Bending Fuchsian representations of fundamental groups of cusped surfaces in PU(2,1).

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Abstract

We describe a family of representations of $\pi_1(\Sigma)$ in PU(2,1), where Σ is a hyperbolic Riemann surface with at least one deleted point, which is obtained by a bending process associated to an ideal triangulation of Σ. We give an explicit description of this family by describing a coordinate system on it which is in the spirit of Fock and Goncharov coordinates on the Teichmüller space. We identify within this family new examples of discrete, faithful and type-preserving representations of $\pi_1(\Sigma)$, and obtain this way a 1-parameter family of embeddings of the Teichmüller space of Σ in the PU(2,1),representation variety. These results generalize to arbitrary Σ the result we obtained in [21] for the 1-punctured torus.

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1 Introduction

Contrary to its real analogue, the systematic study of complex hyperbolic quasi-Fuchsian groups started only recently. The first example of a complex hyperbolic quasi-Fuchsian deformation of a Fuchsian group was given in 1992 by Goldman and Parker in [11], where they deal with complex hyperbolic ideal triangle groups. The family of known examples has expanded since, but complex hyperbolic quasi-Fuchsian representations are still far from being well-understood. An account of what is known in the field is presented in the survey [16]. The main obstruction to constructing new examples is the difficulty of proving the discreteness of a finitely generated subgroup of the group of isometries of the complex hyperbolic n-space $\mathbf{H}^n_{\mathbb{C}}$. Indeed, most of the constructions use the Poincaré Polyhedron theorem, and involve the building of a fundamental domain for the action of the considered group on the complex hyperbolic space (see for instance [4]). The construction of a fundamental domain in $\mathbf{H}^n_{\mathbb{C}}$ is very technical, due to the fact that there are no totally geodesic real hypersurfaces, that would be a natural choice to be the faces of the polyhedron. We will restrict ourselves to $\mathbf{H}^2_{\mathbb{C}}$ in this work.

The standard method to produce non-trivial examples of quasi-Fuchsian representations of surface groups is to start with a representation ρ_0 preserving a totally geodesic subspace V and to deform it. There are two ways to do so since there are two kinds of maximal totally geodesic subspaces in $\mathbf{H}^2_{\mathbb{C}}$, namely complex lines and totally geodesic Lagrangian planes, or real planes. Complex lines are embeddings of $\mathbf{H}^1_{\mathbb{C}}$ with sectionnal curvature -1, and real planes are embeddings of $\mathbf{H}^2_{\mathbb{R}}$ with sectionnal curvature $-1/4$. Their respective stabilizers are the subgroups P(U(1,1)×U(1)) and PO(2,1). A discrete and faithful representation preserving a complex line (resp. a real plane) is called $\mathbb{C}$-Fuchsian (resp. $\mathbb{R}$-*Fuchsian*). The behaviors of $\mathbb{C}$ and $\mathbb{R}$-Fuchsian representations under deformation are different. Indeed, in the case where Σ is closed, Goldman's rigidity theorem (see [9]) asserts that any deformation of a $\mathbb{C}$-Fuchsian representation still preserves a complex line, whereas it is proved in [17] that any $\mathbb{R}$-Fuchsian representation is contained in an open set of maximal dimension in the PU(2,1)-representation variety containing only discrete and faithful representations. In the case of a surface with punctures, Goldman's rigidity theorem does not hold. Several counter examples exist in the literature, such as [3, 11, 12].

The purpose of this work is to study a special kind of deformations of $\mathbb{R}$-Fuchsian representations of fundamental groups of surfaces with cusps.

1. We first describe a new family of deformations obtained by a bending process. These representations arise as holonomies of equivariant mappings from the Farey set of a surface Σ to the boundary of $\mathbf{H}^2_{\mathbb{C}}$. We see the Farey set as the set of vertices of a lift to $\tilde{\Sigma}$ of an ideal triangulation T of Σ and consider applications mapping the triangles of $\tilde{T}$ to ideal triangles contained in a real plane. Roughly speaking, we are bending along the edges of an ideal triangulation of Σ.

2. Identify within this family some discrete, faithful, and type-preserving representations. The proof of discreteness is done by showing the discontinuity of the action of $\rho(\pi_1)$ on $\mathbf{H}^2_{\mathbb{C}}$. We obtain in turn a 1-parameter family of embeddings of the Teichmüller space of Σ into the PU(2,1)-representation variety of Σ.

In the frame of $\mathbf{H}^2_{\mathbb{C}}$, the bending method was first used by Apanasov in [1]. In [18], Platis has described the complex hyperbolic version of Thurston's *quakebending deformations* for deformations of $\mathbb{R}$-fuchsian representations of groups case of compact surfaces. He shows that if ρ_0 is an $\mathbb{R}$-Fuchsian representation of $\pi_1(\Sigma)$, Λ is a finite geodesic lamination with a complex transverse measure μ, then there exists $\epsilon > 0$ such that any quakebend deformation $\rho_{t\mu}$ of ρ_0 is complex hyperbolic quasi-Fuchsian for all $t < \epsilon$. The proof of discreteness in [18] rests on the main result in [17] where the proof of discreteness is done by building a fundamental domain. Our goal here is to obtain a complete and explicit parametrization of representations obtained by bending along the edges of an ideal triangulation. Let us now give some explanations about our method.

In [8], Fock and Goncharov described a coordinate system on the representation variety of $\pi_1(\Sigma)$ in G, where G is a real semi-simple split Lie group. When G is equal to PSL(2,$\mathbb{R}$), they identify the Teichmüller space of Σ within the representation variety, and their coordinates are very similar to Thurston's shear coordinates in this case. Similarly, when $G =$PSL(3,$\mathbb{R}$), they identify the moduli space of real convex projective structures on Σ. These two cases have been our main source of inspiration, and are separatly exposed in [6, 7]. We are using Fock and Goncharov's coordinate on the Teichmüller space to describe our bending deformations.

Throughout this work, we will denote by Σ a oriented surface of genus g with $n > 0$ deleted points, which we denote by $x_1, \cdots x_n$. We will assume that Σ has negative Euler characteristic, that is, $2-2g-n<0$. We will denote by $\pi_1(\Sigma)$ or π_1 the fundamental group of Σ. It admits the following presentation

$$\pi_1(\Sigma) \sim \langle a_1, b_2, \cdots a_g, b_g, c_1 \cdots c_n | \prod [a_i, b_i] \prod c_j = 1 \rangle,$$

where the c_j's are homotopy classes of loops enclosing the punctures of Σ. The universal covering of Σ is an open disc $\tilde{\Sigma}$, with a π_1-invariant family of points on its boundary corresponding to the deleted points of Σ. This set of boundary points is called the *Farey set* of Σ, and denoted by $\mathcal{F}_\infty$. We will denote by $\mathcal{DF}(\Sigma)$ the set of PSL(2,$\mathbb{R}$)-classes of discrete and faithful representations of $\pi_1(\Sigma)$ in PSL(2,$\mathbb{R}$), and by $\mathcal{T}(\Sigma)$ the Teichmüller space of Σ. $\mathcal{T}(\Sigma)$ might be seen as the subset of $\mathcal{DF}(\Sigma)$ consisting of the classes of type-preserving representations, that is, representations mapping the c_j's to parabolic isometries. If T is an ideal triangulation of Σ, a decoration of T is a mapping $\mathtt{d} : e(T) \longrightarrow \mathbb{R}$, where $e(T)$ is the set of unoriented edges of T. Such a decoration is called *positive* whenever $\mathtt{d}(e) > 0$ for all edge e.

An $\mathbf{H}^1_{\mathbb{C}}$-realization of $\mathcal{F}_\infty$ is a pair (ϕ, ρ), where ϕ is an application $\mathcal{F}_\infty \longrightarrow \partial \mathbf{H}^1_{\mathbb{C}}$, ρ is a discrete and faithful representation $\pi_1(\Sigma) \longrightarrow$PSL(2,$\mathbb{R}$), and ϕ is $(\rho, \pi_1(\Sigma))$-equivariant, that is, $\phi(\gamma \cdot m) = \rho(\gamma)\phi(m)$ for all $\gamma \in \pi_1(\Sigma)$ and $m \in \mathcal{F}_\infty$. The group PSL(2,$\mathbb{R}$) acts on the set of $\mathbf{H}^1_{\mathbb{C}}$-realizations of $\mathcal{F}_\infty$ by

$$g \cdot (\phi, \rho) = (g \circ \phi, g\rho g^{-1}), \tag{1}$$

and we will denote by $\mathcal{DF}^+$ the set PSL(2,$\mathbb{R}$)-classes of $\mathbf{H}^1_{\mathbb{C}}$ realizations of $\mathcal{F}_\infty$. The set $\mathcal{DF}^+$ is a $2^n : 1$ ramified cover of of $\mathcal{DF}$, with ramification locus the set of classes of representations ρ such that $\rho(c_i)$ is parabolic for at least one index i. The starting point of our work is the following theorem due to Fock and Goncharov.

Theorem 1. *Let T be an ideal triangulation of Σ. There is a canonical bijection between $\mathcal{DF}^+$ and the set of positive decorations of T.*

The proof of this result goes as follows. To any positively decorated triangulation, it is possible to associate a unique class of $\mathbf{H}^1_{\mathbb{C}}$-realization (ϕ, ρ). This is done by interpreting the positive numbers attached to edges of T as cross-ratios and use it to develop Σ in $\mathbf{H}^1_{\mathbb{C}}$. The mapping ϕ appears as the develloping map, and ρ as the associated holonomy representation of $\pi_1(\Sigma)$. Conversely it is possible to reconstruct the decoration $\mathtt{d}$ from the data (ϕ, ρ). We give a detailled treatment of this material in section 4. The interesting point for us is the fact that the image by a representation ρ associated to a positively decorated triangulation of a class of loop γ is given explicitely as a product of elementary isometries which play the role of transition maps between the charts of $\tilde{\Sigma}$. These elementary isometries are the following. First, an elliptic element of order three cyclically permuting the three point ∞, -1 and 0 in the upper half-plane model of $\mathbf{H}^2_{\mathbb{C}}$. Second, a one parameter family of involutions $(I_x)_{x \in \mathbb{R}_{>0}}$ characterized by the conditions

$$I_x(\infty) = 0, I_x(-1) = x \text{ and } I_x^2 = \mathrm{Id}$$

The class of examples we are interested in is obtained by following this process in the frame of $\mathbf{H}^2_{\mathbb{C}}$. It is possible to give a complete description of the PU(2,1)-representation variety of $\pi_1(\Sigma)$ by a construction similar to Fock and Goncharov's one (see [14]). However, the complexity of the coordinates obtained in this way makes it difficult to identify representations which are likely to be discrete. Therefore, we restrict ourselves to a family of special cases, obtained by doing an additional geometric assumption. More precisely, our first goal is to classify what we call *T-bended realizations* of $\mathcal{F}_\infty$ in $\mathbf{H}^2_{\mathbb{C}}$, that is, pairs (ϕ, ρ), where

- ρ is a representation $\pi_1(\Sigma) \longrightarrow \mathrm{Isom}(\mathbf{H}^2_{\mathbb{C}})$,
- $\phi : \mathcal{F}_\infty \longrightarrow \partial\mathbf{H}^2_{\mathbb{C}}$ is a (π_1, ρ)-equivariant mapping,
- for any face Δ of $\hat{T}$ with vertices $a, b, c \in \mathcal{F}_\infty$, the three points $\phi(a)$, $\phi(b)$ and $\phi(c)$ form a real ideal triangle of $\mathbf{H}^2_{\mathbb{C}}$, that is, they belong to the boundary of a real plane of $\mathbf{H}^2_{\mathbb{C}}$.

In the frame of PSL(2,$\mathbb{R}$), the parameter x decorating edges of an ideal triangulation is a cross-ratio: the unique invariant of a pair of ideal triangles of $\mathbf{H}^1_{\mathbb{C}}$ sharing an edge. To parametrize the isometry classes of T-bended realizations of $\mathcal{F}_\infty$ associated to T, we will decorate the edges of T using the invariant of a pair of oriented real ideal triangles (Δ_1, Δ_2) in $\mathbf{H}^2_{\mathbb{C}}$, which is a complex number $z \in \mathbb{C} \setminus \{-1, 0\}$ denoted by $\mathtt{Z}(\Delta_1, \Delta_2)$. The parameter z is similar to the Koranyi-Reimann cross-ratio on the Heisenberg group (see [10, 13] and remark 6 in section 3.2), and is actually the same parameter used by Falbel in [2] to glue ideal tetrahedra in $\mathbf{H}^2_{\mathbb{C}}$. Note that Falbel needs two z-parameters to describe the gluing of tetrahedra. We only need one such parameter since we only consider pairs of real ideal triangles, which correspond in his terminology to symmetric tetrahedra (see section 4.3 of [2]).

The modulus of z is similar to the cross-ratio in $\mathbf{H}^1_{\mathbb{C}}$, and its argument is the *bending parameter*, which might be seen as the measure of an angle. In particular, if z is real, the two adjacent real ideal triangles are contained in a common real plane. We will therefore call a *bending decoration* of T any application $\mathtt{D} : e(T) \longrightarrow \mathbb{C} \setminus \{-1, 0\}$.

As in the case of PSL(2,$\mathbb{R}$), it is possible to associate T-bended realizations to bending decorations. We obtain again an explicit expression for the images of classes of loops by ρ as products of elementary isometries. This time, one of the elementary isometries is again an elliptic element of order 3, cyclically permuting the three points ∞, $[-1, 0]$ and $[0, 0]$ of $\partial\mathbf{H}^2_{\mathbb{C}}$ seen as the one point compactification of the 3-dimensional Heisenberg group $\mathbb{C} \times \mathbb{R}$. The other kind of elementary isometry is a family of involutions $(\sigma_z)_{z \in \mathbb{C} \setminus \{-1,0\}}$, where σ_z is characterized by the conditions

$$\sigma_z(\infty) = [0, 0], \sigma_z([-1, 0]) = [z, 0] \text{ and } \sigma_z^2 = \mathrm{Id}.$$

It turns out that σ_z is antiholomorphic. This is related to the fact that $\mathtt{Z}$ classifies ordered pairs of triangles (in particular, $\mathtt{Z}(\Delta_1, \Delta_2) = \overline{\mathtt{Z}(\Delta_2, \Delta_1)}$, see section 3.2). Therefore a product of elementary isometries is not always holomorphic. This is why the representation ρ is taken in $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{C}})$ rather than in PU(2,1), which is the index two subgroup of $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{C}})$ containing holomorphic isometries. However, for some special triangulations, the representation is actually in PU(2,1). Namely, we show in section 5.3, that the representation ρ associated to a T-bended realization of $\mathcal{F}_\infty$ is holomorphic if and only if T is *bipartite*, that is if its dual graph is bipartite . Now, any cusped surface Σ admits a bipartite ideal triangulation (this is proposition 15). This bending process produces thus representations of $\pi_1(\Sigma)$ in PU(2,1) for any genus and number of punctures of Σ.

Let $\mathcal{BD}_T$ be the set of bending decorations of an ideal triangulation T and $\mathcal{BR}_T^*$ the quotient of $\mathcal{BD}_T$ by the action of complex conjugation. The first result of our work is the following.

The bending theorem

Theorem 2. *There is a bijection between $\mathcal{BD}_T^*$ and $\mathcal{BR}_T$.*

We will see that to any bending decoration is naturally associated a unique pair (r_1, r_2) of PU(2,1)-classes of bended realizations of $\mathcal{F}_\infty$ which represent the same Isom($\mathbf{H}^2_{\mathbb{C}}$)-class of realization. The complex conjugation of bending representations corresponds to the permutation $(r_1, r_2) \longrightarrow (r_2, r_1)$.

After having classified T-bended realizations, we focus on a special kind: those T-bended realizations corresponding to *regular* bending decorations. A bending decoration is said to be *regular* if it has constant argument, that is, if it might be written $\mathtt{D} = \mathtt{d}e^{i\theta}$, where $\mathtt{d}$ is a positive decoration of T, and $\theta \in]-\pi, \pi]$ is a fixed real number. When $\theta = 0$, we obtain T-bended realizations where all the images of the points of $\mathcal{F}_\infty$ are contained in a real plan. The corresponding representations are $\mathbb{R}$-fuchsian. For any positive decoration $\mathtt{d}$ of T and any $\theta \in]-\pi, \pi]$, $\mathtt{d}e^{i\theta}$ is the regular bending decoration of T associating to any edge $e \in e(T)$ the complex number $\mathtt{d}(e)e^{i\theta}$.

The discreteness theorem

Theorem 3. *Let T be a bipartite ideal triangulation of Σ, and $\theta \in]-\pi, \pi[$ be a real number. For any positive decoration $\mathtt{d}$ of T, let $\rho_{\mathtt{d}}$ be a representative of the unique PSL(2,$\mathbb{R}$)-class of representation $\pi_1(\Sigma) \longrightarrow$PSL(2,$\mathbb{R}$) associated with $\mathtt{d}$, and $\rho^\theta_{\mathtt{d}}$ be a representative of the unique Isom($\mathbf{H}^2_{\mathbb{C}}$)-class of representation $\pi_1(\Sigma) \longrightarrow$PU(2,1) associated to $\mathtt{d}^\theta$. Then*

1. *For any index i, $\rho^\theta_{\mathtt{d}}(c_i)$ is loxodromic (resp. parabolic) if and only if $\rho_{\mathtt{d}}(c_i)$ is hyperbolic (resp. parabolic).*
2. *The representation $\rho^\theta_{\mathtt{d}}$ do not preserve any totally geodesic subspace of $\mathbf{H}^2_{\mathbb{C}}$ unless $\theta = 0$ in which case it is $\mathbb{R}$-fuchsian.*
3. *As long as $\theta \in [\pi/2, \pi/2]$, the representation $\rho^\theta_{\mathtt{d}}$ is discrete and faithful.*

The two theorems 2 and 3 are generalizations to the case of any punctured surface of results we had obtained in [21] in the case of the punctured torus.

In particular, when we restrict to those positive decorations $\mathtt{d}$ such that the associated representations $\rho_{\mathtt{d}}$ are type-preserving, we obtain a 1-parameter family parametrized by $\theta \in [-\pi/2, \pi/2]$ of embeddings of the Teichmüller space of Σ in the PU(2,1)-representation variety $\pi_1(\Sigma)$ of which images contain only classes of discrete, faithful and type-preserving representations.

The proof of theorem 3 goes as follows. The bended realization associated to the bending decoration $\mathtt{d}^\theta$ provides a family F of real ideal triangles in $\mathbf{H}^2_{\mathbb{C}}$. If Δ and Δ' are two adjacent triangles within this family, we define a canonical real hypersurface of $\mathbf{H}^2_{\mathbb{C}}$ called the *splitting surface* of Δ and Δ' and denoted by $\mathrm{Spl}(\Delta, \Delta')$. The main technical point is to show that if Δ is any triangle in F, and $\Delta_i, i = 1, 2, 3$ are its neighbours, the three associated splitting surfaces are mutually disjoint in $\mathbf{H}^2_{\mathbb{C}}$. More precisely, denoting by p_i the unique vertex of Δ

which is also a vertex of Δ_{i+1} and Δ_{i+2}, then the intersection of the closures of $\mathrm{Spl}(\Delta, \Delta_{i+1})$ and $\mathrm{Spl}(\Delta, \Delta_{i+2})$ is $\{p_i\}$. This is theorem 6. This defines a prism $\mathfrak{p}_\Delta$ associated to Δ. If Δ and Δ' are two adjacent triangles, the two prisms $\mathfrak{p}_\Delta$ and $\mathfrak{p}_{\Delta'}$ intersect along the splitting surface $\mathrm{Spl}(\Delta, \Delta')$. This is sufficient in order to show that the action of $\rho(\pi_1)$ is discontinuous on the union of all the $\mathfrak{p}_\Delta$'s, and therefore $\rho(\pi_1)$ is discrete and faithful. Splitting surfaces are examples of what we call *spinal* $\mathbb{R}$*-surfaces*, which are the inverse images of geodesics by the orthogonal projection on real planes (see section 6.2). This terminology refers to spinal surfaces, defined by Mostow in [15], which are the inverse images of geodesic by the projection on a complex line (spinal surfaces are often called *bisectors*, see [10]). Spinal $\mathbb{R}$-surfaces appear in [21] under the name of $\mathbb{R}$-balls. They were used in a generalized form by Parker and Platis in [17] under the name of *packs*. In their terminology, spinal $\mathbb{R}$-surfaces are *flat packs*. In particular, the characterization of spinal $\mathbb{R}$-surfaces given in the lemma 7 is similar to their definition of packs.

To put our results in perspective, let us sum up a few known facts about the PU(2,1)-representation variety of $\pi_1(\Sigma)$.

- The representation variety $\mathfrak{R}$ =Hom($\pi_1(\Sigma)$,PU(2,1))/PU(2,1) has real dimension $16g-16+8n$. The subset $\mathfrak{R}^{\mathrm{par}}$ of $\mathfrak{R}$ contining the classes of type preserving representations has real dimension $16g-16+7n$. If Σ has genus g and n punctures, any ideal triangulation of Σ has $6g-6+3p$ edges. Therefore the dimension of $\mathcal{DF}^+$ is $6g-6+3n$, and $\mathcal{DF}^+$ might be seen as $\mathbb{R}_{>0}^{6g-6+3n}$. The subset corresponding to type preserving representation, which is in fact the Teichmüller space of Σ is a real subvariety of dimension $6g-6+2n$,
- If T is an ideal triangulation of Σ, $\mathcal{BR}_T$ and $\mathcal{BR}_T^*$ have real dimension $12g-12+6n$, and may be seen as $\left(\mathbb{C}\setminus\{-1,0\}\right)^{6g-6+3n}$. The real dimension of the family of the classes of discrete and faithful representations obtained by examining regular bending decorations of T is $6g-6+3n+1$ and falls to $6g-6+2n+1$ if we add the condition of type-preservation.
- An important tool in the study of representations of surface groups in PU(2,1) is the Toledo invariant τ. It is defined for representations of fundamental groups of compact surfaces, and for type preserving representations of surfaces with deleted points. If Σ is closed, τ classifies the connected components of Hom($\pi_1(\Sigma)$,PU(2,1))/PU(2,1). This is not true any more when Σ is not closed (see [12]). All the discrete, faithful and type-preserving representations we obtain have vanishing Toledo invariant.

Our work is organised as follows. We provide in section 2 the necessary background about the complex hyperbolic plane and its isometries. The invariant of a pair of real ideal triangles is described in section 3. In section 4, we expose Fock and Goncharov's coordinates on $\mathcal{DF}^+$, and give a complete proof of theorem 1. Section 5 is devoted to the classification of the T-bended realizations of $\mathcal{F}_\infty$. We prove there theorem 2 in section 5.1, provide an explicit form of the corresponding representations in section 5.2. The characterization of bended realization giving representations on PU(2,1) in terms of bipartite triangulations is given in 5.3, and we study the holonomy of loops around deleted points in 5.4. We turn then to the proof of theorem 6. We define spinal $\mathbb{R}$-surfaces and splitting surfaces in 6.2, and prove the discreteness part of the theorem in 6.3. Section 7 is devoted to some remarks and comments. In particular we draw the connection between our work and the previously known families of examples studied in [3, 12, 21].

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2 The hyperbolic 2-space

2.1 $\mathbf{H}^2_{\mathbb{C}}$ and its isometries

Let $\mathbb{C}^{2,1}$ denote the vector space $\mathbb{C}^3$ equipped with the Hermitian form of signature (2,1) given by the matrix

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{2}$$

The hermitian product of two vectors X and Y is given by $\langle X, Y\rangle = X^T J\bar{Y}$, where X^T denotes the transposed of X. We denote by V^- (resp. V^0) the negative (resp. null) cone associated to the hermitian form.

Definition 1. The complex hyperbolic 2-space $\mathbf{H}^2_{\mathbb{C}}$ is the projectivization of V^- equipped with the distance function d given by

$$\cosh^2 d\left(\frac{m,n}{2}\right) = \frac{\langle \mathbf{m},\mathbf{n}\rangle\langle \mathbf{n},\mathbf{m}\rangle}{\langle \mathbf{m},\mathbf{m}\rangle\langle \mathbf{n},\mathbf{n}\rangle}. \tag{3}$$

Proposition 1. *The isometry group of $\mathbf{H}^2_{\mathbb{C}}$ is generated by PU(2,1), the projective unitary group associated to J and the complex conjugation.*

The group PU(2,1) is the group of holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$, and is the neutral component of Isom($\mathbf{H}^2_{\mathbb{C}}$). The other component contains the antiholomorphic isometries, all of which may be written in the form $\phi\circ\sigma$, where ϕ is a holomorphic isometry and σ is the complex conjugation.

Horospherical coordinates The complex hyperbolic 2-space is biholomorphic to the unit ball of $\mathbb{C}^2$, and its boundary is diffeomorphic to the 3-sphere S^3. The projective model of $\mathbf{H}^2_{\mathbb{C}}$ associated to the matrix J given by (2) is often referred to as the *Siegel model* of $\mathbf{H}^2_{\mathbb{C}}$. In this model, any point m of $\mathbf{H}^2_{\mathbb{C}}$ admits a unique lift to $\mathbb{C}^3$ given by

$$\mathbf{m} = \begin{bmatrix} -|z|^2 - u + it \\ z\sqrt{2} \\ 1 \end{bmatrix}, \text{ with } z\in\mathbb{C},\ t\in\mathbb{R} \text{ and } u>0. \tag{4}$$

The boundary of $\mathbf{H}^2_{\mathbb{C}}$ corresponds to those vectors for which u vanishes together with the point of $\mathbb{C}\mathbf{P}^2$ corresponding to the vector $\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$. It might thus be seen as the one point compactification of $\mathbb{R}^3$.

The triple (z, t, u) given by (4) is called the *horospherical coordinates* of m (the hypersurfaces $\{u = u_0\}$ are the horospheres centered at the point ∞ of $\partial\mathbf{H}^2_{\mathbb{C}}$, which corresponds to the vector $\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$).

The boundary of $\mathbf{H}^2_{\mathbb{C}}$ has naturally the structure of the 3-dimensional Heisenberg group, seen as the maximal unipotent subgroup of PU(2,1) fixing ∞. We will call *Heisenberg coordinates* of the boundary point with horospherical coordinates $(z, t, 0)$ the pair $[z, t]$. In these coordinates the group structure is given by

$$[z, t] \cdot [w, s] = [z + w, s + t + 2\mathrm{Im}\,(z\bar{w})]$$

The ball model of $\mathbf{H}^2_{\mathbb{C}}$ The same contruction can be done using a different hermitian form of signature $(2, 1)$ on $\mathbb{C}^3$. Using the special form associated to the matrix $J_0 = \mathrm{diag}(1, 1, -1)$, we would obtain the so-called *ball model* of the complex hyperbolic 2-space, which lead to a description of $\mathbf{H}^2_{\mathbb{C}}$ as the unit ball $\mathbb{C}^2$.

2.2 Totally geodesic subspaces

The maximal totally geodesic subspaces of $\mathbf{H}^2_{\mathbb{C}}$ have real dimension 2. There are two type of such subspaces: the *complex lines*, and the *real planes*.

The complex lines. These subspaces are the images under projectivization of those complex planes of $\mathbb{C}^3$ intersecting the negative cone V^-. The standard example is the subset C_0 of $\mathbf{H}^2_{\mathbb{C}}$ containing points of horospherical coordinates $(0, t, u)$ with $t \in \mathbb{R}$ and $u > 0$. This is an embedded copy of the usual Poincaré upper half-plane. We will refer to this particular complex line as $\mathbf{H}^1_{\mathbb{C}} \subset \mathbf{H}^2_{\mathbb{C}}$. All the other complex lines are the images of $\mathbf{H}^1_{\mathbb{C}}$ by an element of PU(2,1). Note that any complex line C is fixed pointwise by a unique holomorphic involutive isometry, called the *complex symmetry about C*.

The real planes. These subspaces are the images of the Lagrangian vector subspaces of $\mathbb{C}^{2,1}$ under projectivization. The standard example is the subset containing points of horospherical coordinates $(x, 0, u)$ with $x \in \mathbb{R}$ and $u > 0$. Again, this is an embedded copy of the usual Poincaré upper half-plane and we will refer to this particular real plane as $\mathbf{H}^2_{\mathbb{R}} \subset \mathbf{H}^2_{\mathbb{C}}$. All other real planes are images of the standard one by an element of PU(2,1). There is also a unique involution fixing pointwise a real plane R which is called the *real symmetry about R*. It is antiholomorphic, and, in the case of $\mathbf{H}^2_{\mathbb{R}}$, is the complex conjugation. If σ is a real symmetry, we will call the real plane which is its fixed point set its *mirror*.

Remark 1. In the ball model, the standard complex line is the first axis of coordinates $\{(z, 0), |z| < 1\}$. The standard real plane $\mathbf{H}^2_{\mathbb{R}}$ is again the set of points with real coordinates $\{(x_1, x_2), x_1^2 + x_2^2 < 1\}$.

Computing with real symmetries The following proposition is of great use to work with real symmetries.

Proposition 2. *Let Q be an $\mathbb{R}$-plane, and σ_Q be the symmetry about Q. There exists a matrix $M_Q \in \mathrm{SU}(2,1)$ such that*

$$M_Q\overline{M_Q} = 1 \text{ and } \sigma_Q(m) = \mathbf{P}(M_Q.\bar{\mathbf{m}}) \text{ for any } m \in \mathbf{H}^2_{\mathbb{C}} \text{ with lift } \mathbf{m}. \tag{5}$$

Proof. In the special case where $Q = \mathbf{H}^2_{\mathbb{R}}$, the identity matrix satisfy these conditions. In general, let $\mathbf{Q}$ be a lift to $\mathbb{C}^{2,1}$ of Q, and A be a matrix of SU(2,1) mapping $\mathbb{R}^3$ to $\mathbf{Q}$. The matrix $A\bar{A}^{-1}$ satisfies the above conditions. □

Remark 2. Let σ_1 and σ_2 be real symmetries, with lifts M_1 and M_2 given by proposition 2. The product $\sigma_1\sigma_2$ is a holomorphic isometry, and lifts to the matrix $M_1\overline{M}_2$. Similarly, if h is a holomorphic isometry lifting to H, the conjugation $h\sigma_1h^{-1}$ lifts to $HM_1\overline{H^{-1}}$.

The isometry type of the product of two real symmetries is directly related to the relative position of their mirrors. The following lemma is due to Falbel and Zocca in [5] (see the next section for information about the different isometry types).

Lemma 1. *Let P_1 and P_2 be two real planes, with respective symmetries σ_1 and σ_2. Then*

- *The closures in $\mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$ of P_1 and P_2 are disjoint if and only if the isometry $\sigma_1\sigma_2$ is loxodromic.*
- *The intersection of the closures in $\mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$ of P_1 and P_2 contains exactly one boundary point if and only if $\sigma_1\sigma_2$ is parabolic.*
- *The intersection of the closures in $\mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$ of P_1 and P_2 contains one point of $\mathbf{H}^2_{\mathbb{C}}$ if and only if $\sigma_1\sigma_2$ is elliptic.*

2.3 Classification of isometries.

The non-trivial holomorphic isometries are classified in three different types:

1. An element of PU(2,1) is said to be *elliptic* if it has a fixed point inside $\mathbf{H}^2_{\mathbb{C}}$.
2. An element of PU(2,1) is said to be *parabolic* if it has a unique fixed point on $\partial\mathbf{H}^2_{\mathbb{C}}$.
3. An element of PU(2,1) is said to be *loxodromic* if it has exactly two fixed points on $\partial\mathbf{H}^2_{\mathbb{C}}$.

This exhausts all possibilities. Note that there is still a small ambigouity among elliptic elements. An elliptic isometry will be called a *complex reflection* if one of its lift has two equal eigenvalues, else, it will be said to be *regular elliptic*. As in the case of PSL(2,$\mathbb{R}$), there is an algebraic criterion to determine the type of an isometry according to the trace of one of its lifts to SU(2,1).An element of PU(2,1) admits three lifts to SU(2,1) which are obtained one from another by multiplication by a cube root of 1. Therefore its trace is well-defined up to multiplication by a cube root of 1.

Proposition 3. *Let f be the polynomial given by $f(z) = |z|^4 - 8Re(z^3) + 18|z|^2 - 27$, and h be a holomorphic isometry of $\mathbf{H}^2_{\mathbb{C}}$.*

- *The isometry h is loxodromic if and only $f(\operatorname{tr} h)$ is positive.*

- *The isometry h is regular elliptic if and only $f(\operatorname{tr} h)$ is negative.*

- *If $f(h) = 0$, then h is either parabolic or a complex reflection.*

Proof. Note that f is invariant under multiplication of z by a cube root of 1. The polynomial f is actually the resultant of χ and χ', where χ is the characteristic polynomial of a lift of h to SU(2,1). See [10] (chapter 6) for details. $\square$

Remark 3. The function f of proposition 3 may be written in terms or real coordinates as

$$f(x+iy) = y^4 + y^2\left(x+6-3\sqrt{3}\right)\left(x+6+3\sqrt{2}\right) + (x+1)(x-3)^3,$$

with $x, y \in \mathbb{R}$. It is then a direct consequence that if $\operatorname{Re}(\operatorname{tr}(h)) > 3$, then h is loxodromic.

Loxodromic isometries We give now more informations about loxodromic isometries.

Proposition 4. *Let $h \in PU(2,1)$ be a loxodomic isometry. Then h is conjugate in PU(2,1) to an isometry given by the matrix in SU(2,1)*

$$\mathbf{D}_\lambda = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \bar{\lambda}/\lambda & 0 \\ 0 & 0 & 1/\bar{\lambda} \end{bmatrix} \text{ with } \lambda \in \mathbb{C}, |\lambda| \neq 1. \tag{6}$$

Proof. Since PU(2,1) acts doubly transitively on the boundary of $\mathbf{H}^2_{\mathbb{C}}$, the isometry h is conjugate to a loxodromic isometry fixing the two points ∞, and $[0,0]$. The generic form of the lift of such an isometry is $\mathbf{D}_\lambda$. $\square$

The family $\{\mathbf{D}_t, t > 0\}$ defines a 1-parameter subgroup of PU(2,1) containing only matrices with real trace greater or equal to 3.

Definition 2. Let γ be a geodesic in $\mathbf{H}^2_{\mathbb{C}}$, and g_γ be an isometry mapping the geodesic γ to the geodesic connecting ∞ and $[0,0]$. We denote by R_γ the 1-parameter subgroup of PU(2,1) given by $g_\gamma^{-1}\{\mathbf{D}_t, t > 0\}g_\gamma$, which does not depend on the choice of g_γ.

Remark 4. Let us give another characterization of R_γ. An isometry A belongs to R_γ if and only if for any real plane P containing γ, A preserves P and the two connected components of $P \setminus \gamma$. Indeed, we may normalize the situation in such a way that P is $\mathbf{H}^2_{\mathbb{R}}$ and γ is the geodesic connecting the two points with Heisenberg coordinates $[0,0]$ and ∞, in which case $R_\gamma = (\mathbf{D}_t)_{t>0}$. The two connected components of $\mathbf{H}^2_{\mathbb{R}} \setminus \gamma$ are C^+ and C^-, where, in horospherical coordinates, $C^+ = \{(x,0,u), x > 0 \text{ and } u > 0\}$ and $C^- = \{(x,0,u), x < 0 \text{ and } u > 0\}$. Now, any isometry preserving $\mathbf{H}^2_{\mathbb{R}}$, and fixing both $[0,0]$ and ∞ lifts to SU(2,1) as the diagonal matrix $\operatorname{diag}(t, 1, 1/t)$ with real t. It is then a straightforward computation to check that such an isometry preserves the connected components C^+ and C^- if and only if t is positive, that is, if it belongs to $(\mathbf{D}_t)_{t>0}$.

Parabolic isometries. There are two main types of parabolic isometries.

1. Heisenberg translations are unipotent parabolics. They are conjugate to isometries associated to the matrices SU(2,1)

$$T_{[z,t]} = \begin{bmatrix} 1 & -\bar{z}\sqrt{2} & -|z|^2 + it \\ 0 & 1 & z\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}, \text{ with } z \in \mathbb{C}, t \in \mathbb{R}.$$

   There are two PU(2,1) conjugacy classes of Heisenberg translations. The first one contains *vertical* translations, which correspond to $z = 0$. In this case $T_{[0,t]} - Id$ is nilpotent of order 2. These isometries preserve a complex line. The parabolic elements $T_{[0,t]}$ preserves the complex line $\mathbf{H}^1_{\mathbb{C}}$. The second conjugacy class is when $z \neq 0$, in which case $T_{[z,t]} - Id$ is nilpotent of order 3. These isometries preserve a real plane, which is $\mathbf{H}^2_{\mathbb{R}}$ in the case where z is real and t vanishes.

2. The other type of parabolic isometries are the *screw-parabolic* isometry. They are conjugate to a product $h \circ r$, where h is a vertical Heisenberg translation and r a complex reflection about the invariant complex line of h.

3 Real ideal triangles.

3.1 Ideal triangles

An ideal triangle is an oriented triple of boundary points of $\mathbf{H}^2_{\mathbb{C}}$.

Definition 3. Let (p_1, p_2, p_3) be an ideal triangle. The quantity

$$\mathbb{A}\,(p_1, p_2, p_3) = -\arg\left(\langle \mathbf{p}_1, \mathbf{p}_2 \rangle \langle \mathbf{p}_2, \mathbf{p}_3 \rangle \langle \mathbf{p}_3, \mathbf{p}_1 \rangle\right)$$

does not depends on the choice of the lifts of the p_i's, and is called the Cartan invariant of the ideal triangle (p_1, p_2, p_3).

The Cartan invariant classifies the ideal triangles, as stated in the following proposition (see [10] chapter 7 for a proof).

Proposition 5. *The Cartan invariant enjoys the following properties*

1. *Two ideal triangles are identified by an element of PU(2,1) if and only if they have the same Cartan invariant.*

2. *Two ideal triangles are identified by a antiholomorphic isometry of $\mathbf{H}^2_{\mathbb{C}}$ if and only if they have opposite Cartan invariants.*

3. *An ideal triangle has Cartan invariant $\pm\pi/2$ (resp. 0) if and only if it is contained in a complex line (resp. a real plane).*

Definition 4. We will call *real ideal triangle* any ideal triangle contained in a real plane. Since the three points are contained in a real plane we will as well refer to the 2-simplex determined by three points on the boundary of a real plane as a real ideal triangle.

Remark 5. Up to isometry, there is a unique real ideal triangle, as shown by proposition 5. More precisely, if Δ and Δ' are two real ideal triangles, there are exaclty two isometries mapping Δ to Δ'. One is holomorphic and the other antiholomorphic.

3.2 The invariant of a pair of adjacent ideal real triangles.

We say that two real ideal triangles are *adjacent* if they have a common edge. All the pairs of real ideal triangles we consider are **ordered**.

Lemma 2. *Let Δ_1 and Δ_2 be two adjacent ideal real triangles, sharing a geodesic γ as an edge. There exists a unique complex number $\mathbb{C} \setminus \{-1, 0\}$ such that the ordered pair of real triangles (Δ_1, Δ_2) is PU(2,1)-equivalent to the ordered pair of ideal real triangles (Δ_0, Δ_z) given by the Heisenberg coordinates of its vertices by*

$$\Delta_0 = (\infty, [-1, 0], [0, 0]) \text{ and } \Delta_z = (\infty, [0, 0], [z, 0]) \tag{7}$$

Proof. As shown by proposition 5 and remark 5, there exists a unique holomorphic isometry h mapping Δ_1 to Δ_0 and γ to the geodesic connecting ∞ to $[0, 0]$. The isometry h maps the triangle Δ_2 to an ideal triangle of which vertices are a priori given in Heisenberg coordinates by ∞, $[0, 0]$ and $[z, t]$ with $z \in \mathbb{C}$ and $t \in \mathbb{R}$. The relation $\mathbb{A}(\infty, [0, 0], [z, t]) = 0$ yields $t = 0$. $\square$

Definition 5. Let (Δ_1, Δ_2) be a pair of adjacent real ideal triangles. We will call the complex number z associated to it by lemma 2 the invariant of the pair (Δ_1, Δ_2), and denote it by $\mathtt{Z}(\Delta_1, \Delta_2)$.

Remark 6. It is possible to give another description of the invariant $\mathtt{Z}$ of a pair of ideal triangles. Let p_1, p_2, p_3 and p_4 be four points in $\partial \mathbf{H}^2_{\mathbb{C}}$, such that $\Delta_1 = (p_1, p_2, p_3)$ and $\Delta_2 = (p_3, p_4, p_1)$ are two real ideal triangles. Let C_{13} be the (unique) complex line containing p_1 and p_3. Neither p_2 nor p_3 belong to C_{13}, since the two corresponding ideal triangles are real. The complex line C_{13} lifts to $\mathbb{C}^3$ as a complex plane. Let $\mathbf{c}_{13}$ be a vector in $\mathbb{C}^{2,1}$ Hermitian orthogonal to this complex plane. Then $C_{13} = \mathbf{P}\left(\mathbf{c}_{\mathbf{13}}^{\perp}\right)$. Let $\mathbf{p}_i$ be a lift of p_i for $i = 1, 2, 3, 4$. The invariant $\mathtt{Z}(\Delta_1, \Delta_2)$ is given by

$$\mathtt{Z}(\Delta_1, \Delta_2) = -\frac{\langle \mathbf{p}_4, \mathbf{c}_{13} \rangle \langle \mathbf{p}_2, \mathbf{p}_1 \rangle}{\langle \mathbf{p}_2, \mathbf{c}_{13} \rangle \langle \mathbf{p}_4, \mathbf{p}_1 \rangle} \tag{8}$$

The above quantity does not depend on the various choices of lifts we made. This definition is similar to the one of the complex cross-ratio of Koranyi and Reimann (see [13]). To check that this formula is valid, it is sufficient to check it on the special case $p_1 = \infty$, $p_2 = [-1, 0]$, $p_3 = [0, 0]$ and $p_4 = [z, 0]$. In this case, the choice $\mathbf{c}_{13} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ is convenient. This invariant is the similar to the one used by Falbel in [2], although the form 8 is not used there.

Lemma 3. *Let (Δ_1, Δ_2) be a pair of adjacent real ideal triangles, with $\mathtt{Z}(\Delta_1, \Delta_2) = z$, and f be an antiholomorphic isometry. Then $\mathtt{Z}(f(\Delta_1), f(\Delta_2)) = \bar{z}$.*

Proof. Let σ be the symmetry about the real plane containing Δ. The isometry $f \circ \sigma$ is holomorphic, and therefore preserves the invariant of pairs of adjacent real ideal triangles. As a consequence, it is sufficient to show that $\mathtt{Z}(\sigma(\Delta_1), \sigma(\Delta_2)) = \bar{z}$. We can normalize the situation to the reference pair (Δ_0, Δ_z) given by (7). In this case, the real symmetry σ is just the complex conjugation. It fixes the three points ∞, $[-1,0]$ and $[0,0]$, and maps the point $[z,0]$ to $[\bar{z},0]$. This shows the result. $\square$

Proposition 6. *Let $\Delta_1 = (a,b,c)$ and $\Delta_2 = (a,c,d)$ be two real ideal triangles. There exists a unique real symmetry σ such that $\sigma(a) = c$ and $\sigma(b) = d$.*

Proof. It is sufficient to prove that such a real symmetry exists for the two triangles Δ_0 and Δ_z. More precisely, we have to show that for any $z \in \mathbb{C}$ and, there exists a unique real symmetry σ_z such that

$$\sigma_z([-1,0]) = [z,0] \text{ and } \sigma_z(\infty) = [0,0] \tag{9}$$

If there existed two such symmetries, their product would be a holomorphic isometry having four fixed points on $\partial\mathbf{H}^2_{\mathbb{C}}$, not belonging to the boundary of a complex line. Therefore this product would be the identity. This shows the uniqueness part.
Writing $z = xe^{i\alpha}$, the matrix

$$M_{x,\alpha} = \begin{bmatrix} 0 & 0 & x \\ 0 & e^{i\alpha} & 0 \\ 1/x & 0 & 0 \end{bmatrix} \tag{10}$$

is such that $M_{x,\alpha}\overline{M_{x,\alpha}} = 1$, and the real symmetry associated to it satisfies to (9). This proves the result. $\square$

Definition 6. We call the involution provided by proposition 6 the symetry of the pair (Δ_1, Δ_2) and denote it by $\sigma_{\Delta_1,\Delta_2}$.

Remark 7. As a direct consequence of lemma 3 and proposition 6, we see that for any pair (Δ_1, Δ_2) of adjacent real ideal triangles,

$$\mathsf{Z}(\Delta_1, \Delta_2) = \overline{\mathsf{Z}(\Delta_2, \Delta_1)}.$$

Remark 8. It is a direct consequence of lemma 2 that two real ideal triangles are contained in a common real plane if and only if their invariant is real. To be more precise, let us call γ the geodesic shared by two real ideal triangles Δ_1 and Δ_2 as an edge. Then

- the invariant of (Δ_1, Δ_2) is real and positive if and only if the two triangles are contained in a common real plane P, and belong to opposite connected components of $P \setminus \gamma$.
- the invariant of (Δ_1, Δ_2) is real and negative if and only if the two triangles are contained in a common real plane P, and belong to the same connected component of $P \setminus \gamma$.

We will need the following in section 6.2.

Proposition 7. *Let $\Delta = (a,b,c)$ be an ideal real triangle, and γ the geodesic connecting a and c. Let Δ_1 and Δ_2 be two other real ideal triangles adjacent to Δ along γ. Assume moreover that the invariants of the pairs (Δ, Δ_1) and (Δ, Δ_2) satisfy*

$$\frac{\mathsf{Z}(\Delta, \Delta_1)}{\mathsf{Z}(\Delta, \Delta_2)} \in \mathbb{R}_{>0}.$$

Call d_i the vertex of Δ_i different from a and c, and Q_i the mirror of the real symmetry σ_i given by the proposition 6, such that $\sigma_i(a) = c$ and $\sigma_i(b) = d_i$. Then there exists a unique element $g \in R_\gamma$ such that $g(Q_1) = Q_2$.

Proof. We may normalise the situation so that

$$a = \infty,\ b = [-1, 0],\ c = [0, 0],\ d_1 = [z_1, 0] \text{ and } d_2 = [z_2, 0], \tag{11}$$

where $z_i = \mathtt{Z}(\Delta, \Delta_i)$. In this normalized situation, the two real symmetries associated to d_1 and d_2 are σ_{z_1} and σ_{z_2}, and the one parameter subgroup R_γ corresponds to $(\mathbf{D}_t)_{t>0}$. Using the matrices given above, it is a straightforward computation to check that $M_{z_2} = \mathbf{D}_t M_{z_1} \mathbf{D}_{1/t}$ for t real and positive if and only if $0 < t = z_1/z_2 \in \mathbb{R}_{>0}$. This shows the result. $\square$

4 Fock-Goncharov coordinates on $\mathcal{DF}^+(\Sigma)$

4.1 Notation, definition.

We denote by $\Sigma = \Sigma_g \setminus \{x_1, \cdots x_n\}$ an oriented surface of genus g with n deleted points. We denote by $\pi_1(\Sigma)$ or more simply π_1 the fundamental group of Σ, which is given by the presentation

$$\pi_1 = \langle a_1, b_1, \ldots a_g, b_g, c_1, \ldots c_n \mid \prod_i [a_i, b_i] \prod_j c_j = 1 \rangle,$$

where the c_i's are homotopy classes of loops around the deleted points. Topologically, the universal cover of Σ is an open disk with a familly of boundary points which are all the lifts of the x_i's. This family is called the *Farey set* of Σ, and we will denote it by $\mathcal{F}_\infty$.

Definition 7. We call $\mathcal{DF}$ the set of PSL(2,$\mathbb{R}$)-conjugacy classes of discrete and faithful representations of π_1 in PSL(2,$\mathbb{R}$).

To any class of discrete and faithful representation in $\mathcal{DF}$ is associated a unique class of hyperbolic metric on Σ. The Teichmüller space of Σ corresponds to those classes of metrics on Σ that have finite area. In terms of representations of $\pi_1(\Sigma)$, the Teichmüller space of Σ is given by

$$\mathcal{T}(\Sigma) = \{[\rho] \in \mathcal{DF}, \rho \text{ is type-preserving}\}. \tag{12}$$

Recall that a representation ρ of π_1 in PSL(2,$\mathbb{R}$) is said to be type-preserving when the $\rho(c_i)$'s are all parabolic isometries.

Definition 8. We call $\mathbf{H}^1_{\mathbb{C}}$-realization of $\mathcal{F}_\infty$ any pair (ϕ, ρ) where

- ϕ is an application $\mathcal{F}_\infty \longrightarrow \partial\mathbf{H}^1_{\mathbb{C}}$, ρ is a discrete and faithful representation of π_1 in PSL(2,$\mathbb{R}$),
- ϕ is (π_1, ρ)-equivariant, that is, for any $\gamma \in \pi_1$ and $m \in \mathcal{F}_\infty$, $\phi(\gamma \cdot m) = \rho(\gamma)\phi(m)$.

The group PSL(2,$\mathbb{R}$) acts on the set of $\mathbf{H}^1_{\mathbb{C}}$-realizations of $\mathcal{F}_\infty(\Sigma)$ by

$$g \cdot (\phi, \rho) = (g \circ \phi, g\rho g^{-1}). \tag{13}$$

Definition 9. We will denote by $\mathcal{DF}^+$ the set of PSL(2,$\mathbb{R}$)-classes of $\mathbf{H}^1_{\mathbb{C}}$ realizations of $\mathcal{F}_\infty$ for the action given in (13).

Proposition 8. *$\mathcal{DF}^+$ is a $2^n : 1$ ramified cover of $\mathcal{DF}$. The ramifying locus is the set of class of representations where at least one of the c_i's is mapped to a parabolic isometry.*

Proof. The c_i's are homotopy classes of loops around the deleted point x_i's. For each of the c_i's there exists a unique $\hat{x}_i \in \mathcal{F}_\infty$ such that the action of c_i on the universal cover of Σ fixes $\hat{x}_i$.

Due to the equivariance property, the point $\phi(\hat{x}_i)$ is a fixed point of $\rho(c_i)$: $\phi(c_i \cdot \hat{x}_i) = \rho(c_i)\phi(\hat{x}_i) = \phi(\hat{x}_i)$. Now, if ρ is given and we want to define ϕ, it suffices to define the images of the n points $\hat{x}_1, \ldots, \hat{x}_n$, and to extend it to a mapping defined on $\mathcal{F}_\infty$ using the (π_1, ρ)-equivariance property. If the representation ρ is discrete and faithful, then $\rho(c_i)$ is either hyperbolic or parabolic, and therefore has either exactly two fixed points or a unique fixed point, which lie on the boundary of $\mathbf{H}^1_{\mathbb{C}}$ in both cases.

As a consequence, if $\rho(c_i)$, is parabolic, then $\phi(\hat{x}_i)$ must be the fixed point of $\rho(c_i)$, and if it is hyperbolic, it is one of its two fixed points. Therefore each of the c_i's which is mapped by ρ to a hyperbolic isometry gives rise to a choice of order 2. This shows the result. ☐

Remark 9. We can be more precise about the projection

$$\mathbf{p}_1 : \mathcal{DF}^+ \longrightarrow \mathcal{DF}, \tag{14}$$

given by $[(\phi, \rho)] \longmapsto [\rho]$. Let $[[1, n]]$ be the set of integers comprised between 1 and n. For any subset $I = \{i_1, \cdots, i_k\}$ of $[[1, n]]$, define

$$\mathcal{P}_I = \left\{[\phi, \rho] \in \mathcal{DF}^+ | \rho(c_i) \text{ is parabolic } \Leftrightarrow i \in I\right\}.$$

Then $\mathcal{DF}^+$ decomposes as the disjoint union

$$\mathcal{DF}^+ = \coprod_{I \subset [[1,n]]} \mathcal{P}_I, \tag{15}$$

and the restriction to $\mathcal{P}_I$ of the projection (14) is $2^{n-|I|}$ to 1. In particular, is it 2^n to 1 when restricted to $\mathcal{P}_\emptyset$, which is the set of realisations associated to totally hyperbolic representations, and it a bijection when restricted to $\mathcal{P}_{[[1,n]]}$, which corresponds to the Teichmüller space.

4.2 Coordinates on $\mathcal{DF}^+$: decorated triangulations

Definition 10. A positive decorated triangulation of Σ is pair $(T, \mathtt{d})$, where T is an ideal triangulation of Σ and $\mathtt{d} : e(T) \longrightarrow \mathbb{R}_{>0}$ is an application defined on the set of unoriented edges of T taking real positive values.

Definition 11. Let T be an ideal triangulation of Σ. We will call *modified dual graph* of T and denote by $\Gamma(T)$ the graph obtained from the dual graph of T as follows:

- The vertices of $\Gamma(T)$ are the combinations $1/3x+2/3y$, where x and y are adjacent vertices of the dual graph.

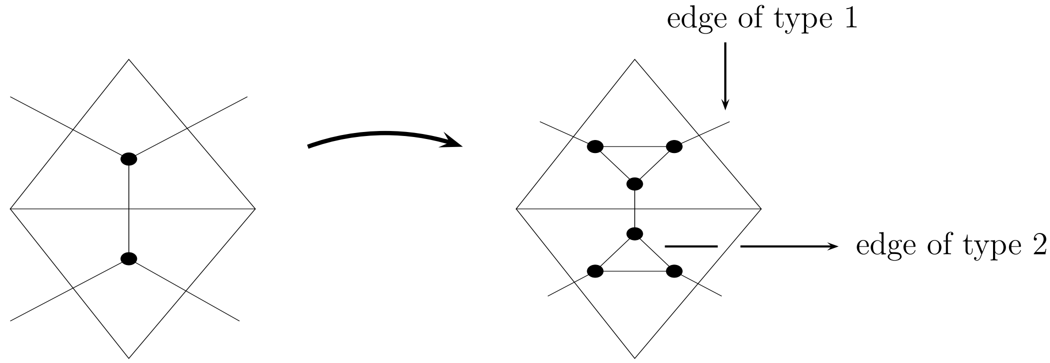


Figure 1: The triangulation, its dual graph and its modified dual graph

- Two vertices v and v' of $\Gamma(T)$ are connected by an edge if and only if there are either of the form $v = 1/3x + 2/3y$ and $v' = 2/3x + 1/3y$ for some adjacent edges of the dual graph, or of the form $v = 1/3x + 2/3y$ and $v' = 1/3z + 2/3y$ where (x, y) and (y, z) are consecutive pairs of adjacent vertices of the dual graph. See figure 1.

We define similarly $\Gamma(\hat{T})$, the modified dual graph of $\hat{T}$, which is the lift of $\Gamma(T)$ to the universal cover of Σ. We will refer to these two modified dual graphs as Γ and $\hat{\Gamma}$ whenever it is clear from the context which triangulation we are dealing with.
The modified dual graph of T or $\hat{T}$ has two different types of edges:

- we will call *edges of type 1* those edges connecting two points of the form $v = 1/3x+2/3y$ and $v' = 1/3x + 2/3y$
- we will call *edges of type 2* those edges connecting two points of the form $v = 1/3x+2/3y$ and $v' = 1/3z + 2/3y$.

The edges of type 1 of Γ intersect the edges of T. The orientation of Σ induces an orientation of $\hat{\Sigma}$ which in turn induces an orientation of the edges of type 2 of the modified dual graph.

Definition 12. Let v be a vertex of $\hat{\Gamma}$ and Δ be the unique face of $\hat{T}$ in which v is contained. The orientation of Σ induces an orientation of the edges of Δ, and we will call a_v the ending vertex of the edge of Δ closest to v. We will then call b_v and c_v the two other vertices of Δ, in such a way that the triple (a_v, b_v, c_v) is positively oriented.

Since three vertices of $\hat{\Gamma}$ are contained in Δ, there are three possible labellings of the vertices of a given Δ.

In the following, we will interpret the positive numbers $\mathtt{d}(e)$ decorating the edges of T as cross-ratios on the boundary of $\mathbf{H}^1_{\mathbb{C}}$. We call r the cross-ratio function on the boundary of $\mathbf{H}^1_{\mathbb{C}}$, which we see as the upper half-plane, defined by

$$r(m_1, m_2, m_3, m_4) = \frac{(m_1 - m_2)(m_3 - m_4)}{(m_1 - m_4)(m_2 - m_3)} \text{ for } m_i \in \mathbb{R}. \tag{16}$$

Remark 10. Note that $r(\infty, -1, 0, x) = x$. This shows that the cross-ratio $r(m_1, m_2, m_3, m_4)$ is positive if and only if the geodesic connecting m_1 and m_3 separates m_2 and m_3.

Definition 13. For any positive real number x, let I_x ∈PSL(2,$\mathbb{R}$) be the isometry given by

$$I_x = \begin{bmatrix} 0 & \sqrt{x} \\ -1/\sqrt{x} & 0 \end{bmatrix}.$$

Define E ∈PSL(2,$\mathbb{R}$) to be the isometry given by

$$E = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Note that I_x is an involution. More precisely, it is a half-turn, that is, a rotation of order 2, which swaps 0 and ∞ on one hand, and -1 and x on the other hand. The isometry E is an order 3 rotation which cyclically permutes the three points ∞, -1 and 0.

Theorem 4 (Fock-Goncharov). *Let T be an ideal triangulation of Σ. There is a bijection between the set of positive decorations of T and $\mathcal{DF}^+$.*

Proof. Let us start with $\mathtt{d}$, a positive decoration of T. We will associate to any vertex v of the modified dual graph $\hat{\Gamma}$ a $\mathbf{H}^1_{\mathbb{C}}$-realisation (ϕ_v, ρ_v) of $\mathcal{F}_\infty(\Sigma)$, obtained by using v as basepoint in the description of $\pi_1(\Sigma)$. We will see a posteriori that the PSL(2,$\mathbb{R}$)-class of the realization is independant from v.

Step 1: Definition of ϕ_v and ρ_v The positive decorated ideal triangulation of Σ lifts to a triangulation of its universal cover of which vertices are the points of $\mathcal{F}_\infty$, and which is positively and π_1-invariantly decorated.
Let us define the mapping ϕ_v. Any vertex v of $\Gamma(\hat{T})$ belongs to a unique face Δ_v of $\hat{T}$, and we label the vertices of Δ_v by a_v, b_v and c_v, as in definition 12.

- First, set $\phi_v(a_v) = \infty$, $\phi_v(b_v) = -1$ and $\phi_v(c_v) = 0$.

- Next, extend ϕ_v recursively to all of $\mathcal{F}_\infty$ as follows. Let $\Delta_1 = (m_1, m_2, m_3)$ and $\Delta_2 = (m_1, m_3, m_4)$ be two (oriented) faces sharing an edge, and assume that $\phi_v(m_i)$ is already defined for $i = 1, 2, 3$. Then $\phi_v(m_4)$ is the unique point z of $\partial\mathbf{H}^1_{\mathbb{C}}$ such that the cross-ratio $r(\phi_v(m_1), \phi_v(m_2), \phi_v(m_3), z) = \mathtt{d}(e(m_1, m_3))$, where $e(m_1, m_3)$ is the edge of T connecting m_2 and m_3.

For any $\gamma \in \pi_1$, these exists a unique $g_{v,\gamma}$ mapping the triangle $(\infty, -1, 0) = \phi_v(a_v, b_v, c_v)$ to the triangle $\phi_v(\gamma \cdot a_v, \gamma \cdot b_v, \gamma \cdot c_v)$. Due to the definition of ϕ_v and the fact that $g_{v,\gamma}$ preserves cross-ratio, it is clear that $\phi_v(\gamma \cdot m) = g_{v,\gamma}\phi_v(m)$ for all $m \in \mathcal{F}_\infty$. Therefore ρ_v is given by $\rho_v(\gamma) = g_{v,\gamma}$.
The fact that ρ_v is discrete is due to the posivity of all the cross ratios involved. This positivity implies that all the triangles are mutually disjoint and therefore, that the action of $\rho(\pi_1)$ is freely discontinuous.

Step 2: The PSL(2,$\mathbb{R}$)-class of (ϕ_v, ρ_v) does not depend on v. It is sufficient to prove that if v and v' are the two vertices of an edge e of $\hat{\Gamma}$, there exists an isometry g_e ∈PSL(2,R)

such that $(\phi_{v'}, \rho_{v'}) = g_e \cdot (\phi_v \rho_v)$. Assume that e is of type 1, and that it intersects an edge of $\hat{T}$ decorated by the positive number x. The two applications ϕ_v and $\phi_{v'}$ are defined by

$$\phi_v : \begin{cases} a_v & \longmapsto & \infty \\ b_v & \longmapsto & -1 \\ c_v & \longmapsto & 0 \end{cases} \text{ and } \phi_v : \begin{cases} a_{v'} = c_v & \longmapsto & \infty \\ b_{v'} & \longmapsto & -1 \\ c_{v'} = a_v & \longmapsto & 0 \end{cases} \tag{17}$$

Moreover, by construction, $\phi_v(b_{v'}) = x = \phi_{v'}(b_v)$. As a consequence, we see that $\phi_{v'} = I_x \circ \phi_v$. It follows that $\rho_{v'} = I_x \rho_v I_x$. By a similar argument, we see that if e is of type 2, then

- $(\phi_v, \rho_v) = E \cdot (\phi_{v'}, \rho_{v'})$ if the orientation of Σ induces the orientation $v \to v'$ on e,
- $(\phi_v, \rho_v) = E^{-1} \cdot (\phi_{v'}, \rho_{v'})$ in the opposite case.

Step 3: Decorating a triangulation from a $\mathbf{H}^1_{\mathbb{C}}$-realization. Conversely, assume we are given a pair (ϕ, ρ), with ρ discrete and faithful and $\phi : \mathcal{F}_\infty(\Sigma) \longrightarrow \partial\mathbf{H}^1_{\mathbb{C}}$ (π_1, ρ)-equivariant. Then we obtain a family of ideal triangles of $\mathbf{H}^1_{\mathbb{C}}$ by connecting $\phi(m)$ and $\phi(n)$ by a geodesic every time m and n are connected by an edge of $\hat{T}$. Since the representation is discrete and faithful, two disjoint triangles of $\hat{T}$ are mapped to two disjoint ideal triangles of $\mathbf{H}^1_{\mathbb{C}}$, and thus this family of triangles tesselates a convex subset of the hyperbolic disc. This convex subset is equal to the whole hyperbolic disc if and only if the representation is type-preserving. We can then decorate the edges of $\hat{T}$ by the corresponding cross-ratios, which are positive. Due to the (π_1, ρ)-equivariance of ϕ, this decoration is π_1-invariant and therefore it projects onto a decoration of T. This is the result. □

Remark 11. In [8] (p. 88-89), Fock and Goncharov prove the discreteness of ρ by studying the coordinate changes associated to changes of triangulation. It boils down to check that the decoration remains positive when one perform a flip move on a pair of adjacent triangles. This point of view seems to be difficult to generalize to PU(2,1).

4.3 An explicit description of the representation associated to a decorated triangulation

Definition 14. We associate to each oriented edge of Γ an isometry A_s in the following way.

- If s is an edge of type 1, intersecting an edge e of the triangulation $\hat{T}$ decorated by the positive number $x = \mathtt{d}(e)$, we set $A_s = I_x$. In this case, A_s does not depend on an orientation of s since I_x is an involution.
- If s is an edge of type 2 oriented positively (resp. negatively) with respect to the orientation of Σ, we set $A_s = E$ (resp. $A_s = E^{-1}$).

Lemma 4. *Let T be a decorated triangulation of Σ The mapping $s \longmapsto A_s$ is a 1-cocycle of Γ with values in PSL(2,$\mathbb{R}$).*

Proof. The only cycles of Γ are the triangles formed by three consecutives edges of type 2. The result follows thus from the fact that E has order 3. □

Proposition 9. *Let v and v' be two vertices of Γ, and $p_{v,v'} = s_1 s_2 \cdots s_k$ be a simplicial path connecting them, oriented toward v'. Then the isometry $B_{v,v'} = A_{s_1} \circ \cdots \circ A_{s_k}$ satisfies*

$$\phi_v = B_{v,v'} \phi_{v'}$$

Proof. This is a direct reccursion using the second step of the proof of theorem 4. ☐

Now, if v and v' satisfy to $v' = \gamma \cdot v$, we obtain by this process an isometry $B_{v,\gamma\cdot v}$. Due to lemma 4, the mapping $\gamma \longmapsto B_{v,\gamma\cdot v}$ does not depend of the choice of the simplicial representative of γ, and thus it is a representation of π_1 in PSL(2,$\mathbb{R}$).

Proposition 10. *The mapping $\gamma \longmapsto B_{v,\gamma\cdot v}$ is equal to the representation ρ_v.*

Proof. Applying proposition 9 to the two points v and $\gamma \cdot v$, we see that $B_{v,\gamma\cdot v}$ satisfies to

$$\phi_v = B_{v,\gamma\cdot v} \phi_{\gamma\cdot v}.$$

As a consequence, the isometry $B_{\gamma\cdot v,v}$ acts as follows:

$$B_{v,\gamma\cdot v} : \begin{cases} \phi_{\gamma\cdot v}(\gamma \cdot a_v) = \infty & \longmapsto \quad \phi_v(\gamma \cdot a_v) \\ \phi_{\gamma\cdot v}(\gamma \cdot b_v) = -1 & \longmapsto \quad \phi_v(\gamma \cdot b_v) \\ \phi_{\gamma\cdot v}(\gamma \cdot c_v) = 0 & \longmapsto \quad \phi_v(\gamma \cdot c_v) \end{cases}$$

This shows the result.

☐

4.4 Preservation of type

Definition 15. Let T be an ideal triangulation, and $\mathtt{d}$ be a decoration of it. We will say that the decoration is *balanced* if for any vertex v of the ideal triangulation the product $\prod_{e\in E_v} \mathtt{d}(e)$ equals 1, where E_v is the set of edges of T adjacent to v.

Proposition 11. *Let $(T, \mathtt{d})$ be a positive decorated triangulation and $[\rho]$ be the associated class of discrete and faithful representations $\pi_1 \longrightarrow$PSL(2,$\mathbb{R}$). Then ρ is type-preserving if and only if the decoration $\mathtt{d}$ is balanced.*

Proof. Assume first that $\mathtt{d}$ is balanced. Fix a deleted point p of Σ, represented by a vertex of the triangulation T. Let c be a loop around p, and call $\nu_1, \cdots \nu_k$ the edges of T adjacent to p, and $x_1, \cdots, x_k$ the positive numbers decorating these edges, that is $x_j = \mathtt{d}(\nu_j)$. Pick a vertex v of $\Gamma(T)$, and represent the class $[\rho]$ by the representation ρ_v associated to to v. Let v' be one of the vertices of one of the edges of $\Gamma(T)$ intersecting one of the ν_j's. The homotopy class c is represented by a simplicial loop $sls^{-}1$, where s in a simplicial path connecting v to v', and l is a simplicial loop enclosing p, based at v' such that $l = t_1^2 t_1^1 \cdots t_j^2 t_j^1 \cdots t_m^2 t_m^1$, where t_j^i is an edge of type i of $\hat{\Gamma}$ (see figure 2).

Then, according to the proposition 10, the image of the homotopy class of c by the representation is conjugate to the product $E^\epsilon I_{x_1} E^\epsilon \cdots E^\epsilon I_{x_k}$, where $\epsilon = 1$ when the orientation of c agrees with the orientation of Σ, and -1 if not. It is then a simple reccursion to check that

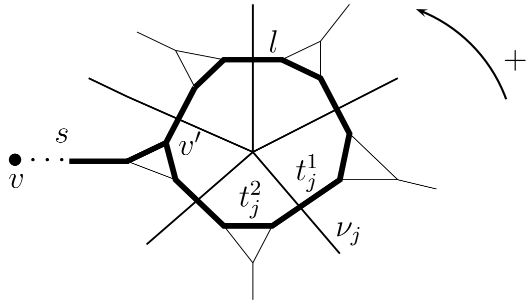


Figure 2: loop around a vertex of the triangulation

- if $\epsilon = 1$, the latter product is proportionnal to

$$\begin{bmatrix} \dfrac{1}{x_1 x_2 \cdots x_k} & -\left(1 + \dfrac{1}{x_2 \cdots x_k} \sum_{i=2}^k x_2 \cdots x_i\right) \\ 0 & 1 \end{bmatrix} \text{ when } \epsilon = 1,$$

- and if $\epsilon = -1$, it is proportionnal to

$$\begin{bmatrix} \dfrac{1}{x_1 x_2 \cdots x_k} & 0 \\ -\sum_{i=}^k \dfrac{1}{x_1 \cdots x_i} & 1 \end{bmatrix}.$$

(We replaced I_x by $\sqrt{x} I_x$ to compute the product.)

The associated element of PSL(2,$\mathbb{R}$) is hyperbolic as long the product $x_1 \cdots x_k$ is not equal to 1. If the product of the x_i's equals 1, then the positivity of the x_i's implies that the associated isometry cannot be the identity. It is therefore parabolic.

Reciprocally, assume that (ϕ, ρ) is a $\mathbf{H}^1_{\mathbb{C}}$-realization of $\mathcal{F}_\infty$ such that ρ is type-preserving. As a consequence, the family of ideal triangles obtained through ϕ from the faces of $\hat{T}$ tesselates $\mathbf{H}^1_{\mathbb{C}}$. Consider the (parabolic) fixed point of $\rho(c_l)$ for some l. Conjugating if necessary, we may assume that it is ∞, and that $\rho(c_l)$ is conjugate to a horizontal translation. The family of ideal triangles of $\mathbf{H}^1_{\mathbb{C}}$ having ∞ as a vertex and corresponding to the triangles of $\hat{T}$ tesselates a horizontal strip $\mathrm{Im}\,(z) > t$ for some great enough real t. The result is then a consequence of the above lemma. $\square$

Lemma 5. *Let $(x_1, \cdots, x_k)$ be a familly of positive numbers. Define from it a periodic sequence $(y_l)_{l \in \mathbb{Z}}$ given by $y_l = x_{l \bmod k}$ for $l \in \mathbb{Z}$, and a sequence of real points $(m_l)_{l \in \mathbb{Z}}$ defined by:*

- *$m_0 = 0$, $m_{-1} = -1$*
- *For $l > 0$, m_l is the unique point such that $r(\infty, m_{l-2}, m_{l-1}, m_l) = y_l$.*
- *For $l < -1$, m_l is the unique point such that $r(\infty, m_l, m_{l+1}, m_{l+2}) = y_{l+1}$.*

Then the family of points $(m_l)_{l \in \mathbb{Z}}$ is unbounded if and only if the product $\pi = x_1 x_2 \ldots x_k$ equals 1.

Proof. It is a simple recursion using the definition of r to check that for $l > 1$, the point m_l is given by

$$m_l = \sum_{i=1}^{l} \prod_{j=1}^{i} x_j.$$

The sequence of real numbers $(m_k)_{k>1}$ is increasing since the $x_i's$ are positive, and converges to

$$\frac{1}{1-\pi} \sum_{i=1}^{k} \prod_{j=1}^{i} x_j,$$

unless $\pi = 1$ in which case it goes to ∞. $\square$

5 The bending theorem

5.1 Bended $\mathbf{H}^2_{\mathbb{C}}$-realizations associated to a bending decoration

Definition 16. Let T be a triangulation of Σ, and $\hat{T}$ be the associated triangulation of $\hat{\Sigma}$. We will call $\mathbf{H}^2_{\mathbb{C}}$*-realization bended along* T, or T*-bended realization* of $\mathcal{F}_\infty(\Sigma)$ any pair (ϕ, ρ) such that

- ρ is a representation $\pi_1(\Sigma) \longrightarrow \mathrm{Isom}(\mathbf{H}^2_{\mathbb{C}})$
- $\phi : \mathcal{F}_\infty(\Sigma) \longrightarrow \partial\mathbf{H}^2_{\mathbb{C}}$ is a $(\pi_1(\Sigma), \rho)$-equivariant mapping.
- for any face Δ of $\hat{T}$ with vertices a, b, and c, the three points $\phi(a)$, $\phi(b)$ and $\phi(c)$ are contained in the boundary of a real plane.

The group PU(2,1) acts on the set of T-bended realizations of $\mathcal{F}_\infty$ by $g \cdot (\phi, \rho) = (g \circ \phi, g\rho g^{-1})$. We will denote by $\mathcal{BR}_T$ the set of $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{C}})$-classes of T-bended realizations for this action. As in the case of $\mathbf{H}^1_{\mathbb{C}}$-realization of $\mathcal{F}_\infty$, we will parametrize $\mathcal{BR}_T$ by decorations of T.

Definition 17. A *bending decoration* of an ideal triangulation T is an application $\mathtt{D} : e(T) \longrightarrow \mathbb{C} \setminus \{-1, 0\}$ defined on the set of unoriented edges of T.

It follows from section 3.2, the case where the invariant $\mathtt{Z}$ of a pair of real ideal triangles equals 0 or -1 corresponds to degenerate pairs of triangles. More precisly, $\mathtt{Z}(\Delta_1\Delta_2) = 0$ if and only if Δ_2 has two identical vertices and -1 if and only if the two triangles are equal. We do not consider these degenerate cases.
We will often refer to the function $\arg(\mathtt{D})$ as the *angular part* of the bending decoration. There is an action of $\mathbb{Z}/2\mathbb{Z}$ on the set of bending decorations of T which is given by the complex conjugation: if $\mathtt{D}$ is a bending decoration of T, the decoration $\overline{\mathtt{D}}$ is given by $\overline{\mathtt{D}}(e) = \overline{\mathtt{D}(e)}$ for any edge e of T.

Definition 18. For any ideal triangulation T of Σ, we denote by $\mathcal{BD}_T$ the set of bending decorations of T, and by $\mathcal{BD}^*_T$ the quotient of $\mathcal{BD}_T$ by the action of $\mathbb{Z}/2\mathbb{Z}$ given above.

The set of bending decoration $\mathcal{BD}_T$ of T is thus a copy of $(\mathbb{C}\setminus\{-1,0\})^{|e(T)|}$. Prior to proving theorem 2, we introduce the following isometries of $\mathbf{H}^2_{\mathbb{C}}$.

Definition 19. For any $z\in\mathbb{C}\setminus\{0,1\}$, we will call σ_z the real symmetry defined by its lift M_z to U(2,1) (see proposition 2), and $\mathcal{E}$ the isometry given by its lift to SU(2,1), where

$$M_z=\begin{bmatrix}0 & 0 & |z|\\ 0 & z/|z| & 0\\ |z|^{-1} & 0 & 0\end{bmatrix}\text{ and }\mathcal{E}=\begin{bmatrix}-1 & \sqrt{2} & 1\\ -\sqrt{2} & 1 & 0\\ 1 & 0 & 0\end{bmatrix}. \tag{18}$$

(We identify $\mathcal{E}$ and its lift).

The symmetry σ_z acts on the complex hyperpolic space by $\sigma_z(m)=\mathbf{P}\left(M_z\bar{\mathbf{m}}\right)$, and swaps respectively the points ∞ and $[0,0]$ and the points $[-1,0]$ and $[z,0]$. In Heisenberg coordinates, the restricted action of σ_z on $\partial\mathbf{H}^2_{\mathbb{C}}$ is given by

$$\sigma_z([\omega,\tau])=\left[\frac{z\bar{\omega}}{|\omega|^4+\tau^2}\left(|\omega|^2-i\tau\right),\frac{\tau|z|^2}{|\omega|^4+\tau^2}\right].$$

The isometry $\mathcal{E}$ is elliptic of order 3 and permutes cyclically the three points ∞, $[-1,0]$ and $[0,0]$.

Proof of theorem 2. To prove the result, we will associate to any bending decoration in $\mathcal{BR}_T$ a unique pair of PU(2,1)-classes of bended realizations of $\mathcal{F}_\infty$, which represent the same Isom($\mathbf{H}^2_{\mathbb{C}}$)- class and correspond to conjugate bending decorations. As in the case of $\mathbf{H}^1_{\mathbb{C}}$, we will first associate to any vertex v of the modified dual graph a bended realization (ϕ_v,ρ_v) of $\mathcal{F}_\infty$ by using v as a basepoint. We will see a posteriori that we obtain this way two PU(2,1) classes of realization which correspond to the same Isom($\mathbf{H}^2_{\mathbb{C}}$)-class.

Step 1: Definition of the mapping ϕ_v. The idea is the same as in the case of PSL(2,$\mathbb{R}$) : to any edge e of T is associated a complex number z with $\mathtt{D}(e)=z$, and we interpret this number as the invariant of a pair of real ideal triangles. The difference comes from the fact that the invariant of a pair of real ideal triangles is oriented, as seen in remark 7. We will take this into account by colorating the triangles of $\hat{T}$ in two colors (say black and white), and interpret the bending decoration of the edges as invariants of ordered pairs (white triangle, black triangle). We do this as follows.

Let v be a vertex of Γ, and a_v, b_v and c_v be the vertices of the face of the triangulation v belongs to, as in definition 12.

- Attribute to the triangle Δ_v the color white, and define $\phi_v(a_v)=\infty$, $\phi_v(b_v)=[-1,0]$ and $\phi_v(c_v)=[0,0]$.
- Colorate all the triangles of $\hat{T}$ in black or white from the one containing v by following the rule that two triangles sharing an edge have opposite color.
- Define the images of all the other points of $\mathcal{F}_\infty$ recursively according to the following principle: if an edge e separates two triangles Δ_w (white) and Δ_b (black), then the number z associated to the edge e should be interpreted as the invariant of the (ordered) pair $\mathtt{Z}(\Delta_w,\Delta_b)$.

Step 2: Definition of the representation ρ_v. For any $\gamma \in \pi_1$, we have to define an isometry g_γ such that $\phi_v(\gamma \cdot m) = g_\gamma \phi_v(m)$ for any m in the Farey set. In particular, such an isometry must map the reference triangle $(\infty, [-1,0], [0,0])$ to the ideal real triangle $(\phi_v(\gamma \cdot a_v), \phi_v(\gamma \cdot b_v), \phi_v(\gamma \cdot c_v))$. Now, for any pair of ideal real triangles, there exists exactly two isometries mapping one onto the other: one of them is holomorphic and the other is antiholomorphic. Define $\rho_v(\gamma)$ according to the following rule (recall that v belongs to a white triangle).

- If $\gamma \cdot v$ belongs to a white triangle, define $\rho_v(\gamma)$ to be the unique holomorphic isometry mapping $(\infty, [-1,0], [0,0])$ to $(\phi_v(\gamma \cdot a_v), \phi_v(\gamma \cdot b_v), \phi_v(\gamma \cdot c_v))$.
- If $\gamma \cdot v$ belongs to a black triangle, choose the antiholomorphic one.

Step 3: ϕ_v is (π_1, ρ_v)-equivariant. If $\rho_v(\gamma)$ is holomorphic, it preserves the invariant $\mathtt{Z}$. As a consequence of the definition of ϕ_v and ρ_v, the identity $\phi_v(\gamma \cdot m) = \rho_v(\gamma)\phi_v(m)$ holds for any m, and for any γ such that $\rho_v(\gamma)$ is holomorphic. If $\rho_v(\gamma)$ is antiholomorphic, then it transforms the invariants of real ideal triangle from z to $\bar{z}$, as seen in lemma 3. The equivariance property in this case is a direct consequence of the choice made in the construction of ϕ_v to interpret the decoration as $\mathtt{Z}(\Delta_w, \Delta_b)$.

Step 4: Description of the class of the realization (ϕ_v, ρ_v). As in the step 2 of the proof of theorem 4, it is sufficient to compare the classes of bended realizations associated to the two vertices v and v' of an edge e of $\hat{\Gamma}$.

1. Assume first that v and v' belong to different faces Δ and Δ' of the triangulation, that is, e is of type 1. These two faces have opposite colors. Then e intersects an edge of $\hat{T}$, such that $\mathtt{D}(e) = z$. Call d_v the vertex of Δ' which is not a vertex of Δ, and the vertices of Δ a_v, b_v and c_v as usual. Then, according to their definitions, ϕ_v and $\phi_{v'}$ satisfy to
$$\phi_v(a_v) = \infty \quad , \quad \phi_v(b_v) = [-1,0] \quad , \quad \phi_v(c_v) = [0,0] \quad \text{and} \quad \phi_v(d_v) = [z,0]$$
$$\phi_{v'}(a_v) = [0,0] \quad , \quad \phi_{v'}(b_v) = [z,0] \quad , \quad \phi_{v'}(c_v) = \infty \quad \text{and} \quad \phi_{v'}(d_v) = [-1,0].$$
The antiholomorphic involution σ_z (definition 19), is the unique isometry exchanging respectively ∞ and $[0,0]$, and $[-1,0]$ and $[z,0]$. Therefore we see $\phi_{v'} = \sigma_z \circ \phi_v$, and $\rho_{v'} = \sigma_z \rho_v \sigma_z$. In this case, the two realization have the same antiholomorphic class, but not the same PU(2,1)-class.
2. By examining similarly what happens when v and v' are connected by an edge of type 2, that is, if they belong to a common face of $\hat{T}$, we see that $(\phi_v, \rho_v) = \mathcal{E} \cdot (\phi_{v'}, \rho_{v'})$ if the orientation induced on e by the orientation of Σ is $v \to v'$, and $(\phi_v, \rho_v) = \mathcal{E}^{-1} \cdot (\phi_{v'}, \rho_{v'})$ in the opposite case. The two realizations have the same holomorphic class in this case.

If v and v' are arbitrary vertices of $\hat{T}$, belonging to the triangles Δ_v and $\Delta_{v'}$ of $\hat{T}$, colorate the faces of $\hat{T}$ starting from Δ_v. The facts *1* and *2* above imply that.

- if Δ_v and $\Delta_{v'}$ have the same color for this choice of coloration, then (ϕ_v, ρ_v) and $(\phi_{v'}, \rho_{v'})$ correspond to the same PU(2,1)-class of T-bended realization,

- if not, then (ϕ_v, ρ_v) and $(\phi_{v'}, \rho_{v'})$ correspond to the same Isom($\mathbf{H}^2_{\mathbb{C}}$)-class, but have opposite PU(2,1)-classes.

Indeed, if Δ_v and $\Delta_{v'}$ have the same color if and only if any simplicial path connecting v and v' contains an even number of edges of type 1. Since the PU(2,1)-class changes every time an edge of type 1 is used, this shows the above assertion.

Step 5: Passing from $\mathtt{D}$ to $\overline{\mathtt{D}}$. We have so far associated to $\mathtt{D}$ a pair of PU(2,1)-classes of T-bended realizations, which we call r_w and r_b. The choice of a starting vertex v of $\hat{\Gamma}$ determines a coloration of the faces of $\hat{T}$. The class r_w corresponds then to white triangles for this choice of coloration, and r_b to black triangles. Now, if we keep the same starting vertex v but construct the classes associated to the decoration $\overline{\mathtt{D}}$, the new equivariant mapping $\psi_v : \mathcal{F}_\infty \longrightarrow \partial\mathbf{H}^2_{\mathbb{C}}$ is defined reccursively from

$$\psi_v(a_v) = \infty \quad , \quad \psi_v(b_v) = [0,0] \quad , \quad \psi_v(c_v) = [-1,0] \quad \text{and} \quad \psi_v(d_v) = [\bar{z}, 0].$$

As a consequence, we see that $\psi_v = \sigma \circ \phi_v$ and the corresponding holonomy representation are conjugate by σ, where σ is the symmetry about the real plane $\mathbf{H}^2_{\mathbb{R}}$, that is, the complex conjugation. Therefore the change $\mathtt{D} \longrightarrow \overline{\mathtt{D}}$ induces the permutation $(r_w, r_b) \longrightarrow (r_b, r_w)$.

Step 6: The reverse operation: decorating a triangulation from a T-bended realization. This goes as in the proof of theorem 4. The only difference is that we have to be slightly more careful about the fact that the invariant $\mathtt{Z}$ is oriented. If e is an edge of $\hat{T}$, to which two triangles are adjacent (say Δ_w, which is white, and Δ_b which is black), we decorate the edge e with the invariant of the *oriented* pair $(\phi(\Delta_w), \phi(\Delta_b))$. We can reconstruct this way the decoration the class $[(\phi,\rho)]$ is to be built from as in the case of theorem 4. There is an order 2 ambigouity: if we start with a given real ideal triangle, and obtain this way a decoration $\mathtt{D}$, starting with an adjacent triangle will produce the decoration $\overline{\mathtt{D}}$. $\square$

As a direct consequence, we obtain the following

Theorem 5. *Let T be an ideal triangulation of Σ, and $\hat{T}$ be its lift to $\hat{\Sigma}$. Fix a coloration of the faces of $\hat{T}$ in black or white, and let $\alpha : e(T) \longrightarrow]-\pi,\pi]$ be a mapping. The application*

$$\begin{array}{ccc} \mathbb{R}^{e(T)}_{>0} & \longrightarrow & (\mathbb{C} \setminus \{-1,0\})^{e(T)} \\ \mathtt{d} & \longmapsto & \mathtt{D} = \mathtt{d}e^{i\alpha} \end{array} \tag{19}$$

induces an injection of $\mathcal{DF}^+$ into $\mathcal{BR}_T$.

Corollary 1. *With the same notations as above, the mapping $\mathtt{d} \longmapsto \mathtt{d}e^{i\alpha}$ induces by restriction to balanced positive decorations an embedding of the Teichmüller space of Σ in $\mathcal{BR}_T$.*

5.2 Explicit computation of the representations

Definition 20. For any oriented edge ν of Γ, let A_ν be the isometry defined by

1. If ν is of type one and intersects an edge e of $\hat{T}$, then A_ν is the real symmetry $\sigma_{\mathtt{D}(e)}$.

2. If ν is of type one, then if it is positively oriented with respect to the orientation of Σ, $A_\nu = \mathcal{E}$, else $A_\nu = \mathcal{E}^{-1}$.

Lemma 6. *The mapping $\nu \longrightarrow A_\nu$ is a 1-cocycle of Γ.*

Proof. The only cycles of Γ are made of three consecutive edges of type 2. The result is therefore due to the fact that $\mathcal{E}$ has order 3. □

The following proposition is proved exactly as its PSL(2,$\mathbb{R}$) analogue (proposition 9).

Proposition 12. *Let T be an ideal triangulation of Σ, with a bending decoration, v and v' be two vertices of Γ, and $p_{v,v'} = s_1 \cdots s_k$ be a simplicial path connecting them.Call r_v and $r_{v'}$ the T-bended realizations associated to v and v', and $B_{v,v'}$ be the isometry $A_{s_1} \cdots A_{s_k}$. Then $B_{v,v'}$ satisfies to*

$$r_v = B_{v,v'} r_{v'}.$$

We now compute the representation interms of the bending decoration.

Proposition 13. *Let γ be a homotopy class of a loop on Σ, and v be a vertex of Γ. We may represent γ as a simplicial path starting at v consisting of a sequence $e_1 \cdots e_k$ of oriented edges of Γ. Associate to γ the isometry $B_{v,\gamma\cdot v} = A_{e_1} \cdots A_{e_n}$. Then*

- *The isometry A_γ does not depend on the choice of the simplicial loop representing γ.*
- *The mapping $\gamma \longmapsto B_{v,\gamma\cdot v}$ is equal to the representation ρ_v.*

Proof. The first assertion is due to the fact that $s \longmapsto A_s$ is a cocycle. To prove the second assertion, we have to show

1. A_γ maps the triple $(\phi_v(\gamma \cdot a_v), \phi_v(\gamma \cdot b_v), \phi_v(\gamma \cdot c_v)$ to the triple $(\infty, [-1,0], [0,0])$
2. A_γ is holomorphic if and only if v and $\gamma \cdot v$ lie in triangles having the same color.

We already know from proposition 12 that $\phi_{\gamma\cdot v} = A_{e_1} \cdots A_{e_n} \phi_v = B_{v,\gamma\cdot v}\phi_v$. As a consequence, the isometry $A_{v,\gamma\cdot v}$ maps the triple $(\phi_v(\gamma \cdot a_v), \phi_v(\gamma \cdot b_v), \phi_v(\gamma \cdot c_v)$ to the triple $(\phi_{\gamma\cdot v}(\gamma \cdot a_v), \phi_{\gamma\cdot v}(\gamma \cdot b_v), \phi_{\gamma\cdot v}(\gamma \cdot c_v)$, which is by definition $(\infty, [-1,0], [0,0])$. This shows the first part.

Now, the isometry A_e attached to an edge e is antiholomorphic if and only if the edge e is of type 1, that is, if e passes from a triangle to another. The isometry $B_{v,\gamma}$ is therefore holomorphic if and only if the simplicial path corresponding to γ contains an even number of type 1 edges. Since the color of the triangle passes from black to white or vice versa for each each edge of type 1, we see that the isometry $B_{v,\gamma\cdot v}$ is holomorphic if and only if the first and last triangles have the same color. This shows the result. □

5.3 When is the representation in PU(2,1)?

5.3.1 Representations in PU(2,1) and bipartite triangulations

Definition 21. Let T be an ideal triangulation of Σ, and F be the set of faces of T. The triangulation T is said to be bipartite if there exist to subset of F, F_1 and F_2 such that

1. $F = F_2 \bigcup F_2$
2. If a face Δ belongs to F_i, then its three neighbours belong to F_{i+1}, where the indices are taken modulo 2.

Remark 12. An ideal triangulation is bipartite if and only if it is possible to colorate its faces in two colors, black and white, in such a way that any white (resp. black) face has three black (resp. white) neighbours. For this reason, we will refer to black or white triangles. Note that a triangulation is bipartite if and only if its dual graph is.

Remark 13. If T is an ideal triangulation of Σ, then its lift $\hat{T}$ to $\hat{\Sigma}$ is always bipartite, if we keep the same definition of bipartiteness for $\hat{T}$. However, this bipartite structure of $\hat{T}$ will project to a bipartite structure on T if and only if it is π_1-invariant, that is, if and only if for any $\gamma \in \pi_1$ and any triangle Δ of $\hat{T}$, the two triangles Δ and $\gamma \cdot \Delta$ have the same color.

Proposition 14. *Let $(T, \mathtt{D})$ be an ideal triangulation of Σ equipped with a bending decoration, and let $\rho : \pi_1(\Sigma) \longrightarrow$ Isom($\mathbf{H}^2_{\mathbb{C}}$) represent the Isom($\mathbf{H}^2_{\mathbb{C}}$)-class of representation of π_1 in Isom($\mathbf{H}^2_{\mathbb{C}}$) associated to $\mathtt{D}$ by theorem 2. Then the following two statements are equivalent.*

1. *The isometry $\rho(\gamma)$ is in PU(2,1) for all $\gamma \in \pi_1(\Sigma)$.*
2. *The triangulation T is bipartite.*

Proof.

- We first prove that the bipartiteness is necessary. Pick a vertex v of Γ to be the basepoint. Let $\nu_1 \cdots \nu_k$ be a simplicial loop based at v representing a homotopy class $\gamma \in \pi_1$. Every ν_l of type 1 (resp. type 2) contributes to $\rho_v(\gamma)$ by an antiholomorphic (resp. holomorphic) isometry. Hence $\rho_v(\gamma)$ is holomorphic if and only if ν_l is of type 1 for an even number of indices l. The number of color changes is equal to the number of edges of type 1, and is even since γ is a loop. Thus $\rho_v(\gamma)$ is holomorphic.

- Assume now that $\rho(\gamma)$ is holomorphic for any $\gamma \in \pi_1$. Pick a homotopy class, and represent it by a simplicial loop –still denoted by γ– based at a vertex v of Γ, belonging to a face Δ_v of T. Attribute to Δ_v the color white. We can colorate every triangles intersected by γ by changing the color every time an edge of type 1 is taken by γ. Since $\rho(\gamma)$ is holomorphic, the color of Δ_v is well-defined (there are an even number of color changes). We have to check now that if two simplicial loops γ_1 and γ_2 based at v intersect at a vertex $w \in \Delta_w$, then they define the same color for Δ_w. Write these two loops

$$\gamma_1 = \nu_1^1 \cdots \nu_{k_1}^1 \text{ and } \gamma_2 = \nu_1^2 \cdots \nu_{k_2}^2.$$

  Let γ_i' one of the two subpathes of γ_i connecting v to w. Then $\gamma_{12} = \gamma_1' \gamma_2'^{-1}$ is a loop based at v, and $\rho_v(\gamma_{12})$ is holomorphic. Therefore the number of edges of type 1 in γ_{12} is even. As a consequence, the numbers of edges of type 1 in γ_1' and γ_2' have the same parity and the color of Δ_w is well-defined.

□

5.3.2 Existence of bipartite triangulations

This section is devoted to the proof of the following proposition.

Proposition 15. *Let $\Sigma_{g,p}$ be a Riemann surface of genus g with p deleted points, such that $2-2g-p<0$. Then $\Sigma_{g,p}$ admits a bipartite ideal triangulation.*

Proof. We prove this proposition recursively, starting with the sphere with three marked points and the torus with one marked point.

Both the 1-marked point torus and the 3-marked points sphere admit ideal triangulations consisting of two triangles, the result is thus trivial in these two cases (see figure 3). We prove the result from these two cases by describing a reccursion process increasing the genus of the surface by one or adding one puncture to the surface, and respecting the bipartiteness of the triangulation. We take the point of view that any triangulated surface is obtained from a triangulated polygon with identifications of the external edges.

First, the bipartite triangulation of the surface corresponds to a bipartite triangulation of the polygon, compatible with the identification of external edges. By this we mean that if two external edges are identified, then one of them should belong to a black triangle, and the other to a white one. We will denote respectively by F, E and V the sets of faces, edges and vertices of the triangulation.

- **Increasing the genus** Pick an internal edge of the triangulated polygon, cut along it to open the polygon and insert four new triangles as on figure 4. Identify the new external edges created this way as indicated on figure 4. During this process, 4 new triangles were created, as well as 6 new edges and no new vertex. As a consequence, the Euler characteristic of the compactified surfaces changes from $\chi=|V|-|E|+|F|$ to $\chi'=|V|-(|E|+6)+(|F|+4)=\chi-2$. Since no new vertex was created, the genus has increased by 1. The bicoloration of the new polygon is compatible with the gluing. Therefore the corresponding triangulation of the surface is also bipartite.

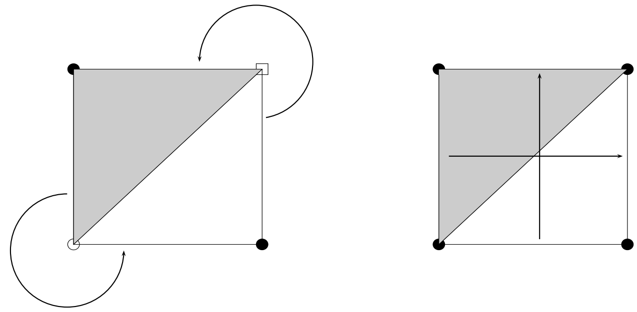
- **Increasing the number of punctures** The method is the same, inserting this time two new triangles, as indicated on figure 5. This time the transformation changes $|V|$ to $|V|+1$, $|E|$ to $|E|+3$ and $|F|$ to $|F|+2$, and preserves χ. As a consequence, the genus of the surface does not change, and we have introduced a new deleted point on the surface.

$\square$

5.4 Loops around holes

Proposition 16. *Let $(T,\mathtt{d})$ be a positively decorated bipartite triangulation of Σ, and let $(\phi_{\mathtt{d}},\rho_{\mathtt{d}})$ be a representative of the class of $\mathbf{H}^1_{\mathbb{C}}$-realizations of $\mathcal{F}_\infty$ associated with $(T,\mathtt{d})$. For any $\alpha: e(T)\longrightarrow]-\pi,\pi]$, let $(\phi^\alpha_{\mathtt{d}},\rho^\alpha_{\mathtt{d}})$ be a representative of the unique Isom($\mathbf{H}^2_{\mathbb{C}}$)-class of bended realization of $\mathcal{F}_\infty$ associated to $\mathtt{d}e^{i\alpha}$. Then, for any homotopy class c_i of loop around one of the deleted points,*

- *the isometry $\rho^\alpha_{\mathtt{d}}(c_i)$ is loxodromic if and only if $\rho_{\mathtt{d}}(c_i)$ is hyperbolic*



The 3 -marked points sphere The 1-marked point torus

Figure 3: Ideal triangulations for surfaces of Euler characteritic -1

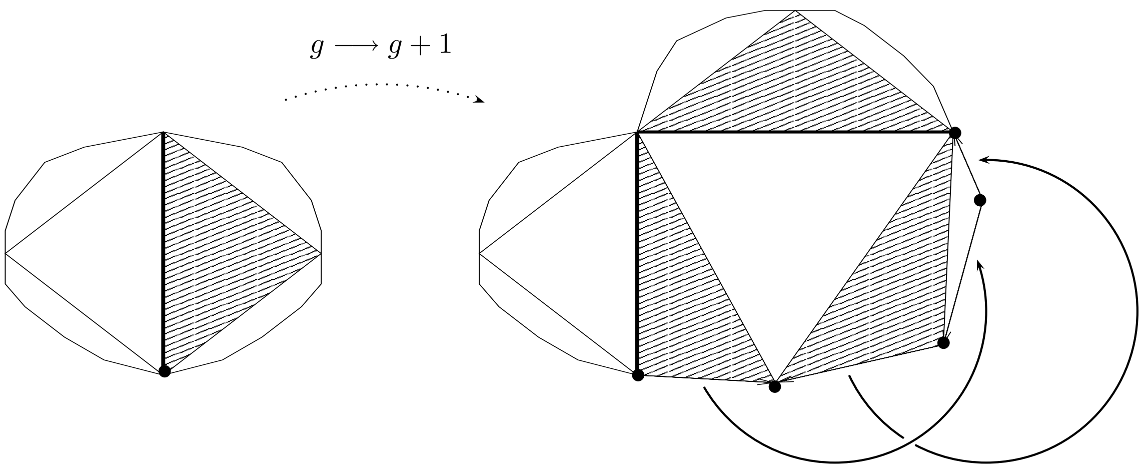


Figure 4: Increasing the genus

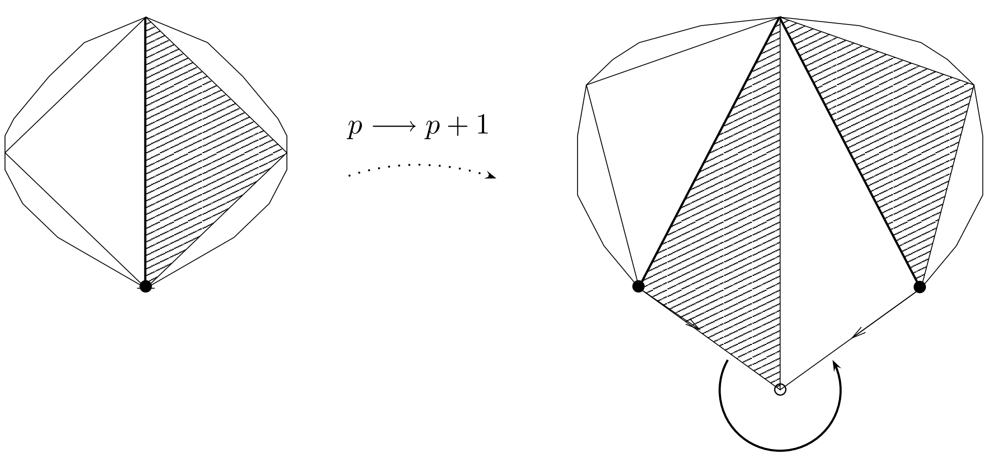


Figure 5: Increasing the number of marked points

- *if $\rho_{\mathtt{d}}(c_i)$ is parabolic, then $\rho_{\mathtt{d}}^{\alpha}(c_i)$ is either parabolic or a complex reflection.*

Proof. As in the proof of proposition 11, we see that the isometry $\rho_{\mathtt{d}}^{\alpha}(c_i)$ is conjugate to the composition $\sigma_{z_1}\circ\mathcal{E}^{\epsilon}\circ\sigma_{z_2}\circ\mathcal{E}^{\epsilon}\circ\cdots\circ\sigma_{z_{2k}}\circ\mathcal{E}^{\epsilon}$, where $2k$ is the number of edges of T adjacent to the deleted point x_i, the z_j's are the complex numbers decorating these edges, and $\epsilon=\pm 1$ according to the relative orientation of c_i and Σ. Note that the number of edges adjacent to x_i has to be even since the triangulation is bipartite. Assume first that $\epsilon=1$, that is c_i is positively oriented with respect to Σ. The involution σ_z being antiholomorphic, this products lifts to U(2,1) as the product of matrices (see remark 2)

$$M_{z_1}\mathcal{E}\overline{M_{z_2}}\mathcal{E}\cdots\overline{M_{z_{2k-1}}}\mathcal{E}M_{z_{2k}}\mathcal{E}=M_{z_1}\mathcal{E}M_{\bar{z}_2,}\mathcal{E}\cdots M_{\bar{z}_{2k-1}}\mathcal{E}M_{z_{2k}}\mathcal{E}=\prod_{j=1}^{2p}M_{z_j^+}\mathcal{E},\tag{20}$$

where z_j^+ is z_j for odd j and $\bar{z}_j$ for even j. For any z, we see that the matrix $M_z\mathcal{E}$ is proportionnal to the element of SU(2,1)

$$M_z\mathcal{E}\underset{\mathrm{SU}(2,1)}{\sim}\begin{bmatrix} w & 0 & 0\\ -\sqrt{2}\bar{w}/w & \bar{w}/w & 0\\ -1/\bar{w} & \sqrt{2}/\bar{w} & 1/\bar{w}\end{bmatrix}\text{ where }w=\bar{z}^2/z.\tag{21}$$

As a consequence, the product (20) has diagonal coefficients $\pi=\prod_{i=1}^{2p}w_i^+$, $\bar{\pi}/\pi$ and $1/\bar{\pi}$.

The matrix (21), and therefore $\rho^{\alpha}(c_i)$ corresponds to a loxodromic isometry if and only if the product π has modulus different from 1, that is, if $\prod_{i=1}^{2k}|z_i|\neq 1$. The latter condition is equivalent to the hyperbolicity of $\rho(c_i)$. If π has modulus 1, then the isometry associated to the above matrix might represent either a parabolic isometry if it is not semi-simple or a complex reflection if it is semi-simple.

In the case where $\epsilon=-1$, the same computation can be done, with the only difference that $M_{x,\theta}\mathcal{E}^{-1}$ is upper triangular. This proves the result. $\square$

Remark 14. For any subset I of $[[1,n]]$ and any bending data α, call $\mathcal{CR}_I^{\alpha}$ the subset of $\mathcal{BR}_T$ containing those classes of T-bended realization such that $\rho_{\mathtt{d}}^{\alpha}(c_i)$ is a complex reflection for all $i\in I$. These representations are either non-discrete or non faithful since complex reflections have fixed points inside $\mathbf{H}^2_{\mathbb{C}}$. According to proposition 16, $\rho_{\mathtt{d}}^{\alpha}\in\mathcal{CR}_I^{\alpha}$ implies that $\rho_{\mathtt{d}}\in\mathcal{P}_I$ (see remark 9). In the next section, we will focus on a special kind of bending data for which these degenerate representations do not appear.

6 The discreteness theorem

6.1 First part of the proof.

The main goal of this section is to focus on those representations associated to a special kind of bending decorations of the triangulation, which we call *regular*.

Definition 22. Let T be a triangulation of Σ. We will say that a bending decoration $\mathtt{D}$ of T is *regular* if there exists $\theta\in[-\pi,\pi[$ such that for all edges e of T, $\arg(\mathtt{D}(e))=\theta$.

The first two parts of theorem 3 follows from what we already know abour the bended representations. We prove them now, and will prove the last part of the result in section 6.3.

Proof of parts 1 and 2 of theorem 3. 1. To prove the first part of the theorem, let us go back to the proof of proposition 16. Consider c_i, one of the homotopy classes of loops around the holes. Without loss of generality, we may assume that c_i is positively oriented with respect to Σ. The fact the decoration is regular implies that this time, $\rho^\theta(c_i)$ is conjugate to the isometry given by the following product of matrices (the assumption on the orientation of c_i implies that $\epsilon = 1$ in the proof of proposition 16).

$$M_{x_1 e^{i\theta}}\mathcal{E} M_{x_2 e^{-i\theta}}\mathcal{E} \cdots M_{\bar{x}_{2k} e^{-i\theta}}\mathcal{E} = \prod_{j=1}^{2k} M_{x_j e^{i(-1)^{j+1}\theta}}\mathcal{E}, \tag{22}$$

It is a direct reccursion using the form of the matrix $M_z\mathcal{E}$ given in (21) to check that the product (22) equals

$$\begin{bmatrix} \prod_{j=1}^{2p} x_j & 0 & 0 \\ -A\sqrt{2} & 1 & 0 \\ * & -\bar{A}\sqrt{2}\prod_{j=1}^{2p} x_j^{-1} & \prod_{j=1}^{2p} x_j^{-1}, \end{bmatrix} \tag{23}$$

where $z_j = x_i e^{i\theta}$ and

$$A = 1 + \underbrace{\sum_{l=1}^{p}\prod_{j=1}^{2l} x_j}_{A_1} + e^{i\theta}\underbrace{\sum_{l=1}^{p-1}\prod_{j=1}^{2l+1} x_j}_{A_2}.$$

The latter matrix corresponds to a loxodromic element if and only if it has one eigenvalue of modulus greater than 1, that is if and only if the product $\prod_{j=1}^{2p} x_j$ is different from 1. The same condition is equivalent to the hyperbolicity of $\rho_{\mathtt{d}}(c_i)$, as seen in proposition 11. Assume now that $\prod_{j=1}^{2p} x_j = 1$. Then the above matrix is either the identity or a unipotent matrix in SU(2,1). If it were the identity, A would to be zero.

- If $e^{i\theta}$ is not real, A is zero if and only if A_1 and A_2 are. The positivity of the x_i's implies that it is not the case.
- If $e^{i\theta}$ is real, then $e^{i\theta} = 0$ since we excluded the case where $\theta = \pi$. Again, the positivity of the x_i's implies that A is not zero in this case.

Therefore the product (22) is unipotent.

2. Assume the representation $\rho_t^\theta d$ preserves a totally geodesic subspace V of $\mathbf{H}^2_{\mathbb{C}}$. We claim that all the fixed points of the isometries $\rho^\theta(c_i)$, $i = 1 \cdots n$ lie in V. Indeed, let us call p_V the orthogonal projection onto V, which is well-defined due to strictly negative curvature. According to the first part of the theorem, $\rho^\theta(c_i)$ is either loxodromic or parabolic. Let

m be one of its fixed points. If m do not belong to V, then $p_V(m)$ is inside $\mathbf{H}^2_{\mathbb{C}}$, and therefore, it is not a fixed point of $\rho^\theta(c_i)$. Since we are dealing with isometries, the geodesic connecting $\rho^\theta(c_i)(p_V(m))$ to m is orthogonal V, and the sum of the angle of the geodesic triangle $(m, p_V(m), \rho^\theta(c_i)(p_V(m)))$ is equal to π. This is absurd.

□

6.2 Spinal $\mathbb{R}$-surfaces.

In order to prove the third part of the theorem 3, we introduce in this section the main tool we will use.

Definition 23. Let P be an $\mathbb{R}$-plane, and γ a geodesic contained in P. The *spinal $\mathbb{R}$-surface* built on γ with respect to P is the hypersurface

$$S_{\gamma,P} = \Pi_P^{-1}(\gamma),$$

where Π_P is the orthogonal projection onto P.

It is a direct consequence of the definition that two spinal $\mathbb{R}$-surfaces are isometric.

Example 1. Using the ball-model of $\mathbf{H}^2_{\mathbb{C}}$, $\mathbf{H}^2_{\mathbb{R}}$ is the real disc containing the points with real coordinates. Then the fiber of the orthogonal projection onto $\mathbf{H}^2_{\mathbb{R}}$ over the point $(0,0)$ is the real plane $i\mathbf{H}^2_{\mathbb{R}} = \{(ix_1, ix_2), x_1^2 + x_2^2 < 1\}$. More information about this projection might be found in [17].

Remark 15.
- In [15], Mostow defined *spinal surfaces*, which are the inverse images of geodesics by the orthogonal projection onto a complex line instead of a real plane. Spinal surfaces are therefore foliated by complex lines. Note that if γ is a geodesic, there exists a unique spinal surface containing it (γ is refered to as its *spine*). In contrast, the set of spinal $\mathbb{R}$-surfaces containing a given geodesic is parametrized by a circle S^1.
- Spinal $\mathbb{R}$-surfaces were already used in [21], where they were called $\mathbb{R}$-balls. They were then generalized to *packs* by Parker and Platis in [17]. In their terminology, spinal $\mathbb{R}$-surfaces are called *flat packs*. The connection between packs and spinal $\mathbb{R}$-surfaces is given above by lemma 7. See also a discussion in the survey [16].

Proposition 17. *The spinal $\mathbb{R}$-surface $S_{\gamma,P}$ is diffeomorphic to a ball of dimension 3, and is foliated by $\mathbb{R}$-planes. It separates $\mathbf{H}^2_{\mathbb{C}}$ in two connected components which are exchanged by the symmetry about any of the leaves of the foliation.*

Proof. The fibers of the orthogonal projection onto P are $\mathbb{R}$-planes (see for instance [17]). Since $\mathbb{R}$-planes are discs, spinal $\mathbb{R}$-surfaces are diffeomorphic to $\mathbb{R} \times \mathbf{H}^2_{\mathbb{R}}$, that is, a 3-dimensional ball. It is clear that a spinal $\mathbb{R}$-surface separates $\mathbf{H}^2_{\mathbb{C}}$ in two connected components. To see that they are exchanged we may normalize so that in the ball model of $\mathbf{H}^2_{\mathbb{C}}$, P is $\mathbf{H}^2_{\mathbb{R}}$, γ connects the two points $(-1,0)$ and $(1,0)$, and $Q = i\mathbf{H}^2_{\mathbb{R}}$. Then the symmetry about Q acts on $\mathbf{H}^2_{\mathbb{R}}$ by $(x_1, x_2) \longmapsto (-x_1, -x_2)$. This proves the result. □

We give now another characterization of spinal $\mathbb{R}$-surfaces. Recall that if γ is a geodesic, R_γ is the 1-parameter subgroup of PU(2,1) associated to γ. It contains the loxodromic isometries of real trace greater than 3 preserving γ (see definition 2).

Lemma 7. *Let Q be a real plane, and γ be a geodesic of which endpoints we denote by p and q. Assume that the real symmetry about Q satisfies $\sigma_Q(p) = q$. Then the union $\cup_{g\in R_\gamma} g \cdot Q$ is a spinal $\mathbb{R}$-surface. Reciprocally, any spinal $\mathbb{R}$-surface may be obtained in this way.*

Proof. We may normalize the situation so that, using the ball model of $\mathbf{H}^2_{\mathbb{C}}$, the points p and q have coordinates $p = (-1, 0)$ and $q = (1, 0)$, and Q is the real plane $i\mathbf{H}^2_{\mathbb{R}}$. The 1-parameter subgroup R_γ preserves the real plane $\mathbf{H}^2_{\mathbb{R}}$ and acts transitively on the geodesic connecting p and q. Since $i\mathbf{H}^2_{\mathbb{R}}$ is the fiber of the orthogonal projection onto $\mathbf{H}^2_{\mathbb{R}}$ above the point $(0, 0)$ which belongs to γ, we see that $\cup_{g\in R_\gamma} g \cdot i\mathbf{H}^2_{\mathbb{R}}$ is the spinal $\mathbb{R}$-surface built on γ with respect to P. □

We will use the following proposition to build the spinal $\mathbb{R}$-surfaces we will need in the proof of theorem 3.

Proposition 18. *Let $\Delta = (m_1, m_2, m_3)$ and $\Delta' = (m_1, m_3, m_4)$ be two ideal real triangles and γ be the geodesic connecting m_1 and m_3. Assume that the argument of $\mathtt{Z}(\Delta, \Delta')$ is not π. Then there exists a unique spinal $\mathbb{R}$-surface S built on the geodesic γ having the mirror of $\sigma_{\Delta,\Delta'}$ as one of its leaves.*

Recall that $\sigma_{\Delta,\Delta'}$ is the symmetry of the pair (Δ, Δ') (see definition 6).

Proof. Let P be the mirror of $\sigma_{\Delta,\Delta'}$. Applying the lemma 7 to the real plane P and the geodesic γ, we obtain a spinal $\mathbb{R}$-surface having the requested property. If there were another spinal $\mathbb{R}$-surface having the same property, the uniqueness part in lemma 6 would show that it would have P as a leave, and contain γ. Thus it would be equal to S by lemma 7. □

Definition 24. Let Δ and Δ be two ideal $\mathbb{R}$-triangles sharing an edge and such that the argument of $\mathtt{Z}(\Delta, \Delta')$ is not π. We will call the spinal $\mathbb{R}$-surface given by proposition 18 the *splitting surface* of Δ and Δ' and denote it by $\mathrm{Spl}(\Delta, \Delta')$.

Remark 16. The definition of the splitting surface implies directly that $\mathrm{Spl}(\Delta_1, \Delta_2)$=$\mathrm{Spl}(\Delta_2, \Delta_1)$.

Proposition 19. *The splitting surface associated to a pair of adjacent $\mathbb{R}$-triangles (Δ_1, Δ_2) is totally determined by the argument of $\mathtt{Z}(\Delta_1, \Delta_2)$.*

Proof. Let Δ be an $\mathbb{R}$-triangle, and γ be one of its edges. Consider Δ_1 and Δ_2 two $\mathbb{R}$-triangles sharing the edge γ with Δ such that $\mathtt{Z}(\Delta, \Delta_j) = x_j e^{i\alpha}$ for $j = 1, 2$. We have to show that the two spinal $\mathbb{R}$-surfaces $\mathrm{Spl}(\Delta, \Delta_1)$ and $\mathrm{Spl}(\Delta, \Delta_2)$ coincide.

Call Q_1 and Q_2 the mirrors of the symmetries of the pairs (Δ, Δ_1) and (Δ, Δ_2). Proposition 7 provides us a unique isometry g belonging to the 1-parameter subgroup G_γ which maps Q_1 to Q_2. In view of lemma 7, this proves the result. □

6.3 Proof of the third part of theorem 3

We will prove now that ρ^θ_d is discrete and faithful for any $\theta \in [-\pi/2, \pi/2]$. It is sufficient to prove that for these values of θ, the action of $\rho^\theta_d(\pi_1)$ is free and discontinuous on some $\rho^\theta(\pi_1(\Sigma))$-invariant subset of $\mathbf{H}^2_{\mathbb{C}}$. The following theorem is the crucial point to prove discreteness.

Theorem 6. *Let Δ be an ideal $\mathbb{R}$-triangles with vertices (p_1, p_2, p_3). For $i = 1, 2, 3$, let γ_i be the geodesic $p_{i+1}p_{i+2}$ (incices taken mod. 3). Let Δ_1, Δ_2 and Δ_3 be real ideal triangles, such that*

- *For $i = 1, 2, 3$, Δ and Δ_i are adjacent, and share the geodesic γ_i as an edge.*
- *There exists $\theta \in [-\pi/2, \pi/2]$ such that the three angular invariants $\mathtt{Z}(\Delta, \Delta_i)$ have argument equal to θ.*

Then, the three splitting surfaces $S_i = \mathrm{Spl}(\Delta, \Delta_i)$ ($i = 1, 2, 3$) enjoy the following properties.

1. *The intersection of S_i and S_{i+1} in $\mathbf{H}^2_{\mathbb{C}}$ is empty.*
2. *The intersection of the closures of S_i and S_{i+1} in $\mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$ is exactly $\{p_{i+2}\}$.*

We postpone the proof of theorem 6, and finish first the proof of theorem 3.

proof of part 3 of theorem 3. Given, $\theta \in [-\pi/2, \pi/2]$, consider a bended realization (ϕ, ρ) associated to a regular bending decoration $\mathtt{d}e^{i\theta}$ of T (we simplify the notation ρ^θ_d used in the statement of the theorem).

To each triangle Δ of T, we associate as follows a prism $\mathfrak{p}_\Delta$, containing the ideal $\mathbb{R}$-triangle $\phi(\Delta)$.

Let Δ_1, Δ_2 and Δ_3 be the three triangles of $\hat{T}$ adjacent to Δ. For $i = 1, 2, 3$, let e_i be the edge of $\hat{T}$ shared by Δ and Δ_i and let S_i be the splitting surface $\mathrm{Spl}(\phi(\Delta), \phi(\Delta_i))$. The theorem 6 shows that for the chosen value of θ, the three splitting surfaces S_i are disjoint in $\mathbf{H}^2_{\mathbb{C}}$. We call $\mathfrak{p}_\Delta$ the connected component of $\mathbf{H}^2_{\mathbb{C}} \setminus (S_1 \cup S_2 \cup S_3)$ containing Δ.

Now, if Δ and Δ' are two adjacent triangles of $\hat{T}$, the intersection $\mathfrak{p}_\Delta \cap \mathfrak{p}_{\Delta'}$ is empty. Therefore the closures of $\mathfrak{p}_\Delta$ and $\mathfrak{p}_{\Delta'}$ intersect exactly along the spinal $\mathbb{R}$-surface $\mathrm{Spl}(\phi(\Delta), \phi(\Delta'))$.

By a direct reccursion, we see that if Δ and Δ' are two disjoint triangles of $\hat{T}$, then the two prisms $\mathfrak{p}_\Delta$ and $\mathfrak{p}_{\Delta'}$ are disjoint. In particular, this shows that for any $\gamma \in \pi_1(\Sigma)$ and any triangle Δ in $\hat{T}$, the two prisms $\mathfrak{p}_\Delta$ and $\rho(\gamma)(p_\Delta)$ are disjoint.

Therefore, the action of $\rho(\pi_1(\Sigma))$ on the union $\cup_{\gamma \in \pi_1(\Sigma)} \rho(\gamma)(\mathfrak{p}_\Delta)$, which is equal to $\cup_{\Delta \in F(\hat{T})} \mathfrak{p}_\Delta$, is free and discontinuous. Since the latter union of prisms is $\rho(\pi_1)$-invariant, this shows that $\rho(\pi_1(\Sigma))$ is discrete and isomorphic to $\pi_1(\Sigma)$. In other words, ρ is discrete and faithful. $\square$

We prove now theorem 6.

Proof of theorem 6.
Note that during this proof, the Δ's are real ideal triangles in $\mathbf{H}^2_{\mathbb{C}}$, and not any more faces of $\hat{T}$.

First step: reduction to a normalized case.
By applying if necessary an isometry, we may assume that Δ is the reference real ideal triangle given by $p_1 = \infty$, $p_2 = [-1, 0]$ and $p_3 = [0, 0]$. The isometry $\mathcal{E}$ given in by (18) in definition 19 cyclically permutes the three latter points, and preserves the invariant of pair of real ideal triangles since it is holomorphic. Due to the fact that the bending decoration is regular, the two invariants $\mathtt{Z}(\Delta, \Delta_{i+1})$ and $\mathtt{Z}(\Delta, \Delta_i)$ have the same argument. Therefore $\mathcal{E}$ maps Δ_i to an ideal $\mathbb{R}$-triangle Δ'_{i+1} (indices taken mod. 3) such that $\mathtt{Z}(\Delta, \Delta_{i+1})/\mathtt{Z}(\Delta, \Delta'_{i+1})$ is real and positive. As a consequence of proposition 19, it maps the splitting surface S_i to S_{i+1}, and in fact, permutes the three splitting surfaces cyclically. Hence it is enough to prove that the two surfaces S_1 and S_2 satisfify *1* and *2*.

Second step : parametrization of the symmetries about the leaves of S_1 and S_2. First, we choose the following lifts for the p_i's:

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{p}_2 = \begin{bmatrix} -1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \text{ and } \mathbf{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{24}$$

We first use lemma 7 to describe the leaves of S_1. Let q_1 be the third point of Δ_1. According to proposition 19, we may assume that q_1 is any point such that $\mathtt{Z}(\Delta, \Delta_1)$ has the form $xe^{i\theta}$ with $x > 0$. We make the choice $q_1 = [e^{i\theta}, 0]$. The unique symmetry about a real plane swapping p_2 and p_3, and p_1 and q_1 is given by $\sigma_1(m) = \mathbf{P}\left(M_1\mathbf{m}\right)$, where M_1 is the matrix

$$M_1^\theta = \begin{bmatrix} 0 & 0 & 1 \\ 0 & e^{i\theta} & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The 1-parameter subgroup R_{γ_1} associated to the geodesic connecting p_2 and p_3 is parametrized by the matrices

$$\mathbf{D}_{r_1} = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/r_1 \end{bmatrix} \text{ with } r_1 > 0. \tag{25}$$

We obtain thus the general form M_{1,r_1}^θ of a lift of the symmetry about a leaf of S_1 by conjugating a lift of the involution associated to M_1^θ by $\mathbf{D}_{r_1}$. Since M_1^θ stands for a antiholomorphic isometry, this yields (see remark 2)

$$\begin{aligned} M_{1,r_1}^\theta &= \mathbf{D}_{r_1} M_1^\theta \overline{\mathbf{D}_{r_1}^{-1}} \\ &= \mathbf{D}_{r_1} M_1^\theta \mathbf{D}_{1/r_1} \ (\mathbf{D}_{1/r_1} \text{ has real coefficients}) \\ &= \begin{bmatrix} 0 & 0 & r_1^2 \\ 0 & e^{i\theta} & 0 \\ 1/r_1^2 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{26}$$

The general form M_{2,r_2}^θ of a lift of the symmetry about a leaf of S_2 is obtained by conjugating the matrix M_{1,r_2}^θ by the order three elliptic element $\mathcal{E}$:

$$\begin{aligned} M_{2,r_2}^\theta &= \mathcal{E}\mathbf{D}_{r_2} M_1^\theta \overline{\mathbf{D}_{r_2}^{-1} E^{-1}} \\ &= \mathcal{E}\mathbf{D}_{r_2} M_1^\theta \mathbf{D}_{1/r_2} \mathcal{E}^{-1} \\ &= \begin{bmatrix} -r_2^2 & \sqrt{2}\left(e^{i\theta} + r_2^2\right) & \dfrac{1 + 2e^{i\theta} r_2^2 + r_2^4}{r_2^2} \\ \sqrt{2} r_2^2 & e^{i\theta} + 2r_2^2 & \sqrt{2}\left(e^{i\theta} + r_2^2\right) \\ r_2^2 & -\sqrt{2} r_2^2 & -r_2^2 \end{bmatrix}. \end{aligned} \tag{27}$$

Third step: proof of the disjonction
Note first that the closures of S_1 and S_2 in $\mathbf{H}^2_{\mathbb{C}}\cup\partial\mathbf{H}^2_{\mathbb{C}}$ both contain the point p_3 as an extremity of the geodesics γ_1 and γ_2. Therefore their intersection should at least contain this point. Now, the result will be proved if we show that the closure of any leaf of S_1 is disjoint from the closure or any leaf of S_2. We do this by showing that the product of the symmetries about these leaves is loxodromic as long as $\theta\in[-\pi/2,\pi/2]$ (see lemma 1). More precisely, we will show that for these values of θ, the isometry associated to the matrix $M^{\theta}_{1,r_1}\overline{M^{\theta}_{2,r_2}}$ is loxodromic for any pair $(r_1,r_2)\in\mathbb{R}^2_{>0}$. Using the above matrix form, it seen that the trace of this matrix is

$$\mathrm{tr}M^{\beta}_{1,r_1}\overline{M^{\theta}_{2,r_2}}=2r_2^2e^{i\theta}+\frac{2}{r_1^2}e^{-i\theta}+1+r1^2r_2^2+\frac{1}{r_1^2r_2^2}+\frac{r_2^2}{r_1^2}. \tag{28}$$

This yields

$$\begin{aligned}\mathrm{Re}\left(\mathrm{tr}M^{\beta}_{1,r_1}\overline{M^{\beta}_{2,r_2}}\right) &= 2r_2^2\cos\theta+\frac{2}{r_1^2}\cos\theta+1+r1^2r_2^2+\frac{1}{r_1^2r_2^2}+\frac{r_2^2}{r_1^2}\\ &\geqslant 1+r1^2r_2^2+\frac{1}{r_1^2r_2^2}\ \text{while}\ \cos\theta\geqslant 0\\ &\geqslant 3 \end{aligned}\tag{29}$$

This implies that the isometry associated to $M^{\beta}_{1,r_1}\overline{M^{\beta}_{2,r_2}}$ is loxodromic as long as $\theta\in[-\pi/2,\pi/2]$ and for any pair $(r_1,r_2)\in\mathbb{R}^2_{>0}$, as shown by remark 3. As a consequence of lemma 1, the corresponding leaves of S_1 and S_2 are disjoint. □

7 Remarks and comments

A stable embedded disc As said in theorem 3, the representations $\rho^{\theta}(\pi_1)$ obtained by bending along a bipartite ideal triangulation do not preserve any totally geodesic subspace, unless $\theta=0$. However, they preserve a non totally geodesic embedded disc, as we will see now.

First, let (ϕ,ρ) be a $\mathbf{H}^1_{\mathbb{C}}$-realization of $\mathcal{F}_\infty$ associated to a decoration $\mathtt{d}$, and for $\theta\in[-\pi/2,\pi/2]$, let $(\phi^{\theta},\rho^{\theta})$ be a T-bended realization associated to $\mathtt{d}e^{i\theta}$. Denote by F the set of faces of $\hat{T}$. The union $\cup_{\Delta\in F}\phi(\Delta)$ is a convex subset $\mathcal{C}$ of $\mathbf{H}^1_{\mathbb{C}}$ which is homotopic to a disc. We have seen that for these values of θ, each of the ideal real triangle $\phi^{\theta}(\Delta)$ is contained in a unique prism $\mathfrak{p}_\Delta$. Therefore the union $\cup_{\Delta\in F}\phi^{\theta}(\Delta)$ is an embedded copy of $\mathcal{C}$, which is not totally geodesic, but "piecewise totally geodesic".

Embeddings of the Teichmüller space. As a direct consequence of theorem 3, we obtain embeddings of the Teichmüller space of Σ into Hom($\pi_1(\Sigma)$,PU(2,1))/PU(2,1).

Theorem 7. *Let $\theta\in[-\pi/2,\pi/2]$ be a real number and T be a bipartite ideal triangulation of Σ. The mapping $\mathtt{d}\longmapsto\mathtt{d}e^{i\theta}$ defined on the set of positive decorations induces a pair of embeddings of $\mathcal{T}(\Sigma)$ in Hom(π_1, PU(2,1))/PU(2,1) of which images contain only classes of discrete, faithful and type-preserving representations.*

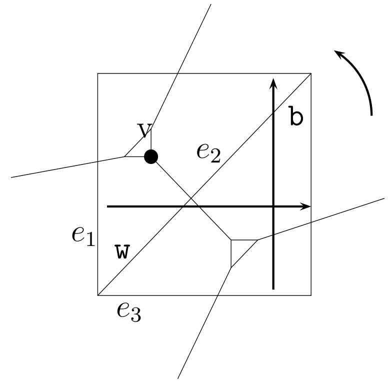


Figure 6: The punctured torus

Proof. Restricting the mapping $\mathtt{d} \longmapsto \mathtt{d}e^{i\theta}$ to balanced decorations of T produces discrete, faithful and type-preserving representations of $\pi_1(\Sigma)$ with images contained in PU(2,1) since T is bipartite. Once a coloration of the faces of $\hat{T}$ is fixed, we obtain two injective applications by mapping the point in $\mathcal{T}(\Sigma)$ associated to $\mathtt{d}$ to the class of representations associated to $\mathtt{d}e^{i\theta}$ corresponding either to white triangles or to black triangles. These two embeddings are identified by the complex conjugation in $\mathbf{H}^2_{\mathbb{C}}$, and correspond in fact to a unique embedding in Hom(π_1,PU(2,1))/Isom($\mathbf{H}^2_{\mathbb{C}}$). □

What happens if the triangulation is not bipartite? In this case, the same result of discreteness holds for regular bending decorations (δ, α_θ), for any $\theta \in [0, \pi/2]$. The image is this time a subgroup of Isom($\mathbf{H}^2_{\mathbb{C}}$) which intersect its connected component not containing the identity.

Link with previously known families of examples.

In this section, we draw the connection betewwen T-bended realizations and families of examples described in the previous works [3, 12, 21].

The case of the 1-punctured torus. In this case T consists of two triangles, as indicated on figure 6. We will use the vertex v as basepoint. There are two faces, of which color is indicated by w and b on figure 6, and three edges, labeled by e_1, e_2 and e_3 on figure 6. In the case of a regular bending decorations,the decoration is given three positive real numbers x_1, x_2 and x_3 and $\theta \in [0, 2\pi[$ such that the edge e_i is decorated by x_i, θ. Following the results of section 5.2, we see that the identifications between opposite faces of the square correspond to the following holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$. Call A and B the isometries associated respectively to the horizontal and vertical identifications of the opposite sides of the square. Following section 5.2, these isometries are given by

$$\begin{cases} A &= \mathcal{E} \circ \sigma_{x_2,\theta} \circ \mathcal{E}^{-1} \circ \sigma_{x_2,\theta} \\ \\ B &= \sigma_{x_2,\theta} \circ \mathcal{E}^{-1} \circ \sigma_{x_3,\theta} \circ \mathcal{E}. \end{cases} \tag{30}$$

As a consequence, we see that the group $\langle A, B \rangle$ has index two in the group generated by the three real symmetries $I_1 = \mathcal{E} \circ \sigma_{x_1,\theta} \circ \mathcal{E}^{-1}$, $I_2 = \sigma_{x_2,\theta}$ and $I_3 = \mathcal{E}^{-1} \circ \sigma_{x_3,\theta} \circ \mathcal{E}$. The group

$\langle I_1, I_2, I_3\rangle$ is an example of a so-called Lagrangian triangle group. This example of bending has been exposed with a different point of view in [21] (see also[20]).

In [21], the discreteness result is stated with an angle $\alpha \in [-\pi/4, \pi/4]$. This angle *alpha* is actually half the bending parameter θ we use here. It may be interprated as an angle between a real ideal triangle Δ and the splitting surface $\mathrm{Spl}(\Delta, \Delta')$, where Δ' is adjacent to Δ. From this point of view, $\mathrm{Spl}(\Delta,\Delta')$ is bisecting the pair (Δ, Δ').

The Toledo invariant and the examples of Gusevskii and Parker The Toledo invariant is a conjugacy invariant defined for representations of fundamental groups of closed surfaces, and for type-preserving representations of cusped surfaces. We refer the reader to [12] and [19] for its definition and main properties. Let us just recall that if ρ is such a representation, then

- if Σ has punctures, then $\tau(\rho)$ is a real number in the interval $[\chi, -\chi]$, where χ is the Euler characteristic of Σ,
- if not, then $\tau(\rho)$ belongs to $2/3\mathbb{Z} \cap [\chi, -\chi]$.

Let (ϕ, ρ) be a T-bended realization of $\mathcal{F}_\infty$, where T is a bipartite triangulation, and Ω be a fundamental domain for the action of $\pi_1(\Sigma)$ on $\tilde{\Sigma}$. We might see Ω as a family of triangles $(\Delta_1, \cdots, \Delta_m)$. Then it follows from [12, 19] that the Toledo invariant of ρ is twice the sum of the Cartan invariants of the ideal triangles $\phi(\Delta_i)$. In our particular case, all the triangles are real. We obtain therefore directly the

Proposition 20. *Let (ϕ, ρ) be a T-bended realization of $\mathcal{F}_\infty$, with ρ type-preserving. The Toledo invariant of ρ is equal to zero.*

In their paper [12], Gusevskii and Parker have described for each genus g and number of punctures n a 1-parameter family $(\rho_t)_{t\in[-\chi,\chi]}$ of non PU(2,1)-equivalent discrete, faithful and type-preserving representations of a Riemann surface of genus g with n punctures having the property that the Toledo invariant of ρ_t equals t. This shows that all the possible values of the Toledo invariant for non-compact surfaces are realised by discrete and faithful representation. To prove this result, Gusevskii and Parker start from discrete and faithful representations of the modular group in PU(2,1) and pass to a finite index subgroup using Millington's theorem (see [12]). In their construction, they show that ρ_0 preserves a real plane (this is a so-called $\mathbb{R}$-fuchsian representation). Therefore ρ_0 corresponds is the unique intersection between Gusevskii and Parker's representations and ours.

The 3-punctured sphere and the examples of Falbel and Koseleff. This time we are using the bipartite triangulation of the 3-punctured sphere showed on figure 7. The representation of the fundamental group associated to the decoration given by $\delta(e_i) = x_i$ and $\alpha(e_i) = \theta_i$ is given by

$$\begin{cases} A &= \mathcal{E}^{-1} \circ \sigma_{x_1,\theta_1} \circ \mathcal{E}^{-1} \circ \sigma_{x_2,\theta_2} \\ B &= \sigma_{x_2,\theta_2} \circ \mathcal{E}^{-1} \circ \sigma_{x_3,\theta_3} \circ \mathcal{E}^{-1}. \\ C &= \mathcal{E} \circ \sigma_{x_3,\theta_3} \circ \mathcal{E}^{-1} \circ \sigma_{x_1,\theta_1} \circ \mathcal{E}, \end{cases} \tag{31}$$

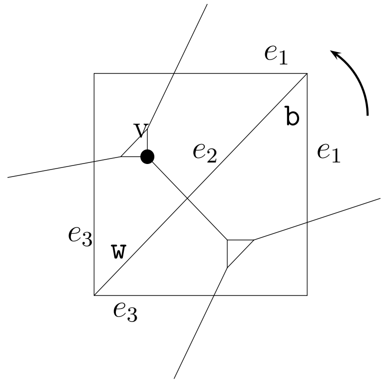


Figure 7: The 3-punctured sphere

It is easily checked that $ABC = 1$. Using the matrices given in section 5.2, we see that the representation is type preserving if and only if $x_1 = x_2 = x_3 = 1$ and none of the θ_i's is equal to π. When $\theta_1 = \theta_2 = \theta_3 \in [-\pi/2, \pi/2]$, this provides through theorem 3 a 1-parameter family of discrete, faithful and type-preserving representations of the fundamental group of the 3-punctured sphere.
Morover, it is possible to prove that in the case where $\delta(e_i) = 1$ and $\alpha(e_i) = \theta$ for all i, then there exists three real symetries s_1, s_2 and s_3 such that $A = s_1 s_2$ and $B = s_2 s_3$. Call Q_i the mirror of s_i. Since A and B are parabolic, the mirrors of the s_i's are mutually asymptotic, that is $Q_i \cap Q_{i+1}$ concists of exactly one point in $\partial \mathbf{H}^2_{\mathbb{C}}$. Therefore these groups belong to the family of groups studied by Falbel and Koseleff in [3]. Note moreover that the discreteness of these groups was not proved in [3], where the focus is on deformations of groups preserving a complex line.

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