CONFORMAL PARACONTACT CURVATURE AND THE LOCAL FLATNESS THEOREM

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Abstract. A curvature-type tensor invariant called para contact (pc) conformal curvature is defined on a paracontact manifolds. It is shown that a paracontact manifold is locally paracontact conformal to the hyperbolic Heisenberg group iff the pc conformal curvature vanishes provided the dimension is bigger than three. A different proof of the Chern-Moser-Webster theorem is given, showing that the vanishing of the Chern-Moser invariant is necessary and sufficient condition a CR-structure to be CR equivalent to a quadric in $\mathbb{C}^{n+1}$ provided $n > 1$. The proof establishes also Cartan's result in dimension three. An explicit formula for the regular part of a solution to the sub-ultrahyperbolic Yamabe equation on the hyperbolic Heisenberg group is shown.

Contents

1. Introduction

A paracontact structure on a real (2n+1)-dimensional manifold M is a codimension one distribution $\mathbb{H}$, locally given as the kernel of a 1-form η, $\mathbb{H} = Ker\,\eta$ and a paracomplex structure I on $\mathbb{H}$, i.e. $I^2 = id$ and the $\pm$ eigen-distributions have equal dimension. We assume in addition that η is a para hermitian contact form in the sense that we have a non-degenerate pseudo-Riemannian metric g, which is defined on $\mathbb{H}$, and compatible with η and I, $\quad d\eta(X,Y) = 2g(IX,Y), \quad g(IX,IY) = -g(X,Y), \quad X,Y \in \mathbb{H}$. The signature of g on $\mathbb{H}$ is necessarily of (signature) type (n,n).

The 1-form η is determined up to a conformal factor and hence $\mathbb{H}$ become equipped with a conformal class [g] of neutral Riemannian metrics of signature (n,n). Transformations preserving a

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given para contact hermitian structure η, i.e. $\bar{\eta} = \mu\eta$ for a non-vanishing smooth function μ are called *para contact conformal (pc conformal for short) transformations* [Zam].

A basic example is provided by a paraSasakian manifold, which can be defined as a $(2n+1)$-dimensional Riemannian manifold whose metric cone is a paraKähler manifold [ACGL]. It was shown in [Zam] that the torsion endomorphism of the canonical connection is the obstruction for a given para contact hermitian structure to be locally para-Sasakian, up to a multiplication with a constant factor.

Any non-degenerate hypersurface in $(R^{2n+2}, \mathbb{I}, g)$ considered with the standard flat parahermitian structure inherits a para-contact hermitian structure. We consider the (2n+1)-dimensional Heisenberg group with a left-invariant para-contact hermitian structure η and call it *hyperbolic Heisenberg group*, denoted by $(\boldsymbol{G}\,(\mathbb{P}), \eta)$. We show that the hyperbolic Heisenberg group is the unique example of a para-contact hermitian structure with flat canonical connection. As a manifold $\boldsymbol{G}\,(\mathbb{P})\ = \mathbb{R}^{2n} \times \mathbb{R}$ with the group law given by

$$(p'', t'') \ = \ (p', t') \circ (p, t) \ = \ (p' \ + \ p, t' + \ t \ - \sum_{k=1}^{n} (u'_k v_k - v'_k u_k),$$

where $p', p \in \mathbb{R}^{2^n}$ with the standard coordinates $(u_1, v_1, \dots, u_n, v_n)$ and $t', t \in \mathbb{R}$. Define the 'standard' para-contact structure by the left-invariant para-contact form

$$\tilde{\Theta} \ = \ -\frac{1}{2}\ dt \ - \sum_{k=1}^{n} (u_k dv_k - v_k du_k).$$

In this paper we find a tensor invariant characterizing locally the para-contact hermitian structures which are para-contact conformally equivalent to the flat structure on $\boldsymbol{G}\,(\mathbb{P})$. To any para-contact hermitian structure we associate a curvature-type tensor W^{pc} defined in terms of the curvature and torsion of the canonical connection by (4.16), whose form is similar to the Weyl conformal curvature in Riemannian geometry (see e.g. [Eis]) and to the Chern-Moser invariant in CR geometry [ChM]. We call W^{pc} *para-contact conformal curvature, pc conformal curvature for short*. The main purpose of this article is to prove

Theorem 1.1. *The pc conformal curvature W^{pc} is invariant under para-contact conformal transformations.*

Theorem 1.2. *Let (M, η) be a $2n + 1$ dimensional para-contact hermitian manifold.*

- *i) If $n > 1$ then (M, η) is locally para-contact conformal to the standard flat para-contact hermitian structure on the hyperbolic Heisenberg group $\boldsymbol{G}\,(\mathbb{P})$ if and only if the para-contact conformal curvature vanishes, $W^{pc} = 0$.*
- *ii) If $n = 1$ then W^{pc} vanishes identically and (M, η) is locally para-contact conformal to the standard flat para-contact hermitian structure on the 3-dimensional hyperbolic Heisenberg group $\boldsymbol{G}\,(\mathbb{P})$ if and only if the differential condition (5.44) holds.*

We define a Cayley transform which establishes a conformal para-contact equivalence between the standard para-Sasaki structure on the hyperboloid

$$HS^{2n+1} = \{(x_1, y_1, \dots, x_{n+1}, y_{n+1}) : x_1^2 + \dots + x_{n+1}^2 - y_1^2 - \dots - y_{n+1}^2 = 1\}$$

and the standard para-contact hermitian structure on $\boldsymbol{G}\,(\mathbb{P})$. As a consequence of Theorem 1.2 and the fact that the Cayley transform is a para-contact conformal equivalence between the hyperboloid and the group $\boldsymbol{G}\,(\mathbb{P})$, we obtain

Corollary 1.3. *Let (M, η) be a $2n + 1$ dimensional para-contact hermitian manifold.*

- *i) If $n > 1$ then (M, η) is locally para-contact conformal to the hyperboloid HS^{2n+1} if and only if the para-contact conformal curvature vanishes, $W^{pc} = 0$.*
- *ii) If $n = 1$ then W^{pc} vanishes identically and (M, η) is locally para-contact conformal to the 3-dimensional hyperboloid HS^3 if and only if the differential condition (5.44) holds.*

A consequence of Theorem 1.2 is the fact that the hyperboloid HS^{2n+1} is para-contact conformally flat, while by the Chern-Moser-Webster results it is not CR flat provided the signature is (n,n) with n-odd.

Our investigations are closed to the classical approach used by H. Weyl (see e.g. [Eis]) and follow the steps of [IV], compare with [ChM] where the Cartan method of equivalence is used. We give a new proof of the Chern-Moser-Webster theorem stating that the vanishing of the Chern-Moser invariant is necessary and sufficient condition a non-degenerate CR manifold M to be locally equivalent to a real hyperquadric in $\mathbb{C}^{n+1}$ of the same signature as M when $n>1$ and recover in a different way the Cartan [Car] result in dimension three.

In the last section we consider the CR-Yamabe equation on a CR manifold of neutral signature. This leads to the non-linear sub ultra-hyperbolic equation (7.1), which coincides with the Yamabe equation for the considered para CR manifolds. Using this relation we show an explicit formula for the regular part of solutions to the Yamabe equation.

The paper uses the canonical connection introduced in [Zam] and the properties of its torsion and curvature described in Section 3.

Convention 1.4. *In the first five sections of the paper we use*

a) $X,Y,Z...$ *will be a horizontal vector fields, i.e.* $X,Y,Z...\in\mathbb{H}$

b) $\{e_1,\dots,e_n,Ie_1,\dots,Ie_n\}$ *denotes an adapted orthonormal basis of the horizontal space* $\mathbb{H}$.

c) The summation convention over repeated vectors from the basis $\{e_1,\dots,e_{2n}\}$ *will be used. For example, for a (0,4)-tensor* P, *the formula* $k=P(e_b,e_a,e_a,e_b)$ *means*

$$k=\sum_{a,b=1}^n P(e_b,e_a,e_a,e_b)-\sum_{a,b=1}^n P(e_b,Ie_a,Ie_a,e_b)-\sum_{a,b=1}^n P(Ie_b,e_a,e_a,Ie_b)+\sum_{a,b=1}^n P(Ie_b,Ie_a,Ie_a,Ie_b).$$

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2. Integrable para-contact manifolds

A para-contact manifold (M^{2n+1},η,I,g) is a (2n+1)-dimensional smooth manifold equipped with a codimension one distribution $\mathbb{H}$, locally given as the kernel of a 1-form η, $\mathbb{H}=Ker\,\eta$ and a paracomplex structure I on $\mathbb{H}$. Recall that a paracomplex structure is an endomorphism I satisfying $I^2=id$ and the $\pm$ eigen-distributions have equal dimension. If in addition there exists a pseudo-Riemannian metric g defined on $\mathbb{H}$ compatible with η and I in the sense that

$$g(IX,IY)=-g(X,Y),\qquad d\eta(X,Y)=2g(IX,Y),\qquad X,Y\in\mathbb{H}, \tag{2.1}$$

we have parahermitian contact manifold. The signature of g restricted to $\mathbb{H}$ is necessarily neutral of type (n,n).

The para-contact Reeb vector field ξ (of length -1) is the dual vector field to η via the metric g, $g(X,\xi)=\eta(X)$, $\eta(\xi)=-1$. The metric g extends to the metric in the whole manifold by requiring $g(\xi,\xi)=-1$. In addition, the 1-form η is a contact form and the fundamental 2-form is defined by

$$2\omega(X,Y)=2g(IX,Y)=d\eta(X,Y). \tag{2.2}$$

The paracomplex structure I on $\mathbb{H}$ is formally integrable [Zam] if the $\pm$ eigen-distrubutions $\mathbb{H}^{\pm}$ of I in $\mathbb{H}$ are formally integrable in the sense that $[\mathbb{H}^{\pm},\mathbb{H}^{\pm}]\in\mathbb{H}^{\pm}$. Using the Nijenhuis tensor $N(X,Y)=[IX,IY]+[X,Y]-I[IX,Y]-I[X,IY]=0$ of I, the formal integrability of I is equivalent to

$$N(X,Y)=0\quad\text{and}\quad[IX,Y]+[X,IY]\in\mathbb{H}. \tag{2.3}$$

A para-contact manifold with an integrable para-contact structure is called a *para CR-manifold*. A para-contact manifold is called para-sasakian if $N(X,Y)=d\eta(X,Y)\xi$.

2.1. **Examples.** Let $\{x_1, y_1, \dots, x_{n+1}, y_{n+1}\}$ be the standard coordinate system in $\mathbb{R}^{2n+2}$. The standard parahermitian structure $(\mathbb{I}, g)$ is defined by

$$\mathbb{I}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}, \quad \mathbb{I}\frac{\partial}{\partial y_j} = \frac{\partial}{\partial x_j}, \quad g\Big(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\Big) = -g\Big(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}\Big) = \delta_{jk}, \quad g\Big(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k}\Big) = 0,$$

where $j, k = 1, \dots, n$. Recall that a smooth map $f = (u_1, v_1, \dots, u_n, v_n) : \mathbb{R}^{2n+2} \longrightarrow \mathbb{R}^{2n+2}$ preserves the paracomplex structure $\mathbb{I}$ iff it is *paraholomorphic*, i.e., satisfies the (para) Cauchy-Riemann equations, see e.g.[Lib], $df \circ \mathbb{I} = \mathbb{I} \circ df$, or,

$$\frac{\partial u_k}{\partial x_j} = \frac{\partial v_k}{\partial y_j}, \qquad \frac{\partial u_j}{\partial y_j} = \frac{\partial v_k}{\partial x_j}. \tag{2.4}$$

Let $(\mathbb{R}^{2n+2}, \mathbb{I}, g)$ be the standard flat parahermitian structure on $\mathbb{R}^{2n+2}$ and M^{2n+1} be a hypersurface with unit normal N such that $T_pM^{2n+1} \oplus N = R^{2n+2}, p \in M^{2n+1}$. Consider the vector field $\xi := \mathbb{I}N$, the dual 1-form $\eta(\xi) = -1$ and denote $\mathbb{H} = \xi^{\perp} = Ker\,\eta$. A para CR-structure on M is defined by $(\mathbb{H}, I = \mathbb{I}_{|\mathbb{H}})$. Moreover

$$d\eta(X, Y) = -\eta([X, Y]) = -d\eta(IX, IY)$$

in view of the integrability condition (2.3). If in addition $d\eta_{|\mathbb{H}}$ is non-degenerate then it necessarily has signature (n, n) and (M, η) is an integrable para-contact hermitian manifold.

Proposition 2.1. *Any non-degenerate hypersurface in $(\mathbb{R}^{2n+2}, \mathbb{I}, g)$ admits an integrable para-contact hermitian structure.*

Similarly to the CR case [ChM], a restriction of a paraholomorphic map $f : R^{2n+2} \longrightarrow R^{2n+2}$ on (M^{2n+1}, η) induces a para conformal transformation of the para-contact hermitian structure $\bar\eta = \mu\eta$ on the hypersurface M^{2n+1}.

a) **Hyperbolic Heisenberg group**

The hyperbolic Heisenberg group is the example of a para-contact hermitian structure with flat canonical connection. Note that the difference between this group and the standard Heisenberg group is in the metric, while the groups are identical. As a group $\boldsymbol{G}\,(\mathbb{P}) = \mathbb{R}^{2n} \times \mathbb{R}$ with the group law given by

$$(p'', t'') = (p', t') \circ (p, t) = \Big(p' + p, t' + t - \sum_{k=1}^{n}(u'_k v_k - v'_k u_k)\Big).$$

where $p', p \in \mathbb{R}^{2n}$, $t', t \in \mathbb{R}$, $p = (u_1, v_1, \dots, u_n, v_n)$ and $p' = (u'_1, v'_1, \dots, u'_n, v'_n)$. A basis of left-invariant vector fields is given by $U_k = \frac{\partial}{\partial u_k} - 2v_k\frac{\partial}{\partial t}$, $V_k = \frac{\partial}{\partial v_k} + 2u_k\frac{\partial}{\partial t}, \xi = 2\frac{\partial}{\partial t}$. Define $\tilde\Theta = -\frac{1}{2}dt - \sum_{k=1}^{n}(u_k dv_k - v_k du_k)$ with corresponding horizontal bundle $\mathbb{H}$ given by the span of the left-invariant horizontal vector fields $\{U_1, ...U_n, V_1...V_n, \}$. We consider an endomorphism on $\mathbb{H}$ by defining $IU_k = V_k$, $IV_k = U_k$, hence $I^2 = \mathrm{Id}$ on H and it is an paracomplex structure on $\mathbb{H}$. The form $\tilde\Theta$ and the para-complex structure I (on H) define a para-contact manifold, which will be called the hyperbolic Heisenberg group. Note that by definition $\{U_1, ...U_n, V_1...V_n, \xi\}$ is an orthonormal basis of the tangent space, $g(U_j, U_j) = -g(V_j, V_j) = 1$, $j = 1, ..., n$.

b) **Hyperboloid of neutral signature.**

Let $\{x_0, y_0, \dots, x_n, y_n\}$ be the standard coordinate system in $(\mathbb{R}^{2n+2}, \mathbb{I}, g)$. Consider the hypersurface

$$HS^{2n+1} = \{(x_0, y_0, \dots, x_n, y_n) \subset \mathbb{R}^{2n+2} \mid x_0^2 + \dots + x_n^2 - y_0^2 \dots - y_n^2 = 1\}.$$

HS^{2n+1} carries a natural para-CR structure inherited from its embedding in $(\mathbb{R}^{2n+2}, \mathbb{I}, g)$. The horizontal bundle $\mathbb{H}$ is the maximal subspace of the tangent space of HS^{2n+1} which is invariant under the (restriction of the) action of $\mathbb{I}$. We take

$$\tilde\eta = -\sum_{j=0}^{n}(x_j dy_j - y_j dx_j)$$

noting that here $N=\sum_{j=0}^{n}\left(x_j\frac{\partial}{\partial x_j}+y_j\frac{\partial}{\partial y_j}\right)$ and $\xi=\sum_{j=0}^{n}\left(x_j\frac{\partial}{\partial y_j}+y_j\frac{\partial}{\partial x_j}\right)$. We will also consider HS^{2n+1} as the boundary of the "ball" $B=\{(x_0,y_0,\dots,x_n,y_n)\subset\mathbb{R}^{2n+2}\ :\ x_0^2+\cdots+x_n^2-y_0^2\cdots-y_n^2<1\}$.

Let $\Sigma_0=\{(x_0,y_0,\dots,x_n,y_n)\in HS^{2n+1}:(1+x_0)^2=y_0^2\}$. The Cayley transform (centered at Σ_0), is defined as follows

$$\mathcal{C}:HS^{2n+1}\setminus\Sigma_0\to \boldsymbol{G}\,(\mathbb{P})$$

$$t=\frac{2y_0}{(1+x_0)^2-y_0^2},\qquad u_k=\frac{x_k(1+x_0)-y_ky_0}{(1+x_0)^2-y_0^2},\qquad v_k=\frac{y_k(1+x_0)-x_ky_0}{(1+x_0)^2-y_0^2}. \tag{2.5}$$

A small calculation shows

$$\mathcal{C}^*\tilde\Theta=\frac{1}{(1+x_0)^2-y_0^2}\tilde\eta.$$

Furthermore, the para-complex structure is preserved. In order to see the last claim, we can consider $G\,(\mathbb{P})$ as the boundary of the domain $D=\{(u_0,v_0,\dots,u_n,v_n)\subset\mathbb{R}^{2n+2}\ :\ u_1^2+\cdots+u_n^2-v_1^2\cdots-v_n^2<v_0\}$ by identifying the point (p,t) with the point $(t,\Sigma_{k=1}^n(u_k^2-v_k^2),u_1,v_1\dots,u_n,v_n)$ and define the diffeomorphism $\mathcal{C}:B\setminus\Sigma_0\to D\setminus\Xi_0$, $\Xi_0=\{(1+u_0)^2-v_0^2=0\}$,

$$\begin{aligned}
u_0&=\frac{2y_0}{(1+x_0)^2-y_0^2}, && v_0=\frac{1-x_0^2+y_0^2}{(1+x_0)^2-y_0^2}\\
u_k&=\frac{x_k(1+x_0)-y_ky_0}{(1+x_0)^2-y_0^2}, && v_k=\frac{y_k(1+x_0)-x_ky_0}{(1+x_0)^2-y_0^2}.
\end{aligned}$$

A calculation shows that the above map is para-holomorphic, i.e., the coordinate functions satisfy the (para) Cauchy-Riemann equations (2.4). Thus, $\mathcal{C}$ preserves the para-CR structure when considered as a map between the boundaries of B and D.

Using hyperbolic rotations, which preserve the para-contact structure, and Cayley maps similar to the above we see that the hyperboloid is locally para-contact conformal to the hyperbolic Heisenberg group.

Finally, it is worth noting that according to Theorem 1.2 the hyperboloid is para-contact flat, while regarded as a CR submanifold of $\mathbb{C}^{n+1}$ the Chern-Moser tensor vanishes if and only if n is even, see also Section 7.

2.2. The canonical connection.

The canonical para-contact connection ∇ on a para-contact manifold is defined in [Zam]. We summarize the properties of ∇ on an integrable para-contact metric manifold from [Zam]).

Theorem 2.2. [Zam] *On an integrable para-contact hermitian manifold (M,η,I,g) there exists a unique linear connection preserving the integrable para-contact hermitian structure, i.e.*

$$\nabla\xi=\nabla I=\nabla\eta=\nabla g=0 \tag{2.6}$$

with torsion tensor $T(A,B)=\nabla_AB-\nabla_BA-[A,B]$ given by

$$T(X,Y)=-d\eta(X,Y)\xi=-2\omega(X,Y)\xi,\quad T(\xi,X)\in\mathbb{H}, \tag{2.7}$$

$$g(T(\xi,X),Y)=g(T(\xi,Y),X)=g(T(\xi,IX),IY)=\frac12\mathcal{L}_\xi g(X,Y). \tag{2.8}$$

It is shown in [Zam] that the endomorphism $T(\xi,.)$ is the obstruction an integrable para-contact hermitian manifold to be parasasakian. We denote the symmetric endomorphism $T_\xi:\mathbb{H}\longrightarrow\mathbb{H}$ by τ and call it *the torsion of the integrable para-contact hermitian manifold.* It follows that the torsion τ is completely trace-free [Zam], i.e.

$$\tau(e_a,e_a)=\tau(e_a,Ie_a)=0. \tag{2.9}$$

3. The Bianchi identities

Let $R = [\nabla, \nabla] - \nabla_{[,]}$ be the curvature of the canonical connection ∇. The Ricci tensor r, the Ricci 2-form ρ and the pc-scalar curvature Scal of ∇ are defined respectively by

$$r(A,B) = g(R(e_a,A)B,e_a), \qquad \rho(A,B) = \frac{1}{2}g(R(A,B)e_a,Ie_a), \qquad \mathrm{Scal} = r(e_a,e_a), \quad A,B \in \Gamma(TM).$$

Proposition 3.1. *Let (M,η,I,g) be an integrable para-contact hermitian manifold. Then:*

i) *The curvature of the canonical connection has the properties:*

$$R(X,Y,IZ,IV) = -R(X,Y,Z,V), \quad R(X,Y,Z,V) = -R(X,Y,V,Z), \quad R(X,Y,Z,\xi) = 0. \tag{3.1}$$

$$\begin{aligned} &\frac{1}{2}\Big[R(X,Y,Z,V) + R(IX,IY,Z,V)\Big] \\ &\quad = g(X,Z)\tau(Y,IV) + g(Y,V)\tau(X,IZ) - g(Y,Z)\tau(X,IV) - g(X,V)\tau(Y,IZ) \\ &\quad +\omega(X,Z)\tau(Y,V) + \omega(Y,V)\tau(X,Z) - \omega(Y,Z)\tau(X,V) - \omega(X,V)\tau(Y,Z); \end{aligned} \tag{3.2}$$

$$R(\xi,X,Y,Z) = (\nabla_Y\tau)(Z,X) - (\nabla_Z\tau)(Y,X). \tag{3.3}$$

ii) *The horizontal Ricci tensor is symmetric, $r(X,Y) = r(Y,X)$ and has the property*

$$r(X,Y) + r(IX,IY) = 4(1-n)\tau(X,IY). \tag{3.4}$$

iii) *The horizontal Ricci 2-from satisfies the relations*

$$\rho(X,IY) = \frac{1}{2}\Big[r(X,Y) - r(IX,IY)\Big] = \frac{1}{2}R(e_a,Ie_a,X,IY). \tag{3.5}$$

iv) *The following differential identity holds*

$$(\nabla_{e_a}r)(e_a,X) = \frac{1}{2}\,dScal(X). \tag{3.6}$$

Proof. Equation (2.6) implies immediately (3.1). The first Bianchi identity

$$\sum_{(A,B,C)}\Big\{R(A,B)C - (\nabla_A T)(B,C) - T(T(A,B),C)\Big\} = 0, \quad A,B,C \in \Gamma(TM). \tag{3.7}$$

together with (2.7) and (2.8) yields

$$\begin{aligned} &R(X,Y,Z,V) - R(Z,V,X,Y) \\ &\qquad = 2\omega(X,Z)\tau(Y,V) + 2\omega(Y,V)\tau(X,Z) - 2\omega(Y,Z)\tau(X,V) - 2\omega(X,V)\tau(Y,Z). \end{aligned} \tag{3.8}$$

$$R(\xi,X,Y,Z) - R(Y,Z,\xi,X) = (\nabla_Y\tau)(Z,X) - (\nabla_Z\tau)(Y,X). \tag{3.9}$$

Combining (3.1) with (3.8) and (3.9) yields (3.2) and (3.3).

When we take the trace of (3.8), use (2.8) and (2.9) we find

$$r(Y,Z) - r(Z,Y) = 2\omega(e_a,Z)\tau(Y,e_a) + 2\omega(Y,e_a)\tau(e_a,Z) = -2\tau(IZ,Y) + 2\tau(Y,IZ) = 0.$$

Furthermore, (3.1) and (3.2) imply

$$\begin{aligned} r(Y,Z) + r(IY,IZ) &= R(e_a,Y,Z,e_a) + R(e_a,IY,IZ,e_a) \\ &= R(e_a,Y,Z,e_a) + R(Ie_a,Y,Z,Ie_a) + 4(1-n)\tau(Y,IZ) = 4(1-n)\tau(Y,IZ). \end{aligned}$$

The first Bianchi identity (3.7) together with (2.6) and (2.8) yields

$$\begin{aligned} 2\rho(X,IY) = R(X,IY,e_a,Ie_a) &= r(X,Y) - r(IY,IX) + 2\tau(X,IY) - 2\tau(IY,X) \\ &= r(X,Y) - r(IX,IY). \end{aligned}$$

The second Bianchi identity reads

$$\sum_{(A,B,C)}\Big\{(\nabla_A R)(B,C,D,E) + R(T(A,B),C,D,E)\Big\} = 0, \qquad A,B,C,D \in \Gamma(TM). \tag{3.10}$$

A suitable trace of (3.10) leads to

$$(\nabla_{e_a}R)(X,Y,Z,e_a)-(\nabla_X r)(Y,Z)+(\nabla_Y r)(X,Z) \\ +2R(\xi,Y,Z,IX)-2R(\xi,X,Z,IY)+2\omega(X,Y)r(\xi,Z)=0. \tag{3.11}$$

Taking the trace of (3.11) we come to

$$2(\nabla_{e_a}r)(X,e_a)-d\mathrm{Scal}(X)+4r(\xi,IX)-4\rho(\xi,X)=0. \tag{3.12}$$

Equation (3.3) implies

$$r(\xi,IX)=(\nabla_{e_a}\tau)(IX,e_a)=\rho(\xi,X). \tag{3.13}$$

Now, the identity (3.6) follows from (3.12) and (3.13). □

Theorem 3.2. *Let (M,η,I,g) be an integrable para-contact hermitian manifold of dimension $2n+1$.*

- *i). If $n>1$ then (M,η,I,g) is locally isomorphic to the hyperbolic Heisenberg group exactly when the canonical connection has vanishing horizontal curvature, $R(X,Y,Z,V)=0$,*
- *ii). If $n=1$ then (M,η,I,g) is locally isomorphic to the 3-dimensional hyperbolic Heisenberg group exactly when the canonical connection has vanishing horizontal curvature and zero torsion.*

Proof. It is easy to see that the canonical connection on the hyperbolic Heisenberg group is the left-invariant connection on the group which is flat and with zero torsion endomorphism. For the converse, we first show that if $n>1$ and the horizontal curvature vanishes then the canonical connection is flat and with zero torsion endomorphism, $R=\tau=0$. Indeed, (3.4) yields $\tau=0$ and (3.3) shows $R(\xi,X,Y,Z)=0$.

Let $\{e_1,\dots,e_n,Ie_1,\dots,Ie_n,\xi\}$ be a local basis parallel with respect to ∇. Then (2.7) and (2.8) show that M has the structure of the Lie algebra of the hyperbolic Heisenberg group, which proves the claim. □

4. Paracontact conformal curvature. proof of Theorem 1.1

In this section we define para-contact conformal invariant and proof Theorem 1.1

4.1. Paracontact conformal transformations.

A conformal para-contact transformation between two para-contact manifold is a diffeomorphism Φ which preserves the para-contact structure i.e. $\Phi^*\eta=\mu\eta$, for a nowhere vanishing smooth function μ.

Let u be a smooth nowhere vanishing function on a para-contact manifold (M,η). Let $\bar\eta=\frac12 e^{-2u}\eta$ be a conformal deformation of η. We will denote the objects related to $\bar\eta$ by over-lining the same object corresponding to η. Thus, $d\bar\eta=-e^{-2u}du\wedge\eta+\frac12 e^{-2u}d\eta,\qquad \bar g=\frac12 e^{-2u}g$. The new para-contact Reeb vector field $\bar\xi$ is [Zam]

$$\bar\xi \;=\; 2e^{2u}\,\xi\;+2e^{2u}\;I\nabla u, \tag{4.1}$$

where ∇u is the horizontal gradient defined by $g(\nabla u,X)=du(X),\quad X\in H$. The horizontal sub-Laplacian and the norm of the horizontal gradient are defined respectively by $\triangle u = tr^g_H(\nabla du) = \nabla du(e_a,e_a)=\sum_{s=1}^n(\nabla du(e_s,e_s)-\nabla du(Ie_s,Ie_s)),\qquad |\nabla u|^2 = du(e_a)^2=\sum_{s=1}^n(du(e_s)^2-du(Ie_s)^2)$. The canonical para-contact connections ∇ and $\bar\nabla$ are connected by a (1,2) tensor S,

$$\bar\nabla_AB=\nabla_AB+S_AB,\qquad A,B\in\Gamma(TM). \tag{4.2}$$

The condition (2.7) yields

$$g(S(X,Y),Z)-g(S(Y,X),Z)=-2\omega(X,Y)Idu(Z)=2\omega(X,Y)du(IZ). \tag{4.3}$$

From $\bar\nabla\bar g=0$ we get

$$g(S(X,Y),Z)+g(S(X,Z),Y)=-2du(X)g(Y,Z). \tag{4.4}$$

The last two equations determine $g(S(X,Y),Z)$ for $X,Y,Z\in H$ due to the equality

$$g(S(X,Y),Z) = -du(X)g(Y,Z) - du(IX)\omega(Y,Z) - du(Y)g(Z,X) + du(IY)\omega(Z,X) + du(Z)g(X,Y) + du(IZ)\omega(X,Y). \tag{4.5}$$

We obtain after some calculations using (4.1) that

$$\bar\tau(X,Y) - 2e^{2u}\tau(X,Y) - g(S(\bar\xi,X),Y) = -2e^{2u}\nabla du(X,IY) - 4e^{2u}du(X)du(IY). \tag{4.6}$$

From (4.6) and (2.8) we find

$$g(S(\bar\xi,X),Y) - g(S(\bar\xi,IX)IY) = 2e^{2u}\Big[\nabla du(X,IY) - \nabla du(IX,Y) + 2du(X)du(IY) - 2du(IX)du(Y)\Big]. \tag{4.7}$$

The condition $\bar\nabla I = \nabla I = 0$ yields $g(S(\bar\xi,X),Y) = -g(S(\bar\xi,IX)IY)$. Substitute the latter into (4.6) and (4.7), use (4.1) and (4.5) to get

$$g(S(\xi,X),Y) = \frac{1}{2}\Big[\nabla du(X,IY) - \nabla du(IX,Y)\Big] - du(X)du(IY) + du(IX)du(Y) + |\nabla u|^2\omega(X,Y), \tag{4.8}$$

$$\bar\tau(X,Y) = e^{2u}\Big[2\tau(X,Y) - \nabla du(X,IY) - \nabla du(IX,Y) - 2du(X)du(IY) - 2du(IX)du(Y)\Big]. \tag{4.9}$$

In addition, the pc-scalar curvature changes according to the formula [Zam]

$$\overline{\mathrm{Scal}} = 2e^{2u}\,\mathrm{Scal} - 8n(n+1)e^{2u}|\nabla u|^2 + 8(n+1)e^{2u}\triangle u. \tag{4.10}$$

The identity $d^2=0$ yields $\nabla du(X,Y) - \nabla du(Y,X) = -du(T(X,Y))$. Applying (2.7), we can write

$$\nabla du(X,Y) = [\nabla du]_{[sym]}(X,Y) + du(\xi)\omega(X,Y), \tag{4.11}$$

where $[.]_{[sym]}$ denotes the symmetric part of the corresponding (0,2)-tensor.

4.2. **Paracontact conformal curvature.** let (M,η,I,g) be a (2n+1)-dimensional integrable paracontact manifold. Let us consider the symmetric (0,2) tensor L defined on H by the equality

$$L(X,Y) = \frac{1}{2(n+2)}\rho(X,IY) - \tau(IX,Y) - \frac{\mathrm{Scal}}{8(n+1)(n+2)}g(X,Y), \qquad X,Y\in H. \tag{4.12}$$

We define the (0,4) tensor PW on H by

$$\begin{aligned} g(PW(X,Y)Z,V) &= g(R(X,Y)Z,V) \\ &+ g(X,Z)L(Y,V) + g(Y,V)L(X,Z) - g(Y,Z)L(X,V) - g(X,V)L(Y,Z) \\ &+ \omega(X,Z)L(Y,IV) + \omega(Y,V)L(X,IZ) - \omega(Y,Z)L(X,IV) - \omega(X,V)L(Y,IZ) \\ &+ \omega(X,Y)\Big[L(Z,IV) - L(IZ,V)\Big] + \omega(Z,V)\Big[L(X,IY) - L(IX,Y)\Big] \end{aligned} \tag{4.13}$$

Denote the trace-free part of L with L_0. Then (4.12) yields $trL = \frac{s}{4(n+1)}$ and

$$L_0 = \frac{1}{2(n+2)}\rho_0(X,IY) - \tau(X,IY), \tag{4.14}$$

where $\rho_0 = \rho + \frac{\mathrm{Scal}}{2n}\omega$ is the trace-free part of ρ. A substitution of (4.12) and (4.14) in (4.13) gives

$$\begin{aligned} g(PW(X,Y)Z,V) &= g(R(X,Y)Z,V) - \frac{s}{2(n+1)}\Big[g(X,Z)g(Y,V) - g(Y,Z)g(X,V)\Big] \\ &+ \frac{s}{2(n+1)}\Big[\omega(X,Z)\omega(Y,V) - \omega(Y,Z)\omega(X,V) + 2\omega(X,Y)\omega_s(Z,V)\Big]\Big\} \\ &+ g(X,Z)L_0(Y,V) + g(Y,V)L_0(X,Z) - g(Y,Z)L_0(X,V) - g(X,V)L_0(Y,Z) \\ &+ \omega(X,Z)L_0(Y,IV) + \omega(Y,V)L_0(X,IZ) - \omega(Y,Z)L_0(X,IV) - \omega(X,V)L_0(Y,IZ) \\ &+ \omega(X,Y)[L_0(Z,IV) - L_0(IZ,V)] + \omega(Z,V)[L_0(X,IY) - L_0(IX,Y)] \end{aligned} \tag{4.15}$$

Proposition 4.1. *The tensor PW is completely trace-free, i.e.*

$$r(PW) = \rho(PW) = 0.$$

Proof. The claim follows after taking the corresponding traces in (4.13) keeping in mind (4.12). □

If we compare (4.13) and (3.3) we obtain the following

Proposition 4.2. *For $n > 1$ the tensor PW has the properties*

$$PW(X,Y,Z,V) + PW(IX,IY,Z,V) = 0,$$

$$\begin{aligned}(4.16)\quad \frac{1}{2}\Big[PW(X,Y,Z,V) - PW(IX,IY,Z,V)\Big] &= \frac{1}{2}\Big[R(X,Y,Z,V) - R(IX,IY,Z,V)\Big] \\ &- \frac{Scal}{8(n+1)(n+2)}\Big[g(X,Z)g(Y,V) - g(Y,Z)g(X,V)\Big] \\ &+ \frac{Scal}{8(n+1)(n+2)}\Big[\omega(X,Z)\omega(Y,V) - \omega(Y,Z)\omega(X,V) + 2\omega(X,Y)\omega_s(Z,V)\Big] \\ &+ \frac{1}{2(n+2)}\Big[g(X,Z)\rho(Y,IV) - g(Y,Z)\rho(X,IV) + g(Y,V)\rho(X,IZ) - g(X,V)\rho(Y,IZ)\Big] \\ &+ \frac{1}{2(n+2)}\Big[\omega(X,Z)\rho(Y,V) - \omega(Y,Z)\rho(X,V) + \omega(Y,V)\rho(X,Z) - \omega(X,V)\rho(Y,Z)\Big] \\ &+ \frac{1}{2(n+2)}\Big[2\omega(X,Y)\rho(Z,V) + 2\omega(Z,V)\rho(X,Y)\Big].\end{aligned}$$

For $n = 1$ the tensor PW vanishes identically.

Definition 4.3. *We denote the tensor $PW(X,Y,Z,V) - PW(IX,IY,Z,V)$ by $2W^{pc}$ and call it* the para-contact conformal curvature.

4.3. **Proof of Theorem 1.1.** First we show

Theorem 4.4. *The para-contact conformal curvature W^{pc} is invariant under conformal para-contact transformations, i.e., if*

$$2\bar\eta = e^{-2u}\eta \quad \textit{for any smooth function } u \textit{ then} \quad 2e^{2u}W^{qc}_{\bar\eta} = W^{qc}_{\eta}.$$

Proof. After a long straightforward computation using (4.2), (4.5) and (4.8) we obtain the formula

$$\begin{aligned}(4.17)\quad 2e^{2u}g(\bar R(X,Y)Z,V) - g(R(X,Y)Z,V) &= -g(Z,V)\Big[M(X,Y) - M(Y,X)\Big] \\ &- g(X,Z)M(Y,V) - g(Y,V)M(X,Z) + g(Y,Z)M(X,V) + g(X,V)M(Y,Z) \\ &- \omega(X,Z)M(Y,IV) - \omega(Y,V)M(X,IZ) + \omega(Y,Z)M(X,IV) + \omega(X,V)M(Y,IZ) \\ &- \omega(X,Y)\Big[M(Z,IV) - M(IZ,V)\Big] - \omega(Z,V)\Big[M(X,IY) - M(Y,IX)\Big].\end{aligned}$$

where the (0,2) tensor M is given by

$$(4.18)\qquad M(X,Y) = \nabla du(X,Y) + du(X)du(Y) + du(IX)du(IY) - \frac{1}{2}g(X,Y)|\nabla u|^2.$$

Let $trM = M(e_a,e_a)$ be the trace of the tensor M. Using (4.18) and (4.11) we obtain

$$(4.19)\qquad trM = \triangle u - n|\nabla u|^2, \quad M(X,Y) + M(IX,IY) = M(Y,X) + M(IY,IX),$$

$$(4.20)\qquad M(e_a,Ie_a) = -2ndu(\xi), \quad M(e_a,Ie_a)\omega(X,Y) = -n\Big[M(X,Y) - M(Y,X)\Big].$$

Taking the trace in (4.17) and using (4.18), (4.19), and (4.20) we come to

$$\begin{aligned}(4.21)\quad &\overline{r}(X,Y) - r(X,Y) \\ &= (n+1)M(X,Y) + nM(Y,X) - M(IX,IY) - 2M(IY,IX) + trM\, g(X,Y), \\ &e^{-2u}\overline{Scal} - 2\,\mathrm{Scal} = 8(n+1)trM.\end{aligned}$$

Proposition 3.1 together with (4.21) imply

$$\overline{r}(X,Y)-r(X,Y)-\Big[\overline{r}(IX,IY)-r(IX,IY)\Big]=2\Big[\overline{\rho}(X,IY)-\rho(X,IY)\Big]= \tag{4.22}$$
$$(n+2)\Big[M(X,Y)+M(Y,X)-M(IX,IY)-M(IY,IX)\Big]+2(trM)\,g(X,Y),$$

$$\overline{r}(X,Y)-r(X,Y)+\Big[\overline{r}(IX,IY)-r(IX,IY)\Big]=4(1-n)\Big[\overline{\tau}(X,IY)-\tau(X,IY)\Big] \tag{4.23}$$
$$=(n-1)\Big[M(X,Y)+M(Y,X)+M(IX,IY)+M(IY,IX)\Big].$$

The equalities (4.21), (4.22) and (4.23) yield

$$M_{[sym]}=\frac{1}{2(n+2)}\overline{\rho}(X,IY)-\overline{\tau}(IX,Y)-\frac{\overline{S}cal}{8(n+1)(n+2)}\overline{g}(X,Y) \tag{4.24}$$
$$-\Big[\frac{1}{2(n+2)}\rho(X,IY)-\tau(IX,Y)-\frac{\mathrm{Scal}}{8(n+1)(n+2)}g(X,Y)\Big]=\overline{L}(X,Y)-L(X,Y)$$

Now, from (4.18) and (4.11) we obtain

$$M(X,Y)=M_{[sym]}(X,Y)+du(\xi)\omega(X,Y). \tag{4.25}$$

Substitute (4.24) into (4.25), the obtained equality insert into (4.17) and use (4.19) to complete the proof of Theorem 4.4 $\square$

At this point a combination of Theorem 4.4 and Proposition 4.2 ends the proof of Theorem 1.1.

5. Converse problem. Proof of Theorem 1.2

Suppose $W^{pc}=0$, hence $PW=0$ by Proposition 4.2. We shall show that in this case there exists (locally) a smooth conformal factor u which changes by a pc-contact conformal transformation the integrable para-contact hermitian structure to a torsion-free flat one.

Let us consider the following system of differential equations with respect to the unknown function u

$$\nabla du(X,Y)=-L(X,Y)-du(X)du(Y)-du(IX)du(IY)+\frac{1}{2}g(X,Y)|\nabla u|^2+du(\xi)\omega(X,Y) \tag{5.1}$$

$$\nabla du(X,\xi)=-\mathbb{B}(X,\xi)-L(X,I\nabla u)+\frac{1}{2}du(IX)|\nabla u|^2-du(X)du(\xi_i) \tag{5.2}$$

$$\nabla du(\xi,\xi)=-\mathbb{B}(\xi,\xi)-\mathbb{B}(I\nabla u,\xi)-\frac{1}{4}|\nabla u|^4-(du(\xi))^2, \tag{5.3}$$

where $\mathbb{B}(X,\xi)$ and $\mathbb{B}(\xi,\xi)$ do not depend on the function u and are determined in (5.9) and (5.28).

In order to prove Theorem 1.2 it is sufficient to show the existence of a local smooth solution to (5.1). Indeed, suppose u is a local smooth solution to (5.1). Then the canonical connection of the para-contact hermitian structure $2\overline{\eta}=e^{-2u}\eta$ has in view of (4.9) zero torsion. Furthermore, the curvature restricted to H vanishes when $W^{pc}=0$ taking into account Proposition 4.2 and the proof of Theorem 4.4. Therefore, we can apply Theorem 3.2 to conclude the result.

The rest of this section is devoted to showing the existence of a smooth solution to the system (5.1)-(5.3).

The integrability conditions for this overdetermined system are the Ricci identities,

$$\nabla du(A,B,C)-\nabla du(B,A,C)=-R(A,B,C,\nabla u)-\nabla du((T(A,B),C),\quad A,B,C\in\Gamma(TM). \tag{5.4}$$

We consider as separate cases the four possibilities for A, B and C.

Case 1: $\Big[A,B,C\in H\Big]$.

The equation (5.4) on H has the following form

$$\nabla du(Z,X,Y)-\nabla du(X,Z,Y)=-R(Z,X,Y,\nabla u)+2\omega(Z,X)\nabla du(\xi,Y), \tag{5.5}$$

after using (2.7).

Let us take a covariant derivative of (5.1) along $Z \in H$, substitute in the obtained equality (5.1) and (4.12), anticommute the covariant derivatives, and then substitute the result in the left hand side of (5.5). The result is

$$
\begin{aligned}
(5.6)\quad & \nabla du(Z,X,Y) - \nabla du(X,Z,Y) = -(\nabla_Z L)(X,Y) + (\nabla_X L)(Z,Y) \\
& + g(Z,Y)L(X,\nabla u) - \Big[\nabla du(X,\xi) - \frac{1}{2}du(IX)|\nabla u|^2 + du(X)du(\xi)\Big]\omega(Z,Y) + L(Z,Y)du(X) \\
& - g(X,Y)L(Z,\nabla u) + \Big[\nabla du(Z,\xi) - \frac{1}{2}du(IZ)|\nabla u|^2 + du(Z)du(\xi)\Big]\omega(X,Y) - L(X,Y)du(Z) \\
& - [2\nabla du(Y,\xi) - 2\nabla du(\xi,Y) + du(Y)du(\xi) - du(IY)|\nabla u|^2 + L(Y,I\nabla u) + L(IY,\nabla u)]\omega(Z,X) \\
& + L(Z,IX)du(IY) - L(X,IZ)du(IY) + L(Z,IY)du(IX) - L(X,IY)du(IZ).
\end{aligned}
$$

Now, let $W^{pc} = 0$ in (4.13) and then insert the result in the right hand side of (5.5) to get

$$
\begin{aligned}
(5.7)\quad & -R(Z,X,Y,\nabla u) + 2\omega(Z,X)\nabla du(\xi,Y) = 2\omega(Z,X)\nabla du(\xi,Y) \\
& + L(Z,Y)du(X) - L(X,Y)du(Z) + L(Z,IX)du(IY) - L(X,IZ)du(IY) \\
& + L(Z,IY)du(IX) - L(X,IY)du(IZ) - g(X,Y)L(Z,\nabla u) + g(Z,Y)L(X,\nabla u) \\
& + \omega(Z,Y)L(X,I\nabla u) - \omega(X,Y)L(Z,I\nabla u) + \omega(Z,X)[L(Y,I\nabla u) - L(IY,\nabla u)].
\end{aligned}
$$

From (5.6), (5.7), using (5.2) and (5.1), we see that the integrability condition (5.5) is

$$(5.8)\qquad (\nabla_Z L)(X,Y) - (\nabla_X L)(Z,Y) = \omega(Z,Y)\mathbb{B}(X,\xi) - \omega(X,Y)\mathbb{B}(Z,\xi) + 2\omega(Z,X)\mathbb{B}(Y,\xi).$$

The 1-forms $\mathbb{B}(X,\xi)$ can be determined by taking traces in (5.8). Thus, we have

$$(5.9)\qquad (\nabla_{e_a} L)(Ie_a, IX) = -(2n+1)\mathbb{B}(IX,\xi) \quad \text{and} \quad (\nabla_X tr\, L) - (\nabla_{e_a} L)(e_a, X) = 3\mathbb{B}(IX,\xi).$$

Notice that the consistences of the first and second equality in (5.9) is equivalent to (3.6).

Lemma 5.1. *Suppose $W^{pc} = 0$ and the dimension of our manifold is bigger than three. Then (5.8) holds.*

Proof. Using (2.7), the second Bianchi identity (3.10) gives

$$(5.10)\qquad \sum_{(X,Y,Z)} \Big[(\nabla_X R)(Y,Z,V,W) - 2\omega(X,Y)R(\xi,Z,V,W)\Big] = 0.$$

From (5.10), we obtain

$$
\begin{aligned}
(5.11)\quad & (\nabla_{e_a} R)(X,Y,Z,e_a) - (\nabla_X r)(Y,Z) + (\nabla_Y r)(X,Z) \\
& + 2R(\xi,Y,Z,IX) - 2R(\xi,X,Z,IY) + 2\omega(X,Y)r(\xi,Z) = 0
\end{aligned}
$$

$$
\begin{aligned}
(5.12)\quad & (\nabla_X \rho)(Y,Z) + (\nabla_Y \rho)(Z,X) + (\nabla_Z \rho)(X,Y) - \\
& - 2\omega(X,Y)\rho(\xi,Z) - 2\omega(Y,Z)\rho(\xi,X) - 2\omega(Z,X)\rho(\xi,Y) = 0
\end{aligned}
$$

$$(5.13)\qquad (\nabla_X \rho)(Y,Z) + (\nabla_{e_a} R)(Ie_a, X, Y, Z) + 2(n-1)R(\xi,X,Y,Z) = 0.$$

We use the condition $W^{pc} = 0$ and (4.12) to express r, ρ and τ in terms of L and $tr\, L$, namely

$$(5.14)\qquad r(X,Y) = (2n+1)L(X,Y) - 3L(IX,IY) + (trL)g(X,Y)$$

$$(5.15)\qquad \rho(X,Y) = (n+2)L(X,IY) - (n+2)L(IX,Y) - (trL)\omega(X,Y)$$

$$(5.16)\qquad \tau(IX,Y) = -\frac{1}{2}[L(X,Y) + L(IX,IY)].$$

Inserting (4.13) and (3.3) in(5.11), and then using (5.14), (5.16) we come after some standard calculations to the following identity

$$\begin{aligned}(5.17)\quad &-3g(Z,X)\mathbb{B}(IY,\xi)+3g(Z,Y)\mathbb{B}(IX,\xi)-(2n+1)\omega(X,Z)\mathbb{B}(Y,\xi)\\ &+(2n+1)\omega(Y,Z)\mathbb{B}(X,\xi)-2(2n+1)\omega(X,Y)\mathbb{B}(Z,\xi)+2n[(\nabla_X L)(Y,Z)-(\nabla_Y L)(X,Z)]\\ &+[(\nabla_{IZ}L)(X,IY)-(\nabla_{IZ}L)(IX,Y)]-[(\nabla_{IX}L)(IY,Z)-(\nabla_{IY}L)(IX,Z)]\\ &-2[(\nabla_{IX}L)(Y,IZ)-(\nabla_{IY}L)(X,IZ)]-3[(\nabla_X L)(IY,IZ)-(\nabla_Y L)(IX,IZ)]=0\end{aligned}$$

Substituting (4.13) and (3.3) in (5.12) together with (5.15) gives, after some calculations,

$$\begin{aligned}(5.18)\quad &(n+2)[(\nabla_X L)(Y,IZ)-(\nabla_Y L)(X,IZ)]-(n+2)[(\nabla_X L)(IY,Z)-(\nabla_Y L)(IX,Z)]\\ &+(n+2)[(\nabla_Z L)(X,IY)-(\nabla_Z L)(IX,Y)]\\ &-2(n+2)[\omega(X,Y)\mathbb{B}(IZ,\xi)+\omega(Y,Z)\mathbb{B}(IX,\xi)+\omega(Z,X)\mathbb{B}(IY,\xi)]=0.\end{aligned}$$

When we plug IZ for Z in (5.18) it follows

$$\begin{aligned}(5.19)\quad &[(\nabla_X L)(Y,Z)-(\nabla_Y L)(X,Z)]-[(\nabla_X L)(IY,IZ)-(\nabla_Y L)(IX,IZ)]+\\ &+[(\nabla_{IZ}L)(X,IY)-(\nabla_{IZ}L)(IX,Y)]-2\omega(X,Y)\mathbb{B}(Z,\xi)+2g(Y,Z)\mathbb{B}(IX,\xi)-2g(Z,X)\mathbb{B}(IY,\xi)=0.\end{aligned}$$

Setting IX and IY, correspondingly, for X and Y in (5.19) and taking the sum of the obtained equality and (5.19) we derive

$$\begin{aligned}(5.20)\quad &[(\nabla_X L)(Y,Z)-(\nabla_Y L)(X,Z)]-[(\nabla_X L)(IY,IZ)-(\nabla_Y L)(IX,IZ)]+\\ &+[(\nabla_{IX}L)(IY,Z)-(\nabla_{IY}L)(IX,Z)]-[(\nabla_{IX}L)(Y,IZ)-(\nabla_{IY}L)(IX,Z)]+\\ &+2g(Y,Z)\mathbb{B}(IX,\xi)-2g(Z,X)\mathbb{B}(IY,\xi)+2\omega(Y,Z)\mathbb{B}(X,\xi)-2\omega(X,Z)\mathbb{B}(Y,\xi)=0.\end{aligned}$$

Insert (4.13) and (3.3) in (5.13), and use (5.15), (5.16) to get

$$\begin{aligned}(5.21)\quad &(n+1)[(\nabla_X L)(Y,IZ)-(\nabla_X L)(IY,Z)]+(n-1)[(\nabla_Z L)(X,IY)+(\nabla_Z L)(IX,Y)\\ &-(\nabla_Y L)(X,IZ)-(\nabla_Y L)(IX,Z)]-(2n+1)\omega(X,Y)\mathbb{B}(IZ,\xi)+(2n+1)\omega(X,Z)\mathbb{B}(IY,\xi)\\ &-(2n+1)g(X,Y)\mathbb{B}(Z,\xi)+(2n+1)g(X,Z)\mathbb{B}(Y,\xi)+2(n-1)\omega(Y,Z)\mathbb{B}(IX,\xi)]=0.\end{aligned}$$

Replacing in (5.21) Y and Z respectively with IY and IZ, and then adding the obtained result to (5.21) allows us to conclude

$$\begin{aligned}(5.22)\quad &(n-1)[(\nabla_{IZ}L)(X,IY)-(\nabla_{IY}L)(X,IZ)]+(n-1)[(\nabla_{IZ}L)(IX,Y)-(\nabla_{IY}L)(IX,Z)]\\ &+(n-1)[(\nabla_Z L)(X,Y)-(\nabla_Y L)(X,Z)]+(n-1)[(\nabla_Z L)(IX,IY)-(\nabla_Y L)(IX,IZ)]=0\end{aligned}$$

Substitute $X\to Z, Z\to X$ into (5.22). The sum of the obtained equality and (5.20) yields

$$\begin{aligned}(5.23)\quad &[(\nabla_X L)(Y,Z)-(\nabla_Y L)(X,Z)]+[(\nabla_{IX}L)(IY,Z)-(\nabla_{IY}L)(IX,Z)]-\\ &-g(X,Z)\mathbb{B}(IY,\xi)+g(Y,Z)\mathbb{B}(IX,\xi)-\omega(X,Z)\mathbb{B}(Y,\xi)+\omega(Y,Z)\mathbb{B}(X,\xi)=0\end{aligned}$$

The cyclic sum in (5.23) gives

$$\begin{aligned}(5.24)\quad &[(\nabla_{IZ}L)(X,IY)-(\nabla_{IZ}L)(IX,Y)]=[(\nabla_{IX}L)(IY,Z)-(\nabla_{IY}L)(IX,Z)]-\\ &-[(\nabla_{IX}L)(Y,IZ)-(\nabla_{IY}L)(X,IZ)]+2\omega(Z,X)\mathbb{B}(Y,\xi)+2\omega(Y,Z)\mathbb{B}(X,\xi)+2\omega(X,Y)\mathbb{B}(Z,\xi).\end{aligned}$$

Now, identity (5.8) follows from (5.17), (5.23) and (5.24). □

Case 2: $\Big[A,B\in H,\quad C=\xi\Big]$.

In this case (5.4) turn into the equation

$$(5.25)\quad \nabla du(Z,X,\xi)-\nabla du(X,Z,\xi)=-R(Z,X,\xi,\nabla u)-\nabla du(T(Z,X),\xi)=2\omega(Z,X)\nabla du(\xi,\xi),$$

where we used (2.7).

Take a covariant derivative of (5.2) along $Z \in H$, substitute (5.1) and (5.2) in the obtained equality and then anticommute the covariant derivatives to get

$$
\begin{aligned}
(5.26)\quad \nabla du(Z,X,\xi) - \nabla du(X,Z,\xi) &= -(\nabla_Z \mathbb{B})(X,\xi) + (\nabla_X \mathbb{B})(Z,\xi)\\
&- (\nabla_Z L)(X, I\nabla u) + (\nabla_X L)(Z, I\nabla u) - L(X, \nabla_Z I\nabla u) + L(Z, \nabla_X I\nabla u)\\
&- \frac{1}{2}|\nabla u|^2[L(Z,IX) - L(X,IZ)] - \frac{1}{2}|\nabla u|^4\omega(Z,X) - 2(du(\xi))^2\omega(Z,X)\\
&- du(IX)L(Z,\nabla u) + du(IZ)L(X,\nabla u) + du(X)L(Z,I\nabla u) - du(Z)L(X,I\nabla u)\\
&= -(\nabla_Z \mathbb{B})(X,\xi) + (\nabla_X \mathbb{B})(Z,\xi) - 2\omega(Z,X)\mathbb{B}(I\nabla u,\xi) - L(X,\nabla_Z I\nabla u) + L(Z,\nabla_X I\nabla u)\\
&- \frac{1}{2}|\nabla u|^2[L(Z,IX) - L(X,IZ)] - du(X)L(Z,I\nabla u) - \frac{1}{2}|\nabla u|^4\omega(Z,X) - 2(du(\xi))^2\omega(Z,X)\\
&- du(IX)L(Z,\nabla u) + du(IZ)L(X,\nabla u) + du(X)L(Z,I\nabla u) - du(Z)L(X,I\nabla u)\\
&= -(\nabla_Z \mathbb{B})(X,\xi) + (\nabla_X \mathbb{B})(Z,\xi) + 2\omega(Z,X)\nabla du(\xi,\xi) - L(Z,IL(X)) + L(X,IL(Z))\\
&- 2\omega(Z,X)\Big[\nabla du(\xi,\xi) + \mathbb{B}(I\nabla u,\xi) + \frac{1}{4}|\nabla u|^4 + (du(\xi))^2\Big].
\end{aligned}
$$

A substitution of (5.26) in(5.25), and a use of the already established (5.8), (5.3) and (4.12) give after some standard calculations that the integrability condition in this case is

$$
(5.27)\qquad (\nabla_Z \mathbb{B})(X,\xi) - (\nabla_X \mathbb{B})(Z,\xi) + L(Z,IL(X)) - L(X,IL(Z)) = 2\mathbb{B}(\xi,\xi)\omega(Z,X).
$$

Here, the function $\mathbb{B}(\xi,\xi)$ is independent of u and is uniquely determined by

$$
(5.28)\qquad \mathbb{B}(\xi,\xi) = -\frac{1}{2n}[(\nabla_{e_a}\mathbb{B})(Ie_a,\xi) + L(e_a, IL(Ie_a))].
$$

Lemma 5.2. *If $W^{pc} = 0$ and the dimension is bigger than three, then (5.27) holds.*

Proof. Differentiate the already proved (5.8), take the corresponding traces and use the symmetry of L to see

$$
\begin{aligned}
(5.29)\quad & (\nabla^2_{e_a,Ie_a}L)(Y,Z) - (\nabla^2_{e_a,Y}L)(Ie_a,Z)\\
&\qquad = (\nabla_Z \mathbb{B})(Y,\xi) - \omega(Y,Z)(\nabla_{e_a}\mathbb{B})(Ie_a,\xi) + 2(\nabla_Y \mathbb{B})(Z,\xi)\\
(5.30)\quad & -(\nabla^2_{e_a,Y}L)(Ie_a,Z) + (\nabla^2_{e_a,Z}L)(Ie_a,Y)\\
&\qquad = -(\nabla_Z \mathbb{B})(Y,\xi) - 2\omega(Y,Z)(\nabla_{e_a}\mathbb{B})(Ie_a,\xi) + (\nabla_Y \mathbb{B})(Z,\xi)\\
(5.31)\quad & (\nabla^2_{Y,e_a}L)(Ie_a,Z) = -(2n+1)(\nabla_Y \mathbb{B})(Z,\xi).
\end{aligned}
$$

A combination of (5.29), (5.31) and (5.30) yield

$$
\begin{aligned}
(5.32)\quad & \Big[(\nabla^2_{Y,e_a}L) - (\nabla^2_{e_a,Y}L)\Big](Ie_a,Z) - \Big[(\nabla^2_{Z,e_a}L) - (\nabla^2_{e_a,Z}L)\Big](Ie_a,Y)\\
&\qquad = 2n(\nabla_Z \mathbb{B})(Y,\xi) - \omega(Y,Z)(\nabla_{e_a}\mathbb{B})(Ie_a,\xi) - 2n(\nabla_Y \mathbb{B})(Z,\xi).
\end{aligned}
$$

The Ricci identities, (2.7) and (3.5) give

$$
\begin{aligned}
(5.33)\quad & \Big[(\nabla^2_{Y,e_a}L) - (\nabla^2_{e_a,Y}L)\Big](Ie_a,Z)\\
&= -R(Y,e_a,Ie_a,e_b)L(e_b,Z) - R(Y,e_a,Z,e_b)L(Ie_a,e_b) - (\nabla_{T(Y,e_a)}L)(Ie_a,Z)\\
&= r(Y,Ie_b)L(e_b,Z) - R(Y,e_a,Z,e_b)L(Ie_a,e_b) + 2(\nabla_\xi L)(Y,Z)\\
&= (2n+1)L(Y,IL(Z)) - 3L(IY,L(Z)) + 2(\nabla_\xi L)(Y,Z) - (trL)[L(Y,IZ) + L(IY,Z)]\\
&\qquad - 3L(IZ,L(Y)) + L(Z,IL(Y)) + \omega(Y,Z)L(e_a,IL(e_a)).
\end{aligned}
$$

$$(5.34)\quad (\nabla^2_{e_a,Ie_a}L)(Y,Z)=\frac{1}{2}\Big[\nabla^2_{e_a,Ie_a}-\nabla^2_{Ie_a,a}\Big]L(Y,Z)$$
$$=-\frac{1}{2}\Big[R(e_a,Ie_a,Y,e_b)L(e_b,Z)+R(e_a,Ie_a,Z,e_b)L(e_b,Y)+(\nabla_{T(e_a,Ie_a)}L)(Y,Z)\Big]$$
$$=-\rho(Y,e_b)L(e_b,Z)-\rho(Z,e_b)L(e_b,Y)-2n(\nabla_\xi L)(Y,Z)=(n+2)[L(IY,L(Z))-L(Y,IL(Z))]$$
$$-(n+2)L(Z,IL(Y))+(n+2)L(IZ,L(Y))-2n(\nabla_\xi L)(Y,Z)+(trL)(L(IY,Z)+L(Y,IZ)).$$

The identity (5.27) follows from (5.32) and (5.33). $\square$

Case 3: $\Big[A=\xi,\quad B,C\in H\Big]$.

In this case (5.4) becomes

$$(5.35)\quad \nabla du(\xi,X,Y)-\nabla du(X,\xi,Y)=-R(\xi,X,Y,\nabla u)-\nabla du(T(\xi,X),Y).$$

Take the covariant derivative of (5.1) along ξ and a covariant derivative of (5.2) along a horizontal direction, and then use (5.2), (5.1) and (5.3) to express the left hand side of (5.35) as follows

$$(5.36)\quad \nabla du(\xi,X,Y)-\nabla du(X,\xi,Y)=-(\nabla_\xi L)(X,Y)+(\nabla_X\mathbb{B})(Y,\xi)+(\nabla_X L)(Y,I\nabla u)$$
$$+L(Y,\nabla_X I\nabla u)+(\nabla_X T)(\xi,Y,\nabla u)+T(\xi,Y,\nabla u)du(Y)+\frac{1}{2}|\nabla u|^2L(X,IY)+\frac{1}{4}|\nabla u|^4\omega(X,Y)$$
$$+du(IY)L(X,\nabla u)-du(\xi)L(X,Y)+(du(\xi))^2\omega(X,Y)+du(X)\mathbb{B}(Y,\xi)+\nabla du(\xi,\xi)\omega(X,Y)$$
$$+du(X)L(Y,I\nabla u)+du(IY)\mathbb{B}(IX,\xi)+du(IY)L(IX,I\nabla u)+du(IY)T(\xi,IX,\nabla u)$$
$$+du(IX)\mathbb{B}(IY,\xi)+du(IX)L(IY,I\nabla u)+du(IX)T(\xi,IY,\nabla u)-g(X,Y)B(\nabla u,\xi)$$
$$=-(\nabla_\xi L)(X,Y)+(\nabla_X\mathbb{B})(Y,\xi)-L(Y,IL(X))-\mathbb{B}(\xi,\xi)\omega(X,Y)-g(T(\xi,Y),L(X))$$
$$+(\nabla_X L)(Y,I\nabla u)+(\nabla_X T)(\xi,Y,\nabla u)+T(\xi,X,\nabla u)du(Y)-T(\xi,X,I\nabla u)du(IY)$$
$$-\frac{1}{2}T(\xi,X,Y)|\nabla u|^2+du(\xi)T(\xi,X,IY)+du(X)\mathbb{B}(Y,\xi)+du(IY)\mathbb{B}(IX,\xi)$$
$$+du(IX)\mathbb{B}(IY,\xi)-g(X,Y)\mathbb{B}(\nabla u,\xi)-\omega(X,Y)\mathbb{B}(I\nabla u,\xi).$$

A substitution of (3.3) in the right hand side of (5.35), and a use of suitable traces of (5.1) and (5.16) gives

$$(5.37)\quad -R(\xi,X,Y,\nabla u)-\nabla du(T(\xi,X),Y)=(\nabla_{\nabla u}T)(\xi,X,Y)-(\nabla_Y T)(\xi,X,\nabla u)$$
$$=-\frac{1}{2}[(\nabla_{\nabla u}L)(IX,Y)+(\nabla_{\nabla u}L)(X,IY)]+\frac{1}{2}[(\nabla_Y L)(IX,\nabla u)+(\nabla_Y L)(X,I\nabla u)]+$$
$$L(T(\xi,X),Y)+T(\xi,X,\nabla u)du(Y)-T(\xi,X,I\nabla u)du(IY)-\frac{1}{2}T(\xi,X,Y)|\nabla u|^2+du(\xi)T(\xi,X,IY).$$

Inserting (5.36) and (5.37) in (5.39), yields with the help of (5.16) and (4.12) the equation

$$(5.38)\quad (\nabla_X\mathbb{B})(Y,\xi)-(\nabla_\xi L)(X,Y)-L(Y,IL(X))-\mathbb{B}(\xi,\xi)\omega(X,Y)-g(T(\xi,Y),L(X))$$
$$-L(T(\xi,X),Y)+du(X)\mathbb{B}(Y,\xi)+du(IY)\mathbb{B}(IX,\xi)+du(IX)\mathbb{B}(IY,\xi)$$
$$-g(X,Y)\mathbb{B}(\nabla u,\xi)-\omega(X,Y)\mathbb{B}(I\nabla u,\xi)=\frac{1}{2}[(\nabla_X L)(\nabla u,IY)-(\nabla_{\nabla u}L)(X,IY)]$$
$$+\frac{1}{2}[(\nabla_Y L)(\nabla u,IX)-(\nabla_{\nabla u}L)(Y,IX)]+\frac{1}{2}[(\nabla_Y L)(X,I\nabla u)-(\nabla_X L)(Y,I\nabla u)].$$

If we apply (5.8) to (5.38) we see that the integrability condition (5.35) becomes

$$(5.39)\quad (\nabla_X\mathbb{B})(Y,\xi)-(\nabla_\xi L)(X,Y)=L(Y,IL(X))+\tau(X,L(Y))+\tau(Y,L(X))+\mathbb{B}(\xi,\xi)\omega(X,Y).$$

Notice that Case 3 implies Case 2 since (5.27) is the skew-symmetric part of (5.39).

Lemma 5.3. *Suppose $W^{pc}=0$ and dimension is bigger than 3. Then (5.39) holds.*

Proof. Combine (5.29), (5.31), (5.30) and the already proved (5.27) to obtain

$$(5.40)\quad (\nabla^2_{e_a,Ie_a}L)(Y,Z)+\Big[(\nabla^2_{Y,e_a}L)-(\nabla^2_{e_a,Y}L)\Big](Ie_a,Z)=-2(n-1)(\nabla_Y\mathbb{B})(Z,\xi)-\\ -2\omega(Y,Z)\mathbb{B}(\xi,\xi)+L(Y,IL(Z))-L(Z,IL(Y))-\omega(Y,Z)(\nabla_{e_a}\mathbb{B})(Ie_a,\xi)$$

Now, (5.33), (5.34) and (5.40) imply (5.39). □

Case 4: $\Big[A\in H,\quad B=C=\xi\Big]$.

In this case (5.4) has the form

$$(5.41)\quad \nabla du(X,\xi,\xi)-\nabla du(\xi,X,\xi)=-R(X,\xi,\xi,\nabla u)+\nabla du(T(\xi,X),\xi)=\tau(X,e_a)\nabla du(X,\xi).$$

Take the covariant derivative of (5.2) along ξ and a covariant derivative of (5.3) along a horizontal direction, then use (5.1) and (5.2) to calculate that the left hand side of (5.41) is equal to

$$(5.42)\quad \nabla du(X,\xi,\xi)-\nabla du(\xi,X,\xi)=-(\nabla_X\mathbb{B})(\xi,\xi)+(\nabla_\xi\mathbb{B})(X,\xi)+(\nabla_\xi L)(X,I\nabla u)\\ -(\nabla_X\mathbb{B})(I\nabla u,\xi)-\mathbb{B}(\nabla_X I\nabla u,\xi)+L(X,\nabla_\xi I\nabla u)+du(IX)\mathbb{B}(\nabla u,\xi)+du(\xi)\mathbb{B}(X,\xi)\\ +\frac{1}{2}|\nabla u|^2\mathbb{B}(IX,\xi)+\frac{1}{2}|\nabla u|^2L(IX,I\nabla u)+\frac{1}{4}|\nabla u|^4du(X)+|\nabla u|^2L(X,\nabla u)+\frac{1}{2}|\nabla u|^2T(\xi,IX,\nabla u)\\ +du(\xi)L(X,I\nabla u)+(du(\xi))^2du(X)-du(\xi)T(\xi,X,\nabla u)+du(X)\nabla du(\xi,\xi).$$

Using (5.39) and (5.2) the equation (5.42) reduces to

$$(5.43)\quad \nabla du(X,\xi,\xi)-\nabla du(\xi,X,\xi)=-(\nabla_X\mathbb{B})(\xi,\xi)+(\nabla_\xi\mathbb{B})(X,\xi)-2\mathbb{B}(e_a,\xi)L(X,Ie_a)\\ -T(\xi,X,e_a)L(e_a,I\nabla u)+\frac{1}{2}T(\xi,X,e_a)|\nabla u|^2du(Ie_a)-T(\xi,X,e_a)du(\xi)du(e_a).$$

A substitution of (5.43) in (5.44) and an application of (5.2) we see that (5.41) is equivalent to

$$(5.44)\qquad (\nabla_\xi\mathbb{B})(X,\xi)-(\nabla_X\mathbb{B})(\xi,\xi)-2\mathbb{B}(e_a,\xi)L(X,Ie_a)+\tau(X,e_a)\mathbb{B}(e_a,\xi)=0.$$

Lemma 5.4. *Suppose $W^{pc}=0$ and dimension is bigger than 3. Then (5.44) holds.*

Proof. Differentiate the already proven (5.27), take the corresponding traces and use the symmetry of L to get

$$(5.45)\quad (\nabla^2_{e_a,Ie_a}\mathbb{B})(Y,\xi)=(n+2)[L(IY,e_b)-L(Y,Ie_b)]\mathbb{B}(e_b,\xi)-(trL)\mathbb{B}(IY,\xi)-2n(\nabla_\xi\mathbb{B})(Y,\xi),$$

$$(5.46)\quad (\nabla^2_{Y,e_a}\mathbb{B})(Ie_a,\xi)=-2n(\nabla_Y\mathbb{B})(\xi,\xi)-(\nabla_Y L)(e_b,IL(Ie_b)),$$

$$(5.47)\quad (\nabla^2_{e_a,Ie_a}\mathbb{B})(Y,\xi)-(\nabla^2_{e_a,Y}\mathbb{B})(Ie_a,\xi)\\ =2(\nabla_Y\mathbb{B})(\xi,\xi)+(2n+1)\mathbb{B}(IL(Y),\xi)+(\nabla_{e_b}L)(Y,IL(Ie_b))$$

The Ricci identities, Proposition 3.1 and (5.14) imply

$$(5.48)\quad (\nabla^2_{Y,e_a}\mathbb{B})\,(Ie_a,\xi)-(\nabla^2_{e_a,Y}\mathbb{B})\,(Ie_a,\xi)=-R(Y,e_a,Ie_a,e_b)\mathbb{B}(e_b,\xi)-(\nabla_{T(Y,e_a)}\mathbb{B})(Ie_a,\xi)\\ =r(Y,Ie_b)\mathbb{B}(e_b,\xi)+2(\nabla_\xi\mathbb{B})(Y,\xi)=(2n+1)L(Y,Ie_b)\mathbb{B}(e_b,\xi)-3L(IY,e_b)\mathbb{B}(e_b,\xi)\\ -(trL)\mathbb{B}(IY,\xi)+2(\nabla_\xi\mathbb{B})(Y,\xi)$$

and also

$$(5.49)\quad (\nabla^2_{e_a,Ie_a}\mathbb{B})(Y,\xi)=\frac{1}{2}\Big[(\nabla^2_{e_a,Ie_a}\mathbb{B})(Y,\xi)-(\nabla^2_{Ie_a,e_a}\mathbb{B})(Y,\xi)\Big]\\ =-\frac{1}{2}\Big[R(e_a,Ie_a,Y,e_b)\mathbb{B}(e_b,\xi)+(\nabla_{T(e_a,Ie_a)}\mathbb{B})(Y,Z)\Big]=-\rho(Y,e_b)\mathbb{B}(e_b,\xi)-2n(\nabla_\xi\mathbb{B})(Y,\xi)\\ =-(n+2)L(Y,Ie_b)\mathbb{B}(e_b,\xi)+(n+2)L(IY,e_b)\mathbb{B}(e_b,\xi)+(trL)\mathbb{B}(IY,\xi)-2n(\nabla_\xi\mathbb{B})(Y,\xi).$$

A small calculation taking into account (5.46) and (5.47) yields

$$\begin{aligned}(5.50)\quad &\Big[\nabla^2_{Y,e_a}-\nabla^2_{Y,e_a}\Big]\mathbb{B}(Ie_a,\xi)+(\nabla^2_{e_a,Ie_a}\mathbb{B})(Y,\xi)=\\ &2(1-n)(\nabla_Y\mathbb{B})(\xi,\xi)-(2n+1)L(Y,Ie_a)\mathbb{B}(e_a,\xi)+(\nabla_{e_a}L)(Y,IL(Ie_a))-(\nabla_Y L)(e_a,IL(Ie_a))\\ &=-2(n-1)(\nabla_Y\mathbb{B})(\xi,\xi)-2(n-1)L(Y,Ie_a)\mathbb{B}(e_a,\xi).\end{aligned}$$

The identity (5.44) follows from (5.48), (5.49) and (5.50). □

If the dimension is equal to 3 then it is easy to check that $W^{pc}=0$ and the integrability conditions (5.8), (5.27) and (5.39) are trivially satisfied. Thus, the existence of a smooth solution depends only on the assumption of the validity of (5.44).

This completes the proof of Theorem 1.2

6. CR structure, Chern-Moser-Webster theorem

In this section we give a proof of the well known Chern-Moser-Webster theorem [ChM, W1] which states that a non-degenerate CR-hypersurface in $\mathbb{C}^{n+1}$, $n>3$, is locally CR equivalent to a hyperquadric in $\mathbb{C}^{n+1}$ iff the Chern-Moser curvature vanishes. Our proof is based on the classical approach used by H.Weyl in Riemannian geometry [Eis], [IV] in quaternionic contact geometry, and is very similar to the arguments applied in this paper in the case of para-contact hermitian structures.

A CR manifold is a smooth manifold M of real dimension 2n+1, with a fixed n-dimensional complex subbbundle H of the complexified tangent bundle $\mathbb{C}TM$ satisfying $H\cap\overline{H}=0$ and $[\mathbb{H},\mathbb{H}]\subset\mathbb{H}$. If we let $\mathbb{H}=Re\,H\oplus\overline{H}$, the real subbundle $\mathbb{H}$ is equipped with a formally integrable almost complex structure J. We assume that M is oriented and there exists a globally defined contact form θ such that $\mathbb{H}=Ker\,\theta$. Recall that a 1-form θ is a contact form if the hermitian bilinear form $2g(X,Y)=-d\theta(JX,Y)$ is non-degenerate. The vector field ζ dual to θ with respect to g is called the Reeb vector field. The almost complex structure J is formally integrable in the sense that $([JX,Y]+[X,JY])\in\mathbb{H}$ and the Nijenhuis tensor $N^J(X,Y)=[JX,JY]-[X,Y]-J[JX,Y]-J[X,JY]=0$. A CR manifold (M,θ,g) with fixed contact form θ is called *a pseudohermitian manifold*. In this case the 2-form $d\theta_{|_{\mathbb{H}}}:=2\Omega$ is called the fundamental form. Note that the contact form is determined up to a conformal factor, i.e. $\bar\theta=\nu\theta$ for a positive smooth function ν, defines another pseudohermitian structure called pseudo-conformal to the original one.

Convention 6.1. *In this section we use the conventions:*

a) $X,Y,Z...$ will be a horizontal vector fields, i.e. $X,Y,Z...\in\mathbb{H}$

b) $\{\epsilon_1,\dots,\epsilon_n,J\epsilon_1,\dots,J\epsilon_n\}$ denotes an adapted orthonormal basis of the horizontal space $\mathbb{H}$.

c) The summation convention over repeated vectors from the basis $\{\epsilon_1,\dots,\epsilon_{2n}\}$ will be used. For example, for a (0,4)-tensor Q, the formula $s=Q(\epsilon_i,\epsilon_j,\epsilon_j,\epsilon_i)$ means

$$k=\sum_{i,j=2n}^{n}Q(\epsilon_i,\epsilon_j,\epsilon_j,\epsilon_i).$$

The Tanaka-Webster connection [Ta, W1, W2] is the unique linear connection ∇^{cr} preserving a given pseudohermitian structure with torsion T^{cr} having the properties

$$\nabla^{cr}\zeta=\nabla^{cr}J=\nabla^{cr}\theta=\nabla^{cr}g=0,\tag{6.1}$$

$$T^{cr}(X,Y)=d\theta(X,Y)\zeta=2\Omega(X,Y)\zeta,\quad T^{cr}(\zeta,X)\in\mathbb{H},\tag{6.2}$$

$$g(T^{cr}(\zeta,X),Y)=g(T^{cr}(\zeta,Y),X)=-g(T^{cr}(\zeta,JX),JY).\tag{6.3}$$

It is well known that the endomorphism $T^{cr}(\zeta,.)$ is the obstruction a pseudohermitian manifold to be Sasakian. The symmetric endomorphism $T^{cr}_\zeta:\mathbb{H}\longrightarrow\mathbb{H}$ is denoted by A and it is call *the torsion of the pseudohermitian manifold.* The torsion A is completely trace-free i.e.

$$A(\epsilon_i,\epsilon_i)=A(\epsilon_i,J\epsilon_i)=0.\tag{6.4}$$

Let R^{cr} be the curvature of the Tanaka-Webster connection. The pseudohermitian Ricci tensor r^{cr}, the pseudohermitian Ricci 2-form ρ^{cr} and the psudohermitian scalar curvature s^{cr} are defined by

$$r^{cr}(A,B)=g(R(\epsilon_i,A)B,\epsilon_i),\quad \rho^{cr}(A,B)=\frac{1}{2}g(R^{cr}(A,B)\epsilon_i,I\epsilon_i),\quad s^{cr}=r(\epsilon_i,\epsilon_i),\quad A,B\in\Gamma(TM).$$

We summarize below the well known properties of the curvature R^{cr} of the Tanaka-Webster connection [W1, W2, L1] using real expression, see also [DT].

$$R^{cr}(X,Y,JZ,JV)=R(X,Y,Z,V)=-R^{cr}(X,Y,V,Z),\quad R^{cr}(X,Y,Z,\zeta)=0 \tag{6.5}$$

$$\begin{aligned}&\frac{1}{2}\Big[R^{cr}(X,Y,Z,V)-R^{cr}(JX,JY,Z,V)\Big]\\ &\quad=-g(X,Z)A(Y,JV)-g(Y,V)A(X,JZ)+g(Y,Z)A(X,JV)+g(X,V)A(Y,JZ)\\ &\qquad-\Omega(X,Z)A(Y,V)-\Omega(Y,V)A(X,Z)+\Omega(Y,Z)A(X,V)+\Omega(X,V)A(Y,Z)\end{aligned} \tag{6.6}$$

$$R^{cr}(\zeta,X,Y,Z)=(\nabla^{cr}_YA)(Z,X)-(\nabla^{cr}_ZA)(Y,X) \tag{6.7}$$

$$r^{cr}(X,Y)=r^{cr}(Y,X)\qquad\qquad r^{cr}(X,Y)-r^{cr}(JX,JY)=4(n-1)A(X,JY) \tag{6.8}$$

$$\rho^{cr}(X,JY)=-\frac{1}{2}\Big[r^{cr}(X,Y)+r^{cr}(JX,JY)\Big]=\frac{1}{2}R^{cr}(\epsilon_i,J\epsilon_i,X,JY), \tag{6.9}$$

$$(\nabla^{cr}_{\epsilon_i}r^{cr})(\epsilon_i,X)=\frac{1}{2}ds^{cr}(X). \tag{6.10}$$

Let v be a smooth function on a pseudohermitian manifold (M,θ,g). Let $\bar\theta=\frac{1}{2}e^{-2v}\theta$ be a pseudoconformal deformation of θ. We will denote the objects related to $\bar\theta$ by over-lining the same object corresponding to θ. Thus, $d\bar\theta=-e^{-2v}dv\wedge\theta+\frac{1}{2}e^{-2v}d\theta$, $\bar g=\frac{1}{2}e^{-2v}g$. The new Reeb vector field $\bar\zeta$ is $\bar\zeta=2e^{2v}\zeta+2e^{2v}J\nabla^{cr}v$, where $\nabla^{cr}v$ is the horizontal gradient, $g(\nabla^{cr}v,X)=dv(X)$. The horizontal sub-Laplacian and the norm of the horizontal gradient are defined respectively by $\triangle v=tr^g_{\mathbb{H}}(\nabla^{cr}dv)=\nabla^{cr}dv(\epsilon_i,\epsilon_i)$, $|\nabla^{cr}v|^2=dv(\epsilon_i)^2$. The connections ∇^{cr} and $\overline{\nabla^{cr}}$ are connected by a (1,2) tensor S, $\overline{\nabla^{cr}}_AB=\nabla^{cr}_AB+S_AB$, $A,B\in\Gamma(TM)$. We obtain similarly as in Section 4.1 that

$$\begin{aligned}g(S(X,Y),Z)&=-dv(X)g(Y,Z)+dv(JX)\Omega(Y,Z)\\ &\quad-dv(Y)g(Z,X)-dv(JY)\Omega(Z,X)+dv(Z)g(X,Y)-dv(JZ)\Omega(X,Y).\end{aligned} \tag{6.11}$$

$$\begin{aligned}g(S(\zeta,X),Y)&=\frac{1}{2}\Big[\nabla^{cr}dv(X,JY)-\nabla^{cr}dv(IX,Y)\Big]\\ &\quad-dv(X)dv(IY)+dv(IX)dv(Y)+|dv|^2\Omega(X,Y).\end{aligned} \tag{6.12}$$

In addition, the torsion changes according to [L2]

$$\begin{aligned}\bar A(X,Y)&=e^{2v}2A(X,Y)\\ &\quad-e^{2v}\Big[\nabla^{cr}dv(X,IY)+\nabla^{cr}dv(IX,Y)+2dv(X)dv(IY)+2dv(IX)dv(Y)\Big].\end{aligned} \tag{6.13}$$

The identity $d^2=0$ together with (6.2) yields

$$\nabla^{cr}dv(X,Y)=[\nabla^{cr}dv]_{[sym]}(X,Y)-dv(\zeta)\Omega(X,Y). \tag{6.14}$$

We consider the symmetric (0,2) tensor C defined on $\mathbb{H}$ by the equality

$$C(X,Y)=-\frac{1}{2(n+2)}\rho^{cr}(X,JY)-\frac{s^{cr}}{8(n+1)(n+2)}g(X,Y)+A(JX,Y) \tag{6.15}$$

and define the (0,4) tensor CW on $\mathbb{H}$ by

$$\begin{aligned}g(CW(X,Y)Z,V)&=g(R(X,Y)Z,V)\\ &\quad+g(X,Z)C(Y,V)+g(Y,V)C(X,Z)-g(Y,Z)C(X,V)-g(X,V)C(Y,Z)\\ &\quad-\Omega(X,Z)C(Y,JV)-\Omega(Y,V)C(X,JZ)+\Omega(Y,Z)C(X,JV)+\Omega(X,V)C(Y,JZ)\\ &\quad-\Omega(X,Y)\Big[C(Z,JV)-C(JZ,V)\Big]-\Omega(Z,V)\Big[C(X,IY)-C(IX,Y)\Big]\end{aligned} \tag{6.16}$$

Take the corresponding traces in (6.16) taking into account (6.15) to verify that the tensor CW is completely trace-free, i.e. $r^{cr}(PW) = \rho^{cr}(PW) = 0$.

Compare (4.13) with (3.3) to obtain the following

Proposition 6.2. *The (2,0)+(0,2)-part of the tensor CW vanishes identically,*

$$CW(X,Y,Z,V) - CW(JX,JY,Z,V) = 0.$$

The (1,1)-part of CW is precisely the Chern-Moser tensor S defined in [ChM] *and it is determined completely by the Ricci 2-form as follows*

$$\begin{aligned}
(6.17)\quad S(X,Y,Z,V) &= CW_{1,1}(X,Y,Z,V) = \frac{1}{2}\Big[CW(X,Y,Z,V) + CW(JX,JY,Z,V)\Big]\\
&= \frac{1}{2}\Big[R(X,Y,Z,V) + R(JX,JY,Z,V)\Big] - \frac{s^{cr}}{8(n+1)(n+2)}\Big[g(X,Z)g(Y,V) - g(Y,Z)g(X,V)\Big]\\
&\quad - \frac{s^{cr}}{8(n+1)(n+2)}\Big[\Omega(X,Z)\Omega(Y,V) - \Omega(Y,Z)\Omega(X,V) + 2\Omega(X,Y)\Omega_s(Z,V)\Big]\\
&\quad - \frac{1}{2(n+2)}\Big[g(X,Z)\rho^{cr}(Y,JV) - g(Y,Z)\rho^{cr}(X,JV) + g(Y,V)\rho^{cr}(X,JZ) - g(X,V)\rho^{cr}(Y,JZ)\Big]\\
&\quad - \frac{1}{2(n+2)}\Big[\Omega(X,Z)\rho^{cr}(Y,V) - \Omega(Y,Z)\rho^{cr}(X,V) + \Omega(Y,V)\rho^{cr}(X,Z) - \Omega(X,V)\rho^{cr}(Y,Z)\Big]\\
&\quad - \frac{1}{n+2}\Big[\Omega(X,Y)\rho^{cr}(Z,V) + \Omega(Z,V)\rho^{cr}(X,Y)\Big].
\end{aligned}$$

The relevance of the tensor CW become clear from the next

Theorem 6.3. *The tensor CW is a pseudohermitian invariant, i.e. if*

$$2\bar\theta = e^{-2v}\theta \quad \textit{for any smooth function } v \textit{ then} \quad 2e^{2v}CW_{\bar\theta} = CW_\theta.$$

Proof. By a straightforward computations using (6.11) and (6.12), we obtain the formula

$$\begin{aligned}
(6.18)\quad 2e^{2v}g(\bar R(X,Y)Z,V) - g(R(X,Y)Z,V) &= -g(Z,V)\Big[N(X,Y) - N(Y,X)\Big]\\
&- g(X,Z)N(Y,V) - g(Y,V)N(X,Z) + g(Y,Z)N(X,V) + g(X,V)N(Y,Z)\\
&+ \Omega(X,Z)N(Y,JV) + \Omega(Y,V)N(X,JZ) - \Omega(Y,Z)N(X,JV) - \Omega(X,V)N(Y,JZ)\\
&+ \Omega(X,Y)\Big[N(Z,JV) - N(JZ,V)\Big] + \Omega(Z,V)\Big[N(X,JY) - N(Y,JX)\Big].
\end{aligned}$$

where the (0,2) tensor N is given by

$$(6.19)\qquad N(X,Y) = \nabla^{cr}dv(X,Y) + dv(X)dv(Y) - dv(JX)dv(JY) - \frac{1}{2}g(X,Y)|\nabla^{cr}v|^2.$$

We denote $trN = N(e_a,e_a)$ the trace of the tensor N. Using (6.19) and (6.14), we obtain

$$(6.20)\qquad trN = \triangle v - n|dv|^2, \quad N(X,Y) - N(JX,JY) = N(Y,X) - N(JY,JX),$$

$$(6.21)\qquad N(\epsilon_i, I\epsilon_i) = -2ndv(\zeta), \quad N(\epsilon_i, I\epsilon_i)\Omega(X,Y) = n\Big[N(X,Y) - N(Y,X)\Big].$$

Taking the traces in (6.18) and using (6.19), (6.20) and (6.21) we obtain

$$\begin{aligned}
&\overline{r}(X,Y) - r(X,Y)\\
(6.22)\qquad &\qquad = (n+1)N(X,Y) + nN(Y,X) + N(JX,JY) + 2N(JY,JX) + trN\, g(X,Y),\\
&e^{-2v}\overline{s}^{cr} - 2s^{cr} = 8(n+1)trN.
\end{aligned}$$

Equation (6.22) implies

$$\begin{aligned}(6.23)\qquad \overline{r}(X,Y)-r(X,Y)+\overline{r}(JX,JY)-r(JX,JY)&=-2\Big[\overline{\rho}(X,JY)-\rho(X,JY)\Big]\\ &=(n+2)\Big[N(X,Y)+N(Y,X)+N(JX,JY)+N(JY,JX)\Big]+2(trN)\,g(X,Y)\end{aligned}$$

$$\begin{aligned}(6.24)\qquad \overline{r}(X,Y)-r(X,Y)-\overline{r}(JX,JY)+r(JX,JY)&=4(n-1)\Big[\overline{A}(X,JY)-A(X,JY)\Big]\\ &=(n-1)\Big[N(X,Y)+N(Y,X)-N(JX,JY)-N(JY,JX)\Big].\end{aligned}$$

The equalities (6.22), (6.23) and (6.24) yield

$$\begin{aligned}(6.25)\quad N_{[sym]}&=-\frac{1}{2(n+2)}\overline{\rho}^{cr}(X,JY)-\frac{\overline{s}^{cr}}{8(n+1)(n+2)}\overline{g}(X,Y)+\overline{A}(JX,Y)\\ &-\Big[-\frac{1}{2(n+2)}\rho^{cr}(X,JY)-\frac{s^{cr}}{8(n+1)(n+2)}g(X,Y)+\overline{A}(JX,Y)\Big]=\overline{C}(X,Y)-C(X,Y)\end{aligned}$$

We obtain from (6.19) and (6.14) that

$$(6.26)\qquad N(X,Y)=N_{[sym]}(X,Y)-dv(\zeta)\omega(X,Y).$$

Substituting (6.25) in (6.26), and then inserting the obtained equality in (6.18), invoking also (6.20), completes the proof of Theorem 6.3 □

Remark 6.4. *If we substitute (6.19), (6.20) and (6.21) in (6.22) one recovers the transformation formulas of the pseudo hermitian Ricci tensor and the pseudo hermitian scalar found in [L2]. In particular, the pseudohermitian scalar and the pc scalar curvature satisfy the same transformation formula (4.10).*

Combine Theorem 6.3 and Proposition 6.2 to recover the result due to Chern and Moser [ChM]

Theorem 6.5. [ChM] *The Chern-Moser tensor S is a pseudo-conformal invariant.*

The rest of the section is devoted to give a new proof of the next theorem due to Chern-Moser [ChM] and Webster [W1] in dimension bigger than three and due to Cartan [Car] in dimension three.

Theorem 6.6. [Car, ChM, W1] *Let (M,θ,g) be a 2n+1-dimensional non-degenerate pseudo-hermitian manifold.*

- *i). [ChM, W1] If $n>1$ then (M,θ,g) is locally pseudoconformal equivalent to a hyperquadric in $\mathbb{C}^{n+1}$ if and only if the Chern-Moser tensor vanishes, $S=0$;*
- *ii). [Car] If $n=1$ then (M,θ,g) is locally pseudoconformal equivalent to a hyperquadric in $\mathbb{C}^2$ iff the condition (6.47) is satisfied.*

Proof. It is well known that a pseudohermitian manifold with flat Tanaka-Webster connection is locally isomorphic to a Heisenberg group. On the other hand, the Cayley transform is a pseudo-conformal equivalence between the Heisenberg group with its flat pseudo-hermitian structure and a hypersphere $h_{\alpha\bar\beta}Z^\alpha Z^{\bar\beta}+W\bar W=1$ in $\mathbb{C}^{n+1}$ [ChM, p.223]. We shall show that the condition $S=0$ is a sufficient condition a given pseudohermitian manifold to be locally pseudoconformally flat provided the dimension is bigger than three. In dimension three S vanishes identically and the sufficient condition remains only (5.44). The scheme is formally very similar to that used in the proof of Theorem 1.2. Namely in all formulas in the proof of Theorem 1.2 one formally replaces I with $\sqrt{-1}J$ and ξ by $\sqrt{-1}\zeta$. Therefore we write down here the most important steps.

Suppose $S=0$. Then $CW=0$ due to Proposition 6.2. We shall show that in this case there locally exists a smooth function v which sends the pseudo-hermitian structure to the flat one by a pseudoconformal transformation.

We consider the following system of differential equations with respect to unknown function v:

$$\nabla^{cr}dv(X,Y) = -C(X,Y) - dv(X)dv(Y) + dv(JX)dv(JY) + \frac{1}{2}g(X,Y)|dv|^2 - dv(\zeta)\Omega(X,Y), \tag{6.27}$$

$$\nabla^{cr}du(X,\zeta) = -\mathbb{D}(X,\zeta) - C(X,J\nabla v) + \frac{1}{2}dv(JX)|dv|^2 - dv(X)du(\zeta), \tag{6.28}$$

$$\nabla^{cr}dv(\zeta,\zeta) = -\mathbb{D}(\zeta,\zeta) - \mathbb{D}(J\nabla v,\zeta) + \frac{1}{4}|dv|^4 - (dv(\zeta))^2, \tag{6.29}$$

where $\mathbb{D}(X,\zeta)$ and $\mathbb{D}(\zeta,\zeta)$ do not depend on the function u and are determined by

$$\begin{aligned} (\nabla^{cr}_{\epsilon_i}C)(J\epsilon_i,JX) &= -(2n+1)\mathbb{D}(JX,\zeta) \\ -(\nabla^{cr}_X tr\, C) + (\nabla^{cr}_{\epsilon_i}C)(\epsilon_i,X) &= 3\mathbb{D}(JX,\zeta), \end{aligned} \tag{6.30}$$

$$\mathbb{D}(\zeta,\zeta) = -\frac{1}{2n}[(\nabla^{cr}_{\epsilon_i}\mathbb{D})(J\epsilon_i,\zeta) + C(\epsilon_i,J\epsilon_j)C(J\epsilon_i,\epsilon_j)]. \tag{6.31}$$

The consistences of the first and second equality in (6.30) is precisely equivalent to (6.10).

To prove Theorem 1.2 it is sufficient to show the existence of a local smooth solution to (6.27) because of (6.13) and the proof of Theorem 6.3.

The integrability conditions for the overdetermined system (6.27)-(6.29) are the Ricci identities,

$$\nabla^{cr}dv(A,B,C) - \nabla^{cr}dv(B,A,C) = - R^{cr}(A,B,C,\nabla^{cr}v) - \nabla^{cr}dv((T^{cr}(A,B),C), \quad A,B,C \in \Gamma(TM). \tag{6.32}$$

We consider all possible cases:

Case 1: $\Big[Z,X,Y \in H\Big]$.

The equation (6.32) on H has the following form

$$\nabla^{cr}dv(Z,X,Y) - \nabla^{cr}dv(X,Z,Y) = -R^{cr}(Z,X,Y,\nabla^{cr}v) - 2\Omega(Z,X)\nabla^{cr}dv(\zeta,Y), \tag{6.33}$$

where we have used (6.2).

Take a covariant derivative of (6.27) along $Z \in H$, substitute in the obtained equality (6.27) and (6.28), anticommute the covariant derivatives, substitute into (6.33) use (6.16) with $S = 0 = CW$ to get that the integrability condition here is

$$(\nabla^{cr}_Z C)(X,Y) - (\nabla^{cr}_X C)(Z,Y) = -\Omega(Z,Y)\mathbb{D}(X,\zeta) + \Omega(X,Y)\mathbb{D}(Z,\zeta) - 2\Omega(Z,X)\mathbb{D}(Y,\zeta). \tag{6.34}$$

Lemma 6.7. *Suppose $S = 0$ and dimension is bigger than 3. Then (6.34) holds.*

Proof. Using (6.2), the second Bianchi identity (3.10) gives

$$\sum_{(X,Y,Z)} \Big[(\nabla^{cr}_X R^{cr})(Y,Z,V,W) + 2\Omega(X,Y)R^{cr}(\zeta,Z,V,W)\Big] = 0. \tag{6.35}$$

Trace in (6.35) yields

$$\begin{aligned}(\nabla^{cr}_{\epsilon_i}R^{cr})(X,Y,Z,\epsilon_i) - (\nabla^{cr}_X r^{cr})(Y,Z) + (\nabla^{cr}_Y r^{cr})(X,Z) \\ -2R^{cr}(\zeta,Y,Z,JX) + 2R^{cr}(\zeta,X,Z,JY) - 2\Omega(X,Y)r^{cr}(\zeta,Z) = 0,\end{aligned} \tag{6.36}$$

$$\begin{aligned}(\nabla^{cr}_X\rho^{cr})(Y,Z) + (\nabla^{cr}_Y\rho^{cr})(Z,X) + (\nabla^{cr}_Z\rho^{cr})(X,Y) \\ +2\Omega(X,Y)\rho^{cr}(\zeta,Z) + 2\Omega(Y,Z)\rho^{cr}(\zeta,X) + 2\Omega(Z,X)\rho^{cr}(\zeta,Y) = 0,\end{aligned} \tag{6.37}$$

$$(\nabla^{cr}_X\rho^{cr})(Y,Z) + (\nabla^{cr}_{e_a}R^{cr})(J\epsilon_i,X,Y,Z,) + 2(n-1)R^{cr}(\zeta,X,Y,Z) = 0. \tag{6.38}$$

We use $S = 0$ and (6.15) to express r^{cr}, ρ^{cr} and A in terms of C and $tr\, C$, namely

$$r^{cr}(X,Y) = (2n+1)C(X,Y) + 3C(JX,JY) + (tr\, C)g(X,Y), \tag{6.39}$$

$$\rho^{cr}(X,Y) = (n+2)C(X,JY) - (n+2)C(JX,Y) - (tr\, C)\Omega(X,Y), \tag{6.40}$$

$$A(JX,Y) = \frac{1}{2}[C(X,Y) - C(JX,JY)]. \tag{6.41}$$

At this moment one could proceed as in the proof of Lemma 5.1, hence we omit the details. $\square$

Case 2: $\Big[Z, X \in H, \quad \zeta\Big]$.

In this case (6.32) reads

$$\nabla^{cr} dv(Z,X,\zeta) - \nabla^{cr} dv(X,Z,\zeta) = \\ - R^{cr}(Z,X,\zeta,\nabla^{cr} v) - \nabla^{cr} dv(T(Z,X),\zeta) = -2\Omega(Z,X)\nabla^{cr} dv(\zeta,\zeta), \tag{6.42}$$

where we used (6.2).

The integrability condition in this case is

$$(\nabla^{cr}_Z \mathbb{D})(X,\zeta) - (\nabla^{cr}_X \mathbb{D})(Z,\zeta) + C(Z,JC(X)) - C(X,JL(Z)) = -2\mathbb{D}(\zeta,\zeta)\Omega(Z,X). \tag{6.43}$$

The proof of the next lemma is similar to the proof of Lemma 5.2 and is omitted.

Lemma 6.8. *Suppose $S = 0$ and dimension is bigger than 3. Then (5.27) holds.*

Case 3: $\Big[\zeta, \quad X, Y \in H\Big]$.

In this case (6.32) reads

$$\nabla^{cr} dv(\zeta,X,Y) - \nabla^{cr} dv(X,\zeta,Y) = -R^{cr}(\zeta,X,Y,\nabla^{cr} v) - \nabla^{cr} dv(T(\zeta,X),Y) \tag{6.44}$$

and reduces to

$$(\nabla^{cr}_X \mathbb{D})(Y,\zeta) - (\nabla^{cr}_\zeta C)(X,Y) = \\ C(Y,JC(X)) + A(X,C(Y)) + A(Y,C(X)) - \mathbb{D}(\zeta,\zeta)\Omega(X,Y). \tag{6.45}$$

Clearly Case 3 implies Case 2 since (6.43) is the skew-symmetric part of (6.45) and we have very similar proof as the proof of Lemma 5.3 of the next

Lemma 6.9. *Suppose $S = 0$ and dimension is bigger than 3. Then (5.39) holds.*

Case 4**:** $\Big[X \in H, \quad \zeta, \zeta\Big]$.

In this case (6.32) has the form

$$\nabla^{cr} dv(X,\zeta,\zeta) - \nabla^{cr} dv(\zeta,X,\zeta) = \\ - R^{cr}(X,\zeta,\zeta,\nabla^{cr} v) + \nabla^{cr} dv(T(\zeta,X),\zeta) = A(X,\epsilon_i)\nabla^{cr} dv(\epsilon_i,\zeta). \tag{6.46}$$

which turns out to be equivalent to

$$(\nabla^{cr}_\zeta \mathbb{D})(X,\zeta) - (\nabla^{cr}_X \mathbb{D})(\zeta,\zeta) - 2\mathbb{D}(\epsilon_i,\zeta)C(X,J\epsilon_i) + A(X,\epsilon_i)\mathbb{D}(\epsilon_i,\zeta) = 0. \tag{6.47}$$

We achieve similarly as Lemma 5.4 the next

Lemma 6.10. *Suppose $S = 0$ and dimension is bigger than 3. Then (5.44) holds.*

If the dimension is equal to 3 then $S = 0$ and it is easy to check that the integrability conditions (5.8), (5.27) and (5.39) are trivially satisfied. Hence, (5.44) implies the the existence of a smooth solution to the system (6.27)-(6.29). $\square$

7. The ultrahyperbolic Yamabe equation

Recall that the CR Yamabe problem is to determine if there exists a pseudohermitian structure compatible with a given CR structure such that the pseudohermitian scalar, i.e. the scalar curvature of the Tanaka-Webster connection is constant. If the CR structure is strongly pseudo-convex, i.e. the Levi form is negative definite then the CR Yamabe problem reduces to a subelliptic PDE which can be solved on a compact manifold [JL1].

Similarly to the CR case one can pose a Yamabe type problem for a para CR manifold. Namely, given a para CR structure is there a compatible para hermitian structure such that the scalar curvature of the canonical connection is a constant.

In the case when the Levi form of a given CR structure has neutral signature of type (n,n) then the CR Yamabe equation is of the same type as the para CR Yamabe equation due to Remark 6.4, i.e. one has to consider the sub ultrahyperbolic equation (4.10) where $\bar{s} = const.$.

Here we show an explicit formula for the regular part of a solution to the ultrahyperbolic Yamabe equation on $G\,(\mathbb{P})$

$$\mathcal{L}\varphi \;\equiv\; \sum_{k=1}^{n} \left(U_k^2 \;-\; V_k^2\right)\varphi \;=\; -\varphi^{2^*-1}, \tag{7.1}$$

where $2^* - 1 = (Q+2)/(Q-2) = (n+2)/n$ with $Q = 2n+2$ the homogenous dimension of the group. When $n = 2m$ (7.1) coincides with the Yamabe equation on the Heisenberg group of (real) signature $(2m, 2m)$ defined by the quadric

$$Q = \{(z,w) \in \mathbb{C}^n \times \mathbb{C} : \ \operatorname{Im} w = H(z,z)\},$$

where $H(z,z) = \sum_{j=1}^m \left(z_j \overline{z}'_j - z_{j+m}\overline{z}'_{j+m}\right)$, with the natural group structure

$$(z'', w'') \;=\; (z', w') \circ (z, w) \;=\; \big(z' \;+\; z, w' +\; w \;+ 2\operatorname{Im} H(z', z)\big).$$

The left-invariant horizontal vector fields are given by

$$X_j = \frac{\partial}{\partial x_j} - 2y_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + 2x_j\frac{\partial}{\partial t}$$
$$X_{m+j} = \frac{\partial}{\partial x_{m+j}} + 2y_{j+m}\frac{\partial}{\partial t}, \quad Y_{m+j} = \frac{\partial}{\partial x_{m+j}} - 2x_{j+m}\frac{\partial}{\partial t}, \quad j = 1, \dots, m,$$

while the left invariant contact form with corresponding metric, for which the above vector fields are an orthonormal frame, is given by

$$\theta = \frac{1}{2}\,dt + \Sigma_{j=1}^m (y_j\,dx_j - x_j\,dy_j) - \Sigma_{j=1}^m (y_{j+m}\,dx_{j+m} - x_{j+m}\,dy_{j+m})$$

so that

$$g(X_j, X_j) = -g(Y_j, Y_j) = -g(X_{j+m}, X_{j+m}) = g(Y_{j+m}, Y_{j+m}) = 1, \quad j = 1, \dots, m.$$

By the Chern-Moser-Webster result, the above quadric is the flat CR structure of (hermitian) signature (m,m). Henceforth, for $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{u} = (u_1, \dots, u_n)$ we set $|\mathbf{u}| = \left(u_1^2 + \dots u_n^2\right)^{1/2}$. We observe the following

Proposition 7.1. *Let $G\,(\mathbb{P})$ be the Heisenberg group of topological dimension $2n+1$. For every $\epsilon > 0$ the function*

$$\varphi_\epsilon(\mathbf{u}, \mathbf{v}, t) \;=\; \left(\frac{4n^2\epsilon^2}{(\epsilon^2 + |\mathbf{u}|^2 - |\mathbf{v}|^2)^2 - t^2}\right)^{\frac{n}{2}}, \qquad g \in \boldsymbol{G}, \tag{7.2}$$

is a solution of the ultrahyperbolic Yamabe equation (7.1) on the set where $|\epsilon^2 + |\mathbf{u}|^2 - |\mathbf{v}|^2| \neq |t|$.

Proof. Let $f = \left((1 + |\mathbf{u}|^2 - |\mathbf{v}|^2)^2 - t^2\right)^{-\frac{n}{2}}$. After a straightforward calculation we find $\mathcal{L}f = -4n^2 f^{2^*-1}$, which implies easily the equation for φ_1. Furthermore, using the dilations on the group $\delta_\lambda(\mathbf{u}, \mathbf{v}, t) = (\lambda u, \lambda v, \lambda^2 t)$ we have that the function $f_\lambda(\mathbf{u}, \mathbf{v}, t) = \lambda^{n/2} f(\lambda u, \lambda v, \lambda^2 t)$ satisfies the same equation as f, which implies the equation for φ_ϵ by taking $\epsilon = 1/\lambda$. □

Since the ultra-hyperbolic Yamabe equation is invariant under translations it follows that we can construct other solutions, each being a regular function on a corresponding set. The question whether there is a global solution, in the sense of distributions, will not be considered here. In this respect we note that [T], [MR] found the fundamental solution of the ultra-hyperbolic operator in the left-hand side of (7.1).

It should be pointed out that there is a correspondence between the regular part of solutions to partial differential equations on the hyperbolic Heisenberg group and solutions of partial differential equations on the Heisenberg group. Let $X_k = \frac{\partial}{\partial x_k} - 2y_k\frac{\partial}{\partial s}$, $Y_k = \frac{\partial}{\partial y_k} + 2x_k\frac{\partial}{\partial s}$ be the horizontal left invariant vector fields on the standard Heisenberg group, note that the difference between this group and the hyperbolic Heisenberg group is in the metric, while the groups are identical. Given a function $f(x,y,t)$, $t\in\mathbb{R}$, $x,y\in\mathbb{R}^n$, let $g(u,v,t)=f(it,u,iv)$, which could be a complex valued function even when f is real-valued. Since

$$(X_kf)(\mathbf{x},\mathbf{y},s)=(U_kg)(\mathbf{u},\mathbf{v},t),\qquad (Y_kf)(\mathbf{x},\mathbf{y},s)=-i(V_kg)(\mathbf{u},\mathbf{v},t) \tag{7.3}$$

we have $\sum_{k=1}^n\left(X_k^2\ +\ Y_k^2\right)f=\sum_{k=1}^n\left(U_k^2\ -\ V_k^2\right)f$. In particular, solutions of the Yamabe equation on the Heisenberg group turn into solutions of the ultra-hyperbolic Yamabe equation on the hyperbolic Heisenberg group outside a corresponding singular set.

Consider the (standard) Heisenberg group of dimension $2n+1$ with typical point (z,s), $z\in\mathbb{C}^n$, $s\in\mathbb{R}$, and let $A=|z|^2+it$. The inversion of the point (z,s) is given by

$$(z',s')\overset{def}{=}\big(-\frac{z}{\bar A},-\frac{s}{A\bar A}\big), \tag{7.4}$$

which can also be written in real coordinates as

$$\mathbf{x}'=\mathbf{x}'(\mathbf{x},\mathbf{y},s),\qquad \mathbf{y}'=\mathbf{y}'(\mathbf{x},\mathbf{y},s),\qquad s'=s'(\mathbf{x},\mathbf{y},s).$$

If $B\overset{def}{=}|z'|^2+is'$, then $AB=1$ as $B=\frac{|z|^2}{A\bar A}-\frac{is}{A\bar A}=\frac{\bar A}{A\bar A}=\frac{1}{A}$.

Based on the above mentioned formal substitution and the preceding paragraph we define an inversion on the hyperbolic Heisenberg group as follows. Let $\Xi=\{p=(\mathbf{u},\mathbf{v},t)\in G\,(\mathbb{P})\,:\,\big||\mathbf{u}|^2-|\mathbf{v}|^2\big|=|t|\}$ and $p=(t,u,v)\in G\,(\mathbb{P})\setminus\Xi$. We define the inversion on the hyperbolic Heisenberg group letting

$$\mathbf{u}'=\mathbf{x}'(\mathbf{u},i\mathbf{v},it),\qquad \mathbf{v}'=-i\mathbf{y}'(\mathbf{u},i\mathbf{v},it),\qquad t'=-is'(\mathbf{u},i\mathbf{v},it) \tag{7.5}$$

using the real form of the "standard" inversion on the Heisenberg group. In other words, for $k=1,\dots,n$ we have

$$u_k'=-\frac{(|u|^2-|v|^2)u_k+tv_k}{\left(|u|^2-|v|^2\right)^2-t^2},\quad v_k'=-\frac{(|u|^2-|v|^2)v_k+tu_k}{\left(|u|^2-|v|^2\right)^2-t^2},\quad t'=-\frac{t}{\left(|u|^2-|v|^2\right)^2-t^2}, \tag{7.6}$$

which defines a point $p'=(\mathbf{u}',\mathbf{v}',t')\in G\,(\mathbb{P})\setminus\Xi$, or, using $\mathbf{w}=\mathbf{u}+e\mathbf{v}$ with $e^2=1$, $|\mathbf{w}|^2=|\mathbf{u}|^2-|\mathbf{v}|^2$,

$$\mathbf{w}'\equiv\mathbf{u}'+e\mathbf{v}'=-\frac{\mathbf{w}}{|\mathbf{w}|^2-et}\qquad t'=-\frac{t}{|\mathbf{w}|^4-t^2}.$$

This map will be called the inversion of $G\,(\mathbb{P})$ centered at Ξ. The inverse transformation is found by taking into account that the inversion is an involution.

Recall, see[K], that for a function $f(z,t)$ defined on a domain Ω in the Heisenberg group we define the Kelvin transform f^* on the image Ω^* of Ω under the inversion by the following formula

$$f^*\overset{def}{=}A^{\frac{n}{2}}\bar A^{\frac{n}{2}}f \text{ i.e. } f^*|B|^n=f. \tag{7.7}$$

Thus, using the preceding considerations we can define a Kelvin transform on the hyperbolic Heisenberg group as follows

$$(\mathcal{K}\varphi)(\mathbf{u},\mathbf{v},t)=\big((|\mathbf{u}|^2-|\mathbf{v}|^2)^2-t^2\big)^{-n/2}\varphi(\mathbf{u}',\mathbf{v}',t'), \tag{7.8}$$

where $(\mathbf{u}'\mathbf{v}',t')$ are given by (7.6). Given a function $\varphi(u,v,t)$ we consider $\psi(x,y,s)=\varphi(x,-iy,-is)$. Thus, using $s=it$, $\mathbf{x}=\mathbf{u}$ and $\mathbf{y}=i\mathbf{v}$, we have from (7.5)

$$\begin{aligned}\varphi(\mathbf{u}',\mathbf{v}',t')=\psi(\mathbf{u}',i\mathbf{v}',it')=\psi\big(\mathbf{x}'(\mathbf{u},i\mathbf{v},it),\mathbf{y}'(\mathbf{u},i\mathbf{v},it),s'(\mathbf{u},i\mathbf{v},it)\big)\\ =\psi\big(\mathbf{x}'(\mathbf{x},\mathbf{y},s),\mathbf{y}'(\mathbf{x},\mathbf{y},s),s'(\mathbf{x},\mathbf{y},s)\big),\end{aligned}$$

which shows that the (hyperbolic) Kelvin transform of φ corresponds to the ("standard" Heisenberg) Kelvin transform of ψ. Due to (7.3) and the properties of the Kelvin transform on the Heisenberg group (in fact any group of Iwasawa type), cf. [CDKR1] and [GV], the hyperbolic Kelvin transform preserves the ultra-hyperbolic functions, i.e., solutions of

$$\mathcal{L}\varphi\ \equiv\ \sum_{k=1}^n\left(U_k^2\ -\ V_k^2\right)\varphi\ =0$$

and the solutions of the ultra-hyperbolic Yamabe equation. Note that the Kelvin transform of the functions in (7.2) is given by the same formula.

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