HEEGNER DIVISORS AND NON-HOLOMORPHIC MODULAR FORMS

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1. Introduction

In their celebrated paper, Hirzebruch and Zagier [7] show that the intersection numbers of certain algebraic cycles on a Hilbert modular surface S occur as the Fourier coefficients of holomorphic modular forms on the upper half plane $\mathbb{H}$. They explicitly compute the intersection numbers of certain curves T_N both in S and at the resolution of the cusp singularities. By a direct computation they then show that these numbers are Fourier coefficients of modular forms.

These results have inspired numerous other people to look at geodesic cycles in other locally symmetric spaces and their relationship to modular forms. Here the case $SO(2,q)$ was of particular interest and was studied by Oda [19], Rallis and Schiffmann [20], Kudla [9], and recently, by Borcherds [1, 2] and Bruinier [3, 4].

Starting in the late 1970s and throughout the 1980s, Kudla and Millson (see e.g. [14]) carried out an extensive program to explain the work of Hirzebruch-Zagier from the point of view of Riemannian geometry and the theory of reductive dual pairs and the theta correspondence. Under some restrictions, they vastly generalize the results of [7] to orthogonal, unitary, and symplectic groups of arbitrary dimension and signature. Tong and Wang (e.g. [24]) ran a parallel program.

Currently, Kudla, Rapoport and Yang undertake a major investigation of the occurrence of arithmetic intersection numbers in certain moduli spaces as Fourier coefficients of modular forms, see e.g. [11].

This paper deals with the case of the real orthogonal group of signature $(p,2)$.

Before we state our results we need to establish the basic notions of the paper. Let $(V(\mathbb{Q}),q)$ be a rational quadratic space of dimension $p+2$ and signature $(p,2)$ and let $(\ ,\ )$ be the associated non-degenerate symmetric bilinear form on $V(\mathbb{Q})$. We have $(v,w) = q(v+w) - q(v) - q(w)$. We let $G = Spin(V(\mathbb{Q}))$ viewed as an algebraic group over $\mathbb{Q}$ and write $D = G(\mathbb{R})/K$ for the associated symmetric space, where K is a maximal compact subgroup of $G(\mathbb{R})$. It is very well known that D is of Hermitian type of complex dimension p; for example, for $p=1$, $D \simeq \mathbb{H}$, the upper half plane, and $D \simeq \mathbb{H}\times\mathbb{H}$ for $p=2$. We identify D with the space of two-dimensional subspaces of $V(\mathbb{R})$ on which the bilinear form $(\ ,\ )$ is negative definite:

$$D \simeq \{z \subset V(\mathbb{R}) : \dim z = 2 \quad \text{and} \quad (\ ,\ )|_z < 0\}. \tag{1.1}$$

Let $L \subset V(\mathbb{Q})$ be an integral $\mathbb{Z}$-lattice of full rank, i.e., $L \subset L^{\#}$, the dual lattice, and Γ be a congruence subgroup of G preserving L (in the main body of the paper we will allow a congruence condition as well). We write $M = \Gamma\backslash D$ for the attached locally symmetric space of finite volume.

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We construct *special* cycles in M as follows. Let $U \subset V$ be a positive definite subspace of V of any dimension $0 \leq n \leq p$. We put

$$D_U = \{z \in D : z \subset U^\perp\}; \tag{1.2}$$

so D_U is a complex submanifold of the (same) orthogonal type $O(p-n,2)$ and codimension n. We can naturally identify $Spin(U^\perp) \simeq G_U$, where G_U is the pointwise stabilizer of U in G. We put $\Gamma_U = \Gamma \cap G_U$ and define the *special* cycle

$$C_U = \Gamma_U \backslash D_U. \tag{1.3}$$

The map $C_U \longrightarrow \Gamma\backslash D$ defines an algebraic cycle in M and is actually an embedding if one passes to a suitable subgroup $\Gamma' \subset \Gamma$ of finite index (see [14]). For $x \in V(\mathbb{Q})^n$, we denote by $U(x)$ the subspace generated by x (of possibly lower dimension) and set $D_x = D_{U(x)}$ and $C_x = \Gamma_x \backslash D_x$.

We will be mainly concerned with the case $n = 1$, when the special cycles are divisors.

For $N \in \mathbb{N}$, Γ acts on $L_N = \{x \in L : q(x) = N\}$ with finitely many orbits, and we define the composite cycle

$$C_N = \sum_{x \in \Gamma\backslash L_N} C_x. \tag{1.4}$$

We call C_N the *Heegner divisor* of discriminant N. For $p = 1$, C_N is the collection of Heegner points of discriminant N in a modular (or Shimura) curve, while for $p = 2$, C_N is a Hirzebruch-Zagier curve T_N([7]) in a Hilbert modular surface (if the $\mathbb{Q}$-rank of G is 1).

Kudla and Millson explicitly construct (in much greater generality) a theta function $\theta_\varphi(\tau, L) = \sum_{x \in L} \varphi(x,\tau)$ $(\tau = u + iv \in \mathbb{H})$ with values in the closed differential $(1,1)$-forms of M attached to a certain Schwartz function $\varphi = \varphi_V$ on $V(\mathbb{R})$ (In Section 2 we review this theta function in more detail). They then consider

$$I_\varphi(\tau, C) = \int_C \theta_\varphi(\tau, L), \tag{1.5}$$

the integral of $\theta_\varphi(\tau, L)$ over a *compact* curve C. $I_\varphi(\tau, C)$ turns out to be a holomorphic modular form of weight $(p+2)/2$, whose N-th Fourier coefficient is given as the (cohomological) intersection numbers of C with the composite cycle C_N. This gives analogues of the original results of Hirzebruch-Zagier in a much more general setting, but actually does not contain their work as they consider the intersection numbers of (in general) *non*compact cycles. It therefore seems quite naturally to study the theta integral (1.5) for possibly noncompact curves C.

We study the theta integral $I_\varphi(\tau, C)$ in the noncompact case, where we restrict our attention to the special curves C_U, with U positive definite of dimension $p-1$, i.e., to embedded quotients of modular curves in M.

This leads to considerable complications because in [14] the assumption that C be compactly supported is quite essential and needed at several places; for example, to guarantee the convergence of the integral (1.5), to show the holomorphicity in $\tau \in \mathbb{H}$, and to verify the vanishing of the negative Fourier coefficients.

In Section 3 we consider the case $p = 1$ when $M \simeq \Gamma\backslash\mathbb{H}$ is itself a modular curve, i.e., V is isotropic, and study the integral $I_\varphi(\tau, M) = \int_M \theta_\varphi(\tau, L)$. Note that the cusps of M correspond to the Γ-equivalence classes of isotropic lines ℓ in V.

Our main result is then that the generating series $P(\tau) = \sum_{N=0}^{\infty} \deg(C_N) q^N$ of the degree of Heegner points in M is the holomorphic part of a non-holomorphic modular form of weight 3/2. Here $\deg(C_N) = \sum_{x\in\Gamma\backslash L_N} \frac{1}{|\Gamma_x|}$ for $N > 0$, while for $N = 0$ we put $\deg(C_0) = \mathrm{vol}(M)$. More precisely:

Theorem 1.1.

$$\int_M \theta_\varphi(\tau, L) = P(\tau) + \frac{v^{-1/2}}{4\pi} \sum_{cusps\ \ell} \epsilon(\ell, L, \Gamma) \sum_{N\in\mathbb{Z}} \beta(4\pi d k_\ell^2 N^2 v) q^{-dk_\ell^2 N^2}$$

is a <u>*non-holomorphic*</u> *elliptic modular form of weight* $3/2$. *Here* $\epsilon(\ell, L, \Gamma)$ *denotes the 'width' of the cusp* ℓ *of* Γ *(see Def. 3.2),* $d \in \mathbb{N}$ *squarefree, the discriminant of the quadratic space* V, k_ℓ *the smallest* $k \in \mathbb{N}$ *such that* $L_{-dk^2,\ell} = \{x \in L_{-dk^2} : x \perp \ell\}$ *is nonempty, and* $\beta(y) = \beta_{\frac{3}{2}}(y) = \int_1^\infty t^{-3/2} e^{-ty} dt$.

For example, specializing to a certain lattice in the quadratic space of discriminant 1, we recover Zagier's [25] well-known Eisenstein series $\mathcal{F}$ of weight 3/2 as a theta integral. One has

$$\mathcal{F}(\tau) = \sum_{N=0}^{\infty} H(N) e^{2\pi i n\tau} + \frac{v^{-1/2}}{16\pi} \sum_{N=-\infty}^{\infty} \beta(4\pi N^2 v) e^{-2\pi i N^2 \tau}, \tag{1.6}$$

where $H(N)$ denotes the class number of positive definite binary quadratic forms of discriminant $-N^2$. From this perspective, we can consider Theorem 1.1 on one hand as a special case of the Siegel-Weil formula expressing the theta integral as an Eisenstein series, and on the other hand as the generalization of Zagier's function to arbitrary lattices of signature $(1, 2)$.

Corollary 1.2. $\int_M \theta_\varphi(\tau, L) - 4\sum_\ell \sqrt{d} k_\ell \epsilon(\ell, L, \Gamma) \mathcal{F}(dk_\ell^2 \tau)$ *is a* <u>*holomorphic*</u> *modular form. Hence* $\#(\Gamma\backslash L_N)$ *can be expressed in terms of class numbers and Fourier coefficients of a holomorphic modular form.*

We also would like to mention the results of [15], where it was shown that the generating series of the degree of certain 0-cycles in an arithmetic curve (namely, the moduli scheme of elliptic curves with complex multiplication over the ring of integers of an imaginary quadratic field) is the holomorphic part of a non-holomorphic modular form of weight 1. Here the negative Fourier coefficients involve the function $\beta_1(y) = \int_1^\infty t^{-1} e^{-ty} dt$. The similarities of the situation over $\mathbb{C}$ and over number fields are quite striking.

We also compute the Mellin transform $\Lambda(s)$ of $\int_M \theta_\varphi(\tau, L)$. It turns out to be closely related to certain Siegel Zeta-functions associated to (split) indefinite quadratic forms of signature $(1, 2)$ which were previously studied by Shintani and F. Sato (see [22, 21]). We have

Theorem 1.3.

$$\Lambda(s) = (2\pi)^{-s} \Gamma(s) \zeta_2(s, L) + 2^{-1-s} \pi^{-\frac{1}{2}-s} \Gamma(s - \frac{1}{2}) \frac{1}{s} F(\frac{3}{2}, s, s+1, -1) \zeta_1(s, L),$$

where $F = {}_2F_1$ *is the hypergeometric function, and the Siegel Zeta-functions are defined by*

$$\zeta_1(s, L) = \sum_{\substack{x\in L \\ x^\perp\ split}} |q(x)|^{-s} \qquad and \qquad \zeta_2(s, L) = \sum_{\substack{x\in L \\ q(x)>0}} \frac{1}{|\Gamma_x|} |q(x)|^{-s}.$$

The proof of Theorem 1.1 is long and occupies Section 4. After showing the convergence of $\int_M \theta_\varphi(\tau, L)$ using a Poisson summation argument, we are reduced to calculating the orbital integrals

$$\int_{\Gamma\backslash D} \sum_{\gamma\in\Gamma_x\backslash\Gamma} \gamma^*\varphi(x) \tag{1.7}$$

for elliptic, hyperbolic, and parabolic stabilizer Γ_x. (In the parabolic case, Γ_x shall mean here the stabilizer of the isotropic line generated by x, and one first has to sum over all isotropic vectors before integrating).

We treat the theta integral (1.5) for general p and a modular curve $C = C_U$ in Section 5. The result is

Theorem 1.4. *Let $U \subset V$ be positive definite of dimension $p-1$ so that $C_U \simeq \Gamma_U\backslash\mathbb{H}$. Assume for simplicity $L = L\cap U + L\cap U^\perp$. Then $\int_{C_U} \theta_{\varphi_V}(\tau, L)$ is non-holomorphic for C_U noncompact and*

$$\int_{C_U} \theta_{\varphi_V}(\tau, L) = \theta(\tau, L\cap U) \int_{C_U} \theta_{\varphi_{U^\perp}}(\tau, L\cap U^\perp),$$

where $\theta(\tau, L\cap U) = \sum_{x\in L\cap U} e^{2\pi i q(x)\tau}$ is the standard theta series attached to the positive definite lattice $L\cap U$ and the second integral is the one considered in Th. 1.4 for the space $U^\perp$, which has signature $(1,2)$.

Moreover, the holomorphic Fourier coefficients can be interpreted as the intersection numbers in (the interior of) M of the curve C_U and the divisor C_N.

This generalizes parts of the results of Hirzebruch-Zagier, namely the ones concerning the intersection numbers of the curves T_N in the 'interior' of the Hilbert modular surface. As an example for Theorem 1.2 we then explicitly derive these parts. From there one can obtain the complete result of Hirzebruch-Zagier by applying the holomorphic projection principle for modular forms to the theta integral. This will then account for the contribution of the cusps to the intersection numbers (This is an idea of van der Geer and (independently) Zagier; see [23]). However, when using this procedure, one still needs explicit formulas for the intersection at the cusps. As this seems unfeasible for the higher dimensional case, a more conceptual approach for the cusps (similar to the treatment of the 'interior' presented here) is still needed.

Our results should be closely related to recent work of Borcherds [2] and Bruinier [3, 4]. They showed that the generating series $\sum_{N=0}^{\infty} C_N q^N$ of Heegner divisors, when considered as elements in the so-called 'Heegner divisor class group', is a *holomorphic* modular form of weight $(p+2)/2$ with values in this group in the sense that application of a linear form on this Heegner divisor class group gives a scalar-valued holomorphic modular form. One should expect that for $p \geq 2$, an extension of the methods used here to the cusps should yield the results of [2, 3, 4] (while it seems that the methods used there will not imply our results). For $p = 1$, the results actually are independent of each other, as taking the degree is the zero map on the 'Heegner divisor class group'.

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2. Work of Kudla and Millson

Kudla and Millson explicitly construct for orthogonal, unitary, and symplectic groups of arbitrary signature the Poincaré dual form of the cycles C_U ([12, 13]).

We denote by $\Omega^{r,s}(D)$ the space of smooth differentials forms of type (r,s) on D.

Theorem 2.1 (Kudla-Millson [12, 13]). *For each n with $0 \leq n \leq p$, there is a nonzero Schwartz form*

$$\varphi^{(n)} \in [S(V(\mathbb{R})^n) \otimes \Omega^{n,n}(D)]^G \tag{2.1}$$

such that

(i)
$$d\varphi^{(n)} = 0;$$
i.e., for each $x \in V^n$, $\varphi^{(n)}(x)$ is a closed (n,n)-form on D which is G_x-invariant:
$$g^*\varphi^{(n)}(x) = \varphi^{(n)}(x)$$
for $g \in G_x$, the stabilizer of x in G.

(ii) *Denote by $\varphi_U^+(x) = e^{-\pi tr(x,x)}$ with $x \in U^n$ and $(x,x)_{i,j} = (x_i, x_j)$ the standard Gaussian on a positive definite subspace U of $V(\mathbb{R})$. Then, under the pullback $i_U^* : \Omega^{n,n}(D) \longrightarrow \Omega^{n,n}(D_U)$ of differential forms, we have*
$$i_U^*\varphi^{(n)} = \varphi_U^+ \otimes \varphi_{U^\perp}^{(n)},$$
where $\varphi_{U^\perp}^{(n)}$ is the n-th Schwartz form for $U^\perp$ and G_U.

(iii) *Assume $U = U(x)$ for a linear independent n-frame in U. Then the Poincaré dual of $\Gamma_U \backslash D_U$ is given by*
$$\left[e^{\pi(x,x)} \sum_{\gamma \in \Gamma_U \backslash \Gamma} \gamma^* \varphi^{(n)}(x) \right].$$

From now on we will restrict our attention to the case $n = 1$. For simplicity we will write φ for the $(1,1)$-form $\varphi^{(1)}$.

Remark 2.2. We do not need the explicit general formula for the Schwartz form right now; for an easily accessible construction see [9]. For signature $(1,2)$, we will give the formula in the next section.

We denote by $\widetilde{SL_2(\mathbb{R})}$ the two-fold cover of $SL_2(\mathbb{R})$. Recall that $\widetilde{SL_2(\mathbb{R})}$ acts on the Schwartz space $S(V(\mathbb{R}))$ via the Weil representation ω associated to the additive character $t \longmapsto \exp(2\pi i t)$, see for example [17]. Let $K' \subset \widetilde{SL_2(\mathbb{R})}$ be the inverse image of the standard maximal subgroup $SO(2)$ in $SL_2(\mathbb{R})$.

Proposition 2.3 ([12]). *The Schwartz form φ is an eigenfunction for K' with respect to the Weil representation:*

$$\omega(k')\varphi = \det(k')^{(p+2)/2}\varphi. \tag{2.2}$$

We associate to φ in the usual way a function on the upper half plane $\mathbb{H}$. For $\tau = u + iv \in \mathbb{H}$ we put

$$g'_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix}, \tag{2.3}$$

and define

$$\varphi(\tau, x) = v^{-(p+2)/4}\omega(g'_\tau)\varphi(x). \tag{2.4}$$

Fix a congruence condition $h \in L^\#$ and assume that Γ fixes the coset $L + h$ under the action on $L^\#/L$. Then the theta kernel

$$\theta_\varphi(\tau) = \theta_\varphi(\tau, L, h) = \sum_{x \in h+L} \varphi(\tau, x) \in \Omega^{1,1}(D)^\Gamma \tag{2.5}$$

defines a closed $(1, 1)$-form in $M = \Gamma\backslash D$. By the usual theta machinery (Poisson summation) it is a *non-holomorphic* modular form for the congruence subgroup $\Gamma(N)$ of $SL_2(\mathbb{Z})$ (where N is the level of the lattice L), of weight $(p + 2)/2$ with values in $\Omega^{1,1}(D)^\Gamma$.

Slightly more general than in the introduction we put $\mathcal{L}_m = \{x \in L+h : q(x) = m\}$ for $m \in \mathbb{Q}$, suppressing the dependance on h. For $m > 0$, we again obtain a composite divisor $C_m = \sum_{x\in\Gamma\backslash\mathcal{L}_m} C_x$ in M.

Let η be a closed $(p-1, p-1)$-form on M and assume that η is rapidly decreasing if M is noncompact. The main result of [14] (in much greater generality) is that

$$I_\varphi(\tau, \eta) = \int_M \theta_\varphi(\tau) \wedge \eta \tag{2.6}$$

is a *holomorphic* modular form whose Fourier coefficients are periods of η over the composite cycles C_m in M. If η now represents the Poincaré dual class of a *compact* curve C in M, we obtain

$$I_\varphi(\tau, C) = \int_C \theta_\varphi(\tau) = I_\varphi(\tau, \eta), \tag{2.7}$$

and the Fourier coeffcients are the (cohomological) intersection numbers of C with C_m.

In the following we consider (2.7) for modular curves C.

3. The Theta Integral Associated to $SO(1, 2)$

3.1. **Preliminaries.** Now assume $\dim V = 3$; hence V has signature $(1, 2)$. Over $\mathbb{R}$ we fix an isomorphism

$$V(\mathbb{R}) \simeq \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \in M_2(\mathbb{R}) \right\} \tag{3.1}$$

such that $q(X) = \det(X) = -x_1^2 - x_2x_3$ and $(X,Y) = -tr(XY)$. So we can view $V(\mathbb{R})$ as the trace zero part $B_0(\mathbb{R})$ of the indefinite quaternion algebra $B(\mathbb{R}) = M_2(\mathbb{R})$ over $\mathbb{R}$. We have $G = Spin(V) = SL_2$ and the action on B_0 is the conjugation:

$$g.X := gXg^{-1} \tag{3.2}$$

for $X \in B_0$ and $g \in G$.

Notation. In this section, we will write $z = x + iy$ for an element in $D \simeq \mathbb{H}$ (the orthogonal variable) and $\tau = u + iv \in \mathbb{H}$ for the symplectic variable. The upper case letters X and Y we reserve for vectors in $V(\mathbb{R})$ with coefficients x_i and y_i.

Here it is more convenient to consider the symmetric space $D \simeq \mathbb{H}$ not as the space of negative two-planes in $V(\mathbb{R})$ but rather as the space of positive lines. We give the following identification with the upper half plane. Picking as base point of D the line z_0 spanned by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, we note that $K = SO(2)$ is its stabilizer in $G(\mathbb{R})$, and we have the isomorphism:

$$\mathbb{H} \simeq G(\mathbb{R})/K \longrightarrow D \tag{3.3}$$

with

$$z \longmapsto gK \longmapsto g.z_0 =: \ell(z) \tag{3.4}$$

(where $g \in G(\mathbb{R})$ such that $gi = z$; the action is the usual linear fractional transformation). We find that $\ell(z)$ is generated by

$$X(z) := y^{-1} \begin{pmatrix} -\frac{1}{2}(z + \bar{z}) & z\bar{z} \\ -1 & \frac{1}{2}(z + \bar{z}) \end{pmatrix}, \tag{3.5}$$

where $z = x + iy$. Note $q(X(z)) = 1$. Moreover, per construction

$$g.X(z) = X(gz). \tag{3.6}$$

For $X = \left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{smallmatrix}\right) \in B_0(\mathbb{R})$ we compute

$$\begin{aligned} (X, X(z)) &= -y^{-1}(x_3 z\bar{z} - x_1(z + \bar{z}) - x_2) \\ &= -y^{-1}(x_3(x^2 + y^2) - 2x_1 x - x_2) \\ &= \frac{(x_3 x - x_1)^2 + q(X)}{-x_3 y} - x_3 y, \end{aligned} \tag{3.7}$$

when $x_3 \neq 0$.

The minimal majorant of $(\ , \ )$ associated to $z \in D$ is given by

$$(X,X)_z = \begin{cases} (X,X) & \text{if } X \in \ell(z) \\ -(X,X) & \text{if } X \in \ell(z)^\perp. \end{cases} \tag{3.8}$$

A little calculation shows

$$\begin{aligned} (X,X)_z &= (X, X(z))^2 - (X,X) \\ &= \frac{(x_3 x^2 - x_2 - 2x_1 x)^2}{y^2} + (x_3 y)^2 + 2(x_3 x - x_1)^2. \end{aligned} \tag{3.9}$$

Proposition 3.1 ([10]). *The Schwartz function $\varphi = \varphi^{(1)}$ on $V(\mathbb{R})$ valued in the $(1,1)$-forms on D is explicitly given by*

$$\varphi(X,z) = \left((X, X(z))^2 - \frac{1}{2\pi} \right) e^{-\pi(X,X)_z}\, \omega. \tag{3.10}$$

Here

$$\omega = \frac{dx \wedge dy}{y^2} = \frac{i}{2}\frac{dz \wedge d\bar{z}}{y^2}, \tag{3.11}$$

the standard G-invariant $(1,1)$-form on $D \simeq \mathbb{H}$.

We will write $\varphi(X)$ for its value at X. Then

$$g^*\varphi(X) = \varphi(g^{-1}X); \tag{3.12}$$

i.e., $\varphi(g.X, gz) = \varphi(X,z)$ for $g \in G(\mathbb{R})$, follows from (3.6); while Proposition 2.3 becomes an exercise using the explicit formulae of the Weil representation and reduces to $\hat{\varphi} = -\varphi$.

Finally, we define

$$\varphi^0(X,z) = e^{\pi(X,X)}\varphi(X,z) \tag{3.13}$$

$$= \left((X, X(z))^2 - \frac{1}{2\pi} \right) e^{-\pi(X,X(z))^2 + 2\pi(X,X)}\, \omega \tag{3.14}$$

and

$$\varphi(\tau, X) = \left(v(X, X(z))^2 - \frac{1}{2\pi} \right) e^{\pi i (X,X)_{z,\tau}}\, \omega, \tag{3.15}$$

where

$$\begin{aligned} (X,X)_{z,\tau} &= u(X,X) + iv(X,X)_z. \\ &= \bar{\tau}(X,X) + iv(X,X(z))^2. \end{aligned} \tag{3.16}$$

3.2. **The Theta Integral.** We put

$$I_\varphi(\tau) := \int_{\Gamma\backslash D} \theta(\tau; L, h) = \int_{\Gamma\backslash D} \sum_{X \in L+h} \varphi(\tau, X). \tag{3.17}$$

Thus, in the notation of the previous section, $I_\varphi(\tau) = \theta_\varphi(\tau, \eta)$ where η is the constant function 1. In particular, η is *not* rapidly decreasing. Alternatively we can interpret (3.17) as the integral $I_\varphi(\tau, C_U)$ in the case of signature $(1,2)$ with $U = 0$.

We assume the convergence of (3.17) for the moment. Then it is again clear that $I(\tau)$ defines a (in general non-holomorphic) modular form on the upper half plane of weight $3/2$.

For $X \in \mathcal{L}_m$, we have

$$\varphi(\tau, X) = q^m \varphi^0(\sqrt{v}X), \tag{3.18}$$

where $q^m = e^{2\pi i m \tau}$, as usual. Define

$$\theta_m(\tau) = \sum_{X \in \mathcal{L}_m} \varphi(\tau, X) \qquad \text{and} \qquad \theta^0_m(v) = \sum_{X \in \mathcal{L}_m} \varphi^0(\sqrt{v}X). \tag{3.19}$$

We then - yet formally - have

$$I_\varphi(\tau) = \int_{\Gamma\backslash D} \sum_{m\in\mathbb{Q}} \theta_m(\tau) = \sum_{m\in\mathbb{Q}} \left(\int_{\Gamma\backslash D} \theta^0_m(v) \right) q^m, \tag{3.20}$$

which is the Fourier expansion of $I_\varphi(\tau)$.

Since $\Gamma\backslash\mathcal{L}_m$ is finite for $m\neq 0$, we can simplify the inner integral in (3.20) in that case:

$$\begin{aligned}\int_{\Gamma\backslash D} \theta^0_m(v) &= \int_{\Gamma\backslash D} \sum_{X\in\Gamma\backslash\mathcal{L}_m} \sum_{\gamma\in\Gamma_X\backslash\Gamma} \varphi^0(\gamma^{-1}\sqrt{v}X, z) \\ &= \sum_{X\in\Gamma\backslash\mathcal{L}_m} \int_{\Gamma\backslash D} \sum_{\gamma\in\Gamma_X\backslash\Gamma} \gamma^*\varphi^0(\sqrt{v}X, z).\end{aligned} \tag{3.21}$$

For $X=0$, we simply have $\varphi(\tau, X) = -\frac{1}{2\pi}\omega$ and therefore

$$\int_{\Gamma\backslash D} \varphi(\tau, 0) = \mu(\Gamma\backslash D), \tag{3.22}$$

the hyperbolic volume of $\Gamma\backslash D$, normalized such that $\mu(SL_2(\mathbb{Z})\backslash\mathbb{H}) = -\frac{1}{6}$. In particular, we see that with this normalization (see e.g.[18])

$$\mu(\Gamma\backslash D) \in \mathbb{Q}. \tag{3.23}$$

If V is isotropic over $\mathbb{Q}$, we can pick the isomorphism (3.1) such that

$$V(\mathbb{Q}) \simeq \left\{ \begin{pmatrix} \sqrt{d}x_1 & x_2 \\ x_3 & -\sqrt{d}x_1 \end{pmatrix} : x_i \in \mathbb{Q} \right\} := B_0(d;\mathbb{Q}) \tag{3.24}$$

as quadratic $\mathbb{Q}$-vector spaces, where d is a square-free positive integer, the discriminant of the quadratic space V. The set of all isotropic lines in V corresponds to the cusps of $G(\mathbb{Q})$ and Γ acts on them with finitely many orbits.

For the constant term in the Fourier expansion of $I_\varphi(\tau)$, we therefore have to proceed differently than in the case $m\neq 0$: Let $\ell_1, ..., \ell_t$ be a set of Γ-representatives of isotropic lines and pick $X_i \in \ell_i$ primitive in L. Define $\delta(\ell_i) = \delta(\ell_i, L, h, \Gamma) = 1$ if ℓ_i intersects the coset $L+h$ and $\delta(\ell_i) = 0$ otherwise. Hence $\ell_i \cap (L+h) = \mathbb{Z}X_i + h_i$ for some $h_i \in \ell_i$ if $\delta(\ell_i) = 1$. Denote by Γ_i the stabilizer of ℓ_i in Γ. We then have

$$\begin{aligned}\int_{\Gamma\backslash D} \sum_{\substack{X\in\mathcal{L}_0\\X\neq 0}} \varphi^0(\sqrt{v}X, z) &= \int_{\Gamma\backslash D} \sum_{i=1}^t \sum_{\substack{X\in\Gamma(\ell_i\cap(L+h))\\X\neq 0}} \varphi^0(\sqrt{v}X, z) \\ &= \int_{\Gamma\backslash D} \sum_{i=1}^t \sum_{\substack{X\in\ell_i\cap(L+h)\\X\neq 0}} \sum_{\gamma\in\Gamma_i\backslash\Gamma} \gamma^*\varphi^0(\sqrt{v}X, z) \\ &= \sum_{i=1}^t \delta(\ell_i) \int_{\Gamma\backslash D} \sum_{\gamma\in\Gamma_i\backslash\Gamma} \sideset{}{'}\sum_{k=-\infty}^{\infty} \gamma^*\varphi^0(\sqrt{v}(kX_i + h_i), z).\end{aligned} \tag{3.25}$$

Here $\sum'$ indicates that we omit $k=0$ in the sum in the case of the trivial coset.

We need to discuss the notion of the width of a cusp for our purposes. For $Y\in V$ isotropic (usually primitive in L), pick $g\in G(\mathbb{R})$ such that $gY = \beta X_0$ with $X_0 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$

and $\beta \in \mathbb{R}^\times$. Put $\Gamma' = g\Gamma g^{-1}$. Hence $\Gamma'_{X_0} = g\Gamma_Y g^{-1}$ is equal to $\{\pm \left(\begin{smallmatrix} 1 & k\alpha \\ 0 & 1 \end{smallmatrix}\right) : k \in \mathbb{Z}\}$ (if $-I \in \Gamma$) for some $\alpha \in \mathbb{R}_+$. Now we could call α the width of the cusp ℓ, however this is *not* well defined, since it depends on the choice of $g \in G(\mathbb{R})$ and hence on β. However, one easily checks that the *ratio* $\frac{\alpha}{|\beta|}$ is constant and only depends on Y and Γ.

Definition 3.2 (Width of a cusp). (i) For Y isotropic, we define

$$\epsilon(Y,\Gamma) = \frac{\alpha}{|\beta|},$$

where α and β are the quantities above. We call the pair (Y,Γ) a cusp and the number $\epsilon(Y,\Gamma)$ its width.

(ii) For a lattice $L \subset V$, $h \in L^\#/L$ and $\Gamma \subset \Gamma(L)$, we define (with the above notation) the total width by

$$\epsilon(L,h,\Gamma) = \sum_{i=1}^{t} \delta(\ell_i)\epsilon(X_i,\Gamma).$$

Remark 3.3. (i) If Γ is a congruence subgroup of $SL_2(\mathbb{Z})$, then there is (up to sign) a unique $g \in SL_2(\mathbb{Z})$ such that $gY = \beta X_0 = \beta \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then both numbers α and β are intrinsic: α is simply the usual width of a cusp of a congruence subgroup of $SL_2(\mathbb{Z})$ while β can be interpreted as the volume of the fundamental domain of $\mathbb{R}X_0/\mathbb{Z}\beta X_0$ with respect to the Lebesgue measure on $\mathbb{R}X_0$.

(ii) We can always arrange $|\beta| = 1$. Then we can interpret $\alpha = \epsilon(Y,\Gamma)$ as the volume of the corresponding component of the Borel-Serre boundary of M.

(iii) It does indeed happen that for a fixed space V we can find two lattices with the same stabilizer Γ such that the "cusp ∞" has different width. Consider the lattices $\left\{\left(\begin{smallmatrix} b & a \\ c & -b \end{smallmatrix}\right) : a,b,c \in \mathbb{Z}\right\}$ and $\left\{\left(\begin{smallmatrix} b & 2a \\ 2c & -b \end{smallmatrix}\right) : a,b,c \in \mathbb{Z}\right\}$. Both have stabilizer $\Gamma = SL_2(\mathbb{Z})$, hence $\alpha = 1$, but $\beta = 1$ and 2 respectively. See also Example 3.9.

We are now ready for the main result:

Theorem 3.4. *With the above notation we have*

(i)

$$\theta_\varphi(\tau) \in L^1(\Gamma\backslash D).$$

(ii)

$$\theta^0_m(v) \in L^1(\Gamma\backslash D)$$

for all $m \in \mathbb{Q}$ and

$$\int_{\Gamma\backslash D} \theta_\varphi(\tau) = \sum_{m\in\mathbb{Q}} \left(\int_{\Gamma\backslash D} \theta^0_m(v)\right) q^m.$$

For the Fourier coefficients, we get

(iii) *for $m > 0$:*

$$\int_{\Gamma\backslash D} \theta^0_m(v) = \sum_{X\in\Gamma\backslash\mathcal{L}_m} \frac{1}{|\Gamma_X|};$$

(iv) *for* $m = 0$*:*

$$\int_{\Gamma\backslash D} \theta^0_m(v) = \mu(\Gamma\backslash D) + \frac{1}{2\pi} v^{-1/2} \epsilon(L, h, \Gamma),$$

where the volume term only occurs for $h \in L$*;*

(v) *for* $m < 0$*:*

$$\int_{\Gamma\backslash D} \theta^0_m(v) = \frac{1}{4\pi\sqrt{-m}} v^{-1/2} \sum_{\substack{X\in\Gamma\backslash\mathcal{L}_m \\ X^\perp \mathit{isotropic}}} \left(\int_1^\infty t^{-3/2} e^{4\pi vmt} dt \right).$$

Note that for $m < 0$ we have

$$\int_1^\infty t^{-3/2} e^{4\pi mt} dt \leq O(e^{4\pi m}) \tag{3.26}$$

so that the negative part of the Fourier expansion has the same convergence behavior as the positive part.

The next section will deal with the proof of the Theorem 3.4. First we discuss the convergence of $I_\varphi(\tau)$. Then we turn our attention to the computation of the individual integrals which define the Fourier coefficients via the analysis (3.19) to (3.25).

For $m > 0$, the composite 0-cycle $C_m = \sum_{X\in\Gamma\backslash\mathcal{L}_m} C_X$ is the collection of Heegner points of discriminant m, and we define its degree by

$$\deg(C_m) = \sum_{X\in\Gamma\backslash\mathcal{L}_m} \frac{1}{|\Gamma_X|}. \tag{3.27}$$

We also put

$$\deg(C_0) = \begin{cases} \mu(\Gamma\backslash D) & \text{if } 0 \in L + h \\ 0 & \text{else.} \end{cases} \tag{3.28}$$

In any case we have $\deg(C_m) \in \mathbb{Q}$. We define the generating series of the degree of the Heegner points by

$$P(\tau) = \sum_{m\geq 0} \deg(C_m) q^m. \tag{3.29}$$

Then as a corollary we obtain the following generalization of the results of Kudla and Millson to the noncompact case:

Theorem 3.5. *Let* V *be a quadratic space over* $\mathbb{Q}$ *of signature* $(1, 2)$*.*

(i) *([14]) If* V *is anisotropic, i.e.,* $\Gamma\backslash D$ *compact, then the generating function* $P(\tau)$ *is a* holomorphic *modular form of weight* $3/2$ *for a suitable congruence subgroup of* $SL_2(\mathbb{Z})$*.*

(ii) *If* V *is isotropic, i.e.,* $\Gamma\backslash D$ *noncompact, then* $P(\tau)$ *is the holomorphic part of the* non-holomorphic *modular form of weight* $3/2$ *given by*

$$\sum_{m\geq 0} \deg(C_m) q^m + \frac{v^{-1/2}}{2\pi} \epsilon(L, h, \Gamma) + \sum_{m>0} \frac{v^{-1/2}}{4\pi\sqrt{m}} \sum_{\substack{X\in\Gamma\backslash\mathcal{L}_{-m} \\ X^\perp \mathit{isotropic}}} \beta(4\pi vm)\, q^{-m}, \tag{3.30}$$

where we set, following Zagier [25] *(up to a factor),* $\beta(s) = \int_1^\infty t^{-3/2} e^{-st} dt$*.*

The following lemma characterizes the values for which the negative Fourier coefficients in the previous theorem are nonzero.

Lemma 3.6. *For $X \in V(\mathbb{Q}) \simeq B_0(d;\mathbb{Q})$ with $q(X) < 0$ the following two statements are equivalent:*

1. *$X^{\perp}$ is split over $\mathbb{Q}$.*
2. *$q(X) \in -d\,(\mathbb{Q}^{\times})^2$.*

Proof. If $q(X) = -dm^2 < 0$ with $m \in \mathbb{Q}$, then by Witt's Theorem we can map $X \mapsto \left(\begin{smallmatrix} m\sqrt{d} & 0 \\ 0 & -m\sqrt{d} \end{smallmatrix}\right) \in B_0(d;\mathbb{Q})$; hence $X^{\perp}$ is split. Conversely, if $X^{\perp}$ is split, we can assume $X \perp \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ (again by Witt's Theorem moving an isotropic vector orthogonal to X to $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$). Thus $X = \left(\begin{smallmatrix} m\sqrt{d} & * \\ 0 & -m\sqrt{d} \end{smallmatrix}\right)$ for some $m \in \mathbb{Q}$. ☐

Let $X \in \mathcal{L}_{-dm^2}$. Then X is orthogonal to two cusps. However, giving an orientation to $\mathbb{Q}X$ we can distinguish between these cusps (if they are not equivalent), since switching the cusps by an element in $G \cap SO(X^{\perp})$ switches X to $-X$.

For a fixed cusp ℓ_i, we write $\mathcal{L}_{-dm^2,i,+} = \{X \in \mathcal{L}_{-dm^2} : X \perp \ell_i;\ X \text{ pos orient.}\}$ and note that Γ_i acts on this set. We have

Lemma 3.7.

$$\#\Gamma\backslash\mathcal{L}_{-dm^2} = \sum_{i=1}^{t} \#\Gamma_i\backslash\mathcal{L}_{-dm^2,i,+}$$

and

$$\#\Gamma_i\backslash\mathcal{L}_{-dm^2,i,+} = 2m\sqrt{d}\epsilon(X_i,\Gamma),$$

if $\mathcal{L}_{-dm^2,i,+}$ is not empty.

Proof. The first assertion is clear. For the second, take $X \in \mathcal{L}_{-dm^2}$, say $X = \left(\begin{smallmatrix} m\sqrt{d} & 0 \\ 0 & -m\sqrt{d} \end{smallmatrix}\right)$. So X is orthogonal to the cusps 0 and ∞. We distinguish them by requiring that the left upper left entry of X is positive, since switching the cusps by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ maps X to $-X$. By our conventions about the cusps we can assume that $\left(\begin{smallmatrix} 0 & \beta \\ 0 & 0 \end{smallmatrix}\right)$ is primitive in L with stabilizer $\Gamma_{\infty}(\alpha) = \{T_{\alpha k} = \pm\left(\begin{smallmatrix} 1 & \alpha k \\ 0 & 1 \end{smallmatrix}\right) \;:\; k \in \mathbb{Z}\}$ in Γ (if $-I \in \Gamma$). Now

$$T_{\alpha k}.X = \begin{pmatrix} m\sqrt{d} & -2m\sqrt{d}\alpha k \\ 0 & -m\sqrt{d} \end{pmatrix}; \tag{3.31}$$

hence

$$X - T_{\alpha k}.X = \begin{pmatrix} 0 & -2m\sqrt{d}\alpha k \\ 0 & 0 \end{pmatrix} \in L. \tag{3.32}$$

Thus $2m\sqrt{d}\alpha \in \beta\mathbb{Z}$. The assertion now follows from the observation that we have showed that we have $2m\sqrt{d}\epsilon$ equivalence classes of vectors in $\mathcal{L}_{-dm^2,i,+}$. ☐

Lemmata 3.6 and 3.7 enable us to rewrite Theorem 3.5 (ii):

Theorem 3.8. *With the above notation we have*

$$I_{\varphi}(\tau,L,h) = \sum_{m\geq 0}\deg(C_m)q^m + \frac{v^{-\frac{1}{2}}}{2\pi}\sum_{\textit{cusps } \ell_i}\epsilon(X_i,\Gamma)\Big(\delta(\ell_i) + \sum_{\substack{m\in\mathbb{Q}_+ \\ \mathcal{L}_{-dm^2,i,+}\neq\emptyset}}\beta(4\pi dm^2v)q^{-dm^2}\Big).$$

This implies Theorem 1.1 as follows: For the constant Fourier coefficient note that for the trivial coset we have $\delta(\ell_i) = 1$ and $\beta(0) = 2$. For the negative coefficients, take a non-isotropic vector X in $\ell_i^{\perp}$, primitive in L. We see $q(X) = -dk_{\ell_i}^2$ for some $k_{\ell_i} \in \mathbb{N}$. Now all other vectors in the sum for the cusp ℓ_i are integral multiples of X.

Example 3.9 (Zagier's Eisenstein Series of weight 3/2)**.** Let V be the space of trace 0 elements of the (indefinite) split quaternion algebra $M_2(\mathbb{Q})$ over $\mathbb{Q}$; i.e., $V \simeq B_0(1, \mathbb{Q})$.

(i). We first consider the isotropic lattice

$$L = \left\{ \begin{pmatrix} b & 2a \\ 2c & -b \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}; \tag{3.33}$$

so $q(a, b, c) = -b^2 - 4ac$. L has level 4. As mentioned above we have $\Gamma = \Gamma(L) = SL_2(\mathbb{Z})$ and one isomorphism class of cusps. The stabilizer of $\ell_0 = \mathbb{Q}\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right)$ is $\Gamma_\infty = \{\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$. Note that the width of this cusp (in the above sense) is $\epsilon = \frac{1}{2}$! By Lemma 3.7 Γ acts on L_{-n^2} with n orbits, and these X are precisely the vectors such that $X^{\perp}$ is split over $\mathbb{Q}$.

For $N > 0$ and $X = \left(\begin{smallmatrix} b & 2a \\ 2c & -b \end{smallmatrix}\right)$ such that $q(X) = N$, observe that the assignment

$$X \longmapsto \begin{pmatrix} -2c & b \\ b & 2a \end{pmatrix} \tag{3.34}$$

defines (for $a > 0$) a binary positive definite quadratic form of discriminant $-N$. One easily deduces that

$$\#\left(\Gamma \backslash L_N\right) = 2H(N), \tag{3.35}$$

where $H(N)$ denotes the class number of binary positive definite integral quadratic forms of discriminant $-N$; we count the classes with nontrivial stabilizer with multiplicities 1/2 and 1/3 respectively. The elements of the form $k\left(\begin{smallmatrix} 0 & 2 \\ -2 & 0 \end{smallmatrix}\right)$ (corresponding to $i \in \mathbb{H}$) are fixed by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and the stabilizer of the elements of the form $k\left(\begin{smallmatrix} 1 & 2 \\ -2 & 1 \end{smallmatrix}\right)$ (corresponding to $(1 + i\sqrt{3})/2$) is generated by $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We set $H(0) = -\frac{1}{12} = \frac{1}{2}\mu(\Gamma \backslash D)$.

Applying Theorem 1.1 we obtain

$$\frac{1}{2} I_\varphi(\tau; L) = \mathcal{F}(\tau) = \sum_{N=0}^{\infty} H(N) q^N \; + \; \frac{v^{-1/2}}{16\pi} \sum_{n \in \mathbb{Z}} \beta(4\pi n^2 v)\, q^{-n^2}, \tag{3.36}$$

Zagier's well known non-holomorphic Eisenstein series of weight 3/2 and level 4, see [25, 7].

(ii). This example we will need later to deduce a case of the results of Hirzebruch and Zagier. We almost repeat (i) considering the trace zero elements in the maximal order $M_2(\mathbb{Z})$ in $M_2(\mathbb{Q})$:

$$K = \left\{ \begin{pmatrix} b & a \\ c & -b \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}; \tag{3.37}$$

so $q(a, b, c) = -b^2 - ac$. We set $X_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and define (the coset)

$$K_j = \frac{j}{2} X_1 \; + \; K \tag{3.38}$$

for $j = 0, 1$. Again we put $\Gamma = SL_2(\mathbb{Z}) = \Gamma(K_j)$. This time we have $\epsilon = 1$ and $\delta = 0$ for $j = 1$. As above we see that Γ acts on the vectors of length $-(n+\frac{j}{2})^2$ with $2n+j$ orbits.

For $N > 0$ and $X = \left(\begin{smallmatrix} b+j/2 & a \\ c & -b-j/2 \end{smallmatrix}\right)$ such that $q(X) = N - \frac{j}{4}$ we assign

$$X \longmapsto \begin{pmatrix} -2c & 2b+j \\ 2b+j & 2a \end{pmatrix}, \tag{3.39}$$

which defines a positive form of discriminant $-4N + j$! We obtain

$$\#\left(\Gamma\backslash (K_j)_{N-\frac{j}{4}}\right) = 2H(4N-j), \tag{3.40}$$

hence

$$\frac{1}{2} I_\varphi(\tau; K_j) = \sum_{N=0}^{\infty} H(4N-j) q^{N-j/4} + \frac{v^{-1/2}}{8\pi} \sum_{n\in\mathbb{Z}} \beta\left(4\pi(n+\tfrac{j}{2})^2 v\right) q^{-(n+\frac{j}{2})^2}. \tag{3.41}$$

The example now implies Corollary 1.2.

As a corollary to the example and as an illustration of Corollary 1.2 we obtain the famous relationship between the representation numbers of integers as a sum of three squares and the class numbers of binary quadratic forms:

Corollary 3.10. *Denote by $r_3(N)$ the representation number of N as a sum of three squares and write $\vartheta(\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 \tau}$ for the classical Jacobi theta series. Then*

$$(\vartheta(\tau))^3 = 12\left(\frac{1}{2} I_\varphi(\tau; K_0) - I_\varphi(\tau; L)\right);$$

i.e.,

$$r_3(N) = 12\left(H(4N) - 2H(N)\right)$$

for all integers N.

Proof. The space of holomorphic modular forms for $\Gamma_0(4)$ of weight $3/2$ of Nebentypus is one-dimensional, spanned by $(\vartheta(\tau))^3$. □

We now compute the Mellin transform $\Lambda(s)$ of $I_\varphi(\tau)$, hence giving the proof of Theorem 1.3:

We write

$$\Lambda(s) = \Lambda_+(s) + \Lambda_-(s) \tag{3.42}$$

with

$$\Lambda_\pm(s) = \int_0^\infty \sum_{\substack{X\in L+h \\ \pm q(X)>0}} \varphi(iv, X) v^s \frac{dv}{v}. \tag{3.43}$$

For the positive part, we obtain in the standard fashion via Theorem 3.4 (iii)

$$\Lambda_+(s) = (2\pi)^{-s}\Gamma(s) \sum_{\substack{X\in L+h \\ q(X)>0}} \frac{1}{|\Gamma_x|} q(X)^{-s} = (2\pi)^{-s}\Gamma(s)\zeta_2(s, L, h). \tag{3.44}$$

For the negative part, we have via Theorem 3.4 (v)

$$\Lambda_-(s) = \int_0^\infty \sum_{\substack{X\in L+h\\ X^\perp \text{ split}}} \frac{v^{-\frac{1}{2}}}{4\pi\sqrt{|q(X)|}} \left(\int_1^\infty u^{-\frac{3}{2}} e^{-4\pi v|q(X)|u} du\right) e^{2\pi|q(X)|v} v^s \frac{dv}{v} \tag{3.45}$$

$$= \sum_{\substack{X\in L+h\\ X^\perp \text{ split}}} \frac{v^{-\frac{1}{2}}}{4\pi\sqrt{|q(X)|}} \int_1^\infty \left(\int_0^\infty e^{-2\pi|q(X)|(2u-1)v} v^{s-\frac{1}{2}} \frac{dv}{v}\right) u^{-\frac{3}{2}} du \tag{3.46}$$

$$= \sum_{\substack{X\in L+h\\ X^\perp \text{ split}}} \frac{v^{-\frac{1}{2}}}{4\pi\sqrt{|q(X)|}} \Gamma(s-\frac{1}{2}) \int_1^\infty (2\pi|q(X)|(2u-1))^{\frac{1}{2}-s} u^{-\frac{3}{2}} du \tag{3.47}$$

$$= 2^{-\frac{3}{2}-s}\pi^{-\frac{1}{2}-s}\Gamma(s-\frac{1}{2})\zeta_1(s,L,h) \int_1^\infty \frac{(2u-1)^{-\frac{1}{2}-s}}{u^{3/2}} du. \tag{3.48}$$

The integral in the last line is equal to $\sqrt{2}\int_0^1 \frac{w^{s-1}}{(w+1)^{3/2}} = \sqrt{2}\frac{1}{s}F(\frac{3}{2},s,s+1,-1)$, see [16].

This concludes the proof of Theorem 1.3.

4. Proof of Theorem 3.4

4.1. Convergence of the Theta Integral.

Proposition 4.1 (Theorem 3.4 (i))**.**

$$\theta_\varphi(\tau; L, h) \in L^1(\Gamma\backslash D).$$

Proof. If $\Gamma\backslash D$ is compact, i.e., $V = V(\mathbb{Q})$ is anisotropic, then there is nothing to show; the existence of the integral is immediate.

On the other hand, if V is isotropic then we can choose the isomorphism of $V(\mathbb{R}) \simeq B_0(\mathbb{R})$ such that $V(\mathbb{Q}) \simeq B_0(d;\mathbb{Q})$ and $X_0 := \left(\begin{smallmatrix} 0 & 1\\ 0 & 0\end{smallmatrix}\right)$ primitive in L.

We then can find $r \in \mathbb{Q}$ such that $L' := rB_0(d;\mathbb{Z}) \subset L$. With this notation we see

$$\theta_\varphi(\tau; L, h) = \sum_{\substack{h'\equiv h(L)\\ \text{mod } L'}} \theta_\varphi(\tau; L', h'). \tag{4.1}$$

So it is sufficient to show that each $\theta_\varphi(\tau; L', h') \in L^1(\Gamma\backslash D)$. Picking a fundamental domain for $\Gamma\backslash D$ we observe that via (3.12) it suffices to show that θ_φ is rapidly decreasing as $y \longrightarrow \infty$.

For $X \in B_0(d;\mathbb{Q})$, we have

$$\begin{aligned} &\varphi(\tau, X)\\ &= \left(v(X,X(z))^2 - \frac{1}{2\pi}\right) e^{-\pi v(X,X(z))^2} e^{2\pi i\bar\tau q(X)}\omega \\ &= \left(\frac{v}{y^2}(x_3 z\bar z - 2\sqrt{d}x_1 x - x_2)^2 - \frac{1}{2\pi}\right) \exp\left(-\pi\frac{v}{y^2}(x_3 z\bar z - 2\sqrt{d}x_1 x - x_2)^2\right)\\ &\quad\times\ e\left(-\bar\tau(dx_1^2 + x_2x_3)\right)\omega. \end{aligned} \tag{4.2}$$

Write $x_2 = x_2' + h_2'$ and let x_2' run over $r\mathbb{Z}$. We will apply Poisson summation to the sum on x_2'. So consider in the above expression the coeffcient of ω as a function f of x_2'. For the Fourier transform of f, we see by changing variables to $-t = \frac{\sqrt{v}}{y}(x_3 z\bar{z} - 2\sqrt{d}x_1 x - x_2' - h_2')$,

$$\begin{aligned}
\hat{f}(w) &= \int_{-\infty}^{\infty} f(x_2') e^{-2\pi i x_2' w} dx_2' \\
&= (y\sqrt{v})^{-1} e\big(-\bar{\tau} d x_1^2\big) e\Big(-[w + x_3\bar{\tau}][x_3 z\bar{z} - 2\sqrt{d}x_1 x - h_2']\Big) \\
&\qquad \times \int_{-\infty}^{\infty} \left(t^2 - \frac{1}{2\pi}\right) \exp(-\pi t^2)\, e\left(-t\frac{y}{\sqrt{v}}(w + x_3\bar{\tau})\right) dt \\
&= *(-1)\left(\frac{y}{\sqrt{v}}(w + x_3\bar{\tau})\right)^2 \exp\left(-\pi\left(\frac{y}{\sqrt{v}}(w + x_3\bar{\tau})\right)^2\right),
\end{aligned} \tag{4.3}$$

since the Fourier transform of $\big(t^2 - \frac{1}{2\pi}\big)\exp(-\pi t^2)$ is $-t^2 \exp\left(-\pi t^2\right)$.

We obtain:

$$\begin{aligned}
&\theta(\tau; L', h')(z) = \\
&- r^{-1}\frac{y}{v^{3/2}} \sum_{\substack{w \in r^{-1}\mathbb{Z} \\ x_1 \in h_1' + r\mathbb{Z} \\ x_3 \in h_3' + r\mathbb{Z}}} (w + x_3\bar{\tau})^2\, e\big(-\bar{\tau} d x_1^2\big) \\
&\qquad \times\ e\Big(-[w + x_3\bar{\tau}][x_3 z\bar{z} - 2\sqrt{d}x_1 x - h_2']\Big) \exp\left(-\pi\frac{y^2}{v}(w + x_3\bar{\tau})^2\right)\omega.
\end{aligned}$$

All terms of this sum are exponentially decreasing as $y \mapsto \infty$ except those for which $w = x_3 = 0$. But the coefficient $(w + x_3\bar{\tau})^2$ vanishes for such terms.

So $|\theta(\tau; L', h')|$ is integrable and $\theta(\tau; L, h)$ is real analytic in τ. $\square$

4.2. **Computation of the Individual Terms.** We will compute for Γ an arbitrary Fuchsian subgroup of $SL_2(\mathbb{R})$ the integrals

$$\int_{\Gamma \backslash D} \sum_{\gamma \in \Gamma_X \backslash \Gamma} \gamma^* \varphi^0(X, z) \tag{4.4}$$

with any non-isotropic $X \in V(\mathbb{R})$ and Γ_X the stabilizer of X in Γ, and

$$\int_{\Gamma \backslash D} \sum_{\substack{X \in \Gamma(\mathbb{Z}Y + h_Y) \\ X \neq 0}} \varphi^0(X, z), \tag{4.5}$$

where $Y \in V(\mathbb{R})$ isotropic such that $\mathbb{Q}Y = \ell$ is a cusp of Γ, i.e, Γ_Y is nontrivial, and $h_Y \in \ell$. Note that via (3.21) and (3.25) the calculation of these integrals yields the Fourier expansion of $I_\varphi(\tau)$ (Theorem 3.4).

There are several cases to consider:

A:	$q(X) > 0$
B1:	$q(X) < 0$, Γ_X nontrivial, infinite cyclic
B2:	$q(X) < 0$, Γ_X trivial
C :	The integral (4.5)

According to the stabilizer Γ_X of X (Γ_ℓ of the isotropic line ℓ) we call A the elliptic, B the hyperbolic and C the parabolic case.

Note that the cases are closely related to each other:

Lemma 4.2. *Let $q(X) < 0$ for $X \in V(\mathbb{Q})$, so $X^\perp$ has signature $(1,1)$. Then Γ_X is trivial if $X^\perp$ splits over $\mathbb{Q}$. Conversely, if $X^\perp$ is nonsplit, i.e., anisotropic over $\mathbb{Q}$, then Γ_X is infinite cyclic.*

Proof. Indeed, we can identify $G_X(\mathbb{R})$ with $Spin(X^\perp) \simeq \mathbb{R}^\times$ which acts on the two isotropic lines of $X^\perp$ as homotheties. Hence $\Gamma_X = \Gamma(X^\perp)$ being a discrete subgroup of $\mathbb{R}^\times$ is either trivial or infinite cyclic. But for a possible isotropic line ℓ over $\mathbb{Q}$, i.e.,a cusp for Γ, we would also have a discrete unipotent subgroup in Γ stabilizing ℓ. This together with $\Gamma(X^\perp)$ has an accumulation point, see [18], p.16. On the other hand, an indefinite anisotropic binary quadratic form over $\mathbb{Q}$ corresponds to the norm form of a real quadratic field K over $\mathbb{Q}$, and the units $\mathcal{O}_K^\times$ act as isometries and are infinite cyclic (up to torsion). $\square$

So the cases B2 and C occur together. Moreover, they occur if and only if $\Gamma\backslash D$ is noncompact. These two cases create the complications when extending the results of Kudla and Millson to the noncompact case.

4.2.1. *A: The Elliptic Case.* Let $X \in V(\mathbb{R})$ such that $q(X) > 0$, hence $\mathbb{R}X \in D$. Therefore the stabilizer Γ_X of X in $G(\mathbb{R}) = SL_2(\mathbb{R})$ is conjugate to $SO_2(\mathbb{R})$, and is in particular compact. Since Γ_X is a discrete subgroup, it is finite cyclic.

Proposition 4.3 (Theorem 3.4 (iii))**.** *Let $q(X) > 0$. Then*

$$\sum_{\gamma\in\Gamma_X\backslash\Gamma} |\gamma^*\varphi(X,z)| \in L^1(\Gamma\backslash D);$$

unfolding in (4.4) is allowed, and we have

$$\int_{\Gamma\backslash D}\sum_{\gamma\in\Gamma}\gamma^*\varphi^0(X,z) = \int_D \varphi^0(X,z) = 1. \tag{4.6}$$

Remark 4.4. This result is certainly already contained in [12, 13]. In fact, it is one of the corner stones of the theory. However, for reasons of selfcontainedness, we give a brief sketch of the proof.

Proof. Write $X = \left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & -x_1\end{smallmatrix}\right)$. Since $q(X) > 0$, we have $x_3 \neq 0$. Hence by the explicit formulae for φ we get

$$\begin{aligned}\varphi^0(X,z) &= e^{2\pi(X,X)} \\ &\times \left[\left(\frac{(x_3x-x_1)^2+q(X)}{-x_3y} - x_3y\right)^2 - \frac{1}{2\pi}\right] e^{-\pi\left(\frac{(x_3x-x_1)^2+q(X)}{-x_3y}-x_3y\right)^2}\frac{dxdy}{y^2}.\end{aligned} \tag{4.7}$$

The integrand is rapidly decreasing for $x\to\pm\infty$ and $y \to 0,\infty$. Note that here one needs $q(X) > 0$. Hence unfolding is allowed and therefore $\varphi^0(X,z) \in L^1(D)$. The proof of $\int_D \varphi^0(X,z) = 1$ can be found in [13], p. 301-302. Note that the Schwartz function considered there differs from ours by a factor of 1/2. $\square$

4.2.2. *B: The Hyperbolic Case.* So let $X \in V(\mathbb{R})$ such that $q(X) < 0$, say $q(X) = -d$ with $d \in \mathbb{R}_+$. The stabilizer of $X_1(d) := \begin{pmatrix} \sqrt{d} & 0 \\ 0 & -\sqrt{d} \end{pmatrix}$ in $G(\mathbb{R}) = SL_2(\mathbb{R})$ is $G_{X_1(d)}(\mathbb{R}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^\times \right\}$. So Γ_X is conjugate to a discrete subgroup Γ'_X of $G_{X_1(d)}$, hence either infinite cyclic or trivial.

Case B1.

With the above notation assume that Γ_X is infinite cyclic, say $g.X = X_1(d)$ and $g\Gamma_X g^{-1} = \Gamma'_X = \left\langle \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \right\rangle$ with some $\epsilon > 1$ (can assume $\epsilon > 0$, since $-I$ acts trivially on D).

Proposition 4.5 (Theorem 3.4(v)). *Let $X \in V(\mathbb{R})$ such that $q(X) < 0$ and Γ_X infinite cyclic. Then*

$$\sum_{\gamma \in \Gamma_X \backslash \Gamma} |\gamma^* \varphi^0(X, z)| \in L^1(\Gamma \backslash D);$$

unfolding in (4.4) is valid and

$$\int_{\Gamma \backslash D} \sum_{\gamma \in \Gamma_X \backslash \Gamma} \gamma^* \varphi^0(X, z) = \int_{\Gamma_X \backslash D} \varphi^0(X, z) = 0. \tag{4.8}$$

Remark 4.6. This orbital integral also appears in the compact quotient case. Since in that case all negative Fourier coefficients vanish, the above proposition also follows from the work of Kudla and Millson. However, as they only sketch the argument in this particular case ([14], p. 138), it seems desirable to give a direct proof.

Proof. We have

$$\begin{aligned} \int_{\Gamma \backslash D} \sum_{\gamma \in \Gamma_X \backslash \Gamma} \gamma^* \varphi^0(X, z) &= \int_{g\Gamma g^{-1} \backslash D} \sum_{\gamma \in \Gamma_X \backslash \Gamma} \varphi^0(g.X, g\gamma g^{-1} z) \\ &= \int_{\Gamma' \backslash D} \sum_{\gamma \in \Gamma'_X \backslash \Gamma'} \gamma^* \varphi^0(X_1(d), z) \end{aligned} \tag{4.9}$$

where $\Gamma' := g\Gamma g^{-1}$.

We will now show the validity of the unfolding which then proves the existence of the original integral as well.

$$\begin{aligned} \int_{\Gamma' \backslash D} \sum_{\gamma \in \Gamma'_X \backslash \Gamma'} \gamma^* \varphi^0(X_1(d), z) &= \int_{\Gamma'_X \backslash D} \varphi^0(X_1(d), z) \\ &= e^{-4\pi d} \int_{\Gamma'_X \backslash D} \left(4d \frac{x^2}{y^2} - \frac{1}{2\pi} \right) e^{-4\pi d \frac{x^2}{y^2}} \frac{dx dy}{y^2}. \end{aligned} \tag{4.10}$$

$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}$ acts on $z \in D$ as $z \to \epsilon^2 z$. So a fundamental domain $\mathcal{F}$ of $\Gamma'_X \backslash D$ is the domain bounded by the semi arcs $|z| = 1$ and $|z| = \epsilon^2 > 1$ in the upper half plane:

$$\mathcal{F} = \left\{ z \in D : 1 \le |z| < \epsilon^2 \right\}. \tag{4.11}$$

But in this domain $\varphi^0(X_1(d), z)$ is clearly rapidly decreasing as z approaches the boundary of D. So all considered integrals actually exist and unfolding is allowed.

Changing to polar coordinates we compute

$$\begin{aligned}
&\int_{\mathcal{F}} \left(4d\frac{x^2}{y^2} - \frac{1}{2\pi}\right) e^{-4\pi d\frac{x^2}{y^2}} \frac{dxdy}{y^2} \\
&= \int_1^{\epsilon^2} \int_0^{\pi} \left(4d\cot^2(\theta) - \frac{1}{2\pi}\right) e^{-4\pi d\cot^2(\theta)} \frac{1}{r} \csc^2(\theta) d\theta dr \\
&= \log(\epsilon^2) \int_{-\infty}^{\infty} \left(4\pi d t^2 - \frac{1}{2\pi}\right) e^{-4\pi d t^2} dt \qquad (t = \cot\theta) \\
&= 0,
\end{aligned} \tag{4.12}$$

since

$$\int \left(t^2 - \frac{1}{2\pi}\right) e^{-\pi t^2} dt = \frac{1}{2\pi} t e^{-\pi t^2} + C. \tag{4.13}$$

□

Case B2.

Proposition 4.7 (Theorem 3.4(v))**.** *Let $q(X)$ be negative and assume that Γ_X is trivial. Then*

$$\sum_{\gamma\in\Gamma} \gamma^*\varphi^0(X,z) \in L^1(\Gamma\backslash D) \tag{4.14}$$

and

$$\int_{\Gamma\backslash D} \sum_{\gamma\in\Gamma} \gamma^*\varphi^0(X,z) = \frac{1}{4\pi\sqrt{|q(X)|}} \int_1^{\infty} t^{-3/2} e^{4\pi q(X)t} dt. \tag{4.15}$$

Proof. Let $q(X) = -d$ and suppose that Γ_X is trivial. Recall that we chose $g \in G(\mathbb{R})$ such that $g.X = X_1(d) = \left(\begin{smallmatrix} \sqrt{d} & 0 \\ 0 & -\sqrt{d} \end{smallmatrix}\right)$ and $g\Gamma g^{-1} = \Gamma'$. As above we have

$$\int_{\Gamma\backslash D} \sum_{\gamma\in\Gamma} \gamma^*\varphi^0(X,z) = \int_{\Gamma'\backslash D} \sum_{\gamma\in\Gamma'} \gamma^*\varphi^0(X_1(d),z). \tag{4.16}$$

Unfolding in this situation will be *not* possible since

$$\varphi^0(X_1(d),z) = e^{-4\pi d} \left(4d\frac{x^2}{y^2} - \frac{1}{2\pi}\right) e^{-4\pi d\frac{x^2}{y^2}} \frac{dxdy}{y^2} \tag{4.17}$$

is not integrable over D. So we have to proceed more carefully.

Γ_X is trivial. Hence $X^\perp$, having signature $(1,1)$, is split. Therefore the stabilizers in Γ of the isotropic lines in $X^\perp$ are nontrivial by Lemma 4.2. By conjugation we consider the isotropic lines in $X_1(d)^\perp$. They are generated by $X_0 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\widetilde{X_0} = \left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}\right)$. We put $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Note that J switches the isotropic lines (i.e. the cusps) orthogonal to $X_1(d)$: $JX_0 = \widetilde{X_0}$.

Hence $\Gamma'_{X_0} = \langle T_\alpha \rangle$ with $T_\alpha = \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$ for some $\alpha \in \mathbb{R}_+$ and $\Gamma'_{\widetilde{X_0}} = \langle J T_\beta J^{-1} \rangle$ for some $\beta \in \mathbb{R}_+$. We write $\Gamma''_{X_0} = \langle T_\beta \rangle$. Note that α and β are *not* intrinsic to the situation since they depend on the choice of g such that $g.X = X_1(d)$.

Define a subset $\mathcal{G}$ of $D = \mathbb{H}$ by

$$\mathcal{G} = \{z \in D : |z| \geq 1\}. \tag{4.18}$$

Note $J\mathcal{G} = \bar{\mathcal{G}} := D - \mathcal{G}$ (up to measure zero).

Fundamental for us is the fact that J essentially fixes $X_1(d)$: $J.X_1(d) = -X_1(d)$; hence for any $z \in D$

$$\varphi^0(X_1(d), z) = \varphi^0(J.X_1(d), z) \tag{4.19}$$

and therefore (up to measure zero)

$$\varphi^0(X_1(d), z) = \chi_{\mathcal{G}}(z)\, \varphi^0(X_1(d), z) + J^*\left(\chi_{\mathcal{G}}(z)\, \varphi^0(X_1(d), z)\right). \tag{4.20}$$

Here $\chi_{\mathcal{G}}$ denotes the characteristic function of $\mathcal{G}$. We have

$$\begin{aligned}
&\int_{\Gamma\backslash D} \Big| \sum_{\gamma\in\Gamma} \gamma^* \varphi^0(X, z) \Big| \\
&= \int_{\Gamma'\backslash D} \Big| \sum_{\gamma\in\Gamma} \chi_{\mathcal{G}}\,(\gamma z) \varphi^0(X_1(d), \gamma z) + \chi_{\mathcal{G}}(J\gamma z)\, \varphi^0(X_1(d), J\gamma z) \Big| \\
&\leq \int_{\Gamma'\backslash D} \sum_{\gamma\in\Gamma'_{X_0}\backslash\Gamma'} \Big| \sum_{k\in\mathbb{Z}} \chi_{\mathcal{G}}(T_\alpha^k \gamma z)\, \varphi^0(X_1(d), T_\alpha^k \gamma z) \Big| \\
&\qquad + \int_{\Gamma'\backslash D} \sum_{\gamma\in\Gamma'_{\widetilde{X_0}}\backslash\Gamma'} \Big| \sum_{k\in\mathbb{Z}} \chi_{\mathcal{G}}(JJT_\beta^k J^{-1}\gamma z)\, \varphi^0(X_1(d), JJT_\beta^k J^{-1}\gamma z) \Big|.
\end{aligned} \tag{4.21}$$

Now we are in position to unfold and obtain

$$\begin{aligned}
&= \int_{\Gamma'_{X_0}\backslash D} \Big| \sum_{k\in\mathbb{Z}} \chi_{\mathcal{G}}(T_\alpha^k z)\, \varphi^0(X_1(d), T_\alpha^k z) \Big| \\
&\qquad + \int_{\Gamma'_{\widetilde{X_0}}\backslash D} \Big| \sum_{k\in\mathbb{Z}} \chi_{\mathcal{G}}(T_\beta^k J^{-1} z)\, \varphi^0(X_1(d), T_\beta^k J^{-1} z) \Big| \\
&= \int_{\Gamma'_{X_0}\backslash D} \Big| \sum_{k\in\mathbb{Z}} \chi_{\mathcal{G}}(T_\alpha^k z)\, \varphi^0(X_1(d), T_\alpha^k z) \Big| \\
&\qquad + \int_{\Gamma''_{X_0}\backslash D} \Big| \sum_{k\in\mathbb{Z}} \chi_{\mathcal{G}}(T_\beta^k z)\, \varphi^0(X_1(d), T_\beta^k z) \Big|
\end{aligned} \tag{4.22}$$

by changing variables $z \longmapsto Jz$ in the second integral.

It is sufficient to show the convergence of the first integral. So let $\mathcal{F}_\infty$ be a fundamental domain for $\Gamma'_{X_0}\backslash D$ and split $\mathcal{F}_\infty = \mathcal{F}_1 \coprod \mathcal{F}_2$, where $\mathcal{F}_1 = \{z \in \mathcal{F}_\infty : \operatorname{Im}(z) \geq 1\}$. For $\mathcal{F}_1$, we have

$$\begin{aligned}
&\int_{\mathcal{F}_1} \Big| \sum_{k\in\mathbb{Z}} \chi_{\mathcal{G}}(T_\alpha^k z) \varphi^0(X_1(d), T_\alpha^k z) \Big| \\
&= \int_{\mathcal{F}_1} \Big| \sum_{k\in\mathbb{Z}} \varphi^0(X_1(d), z + \alpha k) \Big| \\
&= e^{-4\pi d} \int_{\mathcal{F}_1} \Big| \sum_{k\in\mathbb{Z}} \left(4d \frac{(x+\alpha k)^2}{y^2} - \frac{1}{2\pi} \right) e^{-4\pi d \frac{(x+\alpha k)^2}{y^2}} \Big| \frac{dx\,dy}{y^2} \\
&= \frac{1}{8d^{3/2}\alpha} e^{-4\pi d} \int_{\mathcal{F}_1} \Big| \sum_{w\in\alpha^{-1}\mathbb{Z}} y^3 w^2 e^{-\pi y^2 w^2} e^{2\pi i x w} \Big| \frac{dx\,dy}{y^2}
\end{aligned} \tag{4.23}$$

by Poisson summation (see Section 4.1). But the integrand is clearly exponentially decreasing for $y \longrightarrow \infty$. This shows the existence of this part of the integral. M

Moreover, removing the absolute value signs, we see that the integral (4.23) vanishes, as we easily conclude by interchanging the summation and the integration w.r.t. x in the last line of (4.23). However, note that

$$\sum_{k\in\mathbb{Z}} \left(4d\frac{(x+\alpha k)^2}{y^2} - \frac{1}{2\pi}\right) e^{-4\pi d\frac{(x+\alpha k)^2}{y^2}} \frac{dxdy}{y^2} \tag{4.24}$$

is *not* termwise integrable over $\mathcal{F}_1$.

For $\mathcal{F}_2$, we can even unfold further and get

$$\int_{\{z\in\mathcal{G}:y\leq 1\}} \left|\varphi^0(X_1(d),z)\right| = e^{-4\pi d} \int_{\{z\in\mathcal{G}:y\leq 1\}} \left| \left(4d\frac{x^2}{y^2} - \frac{1}{2\pi}\right) e^{-4\pi d\frac{x^2}{y^2}} \right| \frac{dxdy}{y^2} \tag{4.25}$$

which is clearly finite since the integrand is rapidly decreasing at the boundary of the domain of integration. Note that the last expression does not depend on α.

This shows integrability for the first summand in (4.22); the second summand is handled in the same manner!

These considerations give us

$$\int_{\Gamma\backslash D} \sum_{\gamma\in\Gamma} \gamma^*\varphi^0(X,z) = 2\int_{\{z\in\mathcal{G}:y\leq 1\}} \varphi^0(X_1(d),z) \tag{4.26}$$

$$= 2e^{-4\pi d} \int_{\{z\in\mathcal{G}:y\leq 1\}} \left(4d\frac{x^2}{y^2} - \frac{1}{2\pi}\right) e^{-4\pi d\frac{x^2}{y^2}} \frac{dxdy}{y^2} \tag{4.27}$$

Using (4.13) we get

$$\begin{aligned} 2\int_{\{z\in\mathcal{G}:y\leq 1\}} \varphi^0(X_1(d),z) &= 2\int_0^1 2\int_{\sqrt{1-y^2}}^\infty e^{-4\pi d} \left(4d\frac{x^2}{y^2} - \frac{1}{2\pi}\right) e^{-4\pi d\frac{x^2}{y^2}} \frac{dxdy}{y^2} \\ &= 4e^{-4\pi d} \int_0^1 \frac{\sqrt{1-y^2}}{2\pi} e^{-4\pi d\frac{1-y^2}{y^2}} y^{-2} dy \\ &= \frac{1}{\pi} \int_1^\infty \sqrt{w-1}\, e^{-4\pi d w} w^{-1} dw \qquad (w = y^{-2}) \\ &= \frac{1}{\pi} \int_0^\infty w^{\frac{1}{2}} (w+1)^{-1} e^{-4\pi d(w+1)} dw \\ &= \frac{e^{-4\pi d}}{2\sqrt{\pi}} \Psi\left(\frac{3}{2}, \frac{3}{2}; 4\pi d\right), \end{aligned} \tag{4.28}$$

where $\Psi(\alpha,\gamma;z)$ is the confluent hypergeometric function of the second kind which has for $Re(\alpha), Re(z) > 0$ the integral representation

$$\Psi(\alpha,\gamma;z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}(t+1)^{\gamma-\alpha-1} e^{-zt} dt \tag{4.29}$$

([16], 9.11.6). Further note the functional equation

$$\Psi(\alpha,\gamma;z) = z^{1-\gamma}\Psi(1+\alpha-\gamma, 2-\gamma; z) \tag{4.30}$$

for $|arg(z)| < \pi$ ([16], 9.10.8). Thus

$$\begin{aligned} \frac{e^{-4\pi d}}{2\sqrt{\pi}} \Psi\left(\frac{3}{2},\frac{3}{2};4\pi d\right) &= \frac{1}{4\sqrt{d\pi}} e^{-4\pi d}\,\Psi\left(1,\frac{1}{2};4\pi d\right) \\ &= \frac{1}{4\sqrt{d\pi}}\int_0^\infty (t+1)^{-3/2}e^{-4\pi d(t+1)}dt \\ &= \frac{1}{4\sqrt{d\pi}}\int_1^\infty t^{-3/2}e^{-4\pi dt}dt. \end{aligned} \tag{4.31}$$

This proves the proposition. □

4.2.3. *C: The Parabolic Case.* Now let ℓ be an isotropic line in $V(\mathbb{R})$ such that its (pointwise) stabilizer Γ_ℓ is nontrivial. Pick a point $Y \in \ell$. Choose $g \in G(\mathbb{R}) = SL_2(\mathbb{R})$ such that $g.Y = \beta X_0$ with $X_0 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\beta \in \mathbb{R}^\times$. Put $\Gamma' = g\Gamma g^{-1}$. Hence $\Gamma'_{X_0} := g\Gamma_Y g^{-1}$ is equal to $\{\pm\left(\begin{smallmatrix} 1 & k\alpha \\ 0 & 1 \end{smallmatrix}\right) : k \in \mathbb{Z}\}$ (if $-I \in \Gamma$) for some $\alpha \in \mathbb{R}_+$. Recall that we defined the width of the cusp (Y,Γ) by $\epsilon(Y,\Gamma) = \frac{\alpha}{|\beta|} = \epsilon(Y,\Gamma)$.

Proposition 4.8 (Theorem 3.4(iv))**.** *Let $\ell \in V(\mathbb{R})$ isotropic be a cusp of Γ as above, $Y \in \ell$, $h_Y \in \mathbb{Q}Y$ and $r \in \mathbb{R}^\times$. Then*

$$\int_{\Gamma\backslash D} \sum_{\substack{X\in\Gamma(\mathbb{Z}Y+h_Y)\\ X\neq 0}} \varphi^0(rX,z) = \frac{\epsilon(Y,\Gamma)}{2\pi|r|} \tag{4.32}$$

where $\epsilon(Y,\Gamma)$ is the width of the cusp (Y,Γ) (Def. 3.2).

Proof. We can assume $\beta = \pm 1$ and by abuse of notation $h_Y = h = hX_0$ with $h \in [0,1)$. We are interested in

$$\begin{aligned} \int_{\Gamma\backslash D} \sum_{\substack{X\in\Gamma(\mathbb{Z}Y+h)\\ X\neq 0}} \varphi^0(X,z) &= \int_{\Gamma\backslash D} \sum_{\gamma\in\Gamma_Y\backslash\Gamma} \sideset{}{'}\sum_{k=-\infty}^{\infty} \gamma^*\varphi^0(kY+h,z) \\ &= \int_{g\Gamma g^{-1}\backslash D} \sum_{\gamma\in\Gamma_Y\backslash\Gamma} \sideset{}{'}\sum_{k=-\infty}^{\infty} \varphi^0(g.(kY+h),g\gamma g^{-1}z) \\ &= \int_{\Gamma'\backslash D} \sum_{\gamma\in\Gamma'_{X_0}\backslash\Gamma'} \sideset{}{'}\sum_{k=-\infty}^{\infty} \gamma^*\varphi^0(\pm kX_0+h,z), \end{aligned} \tag{4.33}$$

where $\sum_{k=-\infty}^{\prime\infty}$ omits $k=0$ in the case of the trivial coset. Now

$$\varphi^0(\pm r(k+h)X_0),z) = \left(\frac{((k+h)r)^2}{y^2} - \frac{1}{2\pi}\right) e^{-\pi\frac{((k+h)r)^2}{y^2}} \frac{dxdy}{y^2}. \tag{4.34}$$

We unfold and get

$$(4.35)\quad \int_{\Gamma'\backslash D} \sum_{\gamma\in\Gamma'_{X_0}\backslash\Gamma'} \sideset{}{'}\sum_{k=-\infty}^{\infty} \gamma^*\varphi^0(\pm r(k+h)X_0, z)$$

$$= \int_{\Gamma'_{X_0}\backslash D} \sideset{}{'}\sum_{k=-\infty}^{\infty} \left(\frac{((k+h)r)^2}{y^2} - \frac{1}{2\pi}\right) e^{-\pi\frac{((k+h)r)^2}{y^2}} \frac{dxdy}{y^2}$$

$$= \epsilon(Y,\Gamma) \int_0^{\infty} \sideset{}{'}\sum_{k=-\infty}^{\infty} \left(\frac{((k+h)r)^2}{y^2} - \frac{1}{2\pi}\right) e^{-\pi\frac{((k+h)r)^2}{y^2}} y^{-2}dy.$$

The validity of the unfolding in (4.35) (and therefore the existence of the original integral) follows by looking at the last integral: For $y\to 0$ we have exponential decay and for $y\to\infty$ one sees - adding the constant term $k=0$ into the summation if $h=0$ - by Poisson summation (see Section 4.1)

$$(4.36)\quad \sideset{}{'}\sum_{k=-\infty}^{\infty} \left(\frac{(kr)^2}{y^2} - \frac{1}{2\pi}\right) e^{-\pi\frac{(kr)^2}{y^2}} \frac{dxdy}{y^2} = \left(\frac{1}{2\pi} - r^{-1}y^3 \sum_{w=-\infty}^{\infty} (w/r)^2 e^{-\pi(yw/r)^2}\right)\frac{dxdy}{y^2}$$

which is $O(y^{-2})$ as $y\to\infty$. (The same argument works for $h\neq 0$.)

Now in the last expression of (4.35) interchanging summation and integration is *not* allowed in general since

$$(4.37)\qquad \sideset{}{'}\sum_{k=-\infty}^{\infty} \frac{(kr)^2}{y^2} e^{-\pi\frac{(kr)^2}{y^2}} = \sideset{}{'}\sum_{k=-\infty}^{\infty} \frac{1}{2\pi} e^{-\pi\frac{(kr)^2}{y^2}} = O(y) \qquad (y\to\infty)$$

However, we can modify the integrand defining

$$(4.38)\qquad F(s) := \int_0^{\infty} \sideset{}{'}\sum_{k=-\infty}^{\infty} \left(\frac{((k+h)r)^2}{y^2} - \frac{1}{2\pi}\right) e^{-\pi\frac{((k+h)r)^2}{y^2}} y^{-2-s}dy.$$

for $s \in \mathbb{C}$. F is holomorphic for $Re(s) > -1$ and for $Re(s) > 0$ interchanging summation and integration is valid. We obtain

(4.39)
$$F(s) = \sum_{k=-\infty}^{\infty}{}' \int_0^\infty \left(\frac{((k+h)r)^2}{y^2} - \frac{1}{2\pi} \right) e^{-\pi \frac{((k+h)r)^2}{y^2}} y^{-2-s} dy$$

(4.40)
$$= \frac{1}{2}\pi^{\frac{-3-s}{2}} \sum_{k=-\infty}^{\infty}{}' |(k+h)r|^{-1-s} \int_0^\infty \left(w - \frac{1}{2} \right) \frac{w^{(s+1)/2}}{e^w} \frac{dw}{w} \qquad \left(w = \frac{\pi(k+h)r^2}{y^2} \right)$$

(4.41)
$$= \frac{1}{2}\pi^{\frac{-3-s}{2}} |r|^{-1-s} \left(\zeta(1+s,h) + \zeta(1+s,1-h) \right) \left(\Gamma\left(\frac{s+3}{2} \right) - \frac{1}{2}\Gamma\left(\frac{s+1}{2} \right) \right)$$

(4.42)
$$= \frac{1}{2}\pi^{\frac{-3-s}{2}} |r|^{-1-s} \left(\zeta(1+s,h) + \zeta(1+s,1-h) \right) \frac{s}{2}\Gamma\left(\frac{s+1}{2} \right) \quad \longrightarrow \quad \frac{1}{2\pi|r|},$$

as $s \longrightarrow 0$. Here $\zeta(s,x) = \sum_{n=0}^\infty (x+n)^{-s}$ is the Hurwitz Zeta-function which has a simple pole of residue 1 at $s = 1$. □

5. General Signature $(p, 2)$

5.1. **Theta Integral for $\mathbf{SO(p,2)}$.** Now we assume that V has signature $(p, 2)$. Let $U \subset V$ be a positive definite subspace of dimension $p-1$ so that $C_U = \Gamma_U \backslash D_U$ is a quotient of $\mathbb{H}$ in M. We consider the period integral

$$I_{\varphi_V}(\tau, L, C_U) = \int_{C_U} \theta_{\varphi_V}(\tau, L). \tag{5.1}$$

Here we write φ_V for φ to emphasize the domain φ is associated to. For $p = 1$ and $U = (0)$, this is the theta integral considered in the previous sections.

We write

$$L_U = L \cap U \qquad \text{and} \qquad L_{U^\perp} = L \cap U^\perp. \tag{5.2}$$

We can split the lattice L as follows:

$$L = \sum_{i=1}^r (\lambda_i + L_U) \ \perp \ (\mu_i + L_{U^\perp}) \tag{5.3}$$

with $\lambda_i \in L_U^\#$ and $\mu_i \in L_{U^\perp}^\#$.

Theorem 5.1.

$$I_{\varphi_V}(\tau, L, C_U) = \sum_{i=1}^r \vartheta(\tau, \lambda_i + L_U) I_{\varphi_{U^\perp}}(\tau, \mu_i + L_{U^\perp}).$$

Here $\vartheta(\tau, \lambda_i + L_U) = \sum_{x \in \lambda_i + L_U} e^{2\pi i q(x)\tau}$ is the standard theta function of a coset of the positive definite lattice L_U. In particular, the period integral $I_{\varphi_V}(\tau, L, C_U)$ is a non-holomorphic (if C_U is noncompact) modular form of weight $(p+2)/2$.

Proof. By Theorem 2.1 (ii), we have that under the pullback $i_U^* : \Omega^{1,1}(D) \longrightarrow \Omega^{1,1}(D_U)$ of differential forms,

$$i_U^*\varphi_V = \varphi_U^+ \otimes \varphi_{U^\perp}, \tag{5.4}$$

where φ_U^+ is just the Gaussian on U. Then

$$\theta_{\varphi_V}(\tau, L) = \sum_{x\in L} \varphi_V(X,\tau) \tag{5.5}$$

$$= \sum_{i=1}^{r} \sum_{x\in\lambda_i+L_U} \varphi_U^+(x,\tau) \sum_{y\in\mu_i+L_{U^\perp}} \varphi_{U^\perp}(y,\tau). \tag{5.6}$$

integrating over C_U together with the results on the theta integral for signature $(1,2)$ (Theorem 3.4) now gives the theorem. □

5.2. **Intersection Numbers.** We will now show how one can interpret the Fourier coeffcients of $\theta_{\varphi_V}(\tau, L)$ as intersection numbers.

For a special curve $C = C_U$ ($\dim U = p-1$) and a divisor $C' = C_{U'}$ ($\dim U' = 1$), we define the intersection number in (the interior of) M:

$$[C.C']_M := [C.C']^{tr} + vol(C\cap C')^1, \tag{5.7}$$

the sum of the transversal intersection and the (hyperbolic) volume of the one-dimensional intersection of C and C'. For $p = 2$, the Hilbert modular surface case, this follows Hirzebruch and Zagier ([7]). One easily sees that C_U and $C_{U'}$ intersect transversally if and only if $U + U'$ is positive definite of (maximal) dimension p, while C_U and $C_{U'}$ have a one-dimensional intersection if and only if $C_U \subseteq C_{U'}$, i.e., $U' \subseteq U$.

Theorem 5.2. *Assume for simplicity that $L = L_U + L_{U^\perp}$. Also assume that Γ is torsion-free so that M has no quotient singularities.*

For a composite special divisor C_N ($N \in \mathbb{N}$), we have

(i)

$$[C_U.C_N]^{tr} = \sum_{\substack{N_1\geq 0, N_2>0\\ N_1+N_2=N}} r(N_1, L_U)\deg(N_2, C_U);$$

(ii)

$$vol(C_U\cap C_N)^1 = r(N, L_U)vol(C_U).$$

Here $r(N, L_U) = \#\{x \in L_U : q(x) = N\}$ and $\deg(N, C_U) = \#\Gamma_U\backslash\{x \in L_{U^\perp} : q(x) = N\}$, the degree of the Heegner divisor in the modular curve C_U.

Proof. We write $C_N = \sum_{x\in\Gamma\backslash L_N} C_x$. We decompose $x = x_1 + x_2$ with $x_1 \in L_U$ and $x_2 \in L_{U^\perp}$. We have $C_U \subset C_x$ if and only if $x_2 = 0$. In the sum defining C_N therefore exactly the $x = x_1 \in L_U$ of length N contribute to the one-dimensional intersection. This proves (ii). For (i), each x with $q(x_2) = N_2 > 0$ contributes. Taking into account the action of Γ_U on $L_{U^\perp}$ gives the assertion. □

We certainly have a similar theorem if L does not split along U. If Γ is not torsion-free, then we can always pass to a torsion-free subgroup Γ' of finite index to obtain intersection numbers on $\Gamma'\backslash D$. The intersection numbers on $\Gamma\backslash D$ (in the sense of rational homology manifolds) are then obtained by dividing by the degree of the covering $\Gamma'\backslash D \longmapsto \Gamma\backslash D$.

Corollary 5.3. *Write*

$$I_{\varphi_V}(\tau, L, C_U) = \sum_{N=0}^{\infty} c(N)q^N + \sum_{N=-\infty}^{\infty} c(N,v)q^N$$

for the Fourier expansion of the above theta integral. Then

$$c(N) = [C_U.C_N]_M;$$

i.e., the intersection numbers are exactly the Fourier coefficients of the holomorphic part of $I_{\varphi_V}(\tau, L, C_U)$.

Proof. Just write down the Fourier expansion of $I_{\varphi_V}(\tau, L, C_U)$ using Theorem 3.5 and 5.1. □

5.3. **Hirzebruch-Zagier Case.** As an example we will derive the basic case of the results of Hirzebruch and Zagier [7] and its (mild) extensions by Franke [5], Hausmann [6] and van der Geer [23].

Let $D > 0$ be the discriminant of the real quadratic field $K = \mathbb{Q}(\sqrt{D})$ over $\mathbb{Q}$, $\mathcal{O}_K$ its ring of integers. We denote by $x \longmapsto x'$ the Galois involution on K. We let $V \subset M_2(K)$ be the space of of skew-hermitian matrices in $M_2(K)$, i.e., which satisfy the relation ${}^tX' = -X$. V is a vector space over $\mathbb{Q}$ of dimension 4.

We let $L \subset V$ be the integral skew-hermitian matrices; that is

$$L = \left\{ X = \begin{pmatrix} a\sqrt{D} & \lambda \\ -\lambda' & b\sqrt{D} \end{pmatrix} \ : \ a,b \in \mathbb{Z}, \ \lambda \in \mathcal{O}_K \right\}. \tag{5.8}$$

As usual, the determinant defines a quadratic form on L; we have $Q(X) = abD + \lambda\lambda'$, which is of signature $(2,2)$ and has $\mathbb{Q}$-rank 1, i.e., it splits over $\mathbb{Q}$ into a hyperbolic plane and into an anisotropic part of rank 2. $SL_2(\mathcal{O}_K)$ acts on L by $\gamma.X = \gamma X {}^t\gamma'$ as isometries. We let Γ be the Hurwitz-Maass extension of $SL_2(\mathcal{O}_K)$ which is the maximal discrete subgroup of $PGL_2^+(\mathbb{R})^2$ containing $SL_2(\mathcal{O}_K)$ (for a brief discussion of its definition and properties, see [23]; for example, if $D \equiv 1 \mod 4$ is a prime, then $\Gamma = SL_2(\mathcal{O}_K)$. This is actually the case Hirzebruch and Zagier considered). Slightly changing our notation we write $S = \Gamma \backslash D$ for the Hilbert modular surface. The Hirzebruch-Zagier cycles T_N are nothing but our special cycles $C_N = \sum_{X \in L_N} C_X$. T_N has finitely many components, is non-empty if $\left(\frac{N}{p}\right) \neq -1$ and is compact if N is not the norm of an ideal in $\mathcal{O}_K$.

We can compactify S by adding the cusps and resolving the singularities thus created. We call this (up to quotient singularities) nonsingular compact surface $\widetilde{S}$. Now $H_2(\widetilde{S})$ decomposes canonically into as the direct sum of the image of $H_2(S)$ and the subspace generated by the homology cycles of the curves of the cusp resolution. We denote by T_N^c the component in the first factor of T_N. Hirzebruch and Zagier compute the intersection numbers $[T_N^c.T_M]$ $(N, M \in \mathbb{N})$ and by a direct computation they show that these numbers are the Fourier coefficients of a *holomorphic* modular form of weight 2. In fact, $[T_N^c.T_M]$ is the sum of the intersection numbers $[T_N.T_M]_S$ in the interior and the contribution of the cusp resolution, and these numbers individually occur as Fourier coefficients of two *non-holomorphic* modular forms. For $M = 1$, one has

$$(T_1.T_N)_S = H_D(N) = \sum_{\substack{s^2 \leq 4N \\ s^2 \equiv 4N \mod D}} H\left(\frac{4N - s^2}{D}\right), \tag{5.9}$$

where $H(N)$ denotes the class number of positive definite binary quadratic forms, see Example 3.9.

We now recover these results: We consider the curve $T_1 = C_1$ in our setting. Γ acts transitively on L_1, the vectors of length 1, see [6, 23]. So $T_1 = C_{X_0}$ for any $X_0 \in L_1$. We pick $X_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in L$.

Theorem 5.4.

$$\frac{1}{2}\int_{T_1} \theta_{\varphi_V}(\tau, L) = \sum_{N=0}^{\infty} H_D(N) q^N \ + \ \frac{2v^{-1/2}}{\sqrt{D}} \sum_{\lambda \in \mathcal{O}} \beta\left(4\pi (\tfrac{\lambda - \lambda'}{2})^2 v\right) \ q^{\lambda\lambda'}.$$

So the Fourier coefficients of the holomorphic part of the period integral $\int_{T_1} \theta_{\varphi_V}(\tau, L)$ are the intersection numbers of the the cycles T_1 and T_N in the "interior" of $\Gamma \backslash D$.

Proof. First we assume $D \equiv 0 \mod 4$, so $K = \mathbb{Q}(\sqrt{d})$ with $d = D/4$ square free. In this case L splits orthogonally:

$$L = \mathbb{Z}X_0 \perp (L \cap X_0^{\perp}) \tag{5.10}$$

and

$$L_{U^{\perp}} = L \cap X_0^{\perp} = \left\{ \sqrt{d} \begin{pmatrix} 2a & b \\ -b & 2c \end{pmatrix} \ : \ a, b, c \in \mathbb{Z} \right\}, \tag{5.11}$$

which is - up to the scaling - exactly the lattice considered in Example 3.9(i) which gave rise to Zagier's Eisenstein series $\mathcal{F}$!. Hence by Theorem 5.1 and Example 3.9(i) we find

(5.12)

$$\frac{1}{2} I_{\varphi}(\tau, L, T_1) = \vartheta(\tau) I_{\varphi_{U^{\perp}}}(\tau, L_{U^{\perp}})$$

$$= \vartheta(\tau)\mathcal{F}(d\tau) \tag{5.13}$$

$$= \left(\sum_{m \in \mathbb{Z}} q^{m^2}\right) \left(\sum_{N=0}^{\infty} H(N) q^{dN} \ + \ (dv)^{-1/2} \sum_{n \in \mathbb{Z}} \beta(4\pi n^2 dv) \ q^{-dn^2}\right) \tag{5.14}$$

$$= \sum_{\substack{s^2 \leq N \\ s^2 \equiv N \ (d)}} H\left(\frac{N - s^2}{d}\right) q^N + \frac{v^{-1/2}}{\sqrt{d}} \sum_{N=-\infty}^{\infty} \sum_{\substack{n, m \in \mathbb{Z} \\ m^2 - dn^2 = N}} \beta(4\pi d n^2 v) \ q^N \tag{5.15}$$

$$= \sum_{\substack{(2s)^2 \leq 4N \\ (2s)^2 \equiv 4N \ (4d)}} H\left(\frac{4N - (2s)^2}{4d}\right) q^N + \frac{2v^{-1/2}}{\sqrt{4d}} \sum_{\lambda \in \mathcal{O}} \beta\left(4\pi (\tfrac{\lambda - \lambda'}{2})^2 v\right) \ q^{\lambda\lambda'} \tag{5.16}$$

since $\mathcal{O} = \{m + n\sqrt{d} : m, n \in \mathbb{Z}\}$. Noting $D = 4d$, the theorem follows for $D \equiv 0 \mod 4$.

The case $D \equiv 1 \mod 4$ is slightly more complicated since L does not split orthogonally so that one has to deal with cosets. We have

$$L_{U^\perp} = L \cap X_0^\perp = \left\{ \sqrt{D} \begin{pmatrix} a & b \\ -b & c \end{pmatrix} \;:\; a,b,c \in \mathbb{Z} \right\} \tag{5.17}$$

which is the lattice from Example 3.9(ii)! We set $X_1 = \left(\begin{smallmatrix} 0 & \sqrt{D} \\ \sqrt{D} & 0 \end{smallmatrix}\right)$ and obtain

$$L = (\mathbb{Z}X_0 \perp L_{U^\perp}) \;\oplus\; \frac{1}{2}(X_0 + X_1) + (\mathbb{Z}X_0 \perp L_{U^\perp}) \,. \tag{5.18}$$

The holomorphic part is given by

$$\sum_{j=0}^{1} \left(\sum_{m\in\mathbb{Z}} q^{(m+\frac{j}{2})^2} \right) \left(\sum_{n=0}^{\infty} H(4n-j) q^{D(n-\frac{j}{4})} \right) = \sum_{j=0}^{1}\sum_{N=0}^{\infty} \sum_{m,n\in\mathbb{Z}}{}^{*} H(4n-j) q^N, \tag{5.19}$$

where the inner summation extends over all integers m and n such that $(m+\frac{j}{2})^2 + D(n-\frac{j}{4}) = N$; or equivalently $4n - j = \frac{4N-(2m+j)^2}{D}$. So (5.19) becomes

$$\sum_{j=0}^{1}\sum_{N=0}^{\infty} \sum_{\substack{(2m+j)^2\leq 4N \\ (2m+j)^2 \equiv 4N\ (D)}} H\left(\tfrac{4N-(2m+j)^2}{D}\right) q^N = \sum_{N=0}^{\infty} \sum_{\substack{s^2\leq 4N \\ s^2\equiv 4N\ (D)}} H\left(\frac{4N-s^2}{D}\right) q^N. \tag{5.20}$$

For the non-holomorphic part, we first note $\mathcal{O} = \{m + n\frac{1+\sqrt{D}}{2} : m,n \in \mathbb{Z}\}$. For $\lambda = m + n\frac{1+\sqrt{D}}{2}$ we write $\lambda \sim (m,n)$. Then

$$\sum_{j=0}^{1} \left(\sum_{m\in\mathbb{Z}} q^{(m+\frac{j}{2})^2} \right) \left(2(vD)^{-1/2} \sum_{n\in\mathbb{Z}} \beta\left(4\pi(n+\tfrac{j}{2})^2 vD\right) q^{-D(n+\frac{j}{2})^2} \right) \tag{5.21}$$

$$= \frac{2v^{-1/2}}{\sqrt{D}} \sum_{j=0}^{1}\sum_{N=0}^{\infty} \sum_{m,n\in\mathbb{Z}}{}^{*} \beta\left(4\pi(n+\tfrac{j}{2})^2 Dv\right)\, q^N, \tag{5.22}$$

where the inner sum goes over all m,n such that $(m+\frac{j}{2})^2 - D(n-\frac{j}{4}) = N$.

$$= \frac{2v^{-1/2}}{\sqrt{D}} \sum_{j=0}^{1}\sum_{N\in\mathbb{Z}} \sum_{\substack{\lambda\sim(n-m,2n+j) \\ \lambda\lambda'=N}} \beta\left(4\pi(\tfrac{\lambda-\lambda'}{2})^2 v\right)\, q^N \tag{5.23}$$

$$= \frac{2v^{-1/2}}{\sqrt{D}} \sum_{\lambda\in\mathcal{O}} \beta\left(4\pi(\tfrac{\lambda-\lambda'}{2})^2 v\right)\, q^{\lambda\lambda'}, \tag{5.24}$$

as desired.

□

The factor $\frac{1}{2}$ in the previous theorem occurs as $-1 \in \Gamma$, which acts trivially on $D \simeq \mathbb{H}\times\mathbb{H}$.

To obtain the intersection numbers $[T_1^c.T_N]$ and the *holomorphic* modular form of weight 2, one can now apply the holomorphic projection principle onto $\int_{T_1} \theta_{\varphi_V}(\tau, L)$. This is an idea of van der Geer and (independently) Zagier, which has been carried out in [23].

Remark 5.5. We see that Theorem 5.1 is the exact generalization of a part of the results of Hirzebruch-Zagier. One is certainly very interested to obtain complete analogues of these results. Applying holomorphic projection onto the function $\int_{C_U} \theta_{\varphi_V}(\tau, L)$ in Theorem 5.1 for $p > 2$ is certainly possible, but this does not have a priori a geometric interpretation. More promising seems to be an analysis of the boundary along the lines of the theory of Kudla and Millson. It seems likely that Eisenstein cohomology will enter the picture at this stage. We hope to come back to this issue.

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