Supersymmetric integrable systems from geodesic flows on superconformal groups [1]

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Abstract

We discuss the possible relationship between geodesic flow, integrability and supersymmetry, using fermionic extensions of the KdV equation, as well as the recently introduced supersymmetrisation of the Camassa-Holm equation, as illustrative examples.

1. Misha Saveliev was a pioneer of geometric constructions of supersymmetric integrable systems (e.g. [1]). In particular, the interrelationship between integrability, geometry, diffeomorphism invariance and supersymmetry was a special interest of his. We therefore feel that he would have appreciated our recent considerations on a) the meaning of integrability for systems containing both bosonic and fermionic fields and b) the relation between geodesic flow and integrability. Although geodesic flows are *not* integrable in general, many important integrable systems are in fact geodesic flow equations. This raises the question of whether one may geometrically determine integrable flows amongst geodesic ones. This has in fact been a pressing open question for ODEs as well as PDEs ever since Arnold noticed that the Euler flow equations for incompressible fluids, just like the Euler top equations, allow interpretation as geodesic flows on (finite or infinite dimensional) Lie groups. Briefly, an inner-product $\langle .,. \rangle$ on a Lie algebra $\mathfrak{g}$ determines a right (or a left) invariant metric on the corresponding Lie group G. The equation of geodesic motion on G with respect to this metric is determined by the bilinear operator $B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by

$$\Big\langle [V,W]\,,\,U \Big\rangle \;=\; \Big\langle W\,,\,B(U,V) \Big\rangle \quad , \qquad \forall \;\; W \in \mathfrak{g}\;. \tag{1}$$

The geodesic flow equation is then simply the Euler equation

$$U_t = B(U,U)\;. \tag{2}$$

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2. An important class of examples where geodesic flow is indeed integrable concerns the group of diffeomorphisms of the circle, $\mathrm{Diff}(S^1)$. Geodesic motion with respect to a metric induced by an L^2 norm, $\int u^2 dx$, describes Euler flow for a one dimensional compressible fluid, $u_t = -3uu_x$, which has implicit general solution, $ut = \frac{1}{3}x + F(u)$, F an arbitrary function, which describes extremely unstable shock waves. If the group is centrally extended to the Bott-Virasoro group, the celebrated KdV equation arises, which has extremely stable solutions. Now, if the metric is changed to one induced by the H^1 norm, $\int (u^2 + \nu u_x^2) dx$, $\nu \in \mathbb{R}$, one obtains the Camassa-Holm (CH) equation,

$$u_t - \nu u_{xxt} = -3uu_x + \nu(uu_{xxx} + 2u_x u_{xx}) \ . \tag{3}$$

This has drawn considerable interest recently as an integrable system (having bihamiltonian structure), but displaying more general wave phenomena than KdV, such as finite time blow-up of solutions and a class of piecewise analytic weak solutions known as *peakons.* (For more complete references we refer to [2]). Nothing is known about what precisely makes these geodesic flows integrable; the Euler equation for fluid flow in more than one spatial dimension is, in general, *not* integrable. The investigation of families of geodesic flows which include integrable cases may possibly yield clues about the special geometric features required. A convenient way to produce such families, for example containing the KdV and CH equations, is to consider geodesic flows on the superconformal group containing the Bott-Virasoro group as the even part. This is moreover a method of generating couplings to fermionic fields: a remarkably rich generalisation, as we shall see, of the purely bosonic KdV or CH systems. The important question of whether the coupled systems remain integrable has hitherto not been adequately explored.

3. The NSR superconformal algebra consists of triples $(u(x), \varphi(x), a)$, where u is a bosonic field, φ is a fermionic (odd) field and a is a constant. The Lie bracket is given by

$$\begin{aligned} &\Big[(u,\varphi,a)\,,\,(v,\psi,b)\Big] \\ &= \Big(uv_x - u_x v + \tfrac{1}{2}\varphi\psi\,,\, u\psi_x - \tfrac{1}{2}u_x\psi - \varphi_x v + \tfrac{1}{2}\varphi v_x\,,\, \int_{S^1} dx(c_1 u_x v_{xx} + c_2 u v_x + c_1 \varphi_x \psi_x + \tfrac{c_2}{4}\varphi\psi)\Big)\ , \end{aligned} \tag{4}$$

where c_1, c_2 are constants. Geodesic flow on the corresponding group with respect to an L^2 type metric induced by the norm (parametrised by $\alpha \in \mathbb{R}$),

$$\Big\langle (u,\varphi,a)\,,\,(v,\psi,b)\Big\rangle_{L^2} \ = \ \int_{S^1} dx \left(uv + \alpha\varphi\partial_x^{-1}\psi\right) \ + ab \tag{5}$$

yields a 1 parameter fermionic extension of KdV,

$$\begin{aligned} u_t &= u_{xxx} - 3uu_x + 2\xi\xi_{xx} \ , \\ \xi_t &= \tfrac{1}{\alpha}\xi_{xxx} - \tfrac{3}{2}u_x\xi - (1 + \tfrac{1}{2\alpha})u\xi_x \ , \end{aligned} \tag{6}$$

where the fermionic field is defined by $\varphi = \lambda\xi_x$; $\lambda^2 = \frac{4}{3\alpha}$. In general this family of systems is neither integrable nor supersymmetric (in the sense of being invariant under supersymmetry transformations between u and ξ, namely $\delta u = \tau\xi_x$, $\delta\xi = \tau u$, where τ is an odd parameter). Apart from the $\xi = 0$ KdV case, it is well known that this family contains two further integrable cases:

1) $\alpha=\frac{1}{4}:$ the kuperKdV system of Kupershmidt, which is bihamiltonian, but not supersymmetric,

2) $\alpha=1:$ the superKdV system of Mathieu and Manin-Radul, which is supersymmetric, but does not afford an extension of the KdV bihamiltonian structure.

It is interesting that both these are particular cases among the 1-parameter family of geodesic flows. Previously only the bihamiltonian kuperKdV system was thought to occur as a geodesic flow [4].

Extending (5) to the H^1 inner product,

$$\Big\langle (u,\varphi,a)\,,\,(v,\psi,b)\Big\rangle_{H^1} \;=\; \int_{S^1} dx\,\Big(uv+\nu u_x v_x+\alpha\varphi\partial_x^{-1}\psi+\alpha\mu\varphi_x\psi\Big)\;+ab\;, \tag{7}$$

where μ,ν are further constants, gives rise to the Euler equations

$$\begin{aligned} u_t-\nu u_{xxt} &= \kappa_1 u_x+\kappa_2 u_{xxx}-3uu_x+\nu(uu_{xxx}+2u_xu_{xx})+2\xi\xi_{xx}+\tfrac{2\mu}{3}\xi_x\xi_{xxx}\;,\\ \xi_t-\mu\xi_{xxt} &= \tfrac{\kappa_1}{4\alpha}\xi_x+\tfrac{\kappa_2}{\alpha}\xi_{xxx}-\tfrac{3}{2}u_x\xi-(1+\tfrac{1}{2\alpha})u\xi_x+\mu u\xi_{xxx}+\tfrac{3\mu}{2}u_x\xi_{xx}+\tfrac{\nu}{2\alpha}u_{xx}\xi_x\;. \end{aligned} \tag{8}$$

Here κ_1,κ_2 are independent parameters determined by a,c_1,c_2. This is evidently a 5-parameter family of systems containing CH (3) as well as the 1-parameter KdV family (6). It is automatically hamiltonian. Introducing new variables, $m=u-\nu u_{xx}$ and $\eta=\xi-\mu\xi_{xx}$, it may be re-written

$$\begin{pmatrix} m_t\\ \eta_t\end{pmatrix}=\mathcal{P}\begin{pmatrix}\frac{\delta\mathcal{H}}{\delta m}\\ \frac{\delta\mathcal{H}}{\delta\eta}\end{pmatrix} \tag{9}$$

where the graded hamiltonian structure

$$\mathcal{P}=\begin{pmatrix}\kappa_2\partial_x^3+\kappa_1\partial_x-\partial_x m-m\partial_x & \frac{1}{2}\partial_x\eta+\eta\partial_x\\ -\partial_x\eta-\frac{1}{2}\eta\partial_x & \frac{3}{4\alpha}(\frac{\kappa_1}{4}+\kappa_2\partial_x^2)-\frac{3}{8\alpha}m\end{pmatrix} \tag{10}$$

and the hamiltonian functional is given succinctly in terms of $U=(u,\varphi,0)$ by

$$\mathcal{H}_2\;=\;\tfrac{1}{2}\,\Big\langle U\,,\,U\Big\rangle_{H^1}\;=\;\tfrac{1}{2}\int dx\,\Big(u^2+\nu u_x^2+\tfrac{4}{3}(\xi_x\xi+\mu\xi_{xx}\xi_x)\Big)\quad. \tag{11}$$

This generalises the so-called *second Hamiltonian structure* of KdV and its fermionic extensions, as well as that of CH. The *first Hamiltonian structure* of the latter systems does extend to systems of the general form (8), but the single intersection with this 5-parameter family is the kuperKdV case. There are therefore no bihamiltonian extensions of CH amongst the systems (8). The latter however do contain the superCH system,

$$\begin{aligned} u_t-u_{xxt} &= -3uu_x+2\xi\xi_{xx}+uu_{xxx}+2u_xu_{xx}+\tfrac{2}{3}\xi_x\xi_{xxx}\;,\\ \xi_t-\xi_{xxt} &= -\tfrac{3}{2}(u\xi)_x+u\xi_{xxx}+\tfrac{3}{2}u_x\xi_{xx}+\tfrac{1}{2}u_{xx}\xi_x\;, \end{aligned} \tag{12}$$

the unique supersymmetric extension of (3). This is invariant under the transformations $\delta u=\tau\xi_x$, $\delta\xi=\frac{3}{4}\tau u$. We have found sufficiently nontrivial evidence [2] to allow us to conjecture the integrability of this system.

4. The meaning of integrability for such systems with fermions remains somewhat confusing. Usual arguments for integrability have consisted of showing, for instance, the existence of either bihamiltonian structures or an infinite number of conserved quantities. Neither of these are very reliable criteria. In particular, the superKdV example demonstrates that bihamiltonicity is probably sufficient, but by no means necessary for integrability of systems with fermions. Discussions of a generalised Painlevé test for fermionic systems also exist, but the analysis with bosonic and fermionic fields taking values in respectively the even and odd parts of some grassmann algebra has proven difficult to perform carefully enough and there are serious errors in the literature. Usually the underlying grassmann algebra is thought of as an infinite dimensional algebra, generated by infinitely many odd generators. However, integrability should clearly be independent of the choice of this underlying grassmann algebra; and choosing some low dimensional algebra, generated by a small number of odd generators, can yield a great deal of useful information about the general case. In particular, consideration of a series of the simplest grassmann algebras generated by one, two, three,... odd generators has the power of yielding conclusive evidence for the non-integrability of the system in general. With only a finite number of odd generators, the fields afford expansion in a basis of polynomials of the odd generators. We shall talk of the *n-th deconstruction* when referring to coefficients of up to nth-order monomials in the odd generators. For instance, a bosonic and a fermionic field in the *second deconstruction* takes values in a grassmann algebra with basis $\{1, \tau_1, \tau_2, \tau_1\tau_2\}$, where τ_1 , τ_2 are two odd generators. They take the form

$$u = u_0 + \tau_1\tau_2\, u_1 \ , \quad \xi = \tau_1 w_1 + \tau_2 w_2 \ . \tag{13}$$

Such *deconstructed fields* have purely bosonic components, u_0, u_1, w_1, w_2, in terms of which the analysis is considerably more transparent. Manton [5] recently investigated some simple supersymmetric classical mechanical systems in this 'deconstructive' fashion.

We have investigated the somewhat more general family of fermionic extensions of KdV,

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x - \xi\xi_{xx} \\ \xi_t &= -c\xi_{xxx} + a\xi u_x + bu\xi_x \ , \end{aligned} \tag{14}$$

here displayed in rescaled conventions with respect to (6). Now, choosing the simplest grassmann algebra, with single odd generator τ and basis $\{1, \tau\}$, yields the coupling of the KdV field u to another bosonic field w,

$$u_t = -u_{xxx} + 6uu_x \tag{15}$$

$$w_t = -cw_{xxx} + awu_x + buw_x \ , \tag{16}$$

where $\xi{=}\tau w$. This system, being purely bosonic, allows singularity analysis in a manner similar to that of Weiss, Tabor and Carnevale. The WTC-Painlevé analysis for merely this simple system sets strong restrictions on the values of a, b, c in the general system (14) for which integrability remains a possibility.

The WTC algorithm for (15)-(16) is to seek Laurent series solutions in the neighbourhood of an arbitrary singularity manifold, $\phi(x,t){=}0$, of the form

$$u(x,t) = \sum_{n=0}^{\infty} b_n(x,t)\phi(x,t)^{n-2} \ , \quad w(x,t) = \sum_{n=n_1}^{\infty} a_n(x,t)\phi(x,t)^n \ . \tag{17}$$

Here the series for u is the standard WTC series for (15), with b_4, b_6 arbitrary, and the remaining coefficients b_n determined recursively. The series for w is a Frobenius-Fuchs type series, with three arbitrary coefficient functions $a_{n_1}, a_{n_2}, a_{n_3}$ $(n_3 > n_2 > n_1)$ and the remainder of the a_n's determined recursively. It is straightforward to show that $n_1+n_2+n_3 = 3$, and as $N \equiv n_3-n_1$ increases, the number of consistency conditions for existence of the w series increases rapidly. We have examined all possible cases for $N < 10$. For N=8, 9 there are no consistent cases, and the list of cases for smaller N is also very limited; these are tabulated below. It is clear that crucial questions about the general system (14) may already be decided in the very simple 1st deconstruction. Non-integrability, in particular, seems to make itself known early on.

	N	n_i	(a,b,c)	
i	2	0,1,2	(0,0,c)	uncoupled system
ii	4	-1,1,3	(3,6,4)	kuperKdV
iii	5	-2,2,3	(3,3,1)	superKdV
iv			(-6,-6,-2)	
v		-1,0,4	(0,3,1)	potential form of superKdV
vi			(0,-6,-2)	potential form of *iv*
vii	7	-3,2,4	(6,6, 1)	linearisation: $w = \delta u$
viii		-2,0,5	(0,6,1)	potential form of *vii*

Table: P-integrable cases within the family (16)

As we see from this list, most of the cases passing the P-test could have been expected, being 1st deconstructions of kuperKdV (*ii*) and superKdV (*iii*), the so-called *potential form* of the latter having solutions given by $w_{(v)} = \int w_{(iii)}$, and two trivial cases: the uncoupled system (*i*) and the linearisation of KdV with w denoting the variation δu. There is therefore basically only one unexpected case (*iv*), together with its potential form (*vi*).

Proceeding to the 2nd deconstruction, with two odd generators and fields having the structure (13), we note that apart from a multiplicity of w's the only significant change is the enhancement of the system (15),(16) by an additional equation for u_1, the 'soul' of u,

$$u_{1t} = -u_{1xxx} + 6(u_0 u_1)_x - w_{[1}w_{2]xx} \ , \tag{18}$$

where u_0 satisfies (15) and w_1, w_2 are solutions of (16). The P-analysis of this provides a major surprise: u_1 *has a leading term of order* ϕ^{-3}! So coupling fermions to KdV essentially changes the singularities of its solutions. In the literature it has always been erroneously assumed that the Laurent series solution leads with a double pole, as for the simple, uncoupled, KdV equation. This also demonstrates deconstruction to be a very constructive analytical tool for the investigation of systems containing fermions. The only cases in the above table which do not pass the P-test for the 2nd deconstruction are the unexpected cases *iv* and *vi*. In general, therefore, these cases do not correspond to new integrable fermionic extensions of KdV. The 1st deconstructions are, however, certainly P-integrable. This illustrates a further important advantage of this deconstructive method: It provides a simple routine for the construction of integrable couplings of systems known to be integrable.

5. With manifold choices available for the underlying grassmann algebra, both superCH (12) and the integrable cases amongst (14) provide a very rich class of solvable systems. In fact they include some important classical ODEs as reductions. For instance, in the galilean reference frame, with all fields depending on only one variable $z{=}x{-}vt$, CH (just as KdV) corresponds to the familiar equation,

$$p'^2 = 1 - 2c_1 p + c_2 p^2 - \tfrac{2}{v} p^3 \ , \tag{19}$$

where c_1, c_2 are integration constants. This equation has a well-known general solution in terms of the Weierstraß $\wp$-function,

$$p(z) = -2v\wp(z) + \tfrac{1}{6} c_2 v \ , \tag{20}$$

where the periods of $\wp$ are determined by the coefficients c_1, c_2, v. In this galilean reference frame, the first and second deconstructions of super/kuper KdV, for which (19) is the 'body', are special cases of the Lamé equation which have been of interest for over a century. For instance, for the superKdV case, the first deconstruction yields the equation

$$w(z)'' - \left(6\wp(z - z_0) - \tfrac{v}{2}\right) w(z) = d \ , \quad d = const. \tag{21}$$

In the case of our new superCH system, we have shown in [2] that the 1st deconstruction and the galilean reduction of the 2nd deconstruction fulfil the requirements of the P-test. These are therefore nontrivial integrable reductions of the superCH system (12). In the process we have encountered further integrable generalisations of Lamé's equation which seem to be new: In the 1st deconstruction the fermionic equation has galilean reduction,

$$h'' + \tfrac{3}{8}\left(\frac{p}{v} - \frac{c_2}{6} + \frac{c_1}{p} - \frac{7}{2p^2}\right) h \ = \ 0 \ , \tag{22}$$

and the soul of the bosonic equation in the 2nd deconstruction yields

$$k'' + \left(\frac{3p}{v} - \frac{c_2}{4} - \frac{3}{4p^2}\right) k \ = \ 0 \ , \tag{23}$$

where, in both equations, p is given by the linear expression in $\wp$ (20). To conclude we note that integrability for PDEs is certainly not an all-or-nothing affair. Even if superCH turns out to be non-integrable in general, we have shown that it does display nontrivial integrability properties and it would be interesting to understand the geometry underlying our analytical results.

References

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