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CONFORMAL ENTROPY RIGIDITY THROUGH YAMABE FLOWS

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ABSTRACT. We introduce two versions of the Yamabe flow which preserve negative scalar-curvature bounds. First we show existence and smooth convergence of solutions to these flows. We then show that a metric with negative scalar curvature is controlled by the Yamabe metrics in the same conformal class with constant extremal scalar curvatures. This implies that the volume entropy of our original metric is controlled by the entropies of these Yamabe metrics. We eventually use these Yamabe flows to prove an entropy-rigidity result: when the Yamabe metric has negative sectional curvature, the entropy of a metric in the same conformal class is extremal if and only if the metric has constant extremal scalar curvature.

1. INTRODUCTION

Geometric flows are powerful tools when dealing with geometric problems. In this article we present variations of the Yamabe flow and use them to investigate the asymptotic geometry of the underlying manifold.

Let (M, g) be a compact Riemannian manifold whose scalar curvature satisfies $R_{min} \leq R_g \leq R_{max} < 0$. We will say a family of metrics $(g_t)_{t\in[0,T)}$ is an ***increasing Curvature-normalized Yamabe flow***, if it is a solution to the PDE

$$\frac{\partial g_t}{\partial t} = (R_{max}(g_t) - R_{g_t})g_t \quad \text{with initial condition} \quad g_0 = g.$$

It is a conformal flow (it flows along the conformal class), which we will denote by CYF^+. Similarly, we will say a family of metrics $(g_t)_{t\in[0,T)}$ is a ***decreasing Curvature-normalized Yamabe flow*** if it is a solution to the PDE

$$\frac{\partial g_t}{\partial t} = (R_{min}(g_t) - R_{g_t})g_t \quad \text{with initial condition} \quad g_0 = g.$$

It is also a conformal flow and it will be denoted by CYF^-.

Theorem 1. *Let (M, g) be a compact Riemannian manifold whose scalar curvature satisfies $R_{min} \leq R_g \leq R_{max} < 0$. Let g_Y be the unique metric in the conformal class of g whose scalar curvature satisfies $R_{g_Y} \equiv -1$.*

(1) *The increasing Curvature-normalized Yamabe Flow CYF^+ (with initial condition $g_0 = g$) has a unique solution defined for all $t \geq 0$, which we will denote by $(g_t^+)_{t\geq 0}$.*
Similarly, the CYF^- has a unique solution defined for all $t \geq 0$, which we will denote by $(g_t^-)_{t\geq 0}$.

(2) *For all $t \geq 0$, the scalar curvature bounds are preserved along these flows:*

$$R_{min} \leq R_{g_t^+} \leq R_{max},$$

$$R_{min} \leq R_{g_t^-} \leq R_{max}.$$

(3) *For all $x \in M$, the application $t \mapsto g_t^+(x)$ is increasing and $t \mapsto g_t^-(x)$ is decreasing.*

(4) *For all $k \geq 0$ the CYF^+ converges exponentially fast in the C^k topology to*

$$g_{max} = \frac{g_Y}{|R_{max}(g_\infty)|} \leq \frac{g_Y}{|R_{max}|}.$$

For all $k \geq 0$ the CYF^- converges exponentially fast in the C^k topology to

$$g_{min} = \frac{g_Y}{|R_{min}(g_\infty)|} \geq \frac{g_Y}{|R_{min}|}.$$

The Yamabe flow was introduced by R. Hamilton in [Ham89] and item (1) above actually follows from the work of R. Ye [Ye94] and a scaling argument which we present in section 2. This geometric flow has been extensively studied. Notable contributions to the global behaviour of the Yamabe flow in the presence of positive scalar curvature include the recent work of H. Schwetlick and M. Struwe [SchS03] and S. Brendle [Bre05] [Bre07]. However, since we will only be interested in the negative scalar curvature case, we will not need the delicate analysis involved in the positive curvature setting.

Recall that a metric with constant scalar curvature is called a ***Yamabe metric***. The monotonicity of these Curvature-normalized Yamabe flows implies in particular the following Schwarz Lemma, whose first proof is due to S.T. Yau [Yau73].

Corollary 2 (Conformal Schwarz Lemma)**.** *Let (M, g) be a compact Riemannian manifold whose scalar curvature satisfies $R_{min} \leq R_g \leq R_{max} < 0$. Let g_Y be the Yamabe metric in the conformal class of g whose scalar curvature satisfies $R_{g_Y} \equiv -1$. Then*

$$\frac{g_Y}{|R_{min}|} \leq g \leq \frac{g_Y}{|R_{max}|}.$$

We will give several geometric applications of this Schwarz Lemma together with the Curvature-normalized Yamabe flows, and emphasize on what they imply on the entropy of the Riemannian manifold. The ***volume entropy*** of (M, g), which we will simply call its ***entropy***, is defined by

$$h(g) = \liminf_{R\to\infty} \frac{\log \mathrm{Vol}(B_g(\tilde{x}, R))}{R},$$

where $\tilde{x}$ is any point of the universal cover $\tilde{M}$ of M and $B_g(\tilde{x}, R)$ is the ball of radius R centered in $\tilde{x}$ in $\tilde{M}$ for the metric lifted from g.

In this paper, we restrict to the conformal class of a Riemannian manifold with *negative scalar curvature*. Let (M, g) be a Riemannian manifold whose scalar curvature satisfies

$$R_{min} \leq R_g \leq R_{max} < 0,$$

and let g_Y be the Yamabe metric in the conformal class of g with scalar curvature $R_{g_Y} \equiv -1$. The conformal Schwarz Lemma stated above implies that the entropy of the Yamabe metric controls the entropy of g.

Corollary 3. *Let (M, g) be a Riemannian manifold whose scalar curvature satisfies*

$$R_{min} \leq R_g \leq R_{max} < 0,$$

and let g_Y be the unique metric in the conformal class of g whose scalar curvature satisfies $R_{g_Y} \equiv -1$. then

$$\sqrt{|R_{max}|}h(g_Y) \leq h(g) \leq \sqrt{|R_{min}|}h(g_Y).$$

Moreover, using the Curvature-normalized Yamabe flows, in our main result in this paper we will show the following entropy-rigidity in a conformal class for metrics with *negative sectional curvatures*:

Theorem 4 (Conformal entropy-rigidity)**.** *Let (M, g) be a Riemannian manifold with negative sectional curvatures and whose scalar curvature satisfies*

$$R_{min} \leq R_g \leq R_{max} < 0.$$

Let g_Y be the Yamabe metric in the conformal class of g with scalar curvature $R_{g_Y} \equiv -1$.

(1) $h(g) \geq \sqrt{|R_{max}|}h(g_Y)$ *with equality if and only if* $g = \frac{g_Y}{|R_{max}|}$*;*
(2) $h(g) \leq \sqrt{|R_{min}|}h(g_Y)$ *with equality if and only if* $g = \frac{g_Y}{|R_{min}|}$*.*

All the results we will prove are known for compact surfaces. In fact our line of attack is inspired by A. Manning [Man04] who proved that starting with a negatively curved metric on a closed compact Riemann surface the *Volume-normalized Ricci Flow* strictly decreases entropy. In this dimension both the Ricci and Yamabe flows agree. It should also be pointed out that this method will most likely not yield the same results in a non-negatively curved situation. Indeed, D. Jane [Jan07] constructed examples of smooth metrics on the 2–sphere and the 2–torus which have zero topological entropy, *and as they flow along the volume normalized Ricci flow, entropy strictly increases.* Observe that our choice of normalization comes from the scalar curvature bounds and not from the volume. Let us now review some of the known results.

Let (M, g) be a closed Riemannian n–manifold, $n \geq 3$, such that its sectional curvatures satisfy $K_g \leq -1$. Assume that M is homotopically equivalent to a ***locally symmetric manifold*** (M_S, g_S) with sectional curvatures $K_{g_S} \leq -1$ (when such a symmetric space exists, it is unique by Mostow's rigidity theorem). It has been shown by U. Hamenstädt in [Ham90] that the entropy of (M_S, g_S) is a lower bound for the entropy of (M, g):

$$h(M, g) \geq h(M_S, g_S)$$

The equality holds if and only if (M, g) is isometric to (M_S, g_S). This is called the ***entropy rigidity*** of the locally symmetric spaces with negative sectional curvatures. A stronger version of this result was shown using another method by G. Besson, G. Courtois and S. Gallot in [BCG95]. The extension of this result to the case of products of symmetric spaces with negative sectional curvatures was announced in [BCG96] and published by C. Connell and B. Farb in [ConFar03]. Furthermore, it was shown by G. Robert in [Rob94] that a locally symmetric space of *nonpositive* sectional curvatures is the unique minimizer for the entropy *within its conformal class*, see also [Kni05].

When M is not homotopically equivalent to a locally symmetric Riemannian manifold with non-positive sectional curvatures (or a product of them), not much seems to be known about entropy minimizing metrics. In fact, this appears to be the first time that a metric which is **not necessarily** locally symmetric is identified to be an entropy minimizing metric (keep in mind that we are using different normalizations). The first negatively curved smooth 4–manifold which is not diffeomorphic to a locally symmetric space was found by G.D. Mostow and Y.T. Siu in [MS80]. Other interesting examples come from the conformal classes of metrics described by M. Gromov and W. Thurston in [GT87], in which there is a metric with almost-constant negative sectional curvatures but no hyperbolic metric. Our Theorem 4 provides metrics that uniquely minimize the entropy in the conformal class of these metrics. We are not aware of any other method to single out special metrics as minimizers of the entropy on these manifolds.

Maximizing the entropy has been less often dealt with. Bishop's Comparison Theorem implies that when the Ricci curvature of (M, g) satisfies $Ric_g \geq -(n-1)g$, where n is the dimension of M, then its volume entropy satisfies $h(M, g) \leq n-1$. Recently, Ledrappier and Wang proved in [LW09] that under these hypotheses, the upper bound $h(g) = (n-1)$ is attained if and only if g is real-hyperbolic. They also proved a similar entropy-rigidity statement for complex-hyperbolic and quaternionic-hyperbolic closed manifolds relying on a lower bound on the holomorphic-sectional curvature. Our Theorem 4 implies the following result:

Corollary 5. *Given any constant $C > 0$ and any metric g_0 with negative scalar curvature let us consider the set of metrics g conformal to g_0 which satisfy $0 > R_g \geq -C$: on this set, the functional $g \mapsto h(g)$ attains its maximum at the Yamabe metric whose scalar curvature is constant and equal to $-C$. Moreover, if this Yamabe metric has negative sectional curvature, then it is the unique maximum.*

In the next section we explain the details and relevant facts of the CYF. Then in section 3 we prove the conformal version of the Schwarz lemma and give some geometric applications. Finally, in section 4 we focus on finding extrema for the entropy. We have included the maximum principles which underpin this paper's analysis in the appendix.

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2. Curvature-normalized Yamabe Flows

Let $R_{max} < 0$ be a negative constant and (M, g) be a Riemannian n–manifold, $n \geq 3$, whose scalar curvature R_g is at most $R_{max} < 0$. For example, a real-hyperbolic manifold (M, g_H) of dimension $n \geq 3$ satisfies $R_{g_H} \equiv -n(n-1)$. This is a special case of a ***Yamabe metric***, i.e. a metric with constant scalar curvature. Let us first recall the following fundamental result on Yamabe metrics.

Theorem 6 (Yamabe-Trudinger)**.** *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ such that the scalar curvature of g is everywhere nonpositive, and does not vanish everywhere. Then there exist a unique Yamabe metric g_Y conformally equivalent to g with constant scalar curvature $R_{g_Y} = -1$.*

It was claimed by H. Yamabe in [Yam60] that in each conformal class, there exists a metric with constant scalar curvature. N. Trudinger pointed in [Tru68] that the proof had a mistake, and corrected it in the case of non-positive scalar curvature. The remaining cases were settled later by T. Aubin and R. Schoen. A proof of the uniqueness (up to scaling) of the Yamabe metric in the case of non-positive scalar curvature can be found in [Aub82], p. 135. In each conformal class, we will always normalize the Yamabe metric by the value of its (constant) scalar curvature.

We call the ***Curvature-normalized increasing Yamabe Flow*** (in short, CYF^+) on M with initial data g a family of metrics $(g_t)_{t \geq 0}$ on M satisfying :

$$\begin{aligned} \frac{\partial g_t}{\partial t} &= (R_{max}(g_t) - R_{g_t}) g_t \\ g_0 &= g \end{aligned}$$

All this section is devoted to the proof of:

Theorem 7. *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ whose scalar curvature satisfies*

$$R_{min} \leq R_g \leq R_{max} < 0.$$

Then the Curvature-normalized increasing Yamabe Flow with initial data g has a unique maximal solution $(g_t = e^{f_t} g)$, defined for $t \in [0, \infty)$. For all $t \geq 0$, its scalar curvature satisfies

$$R_{g_t} \in [R_{min}, R_{max}].$$

Moreover, there exists a Yamabe metric g_{max} in the conformal class of g with constant scalar curvature $R_{g_{max}} \leq R_{max} < 0$ such that, for all $k \geq 0$, g_t converges exponentially fast to g_{max} in the $\mathcal{C}^k$ topology.

Let us point out that, since along the CYF^+ the scalar curvature is less than the initial R_{max}, then the conformal factor of the metric g_t with respect to the initial metric g is non-decreasing in time.

Proof. Let g be a metric whose scalar curvature satisfies $R_{min} \leq R_g \leq R_{max} < 0$. Let $\tilde{g}_t$ be the solution of the ***Volume-normalized Yamabe Flow*** VYF:

$$\begin{aligned} \frac{\partial \tilde{g}_t}{\partial t} &= (s_{\tilde{g}_t} - R_{\tilde{g}_t}) \tilde{g}_t \\ g_0 &= g, \end{aligned}$$

where

$$s_{\tilde{g}_t} = \frac{1}{\mathrm{Vol}(M, \tilde{g}_t)} \int_M R_{g_t} dv_{\tilde{g}_t}$$

is the ***average scalar curvature*** of $(M, \tilde{g}_t)$. Since $R_{max} < 0$, it follows from Theorem 2 of [Ye94] that this conformal flow has a unique solution, defined for all $t \geq 0$. Moreover, the metric converges exponentially fast when $t \to \infty$ in the $\mathcal{C}^2$ topology to the Yamabe metric in the conformal class of g with same volume as g.

Let us set for all $t \geq 0$

$$\phi(t) = \int_0^t \left(R_{max}(\tilde{g}_\tau) - s_{\tilde{g}_\tau} \right) d\tau,$$

and let $a : (0, \infty) \to (0, \infty)$ be the unique solution of

$$\begin{aligned} a'(t) &= e^{-\phi(a(t))} \\ a(0) &= 0, \end{aligned}$$

The map a is obviously increasing and defined as long as it stays finite. It follows from the exponential convergence of the metric shown by Ye that there exist $C, \epsilon > 0$ such that for all $t \geq 0$, we have

$$|s_{\tilde{g}_t} - R_{max}(\tilde{g}_t)| \leq C e^{-\epsilon t}.$$

Therefore, a exists for all $t \geq 0$, and $\frac{a(s)}{s}$ converges to a positive limit a_∞ when $t \to \infty$.

Lemma 8. *Let us define for all $t \geq 0$,*

$$g_t = e^{\phi(a(t))} \tilde{g}_{a(t)}.$$

Then (g_t) satisfies:

$$\frac{\partial g_t}{\partial t} = (R_{max}(g_t) - R_{g_t}) g_t.$$

Proof. An immediate computation shows that, with the notations of the Lemma, we have

$$\begin{aligned} \frac{\partial g_t}{\partial t} &= \left(R_{max}(\tilde{g}_{a(t)}) - s_{\tilde{g}_{a(t)}} \right) \tilde{g}_{a(t)} + \left(s_{\tilde{g}_{a(t)}} - R_{\tilde{g}_{a(t)}} \right) \tilde{g}_{a(t)} \\ &= \left(R_{max}(\tilde{g}_{a(t)}) - R_{\tilde{g}_{a(t)}} \right) \tilde{g}_{a(t)} \\ &= \left(R_{max}(g_t) - R_{g_t} \right) g_t \end{aligned}$$

because $R_{g_t} = e^{-\phi(a(t))} R_{\tilde{g}_{a(t)}}$. □

Therefore, $(g_t)_{t \geq 0}$ is a solution of the CYF$^+$. By the above argument, since the map a is an increasing bijection on $(0, \infty)$, the uniqueness of the solution to CYF$^+$ follows directly from the (well-known) uniqueness of the solution to VYF. We will now prove that the scalar curvature bounds are preserved along the flow.

Lemma 9 (Scalar curvature remains bounded along CYF$^+$)**.** *For all $t \in [0, T)$, the scalar curvature of g_t satisfies:*

$$\frac{\partial R_{g_t}}{\partial t} = -(n-1) \Delta_{g_t} R_{g_t} + R_{g_t} (R_{g_t} - R_{max}(g_t))$$

and

$$R_{min} \leq R_{g_t} \leq R_{max}.$$

Proof. By Theorem 1.174 of [Bes08], when a metric g varies in the direction of the symetric 2-tensor k, the variation of the scalar curvature is given by

$$R'_g(k) = \Delta_g(tr_g(k)) + \delta_g(\delta_g k) - g(Ric_g, k).$$

Here, at time $t \geq 0$, $k = (R_{max} - R_{g_t}) g_t$, and by definition, $g_t(k, g_t) = tr_{g_t}(k)$. We therefore get:

$$\frac{\partial R_{g_t}}{\partial t} = \Delta_{g_t}(-n R_{g_t}) + \delta_{g_t}(\delta_{g_t}(-R_{g_t} g_t)) + R_{g_t}(R_{g_t} - R_{max}(g_t))$$

Since for all smooth functions $f : M \to \mathbb{R}$, $\delta_g(fg) = -df$ (see [Bes08], 1.59), this eventually becomes

$$\frac{\partial R_{g_t}}{\partial t} = -(n-1)\Delta_{g_t} R_{g_t} + R_{g_t}(R_{g_t} - R_{max}(g_t)), \tag{1}$$

which establishes our first claim. Moreover, $R_{g_0}(x) \leq R_{max}$ for all $x \in M$. Therefore the ***Maximum Principle*** proven in Appendix A (see Proposition 26 (1)) implies that for all $(x,t) \in M \times I$, we have

$$R_{g_t}(x) \leq R_{max}.$$

Now, going back to the variation formula (1), we conclude

$$\frac{\partial R_{g_t}}{\partial t}(x) \geq -(n-1)\Delta_{g_t} R_{g_t}(x).$$

Since $R_{g_0} \geq R_{min}$, the Maximum Principle implies again that $R_{g_t} \geq R_{min}$ for all $t \in I$. Therefore the scalar curvature of a solution to CYF$^+$ stays bounded in $[R_{min}, R_{max}] \subset (-\infty, 0)$. □

Let now $k \geq 2$ be fixed. It follows from Theorem 2 of [Ye94] that the solution $(\tilde{g}_t)$ of the VYF converges exponentially fast in the $\mathcal{C}^k$ topology to the unique Yamabe metric g_V in the conformal class of g with same volume as g. Let recall that the solution $(g_t)_{t\geq 0}$ is given by

$$g_t = e^{\phi(a(t))}\tilde{g}_{a(t)},$$

with a strictly increasing, $\frac{a(t)}{t}$ converging to a positive limit when $t \to \infty$ and $\phi(t)$ converging exponentially fast to a constant l when $t \to \infty$. This implies that g_t converges exponentially fast (in the $\mathcal{C}^k$ topology) to the Yamabe metric

$$g_{max} = e^l g_V.$$

Since for all $t \geq 0$, we have $R_{g_t} \leq R_{max}$, the constant scalar curvature of g_{max} satisfies $R_{g_{max}} \leq R_{max}$. □

We have derived the long time existence and convergence CYF$^+$ from the work of Ye on VYF to shorten the argument. Since the CYF$^+$ equation is rather simple, it can be proven independently by repeating arguments which are very similar to Ye's. However, the smooth exponential convergence of the VYF is deduced in [Ye94] from its $\mathcal{C}^0$ convergence due to *well known facts from the theory of parabolic equations.* It appears that a complete proof of this smooth convergence, which relies on the Schauder Theory for parabolic equations, has never been published. So we will now give a complete proof of this exponential convergence in the case of CYF$^+$. It shows in particular that all derivatives of the flow converge exponentially fast to a Yamabe metric *at the same speed*, given by the upper bound R_{max} on the scalar curvature.

Theorem 10 (Exponential convergence of CYF$^+$)**.** *Let (M,g) be a closed Riemannian manifold whose scalar curvature satisfies $R_g \leq R_{max} < 0$ and $(g_t)_{t\geq 0}$ be the maximal solution of the CYF$^+$ with initial data g. Then there exists a Yamabe metric g_{max}, with scalar curvature $R_{g_{max}} \leq R_{max}$, such that for all $k \geq 0$, there exists a constant $C_{g,k}$ satisfying for all $t \geq 0$:*

$$\|g_t - g_{max}\|_{\mathcal{C}^k} \leq C_{g,k} e^{R_{max} t}.$$

Proof. Let (M,g) be a closed Riemannian manifold whose scalar curvature satisfies $R_g \leq R_{max} < 0$ and $(g_t)_{t\geq 0}$ be the solution of the CYF$^+$ with initial data g on $[0,\infty)$.

Lemma 11. *The scalar curvature R_{g_t} of g_t converges uniformly exponentially fast to $R_{g_{max}} \leq R_{max}$ when $t \to \infty$.*

Proof. We have seen in Lemma 9 that $R_{g_t} \leq R_{max}$ for all $t \geq 0$. Moreover, writing the evolution equation of the scalar curvature given in Lemma 9 at a point where the scalar curvature is maximal, we get that along the flow,

$$\frac{\partial R_{max}(g_t)}{\partial t} \leq 0.$$

Therefore, the evolution equation of the scalar curvature gives for all $(x,t) \in M \times [0,\infty)$,

$$\begin{aligned}\frac{\partial (R_{g_t} - R_{max}(g_t))}{\partial t} \geq \frac{\partial R_{g_t}}{\partial t} &\geq -(n-1)\Delta_{g_t}(R_{g_t} - R_{max}(g_t)) \\ &\quad + R_{g_t}(R_{g_t} - R_{max}(g_t)) \\ &\geq -(n-1)\Delta_{g_t}(R_{g_t} - R_{max}(g_t)) \\ &\quad + R_{max}(R_{g_t} - R_{max}(g_t)).\end{aligned}$$

Using the variable form of the Maximum Principle (Theorem 28), we get for all $(x,t) \in M \times [0,\infty)$:

$$0 \geq R_{g_t} - R_{max}(g_t) \geq (R_{min} - R_{max})e^{R_{max}t} \tag{2}$$

Since $R_{max} < 0$, the curvature converges exponentially fast to $R_{g_{max}} \leq R_{max}$ at all points of M. □

We will need an a priori bound on the conformal factor along the flow. It can be deduced either from Ye's work, or shown independently as indicated below. For all $t \geq 0$, we define $v_t : M \to (0,\infty)$ by $g_t = v_t g_0$. We have seen that the map $t \to v_t$ is increasing.

Lemma 12. *For all $t \geq 0$, we have $1 \leq v_t \leq \left|\frac{R_{min}}{R_{max}}\right|$.*

Proof. Let $g_t = v_t g_0 = u_t^{4/(n-2)} g_0$ be the solution of the CYF$^+$ with initial metric g_0.

It was shown in [Yam60] that if $g = u^{4/(n-2)} g_0$, then its scalar curvature is given by:

$$R_g = u^{-\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2} \Delta_{g_0} u + R_{g_0} u \right) \tag{3}$$

Therefore, the CYF$^+$ evolution equation for the metric is equivalent to the following:

$$\begin{aligned}\frac{\partial}{\partial t}\left(u_t^{\frac{n+2}{n-2}}\right) &= \frac{n+2}{n-2} u_t^{\frac{4}{n-2}} \frac{\partial u_t}{\partial t} \\ &= \frac{n+2}{4} u \frac{\partial \left(u_t^{\frac{4}{n-2}}\right)}{\partial t} \\ &= \frac{n+2}{4} (R_{max}(g_t) - R_{g_t}) u^{\frac{n+2}{n-2}}\end{aligned}$$

Now, writing $N = \frac{n+2}{n-2}$ and using the expression (3) for the scalar curvature, this last equation becomes

$$\frac{\partial}{\partial t}\left(u_t^N\right) = \frac{n+2}{4}\left(R_{max}(g_t)u_t^N - R_{g_0}u_t\right) - \frac{(n+2)(n-1)}{n-2}\Delta_{g_0}u_t. \tag{4}$$

Setting $w_t = u_t^N$, we rewrite this equation as

$$\frac{\partial}{\partial t}(w_t) = c_1(n)(R_{max}w_t - R_{g_0}w_t^{1/N}) - c_2(n)\Delta_{g_0}w_t^{1/N} \leq \tag{5}$$

$$\leq -c_2(n)\Delta_{g_0}w_t^{1/N} + c_1(n)(R_{max}w_t - R_{min}w_t^{1/N})$$

where

$$c_1(n) = \frac{n+2}{4} \quad \text{and} \quad c_2(n) = \frac{(n+2)(n-1)}{n-2}.$$

The positive map w_t satisfies

$$w_0(x) = 1 \leq \left|\frac{R_{min}}{R_{max}}\right|^{\frac{N}{N-1}}$$

for all $x \in M$. A straightforward check from equation (5) shows that for all $(x,t) \in M \times [0,\infty)$ such that $w_t(x) > \left|\frac{R_{min}}{R_{max}}\right|^{\frac{N}{N-1}}$, we have that

$$\frac{\partial}{\partial t}(w_t) \leq -c_2(n)\Delta_{g_0}w_t^{1/N}.$$

Therefore, by the Maximum Principle given in Corollary 27, for all $(x,t) \in M \times [0,\infty)$,

$$w_t(x) \leq \left|\frac{R_{min}}{R_{max}}\right|^{\frac{N}{N-1}}.$$

Now,

$$w_t(x)^{\frac{N-1}{N}} = u_t^{\frac{4}{n-2}} = v_t,$$

therefore we get for all $(x,t) \in M \times I$,

$$v_t \leq \left|\frac{R_{min}}{R_{max}}\right|.$$

□

Schauder estimates for parabolic equations give a classical argument of *bootstrapping* which shows that this $\mathcal{C}^0$ exponential convergence of the curvature actually implies exponential convergence in the $\mathcal{C}^k$ topology at all orders $k \geq 0$ of the conformal factor (and hence also of the scalar curvature and of the metric). We present this now.

For all $k \geq 0$, $t \geq 0$ and for all $\mathcal{C}^k$ maps $f : M \times [t,\infty) \to \mathbb{R}$, we write

$$\|f\|_k(t) = \sum_{p=0}^{k} \sup_{x\in M} \left|\nabla_{g_0}^p f(x,t)\right|,$$

where $\left|\nabla_{g_0}^p f(x,t)\right|$ is the norm of the covariant derivative of order p of the map $x \mapsto f(x,t)$ with respect to the initial metric $g = g_0$. With this notation, for all $t \geq 0$,

$$\|f\|_0(t) = \sup_{x\in M} |f(x,t)|.$$

For all $r > 0$, we also write

$$\|f\|_{k,r}(t) = \sup_{t-r \le s \le t} \|f\|_k(s)$$

Assume that f satisfies a quasilinear uniformly parabolic homogeneous equation on $M \times [0, \infty)$. Schauder theory of Hölder estimates for parabolic equations (which we will not present here, see for example [Lie96], Chapter 4 and 8) implies that for all $k \ge 0$ and all $r > 0$, there exists a constant $C_{k,n,r}$ which depends on k, the dimension n, the range $r > 0$ and the constants of parabolicity such that whenever f is a solution of this parabolic equation, we have

$$\|f\|_k(t) \le C_{k,n,r} \|f\|_{0,r}(t).$$

Let us apply these estimates to the CYF$^+$. Differentiating the evolution equation (4) gives the evolution equation of $\frac{\partial u_t}{\partial t}$:

$$\frac{\partial^2}{\partial t^2}\left(u_t^N\right) = \frac{n+2}{4}\left(R_{max}(g_t)\frac{\partial}{\partial t}(u_t^N) - R_{g_0}\frac{\partial}{\partial t}u_t\right) \tag{6}$$
$$-\frac{(n+2)(n-1)}{n-2}\Delta_{g_0}\frac{\partial}{\partial t}u_t + \frac{n+2}{4}\frac{\partial}{\partial t}\left(R_{max}(g_t)\right)u_t^N.$$

Due to equation (3), the scalar curvature also satisfies a quasilinear parabolic equation with respect to the *variable Laplacian* Δ_{g_t}. However, it follows from equation (4) that for all smooth functions $f : M \to \mathbb{R}$, we have

$$\frac{4(n-1)}{n-2}\Delta_{g_t} f + R_{g_t} f = u_t^{-\frac{n+2}{4}}\left(\frac{4(n-1)}{n-2}\Delta_{g_0}(u_t f) + R_{g_0} u_t f\right).$$

Therefore, since u_t and the scalar curvatures are uniformly bounded and away from 0, equation (4) is uniformly parabolic for the *fixed Laplacian* Δ_{g_0}. Hence, Schauder estimates show that the last term in the equation (6) is bounded. This implies that (6) is also a ***quasilinear uniformly parabolic equation*** for $\frac{\partial u_t}{\partial t}$. Schauder estimates for equation (6) imply that for all $r > 0$ and $k \ge 0$, there exists $C_{k,g_0,r} > 0$ depending only on the initial metric g_0 such that

$$\left\|\frac{\partial u_t}{\partial t}\right\|_k(t) \le C_{k,g_0,r}\left\|\frac{\partial u_t}{\partial t}\right\|_{0,r}(t)$$

Observe that these Schauder estimates work only because all the norms are taken with respect to the fixed metric g_0 (they would no longer work if the norms were taken with respect to the metric g_t).

Since $g_t = v_t g_0 = u_t^{4/(n-2)} g_0$ we have that

$$\begin{aligned}\frac{\partial u_t}{\partial t} &= \frac{\partial(v_t^{(n-2)/4})}{\partial t}\\ &= \frac{n-2}{4}v_t^{\frac{n-2}{4}-1}\frac{\partial v_t}{\partial t}\\ &= \frac{n-2}{4}v_t^{\frac{n-2}{4}-1}(R_{max} - R_{g_t})v_t\end{aligned}$$

by definition of the CYF$^+$. Moreover, it follows from Lemma 12 that v_t is uniformly bounded:

$$v_t^{\frac{n-2}{4}} \le A_{g_0} = \left|\frac{R_{min}}{R_{max}}\right|^{\frac{n-2}{4}}.$$

Applying the bounds from the inequalities in (2), we obtain for all $t > 0$:

$$\begin{aligned} \left|\frac{\partial u_t}{\partial t}\right|(t) &\leq \left|\frac{n-2}{4} v_t^{\frac{n-2}{4}} (R_{max} - R_{g_t})\right| \\ &\leq \frac{n-2}{4} A_{g_0} \left[(R_{max} - R_{min}) e^{R_{max} t}\right] \\ &= \frac{n-2}{4} A_{g_0} \left|R_{min} - R_{max}\right| e^{R_{max} t} \end{aligned}$$

Therefore, since

$$\left\|\frac{\partial u_s}{\partial s}\right\|_{0,r}(s) = \sup_{s-r \leq t \leq s} \left|\frac{\partial u_t}{\partial t}\right|(t),$$

and as $t \mapsto e^{R_{max} t}$ is decreasing, we get

$$\left\|\frac{\partial u_s}{\partial s}\right\|_{0,r}(s) \leq \frac{n-2}{4} A_{g_0} \left|R_{min} - R_{max}\right| e^{(s-r) R_{max}}. \tag{7}$$

Let us now fix $k \geq 2$ and write $\nabla^k = \nabla^k_{g_0}$. For all $r, t, T \geq 0$, we have

$$\begin{aligned} \left|\nabla^k u_{t+T} - \nabla^k u_t\right| &= \left|\int_t^{t+T} \frac{\partial}{\partial s}\left(\nabla^k u_s\right) ds\right| \\ &\leq \int_t^{t+T} \left|\nabla^k \frac{\partial u_s}{\partial s}\right| ds \\ &\leq C_{k,g_0,r} \int_t^{t+T} \left\|\frac{\partial u_s}{\partial s}\right\|_{0,r}(s) ds \end{aligned}$$

where $C_{k,g_0,r}$ is the constant for Schauder estimates of order k for $\frac{\partial u_s}{\partial s}$, which depends only on r, k and the initial metric g_0. From (7) above we find the following estimates:

$$\begin{aligned} \left|\nabla^k u_{t+T} - \nabla^k u_t\right| &\leq C_{k,g_0} \int_t^{t+T} \left\|\frac{\partial u_s}{\partial s}\right\|_{0,r}(s) ds \\ &\leq C_{k,g_0,r} \int_t^{\infty} \frac{n-2}{4} A_{g_0} \left|R_{min} - R_{max}\right| e^{(s-r) R_{max}} ds \\ &\leq C_{k,g_0,r} \frac{n-2}{4} A_{g_0} \left|\frac{R_{min} - R_{max}}{R_{max}}\right| e^{(t-r) R_{max}} \\ &= C_{g_0,k} e^{t R_{max}} \end{aligned}$$

Therefore the family $(u_t)_{t \geq 0}$ is a Cauchy family in the Banach space of $\mathcal{C}^k$ maps from $M \to \mathbb{R}$. So it converges in the $\mathcal{C}^k$ topology to a $\mathcal{C}^k$ map $u_\infty : M \to (0, \infty)$, and this convergence is uniformly exponential because $R_{max} < 0$.

We have seen that the convergence $u_t \to u_\infty$ is uniformly exponential at all orders $k \geq 0$. In particular, together with the convergence of R_{g_t}, this implies that the metric $g_{max} = u_\infty^{4/(n-2)} g$ is a smooth metric with constant scalar curvature $R_{g_{max}} \equiv R_{max} < 0$.

This concludes the proof of Theorem 10. $\square$

Remark 13. Knowing whether $R_{g_{max}} = R_{max}$ is in general a delicate issue. It may be the case if the initial metric contains an open set with constant maximal scalar curvature.

Let us now change the normalization of the flow, and consider the ***Curvature-normalized decreasing Yamabe Flow*** (which we will denote by CYF^-) on M with initial data g. We define it to be the family of metrics $(g_t)_{t\geq 0}$ on M satisfying:

$$\begin{aligned} \frac{\partial g_t}{\partial t} &= (R_{min}(g_t) - R_{g_t})g_t \\ g_0 &= g \end{aligned}$$

We let the reader adapt the method of proof of Proposition 9 to show that, along the CYF^-, the scalar curvature satisfies

$$\frac{\partial(R_{g_t} - R_{min}(g_t))}{\partial t} \leq -(n-1)\Delta_{g_t}(R_{g_t} - R_{min}(g_t)) + R_{max}(R_{g_t} - R_{min}(g_t)).$$

By using analogous arguments to those in the previous results in this section we obtain the following statement for the CYF^-. The appropriate forms of the Maximum principles can be also found in the Appendix.

Theorem 14. *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ whose scalar curvature satisfies*

$$R_{min} \leq R_g \leq R_{max} < 0.$$

Then the Curvature-normalized decreasing *Yamabe Flow with initial data g has a unique maximal solution $(g_t = e^{f_t}g)$, defined for $t \in [0, \infty)$. Moreover, for all $t \geq 0$, its scalar curvatures satisfy*

$$R_{g_t} \in [R_{min}, R_{max}].$$

The conformal factor $t \mapsto e^{f_t}$ is non-increasing *in time t and for all $k \geq 0$, the flow (g_t) converges exponentially fast in the $\mathcal{C}^k$ topology as $t \to \infty$ to a Yamabe metric g_{min} in the conformal class of g with constant scalar curvature $R_{g_{min}} \geq R_{min}$.*

The proof of Theorem 1 follows from Theorem 7 and Theorem 14.

3. Conformal Schwarz Lemma and geometric applications

In this section, we establish Yau's Conformal Schwarz Lemma [Yau73], and give some of its geometric consequences.

Corollary 15 (Yau's Conformal Schwarz Lemma)**.** *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ whose scalar curvature satisfies $R_g \in [R_{min}, R_{max}] \subset (-\infty, 0)$ and $g_Y = v_Y g$ be the Yamabe metric conformally equivalent to g with scalar curvature $R_{g_Y} = -1$. Then in all points of M,*

$$\frac{g_Y}{|R_{min}|} \leq g \leq \frac{g_Y}{|R_{max}|}.$$

Proof. Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ whose scalar curvature satisfies $R_g \in [R_{min}, R_{max}] \subset (-\infty, 0)$. We have seen that the CYF^+ with initial metric g increases the conformal factor, and converges to the metric with constant scalar curvature $R_{g_{max}} \leq R_{max}$, which is $\frac{g_Y}{|R_{g_{max}}|}$. This gives the upper bound stated above. Similarly, the CYF^- with initial data g decreases the conformal factor and converges to $\frac{g_Y}{|R_{g_{min}}|}$ with $R_{g_{min}} \geq R_{min}$, which gives the lower bound. □

This result can be seen as a ***Schwarz Lemma*** in the conformal class of g, it has many interesting geometric applications which seem to have gone unnoticed. For instance, it implies that when the scalar curvature of the initial metric g is pinched enough, then the unique metric g_S with constant scalar curvature in the same conformal class is close to g for the $\mathcal{C}^0$ topology.

By integrating over the manifold we get

$$\frac{\mathrm{Vol}(M, g_Y)}{|R_{min}|^{n/2}} \leq \mathrm{Vol}(M, g) \leq \frac{\mathrm{Vol}(M, g_Y)}{|R_{max}|^{n/2}}.$$

These volume estimates where already shown by 0. Kobayashi in [Kob87], using a study of the Yamabe functional. From the Conformal Schwarz Lemma we also obtain the following *volume-rigidity* result:

Corollary 16 (Volume-rigidity in a conformal class)**.** *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ whose scalar curvature satisfies $R_{min} \leq R_g \leq R_{max} < 0$, and $g_Y = e^f g$ be the Yamabe metric g_Y in the conformal class of g whose scalar curvature satisfies $R_{g_Y} = -1$.*

(1) *If the volume of (M, g) is minimal, in which case* $\mathrm{Vol}(M, g) = \frac{\mathrm{Vol}(M, g_Y)}{|R_{min}|^{n/2}}$, *then $g = \frac{g_Y}{|R_{min}|}$.*

(2) *If the volume of (M, g) is maximal, in which case* $\mathrm{Vol}(M, g) = \frac{\mathrm{Vol}(M, g_Y)}{|R_{max}|^{n/2}}$, *then $g = \frac{g_Y}{|R_{max}|}$.*

Let us recall that the ***systole*** of a Riemannian manifold (M, g), which we write $sys(M, g)$, is the minimal Riemannian length of any loop which is not homotopic to zero. For a pair of smooth metrics which satisfy $g_1 \leq g_2$ it is well known that $sys(M, g_1) \leq sys(M, g_2)$. Therefore we obtain:

Corollary 17. *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ whose scalar curvature satisfies $R_{min} \leq R_g \leq R_{max} < 0$, and $g_Y = e^f g$ be the Yamabe metric g_Y in the conformal class of g whose scalar curvature satisfies $R_{g_Y} = -1$. Then*

$$\frac{sys(M, g_Y)}{\sqrt{|R_{min}|}} \leq sys(M, g) \leq \frac{sys(M, g_Y)}{\sqrt{|R_{max}|}}$$

Recall that the injectivity radius $inj(M, g)$ of a Riemannian manifold (M, g) without conjugate points is half the length of the shortest closed geodesic. Therefore, if (M, g) and (M, g_Y) have no conjugate points, then their injectivity radii satisfy

$$\frac{inj(M, g_Y)}{\sqrt{|R_{min}|}} \leq inj(M, g) \leq \frac{inj(M, g_Y)}{\sqrt{|R_{max}|}}.$$

4. Entropy extrema in a conformal class

We will now apply the Schwarz Lemma established in the previous section to study the entropy in each conformal class. This yields the main results as stated in the introduction.

We first use the Schwarz Lemma we established in Corollary 15 in the previous section to prove that the metric with constant scalar curvature is actually a minimum for the entropy:

Corollary 18 (Entropy bounds in a conformal class)**.** *Let (M,g) be a closed Riemannian manifold of dimension $n \geq 3$ whose scalar curvature satisfies $R_{min} \leq R_g \leq R_{max} < 0$, and g_Y be the Yamabe metric in the conformal class of g with constant scalar curvature $R_{g_Y} = -1$. Then*

$$\sqrt{|R_{max}|}h(g_Y) \leq h(g) \leq \sqrt{|R_{min}|}h(g_Y).$$

Proof. Let (M,g) be a closed Riemannian manifold whose scalar curvature satisfies $R_{min} \leq R_g \leq R_{max} < 0$, and g_Y be the Yamabe metric in the conformal class of g whose scalar curvature satisfies $R_{g_Y} = -1$. The Conformal Schwarz Lemma we proved in Corollary 15 implies that in all points of M we get

$$g_{max} = \frac{g_Y}{|R_{max}|} \leq g \leq g_{min} = \frac{g_Y}{|R_{min}|}.$$

Corollary 18 will now be an immediate consequence of the following well-known lemma.

Lemma 19. *Let g_1 and g_2 be two Riemannian metrics on M such that in all points $x \in M$, we have $g_1(x) \leq g_2(x)$. Then the entropies satisfy $h(g_1) \geq h(g_2)$.*

Proof. Let $\tilde{M}$ be the universal cover of M. We will still denote by g_1 and g_2 the metrics on $\tilde{M}$ lifted respectively from g_1 and g_2 on M. These lifted metrics also satisfy for all $\tilde{x} \in \tilde{M}$,

$$g_1(\tilde{x}) \leq g_2(\tilde{x}).$$

Let $\tilde{x} \in \tilde{M}$ be fixed. By the above inequality, we have for $r > 0$,

$$B_{g_2}(\tilde{x}, r) \subset B_{g_1}(\tilde{x}, r).$$

Moreover, the Riemannian volume-measures dv_{g_1} and dv_{g_2} are absolutely continuous with respect to each other. Therefore, since M is compact, there is a $C > 1$ such that

$$\frac{1}{C} \leq \frac{dv_{g_1}}{dv_{g_2}} \leq C.$$

These estimates are still valid for the lifted volume-measures on $\tilde{M}$: they imply that for all $r > 0$,

$$\mathrm{Vol}_{g_2}\left(B_{g_1}(\tilde{x}, r)\right) \leq C\mathrm{Vol}_{g_1}\left(B_{g_1}(\tilde{x}, r)\right).$$

Therefore, for all $r > 0$

$$\begin{aligned} \frac{\log \mathrm{Vol}_{g_2}\left(B_{g_2}(\tilde{x}, r)\right)}{r} &\leq \frac{\log \mathrm{Vol}_{g_2}\left(B_{g_1}(\tilde{x}, r)\right)}{r} \\ &\leq \frac{\log C}{r} + \frac{\log \mathrm{Vol}_{g_1}\left(B_{g_1}(\tilde{x}, r)\right)}{r}. \end{aligned}$$

Letting $r \to \infty$, this implies $h(g_2) \leq h(g_1)$. □

This ends the proof of Corollary 18. □

Corollary 20. *Let (M,g) be a closed Riemannian manifold of dimension $n \geq 3$ whose scalar curvature satisfies $R_g \leq R_{max} < 0$, and $g_{max} = \frac{g_Y}{|R_{max}|}$ be the Yamabe metric in the conformal class of g with scalar curvature $R_{g_{max}} = R_{max}$. If g_{max} has no conjugate points, then*

$$h_{top}(g) \geq h_{top}(g_{max}),$$

where $h_{top}(g)$ and $h_{top}(g_{max})$ are the topological entropies of the geodesic flows on (M, g) and (M, g_{max}).

Proof. Let (M, g) be a closed Riemannian manifold whose scalar curvature satisfies $R_g \leq R_{max} < 0$, and g_{max} be the unique metric in the conformal class of g with constant scalar curvature $R_{g_{max}} = R_{max}$. By Theorem 18, the volume entropies of (M, g) and (M, g_{max}) satisfy

$$h(g) \geq h(g_{max}).$$

It follows from the work of A. Freire and R. Mañé [FM82] that $h_{top}(g_{max}) = h(g_{max})$. Moreover, it was shown by Manning in [Man79] that in all cases, $h_{top}(g) \geq h(g)$.

Therefore $h_{top}(g) \geq h(g) \geq h(g_{max}) = h_{top}(g_{max})$. □

We now wonder whether—in this setting—the metric with constant maximal scalar curvature is the unique entropy minimum in the conformal class, and whether the metric with constant minimal scalar curvature is the unique entropy maximum. Uniqueness of the metric with minimal (or maximal) entropy is known as ***entropy-rigidity***. We first show that when the scalar curvature is *not maximal everywhere* and the *sectional curvatures are negative*, the increasing curvature-normalized Yamabe flow CYF$^+$ strictly decreases the entropy, and the decreasing curvature-normalized Yamabe flow CYF$^-$ strictly increases it. This will imply our entropy-rigidity theorem.

Proposition 21. (1) *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ with negative sectional curvatures, with scalar curvature $R_g \leq R_{max} < 0$ everywhere and $R_g < R_{max}$ somewhere. Let $(g_t^+)_{t \geq 0}$ be a solution to the CYF$^+$ starting in g. Let $T^* > 0$ be such that $K_{g_t^+} < 0$ for $t \in [0, T^*)$. Then the map $t \mapsto h(g_t^+)$ strictly decreases for $t \in [0, T^*)$.*

(2) *Let (M, g) be a closed Riemannian manifold with negative sectional curvatures, with scalar curvature $R_{min} \leq R_g < 0$ everywhere and $R_{min} < R_g$ somewhere. Let $(g_t^-)_{t \geq 0}$ be a solution to the CYF$^-$ starting in g. Let $T^* > 0$ be such that $K_{g_t^-} < 0$ for $t \in [0, T^*)$. Then the map $t \mapsto h(g_t^-)$ strictly increases for $t \in [0, T^*)$.*

Proof. We will prove the first item, for the Curvature-normalized increasing flow CYF$^+$. The other proof is similar. Let (M, g) be a closed Riemannian manifold with negative sectional curvatures and with scalar curvature $R_{min} \leq R_g \leq R_{max} < 0$. Let (g_t) be the solution of the CYF$^+$ with initial metric g, which we suppose to be non-trivial, and $T^* > 0$ be such that $K_{g_t} < 0$ for $t \in [0, T^*)$.

For all $t \geq 0$, we denote by $S^t M$ the ***unit tangent bundle*** of (M, g_t). Notice that the family of metrics $(g_t)_{t \geq 0}$ is smooth and for all $t \in [0, T^*)$ the sectional curvatures of g_t are negative. Therefore, for all $t \in [0, T^*)$, the entropy $h(g_t)$ is equal to the topological entropy of the geodesic flow on (M, g_t). So it follows from [KKW91] that the map $t \mapsto h(g_t)$ is $\mathcal{C}^1$ and for all $t \in [0, T^*)$, its derivative is given by

$$\left.\frac{\partial}{\partial \lambda}\right|_{\lambda = t} h(g_\lambda) = -\frac{h(g_t)}{2} \int_{S^t M} \left.\frac{\partial}{\partial \lambda}\right|_{\lambda = t} g_\lambda(v, v) d\mu^t(v). \tag{8}$$

Here μ^t is the ***Bowen-Margulis measure*** for the geodesic flow on $S^t M$. By definition of the Curvature normalized Yamabe Flow, for all $x \in M, t \in [0, T^*)$ and

$v \in S^t_x M$,

$$\frac{\partial}{\partial \lambda}\bigg|_{\lambda=t} g_\lambda(v,v) = (R_{max} - R_{g_t}(x)) \geq 0$$

by Proposition 9. Therefore, for all $t \in [0, T^*)$,

$$\frac{\partial}{\partial \lambda}\bigg|_{\lambda=t} h(g_\lambda) \leq 0.$$

Let $t \in [0, T^*)$, since the CYF$^+$ is not trivial, there is an $x_t \in M$ such that $R_{g_t}(x_t) < R_{max}$. Since the CYF$^+$ is smooth, we have $R_{g_s}(x_t) < R_{max}$ for $s \in (t-\epsilon, t+\epsilon)$ for some $\epsilon > 0$. Therefore for all $t \geq 0$ there exists an open set $\mathcal{O}_t \subset M$ containing x_t such that $R_{g_t}(y) < R_{max}$ for all $y \in \mathcal{O}_t$. We will use the following classical lemma:

Lemma 22. *With the previous notations, for all $t \in [0, T^*)$, let $S^t\mathcal{O}_t \subset S^t M$ be the preimage of $\mathcal{O}_t$ by the canonical projection $\pi_t : S^t M \to M$. Then*

$$\mu^t(S^t\mathcal{O}_t) > 0,$$

where μ^t is the Bowen-Margulis measure for the geodesic flow on $S^t M$.

Proof. Let $t \in [0, T^*)$, and $\mathcal{O}_t \subset M$ be an open set. Then $S^t\mathcal{O}_t = \pi_t^{-1}(\mathcal{O}_t)$ is an open set in $S^t M$. Since $K_{g_t} < 0$, the geodesic flow on $S^t M$ is Anosov (see [KH95]), it has an orbit $(v_s)_{s\in(-\infty,\infty)}$ which is dense in $S^t M$ and with full measure. Assume now $\mu^t(\mathcal{O}_t) = 0$, there is a non-empty segment $(a,b) \subset \mathbb{R}$ such that $(v_s)_{s\in(a,b)} \subset \mathcal{O}_t$. Hence $\mu^t\left((v_s)_{s\in(a,b)}\right) = 0$, and since μ^t is invariant by the flow we get $\mu^t(S^t M) = 0$: a contradiction. □

This last lemma and the variational formula (8) imply that if $R_g(x) < R_{max}$, then for all $t \in [0, T^*)$ we obtain

$$\frac{\partial}{\partial \lambda}\bigg|_{\lambda=t} h(g_\lambda) < 0.$$

This ends the proof of Proposition 21. □

We can actually express these properties of the entropy in the following form, which is scale-invariant.

Theorem 23 (Entropy along Unscaled Yamabe Flow)**.** *Let (M, g) be a Riemannian manifold with negative scalar curvature, and (g_t) be the unique solution of the* Unscaled Yamabe Flow

$$\frac{\partial g_t}{\partial t} = -R_{g_t} g_t.$$

Then the map $t \mapsto \sqrt{|R_{max}(g_t)|}h(g_t)$ is non-increasing along the flow. It strictly decreases *whenever g_t has negative sectional curvature. Similarly, the map $t \mapsto \sqrt{|R_{min}(g_t)|}h(g_t)$ is non-decreasing along the flow and* strictly increases *whenever g_t has negative sectional curvature.*

What is the behaviour of the map $t \mapsto \sqrt{|R(g_t)|}h(g_t)$? In other words, how does the entropy behave along the ***Volume-normalized Yamabe Flow*** ? In the case of ***surfaces*** of genus$>$ 1, it has been shown by A. Manning that the Volume-normalized Yamabe-Ricci flow decreases the entropy when it starts from a negatively curved metric [Man04]. For manifolds ***of dimension at least 3***, it is not known whether Volume-normalized Ricci Flow or Volume-normalized Yamabe

Flow decrease the entropy. To our knowledge, the only known fact (claimed in [Tho09]) states that starting in a metric with total scalar curvature less than $-n(n-1)$ and conformally equivalent to a hyperbolic metric, then the Volume-normalized Ricci Flow decreases the entropy for a short time. Theorem 23, obtained from Curvature-normalized Yamabe Flows, gives a first insight to the complexity of this question.

Theorem 23 implies the following entropy-rigidity Theorem.

Theorem 24 (Conformal Entropy-rigidity)**.** *Let (M, g_Y) be a closed Riemannian manifold of dimension $n \geq 3$ where g_Y is a Yamabe metric with negative sectional curvatures and constant scalar curvature $R_{g_Y} = -1$. For any metric $g = vg_Y$ conformally equivalent to g_Y such that $R_{min} \leq R_g \leq R_{max}$, the following holds:*

1. *$h(g) \geq \sqrt{|R_{max}|}h(g_Y)$ with equality if and only if $g = \frac{g_Y}{|R_{max}|}$;*
2. *$h(g) \leq \sqrt{|R_{min}|}h(g_Y)$ with equality if and only if $g = \frac{g_Y}{|R_{min}|}$.*

Proof. Let (M, g_Y) be a closed Riemannian manifold where g_Y is a Yamabe metric with negative sectional curvatures and constant scalar curvature $R_{g_Y} = -1$, and g be a metric in its conformal class. Up to scaling, we can assume that $R_{min} \leq R_g \leq -1$. Let us assume that $g \neq g_Y$, we will show that $h(g) > h(g_Y)$.

Let $(g_t)_{t>0}$ be the solution of the Curvature-normalized increasing Yamabe Flow CYF$^+$. Let $0 < t < s$, we have seen in Theorem 1 that $g_t \leq g_s$. Therefore, it follows from Lemma 19 that $h(g_t) \geq h(g_s)$: the map $t \mapsto h(g_t)$ is non-increasing.

Moreover, it follows from Theorem 7 that the flow g_t converges to $g_{max} \leq g_Y$ in the $\mathcal{C}^2$ topology. Since g_Y has negative sectional curvatures, this implies that there exists a $T > 0$ such that for all $t \geq T$, the sectional curvatures of g_t are negative. Moreover, it follows from the proof of Proposition 21 that since $g \neq g_Y$, we have $R_{g_T} < R_{max}$ at some point of M. Together with Proposition 21, this implies that the entropy $h(g_t)$ does not increase on $[0, T)$ and strictly decreases on $[T, \infty)$. Since it converges to $h(g_{max}) \geq h(g_Y)$, we obtain

$$h(g) = h(g_0) > h(g_Y).$$

A similar proof shows that if g is conformally equivalent to g_Y and $-1 \leq R_g < 0$, if $g \neq g_Y$ then $h(g) < h(g_Y)$. A straightforward scaling argument finishes the proof of Theorem 24. □

Remark 25. Let (M, g) be a metric satisfying $R_{min} \leq R_g \leq R_{max} < 0$. Once the existence of the metric g_Y with constant scalar curvature -1 in the same conformal class as g is proved, and once the Conformal Schwarz Lemma 15 is established, the entropy-rigidity theorem could be proved using other flows. We only need to be sure that the scalar curvature will remain strictly lower than R_{max} on some open set for large time: the Yamabe flow is a natural flow which satisfies this requirement.

Appendix A. Maximum principles for manifolds with variable metrics

We have used as a crucial tool the following forms of the Maximum Principle (see the IHP lectures by Z. Djadli and [CLN06], p101. where a similar version is proved).

Proposition 26 (Weak Maximum Principle for closed manifolds)**.** *Let M be a smooth closed manifold and $(g_t)_{t\in I}$ a smooth time-dependent family of metrics on M*

defined on $I = [0, T)$. Let $b, \eta > 0$ and $C \in \mathbb{R}$ be three constants. Let $f : M \times I \to \mathbb{R}$ be a smooth map.

(1) *If for all (x, t) with $C < f(x, t) \leq C + \eta$, we have*
$$\frac{\partial f}{\partial t}(x, t) \leq -b\Delta_{g_t} f(x, t)$$
and $f(., 0) \leq C$, then for all $(x, t) \in M \times I$ we have $f(x, t) \leq C$.

(2) *If for all (x, t) with $C - \eta \leq f(x, t) < C$, we have*
$$\frac{\partial f}{\partial t}(x, t) \geq -b\Delta_{g_t} f(x, t)$$
and $f(., 0) \geq C$, then for all $(x, t) \in M \times I$ we have $f(x, t) \geq C$.

Proof. We shall prove the first item, the case of an upper bound (the other proof is analogous). Under the assumption of item (1), let $0 < \epsilon < \frac{\eta}{1+T}$ be fixed, we set for all $(x, t) \in M \times I$:
$$v_\epsilon(x, t) = f(x, t) - C - \epsilon(1 + t).$$
We want to prove that v_ϵ is always non-positive on $M \times I$. We have $v_\epsilon(., 0) = f(., 0) - C - \epsilon < 0$. Assume there exists $t \in I$ and $x \in M$ such that $v_\epsilon(x, t) = 0$, which implies that $C < f(x, t) = C + \epsilon(1+t) \leq C + \eta$ by construction of ϵ. Since M is closed, there exists $t_0 \in I$ and $x_0 \in M$ such that $v_\epsilon(x_0, t_0) = 0$ and t_0 is minimal for this property. Then since v_ϵ must be increasing in time at (x_0, t_0) to reach 0, we have
$$0 \leq \frac{\partial v_\epsilon}{\partial t}(x_0, t_0) = \frac{\partial f}{\partial t}(x_0, t_0) - \epsilon \leq -b\Delta_{g_{t_0}} f(x_0, t_0) - \epsilon.$$
Moreover, by construction $v_\epsilon(., t_0)$ is maximal in x_0. Hence $f(., t_0)$ is also maximal in x_0. Therefore $\Delta_{g_{t_0}} f(x_0, t_0) \geq 0$ (because the Hessian of f is non-positive at a point where f is maximum). The previous equality becomes $0 \leq -\epsilon$, a contradiction. Hence, for all $(x, t) \in M \times I$ we obtain
$$v_\epsilon(x, t) = f(x, t) - C - \epsilon(1 + t) \leq 0.$$
As this is valid for all $\epsilon \in (0, \eta/(1 + T))$, it implies that $f \leq C$ on $M \times I$. □

We have also used the following form of the Maximum Principle, whose proof is a direct adaptation of the previous one:

Corollary 27. *Let (M, g_0) be a smooth closed manifold and $(g_t)_{t\in I}$ a smooth time-dependent family of metrics on M. Let $\alpha, b, C > 0$ be three positive constants. Let $f : M \times [0, T) \to \mathbb{R}$ be a smooth* **positive** *map.*

(1) *If for all (x, t) with $f(x, t) > C$, we have*
$$\frac{\partial f}{\partial t}(x, t) \leq -b\Delta_{g_t} f(x, t)^\alpha$$
and $f(., 0) \leq C$, then for all $(x, t) \in M \times I$ we have $f(x, t) \leq C$.

(2) *If for all (x, t) with $f(x, t) < C$, we have*
$$\frac{\partial f}{\partial t}(x, t) \geq -b\Delta_{g_t} f(x, t)^\alpha$$
and $f(., 0) \geq C$, then for all $(x, t) \in M \times I$ we have $f(x, t) \geq C$.

We used the following (stronger) form of the Maximum Principle in our proof that along the CYF the scalar curvature converges exponentially fast to a constant:

Theorem 28 (Variable Maximum Principle for closed manifolds)**.** *Let M be a smooth closed manifold and $(g_t)_{t\in I}$ a smooth time-dependent family of metrics on M defined on an interval $I=[0,T)$. Let $F:\mathbb{R}\to\mathbb{R}$ be a locally Lipschitz function and $b>0$ a constant and $\phi: I\to\mathbb{R}$ be the solution to :*

$$\begin{matrix} \frac{d\phi}{dt}(t) = & F(\phi) \\ \phi(0) = & C \end{matrix} .$$

(1) *If $u: M\times I\to\mathbb{R}$ is a smooth function such that for all $(x,t)\in M\times I$ we have that*
$$\frac{\partial u}{\partial t}(x,t) \geq -b\Delta_{g_t} u(x,t) + F(u(x,t)),$$
and $u(.,0)\geq C$. Then $u(x,t)\geq \phi(t)$ for all $(x,t)\in M\times I$.

(2) *If $u: M\times I\to\mathbb{R}$ is a smooth function such that for all $(x,t)\in M\times I$ we have that*
$$\frac{\partial u}{\partial t}(x,t) \leq -b\Delta_{g_t} u(x,t) + F(u(x,t)),$$
and $u(.,0)\leq C$. Then $u(x,t)\leq \phi(t)$ for all $(x,t)\in M\times I$.

Proof. We shall prove only item (1), since the other proof is similar.

Let M be a smooth closed manifold and $(g_t)_{t\in I}$ a smooth time-depending family of metrics on M defined on an interval $I=[0,T)$. Assume $F:\mathbb{R}\to\mathbb{R}$ is a locally Lipschitz map and $u: M\times I\to\mathbb{R}$ satisfies the hypotheses of Theorem 28. Write $v=u-\phi$, where $\phi:\mathbb{R}_+\to\mathbb{R}$ is the solution of:

$$\begin{matrix} \frac{d\phi}{dt}(t) = & F(\phi) \\ \phi(0) = & C \end{matrix}$$

Let $T_0<T$ be fixed ; then for all $(x,t)\in M\times[0,T_0]$ we have that

$$\frac{\partial v}{\partial t} \geq -b\Delta_{g_t} v + F(u) - F(\phi).$$

Moreover, since F is locally Lipschitz, there exists $C'>0$ such that for all $(x,t)\in M\times[0,T_0]$ we get

$$|F(u(x,t)) - F(\phi(t))| \leq C'\,|u(x,t)-\phi(t)| = C'\,|v(x,t)|\,.$$

Therefore

$$\frac{\partial v}{\partial t} \geq -b\Delta_{g_t} v - C'\,|v|\,. \tag{9}$$

Set $w=e^{-C't}v$, then

$$\frac{\partial w}{\partial t} = -C'w + e^{-C't}\frac{\partial v}{\partial t}.$$

Equation (9) becomes

$$\frac{\partial w}{\partial t} \geq -b\Delta_{g_t} w - C'\,|w| - C'w,$$

So for all (x,t) such that $w(x,t)<0$ we obtain

$$\frac{\partial w}{\partial t} \geq -b\Delta_{g_t} w.$$

Since $w(.,0)\geq 0$, by Proposition 26 this implies that $w(.,t)\geq 0$. Hence $v\geq 0$, for all $t\in[0,T_0]$. Therefore $u(x,t)\geq\phi(t)$ for all t in $[0,T_0]$. Since this is valid for all $T_0\in(0,T)$, it is true on $I=[0,T)$. ◻

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