On Poincaré series on Sp_m

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Introduction

In this paper we compute the Fourier-Jacobi expansion of (holomorphic) Poincaré series on the Siegel modular group $\Gamma_m = Sp(m, \mathbb{Z})$. The case of Γ_2 was treated in [Ko1], where also the possibility of generalisation to Γ_m was indicated.

Let's briefly describe the main result of this paper. Given even integer $k > 2m$, and T a positive definite symmetric half-integral (m, m)-matrix, we denote by $P_{k,T}$ the T^{th} Poincaré series of exponential type and of weight k on Γ_m. Now fix a decomposition $m = n + r$ of m into sum of two integers $n, r \geq 1$. Then we show (Theorem, sect.3) that the Fourier-Jacobi coeficients of $P_{k,T}$ can be written as an infinite linear combination of Jacobi-Poincaré series $P^{J,n,r}$ and $P^{J,r,n}$ on the Jacobi groups $\Gamma^J_{n,r}$ and $\Gamma^J_{r,n}$ respectively; the coefficients of these combinations involve the Fourier coefficients of $P^{J,n,r}$ only.

In preliminary section 1 we recall some basic facts about Siegel- and Jacobi-Poincaré series. In section 2 we compute the Fourier expansion of some Jacobi-Poincaré series; we use it in an appendix to justify the change in order of summation made in the proof of main result. As an application we give, in case $m = 2n$, $n = r$, an explicit description of the adjoints w.r.t. the Petersson scalar products of the maps which send a Siegel modular form to its various Fourier-Jacobi coefficients (sect.4).

We hope that our results will have some further applications as suggested by W. Kohnen, namely to the problem of meromorphic continuation of non-holomorphic Siegel-Poincaré series and to improve the estimates for the Fourier coefficients of Siegel cusp forms.

The starting point for this paper is the coset decomposition for $\Gamma_{m,\infty} \setminus \Gamma_m$ given by S. Böcherer [Bo1,2].

Notations

For a commutative ring A with 1 we denote by $A^{(p,q)}$ the set of (p, q)-matrices with components in A.

The transpose of a matrix X will be denoted by X^t. For matrices X and Y (of appropriate sizes) we set $X[Y] := Y^t X Y$.

If $X, Y \in \mathbb{R}^{(p,p)}$ are symmetric we write $X > Y$ if $X - Y$ is positive definite.

If $Z \in \mathbb{C}^{(p,p)}$ we set $e(Z) := e^{2\pi i\, tr(Z)}$.

We write $S_k(\Gamma_m)$ for the space of Siegel cusp forms of weight $k \in \mathbb{Z}$ on Γ_m.

1. Preliminaries on Siegel- and Jacobi-Poincaré series

We start with recalling some facts about Jacobi groups and Jacobi and Siegel forms. For details the reader is referred to [Kli], [Zie].

Throughout the paper we fix natural numbers m, n, r with $m = n + r$.

Let $\mathbb{H}_m = \{Z \in \mathbb{C}^{(m,m)} | Z = Z^t, Im(Z) > 0\}$ denote the Siegel upper half-space of degree m and let $\Gamma_m = Sp(m, \mathbb{Z})$. We usually write $Z = \begin{pmatrix} \tau & z^t \\ z & \tau' \end{pmatrix}$ with $\tau \in \mathbb{H}_{m-r}$, $\tau' \in \mathbb{H}_r$ and $z \in \mathbb{C}^{(r,m-r)}$. The natural action of Γ_m on $\mathbb{H}_m$ is denoted by $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \langle Z \rangle = (AZ + B)(CZ + D)^{-1}$.

Fix an even integer $k \geq 2m + 1$ and a positive definite symmetric half-integral (m, m)-matrix T.

Let $\Delta \subset \Gamma_m$ be the subgroup consisting of matrices of the form $\begin{pmatrix} 1_m & S \\ 0 & 1_m \end{pmatrix}$ with $S \in \mathbb{Z}^{(m,m)}$ and $S^t = S$.

Let

$$P_{k,T}(Z) := \sum_{\gamma \in \Delta \backslash \Gamma_m} j(\gamma, Z)^{-k} e(T\gamma\langle Z \rangle), \quad (Z \in \mathbb{H}_m) \tag{1}$$

be T^{th} Poincaré series of weight k on Γ_m. It is absolutely uniformly convergent on compact subsets of $\mathbb{H}_m$. Also it is well known that $P_{k,T} \in S_k(\Gamma_m)$.

We consider the Heisenberg group

$$\mathbb{H}_{n,r} := \left\{((\lambda, \mu), \kappa) | \lambda, \mu \in \mathbb{R}^{(r,n)}, \kappa \in \mathbb{R}^{(r,r)}, \kappa + \mu\lambda^t \ symmetric\right\}$$

with group law

$$((\lambda, \mu), \kappa) \cdot ((\lambda', \mu'), \kappa') = \left((\lambda + \lambda', \mu + \mu'), \kappa + \kappa' + \lambda\mu'^t - \mu\lambda'^t\right).$$

The group $Sp(n, \mathbb{R})$ acts on $\mathbb{H}_{n,r}$ from the right by $((\lambda, \mu), \kappa) \circ M = ((\lambda, \mu)M, \kappa)$.

The Jacobi group is by definition the semi-direct product

$$G^J_{n,r} := Sp(n, \mathbb{R}) \ltimes \mathbb{H}_{n,r}$$

The mapping

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ((\lambda, \mu), \kappa)\right) \mapsto \begin{pmatrix} a & 0 & b & \mu' \\ \lambda & 1_r & \mu & \kappa \\ c & 0 & d & -\lambda' \\ 0 & 0 & 0 & 1_r \end{pmatrix}$$

with $(\lambda', \mu') \in \mathbb{R}^{(n,r)} \times \mathbb{R}^{(n,r)}$ given by

$$(\lambda'^t, \mu'^t) = (\lambda, \mu) \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$$

defines an embedding $G^J_{n,r} \to Sp(m,\mathbb{R})$.

We will denote by $M^\dagger$ the image of $M \in G^J_{n,r}$ under this embedding.

Let $Z = \begin{pmatrix} \tau & z^t \\ z & \tau' \end{pmatrix} \in \mathbb{H}_m$. Then

$$(2) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger \langle Z \rangle = \begin{pmatrix} (a\tau+b)(c\tau+d)^{-1} & (c\tau+d)^{-1}z^t \\ z(c\tau+d)^{-1} & \tau' - (c(c\tau+d)^{-1})[z^t] \end{pmatrix},$$

with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n,\mathbb{R})$$

and

$$(3) \qquad ((\lambda,\mu),\kappa)^\dagger \langle Z \rangle = \begin{pmatrix} \tau & z^t + \tau\lambda^t + \mu^t \\ \tau\lambda + z + \mu & \tau[\lambda^t] + z\lambda^t + \mu\lambda^t + \lambda z^t + \tau' + \kappa \end{pmatrix},$$

where $((\lambda,\mu),\kappa) \in \mathbb{H}_{n,r}$

Set $\Gamma^J_{n,r} := G^J_{n,r}(\mathbb{Z})$.

We denote by $J^{cusp}_{k,M,n,r}$ the (finite-dimensional) linear space of Jacobi forms of weight $k \in \mathbb{Z}$ and index a positive definite half-integral matrix M of size r on $\Gamma^J_{n,r}$. It is Hilbert space under the Petersson inner product

$$< f,g >= \int_{\Gamma^J_{n,r}\backslash\mathbb{H}_n \times \mathbb{C}^{(r,n)}} f(\tau,z)\overline{g(\tau,z)}\ det(v)^k e^{-4\pi M v^{-1}[y^t]} dV^J_{n,r}$$

where

$\tau = u+iv$, $z = x+iy$ and $dV^J_{n,r} = det(v)^{-r-2}dudvdxdy$.

Define the T^{th} Poincaré series of weight k on $\Gamma^{J\dagger}_{n,r}$ by

$$(4) \qquad P^{J,n,r}_{k,T}(Z) := \sum_{\gamma\in\Delta\backslash\Gamma^{J\dagger}_{n,r}} j(\gamma,Z)^{-k} e(T\gamma\langle Z\rangle), \quad (Z \in \mathbb{H}_m)$$

It is absolutely uniformly convergent on compact subsets of $\mathbb{H}_m$ [Ko2]. One can prove that $P^{J,n,r}_{k,T} \in J^{cusp}_{k,M,n,r}$.

Set

$$\Gamma^+_{n,\infty} := \Gamma_{n,\infty}/\{\pm I_{2n}\}$$

Then the elements $((\lambda,0),0)^\dagger M^\dagger$, where $\lambda \in \mathbb{Z}^{(m-r,r)}$, $M \in \Gamma^+_{n,\infty} \backslash \Gamma_n$, form a complete set of representatives for $\Delta \backslash \Gamma^{J\dagger}_{n,r}$.

Hence

$$(5) \quad P^{J,n,r}_{k,T}(Z) = \sum_{M\in\Gamma^+_{n,\infty}\backslash\Gamma_n,\ \lambda\in\mathbb{Z}^{(n,r)}} j(M^\dagger,Z)^{-k} e\left(T \begin{pmatrix} 1_r & \lambda^t \\ 0 & 1_{m-r} \end{pmatrix}^\sharp M^\dagger\langle Z\rangle\right)$$

where

$$U \mapsto U^{\sharp} := \begin{pmatrix} U^{t^{-1}} & 0 \\ 0 & U \end{pmatrix}$$

denotes a monomorphism $GL(m, \mathbb{Z}) \to Sp(m, \mathbb{Z})$.

2. Fourier expansion of Jacobi-Poincaré series

Notation: we put, for $n' \leq n$

$$\Gamma_{n',\star} := \{M \in \Gamma_{n'} | C_M \textit{ has maximal rank}\}.$$

The purpose of this section is to compute the Fourier expansion of the series

$$(6\,.)\quad P_{k,T}^{J,n',r,\star}(Z) := \sum_{M \in \Gamma^{+}_{n',\infty} \backslash \Gamma_{n',\star},\ \lambda \in \mathbb{Z}^{(n,r)}} j(M^{\uparrow}, Z)^{-k} e\left(T \begin{pmatrix} 1_r & \lambda^t \\ 0 & 1_n \end{pmatrix}^{\sharp} M^{\uparrow}\langle Z \rangle\right)$$

We use similar arguments as in [GKZ], [BoK], where particular cases were treated.

Proposition 1. *The series $P_{k,T}^{J,n',r,\star}(Z)$ has the following Fourier expansion*

(7)
$$P_{k,T}^{J,n',r,\star}(Z) = e(M\tau') \sum_{\substack{N' \in \mathbb{Z}^{(n,n)},\ R' \in \mathbb{Z}^{(n,r)} \\ N' - M^{-1}[\frac{R'^t}{2}] > 0}} g^{J,n',r,\star}_{k,M;(N,R)}(N', R') e(N'\tau + R'z)$$

where

$$g^{J,n',r,\star}_{k,M;(N,R)}(N', R') := i^{n(n-k-1)} \pi^{n^2} 2^{n^2 - \frac{r}{4}} det(M)^{-\frac{n}{2}}$$
$$\cdot\, det(c)^{k - \frac{r}{2} - 1} det\left((N - M^{-1}[\frac{R^t}{2}])(N' - M^{-1}[\frac{R'^t}{2}])^{-1}\right)^{\frac{r-k-1}{2}}$$
$$\cdot \sum_{c:\ det(c) \neq 0} H_{M,c}(N, R, N', R')$$
$$\cdot\, J_{k - \frac{r}{2} - 1}\left(2\pi \left((N - M^{-1}[\frac{R^t}{2}])(N' - M^{-1}[\frac{R'^t}{2}])\right)^{\frac{1}{2}}\right)$$

and

$$H_{M,c}(N, R, N', R') := det(c)^{-\frac{r}{2} - 1} \sum_{x(c),\ y(c)^*} e_c\left((Mx + Rx + N)\overline{y} + N'y + R'x\right)$$
$$\cdot\, e_{2c}(R' M^{-1} R^t)$$

is a Kloosterman-type sum

(*here x resp. y run over a complete set of representatives for* $\mathbb{Z}^{(n,r)}/c\mathbb{Z}^{(n,r)}$ *resp.* $(\mathbb{Z}^{(n',n')}/c\mathbb{Z}^{(n',n')})^*$)

and

$$J_{k-\frac{r}{2}-1}(X) = \int_{iC-\infty}^{iC+\infty} \cdots \int_{iC-\infty}^{iC+\infty} det(s)^{\frac{r}{2}-k} exp\,(tr\ X(s-s^{-1}))ds$$

is a generalized matrix Bessel function.

Remarks 1. Given $T' = \begin{pmatrix} N' & \frac{R'}{2} \\ \frac{R'^t}{2} & M \end{pmatrix}$, N', R', M as in the Proposition, the condition $N' - M^{-1}[\frac{R'^t}{2}] > 0$ is equivalent to $T' > 0$ as follows from the Jacobi decomposition

$$\begin{pmatrix} N' & \frac{R'}{2} \\ \frac{R'^t}{2} & M \end{pmatrix} = \begin{pmatrix} N' - M^{-1}[\frac{R'^t}{2}] & 0 \\ 0 & M \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ \frac{M^{-1}R'^t}{2} & 1 \end{pmatrix} \right] .$$

2. $J_{k-\frac{r}{2}-1}(X)$ is essentially the matrix Bessel function considered in [Her].

Proof (of Prop.1): First, from (2) and (3) we deduce the explicit expression

(8)
$$\begin{aligned} P_{k,T}^{J,n',r,*}(Z) = e(M\tau') & \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^+_{n',\infty} \backslash \Gamma_{n',*},\ \lambda \in \mathbb{Z}^{(n,r)}} det\,(c\tau+d)^{-k} \\ & \cdot e(N(a\tau+b)(c\tau+d)^{-1} + Rz(c\tau+d)^{-1} \\ & + R\lambda(a\tau+b)(c\tau+d)^{-1} + M(a\tau+b)(c\tau+d)^{-1}[\lambda^t] \\ & + 2\lambda^t zM(c\tau+d)^{-1} - Mc[z^t](c\tau+d)^{-1}) \quad ; \end{aligned}$$

here we identify $X \in Sp(n')$ with its image in $Sp(n)$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Also note the following identities

(*a*) $(a\tau+b)(c\tau+d)^{-1} = ac^{-1} - (c\tau+d)^{t^{-1}}c^{-1}$

(*b*) $tr\,(z(c\tau+d)^{-1} + \lambda(a\tau+b)(c\tau+d)^{-1}) = tr\,((z-\lambda c^{-1})(c\tau+d)^{-1} + \lambda ac^{-1})$

(*c*) $tr\,(M(a\tau+b)(c\tau+d)^{-1}[\lambda^t] + 2\lambda^t zM(c\tau+d)^{-1} - Mc[z^t](c\tau+d)^{-1}) =$

$$tr\left(-Mc(c\tau+d)^{-1}[z-\lambda c^{-1}]+Mac^{-1}[\lambda^t]\right)$$

Replace (d,λ) by $(d+c\alpha,\lambda+c\beta)$, with the new d and λ running over a complete set of representatives for $\left(\mathbb{Z}^{(n',n')}/c\mathbb{Z}^{(n',n')}\right)^*$ resp. $\mathbb{Z}^{(n,r)}/c\mathbb{Z}^{(n,r)}$, and $\alpha\in\mathbb{Z}^{(n',n')}$, $\beta\in\mathbb{Z}^{(n,r)}$. We obtain, for c with $det(c)\neq 0$, the contribution

$$e(M\tau')\sum_{c:\ det(c)\neq 0}\ \sum_{d(c)^*,\lambda(c),\ \alpha\in\mathbb{Z}^{(n',n')},\ \beta\in\mathbb{Z}^{(n,r)}} det(c)^{-k}det(\tau+c^{-1}d+\alpha)^{-k}$$

$$\cdot\ e(-M(\tau+c^{-1}d+\alpha)^{-1}[z-\lambda c^{-1}-\beta]+Mac^{-1}[\lambda^t])$$

$$\cdot\ e(R((z-\lambda c^{-1}-\beta)c^{-1}(\tau+c^{-1}d+\alpha)^{-1}+\lambda ac^{-1})))$$

$$\cdot e(N(ac^{-1}-c^{-2}(\tau+c^{-1}d+\alpha)^{-1}))$$

$$(9)\qquad =e(M\tau')\sum_{c:\ det(c)\neq 0} det(c)^{-k}\sum_{d(c)^*,\ \lambda(c)} e_c\left((M[\lambda^t]+R\lambda+N)\overline{d}\right)$$
$$\cdot\ \mathfrak{F}_{k,M,c;(N,R)}\left(\tau+c^{-1}d,z-\lambda c^{-1}\right)$$

Here

$$\mathfrak{F}_{k,M,c;(N,R)}(\tau,z):=\sum_{\alpha\in\mathbb{Z}^{(n',n')},\ \beta\in\mathbb{Z}^{(n,r)}} det(\tau+\alpha)^{-k}\ e\left(-M(\tau+\alpha)^{-1}[z-\beta]\right)$$
$$\cdot\ e\left(-Nc^{-2}(\tau+\alpha)^{-1}+R(z-\beta)c^{-1}(\tau+\alpha)^{-1}\right)$$

The latter function has period 1 in τ and z, hence an expansion:

$$\sum_{N'\in\mathbb{Z}^{(n,n)},\ R'\in\mathbb{Z}^{(n,r)}}\gamma(N',R')\ e(N'\tau+R'z)$$

where

$$\gamma(N',R'):=\int_{iC_1-\infty}^{iC_1+\infty}\cdots\int_{iC_1-\infty}^{iC_1+\infty} det(\tau)^{-k}\ e(-N'\tau)\int_{iC_2-\infty}^{iC_2+\infty}\cdots\int_{iC_2-\infty}^{iC_2+\infty}$$
$$\cdot\ e\left(-M\tau^{-1}[z]-Nc^{-2}\tau^{-1}+Rzc^{-1}\tau^{-1}-R'z\right)dz\,d\tau$$

We substitute $z\mapsto z+\frac{1}{2}M^{-1}R^tc^{-1}-\frac{1}{2}M^{-1}R'^t\tau$. Then the inner integral becomes

$$e_{2c}\left(-R'M^{-1}R^t\right)\ e(N'\tau)\ e\left((M^{-1}[\frac{R^t}{2}]-N)c^{-2}\tau^{-1}+(M^{-1}[\frac{R'^t}{2}]-N')\tau\right)$$

$$\cdot\int_{iC_2'-\infty}^{iC_2'+\infty}\cdots\int_{iC_2'-\infty}^{iC_2'+\infty} e\left(-\tau^{-1}M[z]\right)dz$$

The latter integral is standard and equals $det(M)^{-\frac{n}{2}} det\left(\frac{\tau}{2i}\right)^{\frac{r}{2}}$ (cf. [Fre, p.21]).
Hence we find

$$\gamma(N',R') = 2^{-\frac{r}{2}} det(M)^{-\frac{n}{2}} e_{2c}(-R'M^{-1}R^t) \int_{iC_1-\infty}^{iC_1+\infty} \dots \int_{iC_1-\infty}^{iC_1+\infty} det\left(\frac{\tau}{i}\right)^{\frac{r}{2}} det(\tau)^{-k}$$
$$\cdot\, e\left((M^{-1}[\frac{R^t}{2}] - N)c^{-2}\tau^{-1} + (M^{-1}[\frac{R'^t}{2}] - N')\tau\right) d\tau$$

If $N' - M^{-1}[\frac{R'}{2}] > 0$, we make the substitution

$$\tau = ic^{-1}\left((N - M^{-1}[\frac{R}{2}])(N' - M^{-1}[\frac{R'}{2}])\right)^{\frac{1}{2}} s$$

We get:

(10)

$$\gamma(N',R') = 2^{-\frac{r}{2}} det(M)^{-\frac{n}{2}} e_{2c}(-R'M^{-1}R^t)\, i^{(-1-k)n} det(c)^{k-\frac{r}{2}-1}$$
$$\cdot\, det\left((N - M^{-1}[\frac{R^t}{2}])(N' - M^{-1}[\frac{R'^t}{2}])^{-1}\right)^{\frac{r}{4}-\frac{k}{2}-\frac{1}{2}} \int_{iC_1'-\infty}^{iC_1'+\infty} \dots \int_{iC_1'-\infty}^{iC_1'+\infty}$$
$$\cdot\, exp\left(2\pi\, tr(((N - M^{-1}[\frac{R^t}{2}])(N' - M^{-1}[\frac{R'^t}{2}]))^{\frac{1}{2}}(s - s^{-1}))\right)$$
$$\cdot\, det(s)^{\frac{r}{2}-k} ds$$

Now, from the theory of the Laplace transform we obtain that the integral on the right-hand side of (10) has the value

$$(2\pi i)^{n^2} J_{k-\frac{r}{2}-1}\left(2\pi\left((N - M^{-1}[\frac{R^t}{2}])(N' - M^{-1}[\frac{R'^t}{2}])\right)^{\frac{1}{2}}\right),$$

hence

$$\gamma(N',R') = i^{n(n-k-1)} \pi^{n^2} 2^{n^2-\frac{r}{2}} det(M)^{-\frac{n}{2}}$$
$$\cdot\, det(c)^{k-\frac{r}{2}-1} det\left((N - M^{-1}[\frac{R^t}{2}])(N' - M^{-1}[\frac{R'^t}{2}])^{-1}\right)^{\frac{r}{2}-\frac{k}{2}-\frac{1}{2}}$$
$$\cdot\, e(-R'M^{-1}R^t)\, J_{k-\frac{r}{2}-1}\left(2\pi\left((N - M^{-1}[\frac{R^t}{2}])(N' - M^{-1}[\frac{R'^t}{2}])\right)^{\frac{1}{2}}\right)$$

This proves (7).

3. Fourier-Jacobi expansion of Poincaré series on Sp_m

In this section we show that the Fourier-Jacobi coefficients of Poincaré series $P_{k,T}(Z)$ on Γ_m can be written as an infinite linear combination of Jacobi-Poincaré series on Jacobi groups $\Gamma^J_{n,r}$ and $\Gamma^J_{r,n}$ (recall we have fixed decomposition $m = n+r$ with n, r natural numbers).

Set

$$\Gamma_{m,\infty} := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_m \mid C = 0 \right\}.$$

Then we have

$$\Gamma_{m,\infty} = \left\{ \begin{pmatrix} 1_m & S \\ 0 & 1_m \end{pmatrix} \left(U^{t^{-1}}\right)^{\sharp} \mid S \in \mathbb{Z}^{(m,m)},\ S = S^t,\ U \in GL(m,\mathbb{Z}) \right\}$$

and

(11)

$$\begin{aligned} P_{k,T}(Z) &= \sum_{U \in GL(m,\mathbb{Z}),\ \gamma \in \Gamma_{m,\infty}\backslash\Gamma_m} j\left(\left(U^{t^{-1}}\right)^{\sharp}\gamma, Z\right)^{-k} e\left(T\left(U^{t^{-1}}\right)^{\sharp}\gamma\langle Z\rangle\right) \\ &= \sum_{U \in GL(m,\mathbb{Z}),\ \gamma \in \Gamma_{m,\infty}\backslash\Gamma_m} j(\gamma, Z)^{-k} e(T[U]\gamma\langle Z\rangle) \end{aligned}$$

For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{n'}$, $1 \le n' \le m$, we put

$$M^{\downarrow} := \begin{pmatrix} 1_{m-n'} & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1_{m-n'} & 0 \\ 0 & c & 0 & d \end{pmatrix}.$$

Now we describe a set of representatives for $\Gamma_{m,\infty} \setminus \Gamma_m$.

Let $\Gamma^{\sim}_{m,n,n',\star} \subset GL(m,\mathbb{Z})$ be the subset consisting of such matrices, for which the blocks determined by the first n' rows and n columns are of maximal rank.

We decompose Γ_m as follows

$$\Gamma_m = \cup_{n'=0}^{n} \Gamma_{m,n,n'}$$

with

$$\Gamma_{m,n,n'} = \{M \in \Gamma_m \mid \textit{the first } n \textit{ columns of } C_M \textit{ are of rank } n'\}.$$

$\Gamma_{m,\infty}$ acts on each $\Gamma_{m,n,n'}$ from the left. For $M \in \Gamma_{m,n,n'}$ let $M^{\sim}$ denote special representative in its $\Gamma_{m,\infty}$-class with the property

$$(*) \qquad C_{M^{\sim}} = \begin{pmatrix} (c_\alpha, c_\beta) & * \\ 0_{m-n',n} & * \end{pmatrix}, \quad \left((c_\alpha, c_\beta) \in \mathbb{Z}^{(n',n')} \times \mathbb{Z}^{(n',n-n')}\right)$$

Let $\Gamma^0_{m,n,n'}$ be the subset of all $M^\sim$ with the property $(*)$.

One can check that

$$\Gamma_{m,\infty} \setminus \Gamma_{m,n,n'} \simeq G_{m,n'} \setminus \Gamma^0_{m,n,n'}$$

where

$$G_{m,n'} = \left\{ M = \begin{pmatrix} A_M & B_M \\ C_M & D_M \end{pmatrix} \in \Gamma_{m,\infty} \mid D_M \in GL(m,n') \right\}$$

with

$$GL(m,n') = \left\{ U \in GL(m,\mathbb{Z}) \mid U = \begin{pmatrix} * & * \\ 0_{m-n',n'} & * \end{pmatrix} \right\}$$

Proposition 2. *[Bo1],[Bo2] A complete set of representatives for* $G_{m,n'} \setminus \Gamma^0_{m,n,n'}$ *is given by* $\{M^\downarrow V^\sharp N^\uparrow\}$, *where* M *resp.* V *resp* N *run through a fixed set of representatives for* $\Gamma_{n',\infty} \setminus \Gamma_{n',\star}$ *resp.* $GL(m,n') \setminus \Gamma^\sim_{m,n,n',\star}$ *resp.* $\Gamma_{r,\infty} \setminus \Gamma_r$.

Therefore we can write

$$P_{k,T} = P^{n,r}_{k,T} + \sum_{n'=1}^{n} P^{n,r,n'}_{k,T} \tag{12}$$

where

$$P^{n,r}_{k,T}(Z) := \sum_{U \in GL(m,\mathbb{Z})} \sum_{\{N\}} j(N^\uparrow, Z)^{-k} e\left(T[U] N^\uparrow \langle Z \rangle\right) \tag{13}$$

and

$$P^{n,r,n'}_{k,T}(Z) := \sum_{U \in GL(m,\mathbb{Z})} \sum_{\{M\},\{V\},\{N\}} j(M^\downarrow V^\sharp N^\uparrow, Z)^{-k} e\left(T[U] M^\downarrow V^\sharp N^\uparrow \langle Z \rangle\right) \tag{14}$$

and M resp. V resp. N run through a fixed set of representatives for $\Gamma_{n',\infty} \setminus \Gamma_{n',\star}$ resp. $GL(m,n') \setminus \Gamma^\sim_{m,n,n',\star}$ resp. $\Gamma_{r,\infty} \setminus \Gamma_r$.

First let's consider $P^{n,r}_{k,T}$.

Set

$$\Gamma^\sim_{n,r,\infty} = \left\{ \pm \begin{pmatrix} 1_r & 0 \\ \lambda & 1_n \end{pmatrix} \mid \lambda \in \mathbb{Z}^{(n,r)} \right\}$$
$$\cup \left\{ \pm \begin{pmatrix} 1_r & 0 \\ \lambda & 1_n \end{pmatrix} \begin{pmatrix} -1_r & 0 \\ 0 & 1_n \end{pmatrix} \mid \lambda \in \mathbb{Z}^{(n,r)} \right\}$$

Write $U = U_1 U_2$ with $U_2 \in \Gamma^\sim_{n,r,\infty}$ and $U_1 \in GL(m,\mathbb{Z})/\Gamma^\sim_{n,r,\infty}$. We obtain

$$P^{n,r}_{k,T}(Z) = 2 \sum_{\{U_1\}} \sum_{\lambda \in \mathbb{Z}^{(n,r)}} \sum_{\{N\}} j(N^\uparrow, Z)^{-k}. \tag{15}$$

$$\cdot\left[e(T[U_1\begin{pmatrix}1&0\\\lambda&1\end{pmatrix}]N^{\dagger}\langle Z\rangle)+e(T[U_1\begin{pmatrix}1_r&0\\\lambda&1_n\end{pmatrix}\begin{pmatrix}-1_r&0\\0&1_n\end{pmatrix}]N^{\dagger}\langle Z\rangle)\right]$$

Now using the following identity

$$tr(S[\begin{pmatrix}-1_r&0\\0&1_n\end{pmatrix}]M^{\dagger}\langle Z\rangle)=tr(S(-M)^{\dagger}\langle Z\rangle)$$

($M\in Sp(r,\mathbb{Z})$, $S\in\mathbb{R}^{(m,m)}$ symmetric)
and (5), we obtain

$$(16)\qquad P^{n,r}_{k,T}(Z)=2\sum_{\{U_1\}}\sum_{\lambda\in\mathbb{Z}^{(r,n)}}\sum_{N\in\Gamma^{\dagger}_{r,\infty}\backslash\Gamma_r}j(N^{\dagger},Z)^{-k}\cdot e(T[U_1]\begin{pmatrix}1_r&\lambda\\0&1_n\end{pmatrix}^{\sharp}N^{\dagger}\langle Z\rangle)$$
$$=2\sum_{U_1\in GL(m,\mathbb{Z})/\Gamma^{\sim}_{n,r,\infty}}P^{J,n,r}_{k,T[U_1]}(Z)$$

Now let's treat $P^{n,r,n'}_{k,T}$.

For $Z\in\mathbb{H}_m$ we set

$$Z^{\wedge,n'}:=Z[\begin{pmatrix}0&1_{n'}\\1_{m-n'}&0\end{pmatrix}]$$

Lemma. *We have the following identities*

$$\text{(i)}\qquad j(M^{\downarrow}V^{\sharp}N^{\dagger},Z)=det\ V\cdot j(N^{\dagger},Z)\cdot j(M,V^{\sharp}N^{\dagger}\langle Z\rangle\left[\begin{pmatrix}0_{m-n',n'}\\1_{n'}\end{pmatrix}\right])$$

$$\text{(ii)}\qquad j(M,Z\left[\begin{pmatrix}0_{m-n',n'}\\1_{n'}\end{pmatrix}\right])=j(M^{\dagger},Z^{\wedge,n'})$$

$$\text{(iii)}\qquad tr(S\begin{pmatrix}1&0\\\lambda&1\end{pmatrix}^{\sharp}M^{\downarrow}\langle Z\rangle)=tr(S^{\wedge,n'}\begin{pmatrix}1&\lambda^t\\0&1\end{pmatrix}^{\sharp}M^{\dagger}\langle Z\rangle)$$

Proof: (i) follows from the cocycle relation for $j(\cdot,Z)$ and the observation that

$$j(M^{\downarrow},Z)=j(M,Z\left[\begin{pmatrix}0_{m-n',n'}\\1_{n'}\end{pmatrix}\right]).$$

(ii) and (iii) are simple calculations.

Set

$$\Gamma^{\sim,\infty}_{n,r} = \left\{ \pm \begin{pmatrix} 1_r & \lambda \\ 0 & 1_n \end{pmatrix} \mid \lambda \in \mathbb{Z}^{(r,n)} \right\}$$
$$\cup \left\{ \pm \begin{pmatrix} 1_r & \lambda \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} -1_r & 0 \\ 0 & 1_n \end{pmatrix} \mid \lambda \in \mathbb{Z}^{(r,n)} \right\}.$$

Using the Lemma, we obtain

$$(17) \qquad P^{n,r,n'}_{k,T}(Z) = 2 \sum_{\{U\}} \sum_{\lambda \in \mathbb{Z}^{(n,r)}} \sum_{\{M\},\{V\},\{N\}} j(N^\dagger, Z)^{-k}$$
$$\cdot j(M, V^\sharp N^\dagger \langle Z \rangle \left[\begin{pmatrix} 0_{m-n',n'} \\ 1_{n'} \end{pmatrix} \right])^{-k}$$
$$\cdot e(T[U] \begin{pmatrix} 1_r & 0 \\ \lambda & 1_{m-r} \end{pmatrix}^\sharp M^\downarrow \langle V^\sharp N^\dagger \langle Z \rangle \rangle)$$
$$= 2 \sum_{\{U\}} \sum_{\{V\},\{N\}} j(N^\dagger, Z)^{-k} P^{J,n',r,\star}_{k,T[U]^{\wedge,n'}} \left((V^\sharp N^\dagger \langle Z \rangle)^{\wedge,n'} \right)$$

where U resp. M resp. V resp. N run through a fixed set of representatives for $GL(m,\mathbb{Z})/\Gamma^{\sim,\infty}_{n,r}$ resp. $\Gamma_{n',\infty} \setminus \Gamma_{n'}$ resp. $GL(m,n') \setminus \Gamma^{\sim}_{m,n,n',\star}$ resp. $\Gamma_{r,\infty} \setminus \Gamma_r$.

Now insert the Fourier expansion (7) of $P^{J,n',r,\star}_{k,T[U]^{\wedge,n'}}$ and interchange the order of summation (see Appendix). We obtain

$$(18) \qquad P^{n,r,n'}_{k,T}(Z) = 2 \sum_{\{U\}} \sum_{T' = \begin{pmatrix} \star & \star \\ \star & T[U]^{\wedge,n'}[\begin{pmatrix} 0_{n,r} \\ 1_r \end{pmatrix}] \end{pmatrix} > 0} g^{J,n',r,\star}_{k,T[U]^{\wedge,n'}}(T')$$
$$\cdot \sum_{\{V\},\{N\}} j(N^\dagger, Z)^{-k} e\left(T'(V^\sharp N^\dagger \langle Z \rangle)^{\wedge,n'} \right).$$

The group

$$\Gamma^{+}_{m,r,\infty} := \left\{ \begin{pmatrix} 1_r & \rho \\ 0 & 1_n \end{pmatrix} \mid \rho \in \mathbb{Z}^{(r,n)} \right\}$$

acts (from the right) on the set $G_{m,n'} \setminus \Gamma^{\sim}_{m,n,n',\star}$.

Note also the identity

$$tr(T'^{\wedge} V^\sharp \langle Z \rangle) = tr(T'[V^{t^{-1}}]Z).$$

Using above remarks, (18) can be rewritten as

(19)

$$P_{k,T}^{n,r,n'}(Z) = 2\sum_{\{U\}} \sum_{T'=\begin{pmatrix} \star & \star \\ \star & T[U]^{\wedge,n'}[\begin{pmatrix} 0_{n,r} \\ 1_r \end{pmatrix}] \end{pmatrix}>0} g_{k,T[U]^\wedge}^{J,n',r,\star}(T')$$

$$\cdot \sum_{V\in GL(m,n')\backslash\Gamma^{\sim}_{m,n,n',\star}/\Gamma^{+}_{m,r,\infty}} \sum_{N\in\Gamma_{r,\infty}\backslash\Gamma_r,\ \rho\in\mathbb{Z}^{(r,n)}}$$

$$\cdot j(N^\dagger, Z)^{-k} e(T'^\wedge V^\sharp \begin{pmatrix} 1_r & \rho \\ 0 & 1_{m-r} \end{pmatrix}^\sharp N^\dagger\langle Z\rangle)$$

$$= 2 \sum_{V\in GL(m,n')\backslash\Gamma^{\sim}_{m,n,n',\star}/\Gamma^{+}_{m,r,\infty}} \sum_{T'} \left(\sum_{\{U\}} g_{k,T[U]^{\wedge,n'}}^{J,n',r,\star}(T')\right) P_{k,T'[V^{t-1}]}^{J,r,n}(Z)$$

Note that we use here the Jacobi-Poincaré series on $\Gamma_{r,n}^J$.

Now from (12),(16),(19) we obtain the following

Theorem. *The Poincaré series $P_{k,T}$ defined by (1) has the expansion*

(20)

$$P_{k,T}(Z) = 2 \sum_{U_1\in GL(m,\mathbb{Z})/\Gamma^{\sim}_{n,r,\infty}} P_{k,T[U_1]}^{J,n,r}(Z) +$$

$$2\sum_{n'=1}^{n} \sum_{V\in GL(m,n')\backslash\Gamma^{\sim}_{m,n,n',\star}/\Gamma^{+}_{m,r,\infty}} \sum_{T'} \left(\sum_{\{U\}} g_{k,T[U]^\wedge}^{J,n',r,\star}(T')\right) P_{k,T'[V^{t-1}]}^{J,r,n}(Z)$$

where U run through a fixed set of representatives for $GL(m,\mathbb{Z})/\Gamma_{n,r}^{\sim,\infty}$ and T' run through all symmetric half-integral matrices of the form

$$T' = \begin{pmatrix} \star & \star \\ \star & T[U]^{\wedge,n'}[(\begin{smallmatrix} 0_{n,r} \\ 1_r \end{smallmatrix})] \end{pmatrix} > 0\,.$$

4. Concluding remarks

In this section we will discuss some application of the Theorem. Namely, we will give an explicit description of the adjoints w.r.t. the Petersson scalar products of the maps which send a Siegel modular form to its various Fourier-Jacobi coefficients. Recall that the case $m = 2$ was treated in [Ko1].

More precisely, let $\rho_M : S_k(\Gamma_m) \to J^{cusp}_{k,M,n,r}$ be the map which sends a Siegel cusp form to its M^{th} Fourier-Jacobi coefficients. Let $\rho^*_M : J^{cusp}_{k,M,n,r} \to S_k(\Gamma_m)$ be the adjoint map w.r.t. the Petersson scalar products.

First let's state the following result.

Proposition 3. *$\forall F(Z) = \sum_T a_F(T)e(TZ) \in S_k(\Gamma_m)$ one has*

(i)
$$< F, P_{k,T} >= c_k\ det(T)^{-k+\frac{m+1}{2}} a_F(T)$$

where $c_k := \pi^{\frac{m(m+1)}{4}}(4\pi)^{\frac{m(m+1)}{2}-mk}\prod_{\nu=1}^{m}\Gamma\left(k-\frac{m+\nu}{2}\right)$.
(ii) $\forall\phi \in J^{cusp}_{k,M,n,r}$ *one has*

$$< \phi, P^{J,n,r}_{k,M;(N,R)} >= \lambda_{k,r,n,M,N,R}\ c_\phi(N,R)$$

where

$$\lambda_{k,r,n,M,N,R} := 2^{-rn}det(M)^{-\frac{n}{2}}det(4N - M^{-1}[R^t])^{-k+2-\frac{n+1-r}{2}} \cdot \pi^{\frac{n(n-1)}{4}-n(k-2+\frac{n+1-r}{2})}\prod_{\nu=1}^{n}\Gamma\left(k-2+\frac{n+1-r}{2}-\frac{\nu-1}{2}\right)$$

Proof: (i) It is well known, cf [Kli];
(ii) Let $\phi \in J^{cusp}_{k,M,n,r}$. Then, by the unfolding argument, we have

$$< \phi, P^{J,n,r}_{k,M;(N,R)} > = \int_{\Gamma^J_{n,r,\infty}\backslash\mathbb{H}_n\times\mathbb{C}^{(r,n)}} \phi(\tau,z)\overline{e_{N,R}(\tau,z)} \cdot det(v)^k exp(-tr(4\pi M v^{-1}[y^t])\ dV^J_{n,r}$$

We choose

$$\{(\tau,z) \in \mathbb{H}_n \times \mathbb{C}^{(r,n)}\,|0 \le u_{ij} \le 1,\ i,j = 1,...,n;\ v > 0;\ 0 \le x_{st} \le 1,\ s = 1,...,r,\ t = 1,...,n;\ y \in \mathbb{R}^{(r,n)}\}$$

as the fundamental domain for the action of $\Gamma^J_{n,r,\infty}$.
Inserting the Fourier expansion of ϕ, we obtain

$$< \phi, P^{J,n,r}_{k,M;(N,R)} > = c_\phi(N,R)\int_0^\infty \cdots \int_0^\infty e^{-4\pi\ tr(Nv)}det(v)^{k-r-2} \cdot \left(\int_{\mathbb{R}^{(r,n)}} e^{-4\pi\ tr(Ry+v^{-1}M[y^t])}dy\right) dv$$

The inner integral equals

$$2^{-rn}det(M)^{-\frac{n}{2}}det(v)^{\frac{r}{2}}e^{\pi\ tr(vM^{-1}[R^t])}$$

Now, using the above and [Kli, p.88], we find

$$< \phi, P^{J,n,r}_{k,M;(N,R)} > = c_\phi(N,R)\ 2^{-rn} det(M)^{-\frac{n}{2}}$$
$$\cdot \int_{v>0} e^{-\pi\ tr\ v(4N - vM^{-1}[R^t])} det(v)^{k-\frac{r}{2}-2} dv$$
$$= c_\phi\ 2^{-rn} det(M)^{-\frac{n}{2}} det(4N - M^{-1}[R^t])^{-k+2-\frac{n+1-r}{2}}$$
$$\cdot \pi^{\frac{n(n-1)}{4} - n(k-2+\frac{n+1-r}{2})} \prod_{\nu=1}^{n} \Gamma\left(k-2+\frac{n+1-r}{2} - \frac{\nu-1}{2}\right)$$

It proves (ii).

Now assume $n = r$, so $m = 2n$. From (i) it follows that the T^{th} Fourier coefficient of $\rho^*_M(\phi)$ is equal to

$$c_k^{-1} det(T)^{k-\frac{m+1}{2}} < \rho^*_M(\phi), P_{k,T} >= c_k^{-1} det(T)^{k-\frac{m+1}{2}} < \phi, \rho_M(P_{k,T}) >$$

Now use (ii) and (20); we find that the T^{th} Fourier coefficient of $\rho^*_M(\phi)$ is expressed as the sum of the following two terms. The first one has the value

$$c_k^{\sim} det(M)^{-\frac{n}{2}} det(4N - M^{-1}[R^t])^{-k+\frac{3}{2}}$$
$$\cdot \sum_{U_1:\ T[U_1][\binom{0_n}{1_n}]=M} c_\phi \left(T[U_1][\binom{1}{0}], (\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} T[U_1])[\binom{0}{1}]\right)$$

for some constant $c_k^{\sim}$ depending on k.

The second term is equal to

$$det(M)^{-\frac{n}{2}} \sum_{T'>0} A_{M,T}(T')\ det(4N - M^{-1}[R^t])^{-k+\frac{3}{2}}$$

where $A_{M,T}(T')$ involve the Fourier coefficients of certain of the Poincaré series on $\Gamma^J_{n,n}$ and the Fourier coefficients

$$c_\phi(T'[V^{t^{-1}}][\binom{1}{0}], \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} T'[V^{t^{-1}}][\binom{0}{1}])$$

and where $T'[V^{t^{-1}}][\binom{0}{1}] = M$.

Appendix

We show that the series

$$(21) \qquad \sum_{\gamma\in\Delta\backslash\Gamma^{J\dagger}_{r,n},\ U,V,T'} g^{J,n',r,*}_{k,T[U]^\wedge,n'}(T') j(\gamma,Z)^{-k} e\left(T'[V^{t^{-1}}]\gamma\langle Z\rangle\right)$$

with U, V, T' as in the Theorem, is absolutely convergent. It will justify the change in order of summation made in the proof above.

First, by the same argument as in [Kol] it is sufficient to show that for any $C > 0$ the series

$$(22) \qquad \sum_{\gamma\in\Delta\backslash\Gamma^{J+}_{r,n}} |g^{J,n',r,\star}_{k,T[U]^\wedge}(T')| \cdot |j(\gamma, i1_m)|^{-k} \cdot exp\left(-C\ tr\ Im(T'[V^{t^{-1}}]\gamma\langle i1_m\rangle)\right)$$

is finite.

<u>*Let's briefly sketch the proof.*</u>

Using (2) and (3) it is easily seen that (22) is equal to

$$(23) \qquad \sum_{M=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma^+_{r,\infty}\backslash\Gamma_r,\ \rho\in\mathbb{Z}^{(n,r)}} \sum_{U,V,T'} |g^{J,n',r,\star}_{k,T[U]^{\wedge,n'}}(T')| \cdot |det\ (ci+d)|^{-k}$$

$$\cdot\ exp\left(-C\ tr((cc^t+dd^t)^{-1}T'[V^{t^{-1}}][\begin{pmatrix}1_r\\ \rho\end{pmatrix}])\right) exp\left(-C\ tr(T'[V^{t^{-1}}]\begin{pmatrix}0&0\\0&1\end{pmatrix})\right)$$

Fix $\epsilon > 0$. Let $y \in \mathbb{R}^{(r,r)}$, $y > 0$ and $Q \in \mathbb{R}^{(m,m)}$, $Q = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4\end{pmatrix} > 0$, $q_1 \in \mathbb{R}^{(r,r)}$, $q_4 \in \mathbb{R}^{(n,n)}$. Then one has, for $\rho \in \mathbb{Z}^{(n,r)}$, $||\rho|| \gg_{y,\epsilon,Q} 0$

$$exp\left(-tr(y\ Q[\begin{pmatrix}1_r\\ \rho\end{pmatrix}])\right) \ll \left(\frac{det\ Q}{det\ q_4}\right)^{1+\frac{r-k}{2}+\epsilon}$$

Also, we have

$$\sum_{\rho\in\mathbb{Z}^{(n,r)}} exp\left(-tr(y\ Q[\begin{pmatrix}1_r\\ \rho\end{pmatrix}])\right) \ll \sum_{\rho\in\mathbb{Z}^{(n,r)}} det(y\ Q[\begin{pmatrix}1_r\\ \rho\end{pmatrix}])^{1-\frac{k}{2}+\epsilon}$$

$$\ll det(y)^{1-\frac{k}{2}+\epsilon}\left(\frac{det\ Q}{det\ q_4}\right)^{1+\frac{r-k}{2}+\epsilon}$$

Now take

$$y = C\ (cc^t+dd^t)^{-1}, \quad Q = T'[V^{t^{-1}}]$$

We obtain that the sum (23) is

$$\ll \sum_{U,V,T'} |g^{J,n',r,\star}_{k,T[U]^{\wedge,n'}}(T')| \cdot \left(\frac{det\ T'}{det\ q_4}\right)^{1+\frac{r-k}{2}+\epsilon} exp\left(-C\cdot tr(T'[V^{t^{-1}}]\begin{pmatrix}0&0\\0&1\end{pmatrix})\right)$$

$$\ll \sum_{U,V,T'} |g^{J,n',r,\star}_{k,T[U]^{\wedge,n'}}(T')| \cdot det(T')^{1+\frac{r-k}{2}+\epsilon} exp\left(-C_1\cdot tr(T'[V^{t^{-1}}]\begin{pmatrix}0&0\\0&1\end{pmatrix})\right)$$

for some $C_1 > 0$.

Now we use the trivial bound

$$|H_{M,c}(N,R,N',R')| \ll det(c)^{1-\frac{r}{2}}$$

the estimate

$$J_{k-\frac{r}{2}-1}(X) \ll det(X)^{k-\frac{r}{2}-1}, \quad (det(X) > 0)$$

and Hadamard's inequality
to give

$$g^{J,n',r,\star}_{k,T[U]^{\wedge,n'}}(T') \ll det(T')^{k-\frac{r}{2}-1} det(T[U]^{\wedge,n'}[\begin{pmatrix}0\\1\end{pmatrix}])^{-k+\frac{r-n}{2}+1}.$$

Therefore the sum in (23) is

$$\ll \sum_{U,V,T'} det(T')^{\frac{k}{2}+\epsilon} det(T[U]^{\wedge,n'}[\begin{pmatrix}0\\1\end{pmatrix}])^{-k+\frac{r-n}{2}+1}$$
$$\cdot exp\left(-C_1\, tr(T'[V^{t^{-1}}]\begin{pmatrix}0&0\\0&1\end{pmatrix}))\right)$$
$$\ll \sum_{U,V,T'} det(T')^{\frac{k}{2}+\epsilon} det(T[U]^{\wedge,n'}[\begin{pmatrix}0\\1\end{pmatrix}])^{-k+\frac{r-n}{2}+1}$$
$$\cdot exp\left(-C_2\, det(\begin{pmatrix}1&0\\0&0\end{pmatrix}+T'[V^{t^{-1}}]\begin{pmatrix}0&0\\0&1\end{pmatrix})^{1/m}\right).$$

The latter can be written as

$$\sum_{a,t>0}\left(\sum det\,(T')^{\frac{k}{2}+\epsilon}\right) t^{-k+\frac{r-n}{2}+1} e^{-C_2 a^{1/m}} \tag{24}$$

where the inner sum is over U, V, T' satisfying (20) and

$$det(T[U]^{\wedge,n'}[(\begin{smallmatrix}0\\1\end{smallmatrix})]) = t,$$
$$det(\begin{pmatrix}1&0\\0&0\end{pmatrix}+T'[V^{t^{-1}}]\begin{pmatrix}0&0\\0&1\end{pmatrix}) = a\,.$$

Now it is sufficient to show that the inner sum is $\ll t^{\frac{k}{2}+\alpha} a^{\beta k}$, for some $\alpha, \beta \in \mathbb{R}$. The latter can be given by similar arguments as in [Ko1]. Namely, first we have to check that

$$t \cdot a \geq det(T') \cdot f(V),$$

for $f(V)$ some positive integer depending on V; then the inner sum in (24) will be

$$\ll \sum_{\nu=1}^{at} \sum_{1\leq\rho\leq\frac{at}{\nu}} \rho^{\delta} t^{\gamma} \nu^{\frac{k}{2}+\epsilon}$$

$$\ll \sum_{\nu=1}^{at} t^{\gamma} \left(\frac{at}{\nu}\right)^{\rho} \nu^{\frac{k}{2}+\epsilon} \ll a^{\frac{k}{2}+\delta+\epsilon} t^{\frac{k}{2}+\gamma+\delta+\epsilon}.$$

We omit the details.

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