroup* $H^0(\widetilde{K}_0(\mathbf{Z}G))$.
5. *The double coboundary* $\delta^2\colon H^1(\widetilde{K}_0(\mathbf{Z}G)) \to H^0(\mathrm{Wh}(\mathbf{Z}G))$ *is injective:*
6. *The maps* $L^s_{2k}(\mathbf{Z}K) \to L^s_{2k}(\mathbf{Z}G)$, $L^p_{2k}(\mathbf{Z}K) \to L^p_{2k}(\mathbf{Z}G)$, *and* $L^{\langle -1\rangle}_{2k}(\mathbf{Z}K) \to L^{\langle -1\rangle}_{2k}(\mathbf{Z}G)$ *are injective when k is even. For k odd and $[G:K]$ even, the kernel is generated by the Arf invariant element.*
7. *In the oriented case,* $\mathrm{Wh}(\mathbf{Z}G)$ *has trivial involution and* $H^1(\mathrm{Wh}(\mathbf{Z}G)) = 0$.

Proof. See [20] for the proof of part (1). Part (2) is due to Bak for L^s and L^h [2]. Part (7) is due to Wall [25] and part(6) follows again from $\mathrm{Res}_K \circ \mathrm{Ind}_K = [G:K]$. The other parts, follow from substituting in the change of K–theory sequence. □

11. The computation of $L^p_1(\mathbf{Z}G, w)$

Here we correct an error in the statement of [16,5.1] (Notice however that Table 2 [16,p.553] has the correct answer).

Proposition 11.1. *Let $G = \sigma \times \rho$, where σ is an abelian 2–group and ρ has odd order. Then $L^p_n(\mathbf{Z}G, w) = L^p_n(\mathbf{Z}\sigma, w) \oplus L^p_n(\mathbf{Z}\sigma \to \mathbf{Z}G, w)$ where $w: G \to \{\pm 1\}$ is an orientation character. For $i = 2k$, the second summand is free abelian and detected by signatures at the type $U(\mathbf{C})$ representations of G which are non–trivial on ρ. For $n = 2k + 1$, the second summand is a direct sum of $\mathbf{Z}/2$'s, one for each type $U(\mathbf{GL})$ representation of G which is non–trivial on ρ.*

Remark 11.2. Note that type $U(\mathbf{C})$ representations of G exist only when $w \equiv 1$, and type $U(\mathbf{GL})$ representations of G exist only when $w \not\equiv 1$. In both cases, the second summand is computed by transfer to cyclic subquotients of order $2^r q$, $q > 1$ odd, with $r \geqq 2$. $\square$

Proof. The given direct sum decomposition follows from the existence of a retraction of the inclusion $\sigma \to G$ compatible with w. It also follows that

$$L^{p,h}_{n+1}(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2 G, w) \cong L^{p,h}_{n+1}(\mathbf{Z}\sigma \to \widehat{\mathbf{Z}}_2 \sigma, w) \oplus L^p_n(\mathbf{Z}\sigma \to \mathbf{Z}G, w)$$

since the map $L^h_n(\widehat{\mathbf{Z}}_2\sigma, w) \to L^h_n(\widehat{\mathbf{Z}}_2 G, w)$ is an isomorphism. The computation of the relative groups for $\mathbf{Z} \to \widehat{\mathbf{Z}}_2$ can be read off from [16,Table 2,Remark 2.14]: for each centre field E of a type $U(GL)$ representation, the contribution is $H^0(C(E)) \cong \mathbf{Z}/2$ if $i \equiv 1 \mod 2$.

The detection of $L^p_i(\mathbf{Z}\sigma \to \mathbf{Z}G, w)$ by cyclic subquotients is proved in [21, 1.B.7, 3.A.6, 3.B.2]. $\square$

Corollary 11.3. *Let $G = C(2^r q)$, for q odd and $r \geqq 2$. If $q = 1$ assume that $r \geqq 3$. Then the group*

$$L^p_1(\mathbf{Z}G, w)(q) = \bigoplus_{i=2}^{r} CL^K_2(E_i) \cong (\mathbf{Z}/2)^{r-1}$$

when $w \not\equiv 1$, where the summand $CL^K_2(E_i) = H^0(C(E_i))$, $2 \leqq i \leqq r$, corresponds to the rational representation with centre field $E_i = \mathbf{Q}(\zeta_{2^i q})$.

Remark 11.4. The calculation of L^p_1 contradicts the assertion in [7,p.733,l.-8] that the projection map $G \to C(2^r)$ induces an isomorphism on L^p_1 in the non–oriented case. This error means that there is a gap in the proofs of the main results of [7] for cyclic groups not of 2–power order. $\square$

We now return to our main calculational device for determining non–linear similarities of cyclic groups, namely the "double coboundary"

$$\delta^2: H^1(\widetilde{K}_0(\mathbf{Z}G)) \to H^1(Wh(\mathbf{Z}G))$$

from the exact sequence

$$0 \to Wh(\mathbf{Z}G) \to Wh(\widehat{\mathbf{Z}}G) \oplus K_1(\mathbf{Q}G) \to K_1(\widehat{\mathbf{Q}}G) \to \widetilde{K}_0(\mathbf{Z}G) \to 0 \ .$$

We recall that the discriminant induces an isomorphism

$$L^h_1(\mathbf{Z}G, w) \cong H^1(Wh(\mathbf{Z}G), w)$$

since $L^s_i(\mathbf{Z}G, w) \cong L'_i(\mathbf{Z}G, w) = 0$ for $i \equiv 0, 1 \mod 4$ by the calculations of [38,3.4.5,5.4].

Proposition 11.5. *The kernel of the map* $L_1^h(\mathbf{Z}G, w) \to L_1^p(\mathbf{Z}G, w)$ *equals the image of the double coboundary* $\delta^2 \colon H^1(\widetilde{K}_0(\mathbf{Z}G)) \to H^1(Wh(\mathbf{Z}G))$ *under the isomorphism* $L_1^h(\mathbf{Z}G, w) \cong H^1(Wh(\mathbf{Z}G), w)$ *induced by the discriminant .*

Proof. We will use the commutative braid

$$\begin{tikzcd}[column sep=small]
H^1(\widetilde{K}_0(\mathbf{Z}G)) \arrow[rr, bend left, "\delta^2"] \arrow[dr] & & H^1(Wh(\mathbf{Z}G)) \arrow[rr, bend left] \arrow[dr] & & L_0^s(\mathbf{Z}G, w) \\
& L_1^h(\mathbf{Z}G, w) \arrow[ur] \arrow[dr] & & H^1(\Delta) \arrow[ur] \arrow[dr] & \\
L_1^s(\mathbf{Z}G, w) \arrow[rr, bend right] \arrow[ur] & & L_1^p(\mathbf{Z}G, w) \arrow[rr, bend right] \arrow[ur] & & H^0(\widetilde{K}_0(\mathbf{Z}G))
\end{tikzcd} \tag{11.6}$$

relating the L^h to L^p and the L^s to L^h Rothenberg sequences. The term $H^1(\Delta)$ is the Tate cohomology of the relative group for the double coboundary. We see that the discriminant of an element $\sigma \in L_1^h(\mathbf{Z}G, w)$ lies in the image of the double coboundary if and only if $\sigma \in \ker(L_1^h(\mathbf{Z}G, w) \to L_1^p(\mathbf{Z}G, w))$. $\square$

Corollary 11.7. *There is an isomorphism* $L_1^p(\mathbf{Z}G, w) \cong H^1(\Delta)$.

Remark 11.8. There is a version of these results for $L_3^p(\mathbf{Z}G, w)$ as well, on the kernel of the projection map $L_3^p(\mathbf{Z}G, w) \to L_3^p(\mathbf{Z}K, w)$, where $K = C(4)$. The point is that $L_i^s(\mathbf{Z}G, w) \cong L_i^s(\mathbf{Z}K, w)$ is an isomorphism for $i \equiv 2, 3 \pmod 4$ as well [38,3.4.5,5.4]. There is also a corresponding braid (15.1) for $L_{2k+1}^s(\mathcal{C}_{W\times \mathbf{R}_-, G}(\mathbf{Z}))$, $L_{2k+1}^h(\mathcal{C}_{W\times \mathbf{R}_-, G}(\mathbf{Z}))$ and $L_{2k+1}^p(\mathcal{C}_{W\times \mathbf{R}_-, G}(\mathbf{Z}))$ involving the double coboundary in bounded K–theory. The cone point inclusion

$$\mathcal{C}_{pt}(\mathbf{Z}G, w) = \mathcal{C}_{pt, G^-}(\mathbf{Z}) \to \mathcal{C}_{W\times \mathbf{R}_-, G}(\mathbf{Z})$$

induces a natural transformation between the two braid diagrams. $\square$

In Section 12 we will need the following calculation. We denote by $L_n^{\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}G^-)$ the L–group of $\mathbf{Z}G$ with the non-oriented involution, and Whitehead torsions allowed in the subgroup $\mathrm{Wh}(\mathbf{Z}H) \subset \mathrm{Wh}(\mathbf{Z}G)$.

Lemma 11.9. $L_1^{\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}G^-) = 0$*, and the discriminant map* $L_0^{\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}G^-) \to H^0(\mathrm{Wh}(\mathbf{Z}H)$ *is an injection.*

Proof. The Rothenberg sequence gives

$$L_n^s(\mathbf{Z}G^-) \to L_n^{\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}G^-) \to H^n(\mathrm{Wh}(\mathbf{Z}H)) \ .$$

For $n \equiv 1 \pmod 4$ the outside terms are zero, and hence $L_1^{\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}G^-) = 0$. For $n \equiv 0 \pmod 4$, $L_0^s(\mathbf{Z}G^-) = 0$ as noted above and the injectivity follows. $\square$

Another specific calculation needed later is:

Lemma 11.10. *For $G = C(2^r)$ and $H < G$ of index 2, the group $L_1^k(\mathbf{Z}H) = 0$, where $\mathbf{k} = \ker(\widetilde{K}_0(\mathbf{Z}H) \to \widetilde{K}_0(\mathbf{Z}G))$.*

Proof. The group $L_1^k(\mathbf{Z}H) = 0$ sits in the arithmetic sequence [20]

$$\cdots \to L_2^{k,h}(\mathbf{Z}H \to \widehat{\mathbf{Z}}_2 H) \to L_1^k(\mathbf{Z}H) \to L_1^h(\widehat{\mathbf{Z}}_2 H) \to \dots$$

and $L_1^h(\widehat{\mathbf{Z}}_2 H) = 0$ since H is a 2–group. The relative group $L_2^{k,h}(\mathbf{Z}H \to \widehat{\mathbf{Z}}_2 H) \cong L_2^{\bar{k},h}(\mathcal{M} \to \widehat{\mathcal{M}}_2)$, where $\mathcal{M}$ denotes an involution invariant maximal order in $\mathbf{Q}H$ containing $\mathbf{Z}H$, and $\bar{\mathbf{k}}$ is the preimage in $K_1(\mathcal{M} \to \widehat{\mathcal{M}}_2)$ of $\mathbf{k}$. By the tables in [20,Appendix], we see that $L_2^{\tilde{k},h}(\mathcal{M} \to \widehat{\mathcal{M}}_2) \cong H^0(\tilde{\mathbf{k}})$, so it remains to compute this group. However, $K_1(\mathcal{M} \to \widehat{\mathcal{M}}_2)$ was computed in [17,4.8] as a module over $\mathbf{Z}/2$. It is the direct sum of copies of $\widehat{\mathbf{Z}}_2$ with -1 action of the involution, and one copy of $\mathbf{Z}[\mathbf{Z}/2]$ generated by $1 + 2i$ in each centre field $\mathbf{Q}(\zeta_{2^k})$, for $k \geqq 2$. It is not difficult to check using 2–adic units $1 + 2t^{2^k}$ (as in [17,§6]) that the elements $1 + 2i$ in each centre field are trivial in $\widetilde{K}_0(\mathbf{Z}H)$. Therefore $\bar{\mathbf{k}}$ has involution -1, and $H^0(\bar{\mathbf{k}}) = 0$. $\square$

In later sections, it will be convenient to stabilize with trivial representations and use the identification

$$L^p_{n+k}(\mathcal{C}_{W \oplus \mathbf{R}^k, G}(\mathbf{Z})(\mathbf{Z})) \cong L_n^{\langle -k \rangle}(\mathcal{C}_{W,G}(\mathbf{Z})(\mathbf{Z})).$$

The composite with the transfer

$$trf_{\mathbf{R}^k} \colon L_n^p(\mathbf{Z}G) \to L^p_{n+k}(\mathcal{C}_{W \oplus \mathbf{R}^k, G}(\mathbf{Z})(\mathbf{Z}))$$

is just the usual "change of K–theory" map, which may be analysed by the Ranicki-Rothenberg sequences [31]. For G a finite group, $K_{-j}(\mathbf{Z}G) = 0$ if $j \geqq 2$ so only the first stablization is needed.

Lemma 11.11. *For $G = C(2^r q)$ and $w \colon G \to \{\pm 1\}$ non–trivial, the map*

$$L^p_{2k+1}(\mathbf{Z}G, w) \to L^{\langle -1 \rangle}_{2k+1}(\mathbf{Z}G, w)$$

is injective.

Proof. The group $K_{-1}(\mathbf{Z}G)$ is a torsion free quotient of $K_0(\widehat{\mathbf{Q}}G)$, and has the involution induced by $[P] \mapsto -[P^*]$ on $K_0(\widehat{\mathbf{Q}}G)$ [16,3.10]. This implies first that $H^0(K_0(\widehat{\mathbf{Q}}G)) = 0$, and so the image of the coboundary

$$H^0(K_{-1}(\mathbf{Z}G)) \to H^1(K_0(\widehat{\mathbf{Z}}G)) \oplus H^1(K_0(\mathbf{Q}G))$$

consists of the classes $(0, [E])$ where E splits at every prime. We need to compare the exact sequences in the following diagram (see [16], [20]):

$$\begin{tikzcd}
 & & L^K_{2k+2}(\widehat{\mathbf{Z}}G, w) \oplus L^K_{2k+2}(\mathbf{Q}G, w) \arrow[r] \arrow[d] & L^K_{2k+2}(\widehat{\mathbf{Q}}G, w) \arrow[d] \\
0 \arrow[r] & L^{\langle -1 \rangle}_{2k+2}(\mathbf{Z}G, w) \arrow[r] \arrow[d] & L^p_{2k+2}(\widehat{\mathbf{Z}}G, w) \oplus L^p_{2k+2}(\mathbf{Q}G, w) \arrow[r] \arrow[d] & L^p_{2k+2}(\widehat{\mathbf{Q}}G, w) \arrow[d] \\
0 \arrow[r] & H^0(K_{-1}(\mathbf{Z}G)) \arrow[r] & H^1(K_0(\widehat{\mathbf{Z}}G)) \oplus H^1(K_0(\mathbf{Q}G)) \arrow[r] & H^1(K_0(\widehat{\mathbf{Q}}G))
\end{tikzcd}$$

The groups $L^K_{2k+2}(\widehat{\mathbf{Z}}G, w)$ reduce to the L–groups of finite fields, which are zero in type U. For each involution invariant field E in the top component of $\mathbf{Q}G$, the group $L^K_{2k+2}(E) = H^0(E^\times)$ which maps injectively into $L^K_{2k+2}(\hat{E}) = H^0(\hat{E}^\times)$, [16]. It follows that the images of $L^K_2(E)$ and $L^{\langle -1\rangle}_{2k+2}(\mathbf{Z}G, w)$ in $L^p_{2k+2}(E)$ have zero intersection , and so the composite map $L^{\langle -1\rangle}_{2k+2}(\mathbf{Z}G, w) \to H^1(K_0(\mathbf{Q}G))$ is an isomorphism. Therefore $L^{\langle -1\rangle}_{2k+2}(\mathbf{Z}G, w) \to H^0(K_{-1}(\mathbf{Z}G))$ is surjective and we conclude that the map $L^p_{2k+1}(\mathbf{Z}G, w) \to L^{\langle -1\rangle}_{2k+1}(\mathbf{Z}G, w)$ is injective. □

Corollary 11.12. *Let $G = C(2^r q)$ and $w: G \to \{\pm 1\}$ the non–trivial orientation. If W is a G–representation with $W^G = 0$, then the map*

$$L^p_{2k+1}(\mathcal{C}_{W,G}(\mathbf{Z})(\mathbf{Z}), w) \to L^{\langle -1\rangle}_{2k+1}(\mathcal{C}_{W,G}(\mathbf{Z})(\mathbf{Z}), w)$$

is injective.

Proof. We first note that the cone point maps $K_0(RG) \to K_0(\mathcal{C}_{W,G}(R))$ are surjective for $R = \widehat{\mathbf{Z}}, \mathbf{Q}$ or $\widehat{\mathbf{Q}}$ since for these coefficients RG has vanishing K_{-1} groups. This shows that $K_{-1}(\mathcal{C}_{W,G}(\mathbf{Z}))$ is again a quotient of $K_0(\widehat{\mathbf{Q}}G)$. To see that $K_{-1}(\mathcal{C}_{W,G}(\mathbf{Z}))$ is also torsion free, consider the boundary map $K_1(\mathcal{C}^{>0}_{W,G}(\widehat{\mathbf{Q}})) \to K_0(\mathcal{C}_W(\widehat{\mathbf{Q}}))$ which is just a sum of induction maps $K_0(\widehat{\mathbf{Q}}K) \to K_0(\widehat{\mathbf{Q}}G)$ from proper subgroups of $K \subset G$. But for G cyclic, these induction maps are split injective. We now complete the argument by comparing the diagram above with the corresponding diagram for the bounded theory, concluding that $H^0(K_{-1}(\mathbf{Z}G)) \to H^0(K_{-1}(\mathcal{C}_{W,G}(\mathbf{Z})))$ is surjective. Since $L^{\langle -1\rangle}_{2k+2}(\mathbf{Z}G, w) \to H^0(K_{-1}(\mathbf{Z}G))$ is also surjective, we are done. □

12. The proof of Theorem A

The condition (i) is equivalent to assuming that $S(V_1)$ and $S(V_2)$ are freely G–homotopy equivalent. Condition (ii) is necessary by Corollary 3.10 which rules out non–linear similarities of semifree representations. Condition (ii) also implies that $S(V_1)$ is s–normally cobordant to $S(V_2)$ by [3,Prop.2.1], which is another necessary condition for topological similarity. Thus under conditions (i) and (ii), there exists a homotopy equivalence $f: S(V_2) \to S(V_1)$, and an element $\sigma = \sigma(f) \in L^h_0(\mathbf{Z}G)$ such that $trf_{\mathbf{R}_-}(\sigma) = 0 \in L^h_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ if and only if $V_1 \oplus \mathbf{R}_- \sim_t V_2 \oplus \mathbf{R}_-$.

Comparing the h- and s- surgery exact sequences it is easy to see that the image of σ in $H^0(\mathrm{Wh}(\mathbf{Z}G))$ is given by the Whitehead torsion $\{\tau(f)\} = \{\Delta(V_1)/\Delta(V_2)\} \in \mathrm{Wh}(\mathbf{Z}G)$ of the homotopy equivalence $S(V_2)/G \simeq S(V_1)/G$.

In Section 2 we gave the short exact sequence

$$0 \to \mathrm{Wh}(\mathbf{Z}G)/\mathrm{Wh}(\mathbf{Z}H) \to \mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G) \to k \to 0$$

where $K_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))/\{\pm G\})$ is denoted by $\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) = \mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G)$ and $\mathbf{k} = \ker(\widetilde{K}_0(\mathbf{Z}H) \to \widetilde{K}_0(\mathbf{Z}G))$. We proved in Theorem 3.6 that the transfer of the torsion element in $\mathrm{Wh}(\mathbf{Z}G)$ in $\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G)$ is given by the same element under the map induced by inclusion $\mathrm{Wh}(\mathbf{Z}G) \to \mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G)$.

It follows that the image of $trf_W(\sigma)$ in $H^1(\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))) = H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$ is given by the image of our well-defined element

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\,\mathrm{Wh}(\mathbf{Z}H))$$

under the cone point inclusion into $H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$.

The necessity of the condition is now easy. To have a non-linear similarity we must have

$$trf_{\mathbf{R}_-}(\sigma) = 0 \in L^h_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))\ .$$

Hence

$$trf_{\mathbf{R}_-}(\Delta(V_1)/\Delta(V_2)) = 0 \in H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$$

must vanish by naturality of the transfer in the Rothenberg sequence. This element comes from $H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\,\mathrm{Wh}(\mathbf{Z}H))$, so to vanish in $H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$ it must be in the image from

$$H^0(\mathbf{k}) \to H^1(\mathrm{Wh}(\mathbf{Z}G^-/\,\mathrm{Wh}(\mathbf{Z}H))$$

under the double coboundary.

To prove sufficiency, we assume that the image of the transferred element

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$$

is zero. Consider the long exact sequence derived from the inclusion of filtered categories

$$\mathcal{C}_{pt}(\mathbf{Z}G) \subset \mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})$$

where $\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})$ has the standard orientation (6.4) inducing the non–trivial orientation at the cone point. The quotient category $\mathcal{C}^{>0}_{\mathbf{R}_-,G}(\mathbf{Z})$ of germs away from 0 is canonically isomorphic to $\mathcal{C}^{>0}_{[0,\infty),H}(\mathbf{Z})$, since by equivariance what happens on the positive half line has to be copied on the negative half line, and what happens near 0 does not matter in the germ category. Since the action of H on $[0,\infty)$ is trivial, this category is precisely $\mathcal{C}^{>0}_{[0,\infty)}(\mathbf{Z}H)$ which has the same K- and L-theory as $\mathcal{C}_{\mathbf{R}}(\mathbf{Z}H)$ by the projection map. By Theorem 6.7 we thus get a long exact sequence

$$\ldots \to L^k_n(\mathbf{Z}H) \to L^h_n(\mathbf{Z}G^-) \to L^h_n(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to L^k_{n-1}(\mathbf{Z}H) \to \ldots$$

where the map is induced by induction. Comparing this sequence with the long exact sequence for the pair $L^{h,k}_n(\mathbf{Z}H \to \mathbf{Z}G^-)$ (see Theorem 7.1) it follows that

$$L^h_n(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \cong L^{k,h}_n(\mathbf{Z}H \to \mathbf{Z}G^-)\ .$$

Consider the following diagram with exact rows and columns:

$$\begin{tikzcd}
L_1^{\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}G^-) \arrow[r] \arrow[d] & L_1^{h,\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}H \to \mathbf{Z}G^-) \arrow[r] \arrow[d] & L^h_0(\mathbf{Z}H) \arrow[d] \\
L^h_1(\mathbf{Z}G^-) \arrow[r] \arrow[d] & L^{k,h}_1(\mathbf{Z}H \to \mathbf{Z}G^-) \arrow[r] \arrow[d] & L^k_0(\mathbf{Z}H) \arrow[d] \\
H^1(\mathrm{Wh}(G^-)/\,\mathrm{Wh}(H)) \arrow[r] & H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-)) \arrow[r] & H^1(\mathbf{k})
\end{tikzcd}$$

We need to show that $trf_{\mathbf{R}_-}(\sigma)$ vanishes in order to produce the non-linear similarity. We know that the image of $\mathrm{Res}_H(\sigma) = 0$, and our assumption is that the image

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$$

vanishes. We will finish the argument by showing:

Lemma 12.1. *Suppose that $\sigma \in L_0^h(\mathbf{Z}G)$.*

(i) *$L_1^{h,\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}H \to \mathbf{Z}G^-)$ is torsion–free, and*
(ii) *the torsion subgroup of $L_1^{k,h}(\mathbf{Z}H \to \mathbf{Z}G^-)$ injects into $H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$.*
(iii) *If $\mathrm{Res}_H(\sigma) = 0 \in L_0^h(\mathbf{Z}H)$, then $trf_{\mathbf{R}_-}(\sigma)$ is a torsion element.*

Proof of Lemma 12.1. For assertion (i), we consider the diagram

$$\begin{tikzcd}
L_0^s(\mathbf{Z}H) \arrow[r] \arrow[d, tail] & L_0^s(\mathbf{Z}G^-) \arrow[d] \\
L_0^h(\mathbf{Z}H) \arrow[r] \arrow[d] & L_0^{\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}G^-) \arrow[d] \\
H^0(\mathrm{Wh}(\mathbf{Z}H)) \arrow[r, equal] & H^0(\mathrm{Wh}(\mathbf{Z}H))
\end{tikzcd}$$

where $L_0^s(\mathbf{Z}G^-) = 0$. Since $L_1^{\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}G^-) = 0$ as well (by Lemma 11.9), it follows that

$$L_1^{h,\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}H \to \mathbf{Z}G^-) = \ker(L_0^h(\mathbf{Z}H) \to L_0^{\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}G^-)) \cong L_0^s(\mathbf{Z}H)$$

where $L_0^s(\mathbf{Z}H)$ is torsion–free. Part (ii) follows from part (i), since the previous term in the exact sequence has exponent two. For assertion (iii) we refer to part of the twisting diagram of Section 7, namely the commutative diagram:

$$\begin{tikzcd}
& L_0^h(\mathbf{Z}G) \arrow[d, tail, "trf_{\mathbf{R}_-}"] \arrow[dr, "\mathrm{Res}_H"] & \\
L_1^h(\mathbf{Z}G^-) \arrow[r] & L_1^{k,h}(\mathbf{Z}H \to \mathbf{Z}G^-) \arrow[r] & L_0^k(\mathbf{Z}H)
\end{tikzcd}$$

and the fact that $\mathrm{Res}_H(trf_{\mathbf{R}_-}(\sigma)) = 0$ since it factors through $\mathrm{Res}_H \colon L_0^h(\mathbf{Z}G) \to L_0^h(\mathbf{Z}H)$. But $L_1^h(\mathbf{Z}G^-) \cong H^1(\mathrm{Wh}(\mathbf{Z}G^-)$ has exponent 2. □

Remark 12.2. Theorem A gives necessary and sufficient conditions for the existence of 5–dimensional similarities. Consider the situation in dimensions $\leqq 5$. By character theory it suffices to consider a cyclic group G, which must be of order divisable by 4, say $4q$, with index 2 subgroup H. It suffices by Lemma 3.1 to consider the following situation

$$V_1 \oplus W \sim_t V_2 \oplus W$$

where V_i are free homotopy equivalent representations which become isomorphic once restricted to H. Therefore $S(V_1)$ is normally cobordant to $S(V_2)$. If V_i are two-dimensional they are determined by one character, so homotopy equivalence implies isomorphism. We may thus assume V_i are at least 4-dimensional, and $\dim W = 1$. According to Theorem 3.5, non-linear similarity is now determined by $trf_W(\sigma)$, where σ is an element of infinite order in $L_0^h(\mathbf{Z}G)$ hitting the element in the structure set determined by the homotopy equivalence of $S(V_2)/G \simeq S(V_1)/G$. In case W is the trivial representation we may identify trf_W with the map

$$L_0^h(\mathbf{Z}G) \to L_0^p(\mathbf{Z}G)$$

which we have seen (in Corollary 3.10) is injective on the elements of infinite order. Hence we are left with the case where W is the non-trivial one-dimensional representation $\mathbf{R}_-$, which is reduced to number theory by Theorem A. In Corollary 19.2 we work out the number theory for $G = C(2^r)$, but the general case remains open. □

13. The proof of Theorem B

The new ingredient in Theorem B is the double coboundary. As in the last section, we may assume that $f\colon S(V_2) \to S(V_1)$ is a G–homotopy equivalence which is freely G–normally cobordant to the identity, giving an element $\sigma = \sigma(f) \in L_0^h(\mathbf{Z}G)$. Then $V_1 \oplus \mathbf{R}_- \oplus \mathbf{R}_+ \sim_t V_2 \oplus \mathbf{R}_- \oplus \mathbf{R}_+$ if and only if

$$trf_{\mathbf{R}_-\oplus\mathbf{R}_+}(\sigma) = 0 \in L_2^h(\mathcal{C}_{\mathbf{R}_-\oplus\mathbf{R}_+,G}(\mathbf{Z}))$$

by Theorem 3.5. But $L_2^h(\mathcal{C}_{\mathbf{R}_-\oplus\mathbf{R}_+,G}(\mathbf{Z})) = L_1^p(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$, so we may regard the criterion as the vanishing of $trf_{\mathbf{R}_-}(\sigma) \in L_1^p(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ instead.

The main commutative diagram is:

$$\begin{tikzcd}[column sep=small]
H^1(\widetilde{K}_0(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))) \arrow[dr] \arrow[rr, bend left, "\delta^2"] & & H^1(K_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \arrow[dr] \arrow[rr, bend left] & & L_0^s(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \\
& L_1^h(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \arrow[ur] \arrow[dr] & & H^1(\Delta_{\mathbf{R}_-}) \arrow[ur] \arrow[dr] & \\
L_1^s(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \arrow[ur] \arrow[rr, bend right] & & L_1^p(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \arrow[ur] \arrow[rr, bend right] & & H^0(\widetilde{K}_0(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})))
\end{tikzcd} \tag{13.1}$$

where $H^1(\Delta_{\mathbf{R}_-})$ denotes the relative group of the double coboundary map. Notice that some of the groups in this diagram already appeared as relative L–groups in the last section. We have

$$L_1^h(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) = L_1^{k,h}(\mathbf{Z}H \to \mathbf{Z}G^-)$$

and
$$L_1^s(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) = L_1^{h,\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}H \to \mathbf{Z}G^-)\ .$$
By Lemma 12.1(ii) we again conclude that the torsion subgroup of $L_1^p(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ injects into $H^1(\Delta_{\mathbf{R}_-})$.

However, since $\mathrm{Res}_H(\sigma) = 0 \in L_0^h(\mathbf{Z}H)$, Lemma 12.1(i) implies that $trf_{\mathbf{R}_-}(\sigma)$ is a torsion element. Furthermore, its image in $H^1(\Delta_{\mathbf{R}_-})$ is zero if and only if

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$$

lies in the image of the double coboundary, and this completes the proof. □

14. Calculations in bounded L–theory

Suppose that $\sigma = \sigma(f) \in L_0^h(\mathbf{Z}G)$ is the surgery obstruction arising from a normal cobordism between $S(V_1)$ and $S(V_2)$, as in the statement of Theorem 3.5. In this section, we establish two important properties of $trf_W(\sigma)$ in preparation for the proof of Theorem C. Unless otherwise mentioned, all bounded categories will have the standard orientation (see Example 6.4).

For a complex G–representation W the standard orientation is trivial, and the cone point inclusion $0 \in W$ induces the map

$$c_*: L_n^h(\mathbf{Z}G) \to L_n^h(\mathcal{C}_{W,G}(\mathbf{Z}))$$

Note that the presence of an $\mathbf{R}_-$ factor introduces a non-trivial orientation at the cone point
$$c_*: L_{n+1}^h(\mathbf{Z}G, w) \to L_{n+1}^h(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$$
where $w: G \to \{\pm 1\}$ is the non-trivial projection. The properties are:

Theorem 14.1. *Suppose W is a complex G–representation with no $\mathbf{R}_+$ summands. If $\mathrm{Res}_H(V_1 \oplus W) \oplus \mathbf{R}_+ \sim_t \mathrm{Res}_H(V_2 \oplus W) \oplus \mathbf{R}_+$, then*

(i) *$trf_{W\times\mathbf{R}_-}(\sigma) \in L_{2k+1}^h(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ is a torsion element, and*

(ii) *there exists a torsion element $\hat{\sigma} \in L_{2k+1}^p(\mathbf{Z}G, w)$ such that*

$$c_*(\hat{\sigma}) = trf_{W\oplus\mathbf{R}_-}(\sigma) \in L_{2k+1}^p(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$$

where $\dim W = 2k$.

Before giving the proof, we need some preliminary results. When the L–theory decoration is not explicitly given, we mean $L^{\langle -\infty\rangle}$.

Lemma 14.2. *Let W be a complex G–representation. Then*

$$L_{2k+1}(\mathcal{C}_{W,G}(\mathbf{Z})) \otimes \mathbf{Q} = 0$$

for $k \geqq 0$

Proof. We argue by induction on $\dim W$, starting with

$$L_{2k+1}(\mathbf{Z}G) = \mathbf{Z}/2 \oplus H^1(K_{-1}(\mathbf{Z}G))$$

which is all 2–torsion. It is enough to prove the result for the top component $L_{2k+1}(\mathcal{C}_{W,G}(\mathbf{Z}))(q)$, and therefore by Theorem 3.8 we may assume that the isotropy groups of W all have 2–power index. Since we are working with $L^{\langle -\infty \rangle}$, we may ignore $\mathbf{R}_+$ summands of W. In the proof, all L–groups will be tensored with $\mathbf{Q}$.

Let $W = W' \oplus W''$ where $\dim W' = 2$ and W' has minimal isotropy group K. We assume the result for W'', and let $\bigcup \ell_\alpha \subset W'$ be a G-invariant set of rays from the origin, dividing W' into fundamental domains for the free G/K-action.

Then

$$L_n(\mathcal{C}^{>W''}_{\bigcup \ell_\alpha \times W'',G}(\mathbf{Z})) = L_{n-1}(\mathcal{C}_{W'',G}(\mathbf{Z})) = L_{n-1-|W''|}(\mathbf{Z}K),$$

which vanishes for n even, and

$$L_n(\mathcal{C}^{>\bigcup \ell_\alpha \times W''}_{W,G}(\mathbf{Z})) = L_{n-2}(\mathcal{C}_{W'',G}(\mathbf{Z})) = L_{n-2-|W''|}(\mathbf{Z}K),$$

which vanishes for n odd. Moreover, we have a long exact sequence

$$\cdots \to L_n(\mathcal{C}^{>W''}_{\bigcup \ell_\alpha \times W'',G}(\mathbf{Z})) \to L_n(\mathcal{C}^{>W''}_{W,G}(\mathbf{Z})) \to L_n(\mathcal{C}^{>\bigcup \ell_\alpha \times W''}_{W,G}(\mathbf{Z})) \to \dots$$

We claim that the first map in this sequence is injective. The previous map in the long exact sequence is

$$L_{n+1}(\mathcal{C}^{>\bigcup \ell_\alpha \times W''}_{W,G}(\mathbf{Z})) \xrightarrow{\partial} L_n(\mathcal{C}^{>W''}_{\bigcup \ell_\alpha \times W'',G}(\mathbf{Z})),$$

which may be identified (using the isomorphisms above and Proposition 8.7) with a geometrically induced map

$$L_{n-1-|W''|}(\mathbf{Z}K) \xrightarrow{1-u} L_{n-1-|W''|}(\mathbf{Z}K)$$

"multiplication by $1 - w(t)$", where $\bar{t} = w(t)t^{-1}$ from the action of t in the anti-structure used to define the L-group.

In the oriented case, $w(t) = +1$ and this boundary map is zero. Therefore, we conclude that

$$L_{2k+1}(\mathcal{C}^{>W''}_{\bigcup \ell_\alpha \times W'',G}(\mathbf{Z})) \xrightarrow{\approx} L_{2k+1}(\mathcal{C}^{>W''}_{W,G}(\mathbf{Z}))$$

is an isomorphism and so

$$L_{2k+1}(\mathcal{C}^{>W''}_{W,G}(\mathbf{Z})) = L_{2k-|W''|}(\mathbf{Z}K)\ .$$

Finally, we will substitute this computation and our inductive assumption into the long exact sequence

$$\cdots \to L_{2k+1}(\mathcal{C}_{W'',G}(\mathbf{Z})) \to L_{2k+1}(\mathcal{C}_{W,G}(\mathbf{Z})) \to$$
$$L_{2k+1}(\mathcal{C}^{>W''}_{W,G}(\mathbf{Z})) \to L_{2k}(\mathcal{C}_{W'',G}(\mathbf{Z})) \to$$

and find that

$$0 \to L_{2k+1}(\mathcal{C}_{W,G}(\mathbf{Z})) \to L_{2k+1}(\mathcal{C}^{>W''}_{W,G}(\mathbf{Z})) \to L_{2k}(\mathcal{C}_{W'',G}(\mathbf{Z})).$$

However, the second map in this sequence can be identified with the inclusion map

$$\mathrm{Ind}_K \colon L_{2k}(\mathcal{C}_{W'',K}(\mathbf{Z})) \to L_{2k}(\mathcal{C}_{W'',G}(\mathbf{Z}))$$

and the composition $\mathrm{Res}_K \circ \mathrm{Ind}_K$ is multiplication by $[G : K]$, which is a rational isomorphism. Therefore Ind_K is injective and $L_{2k+1}(\mathcal{C}_{W,G}(\mathbf{Z})) = 0$. $\square$

Another computation we will need is

Lemma 14.3. *$L_n(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))\otimes\mathbf{Q}=0$ for W a complex G-representation.*

Proof. We may assume that $W^G=0$ and argue by induction on the dimension of W. We write $W=W'\oplus W''$, where $\mathrm{Iso}(W')=K$ is minimal (2–power index isotropy subgroups may be assumed as usual). We have 2 long exact sequences (all L–groups are tensored with $\mathbf{Q}$):

(i) from the inclusion $W''\oplus\mathbf{R}_-\subset W\oplus\mathbf{R}_-$. For short, let $A=\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})$, $B=\mathcal{C}^{>\mathbf{R}_-}_{W''\times\mathbf{R}_-,G}(\mathbf{Z})$, and then $A/B=\mathcal{C}^{>W''\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})$. We need the piece of the L-group sequence:

$$\ldots\to L_n(B)\to L_n(A)\to L_n(A/B)\ldots \tag{14.4}$$

and note that $L_n(B)=0$ by induction.

(ii) we study $L_n(A/B)$ by looking at the usual rays $\bigcup\ell_\alpha\subset W'$ which divide the 2-dim representation W' into G/K chambers. Let

$$D_0=\mathcal{C}^{>W''\times\mathbf{R}_-}_{\bigcup\ell_\alpha\times W''\times\mathbf{R}_-,G}(\mathbf{Z})$$

and

$$D_1=\mathcal{C}^{>\bigcup\ell_\alpha\times W''\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})$$

Then we need the L–group sequence

$$\to L_n(D_0)\to L_n(A/B)\to L_n(D_1)\to L_{n-1}(D_0) \tag{14.5}$$

But the groups $L_n(D_i)$ are the ones we have been computing by using the chamber structure. In particular,

$$L_n(D_0)=L_{n-2}(\mathcal{C}_{W'',K}(\mathbf{Z}))$$

and

$$L_n(D_1)=L_{n-3}(\mathcal{C}_{W'',K}(\mathbf{Z})).$$

But since K is the minimal isotropy group in W, it acts trivially on W'' and these L-groups are just

$$L_n(D_0)=L_{n-2-|W''|}(\mathbf{Z}K)$$

and

$$L_n(D_1)=L_{n-3-|W''|}(\mathbf{Z}K).$$

Now the boundary map

$$L_n(D_1)\to L_{n-1}(D_0) \tag{14.6}$$

in the sequence is just multiplication by $1-w(t)=2$ ($w(t)=-1$ since $\mathbf{R}_-$ is non-oriented), and this is a rational isomorphism. Therefore, $L_n(A/B)=0$ by (14.5) and substituting back into (14.4), the $L_n(A)=0$. $\square$

Now a more precise result in a special case:

Lemma 14.7. *Let W be a complex 2–dimensional G-representation with $W^G = 0$. Then $L_3^s(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) = 0$*

Proof. We have an exact sequence

$$\cdots \to L_3^s(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z})) \to L_3^s(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to \\ L_3^I(\mathcal{C}^{>\bigcup \ell_\alpha\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to L_2^s(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z})) \to \dots$$

arising from the orbit type filtration. Here

$$\begin{aligned} I &= \operatorname{Im}(K_2(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to K_2(\mathcal{C}^{>\bigcup \ell_\alpha\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))) \\ &= \ker(K_2(\mathcal{C}^{>\bigcup \ell_\alpha\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to K_1(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z}))) \\ &= \ker(K_{-1}(\mathbf{Z}K) \xrightarrow{2} K_{-1}(\mathbf{Z}K)) \end{aligned}$$

by Proposition 8.7, where K is the isotropy subgroup of W. But $K_{-1}(\mathbf{Z}K)$ is torsion–free, so $I = \{0\} \subseteq K_2$. Now substitute the computations

$$\begin{aligned} L_3^s(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z})) &= L_1^p(\mathbf{Z}K) = 0 \\ L_3^I(\mathcal{C}^{>\bigcup \ell_\alpha\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) &= L_0^p(\mathbf{Z}K) \\ L_2^s(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z})) &= L_0^p(\mathbf{Z}K) \end{aligned}$$

into the exact sequence. The boundary map

$$L_3^I(\mathcal{C}^{>\bigcup \ell_\alpha\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to L_2^s(\mathcal{C}^{>\mathbf{R}_-}_{\bigcup \ell_\alpha\times\mathbf{R}_-,G}(\mathbf{Z}))$$

is multiplication by 2 so $L_3^s(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) = 0$. $\square$

Remark 14.8. The same method shows that $L_1^s(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) = L_3^p(\mathbf{Z}K) = \mathbf{Z}/2$ for $\dim W = 2$ as above, assuming that $K \neq 1$. $\square$

Our final preliminary result is a Mayer–Vietoris sequence:

Lemma 14.9. *Let W be a complex G–representation with $W^G = 0$. Let $W = W_1 \oplus W_2$ be a direct sum decomposition, where $W_1 = W_{max}$.*

(i) *There is a long exact sequence*

$$\cdots \to L_{n+1}^I(\mathcal{C}^{>W_1\times\mathbf{R}_-\cup W_2\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \xrightarrow{\partial_{n+1}} L_n^s(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to \\ L_n^s(\mathcal{C}^{>W_1\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \oplus L_n^s(\mathcal{C}^{>W_2\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to L_n^I(\mathcal{C}^{>W'\times\mathbf{R}_-\cup W''\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$$

of bounded L-groups, where

$$I = \operatorname{Im}\left[K_2(\mathcal{C}^{>W_2\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to K_2(\mathcal{C}^{>W'\times\mathbf{R}_-\cup W''\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))\right]$$

is the decoration subgroup.

(ii) *For $n \equiv 3 \pmod 4$, the boundary map $\partial_{n+1} = 0$.*

Proof. The Mayer–Vietoris sequence in $L^{\langle -\infty \rangle}$

$$\cdots \to L_{n+1}(\mathcal{C}^{>U\cup V}_{M,G}(\mathbf{Z})) \to L_n(\mathcal{C}^{>U\cap V}_{M,G}(\mathbf{Z})) \to \\ L_n(\mathcal{C}^{>U}_{M,G}(\mathbf{Z})) \oplus L_n(\mathcal{C}^{>V}_{M,G}(\mathbf{Z})) \to L_n(\mathcal{C}^{>U\cup V}_{M,G}(\mathbf{Z}))$$

where U, V are nice G–invariant subspaces of the control space M, follows from an excision isomorphism

$$L_n(\mathcal{C}^{>U}_{U\cup V,G}(\mathbf{Z})) = L_n(\mathcal{C}^{>U\cap V}_{V,G}(\mathbf{Z}))$$

and standard diagram-chasing. We apply this to $M = W \times \mathbf{R}_-$, $U = W_1 \times \mathbf{R}_-$ and $V = W_2 \times \mathbf{R}_-$, where $U \cap V = \mathbf{R}_-$. The decorations follow from Section 6 and Corollary 8.5.

To see that $\partial_0 = 0$, note that this boundary map is the composition of

$$\partial\colon L^I_0(\mathcal{C}^{>W'\times\mathbf{R}_-\cup W''\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to L^s_3(\mathcal{C}^{>W_2\times\mathbf{R}_-}_{W'\times\mathbf{R}_-\cup W''\times\mathbf{R}_-,G}(\mathbf{Z}))$$

and an excision isomorphism

$$L^s_3(\mathcal{C}^{>W_2\times\mathbf{R}_-}_{W'\times\mathbf{R}_-\cup W''\times\mathbf{R}_-,G}(\mathbf{Z})) \cong L^s_3(\mathcal{C}^{>\mathbf{R}_-}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))\ .$$

But $L^s_3(\mathcal{C}^{>\mathbf{R}_-}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})) = 0$ by Lemma 14.7. $\square$

Corollary 14.10. *Suppose W is a complex G–representation with no $\mathbf{R}_+$ summands. Then the image of $trf_{W\times\mathbf{R}_-}(\sigma)$ is zero in $L^s_{2k+1}(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$, where $\dim W = 2k$.*

Proof. First recall from Theorem 3.6 that $trf_{W\times\mathbf{R}_-}(\sigma) \in L^{c_1}_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ where $c_1 = \operatorname{Im}(\mathrm{Wh}(\mathbf{Z}G) \to \mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})))$ under the cone point inclusion. Therefore, the image of $trf_{W\times\mathbf{R}_-}(\sigma)$ "away from $\mathbf{R}_-$" lands in $L^s_{2k+1}(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$. We may therefore apply the Mayer–Vietoris sequence from Lemma 14.9.

As usual, we may assume that W has only 2–power isotropy subgroups, and we will argue by induction on $\dim W$ and on the orbit type filtration of W. The result is true for $\dim W = 2$ (by Lemma 14.7) or for W a free representation (by a direct calculation following the method of Lemma 14.3). For sets of 2–power index subgroups (ordered by inclusion, with repetitions allowed), we say that $\{K'_1,\dots,K'_s\} < \{K_1,\dots,K_t\}$ if $s < t$ or $s = t$, $K'_i = K_i$ for $1 \leqq i < j \leqq s$, and $K'_j \subsetneqq K_j$. Let $\operatorname{Iso}(W) = \{K_1,\dots,K_t\}$ denote the set of isotropy subgroups of W ordered by inclusion. Define an ordering by $W' < W$ if (i) $\dim W' < \dim W$ or (ii) $\dim W' = \dim W$ but $\operatorname{Iso}(W') < \operatorname{Iso}(W)$.

Let $W = W_1 \oplus W_2$, where W_1 has maximal isotropy subgroup K, and suppose the result holds for all $W' < W$. If $\dim W \equiv 2 \pmod 4$, we get an injection

$$L^s_3(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to L^s_3(\mathcal{C}^{>W_1\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \oplus L^s_3(\mathcal{C}^{>W_2\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$$

from Lemma 14.9, where $W_2 < W$ and $W_1 < W$. If $\dim W \equiv 0 \pmod 4$, we get an injection

$$L^s_3(\mathcal{C}^{>\mathbf{R}_-}_{W\times U\times\mathbf{R}_-,G}(\mathbf{Z})) \to L^s_3(\mathcal{C}^{>W_1\times\mathbf{R}_-}_{W\times U\times\mathbf{R}_-,G}(\mathbf{Z})) \oplus L^s_3(\mathcal{C}^{>W_2\times U\times\mathbf{R}_-}_{W\times U\times\mathbf{R}_-,G}(\mathbf{Z}))$$

again from Lemma 14.9, where U is any 2–dimensional free G–representation. Notice that $W_2 \oplus U < W$ in the ordering above. Consider the following commutative diagram (with $\dim W \equiv 2 \pmod 4$):

$$\begin{tikzcd}
L_0^h(\mathbf{Z}G) \arrow[r, "trf_{W_2\times\mathbf{R}_-}"] \arrow[dr, "trf_{W\times\mathbf{R}_-}"'] & L_1^{c_1}(\mathcal{C}_{W_2\times\mathbf{R}_-,G}(\mathbf{Z})) \arrow[r] \arrow[d, "trf_{W_1}"] & L_1^s(\mathcal{C}^{>\mathbf{R}_-}_{W_2\times\mathbf{R}_-,G}(\mathbf{Z})) \arrow[d, "trf_{W_1}"] \\
 & L_3^{c_3}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \arrow[r] & L_3^s(\mathcal{C}^{>W_1\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))
\end{tikzcd}$$

This diagram, and our inductive assumption shows that the image of $trf_{W\times\mathbf{R}_-}(\sigma)$ is zero in $L_3^s(\mathcal{C}^{>W_1\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$. Similarly, by reversing the roles of W_1 and W_2 in the diagram, we see that the image of $trf_{W\times\mathbf{R}_-}(\sigma)$ is zero in $L_3^s(\mathcal{C}^{>W_2\times\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$. Therefore the image of $trf_{W\times\mathbf{R}_-}(\sigma)$ is zero in $L_3^s(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$. If $\dim W \equiv 0 \pmod 4$, we replace W by $W \oplus U$ and make a similar argument, using the observation that $W_2 \oplus U < W$ to justify the inductive step. $\square$

Proof of Theorem 14.1. Part (i) is an immediate consequence of Proposition 5.6, so we consider part (ii). From Corollary 14.10 we know that the image of our obstruction $trf_{W\times\mathbf{R}_-}(\sigma)$ is zero "away from $\mathbf{R}_-$", so comes from a torsion element in $L^h_{2k+1}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ (by Lemma 14.3, $L_{2k+2}(\mathcal{C}^{>\mathbf{R}_-}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ is torsion). But $L^h_{2k+1}(\mathcal{C}^{>0}_{\mathbf{R}_-,G}(\mathbf{Z})) = L^p_{2k}(\mathbf{Z}H)$ is torsion–free (except for the Arf invariant $\mathbf{Z}/2$ which doesn't matter), hence the image of $trf_{W\times\mathbf{R}_-}(\sigma)$ vanishes in $L^h_{2k+1}(\mathcal{C}^{>0}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$.

The exact sequence

$$\cdots \to L^k_{2k+1}(\mathbf{Z}G, w) \to L^h_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to L^h_{2k+1}(\mathcal{C}^{>0}_{W\times\mathbf{R}_-,G}(\mathbf{Z})),$$

with $k = \ker(\widetilde{K}_0(\mathbf{Z}G) \to \widetilde{K}_0(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})))$, now shows that there exists a torsion element

$$\hat{\sigma} \in \operatorname{Im}(L^k_{2k+1}(\mathbf{Z}G, w) \to L^p_{2k+1}(\mathbf{Z}G, w))$$

with $c_*(\hat{\sigma}) = trf_{W\times\mathbf{R}_-}(\sigma) \in L^p_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$. $\square$

Remark 14.11. In Theorem 16.7 we use the sharper result $\hat{\sigma} \in L^k_{2k+1}(\mathbf{Z}G, w)$ to improve our unstable similarity results by removing extra $\mathbf{R}_-$ factors. Since $K_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to K_1(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})$ is injective by Corollary 8.9, the decoration $k = \operatorname{Im}(\widetilde{K}_0(\mathbf{Z}H) \to \widetilde{K}_0(\mathbf{Z}G))$. $\square$

15. The proof of Theorem C

By replacing V_i by $V_i \oplus U$ if necesary, where U is free and 2–dimensional, we may assume that $\dim V_i \oplus W \equiv 0 \pmod 4$. This uses the s–normal cobordism condition. By the top component argument (see Theorem 3.8) and Proposition 9.7, we may also assume that W has only 2–power isotropy. We have a commutative diagram analogous to (13.1).

$$\begin{tikzcd}
H^1(\widetilde{K}_0(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))) \arrow[dr] \arrow[rr, bend left, "\delta^2"] & & H^1(K_1(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \arrow[dr] \arrow[rr, bend left] & & L_0^s(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \\
& L_1^h(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \arrow[ur] \arrow[dr] & & H^1(\Delta_{W\times\mathbf{R}_-}) \arrow[ur] \arrow[dr] & \\
L_1^s(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \arrow[ur] \arrow[rr, bend right] & & L_1^p(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \arrow[ur] \arrow[rr, bend right] & & H^0(\widetilde{K}_0(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})))
\end{tikzcd} \tag{15.1}$$

where $H^1(\Delta_{W\times\mathbf{R}_-})$ denotes the relative group of the double coboundary map.

This diagram for W can be compared with the one for $W_1 = W_{max}$ via the inclusion maps, and we see that the K-theory terms map isomorphically by Lemma 8.2. Now by Theorem 14.1 there exists a torsion element

$$\hat{\sigma} \in \operatorname{Im}(L_1^p(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to L_1^p(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})))$$

which hits $trf_{W\times\mathbf{R}_-}$ under the cone point inclusion. In particular, $\hat{\sigma}$ vanishes "away from $\mathbf{R}_-$".

We have the exact sequence

$$L_2^s(\mathcal{C}^{>\mathbf{R}_-}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})) \to L_1^Y(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to L_1^s(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})) \to L_1^s(\mathcal{C}^{>\mathbf{R}_-}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))$$

and $Y = \ker(\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to \mathrm{Wh}(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))$ is zero, by Corollary 8.9. But $L_2^s(\mathcal{C}^{>\mathbf{R}_-}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))$ is a torsion group, by Lemma 14.3, and $L_1^Y(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) = L_1^s(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ is torsion–free by Lemma 12.1 so $L_1^Y(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \rightarrowtail L_1^s(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))$ is an injection.

Since $L_1^s(\mathcal{C}^{>\mathbf{R}_-}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})) = \mathbf{Z}/2$ by Remark 14.8 we conclude that the previous group $L_1^s(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))$ is torsion–free modulo this $\mathbf{Z}/2$, which injects into $L_1^{\langle -1\rangle}(\mathcal{C}^{>\mathbf{R}_-}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})) = \mathbf{Z}/2 \oplus H^1(K_{-1}(\mathbf{Z}K))$. Since $\hat{\sigma}$ vanishes away from $\mathbf{R}_-$, this extra $\mathbf{Z}/2$ may be ignored.

By substituting this computation into the exact sequence

$$L_1^s(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})) \to L_1^p(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})) \to H^1(\Delta_{W_1\times\mathbf{R}_-})$$

of (13.1), we see that the torsion subgroup of

$$\ker(L_1^p(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})) \to L_1^p(\mathcal{C}^{>\mathbf{R}_-}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z})))$$

injects into the relative group $H^1(\Delta_{W_1\times\mathbf{R}_-})$ of the double coboundary. Therefore $\hat{\sigma} \in L_1^p(\mathcal{C}_{W_1\times\mathbf{R}_-,G}(\mathbf{Z}))$ vanishes if and only if the element $\{\Delta(V_1)/\Delta(V_2)\}$ is in the image of the double coboundary

$$\delta^2\colon H^1(\widetilde{K}_0(\mathcal{C}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z}))) \to H^1(\mathrm{Wh}(\mathcal{C}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z}))) \quad \square$$

A very similar argument can be used to give an inductive criterion for non–linear similarity without the $\mathbf{R}_+$ summand (generalizing Theorem A). In the statement we will use the analogue to $\mathbf{k}$, namely $\mathbf{k}_W = \ker(\mathrm{Wh}(\mathcal{C}^{\geq 0}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \to \widetilde{K}_0(\mathbf{Z}G))$. It follows from our K–theory calculations that $\mathbf{k}_W = \mathbf{k}$ when $K_{-1}(\mathbf{Z}K) = 0$ for $K \in \mathrm{Iso}(W)$.

Theorem 15.2. *Let $V_1 = t^{a_1} + \cdots + t^{a_k}$ and $V_2 = t^{b_1} + \cdots + t^{b_k}$ be free G–representations. Let W be a complex G–representation with no $\mathbf{R}_+$ summands. Then there exists a topological similarity $V_1 \oplus W \oplus \mathbf{R}_- \sim_t V_2 \oplus W \oplus \mathbf{R}_-$ if and only if*

(i) *$S(V_1)$ is s–normally cobordant to $S(V_2)$,*
(ii) *$\mathrm{Res}_H(V_1 \oplus W) \oplus \mathbf{R}_+ \sim_t \mathrm{Res}_H(V_2 \oplus W) \oplus \mathbf{R}_+$, and*
(iii) *the element $\{\Delta(V_1)/\Delta(V_2)\}$ is in the image of the coboundary*

$$\delta\colon H^0(\mathbf{k}_{W_{max}}) \to H^1(\mathrm{Wh}(\mathcal{C}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z})))\ ,$$

where $0 \subseteq W_{max} \subseteq W$ is a 2–dimensional complex subrepresentation with maximal isotropy group among the isotropy groups of W with 2-power index.

Proof. We leave the details to the reader. □

16. Iterated $\mathbf{R}_-$ transfers

We now apply Theorem 7.10 to show that iterated $\mathbf{R}_-$ transfers don't lead to any new similarities.

Theorem 16.1. *Suppose that W is a complex G–representation with no $\mathbf{R}_+$ summands. Then $V_1 \oplus W \oplus \mathbf{R}^l_- \oplus \mathbf{R}_+ \sim_t V_2 \oplus W \oplus \mathbf{R}^l_- \oplus \mathbf{R}_+$ for $l \geqq 1$ implies $V_1 \oplus W \oplus \mathbf{R}_- \oplus \mathbf{R}_+ \sim_t V_2 \oplus W \oplus \mathbf{R}_- \oplus \mathbf{R}_+$.*

The first step in the proof is to show injectivity of certain transfer maps. For any homomorphism $w\colon G \to \{\pm 1\}$, we will use the notation $(\mathcal{C}_{W,G}(\mathbf{Z}), w)$ to denote the antistructure where the involution is $g \mapsto w(g)g^{-1}$ at the cone point. The standard orientation (6.4) has $w = \det(\rho_W)$, but we will need others in this section. Let $\phi = \det(\rho_{\mathbf{R}_-})$ for short, and notice that $\phi\colon G \to \{\pm 1\}$ is non–trivial.

Lemma 16.2. *The transfer map $trf_{\mathbf{R}_-}\colon L^p_{2k+1}(\mathbf{Z}G, w) \to L^p_{2k+2}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w\phi)$ is injective, where w is the non–trivial orientation.*

Proof. Since $L^p_{2k+2}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w) \cong L^p_{2k+2}(\mathbf{Z}H \to \mathbf{Z}G)$, we just have to check that the previous map in (7.6) is zero (using the tables in [20]). □

Next, a similar result for the $\mathbf{R}^2_-$ transfer.

Proposition 16.3. *The bounded transfer*

$$trf_{\mathbf{R}^2_-}\colon L^p_{2k+1}(\mathbf{Z}G, w) \to L^p_{2k+3}(\mathcal{C}_{\mathbf{R}^2_-,G}(\mathbf{Z}), w)$$

is injective, where w is the non–trivial orientation.

Proof. We can relate the iterated $\mathbf{R}_-$ transfer $trf_1 \circ trf_1$ to $trf_2 = trf_{\mathbf{R}^2_-}$ by means of the braid diagram:

$$
\begin{tikzcd}[column sep=small]
L_n(\mathbf{Z}G, w) \arrow[dr, "trf_1"] \arrow[rr, bend left, "trf_2"] & & L_{n+2}(\mathbf{Z}\Gamma_H \to \mathbf{Z}G, w) \arrow[dr] \arrow[rr, bend left] & & LNS_{n+3}(\Phi) \\
& L_{n+1}(\mathbf{Z}H \to \mathbf{Z}G, w\phi) \arrow[ur, "trf_1"] \arrow[dr] & & LS_{n+3}(\Phi) \arrow[ur] \arrow[dr] & \\
LNS_n(\Phi) \arrow[ur] \arrow[rr, bend right] & & LN_{n+3}(\mathbf{Z}H \to \mathbf{Z}G, w\phi) \arrow[ur] \arrow[rr, bend right] & & L_{n+3}(\mathbf{Z}G, w)
\end{tikzcd}
\tag{16.4}
$$

where the new groups $LNS_*(\Phi)$ are the relative groups of the transfer

$$trf_1 = trf_{\mathbf{R}_-} \colon L^p_k(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w\phi) \to L^p_{k+1}(\mathcal{C}_{\mathbf{R}^2_-,G}(\mathbf{Z}), w) \ .$$

The diagram Φ of groups (as in [38,Chap.11]) contains $\Gamma_H \cong \mathbf{Z} \times H$ as the fundamental group of $S(\mathbf{R}_- \oplus \mathbf{R}_-)/G$ (see [19,5.9]).

The groups $LNS_*(\Phi)$ have a geometric bordism description in terms of triples $L^{n-2} \subset N^{n-1} \subset M^n$ where each manifold has fundamental group G, N is characteristic in M, and L is characteristic in N, with respect to the index two inclusion $H < G$. There is also an algebraic description for $LNS_*(\Phi)$ in terms of "twisted antistructures" [22] as for the other groups $LN(\mathbf{Z}H \to \mathbf{Z}G, w\phi)$ [38,12C] and for $LS(\Phi, w)$ [27,7.8.12]. Substituting these descriptions into the braid gives:

$$
\begin{tikzcd}[column sep=small]
L_n(\mathbf{Z}G, \alpha, u) \arrow[dr, "trf_1"] \arrow[rr, bend left, "trf_2"] & & L_n(\mathbf{Z}\Gamma_H \to \mathbf{Z}G, \tilde\alpha, \tilde u) \arrow[dr] \arrow[rr, bend left] & & L_{n-1}(\mathbf{Z}H \to \mathbf{Z}\Gamma_H, \tilde\alpha, \tilde u) \\
& L_n(\mathbf{Z}H \to \mathbf{Z}G, \alpha, u) \arrow[ur, "trf_1"] \arrow[dr] & & L_{n-1}(\mathbf{Z}\Gamma_H, \tilde\alpha, \tilde u) \arrow[ur] \arrow[dr] & \\
L_n(\mathbf{Z}H \to \mathbf{Z}\Gamma_H, \tilde\alpha, \tilde u) \arrow[ur] \arrow[rr, bend right] & & L_{n-1}(\mathbf{Z}H, \alpha, u) \arrow[ur] \arrow[rr, bend right] & & L_{n-1}(\mathbf{Z}G, \alpha, u)
\end{tikzcd}
\tag{16.5}
$$

The antistructure $(\mathbf{Z}G, \alpha, u)$ is the twisted antistructure obtained by scaling with an element $a \in G - H$. The antistructure $\mathbf{Z}\Gamma_H, \tilde\alpha, \tilde u)$ is the one defined by Ranicki [27,p.805], then scaled by $\tilde a \in \Gamma_H$, where $\tilde a$ maps to a under the projection $\Gamma_H \to G$. Since $\Gamma_H \cong \mathbf{Z} \times H$, we have an exact sequence

$$\to L^p_n(\mathbf{Z}H, \alpha, u) \to L^p_n(\mathbf{Z}\Gamma_H, \tilde\alpha, \tilde u) \to L^{\langle -1 \rangle}_{n-1}(\mathbf{Z}H, \alpha, u) \xrightarrow{1-w(a)} L^p_{n-1}(\mathbf{Z}H, \alpha, u)$$

and it follows that $L^p_n(\mathbf{Z}H \to \mathbf{Z}\Gamma_H, \tilde{\alpha}, \tilde{u}) \cong L^{\langle -1\rangle}_{n-1}(\mathbf{Z}H, \alpha, u)$. It is not difficult to see that $L^{\langle -1\rangle}_{2k}(\mathbf{Z}H, \alpha, u)$ is torsion–free using the tables in [20].

Now observe that $L^p_n(\mathbf{Z}G, \alpha, u) \cong L^p_n(\mathbf{Z}G, w)$, and

$$L^p_1(\mathbf{Z}H \to \mathbf{Z}G, \alpha, u) \cong L^p_2(\mathbf{Z}H \to \mathbf{Z}G) \ .$$

The transfer $trf_{\mathbf{R}_-} \colon L^p_1(\mathbf{Z}G, w) \to L^p_2(\mathbf{Z}H \to \mathbf{Z}G)$ is injective by Lemma 16.2, and since $w(a) = -1$, the map

$$L^{\langle -1\rangle}_0(\mathbf{Z}H, \alpha, u) \xrightarrow{1-w(a)} L^p_0(\mathbf{Z}H, \alpha, u)$$

is also injective. Since $L^p_1(\mathbf{Z}G, w)$ is 2–torsion, it must inject into $L^p_1(\mathbf{Z}\Gamma_H \to \mathbf{Z}G, \tilde{\alpha}, \tilde{u}) \cong L^p_3(\mathcal{C}_{\mathbf{R}^2_-,G}(\mathbf{Z}), w)$. $\square$

The final step in the proof is to consider the following commutative square of spectra:

$$\begin{tikzcd}
\mathbb{L}^p(\mathcal{C}_{pt}(\mathbf{Z}G), w) \arrow[r, "c_*"] \arrow[d, "trf_{\mathbf{R}_-}"'] & \mathbb{L}^p(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}), w) \arrow[d, "trf_{\mathbf{R}_-}"] \\
\mathbb{L}^p(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w\phi) \arrow[r, "c_*"] & \mathbb{L}^p(\mathcal{C}_{W\times\mathbf{R}^2_-,G}(\mathbf{Z}), w\phi)
\end{tikzcd}$$

for $l \geq 1$.

Lemma 16.6. *This is a pull–back square.*

Proof. The fibres of the vertical $\mathbf{R}_-$ transfers are

$$\mathbb{LN}^p(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w\phi) \cong \mathbb{L}^p(\mathcal{C}_{pt}(\mathbf{Z}H, \alpha, u)) \ ,$$

and $\mathbb{LN}^p(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}), w\phi) \cong \mathbb{L}^p(\mathcal{C}_{W,H}(\mathbf{Z}), \alpha, u)$ respectively by (7.6). The fibre of the cone point inclusion

$$\mathbb{L}^p(\mathcal{C}_{pt}(\mathbf{Z}H), \alpha, u) \to \mathbb{L}^p(\mathcal{C}_{W,H}(\mathbf{Z}), \alpha, u)$$

is $\mathbb{L}^h(\mathcal{C}^{>0}_{W,H}(\mathbf{Z}), \alpha, u)$, since $K_0(\mathcal{C}^{>0}_{W,H}(\mathbf{Z})) \to K_{-1}(\mathcal{C}_{pt}(\mathbf{Z}H)$ is injective. But Theorem 7.10 shows that $\mathbb{L}^h(\mathcal{C}^{>0}_{W,H}(\mathbf{Z}), \alpha, u)$ is contractible, and therefore the cone point inclusion is a homotopy equivalence. It follows that the fibres of the horizontal maps are also homotopy equivalent. $\square$

Proof of Theorem 16.1. It is enough to prove that $trf_{W\times\mathbf{R}^{l+1}_-}(\sigma) = 0$. for $l \geqq 1$, implies $trf_{W\times\mathbf{R}_-}(\sigma) = 0$ in the top component of $L^p_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$. We may assume that $\mathrm{Iso}(W)$ only contains subgroups of 2–power index, and let $\dim W = 2k$. Now by Theorem C, $trf_{W\times\mathbf{R}^{2t+1}_-}(\sigma) = 0$ implies $trf_{W\times\mathbf{R}^{2t-1}_-}(\sigma) = 0$, provided that $t \geqq 2$. It therefore remains to study $l = 1$ and $l = 2$.

We have the commutative diagram:

$$\begin{tikzcd}
L^p_{2k+1}(\mathbf{Z}G, w) \arrow[r, "c_*"] \arrow[d, "trf_{\mathbf{R}^l_-}"'] & L^p_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}), w) \arrow[d, "trf_{\mathbf{R}^l_-}"] \\
L^p_{2k+l+1}(\mathcal{C}_{\mathbf{R}^l_-,G}(\mathbf{Z}), w\phi^l) \arrow[r, "c_*"] & L^p_{2k+l+1}(\mathcal{C}_{W\times\mathbf{R}^{l+1}_-,G}(\mathbf{Z}), w\phi^l)
\end{tikzcd}$$

In the lower row, the orientation at the cone point is non–trivial for l even, and trivial for l odd. Suppose first that $l = 1$. Then, the transfer map

$$trf_{\mathbf{R}_-}: L^p_{2k+1}(\mathbf{Z}G, w) \to L^p_{2k+2}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w\phi) = L^p_{2k+2}(\mathbf{Z}H \to \mathbf{Z}G)$$

is injective by Lemma 16.2. Since $trf_{W\times\mathbf{R}^l_-}(\sigma) = c_*(\hat{\sigma})$ for some $\hat{\sigma} \in L^p_{2k+1}(\mathbf{Z}G, w)$, we are done by a diagram chase (using Lemma 16.6 to see that the horizontal maps sit in a commutative ladder with isomorphisms every third term).

If $l = 2$, we use the fact that $trf_{\mathbf{R}^2_-}: L^p_{2k+1}(\mathbf{Z}G, w) \to L^p_{2k+3}(\mathcal{C}_{\mathbf{R}^2_-,G}(\mathbf{Z}), w)$ is injective, from Proposition 16.3. A similar diagram chase completes the proof. □

We also have a version without $\mathbf{R}_+$ summands.

Theorem 16.7. *Suppose that W is a complex G–representation with $W^G = 0$. Then $V_1 \oplus W \oplus \mathbf{R}^l_- \sim_t V_2 \oplus W \oplus \mathbf{R}^l_-$ for $l \geqq 1$ implies $V_1 \oplus W \oplus \mathbf{R}_- \sim_t V_2 \oplus W \oplus \mathbf{R}_-$.*

Proof. This follows from a similar argument, using Theorem 15.2 instead of Theorem C. The injectivity results of Lemma 16.2 and Proposition 16.3 also hold for trf_1 and trf_2 on $L^k_{2k+1}(\mathbf{Z}G, w)$, where $k = \mathrm{Im}(\widetilde{K}_0(\mathbf{Z}H) \to \widetilde{K}_0(\mathbf{Z}G))$ as in Remark 14.11. The details are left to the reader. □

17. The proof of Corollary 2.4.

Suppose that $G = C(4q)$, q odd, and that $V_1 \oplus \mathbf{R}_- \oplus \mathbf{R}_+ \sim_t V_2 \oplus \mathbf{R}_- \oplus \mathbf{R}_+$ with $\dim V_i = 4$. We will use Theorem B and the assumption about odd class numbers to proved that $V_1 \cong V_2$. This is the only case we need to discuss to prove Corollary 2.4. If we started instead with $V_1 \oplus \mathbf{R}^2_- \sim_t V_2 \oplus \mathbf{R}^2_-$, then stabilizing with $\mathbf{R}_+$ and applying Theorem 16.1 would reduce to the case above.

We may assume that $q > 1$ is a prime, and that the free representations have the form $V_1 = t + t^i$ and $V_2 = t^{1+2q} + t^{i+2q}$ where $(i, 2q) = 1$. The Reidemeister torsion invariant is $\Delta(V_1)/\Delta(V_2) = U_{1,i} \in \mathrm{Wh}(\mathbf{Z}G)$ in the notation for units of $\mathbf{Z}G$ used in [7,p.732]. With respect to the involution $-: t \mapsto -t^{-1}$, this element defines a class $u_{1,i} = \{U_{1,i}\} \in H^1(\mathrm{Wh}(\mathbf{Z}G^-))$. Since the map $L^p_1(\mathbf{Z}G^-) \to L^p_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ is injective, it is enough to show that the image of $u_{1,i}$ is non–trivial in $H^1(\Delta)$ (see diagram (11.6)).

Let $A = \mathbf{Z}[\zeta_{4q}]$ be the ring of cyclotomic integers., and $B = \mathbf{Z}[\zeta_{2q}]$. We will study the top component $\mathrm{Wh}(\mathbf{Z}G)(q)$ by comparing it to $K_1(\mathcal{M})$, where $\mathcal{M} = A \times B \times B$ is the top component of an involution invariant maximal order in $\mathbf{Q}G$ containing $\mathbf{Z}G$. Notice that the two copies of B are interchanged under the involution, so $H^i(K_1(\mathcal{M})) = H^i(A^\times)$. We have the exact sequence

$$0 \to \mathrm{Wh}(\mathbf{Z}G)(q) \to K_1(\mathcal{M}) \to K_1(\widehat{\mathcal{M}})/\,\mathrm{Wh}(\widehat{\mathbf{Z}}G)(q) \to D(\mathbf{Z}G)(q) \to 0\ .$$

where $D(\mathbf{Z}G) = \ker(\widetilde{K}_0(\mathbf{Z}G) \to \widetilde{K}_0(\mathcal{M}))$. Note that $H^i(D(\mathbf{Z}G)) = H^i(\widetilde{K}_0(\mathbf{Z}G))$ since A has odd class number.

Let $i_*: H^1(\mathrm{Wh}(\mathbf{Z}G^-)(q)) \to H^1(A^\times)$ denote the map on the Tate cohomology induced by the inclusion $i: \mathrm{Wh}(\mathbf{Z}G)(q) \to A^\times$.

Lemma 17.1. *For $(j, 2q) = 1$ and $j \equiv 1 \pmod 4$, the image $i_*(u_{1,j}) = \langle -1 \rangle \in H^1(A^\times)$.*

Proof. Let $\gamma_j = \frac{t^j - 1}{t-1} \in A^\times$, and compute $\bar\gamma_j/\gamma_j$ with respect to the non–oriented involution. We get $\bar\gamma_j/\gamma_j = u_{1,j}/u_{1,1}$, so $i_*(u_{1,j}) = i_*(u_{1,1})$ for all i. Now let $v = \frac{t+1}{t-1}$. Since $t = -t^{1+2q}$, this is a cyclotomic unit in A with $v\bar v = -1$. But $v/\bar v = -u_{1,1}$, so $i_*(u_{1,j}) = \langle -1 \rangle$ in $H^1(A^\times)$. $\square$

Next we need a computation:

Lemma 17.2. $0 \neq \langle -1 \rangle \in H^1(A^\times)$.

Proof. Let $E = \mathbf{Q}(\zeta_{4q})$ be the quotient field of A. Since A has odd class number, $H^i(E^\times/A^\times) \cong H^i(\hat E^\times/\hat A^\times)$. Next, observe that the extension E/F is evenly ramified at q, where F is the fixed field of the involution $\bar{\ }\colon E \to E$. It follows that the map $H^0(\hat E_q^\times) \to H^0(\hat E_q^\times/\hat A_q^\times)$ is zero. We finish by considering the commutative diagram

$$\begin{tikzcd}
H^0(E^\times) \arrow[r] \arrow[d] & H^0(E^\times/A^\times) \arrow[r, "\delta"] \arrow[d, "\approx"] & H^1(A^\times) \\
H^0(\hat E^\times) \arrow[r] & H^0(\hat E^\times/\hat A^\times) &
\end{tikzcd}$$

and the prime element $w = \zeta_q - \zeta_q^{-1} \in E$ over q. Since $\bar w/w = -1$ it defines an element in $H^0(E^\times/A^\times)$ and $\delta(w) = \langle -1 \rangle \in H^1(A^\times)$. On the other hand, w maps under the middle isomorphism to a non–zero element in $H^0(\hat E_q^\times/\hat A_q^\times)$ so $\delta(w) \neq 0$ $\square$

Now we can complete the proof of Corollary 2.4 by considering the commutative braid:

$$\begin{tikzcd}
H^0(A^\times) \arrow[rr, bend left] \arrow[dr] & & H^0\left(\frac{K_1(\widehat{\mathcal{M}})}{\mathrm{Wh}(\widehat{\mathbf{Z}}G)(q)}\right) \arrow[rr, bend left] \arrow[dr] & & H^0(\widetilde{K}_0(\mathbf{Z}G)(q)) \\
& H^0\left(\frac{K_1(\mathcal{M})}{\mathrm{Wh}(\mathbf{Z}G)(q)}\right) \arrow[ur] \arrow[dr] & & H^1(\Delta) \arrow[ur] \arrow[dr] & \\
H^1(\widetilde{K}_0(\mathbf{Z}G)(q)) \arrow[ur] \arrow[rr, bend right, "\delta^2"'] & & H^1(\mathrm{Wh}(\mathbf{Z}G^-)(q)) \arrow[ur] \arrow[rr, bend right, "i_*"'] & & H^1(A^\times)
\end{tikzcd} \tag{17.3}$$

containing the double coboundary and its relative group $H^1(\Delta)$. By commutativity of the lower right triangle, the image of our obstruction element $u_{1,j}$ is non–zero in $H^1(A^\times)$ and therefore non–zero in $H^1(\Delta)$. $\square$

18. The normal invariant.

In this section we collect some results about normal cobordisms of lens spaces with cyclic fundamental group G. In [6,§1] it is asserted that their formula (A') gives necessary and sufficient conditions for homotopy equivalent lens spaces to be s–normally cobordant, when $G = C(2^r)$. However in [40] it is shown that the given conditions (A') are sufficient but not necessary. We only use the sufficiency here.

We want to study the subgroups $\widetilde{R}^{free}(G)$ and $\widetilde{R}_h^{free}(G)$ of $R(G)$.

Lemma 18.1. *Let $G = C(2^r q)$ be a finite cyclic group, and G_2 be the 2–Sylow subgroup. If $q > 1$ there is an exact sequence*

$$0 \to \mathbf{Z}/2 \to \widetilde{R}^{free}(G)/\widetilde{R}_h^{free}(G) \to \widetilde{R}^{free}(G_2)/\widetilde{R}_h^{free}(G_2) \to 0$$

given by the restriction map Res_{G_2}. The kernel is generated by any element $\alpha \in \widetilde{R}^{free}(G)$ with k–invariant $k(\alpha) \in (\mathbf{Z}/|G|)^\times/\{\pm 1\}$ in the coset $k \equiv 1 \,(\mathrm{mod}\, 2^r)$ and $k \equiv -1 \,(\mathrm{mod}\, q)$.

Proof. There is a short exact sequence

$$0 \to R_h^{free}(G) \to \widetilde{R}^{free}(G) \to (cy2^r q)^\times/\{\pm 1\}$$

given by the k–invariant. Moreover, the k–invariants of elements in $\widetilde{R}^{free}(G)$ lie in

$$\ker\bigl((\mathbf{Z}/2^r q)^\times/\{\pm 1\} \to (\mathbf{Z}/q)^\times/\{\pm 1\}\bigr) \cong \{lq + 1 \mid l \in \mathbf{Z}\} \subseteq (\mathbf{Z}/2^r q)^\times$$

which injects under reduction $(\mathrm{mod}\, 2^r)$ into $(\mathbf{Z}/2^r)^\times$ if $q > 1$. Restriction of the k–invariant to G_2 detects only its value in $(\mathbf{Z}/2^r)^\times/\{\pm 1\}$, so the kernel has order 2. Since the restriction map $\tilde{R}^{free}(G) \to \tilde{R}^{free}(G_2)$ is surjective, we have the required exact sequence. $\square$

The situation for the normal invariant is simpler. Recall that we write $V \doteq V'$ or $(V - V') \doteq 0$ if there exists a homotopy equivalence $f\colon S(V)/G \to S(V')/G$ of lens spaces, such that f is s–normally cobordant to the identity.

Lemma 18.2. *Let $G = C(2^r q)$ be a finite cyclic group. Then*

$$\ker\bigl(R_h^{free}(G) \to R_h^{free}(C(q)) \oplus R_h^{free}(C(2^r))\bigr) \subseteq R_n^{free}(G)\ .$$

Proof. We may assume that $r \geq 2$, since $\mathrm{Res}\colon R^{free}(C(2q)) \to R^{free}(C(q))$ is an isomorphism, and consider an element

$$\sum t^{a_i} - \sum t^{b_i} \in \ker\bigl(R^{free}(G) \to R^{free}(C(q)) \oplus R^{free}(C(2^r))\bigr).$$

With a suitable ordering of the indices, $a_i \equiv b_i \,(\mathrm{mod}\, 2^r)$ and $a_i \equiv b_{\tau(i)} \,(\mathrm{mod}\, q)$ for some permutation τ. Since the normal invariants of lens spaces are detected by a cohomology theory, we can check the condition at each Sylow subgroup separately. It follows that $\sum t^{a_i} \doteq \sum t^{b_i}$. $\square$

Example 18.3. For $G = C(24)$ we have $\widetilde{R}^{free}(G) = \{t - t^5, t - t^7, t - t^{11}\}$ and $\ker \operatorname{Res}_{C(8)} \cap \widetilde{R}^{free}(G) = \{t - t^7, t^5 - t^{11}\}$. □

Lemma 18.4. *Let $G = C(2^r q)$ be a finite cyclic group, and G_2 be the 2–Sylow subgroup. If $r \geq 2$ there is an isomorphism*

$$\widetilde{R}_h^{free}(G)/\widetilde{R}_n^{free}(G)) \to \widetilde{R}_h^{free}(G_2)/\widetilde{R}_n^{free}(G_2)$$

given by the restriction map Res_{G_2}.

Proof. We first remark that if $W = t^{a_1} + \cdots + t^{a_n}$ is a free G_2–representation (here we use the assumption that $r \geq 2$), then by choosing integers $b_i \equiv a_i \pmod{2^r}$ and $b_i \equiv 1 \pmod q$ we obtain a free G–representation $V = t^{b_1} + \cdots + t^{b_n}$ with $\operatorname{Res}_{G_2}(V) = W$. It follows that

$$\operatorname{Res}_{G_2} \colon \widetilde{R}_h^{free}(G)/\widetilde{R}_n^{free}(G)) \to \widetilde{R}_h^{free}(G_2)/\widetilde{R}_n^{free}(G_2)$$

is surjective.

Now suppose $\alpha \in \widetilde{R}_h^{free}(G)$ and $\operatorname{Res}_{G_2}(\alpha) \in R_n^{free}(G_2)$. This means that

$$\operatorname{Res}_{G_2}(\alpha) = \sum \beta_i$$

where each $\beta_i = W_i - W_i'$ for some free G_2–representations W_i, W_i' such that $W_i \simeq W_i'$. By the construction of the last paragraph, we can find free G–representations V_i, V_i' such that (i) $\operatorname{Res}_{G_2}(V_i) = W_i$ and $\operatorname{Res}_{G_2}(V_i') = W_i'$, and (ii) $(V_i - V_i') \in \widetilde{R}_h^{free}(G)$. It follows that $V_i \simeq V_i'$ and so $\alpha' = \sum \alpha_i \in \widetilde{R}_n^{free}(G))$, and

$$\alpha - \alpha' \in \ker\big(R_h^{free}(G) \to R_h^{free}(C(q)) \oplus R_h^{free}(C(2^r))\big).$$

By Lemma 18.2, $\alpha - \alpha' \in R_n^{free}(G)$. □

Our final result is a step towards determining $R_h(G)/R_n(G)$ more explicitly.

Lemma 18.5. *Let $b_{i,s} = (t^i - t^{2^{r-s}q-i}) \in \widetilde{R}^{free}(G)$ for $1 \leq i < 2^{r-s-1}q$, $1 \leq s \leq r-1$, and $(i, 2q) = 1$. Let $l(i,s) = s$ for $1 \leq s \leq r-2$, and $l(i, r-1)$ the order of $i/(2q-i) \in (\mathbf{Z}/2^r)^\times$. Then $b_{i,s}$ is an element of order $2^{l(i,s)}$ in $R(G)/R_h(G)$, and $2^{l(i,s)} \cdot b_{i,s} \simeq 0$.*

Proof. If $1 \leq s \leq r-2$ then $k(b_{i,s}) = i/(2^{r-s}q - i) \in (\mathbf{Z}/2^r q)^\times/\{\pm 1\}$ is congruent to $-1 \pmod{4q}$. By Lemma 18.1 the linear span of these elements in $R(G)/R_h(G)$ injects into $R(G_2)/R_h(G_2)$, where G_2 is the 2–Sylow subgroup. By Lemma 18.4 the normal invariant is also detected by restriction to G_2, so it is enough to prove the assertions about these elements ($s \leq r-2$) when $G = C(2^r)$.

For the first part we must show that the k–invariant of $b_{i,s}$ has order 2^s. We will use the expression

$$\nu_2(r!) = r - \alpha_2(r),$$

where $\alpha_2(r)$ is the number of non-zero coefficients in the 2-adic expansion of r, for the 2-adic valuation of r. Then

$$\nu_2(\binom{2^s}{k}) = s - \nu_2(k)$$

for $1 \le k \le 2^s$. These formulas and the binomial expansion show that $k(b_{i,s})^{2^s} \equiv 1 \,(\mathrm{mod}\, 2^r)$ for $1 \le s \le r-2$.

Next we consider the normal invariant. For $1 \le s \le r-2$ we will take $q = 1$ and apply the criterion (A') of [6] to show that $2^s \cdot b_{i,s} \approx 0$. This amounts to a re-labelling of our original elements $b_{i,s}$ without changing the order of their k–invariants.

We must now compute the elementary symmetric functions $\sigma_k(2^s \cdot i^2)$ and $\sigma_k(2^s \cdot (2^{r-s} - i)^2)$, where the notation $2^s \cdot i^2$ means that the weight i^2 is repeated 2^s times in the symmetric function. The formula (A') is

$$\sigma_k(2^s \cdot (2^{r-s} + i)^2) - \sigma_k(2^s \cdot i^2) \equiv 2\big((2^{r-s} - i)^{2^s} - i^{2^s}\big)\binom{2^s - 1}{k-1} \,(\mathrm{mod}\, 2^{r+3})$$

so on the right-hand side we have

$$\begin{cases} +2^{r+1} \,(\mathrm{mod}\, 2^{r+3}) & \text{if } k \text{ odd} \\ -2^{r+1} \,(\mathrm{mod}\, 2^{r+3}) & \text{if } k \text{ even} \end{cases}$$

The formula (A') assumes that the weights are congruent to $1 \,(\mathrm{mod}\, 4)$, or in our case $i \equiv 1 \pmod 4$. Therefore, if $i \equiv 3 \pmod 4$, we must use the equivalent weights $-i$ and $-(2^{r-s} - i)$.

To compute the left-hand side we use the Newton polynomials s_k and their expressions in term of elementary symmetric functions. We need the property $s_k(2^s \cdot i^2) = 2^s i^{2k}$ and the coefficient of σ_k in s_k which is $(-1)^{k+1}k$. By induction, we see that the left-hand side is just

$$\frac{(-1)^{k+1}}{k}\big(s_k(2^s \cdot (2^{r-s} - i)^2) - s_k(2^s \cdot i^2)\big) \equiv \frac{(-1)^{k+1}2^s}{k}\big((2^{r-s} - i)^{2k} - i^{2k}\big) \pmod{2^{r+3}}$$

But by writing

$$(2^{r-s} - i)^{2k} - i^{2k} = (2^{r-s}\theta + i^2)^k - i^{2k}$$

where $\theta = 2^{r-s} - 2i \equiv 2 \,(\mathrm{mod}\, 4)$, our expression becomes

$$\frac{(-1)^{k+1}2^s}{k}(2^{r-s}k\theta) \equiv (-1)^{k+1}2^{r+1} \,(\mathrm{mod}\, 2^{r+3})$$

If $s = r-1$, then $k(b_{i,s}) = i/(2q - i) \equiv -1 \,(\mathrm{mod}\, q)$, and $k(b_{i,s}) \equiv 1 \,(\mathrm{mod}\, 2^r)$ whenever $i \equiv q \,(\mathrm{mod}\, 2^{r-1})$. This is for example always the case for $G = C(4q)$. Such elements $b_{i,r-1}$ lie in the kernel of the restriction map to $R(C(2^r))$. Moreover, if $b_{i,r-1} \in \ker\big(R(G) \to R(C(2^r))\big)$, then $2b_{i,r-1} \in R_h(G)$ and $2b_{i,r-1} \doteq 0$ by Lemma 18.2. Otherwise, the order of $b_{i,r-1} \in R(G)/R_h(G)$ is the same as the order of its restriction $\mathrm{Res}(b_{i,r-1})$ to the 2–Sylow subgroup, and $\mathrm{Res}(b_{i,r-1}) = \pm\,\mathrm{Res}(b_{j,s})$ for $l(i, r-1) = s \le r-2$ and some j. $\square$

19. Cyclic 2–groups.

For $G = C(2^r)$ a cyclic 2–group, we have stronger results because $K_{-1}(\mathbf{Z}G) = 0$.

Theorem 19.1. *Let $V_1 = t^{a_1} + \cdots + t^{a_k}$ and $V_2 = t^{b_1} + \cdots + t^{b_k}$ be free G–representations, where $G = C(2^r)$. Let W be a complex G–representation with no $\mathbf{R}_+$ or $\mathbf{R}_-$ summands. If $\operatorname{Res}_H V_1 \cong \operatorname{Res}_H V_2$ then $V_1 \oplus W \oplus \mathbf{R}_- \sim_t V_2 \oplus W \oplus \mathbf{R}_-$ implies $V_1 \cong V_2$.*

Proof. We will show that the last criterion of Theorem 15.2 can not be fulfilled unless $V_1 \cong V_2$. First, since $G = C(2^r)$ the obstruction map δ in Theorem 15.2 is just the coboundary $\delta\colon H^0(\mathbf{k}) \to H^1(\operatorname{Wh}(\mathbf{Z}G^-)/\operatorname{Wh}(\mathbf{Z}H))$ used in Theorem A. Since $\operatorname{Res}_H V_1 \cong \operatorname{Res}_H V_2$, our Reidemeister torsion element $\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\operatorname{Wh}(\mathbf{Z}G^-))$, and therefore it vanishes under the composite

$$H^1(\operatorname{Wh}(\mathbf{Z}G^-)) \to H^1(\operatorname{Wh}(\mathbf{Z}G^-)/\operatorname{Wh}(\mathbf{Z}H)) \to H^0(\operatorname{Wh}(\mathbf{Z}H)) .$$

Next, we claim that the composite

$$H^0(\mathbf{k}) \xrightarrow{\delta} H^1(\operatorname{Wh}(\mathbf{Z}G^-)/\operatorname{Wh}(\mathbf{Z}H)) \to H^0(\operatorname{Wh}(\mathbf{Z}H))$$

is injective. This map is the double coboundary associated to the sequence

$$0 \to \operatorname{Wh}(\mathbf{Z}H) \to \operatorname{Wh}(\mathbf{Z}G^-) \to \operatorname{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-) \to \mathbf{k} \to 0$$

which by Section 9 equals the composite

$$H^0(\mathbf{k}) \to L_0^h(\mathbf{Z}H) \to H^0(\operatorname{Wh}(\mathbf{Z}H))$$

of maps in Ranicki–Rothenberg sequences. But the group $L_1^k(\mathbf{Z}H) = 0$ by Lemma 11.10, so $H^0(\mathbf{k})$ injects into the torsion subgroup of $L_0^h(\mathbf{Z}H)$, which is mapped injectively into $H^0(\operatorname{Wh}(\mathbf{Z}H))$ since $L_0^s(\mathbf{Z}H)$ is torsion–free. Therefore $\{\Delta(V_1)/\Delta(V_2)\}$ is not in the image of δ and no such non–linear similarity exists. □

Corollary 19.2. *The groups $G = C(2^r)$ have no 5–dimensional non–linear similarities.*

The 6–dimensional case was worked out in detail for cyclic 2–groups in [7], and a solution for the general case was announced in [6]. However, in Example 19.6 we give a counterexample to the necessity of [6,Condition B], and this invaliates the claimed solution.

Theorem 19.3. *Let $V_1 = t^{a_1} + \cdots + t^{a_k}$ and $V_2 = t^{b_1} + \cdots + t^{b_k}$ be free G–representations, where $G = C(2^r)$. Let W be a complex G–representation with no $\mathbf{R}_+$ summands. Then there exists a topological similarity $V_1 \oplus W \oplus \mathbf{R}_- \oplus \mathbf{R}_+ \sim_t V_2 \oplus W \oplus \mathbf{R}_- \oplus \mathbf{R}_+$ if and only if*

(i) *$S(V_1)$ is s–normally cobordant to $S(V_2)$,*
(ii) *$\operatorname{Res}_H(V_1 \oplus W) \oplus \mathbf{R}_+ \sim_t \operatorname{Res}_H(V_2 \oplus W) \oplus \mathbf{R}_+$, and*
(iii) *the element $\{\Delta(V_1)/\Delta(V_2)\}$ is in the image of the double coboundary*

$$\delta^2\colon H^1(\widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G^-)) \to H^1(\operatorname{Wh}(\mathbf{Z}G^-)/\operatorname{Wh}(\mathbf{Z}H)) .$$

Corollary 19.4. *There is a stable topological similarity $V_1 \approx_t V_2$ if and only if $S(V_1)$ is s–normally cobordant to $S(V_2)$ and $\operatorname{Res}_K\{\Delta(V_1)/\Delta(V_2)\}$ is in the image of the double coboundary δ^2_K for all subgroups $K \subseteq G$.*

Remark 19.5. The double coboundary δ^2_K in this statement is the one for the subgroup K with respect to an index two subgroup $K_1 < K$. Corollary 19.4 follows from Theorem 19.3 and Proposition 9.6. Notice that Theorem 19.3 also gives a way to construct the stable similarity, assuming that the conditions are satisfied. In the most complicated case, one would need all proper subgroups of G appearing as isotropy groups in W. $\square$

The proof of Theorem 19.3. By Corollary 8.9 the double coboundary

$$\delta^2_W: H^1(\widetilde{K}_0(\mathcal{C}_{W_{max}\times \mathbf{R}_-,G}(\mathbf{Z}))) \to H^1(\mathrm{Wh}(\mathcal{C}_{W_{max}\times \mathbf{R}_-,G}(\mathbf{Z})))$$

is isomorphic to

$$\delta^2: H^1(\widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G^-)) \to H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$$

under the cone point inclusion. The result now follows from Theorem C. $\square$

We conclude with two examples.

Example 19.6. For $r \geq 5$ the element $(t^9 + t^{1+2^{r-2}} - t - t^{9+2^{r-2}})$ lies in $\widetilde{R}^{free}_t(G)$. On the other hand, the Reidemeister torsion invariant restricted to $\mathrm{Wh}(\mathbf{Z}H)$ is given by $U_{9,1+2^{r-2}} = U_{1,9}(U_{1,1})^{-1}$ and this is non–trivial in $H^0(\mathrm{Wh}(\mathbf{Z}H))$ by the results of Cappell–Shaneson on units (see [7,p.733]).

To prove the first assertion, note that $(t^9+t^{1+2^{r-2}}-t-t^{9+2^{r-2}})$ lies in $\widetilde{R}^{free}_n(G)$ by applying [6,Condition A]. Let W denote the complex 2–dimensional representation with isotropy of index 4. The surgery obstruction in $L^p_3(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ is zero by a similar argument to that given in [7,p.734], based on the fact that $9(1+2^{r-2}) \equiv 1 \pmod 8$ when $r \geq 5$ and invariance under group automorphisms (similar to [7,4.1] since our surgery obstruction comes from the cone point by Theorem 14.1). $\square$

Example 19.7. According to [6,Thm.2], the element $t - t^5$ has order 2^{r-2} in $R_{\mathrm{Top}}(G)$ for $G = C(2^r)$, $r > 3$, and order 4 for $G = C(8)$. Note that $2^{r-2}(t-t^5) \in \widetilde{R}^{free}_h(G)$, for $r \geqq 3$ and this is the smallest multiple that works. A short calculation using [6,Condition A] (done in Lemma 18.5) also shows that $2^{r-2}(t-t^5) \in \widetilde{R}^{free}_n(G)$, and it remains to consider the surgery obstruction.

For $r = 3$ it follows from [7,Cor.(iii)] that $2(t-t^5)$ does not give a 6–dimensional similarity. By Theorem 19.3 it follows that $2(t-t^5) \notin R_t(G)$. Since the obstruction has exponent 2, we get $4(t-t^5) \approx_t 0$ as claimed.

For $r > 3$, the assertion is that $2^{r-2}(t-t^5) \in \widetilde{R}^{free}_t(G)$ already. We prove this by induction starting with $2(t-t^5)$ in $R_n(C(8))$. Suppose that $2^{r-3}(t-t^5) \in R_n(H)$ for $H = C(2^{r-1})$, where $r > 3$. Let $U_1 = 2^{r-3}t$ and $U_2 = 2^{r-3}t^5$ be the free representations over $H = C(2^{r-1})$, and let $V_1 = 2^{r-3}t$ and $V_2 = 2^{r-3}t^5$ be the corresponding free representations over $G = C(2^r)$. Then $\mathrm{Ind}_H(\Delta(U_1)/\Delta(U_2)) = 0$ in $H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$, since we are dividing out exactly the image of Ind_H. It follows from Corollary 19.4 that $\mathrm{Ind}_H(U_1 - U_2) \in R_t(G)$.

However, $\mathrm{Ind}_H(U_1 - U_2) = (V_1 - V_2) + (V_1^\tau - V_2^\tau)$ where τ is the group automorphism $\tau(t) = t^{1+2^{r-1}}$. By [7,Cor(iii)] we have $2t \approx_t 2t^{1+2^{r-1}}$ and $2t^5 \approx_t 2t^{5+2^{r-1}}$, whenever $r \geq 4$. Therefore, $2^{r-2}t = V_1 \oplus V_1 \approx V_1 \oplus V_1^\tau$ and $2^{r-2}t^5 = V_2 \oplus V_2 \approx V_2 \oplus V_2^\tau$. □

20. The proof of Theorem D.

We first summarize our information about $R_{\mathrm{Top}}(G)$, obtained by putting together results from previous section.

Theorem 20.1. *Let $G = C(2^r q)$, with q odd, and $r \geqq 2$.*

(i) *The torsion subgroup of $R^{free}_{\mathrm{Top}}(G)$ is $\widetilde{R}^{free}_{\mathrm{Top}}(G)$.*
(ii) *The rank of $R^{free}_{\mathrm{Top}}(G)$ is $\varphi(q)/2$ for $q > 1$ (resp. rank 1 for $q = 1$), and the torsion is at most 2–primary.*
(iii) *The subgroup $\widetilde{R}^{free}_{n,\mathrm{Top}}(G)$ has exponent two.*

Remark 20.2. In part (ii), $\varphi(q)$ is the Euler function. The precise number of $\mathbf{Z}/2$ summands in $\widetilde{R}^{free}_{n,\mathrm{Top}}(G)$ is determined by working out the conditions in Theorem C on the basis elements of $\widetilde{R}^{free}_n(G)$. In cases where the conditions in Theorem C can actually be evaluated, the structure of $R_{\mathrm{Top}}(G)$ will thus be determined completely. □

Proof. Parts (i) and (ii) of Theorem 20.1 have already been proved in Corollary 5.2, so it remains to discuss part (iii). In fact, the assertion that $\widetilde{R}^{free}_{n,\mathrm{Top}}(G)$ has exponent 2 is an immediate consequence of Theorem C. To see this, suppose that $(V_1 - V_2)$ is any element in $\widetilde{R}^{free}_n(G)$. The obstruction to the existence of a stable non–linear similarity $V_1 \approx_t V_2$ is determined by the class $\{\Delta(V_1)/\Delta(V_2)\}$ in the Tate cohomology group $H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$, which has exponent 2. Since the Reidemeister torsion is multiplicative, $\Delta(V_1 \oplus V_1) = \Delta(V_1)^2$, and we conclude that $V_1 \oplus V_1 \approx_t V_2 \oplus V_2$ by Theorem C. □

More generally, it follows from our results that the structure of $\widetilde{R}^{free}_{n,\mathrm{Top}}(G)$ is determined by working out the criteria of Theorem C on a basis of $\widetilde{R}^{free}_n(G)$. Suppose that V_1 is stably topologically equivalent to V_2. Then there exists a similarity

$$V_1 \oplus W \oplus \mathbf{R}^l_- \oplus \mathbf{R_+}^s \sim_t V_2 \oplus W \oplus \mathbf{R}^l_- \oplus \mathbf{R_+}^s$$

where W has no $\mathbf{R}_+$ or $\mathbf{R}_-$ summands, and $l, s \geqq 1$. But by Corollary 11.12 we may assume that $s = 1$, and by Theorem 16.1 that $l = 1$, so we are reduced to the situation handled by Theorem C.

The proof of Theorem D. We have a basis

$$\mathcal{B} = \{t^i - t^{i+2q} \mid (i, 2q) = 1, i \equiv 1 \pmod 4, 1 \leqq i < 4q\}$$

for $\widetilde{R}^{free}(G)$, so it remains to work out the relations given by topological similarity. Notice that $\widetilde{R}^{free}_h(G) = \widetilde{R}^{free}_n(G)$, and that the sum of any two elements in $\mathcal{B}$ lies in $\widetilde{R}^{free}_n(G)$. Moreover, by Corollary 2.4 there are no 6–dimensional similarities for G.

Now suppose that $V_1 \oplus W \oplus \mathbf{R}_- \oplus \mathbf{R}_+ \sim_t V_2 \oplus W \oplus \mathbf{R}_- \oplus \mathbf{R}_+$, for some complex G–representation W. Then $\mathrm{Res}_H V_1 \cong \mathrm{Res}_H V_2$ since q is odd, and by Theorem C we get a similarity of the form $V_1 \oplus \mathbf{R}_-^l \oplus \mathbf{R}_+ \sim_t V_2 \oplus \mathbf{R}_-^l \oplus \mathbf{R}_+$. But by Theorem 16.1 this implies that $V_1 \oplus \mathbf{R}_- \oplus \mathbf{R}_+ \sim_t V_1 \oplus \mathbf{R}_- \oplus \mathbf{R}_+$. Therefore, for any element $a_i = (t^i - t^{i+2q}) \in \mathcal{B}$ we have $2a_i \notin R_t^{free}(G)$ but $4a_i \in R_t^{free}(G)$. However, in Section 17 we determined the bounded surgery obstructions for all these elements. Since $i_*(u_{1,j}) = \langle -1 \rangle \in H^1(A^\times)$ for all j with $(j, 2q) = 1$ by Lemma 17.1, there are further relations $a_1 + a_j \approx a_1 + a_k$, or $a_1 \approx a_j$ for all i.

It follows that a basis for $\widetilde{R}_{\mathrm{Top}}^{free}(G)$ is given by $\{a_1 \mid 4a_1 \approx 0\}$. $\square$

References.

[1] M.F. Atiyah and R. Bott, *A Lefschetz fixed-point formula for elliptic complexes I*, Ann. of Math. (2) **86** (1967), 374–407.

[2] A. Bak, *Odd dimension surgery groups of odd torsion groups vanish*, Topology **14** (1975), no. 4, 367–374.

[3] S. E. Cappell and J. L. Shaneson, *Linear similarity and non–linear similarity are equivalent below dimension 6*, preprint, 1981.

[4] ———, *Non-linear similarity*, Ann. of Math. (2) **113** (1981), 315–355.

[5] ———, *The topological rationality of linear representations*, Inst. Hautes Études Sci. Publ. Math. **56** (1983), 309–336.

[6] S. E. Cappell, J. L. Shaneson, M. Steinberger, S. Weinberger, and J. West, *The classification of non-linear similarities over* $\mathbf{Z}/2^r$, Bull. Amer. Math. Soc. (N.S.) **22** (1990), 51–57.

[7] S. E. Cappell, J. L. Shaneson, M. Steinberger, and J. West, *Non-linear similarity begins in dimension six*, J. Amer. Math. Soc. **111** (1989), 717–752.

[8] M. Cárdenas and E. K. Pedersen, *On the Karoubi filtration of a category*, K-theory (1995), (To appear).

[9] G. Carlsson and E. K. Pedersen, *Controlled algebra and the Novikov conjectures for K- and L-theory*, Topology **34** (1995), 731–758.

[10] D. Carter, *Localizations in lower algebraic K-theory*, Comm. Algebra **8** (1980), 603–622.

[11] T. tom Dieck, "Transformation groups", de Gruyter Studies in Mathematics, vol. 8, Walter de Gruyter & Co., Berlin, 1987.

[12] A. W. M. Dress, *Induction and structure theorems for orthogonal representations of finite groups*, Ann. of Math. (2) **102** (1975), 291–325.

[13] S. C. Ferry and E. K. Pedersen, *Epsilon surgery Theory*, "Proceedings 1993 Oberwolfach Conference on the Novikov Conjectures, Rigidity and Index Theorems Vol. 2" (S. C. Ferry, J. Rosenberg, and A. A. Ranicki, eds.), London Math. Soc. Lecture Notes, vol. 227, Cambridge University Press, 1995, pp. 167–226.

[14] I. Hambleton, *Projective surgery obstructions on closed manifolds*, "Algebraic K-theory, Oberwolfach 1980,", Lecture Notes in Mathematics, vol. 967, Springer-Verlag, Berlin-New York, 1982, pp. 101–131.

[15] I. Hambleton and I. Madsen, *Actions of finite groups on R^{n+k} with fixed set R^k*, Canad. J. Math. **38** (1986), 781–860.

[16] ———, *On the computation of the projective surgery obstruction group*, K-theory **7** (1993), 537–574.

[17] I. Hambleton and R. J. Milgram, *The surgery obstruction groups for finite 2-groups*, Invent. Math. **61** (1980), no. 1, 33–52.

[18] I. Hambleton and E. K. Pedersen, *Bounded surgery and dihedral group actions on spheres*, J. Amer. Math. Soc. **4** (1991), 105–126.

[19] ———, *Non-linear similarity revisited*, "Prospects in Topology: Proceedings of a conference in honor of William Browder" (Frank Quinn, ed.), Annals of Mathematics Studies, vol. 138, 1995, pp. 157–174.

[20] I. Hambleton and L. R. Taylor, *A guide to the calculations of surgery obstruction groups*, (to appear).

[21] I. Hambleton, L. R. Taylor, and B. Williams, *An introduction to the maps between surgery obstruction groups*, "Algebraic Topology, Aarhus 1982" (J. L. Dupont and I. Madsen, eds.), Lecture Notes in Mathematics, vol. 1051, Springer-Verlag, Berlin-New York, 1984, pp. 49–127.

[22] ———, *Detection theorems in K-theory and L-theory*, J. Pure Appl. Algebra **63** (1990), 247–299.

[23] W-C. Hsiang and W. Pardon, *When are topologically equivalent representations linearly equivalent*, Invent. Math. **68** (1982), 275–316.

[24] I. Madsen and M. Rothenberg, *On the classification of G-spheres I: equivariant transversality*, Acta Math. **160** (1988), 65–104.

[25] R. Oliver, "Whitehead Groups of Finite Groups", London Math. Soc. Lecture Notes, vol. 132, Cambridge U. Press, Cambridge, 1988.

[26] E. K. Pedersen and C. Weibel, *A nonconnective delooping of algebraic K-theory*, "Algebraic and Geometric Topology, Proceedings Rutgers 1983" (A. A. Ranicki, N. Levitt, and F. Quinn, eds.), Lecture Notes in Mathematics, vol. 1126, Springer-Verlag, Berlin-New York, 1985, pp. 166–181.

[27] A. A. Ranicki, *The double coboundary*, letter to B. Williams, 1981.

[28] ______, "Exact Sequences in the Algebraic Theory of Surgery", Math. Notes, vol. 26, Princeton University Press, Princeton N. J., 1981.

[29] ______, *The L-theory of twisted quadratic extensions*, Canad. J. Math. **39** (1987), 345–364.

[30] ______, *Additive L-theory*, K-theory **3** (1989), 163–195.

[31] ______, "Algebraic L-theory and Topological Manifolds", Cambridge Tracts in Math., vol. 102, Cambridge University Press, Cambridge, 1992.

[32] Andrew Ranicki, "Lower K- and L-theory", London Mathematical Society Lecture Note Series, vol. 178, Cambridge University Press, Cambridge, 1992.

[33] G. de Rham, *Sur les nouveaux invariants topologiques de M. Reidemeister*, Mat. Sbornik **I (43)** (1936), 737–743, Proc. International Conference of Topology (Moscow, 1935).

[34] ______, *Reidemeister's torsion invariant and rotations of S^n*, "Diff. Analysis", Oxford University Press, Oxford, 1964, pp. 27–36.

[35] J.-P. Serre, "Linear Representations of finite groups", Graduate Texts in Mathematics, vol. 42, Springer Verlag, 1977.

[36] R. G. Swan and E. G. Evans, "K-theory of finite groups and orders", Lecture Notes in Mathematics, vol. 149, Springer-Verlag, Berlin-New York, 1970.

[37] Richard G. Swan, *Induced representations and projective modules*, Ann. of Math. (2) **71** (1960), 552–578.

[38] C. T. C. Wall, "Surgery on Compact Manifolds", Academic Press, 1970.

[39] C. T. C. Wall, *On the classification of Hermitian forms. II. Semisimple rings*, Invent. Math. **18** (1972), 119–141.

[40] C. Young, *Normal invariants of lens spaces*, Master's thesis, McMaster University, 1995.

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