On Real Quantum Groups

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ABSTRACT. Poisson-Lie structures on real simple Lie and Kac-Moody groups are considered. The symplectic leaves of the inner ones are classified. An algebraic version of the Weinstein's Darboux theorem for Poisson manifolds is proved. The algebra of strongly regular functions on a quantum Kac-Moody group is defined.

§0. Introduction.

0.1. Let G be a complex Poisson Lie group, (g, δ) the corresponding Lie bialgebra. Assume that (g, δ) admits a quantization (in the sense of [D]) $U_\hbar \mathfrak{g}$ and that $U_\hbar \mathfrak{g}$ gives rise to an "algebra of functions on the quantum version of G", a suitable Hopf algebra $\mathbb{C}_\hbar[G]$ dual to $U_\hbar \mathfrak{g}$. Now let $G_\mathbb{R}$ be a real form of G, corresponding to a real form $\mathfrak{g}_\mathbb{R}$ of $\mathfrak{g}$, and suppose that the involution of $\mathfrak{g}$ whose fixed point set is $\mathfrak{g}_\mathbb{R}$ "lifts" to an involution of $U_\hbar \mathfrak{g}$, providing this algebra, and *a fortiori* $\mathbb{C}_\hbar[G]$, with a $*$-Hopf algebra structure. Assume also that the formal parameter $\hbar$ can be specialized, at least in an open neighborhood of 0. It is certainly of interest to understand the irreducible $*$ representations of $\mathbb{C}_\hbar[G]$, at least for some values of $\hbar$. (Compare with [F]).

0.2. Suppose that G is a finite dimensional connected simply connected simple Lie group, that $G_\mathbb{R} = K$ is a compact form and that $\delta(x) = \operatorname{ad} x R$, where R is the famous solution of the classical Yang-Baxter equation

$$R = \sum_{\alpha \in \Delta_+} X_\alpha \wedge X_{-\alpha}.$$

Then $U_\hbar \mathfrak{g}$ is well-known to exist [D1] and the formal parameter can be specialized to the q-version $U_q \mathfrak{g}$ of $U_\hbar \mathfrak{g}$ [J]; the construction of $\mathbb{C}_q[G]$ is not difficult to carry over (see for example [L2], [A3], [LS]); if q is positive, the Cartan involution defining K has a quantum analogue and hence $\mathbb{C}_q[G]$ is a $*$-Hopf algebra (in fact, a compact matrix pseudogroup in the sense of [W]) which is denoted $\mathbb{C}_q[K]$. The description of all the irreducible $*$-representations of $\mathbb{C}_q[K]$ was achieved in [S] (for $K = SU(2)$ previously in [VS] and for $K = U(n)$ independiently in [Ko]). It turns out that the representation theory of $\mathbb{C}_q[K]$ is closely related to the space of symplectic leaves of the Poisson structure of K: there is a bijection between this space and the set of (isomorphy classes of) irreducible $*$-representations of $\mathbb{C}_q[K]$. This is however an optimal case: if we consider, instead of

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δ, the Lie cobracket $\delta_u(x) = \operatorname{ad} x(R + u)$, for a suitable $u \in \Lambda^2\mathfrak{h}$, then the irreducible $*$-representations of $\mathbb{C}_{q,u}[K]$ are still related to the symplectic leaves, but that bijection does not exist any more (see [L], [LS]).

0.3. It would be very interesting to extend these results to another situations, for example to non-compact forms of G, or to G of Kac-Moody type. This presents difficulties of various sorts. The simplest non-compact case, namely $SU(1,1)$, was treated in [AE]. There, it was introduced the notion of admissible representation of $\mathbb{C}_q[SL(2,\mathbb{C})]$; the admissible irreducible representations were classified and those supporting an inner product, invariant for the $*$ structure corresponding to $SU(1,1)$, were detected; and a relation between admissible irreducible representations and symplectic leaves in $SL(2,\mathbb{C})$ was established. Note that it was not proved that any unitary irreducible representation of $\mathbb{C}_q[SU(1,1)]$ is admissible.

0.4. This paper is concerned with technical background in order to make some progress in the directions suggested in *0.3.* Let us briefly describe the contents. First we prove that any real form of G carries a Poisson structure, that at the infinitesimal level corresponds to a Manin triple originated by an analogous of the Iwasawa decomposition. This construction is also available for real forms of second kind of Kac-Moody groups. Second, we discuss Poisson-Lie structures on Kac-Moody groups. We follow here the approach in [KP1], [KP2], [KP3], see also [Sl].The more straightforward approach would be to consider a Poisson algebra of functions on the group. We prove that the algebra considered in [KP2] is a Poisson algebra. However, this algebra has a disadvantage: it lacks the real form corresponding to the algebra of real functions on the real Kac-Moody groups we are interested in. This is certainly connected with the fact that G has attached a twin building, cf [T1], [Rou]. We propose to consider a bigger algebra as the algebra of rational functions on G, which do have such real subalgebras. The precise structure of that algebras is however not clear to us: the problem is in the neighborhood of the determination of the complete reducibility of the tensor product of irreducibles highest and lowest weight modules. Moreover, these bigger algebras seems to have no Poisson structure. However, we are able to classify the simplectic leaves of these "would-be" Poisson Lie groups (under a technical hypothesis). By the way, the method utilised gives also the classification of the symplectic leaves in all the real forms of inner type of a complex simple connected simple Lie group.

Observe now that the symplectic leaves in the $SL(2,\mathbb{C})$ case (cf. **0.3**) are understood in the complex sense. We prove an algebraic analogue of Weinstein's "Darboux" theorem describing locally a Poisson manifold. The existence of symplectic leaves in Poisson regular affine varieties follows from this. (Compare with [We], [Ki].)

Going to the quantum side, we define the algebra of "strongly regular functions on the quantum Kac-Moody group"; it is of course not a Hopf algebra, but worst, it is not a *-algebra, by the reasons evoked below in the "classical" case. Finally, we introduce the notion of admissible representations for the algebra of functions on the quantum group, in the finite case. Of course, the inspiration for this definition comes from Harish-Chandra.

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REAL POISSON LIE GROUPS

§1. Bialgebra structures on real Kac-Moody algebras. Let $A \in \mathbb{Z}^{N\times N}$ be a symmetrizable generalized Cartan matrix and let $(\mathfrak{h}_{\mathbb{R}}, \Pi, \Pi^{\vee})$ be a real realization of A, see [K, 1.1] and also [T2]. Let $\mathfrak{g}_{\mathbb{R}}$ be the real Kac-Moody algebra coresponding to A, with generators e_i, f_i $(1 \le i \le N)$ and $\mathfrak{h}_{\mathbb{R}}$, cf. [K, 1.3]. Now let $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{R}} \otimes \mathbb{C}$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$; we shall identify $\mathfrak{g}_{\mathbb{C}}$ with the complex Kac-Moody algebra corresponding to A. Let us fix $J \subseteq \{1, \dots, N\}$. Let σ be the linear Lie algebra involution of $\mathfrak{g}_{\mathbb{R}}$ determined by

$$\sigma(e_i) = (-1)^{\chi(i)} f_i, \quad \sigma(f_i) = (-1)^{\chi(i)} e_i, \quad \sigma(h) = -h, \forall\, h \in \mathfrak{h}_{\mathbb{R}},$$

where χ is the characteristic function of J. Let $\theta : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ be the induced antilinear involution, i.e. $\theta = \sigma \otimes ^-$, where the bar denotes complex conjugation. Now let $\mathfrak{g}_J$ be the fixed point set of θ; $\mathfrak{g}_J$ is a real Lie algebra, whose complexification is isomorphic to $\mathfrak{g}_{\mathbb{C}}$. Let $(\,|\,)_0$ be a non-degenerate invariant bilinear form on $\mathfrak{g}_J$ whose extension $(\,|\,)$ to $\mathfrak{g}_{\mathbb{C}}$ is the given in [K, Th. 2.2] (cf. [A2, Th. 2], [AR]).

Remark 1.1. If $J = \{1, \dots, N\}$, then we shall denote ω instead of θ and $\mathfrak{k}$ instead of $\mathfrak{g}_J$; ω is usually called the "compact" involution, and $\mathfrak{k}$ the "compact" form, of $\mathfrak{g}_{\mathbb{C}}$.

Now let $\mathfrak{p} = \mathfrak{g}_{\mathbb{C}}$, $\mathfrak{n}_+ = \mathfrak{n}_+$ considered as real Lie algebras, $\mathfrak{p}_1 = \mathfrak{g}_J$ and $\mathfrak{p}_2$ be the direct sum of $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{n}_+$. (Here $\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}_+ \oplus \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{n}_-$ is the triangular decomposition, i.e. $\mathfrak{n}_+$ is the Lie subalgebra spanned by the e_i's, etc. We shall also denote $\mathfrak{b}_\pm = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{n}_\pm$.) Let $\mathrm{Im}(\,|\,)$ be the imaginary part of $(\,|\,)$; it is a real bilinear form on $\mathfrak{p}$, invariant and non-degenerate.

Lemma 1.1. *$(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ is a Manin triple.*

Proof. a) $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_2$: As $\sqrt{-1}\mathfrak{h}_{\mathbb{R}} \subseteq \mathfrak{g}_J$, $\mathfrak{n}_+ \oplus \mathfrak{h}_{\mathbb{C}} \subseteq \mathfrak{p}_1 + \mathfrak{p}_2$. Let $i_1, \dots i_j$ be a sequence of indices in $\{1, \dots, N\}$. We have $\theta([f_{i_1}, [f_{i_2}, [\dots, f_{i_j}] \dots]]) = \pm [e_{i_1}, [e_{i_2}, [\dots, e_{i_j}] \dots]]$ and hence

$$[f_{i_1}, [f_{i_2}, [\dots, f_{i_j}] \dots]] \pm [e_{i_1}, [e_{i_2}, [\dots, e_{i_j}] \dots]]$$

and

$$\sqrt{-1}([f_{i_1}, [f_{i_2}, [\dots, f_{i_j}] \dots]] \mp [e_{i_1}, [e_{i_2}, [\dots, e_{i_j}] \dots]])$$

belong to $\mathfrak{g}_J$. This shows that $\mathfrak{n}_- \subseteq \mathfrak{p}_1 + \mathfrak{p}_2$.

b) Clearly, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are isotropic with respect to $\mathrm{Im}(\,|\,)$. Thus $\mathfrak{p}_1 \cap \mathfrak{p}_2 = 0$ and the Lemma follows. □

Remark 1.2. Note that we have in fact a decomposition of real vector spaces $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_J \oplus \mathfrak{h}_{\mathbb{R}} \oplus \mathfrak{n}_+$, which can be viewed as a generalization of the (infinitesimal) Iwasawa decomposition.

Now let $(\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}, \operatorname{diag}(\mathfrak{g}_{\mathbb{C}}), \mathfrak{P}_2)$ be the Manin triple considered in [D1, §3]. (Recall that $\mathfrak{P}_2 = \mathfrak{n}_+ \times \mathfrak{n}_- \oplus \{(h, -h) : h \in \mathfrak{h}_{\mathbb{C}}\}$.) Assume that the non-degenerate invariant bilinear form fixed in [D1, Example 3.2] is $(\,|\,)$. Let Δ^+ be the set of positive roots of $\mathfrak{g}_{\mathbb{C}}$ and let $\{X_{\alpha,i} : i \in I_\alpha\}$ be a basis of $(\mathfrak{g}_{\mathbb{C}})_\alpha$, such that $I_\alpha = I_{-\alpha}$ and $(X_{\alpha,i}|X_{-\alpha,j}) = \delta_{i,j}$. Let $\{h_\ell\}$ be an orthogonal basis of $\mathfrak{h}_{\mathbb{C}}$. Let us consider the formal expression

$$r = \frac{1}{2} \sum_{\alpha \in \Delta^+, i \in I_\alpha} X_{\alpha,i} \otimes X_{-\alpha,i} - X_{-\alpha,i} \otimes X_{\alpha,i}. \tag{1.1}$$

$(\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}, \operatorname{diag}(\mathfrak{g}_{\mathbb{C}}), \mathfrak{P}_2)$ gives rise to a Lie bialgebra structure on $\mathfrak{g}_{\mathbb{C}}$; let δ be the corresponding cobracket. Then $\delta(x) = \operatorname{ad} x r$. In fact, (cf. [D1, §4]) $r = r_0 - \frac{1}{2}\rho$, where $\rho = r_0^{12} + r_0^{21}$ and

$$r_0 = \sum_{\alpha \in \Delta^+, i \in I_\alpha} X_{\alpha,i} \otimes X_{-\alpha,i} + \frac{1}{2} \sum_\ell h_\ell.$$

Lemma 1.2. *The Manin triple $(\mathfrak{p} \otimes \mathbb{C}, \mathfrak{p}_1 \otimes \mathbb{C}, \mathfrak{p}_2 \otimes \mathbb{C})$ (cf. Lemma 1.1) is isomorphic to $(\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}, \operatorname{diag}(\mathfrak{g}_{\mathbb{C}}), \mathfrak{P}_2)$, modulo multiplying the invariant bilinear form by a scalar.*

Proof. Let $\Upsilon : \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathfrak{p} \otimes \mathbb{C}$ be the application given by

$$\Upsilon(x, y) = \frac{1}{2}(x - \sqrt{-1}x' + \theta(y) + \sqrt{-1}\theta(y)').$$

Here $x \mapsto x'$ is the multiplication by $\sqrt{-1}$ in $\mathfrak{p}$ with respect to its real form $\mathfrak{g}_{\mathbb{R}}$. It is easy to see that Υ is an isomorphism of complex Lie algebras and that $\Upsilon(\operatorname{diag}(\mathfrak{g}_{\mathbb{C}})) = \mathfrak{p}_1 \otimes \mathbb{C}$. Let us show that $\Upsilon(\mathfrak{P}_2) = \mathfrak{p}_2 \otimes \mathbb{C}$. Let $h \in \mathfrak{h}_{\mathbb{R}}$. Then $\Upsilon(h, -h) = h$ hence $\Upsilon(\{(h, -h) : h \in \mathfrak{h}_{\mathbb{C}}\}) = \mathfrak{h}_{\mathbb{R}} \otimes \mathbb{C}$. On the other hand, from $\Upsilon(e_i, 0) = \frac{1}{2}(e_i - \sqrt{-1}e_i')$ and $\Upsilon(0, f_i) = \frac{(-1)^{\chi(i)}}{2}(e_i + \sqrt{-1}e_i')$ follows that $\Upsilon(\mathfrak{n}_+ \times \mathfrak{n}_-) = \mathfrak{n}_+ \otimes \mathbb{C}$. Finally, a straightforward computation shows that

$$\operatorname{Im}(\Upsilon(x, y)|\Upsilon(u, v)) = \frac{-\sqrt{-1}}{2}((x|u) - (y|v)),$$

for any $x, y, u, v \in \mathfrak{g}_{\mathbb{C}}$. $\square$

Remark 1.3. The preceding Lemma is known when the Cartan matrix is finite and $\theta = \omega$, see [M] and also [LW]. When this work was in its final stage, the author noticed the interesting article [LQ], where Lie bialgebra structure on such forms where constructed, for finite Cartan matrices, by means of Koszul operators.

Lemma 1.3. *The Manin triple $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ gives rise to a Lie bialgebra structure on $\mathfrak{p}_1$ and the corresponding cobracket is given by $\delta_0(x) = \operatorname{ad} x \sqrt{-1} \sum_{\alpha \in \Delta^+, i \in I_\alpha} X_{\alpha,i} \otimes X_{-\alpha,i} - X_{-\alpha,i} \otimes X_{\alpha,i}$.*

Proof. Let (p, p_1, p_2) be an arbitrary Manin triple that gives rise to a Lie cobracket d on p_1, where the fixed bilinear invariant form on p is denoted by $(\,|\,)$. Notice that choosing $(\,|\,)_t = t(\,|\,)$ instead of $(\,|\,)$ is equivalent to having the Lie cobracket $t^{-1}d$ instead of d. $\square$

Now let $\tau : \{1, \dots n\} \to \{1, \dots n\}$ be a diagram automorphism, i.e. a bijection satisfying $a_{ij} = a_{\tau(i)\tau(j)}$ for any i, j. Assume for simplicity of the exposition that A is non-degenerate; then $\mathfrak{h}_{\mathbf{R}}$ is spanned by $h_i = [e_i, f_i]$. Let σ_τ be the linear Lie algebra involution of $\mathfrak{g}_{\mathbb{R}}$ determined by

$$\sigma(e_i) = (-1)^{\chi(i)} f_{\tau(i)}, \quad \sigma(f_i) = (-1)^{\chi(i)} e_{\tau(i)}, \quad \sigma(h_i) = -h_{\tau(i)},$$

where χ, we recall, is the characteristic function of J. Let as above $\theta_\tau : \mathfrak{g}_{\mathbf{C}} \to \mathfrak{g}_{\mathbf{C}}$ be the induced antilinear involution, and $\mathfrak{g}_{J,\tau}$, a real Lie algebra whose complexification is isomorphic to $\mathfrak{g}_{\mathbf{C}}$, be the fixed point set of θ_τ. *Assume that there exists a non-degenerate invariant bilinear form* $(\,|\,)_0$ *on* $\mathfrak{g}_{J,\tau}$ *whose extension* $(\,|\,)$ *to* $\mathfrak{g}_{\mathbf{C}}$ *is the given in* [K, Th. 2.2]. (The author ignores the existence of a proof of this fact, but there should be no substantial difficult in checking it; in particular, this is obviously true for finite dimensional $\mathfrak{g}$'s.) Then the preceding discussion applies to this new situation; one has to replace $\mathfrak{p}_2$ by the direct sum of $\mathfrak{h}_{-1}$ and $\mathbf{n}_+$, where $\mathfrak{h}_{-1}$ is the subspace of θ_τ eigenvectors of eigenvalue -1. We get as before

Lemma 1.4. $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ *is a Manin triple.*

Moreover, Lemmas 1.2 and 1.3 still hold, with analogous proofs.

It follows from [KP3, Prop. 3.7] that we have covered all the real forms of $\mathfrak{g}_{\mathbf{C}}$ of second kind.

§2. Symplectic leaves.

2.1 Symplectic leaves on $SU(1,1)$. In this subsection we shall treat, by elementary computations, the simplest non-compact case, namely $SU(1,1)$. Some remarks in this section were stated without proof in [AE]. Notice first that we can not "globalize" the decomposition

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{g}_J \oplus \mathfrak{h}_{\mathbf{R}} \oplus \mathfrak{n}_+. \tag{2.1}$$

Let $H_{\mathbb{R}}$ be the subgroup of $SL(2,\mathbb{C})$ of diagonal matrices with real non-negative entries and N the unipotent subgroup of upper diagonal matrices. Given $x \in SL(2,\mathbb{C})$ it is not always possible to express it as a product

$$x = x_0 h n, \quad \text{where} \quad x_0 \in SU(1,1),\ h \in H_{\mathbb{R}},\ n \in N. \tag{2.2}$$

(The uniqueness of such decomposition, when it exists, follows from $SU(1,1) \cap H_{\mathbf{R}} N = 1$.) (2.2) fails for example for

$$x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

(Thus the method for computing the symplectic leaves via dressing transformations [Se], [LW] seems to do not apply here.) On the other hand, let

$$x = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be such that $|\alpha|^2 - |\gamma|^2 > 0$ and take

$$x_0 = \begin{pmatrix} \alpha t^{-1} & t(\beta - \alpha\lambda) \\ \gamma t^{-1} & t(\delta - \gamma\lambda) \end{pmatrix}, \quad h = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad n = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

where

$$t = \sqrt{|\alpha|^2 - |\gamma|^2}, \quad \lambda = \frac{\overline{\alpha}\beta - \overline{\gamma}\delta}{|\alpha|^2 - |\gamma|^2}.$$

A straightforward computation shows that $x = x_0 h n$ and $x_0 \in SU(1,1)$. (It is not always true that a product hu, $h \in H_{\mathbb{R}}N$, $u \in SU(1,1)$, satisfies the above condition.)

Now let us complexify (2.1); we get $s\ell(2,\mathbb{C}) \times s\ell(2,\mathbb{C}) = \operatorname{diag}(s\ell(2,\mathbb{C})) \oplus \{(x,y) \in \mathfrak{b}_+ \times \mathfrak{b}_- : x_0 + y_0 = 0\}$. Let $\mathbf{H}$ (resp., $\mathbf{B}_+$, $\mathbf{B}_-$) be the diagonal torus (resp., the Borel subgroup of upper, resp. lower, triangular matrices) of $SL(2,\mathbb{C})$. Again, we do not have a diffeomorphism from $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ onto $\operatorname{diag}(SL(2,\mathbb{C})) \times P_2$, where $P_2 = \{(x,y) \in \mathbf{B}_+ \times \mathbf{B}_- : x_0 y_0 = 1\}$. Moreover, even the uniqueness fails, because $\operatorname{diag}(SL(2,\mathbb{C})) \cap P_2 = \{\pm 1\}$. Nonetheless, let x_{ij}, $1 \leq i,j \leq 2$ be the matrix coefficients of the 2-dimensional representation of $SL(2,\mathbb{C})$. It is known [VS] that the Poisson bracket is given by

$$\begin{aligned} &\{x^{11}, x^{12}\} = -x^{11}x^{12}, \quad \{x^{11}, x^{21}\} = -x^{11}x^{21}, \quad \{x^{12}, x^{22}\} = -x^{12}x^{22} \\ &\{x^{21}, x^{22}\} = -x^{21}x^{22} \quad \{x^{12}, x^{21}\} = 0 \quad \{x^{11}, x^{22}\} = -2x^{12}x^{21}. \end{aligned}$$

It follows from this that the sets $\{t\}$, for t in $\mathbf{H}$, and

$$\Sigma_{(\xi_1,\xi_2)} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) : \xi_1 b + \xi_2 c = 0, (b,c) \neq 0\},$$

where $(\xi_1, \xi_2) \neq 0$, are connected (complex) Poisson submanifolds of the (complex) Poisson manifold $SL(2,\mathbb{C})$ (since the corresponding ideals in the ring of rational functions on $SL(2,\mathbb{C})$ are Poisson ideals, and Zariski open subsets of an algebraic Poisson manifold are still Poisson). Moreover the Poisson rank at any of their points equals their dimension and they constitue a partition of $SL(2,\mathbb{C})$. In other words, they are the symplectic leaves of $SL(2,\mathbb{C})$. It follows easily that the symplectic leaves of $SU(1,1)$ are the points of the diagonal (compact) torus and the submanifolds

$$\Sigma_{(\phi)} = \{\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SU(1,1) : \arg b = \phi\}.$$

For any symplectic leave $\Sigma_{(\phi)}$ there exists a unique t in the diagonal compact torus such that $\Sigma_{(\phi)} = t\Sigma_{(\pi/2)}$. On the other hand, observe that the intersection of a symplectic leave in $SL(2,\mathbb{C})$ with $SU(1,1)$ splits as a disjoint union of two (real) leaves.

Note that the symplectic leaves of $SL(2,\mathbb{C})$ are of three different types:

(1) $\Sigma_{(\xi_1,\xi_2)} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \Sigma_{(1,1)}$, where t is a square root of ξ_1/ξ_2 whenever $\xi_1\xi_2 \neq 0$;

(2) $\Sigma_{(1,0)} = \mathbf{B}_- - \mathbf{H}$ and $\Sigma_{(0,1)} = \mathbf{B}_+ - \mathbf{H}$, which are isomorphic, and

(3) the points of $\mathbf{H}$.

2.2. Let $\mathbf{G}$ be the Kac-Moody group corresponding to the matrix A, cf. [KP1, §1] and let $\mathbf{G}_i$ be the image of $SL(2,\mathbb{C})$ under the morphism φ_i defined *loc cit.* Let $\mathcal{W}$ be the Weyl group of $\mathfrak{g}_{\mathbb{C}}$ and w_i the reflection corresponding to the index i. Let $\mathbf{H}$ (resp., $\mathbf{B}_+$, $\mathbf{B}_-$) be the subgroup generated by the images under φ_i of the diagonal torus (resp., of the Borel subgroup of upper, resp. lower, triangular matrices) of $SL(2,\mathbb{C})$.

It follows from the axiomatic definition of $\mathbf{G}$ given in [KP2, **2**] that there exists a unique automorphism Θ of $\mathbf{G}$ such that $\Theta\varphi_i = \varphi_i\epsilon_i$, where ϵ_i is the automorphism of $SL(2,\mathbb{C})$ defined by $\epsilon_i(a) = x_i{}^t\bar{a}^{-1}x_i$,

$$x_i = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{\chi(i)+1} \end{pmatrix}.$$

Let G_J be the fixed-point set of Θ.

Remark 2.1. This definition of G_J is known and was pointed out to me by G. Rousseau. Another definition was proposed in [A1]. Let $\mathbf{G}_J$ be the group defined in [A1]; recall that it is the quotient of the free product of the groups S_i (the fixed-point sets of each ϵ_i) by the intersection of the kernels of all the integrable representations. The inclusions $S_i \hookrightarrow SL(2,\mathbb{C})$ induce a monomorphism $\mathbf{G}_J \hookrightarrow G_J$. Obviuosly, S_i is isomorphic to $SU(2)$ (resp., $SU(1,1)$) if i belongs (resp., does not belong) to J. If $\mathbf{G}$ is finite dimensional, then $\mathbf{G}_J$ is the connected component of G_J. Indeed, the Lie algebra of $\mathbf{G}_J$ contains the Lie algebra of each S_i and by [A1], also contains $\mathfrak{g}_{\mathbb{R}}$. On the other hand, $\mathbf{G}_J$ is generated by exponentials of elements in the Lie algebras of the S_i's, hence it is contained in the component of G_J. We will be more concerned here with $\mathbf{G}_J$.

Now let $\mathbf{G} = \bigcup_{w\in\mathcal{W}} \mathbf{B}w\mathbf{B}$ be the Bruhat decomposition of $\mathbf{G}$ (cf. for example [KP2, **3**]) and let $\mathbf{G}_J^w = \mathbf{G}_J \cap \mathbf{B}w\mathbf{B}$. Let

$$\mathcal{D}_2 = \{\begin{pmatrix} a & b \\ -b & \bar{a} \end{pmatrix} \in SU(2) : b < 0\},$$

$$\mathcal{D}_{1,1} = \{\begin{pmatrix} a & \sqrt{-1}b \\ -\sqrt{-1}b & \bar{a} \end{pmatrix} \in SU(1,1) : b > 0\},$$

$\Sigma_i = \varphi(\mathcal{D}_{1,1})$ (resp., $\varphi(\mathcal{D}_2)$), if $\chi(i) = 0$ (resp., 1). In what follows we shall omit to write φ_i.

Proposition 2.1. *Let $w \in \mathcal{W}$ and let $w = w_{i_1}\dots w_{i_{\ell(w)}}$ be a reduced decomposition of w. Any $x \in \mathbf{G}_J^w$ can be uniquely expressed as a product $x = z_{i_1}\dots z_{i_{\ell(w)}}t$, where $z_{i_j} \in \Sigma_{i_j}$ and $t \in \mathbf{T} = \mathbf{H} \cap \mathbf{G}_J$.*

Proof. Let $x \in \mathbf{G}_J$ and decompose it as a product $x = x_{i_1}\dots x_{i_e}$, where $x_{i_j} \in S_j$. We shall first prove by induction on e that x can be expressed as a product $x = z_{i_1}\dots z_{i_d}t$, where $z_{i_j} \in \Sigma_{i_j}$ and $t \in \mathbf{T}$. The case $e = 1$ is covered by *2.1* or is well known, so we can assume that $e > 1$. Write $x_{i_j} = u_{i_j}t_{i_j}$ for some $u_{i_j} \in \Sigma_{i_j} \cup \{1\}$, $t_{i_j} \in \mathbf{T}$. Applying the inductive hypothesis, we get $x = u_{i_1}t_{i_1}\dots u_{i_e}t_{i_e} = z_{i_1}\dots z_{i_d}tu_{i_e}t_{i_e}$. But then $tu_{i_e} \in \mathbf{G}_J^{w_i}$ and the claim follows from [KP2, **4.6**], [St] and the remarks in *2.1*.

It follows easily that any $x \in \mathbf{G}_J^w$ can be expressed as a product $x = z_{i_1}\dots z_{i_{\ell(w)}}t$. The uniqueness is proved exactly as in [St, Th. 15].

Now for any $w \in \mathcal{W}$, let $w = w_{i_1} \dots w_{i_{\ell(w)}}$ be a reduced decomposition and let $\Sigma_w = \Sigma_{i_1} \dots \Sigma_{i_{\ell(w)}}$. As in the compact case [S], we deduce from Proposition 2.2, [VS] and *2.1* the following fact.

Proposition 2.2. *The family $\Sigma_w t$, $w \in \mathcal{W}$, $t \in \mathbf{T}$, constitutes a classification of the symplectic leaves in $\mathbf{G}_J$.*

Remark 2. 2. The preceding proposition, in the compact Kac-Moody case, can be also deduced from [KP2, Proposition 5.1].

§3. **Poisson structures on real Kac-Moody groups.** In this section we want to discuss the passage from the Lie bialgebra structure on $\mathfrak{g}_{\mathbb{C}}$ (resp., $\mathfrak{g}_J$) discussed in [D1, §3] (resp., §1) to a Poisson structure on the corresponding Kac-Moody group. We shall assume from now on that the generalized Cartan matrix A is not finite; in the finite case, the passage is well-known: see [D3]. It is not clear, to my knowledge, which is the definition of a Poisson group, or even a Poisson manifold, in the infinite dimensional case. We shall adopt the provisional point of view, of considering a structure of Poisson algebra on some algebra of functions on the space in question, but see the remarks after Lemma 3.1.

Let $\mathbb{C}[\mathbf{G}]$ be the ring of strongly regular functions on $\mathbf{G}$ [KP1, §2].Let $\mathfrak{l}(\lambda)$ ($\mathfrak{l}^*(\lambda)$) be the irreducible highest (lowest) weight $\mathfrak{g}_{\mathbb{C}}$-module defined as in [K, Ch. 9]. Then, as a $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$-module, we have

$$\mathbb{C}[\mathbf{G}] \simeq \oplus_{\lambda \in P^+} \mathfrak{l}(\lambda) \otimes \mathfrak{l}^*(\lambda), \tag{3.1}$$

cf. *loc cit.* Moreover, $\mathbb{C}[\mathbf{G}]$ can be also identified with a subalgebra of $U(\mathfrak{g}_{\mathbb{C}})^*$, and then it is a $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$-submodule, where $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$ acts on $U(\mathfrak{g}_{\mathbb{C}})^*$ in the following way: the first (resp; second) by the transpose of the right action (resp., the left action composed with the antipode). In the following, we shall always consider this action. Recall that this identification factorizes through the right hand side of (3.1) thanks to the morphisms of $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$-modules $\phi_\lambda : \mathfrak{l}(\lambda) \otimes \mathfrak{l}^*(\lambda) \to U(\mathfrak{g}_{\mathbb{C}})^*$ given by

$$\langle \phi_\lambda(v \otimes \alpha), x \rangle = \langle \alpha, xv \rangle, \quad x \in U(\mathfrak{g}_{\mathbb{C}}).$$

Note that the image of ϕ_λ is the isotypic component, with respect to the $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$ action, of type $\mathfrak{l}(\lambda) \otimes \mathfrak{l}^*(\lambda)$. Indeed, let $\phi : \mathfrak{l}(\lambda) \otimes \mathfrak{l}^*(\lambda) \to U(\mathfrak{g}_{\mathbb{C}})^*$ be a morphism of $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$-modules and set $b(v, \alpha) = \langle \phi(v \otimes \alpha), 1 \rangle$. Then $b(v, \alpha) = c\langle \alpha, v \rangle$ for some scalar c; hence $\phi = c\phi_\lambda$.

It is suggestive to consider some other subspaces of $U(\mathfrak{g}_{\mathbb{C}}))^*$. First let $\mathbb{C}_-[\mathbf{G}] = \omega^t(\mathbb{C}[\mathbf{G}])$; second, let $\mathbb{C}_\pm[\mathbf{G}]$ let the subalgebra generated by $\mathbb{C}_-[\mathbf{G}] + \mathbb{C}[\mathbf{G}]$. Note that $\mathbb{C}_-[\mathbf{G}] \cap \mathbb{C}[\mathbf{G}]$ is one-dimensional and $\mathbb{C}_-[\mathbf{G}] \simeq \oplus_{\lambda \in P^+} \mathfrak{l}^*(\lambda) \otimes \mathfrak{l}(\lambda)$. Next let $\mathfrak{O}[\mathbf{G}]$ be the sum of all integrable $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$-submodules of $U(\mathfrak{g}_{\mathbb{C}}))^*$. As the multiplication $U(\mathfrak{g}_{\mathbb{C}})^* \otimes U(\mathfrak{g}_{\mathbb{C}})^* \to U(\mathfrak{g}_{\mathbb{C}})^*$ is a morphism of $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$-modules, $\mathfrak{O}[\mathbf{G}]$ is a subalgebra of $U(\mathfrak{g}_{\mathbb{C}})^*$.

Now let r be the formal expression defined by (1.1).

Lemma 3.1. (a) $\mathbb{C}[\mathrm{G}]$ *is a Poisson algebra, with the bracket determined by*

$$\langle\{\phi_\lambda(v\otimes\eta),\phi_\mu(w\otimes\zeta)\},x\rangle=\langle\eta\otimes\zeta,[x\otimes 1+1\otimes x,r]v\otimes w\rangle. \tag{3.2}$$

(b) $\mathbb{C}_-[\mathrm{G}]$ *is also a Poisson algebra, with the bracket defined again by (3.2).*

Let $\{\,,\,\}:(U(\mathfrak{g}_\mathbb{C})\otimes U(\mathfrak{g}_\mathbb{C}))^*\to U(\mathfrak{g}_\mathbb{C})^*$ be the transpose of the extension to $U(\mathfrak{g}_\mathbb{C})$ of the cobracket defined by r. We only need to show that $\{\mathbb{C}[\mathrm{G}],\mathbb{C}[\mathrm{G}]\}\subseteq\mathbb{C}[\mathrm{G}]$. Let $x\in U(\mathfrak{g}_\mathbb{C})$, $\lambda,\mu\in P^+$, $v\in \mathfrak{l}(\lambda)$, $w\in\mathfrak{l}(\mu)$, $\eta\in\mathfrak{l}^*(\lambda)$, $\zeta\in\mathfrak{l}^*(\mu)$. Then

$$\begin{aligned}\langle\{\phi_\lambda(v\otimes\eta),\phi_\mu(w\otimes\zeta)\},x\rangle&=\langle\phi_\lambda(v\otimes\eta)\otimes\phi_\mu(w\otimes\zeta),\operatorname{ad}x(r)\rangle=\langle\eta\otimes\zeta,[x\otimes1+1\otimes x,r]v\otimes w\rangle\\&=\langle\eta\otimes\zeta,(x\otimes1+1\otimes x)rv\otimes w\rangle+\langle r\eta\otimes\zeta,(x\otimes1+1\otimes x)v\otimes w\rangle.\end{aligned}$$

Now $r.v\otimes w=\frac{1}{2}\sum_{\alpha\in\Delta^+,i\in I_\alpha}X_{\alpha,i}v\otimes X_{-\alpha,i}w-X_{-\alpha,i}v\otimes X_{\alpha,i}w$ is a well-defined element of $\mathfrak{l}(\lambda)\otimes\mathfrak{l}(\mu)$. Indeed, we can assume that v is a weight vector of weight $\lambda-\tau$, for some $\tau\in Q^+$ (cf. §4 for the notation); if $X_{\alpha,i}v\neq0$, then $\lambda-\tau+\alpha=\lambda-\tau'$ for some $\tau'\in Q^+$; hence the set of α's such that $X_{\alpha,i}v\neq0$ is finite.

The proof of (b) does not differ from the proof of (a). □ enddemo

Let us now discuss the real case. We want to consider a Poisson algebra of functions on G_J. The (antilinear) Lie algebra automorphism θ extends to an (antilinear) Hopf algebra automorphism of $U(\mathfrak{g}_\mathbb{C})$, still denoted θ, whose fixed-point set is the universal enveloping algebra of $\mathfrak{g}_J$. However, its transpose does not preserve $\mathbb{C}[\mathrm{G}]$, nor $\mathbb{C}_-[\mathrm{G}]$. On the other hand, $\mathbb{C}_\pm[\mathrm{G}]$ and $\mathfrak{O}[\mathrm{G}]$ are both θ^t stable, but the Poisson bracket seems to be not well defined on them. The author imagines two possibilities of circunvecting this problem: one is to consider a Poisson algebraic infinite dimensional variety as a limit of finite dimensional ones, as in [Sh]; the other is not to force ourselves to consider a Kac-Moody group as an "affine" one, but perhaps to take into account simultaneously two rings of functions on it; i.e. an algebraic analogue of twin buidings [T1].

Algebraic Poisson Geometry

The following sections are concerned with a systematic study of algebraic varieties carrying and additional structure-wich is locally described by a Poisson bracket.

§4. **Poisson algebras.** The material of this section, with the exception of two or three points, is taken from [Be]; the author decided to include it because that paper is not easily available.

Let k be a commutative ring. A Poisson algebra over k is a pair $(A,\{\,,\,\})$ where A is a commutative algebra over k and a Lie algebra over k with bracket $\{\,,\,\}$, both structures being related by the Leibnitz rule: $\{f,gh\}=\{f,g\}h+g\{f,h\}$. In what follows, all algebras are understood to be over k, and commutative unless explicitly stated.

A subset of a Poisson algebra which is simultaneously an ideal for both structures is called a *Poisson ideal.* Poisson subalgebras are defined in the same vein. The quotient of a Poisson algebra by a Poisson ideal is obviously a Poisson algebra, wich satisfies the expected universal property. We shall reserve the words "ideal", "subalgebra", ... , for the ideals ,subalgebras, ... , of the associative algebra structure. The sum and the intersection of an arbitrary collection of Poisson ideals are still Poisson. In particular, the greatest Poisson ideal contained in an arbitrary ideal I is denoted $\pi(I)$.

Let us fix a Poisson algebra $(A, \{\ ,\ \})$ and let I be an ideal of $A, a \in A$. Set

$$\sigma_a(I) = \{s \in A : \exists m \in \mathbf{N} \text{ such that } a^m s \in I\}.$$

The following properties can be easily verified:

(a) $\sigma_a(I)$ is an ideal of A, $\sigma_a(I)\ I$. If I is Poisson, then so is $\sigma_a(I)$.

(b) $\sigma_a(I) = A$ if and only if $a \in \sqrt{I}$.

(c) P a prime ideal of A, $a \notin \sqrt{\pi(p)}$. Then $\sigma_a(\pi(P)) = \pi(P)$.

If $a \notin P$, then $\pi_a(P) \subseteq \sqrt{\pi_a(P)} \subseteq P$ and we are done. Now suppose that $\sqrt{\pi_a(P)} \not\subseteq P$ and let $s \in \sqrt{\pi_a(P)} - P$. Take $m \in \mathbb{N}$ such that $a^m s \in \pi(P)$. Then a^m belongs to $\sigma_s(\pi(P)) = \pi(P)$.

(d) P prime implies $\pi(P)$ primary.

(e) Localization and tensor product of Poisson algebras are still Poisson.

Let S be a multiplicatively closed subset contained in a Poisson algebra A and let $t, s \in S$, $a, b \in A$. Then the formula for the Poisson bracket in A_S is

$$\{\frac{a}{s}, \frac{b}{t}\} = \frac{st\{a,b\} - sb\{a,t\} - at\{s,b\} + ab\{s,t\}}{s^2t^2}$$

Let B be another Poisson algebra. Then the Poisson structure on $A \otimes B$ is determined by $\{a \otimes b, a' \otimes b'\} = \{a, a'\} \otimes bb' + aa' \otimes \{b, b'\}$.

(f) "Sheaf" property.

Let $(s_i)_{i\in I}$ be a family of elements in an associative algebra A such that each localized algebra A_{s_i} is provided with a Poisson structure. $A_{s_i s_j}$ inherits then two Poisson structures, by localizing from A_{s_i} or A_{s_j}. Assume that they coincide. If $A = \langle s_i \rangle_{i\in I}$, then A admits a unique Poisson structure inducing the given in A_{s_i} by localization.

Apply [AM, Ex. 7.1] to $\{a, b\}_i \in A_{s_i}$ to guarantee that $\{a, b\} \in A$ whenever a, $b \in A$.

(g) Assume that k is $\mathbb{Q}$-algebra. Then P prime implies $\pi(P)$ prime.

We need to prove that $\sqrt{\pi(P)}$ is Poisson. Let $x \in \sqrt{\pi(P)}$, $a \in A$. Let n be the minimal positive integer such that $x^n \in \pi(P)$; we can assume that $n > 1$. Then $\{a, x^n\} = nx^{n-1}\{a, x\} \in \pi(P)$, hence $x^{n-1}\{a, x\} \in \pi(P)$ and by (d), $\{a, x\} \in \sqrt{\pi(P)}$.

(h) Without the assumption $\mathbb{Q} \hookrightarrow k$, (g) is false.

Let $k[D, \epsilon]$ be a polynomial ring in two variables over k. Then the bracket defined by

$$\{\epsilon^j D^n, \epsilon^q D^m\} = (nq - mj)\epsilon^{q+j-1} D^{m+n-1}$$

provides $k[D,\epsilon]$ with a Poisson algebra structure. Assume that $p = 0$ in k for some integer p. Then $\langle\epsilon^p\rangle$ is a Poisson ideal and hence $A = k[D,\epsilon]/\langle\epsilon^p\rangle$ is a Poisson algebra. Let I be the principal ideal of A generated by the image of ϵ. If k is an integral domain, then I is prime. Now assume that p is a prime number. Then we claim that $\pi(I) = 0$, which is not a prime ideal of A. Indeed, let $0 \neq P(\epsilon, D) \in \pi(I)$, where by abuse of notation we still name D, ϵ, their images in A. Write $P(\epsilon, D) = \sum_{j\leq i\leq p-1} P_i(D)\epsilon^i$, with $P_i(D) \neq 0$, and take j minimal. Obviously, $j > 0$. But $\{D, P(\epsilon, D)\} = \sum_{j\leq i\leq p-1} P_i(D)i\epsilon^{i-1}$. As j is a unit, $jP_j(D) \neq 0$, a contradiction.

On the other hand, let $k = \mathbb{Z}$ and $I \subset \mathbb{Z}[D,\epsilon]$ the ideal generated by 2 and ϵ. Then $\pi(I) = \langle 2, \epsilon^2\rangle$, which is not prime. (If $\epsilon P(D)$ belongs to $\pi(I)$, then so does $\{D, \epsilon P(D)\} = P(D)$; hence $2|P(D)$).

Remark 4.1. (g) was proved in [Be], but claimed in [VK] without the assumption above. The first part of (h) is inspired in [Kp, p. 12, Example].

Now we state some consequences of (g). Assume that $\mathbb{Q} \hookrightarrow k$.

(i) Any prime ideal of A is Poisson.
(j) Let I be an ideal which is maximal in the set of Poisson ideals of A. Then I is prime.
(k) The radical of a Poisson ideal is the intersection of all the Poisson prime ideals containing it. It is then also a Poisson ideal.
(l) If I is a radical ideal, then so is $\pi(I)$.

Example. Let F be a finite set and let A be the algebra of functions on F with values in k; then A admits no non-trivial Poisson bracket. In fact, A has no non-trivial derivation.

§5. A Darboux type theorem. Let A be a k-algebra and let $\mathrm{Der}(A)$ (resp., $\Omega(A)$) denote the A-module of k-derivations (resp. the differential module of A); recall that $\mathrm{Der}(A)$ is isomorphic, as A-module, to $\mathrm{Hom}_A(\Omega(A), A)$. Assume that A is a Poisson algebra. Then there exists $F \in \mathrm{Hom}_A(\Lambda^2\Omega(A), A)$ such that

$$\{a,b\} = F(\mathrm{d}\,a \wedge \mathrm{d}\,b). \tag{5.1}$$

Now assume that the Poisson algebra A is local and let $\mathfrak{M}$ denote its maximal ideal, $K = A/\mathfrak{M}$ its residual field, $y \mapsto \bar{y}$ the projection $\mathfrak{M} \to \mathfrak{M}/\mathfrak{M}^2$. The composition of the Poisson bracket with the canonical projection $A \to K$ induces an antisymmetric bilinear form $B : \mathfrak{M}/\mathfrak{M}^2 \times \mathfrak{M}/\mathfrak{M}^2 \to K$. If $\dim_K(m/\mathfrak{M}^2)$ is finite, B can be expressed in some basis as

$$\begin{pmatrix} 0 & I_r & 0 \\ -I_r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where I_r is the $r \times r$ identity matrix. The following auxiliary result should be well known.

Lemma 5.1. *Let A be a local ring, M a free A-module of finite rank, $B : M \times M \to A$ an antisymmetric bilinear form. Then there exists a basis of M such that the matrix of B in it has the form*

$$\begin{pmatrix} 0 & I_r & 0 \\ -I_r & 0 & 0 \\ 0 & 0 & X \end{pmatrix},$$

where the entries of X belong to the maximal ideal $\mathfrak{M}$ of A. (Clearly, the integer r depends only of B.)

Proof. Let $\{e_i\}$ be a basis of M. If $B(e_i,e_j) \in \mathfrak{M}$ for all i, j, there is nothing to prove. Otherwise we can assume, after reordering the index set and multiplying by a unit, that $B(e_1,e_2)=1$. Take $u_i = e_i + B(e_i,e_1)e_2 - B(e_i,e_2)e_1$ and proceed by induction. □

Theorem 5.2. *Let A be a local regular Poisson algebra over a field k of char 0 and suppose that the residual field $K = A/\mathfrak{M}$ is isomorphic to 0 and that A is a localization of a finitely generated k-algebra. Then A admits a regular system of parameters $p_1,\dots,p_r,q_1,\dots,q_r,y_1,\dots,y_s$ such that*

(5.2)

$$\{p_i,q_j\}=\delta_{ij},\quad \{p_i,p_j\}=\{q_i,q_j\}=\{p_i,y_h\}=\{q_i,y_h\}=0,\quad \{y_h,y_\ell\}\in\langle y_1,\dots,y_s\rangle.$$

In particular, $\pi(\mathfrak{M})$ is the ideal generated by $y_1,\dots,y_s$.

Proof. By hypothesis ΩA is a free A-module of rank equal to $\dim A$ [Ha, Th. II.8.8]. We shall apply the Lemma to the bilinear form $B : \Omega A \times \Omega A \to A$ induced by the Poisson bracket. Let $p_1,\dots,p_r,q_1,\dots,q_r,y_1,\dots,y_s$ be elements of $\mathfrak{M}$ such that their images in $\mathfrak{M}/\mathfrak{M}^2$ coincide with the images of the basis $\mathcal{B}$ provided by the Lemma, after tensoring ΩA with k (see [Ha, II.8.7]). By Nakayama Lemma, $\mathcal{B}=\{\mathrm{d}\,p_1,\dots,\mathrm{d}\,p_r,\dots,\mathrm{d}\,y_s\}$. Assume now that

$$\{y_h,y_\ell\}=\sum a_ip_i+\sum b_jq_j+\sum c_ty_t \mod \mathfrak{M}^2,$$

for some $a_i,b_j,c_t\in k$. Then $0=\{p_i,\{y_h,y_\ell\}\}=b_i \mod \mathfrak{M}$ (by Jacobi) and similarly, $a_j=0$. A similar argument shows that $\{y_h,y_\ell\}$ belongs to $\langle y_1,\dots,y_s\rangle \mod \mathfrak{M}^n$ for all n. Passing to the quotient by $\langle y_1,\dots,y_s\rangle$ and using [Ma, 14.D and Th. 36] we conclude that $\{y_h,y_\ell\}\in\langle y_1,\dots,y_s\rangle$. Hence, $\langle y_1,\dots,y_s\rangle$ is a Poisson ideal and is contained in $\pi(\mathfrak{M})$. To prove the equality, we can assume that $s=0$; that is, we want now to prove that $\pi(\mathfrak{M})=0$. Let n be the minimal integer such that there exists $x\in(\mathfrak{M}^n\cap\pi(\mathfrak{M}))-\mathfrak{M}^{n+1}$. Assume for example that

$$x=\sum_{\ell\geq 0}P_\ell p_1^\ell \mod \mathfrak{M}^{n+1},$$

where P_ℓ is an homogeneous polynomial in $p_2,\dots q_r$ of total degree $n-\ell$, and for some $\ell>0$, $P_\ell\neq 0$ [Ma, Th. 35]. Then $\{q_1,x\}=\sum_{\ell>0}\ell P_\ell p_1^{\ell-1} \mod \mathfrak{M}^n\in\pi(\mathfrak{M})$, contradicting the minimality of n. □

Now let $\mathfrak{X}$ be a Poisson non-singular variety over an algebraically closed field k of char 0. This means (1) $\mathfrak{X}$ is a non-singular variety as defined in [Ha, pp. 105, 177],

(2) the structural sheaf is in addition a sheaf of Poisson algebras.

Let $\mathcal{O}_x$ be the local ring at a closed point x with the inherited Poisson structure and let $p_1,\dots,p_r,q_1,\dots,q_r,y_1,\dots,y_s$ be as in the conclusion of the Theorem. Let U be an affine neighborhood of x such that $p_1,\dots,p_r,q_1,\dots,q_r,y_1,\dots,y_s$ belong to $k[U]$. Let $F_U(x)$ be the closed reduced subvariety of U defined by the ideal generated by $y_1,\dots,y_s$. Observe that $F_U(x)$ is an irreducible subvariety of dimension $2r$. As the application $k[U]\to\mathcal{O}_x$ is injective, we have that (5.2) is still true on $k[U]$; it follows that the Poisson rank of $F_U(x)$ at any closed point z is $2r$ and hence $F_U(x)$ is a symplectic subvariety of U.

Now let us define an equivalence relation on $\mathfrak{X}(k)$ as follows: x and y are declared equivalent if there exists a family of closed points $(x_i)_{1\leq i\leq N}$ and for each x_i an open affine neighborhood U_i and a closed irreducible symplectic subvariety $F_{U_i}(x_i)$ of U_i as above such that $x_1 = x$, $x_N = y$ and $x_i \in F_{U_{i-1}}(x_{i-1})$, $2 \leq i \leq N$. Clearly, x is equivalent to y for any $y \in F_U(x)$. It is easy to see that the equivalence class of x is a symplectic irreducible subvariety of $\mathfrak{X}$.

Algebras of functions on quantum groups

§6. **Quantum Kac-Moody groups.** In this section, we shall discuss the "algebra of strongly regular functions" on a quantum Kac-Moody group, cf. [KP1]. For simplicity, we shall assume that $\det A \neq 0$, cf. [L1, 4.14].

6.1. Let $\mathbf{q} \neq 1$ be a positive real number. Following Gauss, we set for $m \in \mathbf{N}$

$$[m] = \frac{\mathbf{q}^m - \mathbf{q}^{-m}}{\mathbf{q} - \mathbf{q}^{-1}}, \quad [m]! = [m][m-1]\dots[1], \quad \begin{bmatrix} m \\ j \end{bmatrix} = \frac{[m]!}{[j]![m-j]!}.$$

The quantized enveloping algebra $\mathbf{U} = \mathbf{U}_q(A)$ is defined as the associative $\mathbb{C}$-algebra given by generators E_i, F_i, $K_i^{\pm 1}$ and relations

$$K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad K_iK_j = K_jK_i, \tag{6.1}$$

$$K_iE_j = q^{d_ia_{ij}}E_jK_i, \quad K_iF_j = q^{-d_ia_{ij}}F_jK_i, \tag{6.2}$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}, \tag{6.3}$$

and if $i \neq j$

$$\sum_{h+\ell=1-a_{ij}} (-1)^h E_i^{(\ell)}E_jE_i^{(h)} = 0, \qquad \sum_{h+\ell=1-a_{ij}} (-1)^h F_i^{(\ell)}F_jF_i^{(h)} = 0. \tag{6.4}$$

Here $E_i^{(N)}$ denotes E_i^N divided by $[N]_{d_i}^!$ (idem for $F_i^{(N)}$).$\mathbf{U}$ is a Hopf algebra with comultiplication Δ, antipode S and counit ϵ defined by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i \quad \Delta K_i = K_i \otimes K_i \tag{6.5}$$

$$S(E_i) = -K_i^{-1}E_i \quad S(F_i) = -F_iK_i \quad S(K_i) = K_i^{-1} \tag{6.6}$$

$$\epsilon(E_i) = 0 \qquad \epsilon(F_i) = 0 \qquad \epsilon(K_i) = 1. \tag{6.7}$$

Let $\mathbf{U}^0$ (resp.,$\mathbf{U}^+$, $\mathbf{U}^-$) be the subalgebra of $\mathbf{U}$ generated by all the $K_i^{\pm 1}$ (resp., E_i, F_i).

Let R (resp., L) be the right (resp., left) action of $\mathbf{U}$ on itself: $R_x(y) = yx$ (resp., $L_x(y) = xy$.

Let P (resp. $Q^\vee$) be the free abelian group with basis ω_i (resp. $\alpha_i^\vee$), $1 \leq i \leq n$. Let $P^+ = \sum \mathbb{Z}_+\omega_i$. Let $\langle , \rangle : P \times Q^\vee \to \mathbb{Z}$ be the bilinear pairing defined by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. Let $\alpha_j \in P$ be defined by $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$ and let $Q = \sum \mathbb{Z}\alpha_i$, $Q^+ = \sum \mathbb{Z}_+\alpha_i$.

As usual, we shall denote by ρ the element $\sum_{1\leq i\leq n} \omega_i \in P$.

Let $\geq$ be the partial ordering on P defined by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q^+$.

Let $(|) : Q^\vee \times Q^\vee \to \mathbb{Z}$ be the symmetric bilinear non-degenerate form defined by $(\alpha_i^\vee | \alpha_j^\vee) = d_j^{-1} a_{ij} = d_i^{-1} a_{ji}$. Let $\nu : Q^\vee \to P$ be the morphism defined by $\langle \nu(\alpha^\vee), \beta^\vee \rangle = (\alpha^\vee | \beta^\vee)$; we have $Q \subset \nu(Q^\vee)$ because $\alpha_i = d_i \nu(\alpha_i^\vee)$. Thanks to ν, we have a symmetric bilinear non-degenerate form, still denoted $(|)$, on Q; it follows that $(\alpha_i | \alpha_j) = d_i a_{ij} = d_j a_{ji}$. We can even extend this form to $(|) : P \times P \to \mathbb{Q}$.

Let M be an $\mathbf{U}^0$-module and let $\tau \in P$. Then M_τ denotes the vector subspace of M consisting of all v such that $K_i v = q^{(\tau|\alpha_i)} v$. We shall say M is $\mathbf{U}^0$-*diagonalizable* if $M = \oplus_{\tau \in P} M_\tau$; and $\mathbf{U}^0$-*admissible* if in addition $\dim M_\tau < \infty$, for any τ. (We do not need to consider a more general notion of diagonalizable $\mathbf{U}^0$-modules than this one.) The weights of M are the elements of the set

$$\Pi(M) = \{\tau \in P : M_\tau \neq 0\}.$$

Now let M be an $\mathbf{U}$-module. We shall say that M is *integrable* [L1, 3.1] if it is $\mathbf{U}^0$-diagonalizable and E_i, F_i act as locally nilpotent endomorphisms on M, for all i. The direct sum (resp., tensor product) of an arbitrary (resp., finite) family of integrable modules is again integrable.

Let $\lambda \in P$ and let $\mathcal{L}(\lambda)$ be the unique irreducible $\mathbf{U}$-module having a vector $v_\lambda \in \mathcal{L}(\lambda)_\lambda - 0$ such that $\mathbf{U}^+ v_\lambda = 0$. Then [L, Th. 4.12] $\mathcal{L}(\lambda)$ is a graded $\mathbf{U}^0$-module; more precisely, $\mathcal{L}(\lambda) = \oplus_{\tau\in P} \mathcal{L}(\lambda)_\tau$ and the dimension of $\mathcal{L}(\lambda)_\tau$ is given by the Kac-Weyl formula. Now let us identify $\oplus_{\tau \in P} \mathcal{L}(\lambda)_\tau^*$ with a subspace of $\mathcal{L}(\lambda)^*$ and let us consider the latter as a $\mathbf{U}$-module by the formula $\langle x\alpha, v \rangle = \langle \alpha, S(x)v \rangle$, $x \in \mathbf{U}$, $\alpha \in \mathcal{L}(\lambda)^*$, $v \in \mathcal{L}(\lambda)$. Let $\alpha_\lambda \in \mathcal{L}(\lambda)_\lambda^* - 0$ and let $\mathcal{L}^*(\lambda)$ be the $\mathbf{U}$-submodule generated by α_λ. Note that $\alpha_\lambda \in \mathcal{L}(\lambda)_{-\lambda}^* - 0$ and $\mathbf{U}^- \alpha_\lambda = 0$; moreover $\mathcal{L}^*(\lambda) \subseteq \oplus_{\tau\in P} \mathcal{L}(\lambda)_\tau^*$. $\mathcal{L}^*(\lambda)$ has a unique irreducible submodule; applying again [L, Th. 4.12] we see that $\mathcal{L}^*(\lambda)$ is in fact irreducible and equals $\oplus_{\tau\in P} \mathcal{L}(\lambda)_\tau^*$.

6.2. The known (to me) proofs of the "complete reducibility theorem" for quantum groups with finite Cartan matrix are based on an argument of Borel, in order to avoid the use of the Casimir [R], [L2, 7.2]. However, a quantum analogue of the Casimir was constructed in [D2]. The purpose of this subsection is to prove a "complete reducibility theorem" for quantized Kac-Moody algebras, using this quantum Casimir and following the lines of the proof in [K, 10.7].

The category $\mathcal{O}$ and primitive vectors are defined as in [K, §9].

The following important fact was proved in [D2, §5], see the remarks after Proposition 5.2. (It is not difficult to pass from the formal version of the quantized enveloping algebra considered by Drinfeld to the present setting.)

Proposition 6.1. *Let V be an $\mathbf{U}$-module in the category $\mathcal{O}$. Then there exists an operator $\Omega : V \to V$ which commutes with the action of $\mathbf{U}$ and which is the image of a certain element in a suitable completion of $\mathbf{U}$. (In particular, Ω is preserved by morphisms of $\mathbf{U}$-modules.) Moreover, if V is generated by an element v such that $\mathbf{U}^+v = 0$, $v \in V_\tau$ for some $\tau \in P$, then Ω acts on V as multiplication by $q^{(\tau|\tau+2\rho)}$.*

Theorem 6.2. *Let M be a module in the category $\mathcal{O}$. Then the following statements are equivalents:*

(a) *M is integrable.*
(b) *M is isomorphic to a direct sum of modules $\mathcal{L}(\lambda)$, $\lambda \in P^+$.*

Proof. (b) $\implies$ (a) follows from [L1, 3.2]. We shall trace from [K] the steps of the proof of the remaining implication.

Step 1. (Compare with [K, Prop. 9.5].) Let V be a module in the category $\mathcal{O}$ and suppose that for any two primitive vectors λ and μ of V, $\lambda \geq \mu$ implies $\lambda = \mu$. Then V is completely reducible.

Step 2. (Compare with [K, Prop. 9.9 b)].)Let V be a module in the category $\mathcal{O}$ and suppose that for any two primitive vectors λ and μ of V, $\lambda \geq \mu$ implies $2(\lambda + \rho|\lambda - \mu) \neq (\lambda - \mu|\lambda - \mu)$. Then V is completely reducible.

Step 3. Now the proof of (a) $\implies$ (b) is the same as the proof of [K, Th. 10.7 b)], taking into account [L1, Lemma 3.3]. □

6.3. Let λ, $\mu \in P^+$. Then $\mathfrak{l}(\lambda) \otimes \mathfrak{l}(\mu)$ is a completely irreducible $\mathfrak{g}_{\mathbb{C}}$-module and one has a decomposition

$$\mathfrak{l}(\lambda) \otimes \mathfrak{l}(\mu) \simeq \oplus_{i \in I} \mathfrak{l}(\tau_i), \tag{6.8}$$

for some family $(\tau_i)_{i \in I}$ of elements in P^+ [K, Corollary 10.7.b].

Now let us consider $\mathcal{L}(\lambda) \otimes \mathcal{L}(\mu)$ as an $\mathbf{U}$-module via Δ. We shall deduce from [L1] that the Grothendieck rings of the categories of "integrable modules in the category $\mathcal{O}$" coincide in the classical and quantum cases.

Theorem 6.3. *$\mathcal{L}(\lambda) \otimes \mathcal{L}(\mu)$ is completely reducible and*

$$\mathcal{L}(\lambda) \otimes \mathcal{L}(\mu) \simeq \oplus_{i \in I} \mathcal{L}(\tau_i),$$

where $(\tau_i)_{i \in I}$ is the same family as in (6.8).

Proof. The first part follows from the Theorem and the second, from the first and [L1, Th. 4.12]. □

6.4. Now let ϕ_λ be the application $\mathcal{L}(\lambda) \otimes \mathcal{L}^*(\lambda) \to \mathbf{U}^*$ given by

$$\langle \phi_\lambda(v \otimes \alpha), x \rangle = \langle \alpha, xv \rangle, \quad x \in \mathbf{U}.$$

Let us consider $\mathbf{U}^*$ as a $\mathbf{U}\otimes\mathbf{U}$-module via $R^t\otimes(L\circ S)^t$, i. e. $\langle (R^t\otimes(L\circ S)^t)(y\otimes z)T, x\rangle = \langle T, S(z)xy\rangle$. Then ϕ_λ is a morphism of bimodules:

$$\langle (R^t\otimes(L\circ S)^t)(y\otimes z)\phi_\lambda((v\otimes\alpha), x\rangle = \langle \phi_\lambda(v\otimes\alpha), S(z)xy\rangle = \langle z\alpha, xyv\rangle = \langle \phi_\lambda(yv\otimes z\alpha), x\rangle.$$

Let $M_{(\lambda)}$ be the isotypic component of type $\mathcal{L}(\lambda)$ of an $\mathbf{U}$-module M, i.e. the sum of all its submodules isomorphic to $\mathcal{L}(\lambda)$. We have $\phi_\lambda(\mathcal{L}(\lambda)\otimes\mathcal{L}^*(\lambda))\subseteq \mathbf{U}^*_{(\lambda)}$, considering the $\mathbf{U}$-module structure on $\mathbf{U}^*$ given by R^t.

In analogy with the Peter-Weyl theorem and [KP1, Th. 1], we introduce now the "ring of strongly regular functions on a quantum Kac-Moody group". Let $\mathbb{C}_q[G]$ be the image of the morphism $\Phi = \oplus\phi_\lambda : \oplus_{\lambda\in P^+}\mathcal{L}(\lambda)\otimes\mathcal{L}^*(\lambda)\to\mathbf{U}^*$.

Proposition 6.4. (a) *Φ is a monomorphism and hence* $\mathbb{C}_q[G]\simeq\oplus_{\lambda\in P^+}\mathcal{L}(\lambda)\otimes\mathcal{L}^*(\lambda)$.
(b) *$\mathbb{C}_q[G]$ is a subalgebra of* $\mathbf{U}^*$.

Proof. (a). Each ϕ_λ is injective since $\mathcal{L}(\lambda)\otimes\mathcal{L}^*(\lambda)$ is an irreducible bimodule and $\phi_\lambda(v_\lambda\otimes\alpha_\lambda)\neq 0$. As $\mathbb{C}_q[G] = \oplus_{\lambda\in P}\mathbb{C}_q[G]_{(\lambda)}$, Φ is injective.

(b). Let λ, $\mu\in P^+$, $v\in\mathcal{L}(\lambda)$, $\alpha\in\mathcal{L}^*(\lambda)$, $w\in\mathcal{L}(\mu)$, $\beta\in\mathcal{L}^*(\mu)$. Let (τ_i) be a family in P^+ such that $\mathcal{L}(\lambda)\otimes\mathcal{L}(\mu)\simeq\oplus_{i\in I}\mathcal{L}(\tau_i)$; then $\mathcal{L}^*(\lambda)\otimes\mathcal{L}^*(\mu)\simeq\oplus_{i\in I}\mathcal{L}^*(\tau_i)$. Now let $u_i\in\mathcal{L}(\tau_i)$, $\gamma_j\in\mathcal{L}^*(\tau_i)$ such that

$$v\otimes w = \sum_{i\in I} u_i,\quad \alpha\otimes\beta = \sum_{j\in I}\gamma_j.$$

Then

$$\phi_\lambda(v\otimes\alpha)\phi_\mu(w\otimes\beta) = \sum_{i\in I}\phi_\lambda(u_i\otimes\gamma_j).\quad\square$$

§7. **Admissible representations.** In this section we propose a definition of *admissible* modules over $\mathbb{C}_q[G]$. From now on we assume that G is finite dimensional. Let $w\in\mathcal{W}$, $\lambda\in P^+$. Following [LS] we introduce the following elements of $\mathbb{C}_q[G]$:

$$C^+_{w,\lambda} = \phi_\lambda(v_\lambda\otimes\alpha_{-w(\lambda)}),$$

where v_λ (resp., $\alpha_{-w(\lambda)}$) is a non-zero weight vector of $\mathcal{L}(\lambda)$ (resp., $\mathcal{L}^*(\lambda)$) of weight λ (resp., $-w(\lambda)$). Obviously, $C^+_{w,\lambda}$ is defined up to a constant. Let $w_0\in\mathcal{W}$ be the unique element sending the set of all positive roots in the set of negative ones; then $\mathcal{L}^*(\lambda)\simeq\mathcal{L}(-w_0(\lambda))$. Let us also denote

$$C^-_{w,\lambda} = \phi_{-w_0(\lambda)}(\alpha_{-\lambda}\otimes v_{w(\lambda)}),$$

where $v_{w(\lambda)}$, $\alpha_{-\lambda}$ have a similar meaning as above. Let now $\mathcal{A}_w$ be the subalgebra of $\mathbb{C}_q[G]$ generated by $C^\pm_{w,\omega_i}$ for all i, where the ω_i's are the fundamental weights.

Let J, χ be as in §1. Let θ^q be the unique antilinear involution of U_q such that

$$\theta^q(E_i) = (-1)^{\chi(i)}F_i,\quad \theta^q(F_i) = (-1)^{\chi(i)}E_i,\quad \theta^q(K_i) = K_i^{-1}.$$

(If $J=\{1,\dots,N\}$, then we shall denote ω^q instead of θ^q.)

It is known that the formula $x^* = S\theta^q(x)$ provides U_q (and *a fortiori* $\mathbb{C}_q[G]$) with a Hopf *-algebra structure. The following fact generalizes [S].

Lemma 7.1. *(a) Each $\mathcal{L}(\lambda)$, $\lambda \in P^+$, carries a non-degenerate sesquilinear form, invariant with respect to* $*$.

(b) $\phi_\lambda(v_\lambda \otimes \alpha_{-w(\lambda)})^* \in \mathbb{C}\phi_{-w_0(\lambda)}(\alpha_{-\lambda} \otimes v_{w(\lambda)})$.

(c) $\mathcal{A}_w$ *is a* $*$*-subalgebra of* $\mathbb{C}_q[G]$.

Proof. (a) follows as in[K, Ch. 11] and (b), from (a) (see [A3]); (c) is now obvious.

Definition. A $\mathbb{C}_q[G]$-module is called w-admissible if as $\mathcal{A}_w$-module, is a direct sum of one-dimensional submodules. It is called admissible if it w-admissible for some $w \in \mathcal{W}$.

Remark 7.1. It was proved in [LS] that each unitary (with respect to ω^q) irreducible representation is admissible.

Lemma 7.2. $\mathcal{A}_{w_0}$ *is a commutative subalgebra of* $\mathbb{C}_q[G]$.

Proof. This follows from [LS, 3.4.1].

The author suspects that this subalgebra plays an important role in the representation theory of the algebras of functions on quantum groups.

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