RAMIFICATIONS ON ARITHMETIC SCHEMES

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Introduction

It is well known that ramification theory on extensions of Dedekind domains is a very classical topic in algebraic number theory. There are also many works on ramification theory in Noetherian rings ([1],[6],[7]). This paper is interested in applying them to study the geometry of arithmetic schemes.

Let $f : X \to Y$ be a morphism of finite type between regular schemes. For any point $x \in X$, one can define the ramification index $r(\mathcal{O}_x/\mathcal{O}_{f(x)})$ of f at x by using Fitting ideals of $\Omega^1_{\mathcal{O}_x/\mathcal{O}_{f(x)}}$, and the reduced ramification index $e(\mathcal{O}_x/\mathcal{O}_{f(x)})$ of f at x (see the following definitions). When X and Y are of dimension 1, a very classical theorem of Dedekind gives the relation of $r(\mathcal{O}_x/\mathcal{O}_{f(x)})$ and $e(\mathcal{O}_x/\mathcal{O}_{f(x)})$: $r(\mathcal{O}_x/\mathcal{O}_{f(x)}) \geq e(\mathcal{O}_x/\mathcal{O}_{f(x)}) - 1$. This theorem was generalized to the case of birational extensions of regular local rings ([10]) and had been used to study birational morphisms of regular schemes ([9],[12]). But all works only concerned the case that the extension of function fields of X and Y is finite, thus 0-Fitting ideal is enough for the story. Basically, 0-Fitting ideal is a principal ideal and satisfies transitive law, which make everything works well. In this paper, we are going to consider the case that X/Y is a family of algebraic varieties, especially a regular arithmetic scheme. We shall formulate the notation of ramification locus of f by using high order Fitting ideals, which can be considered as the degeneracy loci of a morphism between vector bundles. When $f : X \to Y$ is a fibration of algebraic surface over an algebraically closed field of characteristic zero, the ramification locus of f is nothing but the zero subscheme of a section of vector bundle $\Omega^1_X \otimes (f^*\Omega^1)^\vee$. Since this fact, Iversen ([5]) can prove a formula expressing the diference of Euler characteristics of singular fibre $X_s = \sum m_i\Gamma_i$ and a smooth fibre by

$$K_{X/Y} \cdot \sum_i (m_i - 1)\Gamma_i - (\sum_i (m_i - 1)\Gamma_i)^2 + \sum_{x \in X_s} \mu_x(f).$$

There is no such vector bundle available in the case of arithmetic surfaces. However, Bloch's formula can express Artin conductor by localized Chern class of $\Omega^1_{X/Y}$ (see [2] for the definition). By using Bloch's work ([2],[3]), we can generalize the above formula to arithmetic surface replacing $\sum(m_i - 1)\Gamma_i$ by ramification divisor $R(f)$ of f (theorem 3).

We collected some facts of commutative algebra and recalled some notations of [11] in §0. In section 1, we firstly proved a theorem on ramification index of discrete

This work was done during my staying in Max-Planck-Institut für Mathematik, I thank its hospitality and financial support.

valuation rings, which is a generalization of Dedekind's theorem. After that, we generalized a result of [11] to high dimensional arithmetic schemes, which gave the relations of relative canonical sheaf with ramifications and differentials. In section 2, we proved a formula expressing Artin conductor by ramification locus of f, which should be considered as a corollary of Bloch's theorems. The section 3 is a complement to §2 of [11] about base extensions of arithmetic surfaces. One observation here is that the changes of invariants of arithmetic surfaces caused by a base extension are determined by the difference between the base extension's ramification and the ramification of morphism induced by the base extension. Applying this observation to the case of function fields, we can give a very simple treatment for some known results and drive out a sharper height inequality of algebraic points than [13].

All the morphisms and algebras in this paper are of finite type, and all the rings are noetherian domain. We use some results of [4] such as Riemann-Roch theorem and Serre duality theorem for curves on surface without mention.

Acknowledgement. *I would express my heart thanks to Professor F. Hirzebruch who invited me visit Max-Planck-Institut für Mathematik. I also thank R. Hübl for the communications about theorem 1.*

§0 Preliminary.

Let us recall some notations of [11] in this section, the detail proofs can be found in [6] and [7]. We first recall the notations of Fitting ideals and ramifications.

Let R be a ring, M a finite R-module, and $\{m_1, ..., m_n\}$ a system of generators of M. The exact sequence

$$0 \to K \to R^n \xrightarrow{\alpha} M \to 0$$

is called the presentation of M defined by $\{m_1, ..., m_n\}$, where α maps the i-th canonical basis element e_i onto $m_i (i = 1, ..., n)$and $K = ker\alpha$. Let $\{v_\lambda\}_{\lambda \in \Lambda}$ be a system of generators of K with $v_\lambda = (x_1^\lambda, ..., x_n^\lambda) \in R^n$ $(\lambda \in \Lambda)$. Then

$$(x_i^\lambda)_{\substack{i=1,...,n \\ \lambda\in\Lambda}}$$

is called a relation matrix of M with respect to $\{m_1, ..., m_n\}$.

Given such a matrix, let $F_i(M)$ denote the ideal of R generated by all $(n-i)$-rowed subdeterminants of the relation matrix $(i = 0, 1, .., n-1)$, and let $F_i(M) = R$ for $i \geq n$. One can prove that $F_i(M)$ does not depend on the special choice of the relation matrix and the choice of the generating system $\{m_1, ..., m_n\}$ of M. We call $F_i(M)$ the i-th Fitting ideal of M.

Let A, B be two local rings with $tr.deg(Q(A)/Q(B)) = d$ and $m_A \cap B = m_B$, where m_A and m_B are maximal ideals of A and B. For any ideal I of A, we define $v_A(I)$ to be the largest integer such that $I \subseteq m_A^{v_A(I)}$. We call

$$r(A/B) := v_A(F_d(\Omega^1_{A/B}))$$

the ramification index of A over B, and

$$e(A/B) := \max_{(x_1,...,x_r)} \{v_A(\prod_{i=1}^{r} x_i) \mid (x_1, ..., x_r) \text{ are the generators of } m_B\}$$

the reduced ramification index of A over B

Proposition 1. *Let M be a finite R-module, $F_i(M)$ the i-th Fitting ideal of M. Then*

(1) *For each algebra S/R we have*

$$F_i(S \otimes_R M) = S \cdot F_i(M).$$

(2) *If $N \subset R$ is a multiplicatively closed subset, then*

$$F_i(M_N) = F_i(M)_N.$$

(3) *If M has rank $r := dim_K(K \otimes_R M)$, then $F_i(M) = \{0\}$ for $i = 0, ..., r-1$, and $F_i(M) \neq \{0\}$ for $i \geq r$.*

From above proposition, we can see easily that to study $r(A/B)$ and $e(A/B)$ one can always pass to the completions of A and B. For the convenience, we collect some facts of completion as a proposition without proof, which can be found in books of H.Matsumura and O.Zariski.

Proposition 2. *Let A be a noetherian ring, I an ideal of A and $\widehat{A}$ the I-adic completion of A. If M is an A-module such that M/IM is a finite A/I-module, then we have*

(1) *$\widehat{A}$ is regular if A_P is regular for every prime ideal P of A containing I.*
(2) *The I-adic completion of M is a finite $\widehat{A}$-module. In particular, if A is complete and M is Hausdoff for the I-adic topology, then M is a finite complete A-module.*
(3) *Let $P \supset I$ be a prime ideal of A, and $\widehat{P} = P\widehat{A}$, then*

$$(A_P)^\wedge = \varprojlim A_P/P^nA_P \cong \varprojlim \widehat{A}_{\widehat{P}}/\widehat{P}^n\widehat{A}_{\widehat{P}} = (\widehat{A}_{\widehat{P}})^\wedge$$

Then we want to recall the notations of higher modules of differential forms, all of which can be generalized to global case, i.e., sheaves on schemes. Let $K := Q(R)$, and M a finite R-module such that $M_K := K \otimes_R M$ is a free K-module of some rank r. For a system of generators $\{x_1, ..., x_n\}$ of M, let

$$0 \to U \xrightarrow{\alpha} R^n \xrightarrow{\beta} M \to 0$$

be the presentation of M corresponding to $\{x_1, ..., x_n\}$, i.e. $\beta(e_i) = x_i$ for $i = 1, ..., n$ and $U := ker\,\beta$. Clearly $\Lambda^{n-r+1}\alpha = 0$, since $U_K := K \otimes_R U$ of rank $n-r$. Then, for each $m \in \mathbb{N}$, there is a canonical R-linear map (write $F := R^n$)

$$\varphi^m : \quad \Lambda^m M \longrightarrow Hom_R(\Lambda^{n-r}U\,, \quad \Lambda^{n-r+m}F),$$

which is defined as the following: For $\omega \in \Lambda^m M$ choose a preimage $\bar{\omega} \in \Lambda^m F$ with respect to $\Lambda^m\beta$. Then

$$\varphi^m(\omega) : \quad \Lambda^{n-r}U \longrightarrow \Lambda^{n-r+m}F$$

takes any $u \in \Lambda^{n-r}U$ to $\Lambda^{n-r}\alpha(u) \wedge \bar{\omega} \in \Lambda^{n-r+m}F$. Therefore, there is a canonical commutative diagram

$$\begin{array}{ccc} \Lambda^m M & \xrightarrow{\varphi^m} & Hom_R(\Lambda^{n-r}U, \quad \Lambda^{n-r+m}F) \\ \downarrow & & \downarrow \chi^m \\ \Lambda^m M_K & \xrightarrow{\varphi^m_K} & Hom_K(\Lambda^{n-r}U_K, \quad \Lambda^{n-r+m}F_K), \end{array}$$

where $\chi^m(l) = id_K \otimes l$ for any $l \in Hom_R(\Lambda^{n-r}U, \quad \Lambda^{n-r+m}F)$. One can prove that the R-submodule $(\varphi^m_K)^{-1}(im\,\chi^m)$ of $\Lambda^m M_K$ is independent of the choice of the system of generators of M. If S is a R-algebra, we take $M = \Omega^1_{S/R}$, the relative differential module, then we call

$$\Delta^m(S/R) := (\varphi^m_K)^{-1}(im\,\chi^m)$$

the m-th module of integral differential forms of S/R.

§1 Ramification indexs and canonical sheaves.

In this section, we shall prove a theorem on ramification index of extensions of discrete valuation rings at first, which was known as Dedekind's ramification main theorem in the case of finite extensions, our result is a generalization of Dedekind's theorem to higher dimension (i.e. the extensions may have transcendental degree). Then we will discuss the relation of canonical sheaf of an arithmetic scheme with its ramifications.

Theorem 1. *Let A/B be an extension of discrete valuation rings, essentially of finite type with residual fields $k(A)$ and $k(B)$. If $k(A)$ is separably generated over $k(B)$, we have*

$$r(A/B) \geq e(A/B) - 1,$$

and the equality holds if and only if $e(A/B)$ is not a multiple of $char(k(B))$.

Proof. Let $m_B = (t)B$ and $m_A = (u)A$ be the maximal ideals of B and A, v_B and v_A the valuations of $Q(B)$ and $Q(A)$ determined by B and A. Without lost generality, we suppose that A and B are complete and write $t = a_0u^e$, where $e = e(A/B)$ and $v_A(a_0) = 0$.

Since $k(A)$ is separably generated over $k(B)$, we can choose $x_1, ..., x_d$, in A such that $k(A)/k(B)(\bar{x}_1, ..., \bar{x}_d)$ is a finite separable extension. Since B is a discrete valuation ring, it is easy to see that $x_1, ..., x_d$ are algebraic independent over B. Let $P = (t)B[x_1, ..., x_d]$ and $R = (B[x_1, ..., x_d])_P$, it is not hard to prove that $A \supseteq R \supseteq B$ are extensions of discrete valuation rings with $m_R = (t)R$. Since $k(A)/k(R)$ is a finite separable extension, there exists a $\bar{y} \in k(A)$ and a separable minimal polynomial $\bar{f}(Y) \in k(R)[Y]$ of degree r such that $k(A) = k(R)(\bar{y})$ and $\bar{f}(\bar{y}) = 0$. We can assume that R is complete by passing to its completion, so there is a lifting of $\bar{y}$ and $\bar{f}(Y)$, say $y \in A$ and $f(Y) \in R[Y]$, such that $f(y) = 0$. By Proposition 2, A is finite over R. On the other hand, $A/(t)A$ is generated by

$$\{y^i u^j \mid 0 \leq i \leq r-1,\, 0 \leq j \leq e-1\}$$

as a $k(R)$-module. Thus, by Nakayama's lemma, we have

$$A = \frac{R[[u,y]]}{(f(y), t - a_0 u^e)}.$$

So $\Omega^1_{A/B}$ is generated by $dy, du, dx_1, \cdots, dx_d$ with relations

$$\sum_{i=1}^{d} \frac{\partial f(y)}{\partial x_i} dx_i + f'(y)dy = 0$$
$$\sum_{i=1}^{d} u^e \frac{\partial a_0}{\partial x_i} dx_i + u^e \frac{\partial a_0}{\partial y} dy + (u^e \frac{\partial a_0}{\partial u} + e u^{e-1} a_0) du = 0$$

Thus the relation matrix of $\Omega^1_{A/B}$ is

$$M = \begin{pmatrix} \frac{\partial f(y)}{\partial x_1}, & \ldots, \frac{\partial f(y)}{\partial x_d}, & f'(y), & 0 \\ u^e \frac{\partial a_0}{\partial x_1}, & \ldots, u^e \frac{\partial a_0}{\partial x_d}, & u^e \frac{\partial a_0}{\partial y}, & u^e \frac{\partial a_0}{\partial u} + e u^{e-1} a_0 \end{pmatrix}$$

By the definition of $F_d(\Omega^1_{A/B})$, which is generated by all subdeterminants of M. Thus we have

$$r(A/B) = min\{v_A(\text{subdeterminants of } M)\} \geq e - 1,$$

and the equality holds if and only if e is not a multiple of $char(k(B))$, we have done.

Remark 1. *From the proof, we know that $r(A/B) \leq r(A/R)$. In fact, consider the following commutative diagram (for simplity, we assume that $R' = R$)*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1_{R/B} \otimes_R A & \longrightarrow & \Omega^1_{A/B} & \longrightarrow & \Omega^1_{A/R} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & A^{\oplus d} & \longrightarrow & A^{\oplus (d+2)} & \longrightarrow & A^{\oplus 2} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & \longrightarrow & K & \xrightarrow{\alpha} & K' & \longrightarrow & 0 \end{array}$$

we can see that α is an isomorphism and the images

$$\alpha(\sum_{i=1}^{d} \frac{\partial f(y)}{\partial x_i} dx_i + f'(y)dy) = f'(y)dy$$
$$\alpha(\sum_{i=1}^{d} u^e \frac{\partial a_0}{\partial x_i} dx_i + u^e \frac{\partial a_0}{\partial y} dy + (u^e \frac{\partial a_0}{\partial u} + e u^{e-1} a_0) du)$$
$$= u^e \frac{\partial a_0}{\partial y} dy + (u^e \frac{\partial a_0}{\partial u} + e u^{e-1} a_0) du$$

are generators of K', so $r(A/R) = v_A(u^e\frac{\partial a_0}{\partial u} + eu^{e-1}a_0)$ (one simply remarks that $v_A(f'(y)) = 0$ since $\bar{y}$ is a separable element over $k(B)$). On the other hand, we know that

$$r(A/B) = min\{v_A(u^e\frac{\partial a_0}{\partial u} + eu^{e-1}a_0), v_A(other\ subdeterminants\ of\ M)\}.$$

When e is not a multiple of $char(k(B))$, one can see easily that

$$v_A(u^e\frac{\partial a_0}{\partial u} + eu^{e-1}a_0) = e - 1 < v_A(other\ subdeterminants\ of\ M),$$

hence $r(A/B) = r(A/R) = e - 1$. But when e is a multiple of $char(k(B))$, there is no evidence that $v_A(u^e\frac{\partial a_0}{\partial u} + eu^{e-1}a_0)$ must be smaller than

$$v_A(other\ subdeterminants\ of\ M).$$

Let $S = Spec(\Lambda)$ be the spectrum of a Dedekind domain whose residual field at each prime ideal is perfect or of chacteristic zero, and $f : X \to S$ a flat and projective scheme over S. By an arithmetic scheme of dimension $d+1$, we mean that X is regular, generic smooth over S and $tr.(Q(X)/Q(S)) = d$. The relative canonical sheaf $\mathcal{K}_{X/S}$ and the Fitting ideal sheaf $\mathcal{F}(X/S)$ can be introduced as the following.

Definition 1. *The presheaves of $\mathcal{K}_{X/S}$ and $\mathcal{F}(X/S)$ are defined as the following: For any affine open set $U = Spec\,B$ of X, let*

$$\mathcal{K}_{X/S}(U) = \Delta^d(B/\Lambda) \quad \mathcal{F}(X/S)(U) = F_d(\Omega^1_{B/\Lambda}).$$

Let

$$i : X \hookrightarrow P = \mathbb{P}^n_S$$

be an embedding and $\mathcal{I}$ the ideal sheaf of X in P, then we have the following commutative diagram

$$(*)\qquad \begin{array}{ccc} \Lambda^d\Omega^1_{X/S} & \xrightarrow{\varphi^d} & Hom_{\mathcal{O}_X}(\Lambda^{n-d}\mathcal{I}/\mathcal{I}^2, \quad \Lambda^n i^*\Omega^1_{P/S}) \\ \downarrow{\gamma^d} & & \downarrow{\chi^d} \\ \Lambda^d\Omega^1_{L/K} & \xrightarrow{\varphi^d_L} & Hom_L(\Lambda^{n-d}(\mathcal{I}/\mathcal{I}^2)_L, \quad \Lambda^n(i^*\Omega^1_{P/S})_L), \end{array}$$

where γ^d, χ^d are the canonical maps, and φ^d, φ^d_K are defined as in §0. For a coherent sheaf $\mathcal{G}$ on, we always denote $\mathcal{G} \otimes_{\mathcal{O}_X} L$ by $\mathcal{G}_L$, and consider L here as a constant sheaf. The relative canonical sheaf of X/S is

$$\mathcal{K}_{X/S} \cong Hom_{\mathcal{O}_X}(\Lambda^{n-d}\mathcal{I}/\mathcal{I}^2, \quad \Lambda^n i^*\Omega^1_{P/S}),$$

which can be seen as a subsheaf of the constant sheaf $\Lambda^d\Omega^1_{L/K}$. If $\mathcal{F}(X/S)$ is the Fitting ideal sheaf of X/S, then we can write $\mathcal{F}(X/S) = I_R \cdot I_D$, where I_R is the ideal sheaf of ramification divisor $R(f)$, and I_D denotes the ideal of *residual scheme* D.

Theorem 2. *Let $f : X \to S$ be an arithmetic scheme of dimension $d+1$, and $R := \sum r_i C_i$ the ramification divisor of f. Then we have the following exact sequence*

$$0 \to (\Lambda^d \Omega^1_{X/S})_{tors} \to \Lambda^d \Omega^1_{X/S} \to \mathcal{K}_{X/S} \otimes \mathcal{O}_X(-R) \to \mathcal{O}_D(-R) \otimes \mathcal{K}_{X/S} \to 0$$

In particular, we have

$$\mathcal{K}_{X/S} \cong \mathcal{O}_X(R) \otimes_{\mathcal{O}_X} \Lambda^d \Omega^{1\vee\vee}_{X/S}.$$

If $m_i = m(C_i)$ denotes the multiplicity of C_i in the fibre containing C_i, then

$$r_i \geq m_i - 1.$$

The equality holds if and only if m_i is not a multiple of the characteristic of residual field of $f(C_i)$.

Proof. By using theorem 1, it is easy to see that theorem 2 is a corollary of the following lemma, which gives the relation of $\mathcal{K}_{X/S}$ and the image of $\Lambda^d \Omega^1_{X/S}$ under γ^d.

Lemma 1. *Let $f : X \to S$ be a projective scheme of dimension $d+1$. Then*

$$\Lambda^d \Omega^1_{X/S} / (\Lambda^d \Omega^1_{X/S})_{tors} \cong \mathcal{K}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{F}(X/S).$$

Proof. It is enough to prove the lemma locally, let $U = Spec\, B$ and $B = P/I$, where $P = \Lambda[x_1, ..., x_n]_N$. We have the exact sequence

$$I/I^2 \xrightarrow{\alpha} B \otimes_P \Omega^1_{P/\Lambda} \xrightarrow{\beta} \Omega^1_{B/\Lambda} \to 0.$$

Then the diagram $(*)$ becomes

$$\begin{array}{ccc} \Lambda^d \Omega^1_{B/\Lambda} & \xrightarrow{\varphi^d} & Hom_B(\Lambda^{n-d} I/I^2, \quad \Lambda^n(B \otimes_P \Omega^1_{P/\Lambda})) \\ \downarrow \gamma^d & & \downarrow \chi^d \\ \Lambda^d \Omega^1_{L/K} & \xrightarrow{\varphi^d_K} & Hom_L((\Lambda^{n-d}(I/I^2)_L, \quad \Lambda^n(L \otimes_P \Omega^1_{P/\Lambda})). \end{array}$$

We only need to determine $Im\, \varphi^d$.

Let $\{b_1, ..., b_n\}$ be a basis of $B \otimes_P \Omega^1_{P/\Lambda}$ such that $\{\omega_i = \beta(b_i) : i = 1, ..., n\}$ generates $\Omega^1_{B/\Lambda}$, and let

$$I/I^2 = B\, t_1 + \cdots + B\, t_r + B\, u_1 + \cdots + B\, u_m \quad (m \geq n - d)$$

such that $\alpha(u_1), ..., \alpha(u_m)$ form a system of generators of $\alpha(I/I^2)$ and $t_i\ (i = 1, ..., r)$ are torsion elements (i.e. $\alpha(t_i) = 0$). Then we have

$$\Lambda^{n-d} I/I^2 = \sum_{s+\mu=n-d} B(t_{j_1} \wedge \cdots \wedge t_{j_s} \wedge u_{i_1} \wedge \cdots \wedge u_{i_\mu}),$$

$$\alpha(\Lambda^{n-d}I/I^2) = \sum_{j_1,\dots,j_{n-d}} B\,(\alpha(u_{j_1}) \wedge \cdots \wedge \alpha(u_{j_{n-d}})),$$

$$\alpha(u_i) = \sum_{j=1}^{n} a_{ij}b_j \quad (i = 1, \dots, m).$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

be the relation matrix of $\Omega^1_{B/\Lambda}$ with respect to $\{\omega_i = \beta(b_i) : i = 1, \dots, n\}$. Then, by the definition of φ^d,

$$\begin{aligned} (1.1) \qquad & \varphi^d(\omega_{i_1} \wedge \cdots \wedge \omega_{i_d})(\alpha(u_{j_1}) \wedge \cdots \wedge \alpha(u_{j_{n-d}})) \\ &= \alpha(u_{j_1}) \wedge \cdots \wedge \alpha(u_{j_{n-d}}) \wedge b_{i_1} \wedge \cdots \wedge b_{i_d} \\ &= \pm |A^{i_1,\cdots,i_d}_{j_1,\dots,j_{n-d}}|\, b_1 \wedge \cdots \wedge b_n, \end{aligned}$$

where $A^{i_1,\cdots,i_d}_{j_1,\dots,j_{n-d}}$ is obtained from A by keeping the raws with numbers $j_1, \dots, j_{n-d}$ and deleting the columns with numbers $i_1, \dots, i_d$. So, by the definition of Fitting ideal, we get

$$Im\,\varphi^d \subseteq F_d(\Omega^1_{B/\Lambda}) \cdot Hom_B(\Lambda^{n-d}I/I^2 \;\; \Lambda^n(B \otimes_P \Omega^1_{P/\Lambda})) \cong F_d(\Omega^1_{B/\Lambda}) \cdot \Delta^d(B/\Lambda).$$

Note that $dim_L(L \otimes_B \alpha(I/I^2)) = n - d$, we can assume that $\alpha(u_1), \dots, \alpha(u_{n-d})$ consist a basis of $L \otimes_B \alpha(I/I^2)$, which means that there is a nozero element $a \in B$ such that

$$a \cdot \alpha(\Lambda^{n-d}I/I^2) \subseteq B \cdot (\alpha(u_1) \wedge \cdots \wedge \alpha(u_{n-d})).$$

Thus every homomorphism

$$h \in Hom_B(\Lambda^{n-d}I/I^2 \;\; \Lambda^n(B \otimes_P \Omega^1_{P/\Lambda}))$$

is determined by its image $h(\alpha(u_1) \wedge \cdots \wedge \alpha(u_{n-d}))$, since $\Lambda^n(B \otimes_P \Omega^1_{P/\Lambda})$ is a free B-module of rank one. This fact and the above (1.1) show that

$$Im\,\varphi^d = F_d(\Omega^1_{B/\Lambda}) \cdot Hom_B(\Lambda^{n-d}I/I^2 \;\; \Lambda^n(B \otimes_P \Omega^1_{P/\Lambda})) \cong F_d(\Omega^1_{B/\Lambda}) \cdot \Delta^d(B/\Lambda).$$

But

$$\Lambda^d\Omega^1_{B/\Lambda}/(\Lambda^d\Omega^1_{B/\Lambda})_{tors} \cong Im\,\gamma^d \cong Im\,\varphi^d,$$

which completes the proof.

§2 Ramification number and Artin conductor.

In this section, we restrict ourself to the case of dimension 2. More precisely, let $S = Spec(\Lambda)$ be the spectrum of a complete discrete valuation ring with algebraically closed residue field k, and $f : X \to S$ a flat, proper scheme over S. We assume that X is regular and of dimension 2 with special fibre $X_s = \sum m_i\Gamma_i$ and smooth generic fibre X_η, where s and η denote the closed point and generic point of S. Let Z be the subscheme of X determined by the Fitting ideal sheaf of X/S. We denote $\mathcal{F}(X/S)$ by I_Z and write $I_Z = I_R \cdot I_D$, where I_R is the ideal of a Cartier divisor in X whose local equation is the g.c.d. of generators of I_Z, and I_D is the ideal sheaf of residual scheme of Z. In the Chow groups $CH^1(X_s)$ and $CH^0(X_s)$, we have

$$[R] = \sum r_i\Gamma_i, \quad [D] = \sum \mu_x(f)[x].$$

We define the ramification number of f as

$$r(f) = K_{X/S} \cdot R - R^2 + \sum_{x\in X_s} \mu_x(f).$$

One can prove easily that R is the ramification divisor $R(f)$, and $\mu_x(f)$ are nothing but the Milnor numbers in the geometric case. Precisely, let $m_x = (u,v)\mathcal{O}_x$ and $(t)\Lambda$ be the maximal ideal of Λ, then we have

$$\hat{\mathcal{O}}_x \cong \frac{\Lambda[[u,v]]}{(t-f(u,v))}, \quad I_Z = (\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v})\hat{\mathcal{O}}_x.$$

Let d_x be the g.c.d. of $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$, then, by the defintion, we have

$$\mu_x(f) = l_{\hat{\mathcal{O}}_x}(\frac{\hat{\mathcal{O}}_x}{(\frac{\partial f}{\partial u}/d_x, \frac{\partial f}{\partial v}/d_x)}) = dim_k \frac{k[[u,v]]}{(\frac{\partial f}{\partial u}/d_x, \frac{\partial f}{\partial v}/d_x)}.$$

When X_s is reduced, $\mu_x(f)$ was defined by Deligne and called Milnor number (see [4]). However, when m_i is a multiple of the characteristic $p = char(k)$, we even failure to prove that $\mu_x(f) = 0$ at regular points of Γ_i. We can give the following simple description of $r(f)$ under the assumpation $p \nmid \prod m_i$.

Propositon 3. *Let $s(X_s)$ denote the set of singularities of $X_{s,red}$ and $p \nmid \prod m_i$. Then*

$$r(f) = 2(g - p_a(X_{s,red})) + \sum_{x\in s(X_s)} \mu_x(f),$$

where g is the genus of X_η and $p_a(X_{s,red})$ the arithmetic genus of $X_{s,red}$. In particular, when $X_{s,red}$ is a semi-stable curve, we have

$$r(f) = 2(g - p_a(X_{s,red})) + \sharp s(X_s).$$

Proof. Since $p \nmid \prod m_i$, by our theorem 1, we have

$$R(f) = \sum (m_i - 1)\Gamma_i = X_s - X_{s,red},$$

which implies that

$$K_{X/S} \cdot R(f) - R(f)^2 = 2(g - p_a(X_{s,red})).$$

The local computaion shows that $\mu_x(f) = 0$ at $x \notin s(X_s)$ and $\mu_x(f) = 1$ when x is a rational double point, so we have done.

It is clear that $r(f)$ is determined by local properties of f and can be computed locally. Now we want to relate $r(f)$ with the Artin conductor $Art(X/S)$ of X/S, which was defined as

$$Art(X/S) := \chi(X_s) - \chi(X_{\bar{\eta}}) - sw(X/S),$$

where $\chi(X_s)$ and $\chi(X_{\bar{\eta}})$ are the étale Euler characteristic of X_s and geometric fibre $X_{\bar{\eta}}$, $sw(X/S)$ is the Swan conductor (see [2] and [3] for the details).

Theorem 3. *$Art(X/S) = r(f)$.*

The proof heavily depends on Bloch's formula and should be considered as a remark of [2] and [3]. Firstly, we shall prove two lemmas.

Lemma 1. *Let $\Omega^1_{X/S,tors}$ be the torsion subsheaf of $\Omega^1_{X/S}$. Then*

(1) *$\Omega^1_{X/S,tors}$ is an invertible $\mathcal{O}_R$-module. $\Omega^1_{X/S}\otimes\mathcal{O}_Z$ is a locally free $\mathcal{O}_Z$-module of rank 2.*

(2) *We have the following two exact sequences*

$$0 \to \Omega^1_{X/S,tors} \to \Omega^1_{X/S} \otimes \mathcal{O}_Z \to (I_Z K_{X/S}) \otimes \mathcal{O}_Z \to 0 \tag{2.1}$$

$$0 \to \Omega^1_{X/S,tors} \to \Omega^1_{X/S} \otimes \mathcal{O}_R \to (I_Z K_{X/S}) \otimes \mathcal{O}_R \to 0. \tag{2.2}$$

Proof. Locally, we have

$$(\Omega^1_{X/S})_x \cong \frac{\mathcal{O}_x du \oplus \mathcal{O}_x dv}{(\frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv)}.$$

Write $\omega = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv$ and $\omega' = d_x^{-1}\frac{\partial f}{\partial u}du + d_x^{-1}\frac{\partial f}{\partial v}dv$, we claim that

$$(\Omega^1_{X/S,tors})_x = \mathcal{O}_x \cdot \bar{\omega}', \quad \text{where } \bar{\omega}' \text{ denote the image of } \omega' \text{ in } (\Omega^1_{X/S})_x.$$

It is clear that $\mathcal{O}_x \cdot \bar{\omega}' \subseteq (\Omega^1_{X/S,tors})_x$ since $d_x\bar{\omega}' = d_x\bar{\omega}' = \bar{\omega} = 0$. If $\bar{\omega}_1 \in (\Omega^1_{X/S,tors})_x$, then there exist $a \in \mathcal{O}_x$ such that $a\omega_1 \in \mathcal{O}_x \cdot \omega$. Write $\omega_1 = a_1 du + a_2 dv$, then there is an element $b \in \mathcal{O}_x$ such that

$$a\omega_1 = b\omega = b\frac{\partial f}{\partial u}du + b\frac{\partial f}{\partial v}dv,$$

which implies that $a|b\frac{\partial f}{\partial u}$, $a|b\frac{\partial f}{\partial v}$. We can assume that a and b have no common divisor in $\mathcal{O}_x$, thus $a|d_x$, which implies that $\omega_1 \in \mathcal{O}_x \cdot \omega'$, we got the claim. By the above claim, it is easy to prove that $\Omega^1_{X/S,tors}$ is an invertible $\mathcal{O}_R$-module. On the other hand, we have an exact sequence

$$0 \to \mathcal{O}_x \cdot \omega \xrightarrow{i} \mathcal{O}_x du \oplus \mathcal{O}_x dv \to (\Omega^1_{X/S})_x \to 0,$$

which induce an exact sequence

$$\mathcal{O}_x \cdot \omega \otimes \mathcal{O}_{Z,x} \xrightarrow{i\otimes 1} \mathcal{O}_{Z,x} du \oplus \mathcal{O}_{Z,x} dv \to (\Omega^1_{X/S} \otimes \mathcal{O}_Z)_x \to 0.$$

But the image of $i \otimes 1$ is zero, we have

$$(\Omega^1_{X/S} \otimes \mathcal{O}_Z)_x \cong \mathcal{O}_{Z,x} \oplus \mathcal{O}_{Z,x},$$

namely, $\Omega^1_{X/S} \otimes \mathcal{O}_Z$ is a locally free $\mathcal{O}_Z$-module of rank 2. We have shown (1).

By our theorem 2 and $\Omega^1_{X/S,tors} \cong \Omega^1_{X/S,tors} \otimes \mathcal{O}_Z$ as a $\mathcal{O}_X$-module, we have the exact sequence

$$\Omega^1_{X/S,tors} \to \Omega^1_{X/S} \otimes \mathcal{O}_Z \to (I_Z K_{X/S}) \otimes \mathcal{O}_Z \to 0.$$

We only need to check that $\Omega^1_{X/S,tors} \to \Omega^1_{X/S} \otimes \mathcal{O}_Z$ is locally injective, namely

$$(\Omega^1_{X/S,tors})_x = \mathcal{O}_x \cdot \bar{\omega}' \to \frac{(\Omega^1_{X/S})_x}{I_Z(\Omega^1_{X/S})_x}$$

is injective. If $a\bar{\omega}' \in I_Z(\Omega^1_{X/S})_x$, then $a\omega' \in I_Z \cdot (\mathcal{O}_X du \oplus \mathcal{O}_X dv)$. Thus

$$d_x | a\frac{\partial f}{\partial u} \cdot d_x^{-1}, \quad d_x | a\frac{\partial f}{\partial v} \cdot d_x^{-1}.$$

But d_x is the greatest common divisor of $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$, we have $d_x | a$, namely, $a\bar{\omega}' = 0$ and (2.1) is exact. The same argument implies that (2.2) is exact.

Before stating lemma 2, let us recall some notations of [4]. If $\mathcal{L}$ is a coheren $\mathcal{O}_X$-module with support contained in X_s, we define the Euler-Poincaré characteristic $\chi(\mathcal{L})$ of $\mathcal{L}$ as

$$\chi(\mathcal{L}) = dim_k H^0(\mathcal{L}) - dim_k H^1(\mathcal{L}).$$

If $\mathcal{K}^\cdot$ is an $\mathcal{O}_X$-module complex whose cohomology sheaf have supports in X_s, we define that

$$\chi(\mathcal{K}^\cdot) = \sum (-1)^i \chi(H^i(\mathcal{K}^\cdot)).$$

Lemma 2. *With above notations, we have*

(1) $c_1(\Omega^1_{X/S,tors}) = R$.
(2) $\chi(\Omega^1_{X/S,tors}) = R^2 + \chi(\mathcal{O}_R)$.

Proof. Applying our theorem 2 to $d = 1$, we have exact sequence

$$0 \to \Omega^1_{X/S,tors} \to \Omega^1_{X/S} \to I_Z\mathcal{K}_{X/S} \to 0,$$

which and $c_1(\Omega^1_{X/S}) = K_{X/S}$ implies that

$$c_1(\Omega^1_{X/S,tors}) = K_{X/S} - c_1(I_Z\mathcal{K}_{X/S}).$$

But $I_Z\mathcal{K}_{X/S} = I_D\mathcal{O}_X(K_{X/S} - R)$ and

$$0 \to I_D\mathcal{O}_X(K_{X/S} - R) \to \mathcal{O}_X(K_{X/S} - R) \to \mathcal{O}_D(K_{X/S} - R) \to 0,$$

we have

$$c_1(I_Z\mathcal{K}_{X/S}) = K_{X/S} - R - c_1(\mathcal{O}_D(K_{X/S} - R)).$$

Note that $c_1(\mathcal{O}_D \otimes \mathcal{L}) = 0$ for any invertible sheaf $\mathcal{L}$ on X, we get (1).

By the exact sequence (2.2) of lemma 1, we have

$$\chi(\Omega^1_{X/S,tors}) = \chi(\mathcal{O}_R \otimes \Omega^1_{X/S}) - \chi(\mathcal{O}_R \otimes I_Z\mathcal{K}_{X/S}).$$

By using a lemma of Bloch ([2], lemma 7.4), we claim that

$$\chi(\mathcal{O}_R \otimes \Omega^1_{X/S}) = K_{X/S}R + 2\chi(\mathcal{O}_R).$$

In fact, for any irreducible component Γ_i of R, Bloch's lemma says that, when one considers $\mathcal{O}_{\Gamma_i} \otimes \Omega^1_{X/S}$ as a locally free $\mathcal{O}_{\Gamma_i}$-module on Γ_i, one has that

$$c_1(\mathcal{O}_{\Gamma_i} \otimes \Omega^1_{X/S}) = \Gamma_i K_{X/S}.$$

Thus Riemann-Roch theorem on Γ_i implies that

$$\chi(\mathcal{O}_{\Gamma_i} \otimes \Omega^1_{X/S}) = \Gamma_i K_{X/S} + 2\chi(\mathcal{O}_{\Gamma_i}).$$

On the other hand, we have exact sequence

$$0 \to \mathcal{O}_{\Gamma_i}(-R + \Gamma_i) \to \mathcal{O}_R \to \mathcal{O}_{R-\Gamma_i} \to 0, \tag{2.3}$$

which induces the following exact sequence

$$0 \to \mathcal{O}_{\Gamma_i}(-R + \Gamma_i) \otimes \Omega^1_{X/S} \to \mathcal{O}_R \otimes \Omega^1_{X/S} \to \mathcal{O}_{R-\Gamma_i} \otimes \Omega^1_{X/S} \to 0,$$

since $\mathcal{O}_R \otimes \Omega^1_{X/S}$ is a locally free $\mathcal{O}_R$-module. Thus

$$\chi(\mathcal{O}_R \otimes \Omega^1_{X/S}) = \chi(\mathcal{O}_{R-\Gamma_i} \otimes \Omega^1_{X/S}) + \chi(\mathcal{O}_{\Gamma_i} \otimes \Omega^1_{X/S}) - 2\Gamma_i(R - \Gamma_i).$$

By induction for $\sum r_i$, we have

$$\chi(\mathcal{O}_R \otimes \Omega^1_{X/S}) = K_{X/S}R + 2\chi(\mathcal{O}_{R-\Gamma_i}) + 2\chi(\mathcal{O}_{\Gamma_i}) - 2\Gamma_i(R - \Gamma_i).$$

Use (2.3) again, we get

$$\chi(\mathcal{O}_R) = \chi(\mathcal{O}_{R-\Gamma_i}) + \chi(\mathcal{O}_{\Gamma_i}) - \Gamma_i(R - \Gamma_i),$$

namely, we have the claim.

Now we want to compute $\chi(\mathcal{O}_R \otimes I_Z \mathcal{K}_{X/S})$. Since $I_Z = I_R \cdot I_D$, we have

$$0 \to \mathcal{O}_R \otimes I_D \to \mathcal{O}_Z \to \mathcal{O}_D \to 0,$$

which induces

$$0 \to \mathcal{O}_R \otimes I_D \mathcal{O}(K_{X/S} - R) \to \mathcal{O}_Z \otimes \mathcal{O}(K_{X/S} - R) \to \mathcal{O}_D \otimes \mathcal{O}(K_{X/S} - R) \to 0.$$

But $\mathcal{O}_R \otimes I_Z \mathcal{K}_{X/S} \cong \mathcal{O}_R \otimes I_D \mathcal{O}(K_{X/S} - R)$, we have

$$\chi(\mathcal{O}_R \otimes I_Z \mathcal{K}_{X/S}) = \chi(\mathcal{O}_Z \otimes \mathcal{O}(K_{X/S} - R)) - \chi(\mathcal{O}_D \otimes \mathcal{O}(K_{X/S} - R)).$$

On the other hand, one has exact sequence

$$0 \to \mathcal{O}_D \otimes \mathcal{O}(-R) \to \mathcal{O}_Z \to \mathcal{O}_R \to 0,$$

which implies that

$$\chi(\mathcal{O}_Z \otimes \mathcal{O}(K_{X/S} - R)) = \chi(\mathcal{O}_R(K_{X/S} - R)) + \chi(\mathcal{O}_D \otimes \mathcal{O}(K_{X/S} - 2R)).$$

Note that $\chi(\mathcal{O}_D \otimes \mathcal{L}) = \chi(\mathcal{O}_D)$ for any invertible sheaf $\mathcal{L}$ on X, we get

$$\chi(\Omega^1_{X/S,tors}) = R^2 + \chi(\mathcal{O}_R).$$

Proof of Theorem 3.

Let $\mathcal{K}^{\cdot}$ denote the following two terms complex

$$\Omega^1_{X/S} \to \mathcal{K}_{X/S}.$$

Bloch's theorem ([3], theorem 2.3) tell us that

$$Art(X/S) = \chi(\mathcal{K}^{\cdot}).$$

Since he define the first term as degree -1, we have

$$Art(X/S) = \chi(H^1(\mathcal{K}^{\cdot})) - \chi(H^0(\mathcal{K}^{\cdot})),$$

and

$$H^0(\mathcal{K}^{\cdot}) = \Omega^1_{X/s,tors}$$
$$H^1(\mathcal{K}^{\cdot}) = \frac{\mathcal{K}_{X/S}}{I_Z \mathcal{K}_{X/S}} \cong \mathcal{O}_Z \otimes \mathcal{K}_{X/S}.$$

By using the exact sequence

$$0 \to \mathcal{O}_D \otimes \mathcal{O}(K_{X/S} - R) \to \mathcal{O}_Z \otimes \mathcal{K}_{X/S} \to \mathcal{O}_R \otimes \mathcal{K}_{X/S} \to 0$$

and $\chi(\mathcal{O}_D \otimes \mathcal{O}(K_{X/S} - R)) = \chi(\mathcal{O}_D) = \sum \mu_x(f)$, one has

$$\chi(H^1(\mathcal{K}^{\cdot})) = \sum \mu_x(f) + K_{X/S}R + \chi(\mathcal{O}_R).$$

Thus we have the formula

$$Art(X/S) = K_{X/S}R - R^2 + \sum_{x \in X_s} \mu_x(f).$$

Corollary 1. *Let $f : X \to S$ be a regular arithmetic surface and $f' : X' \to S$ the blowing up of X at a closed point. Then*

(1) $r(f') = r(f) + 1$

(2) $r(f) = 0$ *iff X/S smooth or X is of genus 1 and of type I_0.*

§3. Some remarks on base changes.

In this section, we shall give a few remarks about base changes. Applying our arguments to the case of function fields of characteristic zero, some known results can be drived out easily. Let K be a number field, $\mathcal{O}_K$ the ring of algebraic integers of K, and let $f : X \to B = Spec\,\mathcal{O}_K$ be a regular arithmetic surface of genus $g \geq 2$ over B, namely, X is a regular projective scheme of dimension 2, X_K is geometrically irreducible of genus $g \geq 2$. If $L \supset K$ is a finite extension of degree λ, then the natural morphism $\pi : \widetilde{B} = Spec\,\mathcal{O}_L \to B$ is called a base change of X/B. As the same as [11], we consider the following commutative diagram:

$$\begin{array}{ccccccccc} \widetilde{X} & \xleftarrow{\rho} & X_2 & \xrightarrow{\pi_2} & X_1 & \xrightarrow{\pi_1} & X \times_B \widetilde{B} & \xrightarrow{p_1} & X \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle f} \\ \widetilde{B} & = & \widetilde{B} & = & \widetilde{B} & = & \widetilde{B} & \xrightarrow{\pi} & B, \end{array}$$

where π_1 is the normalization of $X \times_B \widetilde{B}$, π_2 is the minimal desingularization of X_1 and ρ is the contraction of (-1)-curves in the singular fibres of f_2.

Let $\phi = p_1 \circ \pi_1$ and $\varphi = \phi \circ \pi_2$, we call $\tilde{f} : \widetilde{X} \to \widetilde{B}$ the induced arithmetic surface of π. Let $K_{\widetilde{X}/\widetilde{B}}$ and $K_{X/B}$ denote the Weil divisors of $\mathcal{K}_{\widetilde{X}/\widetilde{B}}$ and $\mathcal{K}_{X/B}$ and write $V = \varphi^* K_{X/B} - \rho^* K_{\widetilde{X}/\widetilde{B}}$, we have known that $V = f_2^* R(\pi) - R(\varphi) + R(\rho)$ and $\widetilde{V} = f_2^* R(\pi) - R(\varphi)$ is an effective vertical divisor ([11]), where $R(\pi)$, $R(\varphi)$ and $R(\rho)$ are ramification divisors of π, φ and ρ. Our first remark is an elementary lemma

Lemma 3. *Let $P_1, \cdots, P_s$ be the points of $\widetilde{B}$ where f_2 has bad reductions and let*

$$\widetilde{V} = \widetilde{V}_1 + \cdots + \widetilde{V}_s,$$

where $\widetilde{V}_i \subseteq f_2^{-1}(P_i)$. Then we have

(1) *$R' := R(\varphi) - R(\rho)$ is an effective divisor.*

(2) *Let $\chi_\pi = deg f_* \mathcal{K}_{X/B} - \frac{1}{\lambda} deg \tilde{f}_* \mathcal{K}_{\widetilde{X}/\widetilde{B}}$, then*

$$\chi_\pi = \frac{1}{\lambda} \sum_{i=1}^{s} dim_{k(P_i)} H^0(\mathcal{O}_{\widetilde{V}_i}(K_{X_2/\widetilde{B}} + \widetilde{V}_i)).$$

Proof. We can write $R' = D_1 - D_2$ such that D_1 and D_2 are effective divisors having no common components and $D_2 \subseteq R(\rho)$. Thus

$$D_2^2 = \varphi^* K_{X/B} \cdot D_2 + D_1 \cdot D_2 \geq 0$$

since $\rho^* K_{\widetilde{X}/\widetilde{B}} \cdot D_2 = 0$ and $f_2^* R(\pi) \cdot D_2 = 0$. But $D_2 \subseteq R(\rho)$, D_2 has to be zero. We have shown (1).

From [11], we known that $\varphi^* K_{X/B} - K_{X_2/\widetilde{B}} = \widetilde{V}$. Thus we have

$$0 \to \mathcal{K}_{X_2/\widetilde{B}} \to \varphi^* \mathcal{K}_{X/B} \to \mathcal{O}_{\widetilde{V}}(K_{X_2/\widetilde{B}} + \widetilde{V}) \to 0.$$

Since $f_{2*}\mathcal{O}_{\widetilde{V}}(K_{X_2/\widetilde{B}} + \widetilde{V})$ is a torsion $\mathcal{O}_L$-module and $R^1\mathcal{K}_{X_2/\widetilde{B}}$ is a locally free $\mathcal{O}_L$-module, we have exact sequence of $\mathcal{O}_L$-modules

$$0 \to f_{2*}\mathcal{K}_{X_2/\widetilde{B}} \to f_{2*}\varphi^* \mathcal{K}_{X/B} \to f_{2*}\mathcal{O}_{\widetilde{V}}(K_{X_2/\widetilde{B}} + \widetilde{V}) \to 0,$$

which is also volume exact ([8]). By Riemann-Roch theorem on arithmetic curves, we have

$$deg f_* \mathcal{K}_{X/B} - \frac{1}{\lambda} deg \tilde{f}_* \mathcal{K}_{\widetilde{X}/\widetilde{B}} = \frac{1}{\lambda} \chi(f_{2*}\mathcal{O}_{\widetilde{V}}(K_{X_2/\widetilde{B}} + \widetilde{V})),$$

namely, $\chi_\pi = \frac{1}{\lambda} \sum l_{\mathcal{O}_{L,P}}((f_{2*}\mathcal{O}_{\widetilde{V}}(K_{X_2/\widetilde{B}} + \widetilde{V}))_P)$. Note that

$$(f_{2*}\mathcal{O}_{\widetilde{V}}(K_{X_2/\widetilde{B}} + \widetilde{V}))_P = 0$$

if f_2 has good reduction at P, we get (2).

Theorem 4. *Let $f : X \to B = Spec\, \mathcal{O}_K$ be a regular arithmetic surface of genus $g > 1$, let $L \supset K$ be a finite extension of degree λ, $\mathcal{O}_L$ the ring of integers of L and $\widetilde{B} = Spec\, \mathcal{O}_L$. Then we have*

(1) $\frac{1}{\lambda}\mathcal{K}^2_{\widetilde{X}/\widetilde{B}} \leq \mathcal{K}^2_{X/B}$, *and* $\frac{1}{\lambda} deg \tilde{f}_* \mathcal{K}_{\widetilde{X}/\widetilde{B}} \leq deg f_* \mathcal{K}_{X/B}$, *where the second inequality is valid for any metric on* $f_*\mathcal{K}_{X/B}$.

(2) $\frac{1}{\lambda}\mathcal{K}^2_{\widetilde{X}/\widetilde{B}} = \mathcal{K}^2_{X/B}$ *if and only if all fibres of X/B are reduced and $\widetilde{X} = X_2$, X_1 has only rational double points.*

(3) $\frac{1}{\lambda}\mathcal{K}^2_{\widetilde{X}/\widetilde{B}} = \mathcal{K}^2_{X/B}$ *if and only if* $\frac{1}{\lambda} deg \tilde{f}_* \mathcal{K}_{\widetilde{X}/\widetilde{B}} = deg f_* \mathcal{K}_{X/B}$.

(4) *Let $R(\pi) = \sum r_P[P]$ be the ramification divisor of $\pi : \widetilde{B} \to B$ and $S = \{b \in B | f$ has bad reduction at $b\}$. If D is an effective horizontal divisor on X, and $\widetilde{D}$ is its proper transform on X_2. Then*

$$K^2_{X/B} - \frac{1}{\lambda} K^2_{\widetilde{X}/\widetilde{B}} \leq \frac{4g-4}{\lambda} \sum_{\pi(P) \in S} r_P \tag{3.1}$$

$$K_{X/B} \cdot D - \frac{1}{\lambda} K_{\widetilde{X}/\widetilde{B}} \cdot \rho_* \widetilde{D} \leq \frac{deg(D)}{\lambda} \sum_{\pi(P) \in S} r_P. \tag{3.2}$$

Proof. (1) and (2) have been proved in [11]. For (3), we only need to show that $\frac{1}{\lambda} deg \tilde{f}_* \mathcal{K}_{\widetilde{X}/\widetilde{B}} = deg f_* \mathcal{K}_{X/B}$ implies $V = 0$, which is equivalent to $\widetilde{V} = 0$ because

$V = R(\rho) + \widetilde{V}$, and $\widetilde{V} = 0$ will imply $R(\rho) = 0$. By lemma 3 (2), if $\frac{1}{\lambda} deg \tilde{f}_* \mathcal{K}_{\widetilde{X}/\widetilde{B}} = deg f_* \mathcal{K}_{X/B}$, we have

$$\sum_{i=1}^{s} dim_{k(P_i)} H^0(\mathcal{O}_{\widetilde{V}_i}(K_{X_2/\widetilde{B}} + \widetilde{V}_i)) = 0,$$

namely, for every i, one has

$$-dim_{k(P_i)} H^1(\mathcal{O}_{\widetilde{V}_i}(K_{X_2/\widetilde{B}} + \widetilde{V}_i)) = \chi(\mathcal{O}_{\widetilde{V}_i}(K_{X_2/\widetilde{B}} + \widetilde{V}_i) = \frac{1}{2}\varphi^* K_{X/B} \cdot \widetilde{V}_i \geq 0.$$

Thus $H^0(\mathcal{O}_{\widetilde{V}_i}) \cong H^1(\mathcal{O}_{\widetilde{V}_i}(K_{X_2/\widetilde{B}} + \widetilde{V}_i)) = 0$, which implies that $\widetilde{V}_i = 0$ for any i, we get (3).

By lemma 3, we have the following two equalities, which will imply (3.1) and (3.2) of (4) respectively if we remark that R' contains $r_P f_2^{-1}(P)$ when $\pi(P) \notin S$ and $deg(\widetilde{D}) = deg(D)$,

$$\lambda K^2_{X/B} - K^2_{\widetilde{X}/\widetilde{B}} = (4g-4)\sum_P r_P - 2\rho^* K_{\widetilde{X}/\widetilde{B}} \cdot R' + V^2$$

$$\lambda K_{X/B} \cdot D - K_{\widetilde{X}/\widetilde{B}} \cdot \rho_* \widetilde{D} = deg(D) \sum_P r_P - R' \cdot \widetilde{D}.$$

In the following example, we shall apply above theorem to the case of function fields of characteristic zero and drive out some known results (due to Tan, S-L.). However, our argument is very simple.

Example. *Let $f : S \to C$ be a non-isotrivial fibration of complex algebraic surface of genus g with $b = g(C)$. For any irreducible horizontal divisor D, we fix the following notations*

$$h_K(D) = \frac{K_{S/C} \cdot D}{deg(D)}, \quad d(D) = \frac{2g(\widetilde{D}) - 2}{deg(D)}$$

where $\widetilde{D}$ denotes the normalization of D. If s denotes the number of points of C at which f has bad reduction. Then one has

(1) $K^2_{S/C} \leq (2g-2)(2b-2+3s)$

(2) $h_K(D) \leq (2g-1)(d(D)+3s) - s - K^2_{S/C}$.

For any natural numbers d and e, a refinement of Kodaira-Parshin construction asserts that there is a cover $\pi : \widetilde{C} \to C$ of degree de such that π is ramified to order exactly e at all points lying over points of C of bad reduction. Applying above theorem (4) to this base change π, we have

$$K^2_{S/C} \leq \frac{1}{de} K^2_{\widetilde{S}/\widetilde{C}} + \frac{e-1}{e}(4g-4)s.$$

It is well known that $\tilde{f} : \widetilde{S} \to \widetilde{C}$ will be a semistable fibration when e becomes very large. Thus one can use Vojta's inequality ([8])

$$K^2_{\widetilde{S}/\widetilde{C}} \leq (2g-2)(2g(\widetilde{C}) - 2 + \tilde{s}).$$

Note that $\tilde{s} = ds$ and $2g(\widetilde{C}) - 2 = de(2b-2) + d(e-1)s$, one has

$$K^2_{S/C} \leq (2g-2)(2b-2+3s) - \frac{(4g-4)s}{e}.$$

This is (1).

As it was pointed out in [13], when $f : S \to C$ is semistable, the following inequality can be obtained by using Miyaoka-Yau inequality,

$$h_K(D) \leq (2g-1)(d(D)+s) - K^2_{S/C}.$$

The second step (main part of [13]) was devoted to show the following inequality for nonsemistable case,

$$h_K(D) < (2g-1)(d(D)+3s) - K^2_{S/C}.$$

We would like to present an alternative treatment for the second step of [13] by considering the commutative diagram

$$\begin{array}{ccccccc}
\widetilde{S} & \xleftarrow{\rho} & S_2 & \xrightarrow{\varphi_1} & S_1 & \xrightarrow{\varphi_1} & S \\
\downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} \\
\widetilde{C} & = & \widetilde{C} & \xrightarrow{\pi_2} & \widetilde{D} & \xrightarrow{\pi_1} & C,
\end{array}$$

where $\pi_1 : \widetilde{D} \to C$ is the normalization of D and $\pi_2 : \widetilde{C} \to \widetilde{D}$ is a cover of degree de such that π_2 is ramified to order exactly e at all points lying over points of $\widetilde{D}$ of bad reduction, S_1 and S_2 are minimal desingularizations of $S \times_C \widetilde{D}$ and $S_1 \times_{\widetilde{D}} \widetilde{C}$, ρ is the contraction of (-1)-curves in the singular fibres of f_2.

Write $\pi = \pi_1\pi_2$ and $\varphi = \varphi_1\varphi_2$, let E be a section of f_1 such that $\varphi_{1*}E = D$ and $\widetilde{E}$ the proper transform of E on S_2. Applying (3.2) of theorem 4 (4) to base change π, since $\varphi_*\widetilde{E} = deg(\pi_2)D$ and $deg(\pi) = de \cdot deg(D)$, we have

$$\frac{K_{S/C} \cdot D}{deg(D)} - \frac{1}{de \cdot deg(D)} K_{\widetilde{S}/\widetilde{C}} \cdot \rho_*\widetilde{E} \leq \frac{1}{de \cdot deg(D)} \sum_{\pi(P) \in S} r_P.$$

Let s_1 be the number of points of $\widetilde{D}$ where f_1 has bad reduction, and take e big enough so that $\tilde{f} : \widetilde{S} \to \widetilde{C}$ is semistable. Then, note that $d(\rho_*\widetilde{E}) = 2g(\widetilde{C}) - 2$, by using the inquality of semistable case and (3.1) of theorem 4 (4), one has

$$h_K(D) \leq (2g-1)(d(D) + \frac{s_1}{deg(D)}) - K^2_{S/C} + \frac{4g-3}{de \cdot deg(D)} \sum_{\pi(P) \in S} r_P.$$

The elementary computations tell us that $\sum_{\pi(P) \in S} r_P = de \cdot deg(D)s - ds_1$. Thus we can rewrite above equality as the following

$$h_K(D) \leq (2g-1)(d(D) + 2s + \frac{s_1}{deg(D)}) - K^2_{S/C} - s - \frac{4g-3}{e} \cdot \frac{s_1}{deg(D)}.$$

It is clear that we have done by the remark $s \leq s_1 \leq deg(D)s$.

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