Rationality problem for generic tori in simple groups

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Abstract

We prove that except for several already known cases, the generic torus of a simple (adjoint or simply connected) group is not stably rational. This confirms a conjecture by Le Bruyn on generic norm tori.

0 Introduction

A possible approach to the rationality problem for a connected linear algebraic group G defined over a field k is based on the representation of the function field $k(G)$ in the form $F(T_x)$ where $F = k(X)$ is the field of rational functions of the variety of maximal tori of G and T_x is a "generic" maximal torus of G. Indeed, since X is a k-rational variety [?], [?, XIV.6.1], the F-rationality of T_x would immediately imply the k-rationality of G. This method allows one to prove that all k-forms of a semisimple adjoint group of type A_{2l} are rational [?].

Recall that if k is a number field, generic tori possess excellent arithmetic properties (the Hasse principle for principal homogeneous spaces, weak approximation) whenever one restricts oneself to simply connected, adjoint, or absolutely almost simple groups [?].

*Research of the second named author partially supported by the Ministry of Absorption (Israel) and the Israel Science Foundation founded by the Israel Academy of Sciences and Humanities.

Note that although there are numerous cases where one can prove rationality for particular groups and/or for particular ground fields (see [?], [?] and the references there), there are not many examples of non-rational semisimple groups. One can mention Serre's classic examples [?] (where the groups are neither simply connected nor adjoint); Platonov's examples for certain simply connected groups of type A_l [?] (generalized by Rost and Merkurjev [?]) and of type D_{2l+1} [?]; and, finally, examples by Merkurjev [?] and Gille [?] for adjoint groups of type D_l. One might hope to obtain some results for the remaining types using properties of generic tori.

Our goal is to show the limitations of this method. Our main result (Theorem **??**) says that there are no rational generic tori apart from already known cases. A particular case of that theorem for inner forms of simply connected groups of type A_l (Proposition **??**) confirms a conjecture by Le Bruyn [?] that there are no rational generic norm tori of dimension greater than two.

We are now going to state our main results.

Let k be a field, G a reductive k-group, T_0 a maximal k-torus of G, $N = \mathrm{N}_G(T_0)$ the normalizer of T_0. The homogeneous variety $X = G/N$ is called the variety of maximal tori of G: indeed, since all maximal tori are conjugate, one can associate to each maximal torus T a $g \in G$ such that $g^{-1}Tg = T_0$, and such a g is defined up to a factor in N. Conversely, for all $g \in G$ there exists a maximal k-torus containing g. If g is regular then such a torus is determined uniquely. For a separable extension K/k, we thus obtain a one-to-one correspondence between maximal K-tori of G and K-points of $X_0 = G_0/N$ where G_0 denotes the set of regular points of G.

To be more precise, one constructs a "tautologic" fibration $\pi\colon H \longrightarrow X$ where H is the image of the morphism $\alpha\colon G \times_k T_0 \longrightarrow X \times_k G$, $(g,t) \mapsto (gN, gtg^{-1})$ and π stands for the first projection. One can show that H is birationally equivalent to G [?] (see also [?, 4.1]). Let now x be a generic point of X, so that $k(x) = k(X) = F$. The fibre $\pi^{-1}(x) = H_x$ is called the generic torus of G.

In this paper, we are interested in the rationality problem for generic tori. We restrict our attention to the case where G is an (absolutely almost) simple group, either adjoint or simply connected. Recall that an F-torus T is called stably rational if there is an F-rational variety T' such that $T \times T'$ is F-rational.

Theorem 0.1 *Let G be a simple k-group, either adjoint or simply connected, and let T be the generic torus of G. If G is of one of the following types:*

1. $\operatorname{rk} G \leq 2$,
2. *G is an inner form of an adjoint group of type A_l,*
3. *G is a form of an adjoint group of type A_{2l},*
4. *G is a form of an adjoint group of type B_l,*

5. G is a form of a simply connected group of type C_l,

then T is rational. Otherwise, T is not stably rational.

Let us note an important particular case of the above theorem. Let G be an inner form of a simply connected group of type A_l, and let T be the generic torus of G. Denote by L the splitting field of T, and let $\Gamma = \mathrm{Gal}(L/F)$. Then the character module M of T is isomorphic to the weight lattice $\mathrm{P}(A_l)$, and Γ acts on M as the Weyl group $\mathrm{W}(A_l)$, which is the symmetric group S_{l+1}. The Γ-module M is isomorphic to $\mathbb{Z}[S_{l+1}/S_l]/\mathbb{Z}$. The torus T is none other than the norm torus corresponding to a generic separable extension K/F of degree $l+1$, i.e. a separable extension of degree $l+1$ whose normal closure has the symmetric group S_{l+1} as Galois group. Such a torus is called a generic norm torus and is denoted by T_{l+1}.

In [**?**], Le Bruyn proved that the generic norm torus T_n is not stably rational over F provided n is prime, and stated a conjecture that T_n is never stably rational for $n > 3$ (except possibly for $n = 6$). In this paper, we prove the above conjecture.

Proposition 0.2 *The generic norm torus T_n is not stably rational for $n > 3$.*

The structure of this paper is as follows. In Section **??**, we collect necessary informations on tori and Galois cohomology. In Section **??**, we present a general plan of the proof of the main theorem. in Section **??** we review the cases where the generic tori are rational and we analyze the remaining one s case by case in Sections **??** to **??**. The three-dimensional case, serving as the induction base, is considered separately (Section **??**). Among the inductive branches of the proof, the case $\mathrm{P}(A_l)$ (Section **??**) plays a special rôle: it contains the proof of Le Bruyn's conjecture (Proposition **??**). In the Appendix, we present, for the reader's convenience, a self-contained proof of a technical lemma needed for the proof of Proposition **??**.

Notation and conventions

Given a field F, we denote by $\overline{F}$ a fixed separable closure of F, $\mathfrak{g} = \mathrm{Gal}(\overline{F}/F)$ is the absolute Galois group of F, F^* stands for the multiplicative group of F. We denote by K/F a finite separable extension. All algebraic groups under consideration are assumed to be connected.

An algebraic F-torus T is called quasi-trivial if it is a direct product of tori of the form $R_{K/F}\mathbb{G}_{\mathrm{m}}$ where K/F is a finite extension and $R_{K/F}$ stands for the Weil restriction of scalars. A norm torus is the kernel of the norm map $R_{K/F}\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m},F}$, we often denote it by $T_{K/F}$. Let $\hat{T}$ denote the group of characters of a torus T; viewed as a $\mathfrak{g}$-module, $\hat{T}$ is a $\mathbb{Z}$-free $\mathfrak{g}$-module of finite rank. If T is quasi-trivial, then $\hat{T}$ is a permutation module (i.e. it has a $\mathbb{Z}$-basis permuted by $\mathfrak{g}$). A torus T is called anisotropic if it has no character defined over F; in other words, the group of invariants $\hat{T}^{\mathfrak{g}}$ is zero.

If M is a Galois module (i.e. a discrete continuous $\mathfrak{g}$-module of finite rank), we denote by $H^i(F, M)$ (or by $H^i(\mathfrak{g}, M)$) the i-th Galois cohomology group, and by $\text{Ш}^i_\omega(\mathfrak{g}, M)$ the kernel of the restriction map

$$H^i(\mathfrak{g}, M) \to \prod_{\gamma} H^i(\gamma, M)$$

where γ runs over all closed procyclic subgroups of $\mathfrak{g}$. The dual module $\mathrm{Hom}(M, \mathbb{Z})$ is denoted by M°.

Two modules M_1 and M_2 are called similar if there are permutation modules P_1 and P_2 such that $M_1 \oplus P_1 \cong M_2 \oplus P_2$; Let M denote the similarity class of M.

For a smooth projective F-variety X we denote $X \times_F \overline{F}$ by $\overline{X}$, $\mathrm{Pic}\, X = H^1_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})$ is the Picard group.

We use the notation in [?] for all objects related to a root system R. In particular, $\mathrm{W}(R)$ is the Weyl group, $\mathrm{A}(R)$ is the automorphism group of R; the group $\mathrm{A}(R)$ is a semi-direct product $\mathrm{W}(R) \rtimes \mathrm{Sym}(R)$ where $\mathrm{Sym}(R)$ stands for the group of symmetries of the corresponding Dynkin diagram. Furthermore, $\mathrm{Q}(R)$ is the root lattice, $\mathrm{P}(R)$ is the weight lattice, $R^\vee$ is the dual root system. For an irreducible root system R of rank n, we denote by $\{\alpha_1, \dots, \alpha_n\}$ (resp. $\{\omega_1, \dots, \omega_n\}$) the basis of $\mathrm{Q}(R)$ (resp. $\mathrm{P}(R)$) presented in the corresponding table in [?] in terms of the standard basis $\{\varepsilon_i\}$ of the vector space spanned by R over $\mathbb{R}$.

If k is a field and G a semisimple k-group, let $F = k(X)$ denote the function field of the variety of maximal tori of G and let T be a generic torus of G. Let R be the corresponding root system, then $\hat{T} = \mathrm{P}(R)$ if G is simply connected, and $\hat{T} = \mathrm{Q}(R)$ if G is adjoint. We often shorten "absolutely almost simple group" to "simple group".

Acknowledgements. The work on this paper started while the second named author visited the Université de Franche-Comté (Besançon). It was finished during the visit of the first named author at the University of Chicago and the visit of the two authors at the Max-Planck-Institut für Mathematik (Bonn). The authors are grateful to all these institutions (and especially to E. Bayer-Flückiger and J. Alperin) for their hospitality and support. We also thank C. Bonnafé and E. Plotkin for their very valuable suggestions.

1 Preliminaries

In this section, we collect information on algebraic tori (in particular, on generic tori in semisimple groups) and Galois cohomology which will be systematically used in what follows.

1.1 Birational invariants of algebraic tori

Let T be an F-torus, $\hat{T}$ its character module. There is a canonical exact sequence of $\mathfrak{g}$-modules [?, 4.45, 4.52], [?, 4.6]:

$$0 \to \hat{T} \to S \to P \to 0$$

where S is a permutation module and P is a so-called flasque module (i.e. for all closed subgroups $\mathfrak{h} \subseteq \mathfrak{g}$ one has $H^1(\mathfrak{h}, P^\circ) = 0$). The similarity class $[P]$ turns out to be a birational invariant of T; to be more precise, two tori T_1 and T_2 are stably equivalent if and only if $[P_1] = [P_2]$ [?, 4.60], [?, 4.7]. This shows that stable rationality depends on the character module $\hat{T}$ rather than on the torus T. The class $[P]$ is denoted by $p(T)$ and is called the Picard class (indeed, if F is of characteristic zero, one can take for P the Picard module $\operatorname{Pic}\overline{V}$ where V is a smooth projective variety containing T as an open subset).

Rougher (but very useful and computable) invariants arise from Galois cohomology: for every closed subgroup $\mathfrak{h} \subseteq \mathfrak{g}$, the group $H^1(\mathfrak{h}, P)$ is a birational invariant of T. In particular, to prove that T is not stably rational, it is enough to find a subgroup $\mathfrak{h}$ with $H^1(\mathfrak{h}, P) \neq 0$. This will be one of our main devices. We use another characterisation for the above invariant which is, in a sense, more intrinsic [?, Prop. 9.5(ii)] :

$$H^1(\mathfrak{h}, P) = Ш^2_\omega(\mathfrak{h}, \hat{T}).$$

The invariants of the above paragraph can be explained somewhat simpler by passing to a certain finite level. Namely, let L denote the splitting field of T (i.e. the minimal Galois extension of F such that $T \times_F L \cong \mathbb{G}^d_{\mathrm{m},L}$), and let $\Gamma = \operatorname{Gal}(L/F)$. Then, since P is a torsion-free module and $\operatorname{Gal}(\overline{F}/L)$ acts trivially on $\hat{T}$ (and hence on P), one has $H^1(\operatorname{Gal}(\overline{F}/L), P) = 0$, and the restriction-inflation exact sequence gives $H^1(\mathfrak{g}, P) = H^1(\Gamma, P)$. We shall freely use this remark below.

The most important example here is a norm torus $T = T_{K/F}$. If K/F is a Galois extension with group Γ, one has $\hat{T} = J_\Gamma = \mathbb{Z}[\Gamma]/\mathbb{Z}$, and $Ш^2_\omega(\Gamma, \hat{T}) = H^3(\Gamma, \mathbb{Z})$; in particular, if Γ contains a bicyclic subgroup $\Gamma' = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, one has $Ш^2_\omega(\Gamma', \hat{T}) = \mathbb{Z}/p\mathbb{Z}$ and hence the torus $T_{K/F}$ is not stably rational [?, 6.46], [?, 4.8].

Another important tool in studying birational properties of tori is the passage to the anisotropic factor.

Lemma 1.1 [?, 4.22] *Let $1 \to \mathbb{G}^r_{\mathrm{m}} \to T \to T_a \to 1$ be an exact sequence of F-tori, then it admits a rational F-section $s\colon T_a \to T$ which gives rise to a birational equivalence $T \sim T_a \times_F \mathbb{G}^r_{\mathrm{m}}$.*

This lemma is especially useful if T_a is anisotropic since the module $\hat{T}_a$ is often much simpler than $\hat{T}$.

1.2 Generic tori in semisimple groups

From now on, let T be the generic torus of a semisimple k-group G (see Introduction), Content-Length: 47076 X-Lines: 1157 Status: O
$F = k(X)$ the field of definition of T, L the splitting field of T, $\Gamma = \operatorname{Gal}(L/F)$ the splitting group. Let R denote the root system of $G \times \overline{F}$ with respect to $T \times \overline{F}$. The Galois group $\mathfrak{g} = \operatorname{Gal}(\overline{F}/F)$ permutes the roots; we thus obtain an action of the splitting group Γ on R. Let $\rho\colon \Gamma \to \mathrm{A}(R)$ denote the corresponding representation. It turns out that the image of ρ is "as big as possible"; to be more precise, we cite the following theorem [?], [?, 4.2], which in fact goes back to E. Cartan:

Theorem 1.2 *Let G be a semisimple group, T the generic torus of G, R the corresponding root system, Γ the splitting group of T, and $\rho: \Gamma \to \mathrm{A}(R)$. Then*

$$\mathrm{W}(R) \subseteq \rho(\Gamma) \subseteq \mathrm{A}(R).$$

Moreover, if G is of inner type, then $\rho(\Gamma) = \mathrm{W}(R)$.

2 Plan of the proof

In this section, we outline the proof of Theorem **??**. The first step concerns the "good" cases of the theorem, i.e. those where generic tori are rational. All these cases are already known, and we simply give the necessary references in Section **??**. We thus only have to prove that all the other cases are "bad", i.e. the corresponding generic tori are not stably rational. We proceed case by case using the Killing–Cartan classification, and in each case we show that $p(T) \neq 0$ for the generic torus T.

Except for the case $\mathrm{P}(A_l)$, our main device is as follows: we find a finite extension K/F such that the torus $T_K = T \times_F K$ is stably equivalent to a three-dimensional torus T' which is shown in Section **??** to be non-stably rational. Here we build upon the birational classification of three-dimensional algebraic tori which can be found in [**?**]. In most cases, this approach works in a surprisingly easy way: in fact, it turns out that one can find an extension K/F such that L/K is biquadratic and T' is the norm torus $T_{L/K}$. The only exception is the case $\mathrm{P}(D_4)$ (and those deduced from it by induction) where one has to use a more complicated non-rational torus T' with a triquadratic splitting field.

For $\mathrm{P}(A_l)$, the tori under consideration are generic norm tori. The analysis of this case heavily uses a result of Le Bruyn [**?**] establishing non-rationality of the generic norm torus $T_{L/K}$ where L/K is an extension of prime degree $p \geq 5$. Having this result at our disposal and using the reduction of the general case to the case where the degree of L/K is square-free, we are led to the consideration of only one case, namely $(L:K) = 6$. This last case can be treated using results by Drakokhrust and Platonov [**?**]; for the reader's convenience, we also present a (rather technical) self-contained argument in the Appendix.

The general scheme of the proof is depicted on the following tripartite diagram which gives rise to Sections **??**, **??**, and **??**. We did not include the separate case $\mathrm{P}(A_l)$ which is treated in Section **??**.

$$
\begin{tikzcd}
\mathrm{Q}(A_3)_{out} \arrow[rr, Rightarrow] \arrow[d, equal] & & \mathrm{Q}(A_{2l+1})_{out} & & \mathrm{P}(B_3) \arrow[r, Rightarrow] & \mathrm{P}(B_{2l+1}) \\
\mathrm{Q}(C_3) \arrow[rr, Rightarrow] \arrow[d, equal] & & \mathrm{Q}(C_l) & \mathrm{P}(A_3) \arrow[r, equal] & \mathrm{P}(D_3) \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \mathrm{P}(D_{2l+1}) \arrow[u, Rightarrow] \\
\mathrm{Q}(D_3)_{out} \arrow[r, Rightarrow] & \mathrm{Q}(D_4) \arrow[r, Rightarrow] \arrow[dd, Rightarrow] \arrow[dr, Rightarrow] & \mathrm{Q}(D_l) \arrow[u, Rightarrow] & & \mathrm{P}(B_4) \arrow[r, Rightarrow] & \mathrm{P}(B_{2l}) \\
& & \mathrm{Q}(E_6) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathrm{Q}(E_7) \arrow[d, Rightarrow] & \mathrm{P}(D_4) \arrow[r, Rightarrow] \arrow[u, Rightarrow] \arrow[d, Rightarrow] & \mathrm{P}(D_{2l}) \arrow[u, Rightarrow] \\
& \mathrm{P}(E_6) & \mathrm{P}(E_7) & \mathrm{Q}(E_8) & \mathrm{P}(F_4) \arrow[r, equal] & \mathrm{Q}(F_4)
\end{tikzcd}
$$

We explain the notations. First, the subscript "out" refers to groups of the outer type when the corresponding splitting group Γ maps onto $\mathrm{A}(R)$. Secondly, an arrow means that due to the existence of an extension K_0/F at the tail of the arrow with the property "$T_0 \times_F K_0$ is not stably rational", one can "naturally" find an extension K_1/F at the head of the arrow with the property "$T_1 \times_F K_1$ is not stably rational".

To be more precise, we shall often make use of the following induction argument. We say that R' is a root subsystem of R if $R' \subset R$ and there exist a basis Δ of R and a basis Δ' of R' such that Δ' is a part of Δ. Denoting by V (resp. V') the vector space spanned by Δ (resp. Δ') over $\mathbb{R}$, we get $R' = R \cap V'$ and $\mathrm{Q}(R') = \mathrm{Q}(R) \cap V'$. In particular, $\mathrm{Q}(R')$ is a direct factor of $\mathrm{Q}(R)$ (as a $\mathbb{Z}$-module). Moreover, since $\mathrm{W}(R')$ is generated by the reflections orthogonal to the hyperplanes H_x with $x \in R'$, it can be naturally viewed as a subgroup of $\mathrm{W}(R)$; $\mathrm{W}(R')$ acts trivially on V/V' and hence on $\mathrm{Q}(R)/\mathrm{Q}(R')$. We write $\mathrm{Q}(R)/\mathrm{Q}(R') = \mathbb{Z}^{l-l'}$, the trivial $\mathrm{W}(R')$-module (here $l = \dim V$ and $l' = \dim V'$).

Lemma 2.1 *Let R' be a root subsystem of R. Suppose that there is a subgroup $U \subset \mathrm{W}(R')$ such that the $\mathbb{Z}[U]$-module $\mathrm{Q}(R')$ is the character module of a torus T', where T' is defined over $K = L^U$, split over L, and is not stably rational over K.*

(i) *Let G be an adjoint form of type R. Then its generic torus T is not stably rational.*

(ii) *Suppose, in addition, that $\mathrm{Q}(R') = \mathrm{P}(R) \cap V'$. If G is a simply connected form of type R, its generic torus is not stably rational.*

Proof. The exact sequence of $\mathrm{W}(R')$-modules (and hence of $\mathbb{Z}[U]$-modules)

$$0 \to \mathrm{Q}(R') \to \mathrm{Q}(R) \to \mathbb{Z}^{l-l'} \to 0$$

induces the exact sequence of K-tori

$$1 \to \mathbb{G}_{\mathrm{m}}^{l-l'} \to T_K \to T' \to 1$$

where $T_K = T \times_F K$. By Lemma **??**, T_K is stably equivalent to T', whence (i). With the additional assumption of (ii), $\mathrm{Q}(R') = \mathrm{P}(R) \cap V'$ naturally embeds into $\mathrm{P}(R)$ in such a way that the quotient $C = \mathrm{P}(R)/\mathrm{Q}(R')$ has no torsion (so the $\mathbb{Z}$-module $\mathrm{P}(R)$ can be decomposed into a direct sum $\mathrm{Q}(R') \oplus C$), and $\mathrm{W}(R')$ acts trivially on C. We thus get an exact sequence of $\mathbb{Z}[U]$-modules

$$0 \to \mathrm{Q}(R') \to \mathrm{P}(R) \to \mathbb{Z}^{l-l'} \to 0$$

and proceed as in (i).

$\square$

3 Positive cases

Here we just have to make references for each of cases 1–5.

1) All two-dimensional tori are F-rational [**?**, 4.73, 4.74], [**?**, 4.9].

2) If $M = \mathrm{Q}(A_l)$ and $\Gamma = \mathrm{W}(A_l) = S_{l+1}$, then the $\mathbb{Z}[S_{l+1}]$-module M is the augmentation ideal $I_{l+1} = \ker[\mathbb{Z}[S_{l+1}/S_l] \to \mathbb{Z}]$. The corresponding torus is rational [**?**, Ex. 4.8].

3) If $M = \mathrm{Q}(A_{2l})$ then $M = I_l \otimes I_2$. The generic torus is rational [**?**, Corollary of Th. 8].

4) and 5) In each of these two dual cases, the representation of Γ in $\mathrm{A}(R)$ is orthogonal (i.e. respects the quadratic form $x_1^2 + \ldots + x_l^2$), and the generic torus is rational [**?**], [**?**, 8.2].

4 Three-dimensional tori

Let first $M = \mathrm{P}(A_3)$. Then $\Gamma = \mathrm{W}(A_3) = S_4$ acts on $M = \mathbb{Z}[S_4/S_3]/\mathbb{Z}$ via the standard permutation action of S_4 on S_4/S_3. Choose $U = \langle (14)(23), (13)(24) \rangle$ to be a Klein's four-subgroup. The elements of U represent the cosets of S_4/S_3. Hence $M \cong \mathbb{Z}[U]/\mathbb{Z}$ as a U-module. If now T is the generic torus defined over F and split over L with $\mathrm{Gal}(L/F) = S_4$, set $K = L^U$. Then T_K is the norm torus defined over K with biquadratic splitting field L. Since T_K is not stably rational (see Section **??**), we conclude that T is not stably rational.

Let now $M = \mathrm{Q}(A_3)$. If $\Gamma = \mathrm{W}(A_3)$, the generic torus is rational (see Section **??**). So let $\Gamma = \mathrm{A}(A_3) = S_4 \times \mathbb{Z}/2\mathbb{Z}$. The module M is isomorphic to $I_4 \otimes I_2$ where for any n we denote by $I_n = \ker[\mathbb{Z}[S_n/S_{n-1}] \to \mathbb{Z}]$ the augmentation ideal. One can then take $U = \langle c(12), c(34) \rangle$, where c sends ε_i to $-\varepsilon_i$ ($i = 1, \ldots, 4$), and show (see [**?**]) that $M \cong \mathbb{Z}[U]/\mathbb{Z}$ as a U-module. For reader's convenience here is a proof of this assertion.

Recall that one can take $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1, 2, 3$, as a basis of $\mathrm{Q}(A_3)$. The group $U = \langle c(12), c(34)\rangle = \langle a, b\rangle$ acts as follows:

$$a\colon \begin{cases} \alpha_1 & \mapsto & \alpha_1 \\ \alpha_2 & \mapsto & -\alpha_1 - \alpha_2 \\ \alpha_3 & \mapsto & -\alpha_3 \end{cases} \qquad b\colon \begin{cases} \alpha_1 & \mapsto & -\alpha_1 \\ \alpha_2 & \mapsto & -\alpha_2 - \alpha_3 \\ \alpha_3 & \mapsto & \alpha_3 \end{cases} \tag{1}$$

In order to show that $M \cong J_U = \mathbb{Z}[U]/\mathbb{Z}$, we have to compare the action of U given by formulas (**??**) with the standard action of U on the module J_U. Let $0 \to \mathbb{Z} \to \mathbb{Z}[U] \to J_U \to 0$ be the exact sequence defining J_U, and denote by β_i ($i = 1, \ldots, 4$) the images in J_U of the standard generators of $\mathbb{Z}[U]$. Choosing $\{\beta_1, \beta_2, \beta_3\}$ as a basis of J_U and taking into account that $\beta_1 + \ldots + \beta_4 = 0$, we obtain the following formulas for the action of $U = \langle a, b\rangle$:

$$a\colon \begin{cases} \beta_1 & \mapsto & -\beta_1 - \beta_2 - \beta_3 \\ \beta_2 & \mapsto & \beta_3 \\ \beta_3 & \mapsto & \beta_2 \end{cases} \qquad b\colon \begin{cases} \beta_1 & \mapsto & \beta_3 \\ \beta_2 & \mapsto & -\beta_1 - \beta_2 - \beta_3 \\ \beta_3 & \mapsto & \beta_1 \end{cases} \tag{2}$$

By setting $\beta_1 = \alpha_1 + \alpha_2 + \alpha_3$, $\beta_2 = \alpha_2$, $\beta_3 = -\alpha_1 - \alpha_2$, we show the equivalence of (**??**) and (**??**) and thus identify M with J_U.

Formally speaking, we are finished with the three-dimensional case since $D_3 \cong A_3$. However, it is convenient not simply to appeal to the above isomorphism, but rather to exhibit an explicit subgroup $U \subset \mathrm{W}(D_3)$ (resp. $U \subset \mathrm{A}(D_3)$) such that $\mathrm{P}(D_3)$ (resp. $\mathrm{Q}(D_3)$) is isomorphic to $\mathbb{Z}[U]/\mathbb{Z}$ as a U-module. This will simplify our induction arguments in forthcoming sections.

Recall that $\mathrm{A}(D_n)$ is a semidirect product $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$, we denote by $c_1, \ldots, c_n$ the generators of $(\mathbb{Z}/2\mathbb{Z})^n$. For $g = c_{i_1} \ldots c_{i_k}\sigma \in \mathrm{A}(D_n)$, set $\mathrm{sign}(g) = (-1)^k$, we then identify $\mathrm{W}(D_n)$ with the subgroup of $\mathrm{A}(D_n)$ consisting of the elements g with $\mathrm{sign}(g) = 1$.

We now show that in the case $\mathrm{P}(D_3)$, $U = \langle c_1c_2, c_2c_3\rangle = \langle a, b\rangle \subset \mathrm{W}(D_3)$ is a required subgroup. Indeed, U acts on $M = \mathrm{P}(D_3)$ as follows. Let $\omega_1 = \varepsilon_1$, $\omega_2 = (\varepsilon_1 + \varepsilon_2 - \varepsilon_3)/2$, $\omega_3 = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)/2$ be a standard basis of $\mathrm{P}(D_3)$. Then we have

$$a\colon \begin{cases} \omega_1 & \mapsto & -\omega_1 \\ \omega_2 & \mapsto & -\omega_3 \\ \omega_3 & \mapsto & -\omega_2 \end{cases} \qquad b\colon \begin{cases} \omega_1 & \mapsto & \omega_1 \\ \omega_2 & \mapsto & \omega_1 - \omega_2 \\ \omega_3 & \mapsto & \omega_1 - \omega_3 \end{cases} \tag{3}$$

By setting $\beta_1 = \omega_3 - \omega_1$, $\beta_2 = \omega_2$, $\beta_3 = -\omega_3$, we identify M with J_U, as required.

In the case $\mathrm{Q}(D_3)$, we choose $U = \langle c_1c_3, c_2(13)\rangle = \langle a, b\rangle \subset \mathrm{A}(D_3)$. In the standard basis $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, $\alpha_3 = \varepsilon_2 + \varepsilon_3$, the group U acts on $M = \mathrm{Q}(D_3)$ as follows:

$$a\colon \begin{cases} \alpha_1 & \mapsto & -\alpha_1 - \alpha_2 - \alpha_3 \\ \alpha_2 & \mapsto & \alpha_3 \\ \alpha_3 & \mapsto & \alpha_2 \end{cases} \qquad b\colon \begin{cases} \alpha_1 & \mapsto & \alpha_3 \\ \alpha_2 & \mapsto & -\alpha_1 - \alpha_2 - \alpha_3 \\ \alpha_3 & \mapsto & \alpha_1 \end{cases}$$

This coincides with the standard formulas (**??**) for J_U.

To conclude the consideration of three-dimensional tori, it remains to note that
$\mathrm{Q}(C_3) = \mathrm{Q}(D_3)$, $\mathrm{P}(B_3) = \mathrm{P}(D_3)$ (as $\mathbb{Z}$-modules), and $\mathrm{W}(D_3) \subset \mathrm{W}(C_3) = \mathrm{W}(B_3) = \mathrm{A}(D_3)$.

5 Cases deducible from $\mathrm{Q}(A_3)_{\mathrm{out}}$

5.1 Case $\mathrm{Q}(A_{2k+1})_{\mathrm{out}}$

In this case $R = A_l$ with $l = 2k+1$, $\Gamma = \mathrm{A}(R) = S_{l+1} \times \mathbb{Z}/2\mathbb{Z}$. We denote by c the generator of $\mathbb{Z}/2\mathbb{Z}$ sending any $r \in R$ to $-r$. The elements $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1, \dots, l$, form a standard basis of $M = \mathrm{Q}(R)$.

We choose $U = \langle a, b \rangle$ with $a = c(12)(34)\dots(2k-1 \quad 2k)$, $b = c(2k+1 \quad 2k+2)$, and write down the formulas for the action of U on M:

$$a\colon \begin{cases} \alpha_1 & \mapsto \quad \alpha_1 \\ \alpha_2 & \mapsto \quad -\alpha_1 - \alpha_2 - \alpha_3 \\ \alpha_3 & \mapsto \quad \alpha_3 \\ \alpha_4 & \mapsto \quad -\alpha_3 - \alpha_4 - \alpha_5 \\ \dots & \dots \quad \dots \\ \alpha_{2k-1} & \mapsto \quad \alpha_{2k-1} \\ \alpha_{2k} & \mapsto \quad -\alpha_{2k-1} - \alpha_{2k} \\ \alpha_{2k+1} & \mapsto \quad -\alpha_{2k+1} \end{cases} \qquad b\colon \begin{cases} \alpha_1 & \mapsto \quad -\alpha_1 \\ \alpha_2 & \mapsto \quad -\alpha_2 \\ \alpha_3 & \mapsto \quad -\alpha_3 \\ \alpha_4 & \mapsto \quad -\alpha_4 \\ \dots & \dots \quad \dots \\ \alpha_{2k-1} & \mapsto \quad -\alpha_{2k-1} \\ \alpha_{2k} & \mapsto \quad -\alpha_{2k} - \alpha_{2k+1} \\ \alpha_{2k+1} & \mapsto \quad \alpha_{2k+1} \end{cases}$$

Set $\beta_i = \alpha_i + \alpha_{i+1} + \alpha_{i+2}$ for odd $i < 2k-1$ and $\beta_i = \alpha_i$ otherwise. We write the action of a and b in the matrix form with respect to this new basis:

$$A = \begin{pmatrix} 0 & -1 & & & & & & \\ -1 & 0 & & & & & & \\ & & 0 & -1 & & & & \\ & & -1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & -1 & 0 \\ & & & & & 0 & -1 & 0 \\ & & & & & 0 & 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & & & & & & & \\ & -1 & & & & & & \\ & & -1 & & & & & \\ & & & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & -1 & 0 & 0 \\ & & & & & 0 & -1 & 0 \\ & & & & & 0 & -1 & 1 \end{pmatrix}$$

where the empty places correspond to zero entries. We see that the module M restricted to U decomposes into a direct sum of two-dimensional modules and the three-dimensional module J corresponding to the bottom right corner of the matrices. Looking at equations (**??**), we observe that J coincides with J_U. Since the norm torus corresponding to J is not stably rational, this allows us to conclude that the generic torus is not stably rational.

5.2 Case $\mathrm{Q}(C_l)$

As already shown in Section **??**, the generic torus of an adjoint group of type C_3 is not stably rational. Since C_3 is a root subsystem of C_l for any $l > 3$, Lemma **??** implies that the corresponding generic torus is not stably rational.

5.3 Case $\mathrm{Q}(D_4)$

In $\mathbb{R}^4$ equipped with the standard basis $\varepsilon_1, \dots, \varepsilon_4$, we consider $M = \mathrm{Q}(D_4)$ with $\mathbb{Z}$-basis $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, $\alpha_3 = \varepsilon_3 - \varepsilon_4$, $\alpha_4 = \varepsilon_3 + \varepsilon_4$. We choose $U = \langle c_3c_4, c_1c_2(34)\rangle$. The group U acting on M respects $V' = \langle \varepsilon_2, \varepsilon_3, \varepsilon_4\rangle = \langle \alpha_2, \alpha_3, \alpha_4\rangle$, and $R' = D_4 \cap V' \cong D_3$. Moreover, U respects the one-dimensional $\mathbb{Z}$-module generated by α_1: indeed, c_3c_4 fixes α_1 and $c_1c_2(34)$ sends α_1 to $-\alpha_1$. Therefore the U-module M decomposes into a direct sum of a one-dimensional module and a three-dimensional module. It remains to note that the latter three-dimensional module J is isomorphic to J_U. To see that, we observe that the action of $c_1c_2(34)$ on J coincides with the action of $c_2(34) \in \mathrm{W}(C_3)$ on $\mathrm{Q}(C_3) = \mathrm{Q}(D_3)$, and we are led (up to permutation of indices) to the subgroup considered in Section **??**. We thus proved that for $K = L^U$, we have $T_K = T_1 \times T_{L/K}$ where T_1 is a one-dimensional torus and $T_{L/K}$ is the norm torus corresponding to the biquadratic extension L/K. Hence T_K is not stably rational and therefore T is not stably rational.

5.4 Cases $\mathrm{Q}(D_l)$, $\mathrm{Q}(E_l)$, and $\mathrm{P}(E_l)$

The system D_4 is a root subsystem of D_l, $l \geq 5$ (generated by the four last roots of the standard basis), and it is also a root subsystem of E_l, $l = 6, 7, 8$ (generated by α_i, $i = l-4, \dots, l-1$, with the standard notation of [?] for exceptional root systems). By Lemma **??**(i) and the results of Section **??**, we conclude that in the cases $\mathrm{Q}(D_l)$ and $\mathrm{Q}(E_l)$ the generic tori are not stably rational.

In order to treat the cases $\mathrm{P}(E_6)$ and $\mathrm{P}(E_7)$, we have to apply Lemma **??**(ii). Let R' be a root subsystem of R in the sense of Lemma **??**, we have

$$\mathrm{P}(R) \cap V' = \{x \in V' \, :< x, \alpha^\vee > \in \mathbb{Z} \quad \forall \alpha \in R\},$$

$$\mathrm{P}(R') = \{x \in V' \, :< x, \alpha^\vee > \in \mathbb{Z} \quad \forall \alpha \in R'\},$$

so that $\mathrm{P}(R) \cap V' \subset \mathrm{P}(R')$. Therefore, there are two injections:

$$(\mathrm{P}(R) \cap V')/(\mathrm{Q}(R) \cap V') \to \mathrm{P}(R)/\mathrm{Q}(R)$$

and

$$(\mathrm{P}(R) \cap V')/(\mathrm{Q}(R) \cap V') \to \mathrm{P}(R')/\mathrm{Q}(R').$$

For $R = E_6$ and $R' = D_4$, we have $\mathrm{P}(R)/\mathrm{Q}(R) = \mathbb{Z}/3\mathbb{Z}$ and $\mathrm{P}(R')/\mathrm{Q}(R') = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus the only common subgroup is the trivial one. This means that $\mathrm{P}(R) \cap V' = \mathrm{Q}(R) \cap V' = \mathrm{Q}(R')$. Hence $\mathrm{Q}(D_4) = \mathrm{P}(E_6) \cap V'$, i.e. the hypotheses of Lemma **??**(ii) are satisfied. For $R = E_7$ and $R' = E_6$, we have $\mathrm{P}(R)/\mathrm{Q}(R) = \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{P}(R')/\mathrm{Q}(R') = \mathbb{Z}/3\mathbb{Z}$. This implies that $\mathrm{Q}(E_6) = \mathrm{P}(E_7) \cap V'$, and we are once again in the conditions of Lemma **??**(ii). We thus conclude that the generic tori corresponding to $\mathrm{P}(E_6)$ and $\mathrm{P}(E_7)$ are not stably rational.

Remark. The above argument uses the inclusion $i\colon \mathrm{Q}(R')\hookrightarrow\mathrm{P}(R)$ constructed in Lemma **??**(ii) which is different from the "natural" composite map $j\colon \mathrm{Q}(R')\hookrightarrow\mathrm{Q}(R)\hookrightarrow\mathrm{P}(R)$: the latter one does not induce the required direct decomposition of $\mathbb{Z}$-modules since the quotient $\mathrm{P}(R)/\mathrm{Q}(R)$ is not torsion-free, and hence so is the quotient $\mathrm{P}(R)/j(\mathrm{Q}(R'))$.

6 Cases deducible from $\mathrm{P}(D_3)$

6.1 Case $\mathrm{P}(D_{2k+1})$

We consider the case when $M=\mathrm{P}(D_l)$ with $l=2k+1$. Recall that the case $M=\mathrm{P}(D_3)$ was already treated in Section **??**. We mimic the three-dimensional case and take

$$U=\langle c_1c_2\dots c_{l-1},c_{l-1}c_l\rangle=\langle a,b\rangle\,.$$

Since l is odd, U lies in $\mathrm{W}(D_l)$. Let

$$\begin{array}{rcl}\omega_1&=&\varepsilon_1,\\ \omega_2&=&\varepsilon_1+\varepsilon_2,\\ \dots&\dots&\dots\\ \omega_{l-2}&=&\varepsilon_1+\dots+\varepsilon_{l-2},\\ \omega_{l-1}&=&(\varepsilon_1+\dots+\varepsilon_{l-2}+\varepsilon_{l-1}-\varepsilon_l))/2,\\ \omega_l&=&(\varepsilon_1+\dots+\varepsilon_{l-2}+\varepsilon_{l-1}+\varepsilon_l))/2\end{array}\tag{4}$$

form a standard basis of $\mathrm{P}(D_l)$. The action of U can then be written down as follows:

$$a\colon\left\{\begin{array}{rcl}\omega_1&\mapsto&-\omega_1\\ \dots&\dots&\dots\\ \omega_{l-3}&\mapsto&-\omega_{l-3}\\ \omega_{l-2}&\mapsto&-\omega_{l-2}\\ \omega_{l-1}&\mapsto&-\omega_l\\ \omega_l&\mapsto&-\omega_{l-1}\end{array}\right.\qquad b\colon\left\{\begin{array}{rcl}\omega_1&\mapsto&\omega_1\\ \dots&\dots&\dots\\ \omega_{l-3}&\mapsto&\omega_{l-3}\\ \omega_{l-2}&\mapsto&\omega_{l-2}\\ \omega_{l-1}&\mapsto&\omega_{l-2}-\omega_{l-1}\\ \omega_l&\mapsto&\omega_{l-2}-\omega_l\end{array}\right.$$

The above formulas show that the U-module M decomposes into a direct sum of $l-3$ one-dimensional modules generated by the first $l-3$ elements of the basis and a three-dimensional module which we shall denote by J. We have to prove that $J\cong J_U$. This can be easily done by comparing the action of U on the module spanned by $\{\omega_{l-2},\,\omega_{l-1},\,\omega_l\}$ with formulas (**??**).

6.2 Case $\mathrm{P}(B_{2k+1})$

Since $\mathrm{P}(B_l)$ coincides with $\mathrm{P}(D_l)$ and $\mathrm{W}(D_l)\subset\mathrm{W}(B_l)$, we can take the same subgroup U as in Section **??** (viewed as a subgroup of $\mathrm{W}(B_l)$) in order to show that the corresponding torus is not stably rational.

7 Cases deducible from $\mathrm{P}(D_4)$

7.1 Case $\mathrm{P}(D_4)$

In this case, we cannot play the same game with biquadratic norm tori and have to use more subtle arguments.

Let $M = \hat{T} = \mathrm{P}(D_4)$, $U = \langle c_3c_4, (12), c_1c_2c_3c_4\rangle \subset \mathrm{W}(D_4)$. Let us show that the torus T corresponding to the U-module $\hat{T}$ is not stably rational.

In the standard basis $\omega_1 = \varepsilon_1$, $\omega_2 = \varepsilon_1+\varepsilon_2$, $\varepsilon_3 = (\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4)/2$, $\varepsilon_4 = (\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)/2$, the action of $U = \langle a, b, c\rangle$ can be written down as follows:

$$a\colon \begin{cases} \omega_1 & \mapsto & \omega_1 \\ \omega_2 & \mapsto & \omega_2 \\ \omega_3 & \mapsto & \omega_2 - \omega_3 \\ \omega_4 & \mapsto & \omega_2 - \omega_4 \end{cases} \qquad b\colon \begin{cases} \omega_1 & \mapsto & -\omega_1 + \omega_2 \\ \omega_2 & \mapsto & \omega_2 \\ \omega_3 & \mapsto & \omega_3 \\ \omega_4 & \mapsto & \omega_4 \end{cases} \qquad c\colon \omega_i \mapsto -\omega_i \quad (i = 1, \dots 4)$$

After the base change given by $h_1 = \omega_1$, $h_2 = \omega_1 - \omega_2 + \omega_3$, $h_3 = \omega_3$, $h_4 = \omega_3 - \omega_4$, we obtain

$$a\colon \begin{cases} h_1 & \mapsto & h_1 \\ h_2 & \mapsto & h_1 - h_3 \\ h_3 & \mapsto & h_1 - h_2 \\ h_4 & \mapsto & -h_4 \end{cases} \qquad b\colon \begin{cases} h_1 & \mapsto & -h_2 + h_3 \\ h_2 & \mapsto & -h_1 + h_3 \\ h_3 & \mapsto & h_3 \\ h_4 & \mapsto & h_4 \end{cases} \qquad c\colon h_i \mapsto -h_i \quad (i = 1, \dots 4)$$

Hence $T \cong T_3 \times T_1$ where $\dim T_1 = 1$, $\dim T_3 = 3$. Denote by $H \subset \mathrm{GL}(3, \mathbb{Z})$ the subgroup corresponding to the module $\hat{T}_3$, we have

$$H = \left\langle \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle.$$

This torus was studied in [?] and turned out to be non stably rational ($H = W_2$ in the notation of Theorem 1 of [?]).

7.2 Case $\mathrm{P}(D_{2k})$

Just as in Section **??**, we now proceed by induction in order to treat the case $\mathrm{P}(D_{2k})$.

Let $M = \hat{T} = \mathrm{P}(D_l)$ with l even,

$$U = \langle c_1c_2 \dots c_{l-4}c_{l-1}c_l, (l-3 \quad l-2), c_1c_2 \dots c_l\rangle \subset \mathrm{W}(D_l).$$

Let us show that the torus T corresponding to the U-module $\hat{T}$ is not stably rational.

We write down the action of $U = \langle a, b, c\rangle$ in the basis $\omega'_1 = \varepsilon_1, \dots, \omega'_{l-3} = \varepsilon_{l-3}$, $\omega'_{l-2} = \varepsilon_{l-3} + \varepsilon_{l-2}$, $\omega'_{l-1} = (\varepsilon_1 + \ldots + \varepsilon_{l-2} + \varepsilon_{l-1} - \varepsilon_l)/2$, $\omega'_l = (\varepsilon_1 + \ldots + \varepsilon_{l-2} + \varepsilon_{l-1} + \varepsilon_l)/2$:

$$a\colon \begin{cases} \omega'_1 & \mapsto & -\omega'_1 \\ \dots & \dots & \dots \\ \omega'_{l-4} & \mapsto & -\omega'_{l-4} \\ \omega'_{l-3} & \mapsto & \omega'_{l-3} \\ \omega'_{l-2} & \mapsto & \omega'_{l-2} \\ \omega'_{l-1} & \mapsto & \omega'_{l-2} - \omega'_{l-1} \\ \omega'_{l} & \mapsto & \omega'_{l-2} - \omega'_{l} \end{cases} \qquad b\colon \begin{cases} \omega'_1 & \mapsto & \omega'_1 \\ \dots & \dots & \dots \\ \omega'_{l-4} & \mapsto & \omega'_{l-4} \\ \omega'_{l-3} & \mapsto & -\omega'_{l-3} + \omega'_{l-2} \\ \omega'_{l-2} & \mapsto & \omega'_{l-2} \\ \omega'_{l-1} & \mapsto & \omega'_{l-1} \\ \omega'_{l} & \mapsto & \omega'_{l} \end{cases} \qquad c\colon \omega_i \mapsto -\omega_i$$

We get $\hat{T} = \hat{T}_{l-4} \oplus \hat{T}_4$ where T_{l-4} is a direct product of $l-4$ one-dimensional tori and T_4 is isomorphic to the torus considered in Subsection **??**. Since T_4 is not stably rational, so is T.

7.3 Cases $\mathrm{P}(B_{2k})$ and $\mathrm{P}(F_4)$

We can now easily deduce the remaining cases $\mathrm{P}(B_{2k})$ and $\mathrm{P}(F_4)$ from the already treated ones.

Let $M = \hat{T} = \mathrm{P}(B_l)$ with l even. Then the torus T corresponding to the module $\hat{T}$ is not stably rational. Indeed, one has to use the same argument as in Section **??**.

To prove that the torus T corresponding to the module $\hat{T} = \mathrm{P}(F_4)$ is not stably rational, we observe that $\mathrm{P}(F_4)$ coincides with $\mathrm{P}(D_4)$ (as $\mathbb{Z}$-modules) and $\mathrm{W}(D_4)$ is a subgroup in $\mathrm{W}(F_4)$.

8 Case $\mathrm{P}(A_l)$: Le Bruyn's conjecture

Let G be an inner form of a simply connected group of type A_l. Then the group of characters of the generic torus T is the $\mathbb{Z}$-lattice $M = \mathrm{P}(A_l)$ and by Theorem **??** the splitting group of T is $\Gamma = \mathrm{W}(A_l) = S_{l+1}$. As $\mathbb{Z}[S_{l+1}]$-module, M is isomorphic to $\mathbb{Z}[S_{l+1}/S_l]/\mathbb{Z}$, and we may (and shall) consider the generic torus T as a generic norm torus. Recall the definition of such a torus.

Let F be a field. A separable extension K/F of degree n is said to be generic if the Galois group of the normal closure L of K/F is the symmetric group S_n. Let $T_{K/F} = R^1_{K/F}\mathbb{G}_{\mathrm{m}}$ be the corresponding norm torus, i.e. the kernel of the norm map $N_{K/F}\colon R_{K/F}\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m},F}$ where $R_{K/F}$ stands for Weil's restriction of the ground field from K to F. The F-points of $T_{K/F}$ are the elements of K^* with norm one. The torus $T_{K/F}$ is called generic norm torus. We shall often denote it by T_n if it does not lead to any confusion.

The following result is a cornerstone for what follows.

Lemma 8.1 (Le Bruyn [?]) *Let $n > 3$ be a prime number. Then T_n is not stably rational.*

Note that this fact is surprising enough in view of a theorem by Colliot-Thélène and Sansuc [?] stating that for a prime n the torus T_n is a direct factor of a rational variety. In the same paper [?], Le Bruyn made a conjecture that T_n is never stably rational if $n > 3$ (except, possibly, for $n = 6$). Saltman and Snider proved this fact for n divisible by a square. We are going to prove here Proposition **??** (see Introduction) confirming the above conjecture (without any exceptions).

We shall deduce Proposition **??** from Lemma **??** using the following key lemma.

Lemma 8.2 *Let $n = rs$ with arbitrary $r, s > 1$, and let K/F be a generic extension of degree n. If $T_{K/F} = T_n$ is stably rational over F, there exist an extension E/F and a generic extension K'/E of degree r such that $T_{K'/E} = T_r$ is stably rational over E.*

Proof. We regard S_r as a subgroup of S_n embedded diagonally: if $i \in \{0, \ldots, s-1\}$ then S_r acts naturally on $\{ir+1, \ldots, (i+1)r\}$ by $\sigma \cdot (ir+k) = ir + \sigma(k)$ where $\sigma \in S_r$ and $k \in \{1, \ldots, r\}$. This defines an action of S_r on $\{1, \ldots, sr\}$ and hence an embedding of S_r into $S_{sr} = S_n$. We denote by U_r the image of S_r under this embedding.

Let P be the character module of the torus $R_{K/F}\mathbb{G}_{\mathrm{m}}$. It is a S_n-module isomorphic to $\mathbb{Z}[S_n/S_{n-1}]$. As a U_r-module, it decomposes into a direct sum:

$$P \cong \bigoplus_{i=0}^{s-1} \mathbb{Z}[S_r/S_{r-1}]. \tag{5}$$

Indeed, the action of $\sigma \in S_r$ on the coset $(n\ ir+k)S_{n-1} \in S_n/S_{n-1}$ (where $i \in \{0, \ldots, s-1\}$ and $k \in \{1, \ldots, r\}$) is given by

$$\sigma \cdot (n\ ir+k)S_{n-1} = (n\ \sigma \cdot (ir+k))S_{n-1} = (n\ ir + \sigma(k))S_{n-1}.$$

For a fixed i, we thus obtain an isomorphism of $\mathbb{Z}[U_r]$-modules

$$\langle (n\ ir+k)S_{n-1}, \quad k = 1, \ldots, r\rangle_{\mathbb{Z}} \cong \langle (r\ k)S_{r-1}, \quad k = 1, \ldots r\rangle = \mathbb{Z}[S_r/S_{r-1}],$$

whence the required isomorphism (**??**). Let us rewrite this decomposition as $P = \bigoplus_{i=0}^{s-1} P_i$.

Consider the exact sequence of F-tori split over L:

$$1 \to T_n \to R_{K/F}\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m},F} \to 1.$$

It induces the exact sequence of character modules of these tori

$$0 \to \mathbb{Z} \xrightarrow{N} P \to M \to 0 \tag{6}$$

which is an exact sequence of S_n-modules; we may then view it as an exact sequence of U_r-modules. Here M is the group of characters of the torus T_n and N is the norm map defined by

$$1 \mapsto \sum_{\sigma S_{n-1} \in S_n/S_{n-1}} \sigma S_{n-1} = N_{S_n/S_{n-1}}(1).$$

According to the above decomposition of P,

$$N_{S_n/S_{n-1}}(1) = \sum_{i=0}^{s-1} \sum_{\sigma S_{r-1} \in S_r/S_{r-1}} \sigma S_{r-1} = \sum_{i=0}^{s-1} N_{S_r/S_{r-1}}(1).$$

We deduce from (**??**) and (**??**) that

$$M \cong P/N(\mathbb{Z}) = \left(\bigoplus_{i=0}^{s-1} P_i \right) /N(\mathbb{Z}).$$

Consider

$$\varphi\colon \bigoplus_{i=0}^{s-1} P_i \to P_0/N_{S_r/S_{r-1}}(\mathbb{Z}) \oplus \bigoplus_{i=1}^{s-1} P_i$$

given by $\varphi(a_i) = (\bar{a}_0, a_i - a_0)$. This is an epimorphism of $\mathbb{Z}[S_r]$-modules with kernel $N(\mathbb{Z})$. Hence

$$M \cong P_0/N_{S_r/S_{r-1}}(\mathbb{Z}) \oplus \bigoplus_{i=1}^{s-1} P_i = M_r \oplus \bigoplus_{i=1}^{s-1} P_i,$$

where M_r is defined by the following exact sequence of S_r-modules:

$$0 \to \mathbb{Z} \xrightarrow{N_{S_r/S_{r-1}}} \mathbb{Z}[S_r/S_{r-1}] \to M_r \to 0.$$

Denote by $E = L^{U_r}$ the fixed field of U_r. Then $\mathrm{Gal}(L/E) \cong S_r$. Let $K' = L^{S_{r-1}}$. Then K'/E is a generic extension of degree r and M_r is the character module of the generic torus $T_r = T_{K'/E}$. Denote $S = \prod_{i=1}^{s-1} R_{K'/E}\mathbb{G}_{\mathrm{m}}$, it is a quasi-trivial torus whose character module is the $\mathbb{Z}[S_r]$-module $\bigoplus_{i=1}^{s-1} P_i$. We then have an isomorphism of E-tori

$$T_n \times_F E \cong T_r \times S. \tag{7}$$

Since any quasi-trivial torus is rational, the isomorphism (**??**) proves the lemma: indeed, if T_n is stably rational over F, $T_n \times_F E$ is stably rational over E and so is T_r.

□

Lemma **??** allows us to reprove the above mentioned result by Saltman and Snider. It suffices to combine Lemma **??** with the following fact.

Lemma 8.3 *If n is a square, T_n is not stably rational.*

Proof. Denote $n = m^2$, $P = \mathbb{Z}[S_n/S_{n-1}]$, and let U be the subgroup of S_n generated by

$$\sigma = (1 \quad 2 \dots m)(m+1 \quad m+2 \dots 2m)\dots(n-m+1 \quad n-m+2 \dots n),$$
$$\tau = (1 \quad m+1 \dots n-m+1)(2 \quad m+2 \dots n-m+2)\dots(m \quad 2m \dots m^2).$$

The group U is isomorphic to $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ (σ and τ are of order m and commute). We have

$$\sigma^i(m^2) = (m-1)m + i \quad \text{for } 1 \le i \le m,$$
$$\tau^j\sigma^i(m^2) = (j-1)(m-1) + i \quad \text{for } 1 \le j \le m.$$

Thus $\tau^j\sigma^i S_{n-1} = ((j-1)(m-1)+i \quad m^2)S_{n-1}$ in S_n/S_{n-1}. This defines a bijection of U to S_n/S_{n-1} and an isomorphism of U-modules $\mathbb{Z}[S_n/S_{n-1}] \cong \mathbb{Z}[U]$. Therefore the exact sequence of $\mathbb{Z}[S_n]$-modules

$$0 \to \mathbb{Z} \to \mathbb{Z}[S_n/S_{n-1}] \to M \to 0$$

corresponding to the exact sequence of F-tori

$$1 \to T_n \to R_{K/F}\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m},F} \to 1$$

can be rewritten as an exact sequence of U-modules as follows:

$$0 \to \mathbb{Z} \to \mathbb{Z}[U] \to M \to 0. \tag{8}$$

We thus obtain that M is isomorphic to $\mathbb{Z}[U]/\mathbb{Z}$ as a U-module. From (**??**) we deduce that $Ш^2_\omega(U, M) = \mathbb{Z}/m\mathbb{Z} \neq 0$ (see Section **??**). Therefore if $E = L^U$, the E-torus $T_n \times_F E$, whose character module is M, is not stably rational, and hence so is T_n.

□

Lemma 8.4 *The torus T_6 is not stably rational.*

Proof. Denote $M = \hat{T}_6$, and let U be the subgroup of S_6 generated by $\sigma = (12)(34)$ and $\tau = (34)(56)$. Let us show that $Ш^2_\omega(U, M) \neq 0$.

Denote by $H = A_4$ the alternating group of degree 4 viewed as a transitive subgroup of S_6 (in other words, we regard H as the group of motions of tetrahedron in its action on the edges). We can view U as a subgroup of H, moreover, U is the Sylow 2-subgroup of H. In Lemma 13 of [**?**], it is shown (implicitly) that $Ш^2_\omega(H, M) = \mathbb{Z}/2\mathbb{Z}$. This immediately proves the lemma.

Indeed, suppose that $Ш^2_\omega(U, M) = 0$. For a Sylow 3-subgroup V of H, we have $Ш^2_\omega(V, M) = 0$ because V is cyclic. Since the order of H equals 12, this gives $Ш^2_\omega(H, M) = 0$ (here we use the equality $Ш^2_\omega(H, M) = H^1(H, P)$, see Section **??**, and the fact that for any Γ-module A, $H^1(\Gamma, A) = 0$ if and only if $H^1(\Gamma^{(p)}, A) = 0$ for all Sylow p-subgroups $\Gamma^{(p)}$ of Γ). The obtained contradiction proves the lemma.

$\square$

Remark. For reader's convenience, we present a self-contained proof of Lemma **??** in the Appendix.

Lemmas **??** to **??** prove Proposition **??**. Indeed, Lemmas **??** and **??** reduce the general case to the case where n is squarefree. Applying Lemma **??** and once again Lemma **??**, we are reduced to the case $n = 6$. This last case is treated by Lemma **??**. Proposition **??** is proved.

$\square$

If now G is an arbitrary form of a simply connected group of type A_l and $\Gamma = \mathrm{Gal}(L/E)$ is the splitting group of the generic torus T of G, we have $\Gamma \supseteq \mathrm{W}(A_l) = S_{l+1}$ (see Section **??**). Setting $E = L^{S_{l+1}}$, we see that $T \times_F E$ is a generic norm torus. Since it is not stably rational for $l > 2$, so is T.

This finishes the proof of Theorem **??**.

Appendix

We present here a proof of Lemma **??** not appealing to Lemma 13 of [**?**] (but using the methods of this paper, see also [**?**], Ch. 6.3). We recall here the main points.

The exact sequence of $\mathbb{Z}[S_n]$-modules defining $M = \hat{T}_n$:

$$0 \to \mathbb{Z} \to \mathbb{Z}[S_n/S_{n-1}] \to M \to 0 \tag{9}$$

induces a commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^2(U,\mathbb{Z}) & \overset{\varphi_1}{\to} & H^2(U,\mathbb{Z}[S_n/S_{n-1}]) & \overset{\varphi_2}{\to} & H^2(U,M) & \overset{\varphi_3}{\to} & H^3(U,\mathbb{Z}) \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} \\
\prod\limits_{g\in U} H^2(\langle g\rangle,\mathbb{Z}) & \overset{\psi_1}{\to} & \prod\limits_{g\in U} H^2(\langle g\rangle,\mathbb{Z}[S_n/S_{n-1}]) & \overset{\psi_2}{\to} & \prod\limits_{g\in U} H^2(\langle g\rangle,M) & \overset{\psi_3}{\to} & \prod\limits_{g\in U} H^3(\langle g\rangle,\mathbb{Z}) = 0
\end{array} \tag{10}$$

where the rows are the cohomology exact sequences corresponding to (**??**) with respect to U and $\langle g\rangle$, and the vertical arrows are restriction maps. Since $\langle g\rangle$ is a cyclic group acting trivially on $\mathbb{Z}$, we have $H^3(\langle g\rangle,\mathbb{Z}) = H^1(\langle g\rangle,\mathbb{Z}) = 0$.

Then $Ш^2_\omega(U,M) = \ker\alpha_3 \supset \alpha_2^{-1}(\mathrm{im}\,\psi_1)/\mathrm{im}\,\varphi_1$. We denote the latter group by $Ч^2_\omega(U,M)$ and prove that it is not zero.

We have a decomposition of U-modules

$$\mathbb{Z}[S_n/S_{n-1}] = \bigoplus_{i=1}^{r} \mathbb{Z}[K_i/S_{n-1}] = \bigoplus_{i=1}^{r} \mathrm{Ind}_{U_i}^{U}\mathbb{Z}$$

where $K_i = Ux_iS_{n-1}$ runs over the set of double cosets $U\backslash S_n/S_{n-1}$ and $U_i = (x_iS_{n-1}x_i^{-1}) \cap U$. By Shapiro's lemma, $H^2(U, \mathbb{Z}[S_n/S_{n-1}]) = \bigoplus_{i=1}^r H^2(U_i, \mathbb{Z})$. One can get a similar description of $H^2(\langle g\rangle, \mathbb{Z}[S_n/S_{n-1}])$ for $g \in U$.

For $n = 6$, let us give an explicit expression for the above decomposition. We write $S_6/S_5 = \{(j\ 6)S_5, \quad j = 1, \dots 6\}$ (with abusive notation $(6\ 6) = \mathrm{Id}$), and $U\backslash S_6/S_5$ is in one-to-one correspondence with the orbits of U in S_6/S_5. If $U = \langle(12)(34), (34)(56)\rangle = \langle\sigma, \tau\rangle$, as in Lemma **??**, there are three such orbits. We choose $x_1 = \mathrm{Id}$, $x_2 = (26)$, $x_3 = (36)$ as representatives of these double cosets. Then $U_1 = \langle\sigma\rangle$, $U_2 = \langle\tau\rangle$, $U_3 = \langle\sigma\tau\rangle$. We choose Id, (26), (36), (56) as representatives of the double cosets from $\langle\sigma\rangle\backslash S_6/S_5$. The double cosets $K_i = Ux_iS_5$ decompose as follows:

$$K_1 = \langle\sigma, \tau\rangle \cdot \mathrm{Id} \cdot S_5 = K_{11}^\sigma \cup K_{12}^\sigma$$

with $K_{11}^\sigma = \langle\sigma\rangle \cdot S_5 = \langle\sigma\rangle \cdot \mathrm{Id} \cdot S_5$ and $K_{12}^\sigma = \langle\sigma\rangle \cdot \tau \cdot S_5$, we set $y_{11}^\sigma = \mathrm{Id}$, $y_{12}^\sigma = \tau$, and $s_{1\sigma} = 2$. Similarly,

$$K_2 = \langle\sigma\ , \tau\rangle \cdot (26) \cdot S_5 = K_{21}^\sigma = \langle\sigma\rangle \cdot (26) \cdot S_5 = \langle\sigma\rangle \cdot \mathrm{Id} \cdot x_2 \cdot S_5,$$

we set $y_{21}^\sigma = \mathrm{Id}$ and $s_{2\sigma} = 1$. Finally,

$$K_3 = \langle\sigma, \tau\rangle \cdot (36) \cdot S_5 = K_{31}^\sigma = \langle\sigma\rangle \cdot (36) \cdot S_5 = \langle\sigma\rangle \cdot \mathrm{Id} \cdot x_3 \cdot S_5;$$

we set $y_{31}^\sigma = \mathrm{Id}$ and $s_{3\sigma} = 1$. With this notation, $K_{ij}^\sigma = \langle\sigma\rangle\, y_{ij}^\sigma x_i S_5$ and $y_{ij}^\sigma \in U$. We then have a decomposition of $\langle\sigma\rangle$-modules

$$\mathbb{Z}[S_6/S_5] = \bigoplus_{i=1}^{3} \bigoplus_{j=1}^{s_{\sigma,i}} \mathbb{Z}[K_{ij}^\sigma/S_5] = \bigoplus_{i=1}^{3} \bigoplus_{j=1}^{s_{\sigma,i}} \mathrm{Ind}_{U_{ij}^\sigma}^{\langle\sigma\rangle} \mathbb{Z}$$

where

$$\begin{aligned} U_{ij}^\sigma &= (y_{ij}^\sigma x_i)S_5(y_{ij}^\sigma x_i)^{-1} \cap \langle\sigma\rangle = y_{ij}^\sigma(x_iS_5x_i^{-1})y_{ij}^\sigma \cap U \cap \langle\sigma\rangle \\ &= y_{ij}^\sigma(x_iS_5x_i^{-1} \cap U)(y_{ij}^\sigma)^{-1} \cap \langle\sigma\rangle = y_{ij}^\sigma U_i(y_{ij}^\sigma)^{-1} \cap \langle\sigma\rangle \end{aligned}$$

(here we observed that $y_{ij}^\sigma \in U$). Hence $U_{11}^\sigma = \langle\sigma\rangle = U_{12}^\sigma$ and $U_{21}^\sigma = 1 = U_{31}^\sigma$.

Let us repeat the above calculation for τ and $\sigma\tau$. We get

$$\begin{array}{rlll} & y_{11}^\tau = 1, & s_{1\tau} = 1, & U_{11}^\tau = 1; \\ y_{21}^\tau = 1, & y_{22}^\tau = \sigma, & s_{2\tau} = 2, & U_{21}^\tau = U_{22}^\tau = \langle\tau\rangle\,; \\ & y_{31}^\tau = 1, & s_{3\tau} = 1, & U_{31}^\tau = 1, \end{array}$$

and

$$\begin{array}{rlll} & y_{11}^{\sigma\tau} = 1, & s_{1,\sigma\tau} = 1, & U_{11}^{\sigma\tau} = 1; \\ & y_{21}^{\sigma\tau} = 1, & s_{2,\sigma\tau} = 1, & U_{21}^{\sigma\tau} = 1; \\ y_{31}^{\sigma\tau} = 1, & y_{32}^{\sigma\tau} = \tau, & s_{3,\sigma\tau} = 2, & U_{31}^{\sigma\tau} = U_{32}^{\sigma\tau} = \langle\sigma\tau\rangle\,. \end{array}$$

We can now rewrite the left square of diagram (**??**):

$$\begin{array}{ccc} H^2(U,\mathbb{Z}) & \stackrel{\varphi_1}{\to} & \prod_{i=1}^{3} H^2(U_i,\mathbb{Z}) \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} \\ \prod_{g\in U} H^2(\langle g\rangle,\mathbb{Z}) & \stackrel{\psi_1}{\to} & \prod_{g\in U}\prod_{i=1}^{3}\prod_{j=1}^{s_{ig}} H^2(U_{ij}^g,\mathbb{Z}) \end{array}$$

Here the arrows are the natural homomorphisms of restriction and conjugation. Consider the Pontryagin dual of the above diagram (note that if G is a finite group, $H^2(G,\mathbb{Z}) = H^1(G,\mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G,\mathbb{Q}/\mathbb{Z})$ is dual to $G^{\mathrm{ab}} = G/[G,G]$):

$$\begin{array}{ccc} U & \stackrel{\lambda}{\longleftarrow} & \prod_{i=1}^{3} U_i \\ \uparrow{\scriptstyle \gamma} & & \uparrow{\scriptstyle \delta} \\ \prod_{g\in U} \langle g\rangle & \stackrel{\eta}{\longleftarrow} & \prod_{g\in U}\prod_{i=1}^{3}\prod_{j=1}^{s_{ig}} U_{ij}^g. \end{array}$$

We deduce that $Ч^2_\omega(U,M)$ is dual to $\ker\lambda/\delta(\ker\eta)$. Let us now rewrite the above diagram substituting the results of our preceding calculations:

$$\begin{array}{ccc} U & \stackrel{\lambda}{\longleftarrow} & \langle\sigma\rangle\times\langle\tau\rangle\times\langle\sigma\tau\rangle \\ \uparrow{\scriptstyle \gamma} & & \uparrow{\scriptstyle \delta} \\ \langle\sigma\rangle\times\langle\tau\rangle\times\langle\sigma\tau\rangle & \stackrel{\eta}{\longleftarrow} & \langle\sigma\rangle\times\langle\sigma\rangle\times\langle\tau\rangle\times\langle\tau\rangle\times\langle\sigma\tau\rangle\times\langle\sigma\tau\rangle\,. \end{array}$$

Note that η and δ are the same homomorphisms, hence $\delta(\ker\eta) = 1$. Moreover, $\lambda(\sigma,\tau,\sigma\tau) = \lambda(\sigma)\lambda(\tau)\lambda(\sigma\tau) = \sigma\tau\sigma\tau = 1$, so $(\sigma,\tau,\sigma\tau)\in\ker\lambda$, and this is the only non-zero element of $\ker\lambda$. We thus obtain $Ч^2_\omega(U,M)\cong\mathbb{Z}/2\mathbb{Z}\neq 0$. This finishes the proof of Lemma **??**.

□

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