PROBLEMS ON RATIONAL POINTS AND RATIONAL CURVES ON ALGEBRAIC VARIETIES

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0. Introduction

0.1. Basic problems. In this report, we review some recent results, conjectures, and techniques related to the following questions.

Question 1. Let V be a (quasi)projective algebraic variety defined over a number field k. How large is the set of rational points $V(k)$?

Question 2. Let V be a compact Kähler manifold. How large is the set of rational curves in V, or the space of analytic maps $\mathbf{P}^1 \to V$?

More precisely, in the arithmetic setting we choose a height function $h_L : V(k) \to \mathbf{R}$, and we want to understand the behavior of

$$N_V(H) := \text{card}\,\{x \in V(k) \mid h_L(x) \leq H\} \tag{0.1}$$

as $H \to \infty$.

In the geometric setting, we replace the (logarithmic) height by the degree of the curve with respect to the Kähler class, coinciding with its volume with respect to the Kähler metric (Wirtinger's theorem). If the degree is bounded by H, the space of rational curves is a finite-dimensional complex space, and we might be interested in the number of its irreducible components, their dimensions, their characteristic numbers, etc.

0.2. A heuristic reasoning. In order to see what geometric properties of V influence the behavior of the two sets, let us start with the following naive reasoning.

Let $V = V(n; d_1, \dots, d_r)$ be a smooth complete intersection in $\mathbf{P}^n$ given by the equations $F_i(x_0, \dots, x_n) = 0,\ i = 1, \dots, r$, where F_i is a form of degree d_i.

0.2.1. Arithmetic setting. Assuming that F_i have integral coefficients we take $\mathbf{Q}$ as the ground field. Every rational point is represented by a primitive $(n+1)$–ule of integer-valued coordinates $x = (x_0, \dots, x_n) \in \mathbf{Z}^{n+1}_{\text{prim}}$. A standard (exponential) height function is $h(x) = \max_i(|x_i|)$.

There are about H^{n+1} primitive $(n+1)$–ples of height $\leq H$. A form F_i takes about H^{d_i} values on this set. Assume that the probability of taking the zero value is about H^{-d_i}, and that the conditions $F_i = 0$ are statistically independent. Then we get a conjectural growth order

$$N_V(H) \sim H^{n+1-\sum_i d_i} \qquad (?) \tag{0.2}$$

for the number of points of the height $\leq H$ in $V(\mathbf{Q})$.

0.2.2. Geometric setting. Now we will allow F_i to have complex coefficients, and endow $V(\mathbf{C})$ with the metric induced by the Fubini-Study metric on $\mathbf{P}^n$. We normalize it in such a way that a line in $\mathbf{P}^n$ has degree (volume) 1.

Consider a projective line $\mathbf{P}^1 = \text{Proj}\ \mathbf{C}[t_0, t_1]$. Any map $\varphi : \mathbf{P}^1 \to \mathbf{P}^n$ can be written as

$$(t_0 : t_1) \mapsto (f_0(t_0, t_1) : \cdots : f_n(t_0, t_1))$$

where f_i are forms of some degree $k \geq 0$ not vanishing identically and relatively prime.

Denote by $\tilde{M}_k(\mathbf{P}^n)$ the space of all $(n+1)$–ples of forms of degree k (except $(0, \ldots, 0)$) up to a common scalar factor. Obviously,

$$\tilde{M}_k(\mathbf{P}^n) \cong \mathbf{P}^{(n+1)(k+1)-1}$$

The space $M^k(\mathbf{P}^n) \subset \tilde{M}_k(\mathbf{P}^n)$ is Zariski open and dense.

Similarly, denote by $M_k(V)$ the space of maps $\mathbf{P}^1 \to V$ of degree k. Its closure $\tilde{M}_k(V) \subset \tilde{M}_k(\mathbf{P}^n)$ is defined by a system of polynomial equations on the coefficients of f_i's derived from

$$F_i(f_0(t_0, t_1), \ldots, f_n(t_0, t_1)) = 0; \; i = 1, \ldots, r. \tag{0.3}$$

Clearly, (0.3) furnishes $kd_i + 1$ homogeneous equations of degree d_i corresponding to the monomials $t_0^a t_1^{kd_i+1-a}$. It follows that

$$\dim \tilde{M}_k(V) \geq (n+1)(k+1) - 1 - \sum_{i=1}^{r} (kd_i + 1) = k(n + 1 - \sum_{i=1}^{r} d_i) + \dim V; \tag{0.4}$$

$$\deg \tilde{M}_k(V) \leq \prod_{i=1}^{r} d_i^{kd_i+1}. \tag{0.5}$$

0.3. Discussion. a). Since the geometric degree of a curve corresponds to the logarithmic height of a point (with respect to the same ample class), the r.h.s. of (0.2) and (0.4) predict the same qualitative behavior of the number of points, resp. of the dimension of the space of maps, depending on the sign of $n + 1 - \sum_{i=1}^{r} d_i$. Now, this last number is essentially the anticanonical class of V:

$$-K_V \cong \mathcal{O}_V(n + 1 - \sum_{i=1}^{r} d_i) \tag{0.6}$$

in the Picard group of V.

Boldly extrapolating from the complete intersection case, we may expect many rational curves and points when $-K_V$ is ample (V is a Fano manifold), and few when K_V is ample. The intermediate case $K_V = 0$ must be more subtle.

For example, if we disregard the difference between $\tilde{M}_k(V)$ and $M_k(V)$ and assume that (0.4) is an exact equality, we expect a $\dim(V)$–dimensional family of parametrized rational curves on V of any degree k. And if in addition $\dim V = 3 = \dim \mathrm{Aut}\, \mathbf{P}^1$, we expect only a finite number n_k of rational (unparametrized) curves of degree k belonging to V for all $k \geq 1$. For quintics in $\mathbf{P}^5$, this was conjectured by Clemens (cf. below).

b). These expectations are fulfilled when $\dim V = 1$ that is, when V is a smooth compact curve. More precisely, when $-K_V$ is ample, genus of V is zero, V may be a non-trivial form of $\mathbf{P}^1$ over a non-closed field k which has no k–points. However, after a quadratic extension of k, V will become $\mathbf{P}^1$, and the point count with respect to an anticanonical height gives an asymptotic formula agreeing with (0.2). And the count of maps $\mathbf{P}^1 \to \mathbf{P}^1$ is unconstrained.

When $K_V = 0$, one gets $N_V(H) \sim c(\log H)^{r/2}$ in view of the Mordell-Weil theorem for elliptic curves, so that (0.2) is still valid if one interprets the r.h.s. as "$O(H^\varepsilon)$ *for any* $\varepsilon > 0$". And there are no maps $\mathbf{P}^1 \to V$ of degree $k \geq 1$.

Finally, when $K_V > 0$ one gets $N_V(H) = O(1)$ (Faltings' theorem), and any parametri-zed rational curve is constant.

c). Starting with dimension two, the situation becomes much more complex and problematic. Let us start with geometry.

For smooth m–dimensional Fano varieties, Mori proved that through every point passes a rational curve of $(-K_V)$–degree $\leq m+1$). Moreover, any two points can be connected by a chain of rational curves. But a quantitative picture of the space $\mathrm{Map}(\mathbf{P}^1, V)$ remains unknown.

For general type varieties (K_V ample), we expect only a finite dimensional family of unparametrized rational curves. However, this was proved only for varieties with ample cotangent sheaf which is a considerably stronger assumption.

Finally, for manifolds with $K_V = 0$ (and Kähler holonomy group SU), physicists recently suggested a fascinating conjectural framework for the curve count which we will review in the second part of this report.

Passing to the arithmetic case, let us notice first that (0.2) can be proved by the circle method over $\mathbf{Q}$ when $n+1$ is large in comparison with $\sum d_i$ and when the necessary local conditions are satisfied (see below).

On the other hand, already for $n = 3, r = 1, d = 3$, (0.2) may fail for the following reason: it predicts the linear growth for $N_V(H)$, but V may contain a projective line defined over $\mathbf{Q}$ (there are 27 lines over $\bar{\mathbf{Q}}$) in which case counting points only on this line we already get $N_V(H) \geq cH^2$. Therefore, if anything like (0.2) may be expected in general, we must at least stabilize the situation by allowing ground field extensions and deleting some proper subvarieties tending to accumulate points. And in the case $K_V = 0$ we may have to delete infinitely many subvarieties to achieve the predicted $O(H^\varepsilon)$ estimate.

We elaborate this program in Chapter 1 below. Its goal, roughly speaking, lies in establishing a (conjectural) *direct relation* between the distribution of rational points on V and the geometry of rational curves on V.

In addition, there exists a well known *analogy* between rational curves and rational curves. In Arakelov geometry, rational points on V become "horizontal arithmetical curves" on a $\mathbf{Z}$–model of V, endowed with an Hermitean metric at arithmetical infinity. In the framework of this analogy, the height becomes literally an arithmetical intersection index.

We want to draw attention to an unexplored aspect of this analogy: what in arithmetics corresponds to the local deformation theory of embedded curves?

Here is a relevant fragment of the geometric deformation theory. Below V denotes a quasiprojective variety defined over an algebraically closed field k, and

Map $(\mathbf{P}^1, V)$ is the locally closed finite quasiprojective scheme parametrizing morphisms $\mathbf{P}^1 \to V$. For simplicity, in the next Proposition we consider only the unobstructed case.

0.4. Proposition. *Let φ be a morphism $\mathbf{P}^1 \to V$, $[\varphi] \in Map\ (\mathbf{P}^1, V)$ the corresponding closed point, $\mathcal{T}_V$ the tangent sheaf to V.*

If $H^1(\mathbf{P}^1, \varphi^*(\mathcal{T}_V)) = 0$, then $[\varphi]$ is a smooth point, and the local dimension of $Map(\mathbf{P}^1, V)$ at $[\varphi]$ equals $\dim H^0(\mathbf{P}^1, \varphi^*(\mathcal{T}_V))$.

For a proof of a more general statement, see Mori [20].

Assume now that φ is an immersion, and V is smooth in a neighbourhood of $\varphi(\mathbf{P}^1)$. Then we have the following sequence of locally free sheaves on $\mathbf{P}^1$:

$$0 \to \mathcal{T}_{\mathbf{P}^1} \to \varphi^*(\mathcal{T}_V) \to N_{[\varphi]} \to 0 \tag{0.7}$$

where $N_{[\varphi]}$ is the normal sheaf. Hence $N_{[\varphi]} \cong \oplus_{i=1}^{s-1} \mathcal{O}(m_i)$, $s = \dim(V)$. Recall also that $\mathcal{T}_{\mathbf{P}^1} \cong \mathcal{O}(2)$.

We can now prove that (0.4) becomes exact equality locally on *Map* $(\mathbf{P}^1, V)$ if $\varphi(V)$ is nicely immersed infinitesimally:

0.4.1. Corollary. *Assume in addition that $m_i \geq -1$ for all $i = 1, \dots, s-1$. Then $[\varphi]$ is smooth, and*

$$\dim_{[\varphi]} Map(\mathbf{P}^1, V) = \deg \varphi^*(-K_V) + \dim V \tag{0.8}$$

which coincides with the r.h.s. of (0.4) in the complete intersection case.

Proof. The smoothness of $[\varphi]$ follows from Prop. 0.4. Put now

$$A = \{i | m_i = -1\},\ a = \mathrm{card}(A),$$

$$B = \{i | m_i \geq 0\},\ b = \mathrm{card}(B).$$

We have $a + b = s - 1$; $\deg \varphi^*(-K_V) = 2 + \sum_A m_i + \sum_B m_j = 2 - a + \sum_B m_i$ (take the determinant of (0.7)), and, again from (0.7),

$$\dim_{[\varphi]} Map\ (\mathbf{P}^1, V) = \dim H^0(\mathcal{T}_{\mathbf{P}^1}) + \dim H^) (N_{[\varphi]}) =$$

$$= 3 + \sum_B (m_i + 1) = 3 + b + \sum_B m_i = 3 + b + \deg \varphi^*(-K_V) - 2 + a =$$

$$\dim V + \deg\ \varphi^*(-K_V).$$

In particular, when $\dim V = 3$ and $-K_V = 0$, every immersed curve with normal sheaf $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ must be isolated because the local dimension of the map space equals $\dim V = 3$ and this is accounted for by reparametrizations.

The simplest example when this may occur generically is that of a smooth quintic threefold V. In fact, H. Clemens conjectured that a generic smooth quintic contains only finitely many smooth rational curves of arbitrary degree k, and that all of them have normal sheaf $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Sh. Katz proved partial results in this direction: see [14], [15].

0.5. Problem. *Establish an analog of the geometric deformation theory for embedded arithmetical curves.*

Specifically,

0.6. Problem. *Find conditions on arithmetical normal sheaf (or higher order infinitesimal neighborhoods) of an arithmetical curve which are necessary for the generic point of this curve to lie on a rational curve.*

(We want to find an exact expression of the feeling that an arithmetical curve is deformable only if its generic point lies on a rational curve).

0.7. Rational curves in other contexts. Besides algebraic geometry and number theory, the study of rational curves was recently motivated by quantum field theory and symplectic geometry. We will finish this Introduction with a brief discussion of some relevant ideas.

0.7.1. Physics. Physicists start with a space of maps $Map\ (S^2, V)$ where the target space V is endowed with a Riemannian metric g, and an action functional $S:\ Map\ (S^2, V) \to \mathbf{R}$.

V can be thought of as a space-time with a possibly non-trivial gravity field and topology. Any $\varphi:\ S^2 \to V$ defines a world-sheet of an one-dimensional object, a "string", which replaces the classical image of point-particle. Alternatively, one can think about S^2 as a two-dimensional space-time in its own right. Then (V, g) in a neighborhood of $\varphi(S)$ represents classical fields on S.

Action of a virtual world-sheet $\varphi:\ S^2 \to V$ is usually given by a Lagrangian density which must be integrated over S^2. Here we will look only at the simplest action functional

$$S(\varphi) = \int_{S^2} vol\ (\varphi^*(g)). \tag{0.9}$$

In other words, $S(\varphi)$ is just the surface of the world sheet. Non-trivial stationary points of this action are just minimal surfaces. The path integral quantization of this theory in the stationary phase approximation involves a summation over these minimal surfaces

Imagine now that (V, g) is not just a Riemannian manifold, but a complex Kähler one. It is well known that in this case minimal surfaces in V (actually, minimal submanifolds of any dimension) are precisely complex subvarieties (Wirtinger's theorem).

A physical context in which V acquires a natural Kähler structure arises in string compactification models where V appears as a Planck size compact chunk of space-time adding missing six real dimensions to the classical four-dimensional space-time.

0.7.2. Symplectic geometry. The basic mathematical structure of the classical mechanics is a triple (V^{2n}, ω, H) where V^{2n} is a smooth manifold, ω is a closed non-degenerate 2–form on V^{2n}, and H is a function on V called Hamiltonian. Given such a triple, we want to understand the geometry of the flow defined by the vector field X on V such that $dH = i_X(\omega)$. In particular, we want to know how a domain of initial positions $B \subset V$ may change with time.

Any Hamiltonian flow preserves the symplectic volume $v(B) = \int_B \omega^n$. On the other hand, certain unstable flows like geodesic flows on hyperbolic manifolds severely distort B: a small ball eventually becomes spread all over V forming a fractal-like structure. Nevertheless, $(\exp(tX)B, \omega)$ remains symplectomorphic to B because $\mathrm{Lie}_X(\omega) = di_X(\omega) + i_X d\omega = 0$.

V.I.Arnold in the sixties suggested that $\exp(tX)B$ should satisfy some additional constraints displaying then unknown "symplectic rigidity" properties.

M.Gromov's work confirmed these expectations. He proved in particular that the unit ball

$$(B_1 = \{x | \sum_{i=1}^{2n} x_i^2 < 1\},\ \omega = \sum_{i=1}^{n} dx_i \wedge dx_{i+n})$$

is not symplectomorphic to any open subset of

$$(V_{1-\varepsilon} = \{x|\ |x| < 1 - \varepsilon\},\ \omega = \sum_{i=1}^{n} dx_i \wedge dx_{i+n}).$$

Gromov's argument involves rational curves in the following ingenious way. Notice first that in the example above we envision the two symplectic spaces B_1 and $V_{1-\varepsilon}$ not in terms of ω but rather in terms of the standard Euclidean metric $ds^2 = \sum (dx_i)^2$. But if we are considering pairs (g, ω) consisting of a quadratic and an alternate form, say, on a linear space E, there is a natural subclass of such pairs corresponding to Hermitean forms, which can be characterized by the existence of a complex structure $J : E \to E$, $J^2 = -1$ such that $\omega(Jx, y) = g(x, y)$, $g(Jx, y) = -\omega(x, y)$.

Applying this to tangent spaces of a symplectic manifold (V, ω) and shifting attention from (ω, g) to (ω, J) we come to the following notion due to Gromov.

An almost complex structure J on V is *tamed* by ω, if $g(x, x) := \omega(Jx, x) > 0$ for any tangent vector x, that is, if $g + i\omega$ define a Hermitean metric on the tangent bundle to V. Now, even though J may be non-integrable, its restriction on surfaces is integrable, so that it makes perfect sense to speak about holomorphic maps $\mathbf{P}^1 \to (V, J)$.

M.Gromov derives his results from a thorough study of such rational curves, establishing existence of curves of small volume. (In a similar vein, rational curves of small degrees play the crucial role in the Mori theory).

E.Witten used Gromov's construction as a deformation device allowing one to correctly count the number of rational curves on Calabi-Yau manifolds: cf. also [16].

This paper is structured as follows. §1 is devoted to the analytic methods to count rational points on projective varieties, whereas §2 reviews the algebro-geometric approach. In §3 we turn to the curve count, explaining the simplest example of Calabi-Yau mirrors. Finally, §4 is devoted to the explanation of toric mirror constructions. For the most part, proofs are omitted.

CHAPTER I
COUNTING RATIONAL POINTS

§1. Analytic methods

1.1. Heights on projective varieties. Let k be an algebraic number field. Denote by M_k the set of all places of k; for $v \in k$, let k_v be the completion of k at v. Define the local norm $|.|_v : k_v^* \to \mathbf{R}^*$ by the following condition: if μ is a Haar measure on k_v^+, then $\mu(aU) = |a|_v\mu(U)$ for each measurable subset U.

Let $x \in \mathbf{P}^n(k)$ be a point in a projective space endowed with a homogeneous coordinate system. If coordinates of x are $(x_0, \dots, x_n), x_i \in k$, put

$$h(x) = \prod_{v \in M_k} \max_i(|x_i|_v). \tag{1.1}$$

The product formula shows that this is well defined.

More generally, let V be a projective variety defined over k, $\mathbf{L} = (L, s)$ a pair consisting of a very ample invertible sheaf L and a finite set of sections $s = \{s_0, \dots s_n\} \subset \Gamma(V, L)$ generating L. For a point $x \in V(k)$ and an arbitrary choice of a local section σ of L non-vanishing at x we put

$$h_{\mathbf{L}}(x) = \prod_{v \in M_k} \max_i(|(s_i/\sigma)(x)|_v). \tag{1.2}$$

For $\mathbf{L}_1 = (L_1, \{s_i^1\}), \mathbf{L}_2 = (L_2, \{s_j^2\})$, put $\mathbf{L}_1 \otimes \mathbf{L}_2 = (L_1 \otimes L_2, \{s_i^1 \otimes s_j^2\})$. Then

$$h_{\mathbf{L}_1 \otimes \mathbf{L}_2}(x) = h_{\mathbf{L}_1}(x) h_{\mathbf{L}_2}(x). \tag{1.3}$$

In particular, consider the anticanonical height $h_{\omega^{-1}}$ on $\mathbf{P}^n(k)$ defined by the $(n+1)$–th tensor power of $(\mathcal{O}(1); \{x_0, \dots, x_n\})$. Then $h_{\omega^{-1}}(x) = h(x)^{n+1}$ where $h(x)$ is given by (1.1).

When s in the definition of $\mathbf{L}$ is replaced by another generating set of sections, $h_{\mathbf{L}}$ is multiplied by $\exp(O(1))$. The resulting set of height functions consists of *Weil's heights.* There is a different choice of additional structure allowing one to define height functions directly for non necessarily ample sheaves: the *Arakelov heights* are obtained by choosing an appropriate set of v–adic metrics $\|.\|_v$ on all $L \otimes k_v$ and putting, for $\mathbf{L} = (L, \{\|.\|_v\})$,

$$h_{\mathbf{L}}(x) = \prod_{v \in M_k} \|\sigma(x)\|_v^{-1}.$$

These heights are also multiplicative with respect to the obvious tensor product, and up to $\exp O(1)$ are independent on the choice of local metrics and coincide with the respective Weil heights.

For a subset $U \subset V(k)$, put

$$N_U(\mathbf{L}; H) = \text{card}\,\{x \in U | h_{\mathbf{L}}(x) \le H\}. \tag{1.4}$$

For ample $\mathbf{L}$, this number is always finite. We want to understand its behavior as $H \to \infty$. In this section, we review main situations when an asymptotic formula for (1.4) is known. In all cases which I am aware of, such a formula is of the type

$$N_U(\mathbf{L}; H) = cH^{\beta_U(\mathbf{L})}(\log H)^{t_U(\mathbf{L})}(1 + o(1)) \tag{1.5}$$

for some constants $c > 0, \beta_U(\mathbf{L}) \ge 0, t_U(\mathbf{L}) \ge 0$. The archetypal result is the following theorem due to Schanuel:

1.2. Theorem. *Put $d = [k : \mathbf{Q}]$. Then*

$$N_{\mathbf{P}^n(k)}(\omega^{-1}; H) = c(n,k)H + \begin{cases} O(H^{1/2}\log H) & \text{for} \quad d = n = 1, \\ O(H^{1-1/d(n+1)}) & \text{otherwise;} \end{cases} \tag{1.6}$$

$$c(n,k) = \frac{h}{\zeta_k(n+1)} \left(\frac{2^{r_1+r_2}\pi^{r_2}}{D^{1/2}} \right)^{n+1} \frac{R}{w}(n+1)^{r_1+r_2-1}. \tag{1.7}$$

Here h denotes the class number of k, ζ_k its Dedekind zeta, r_1 (resp. r_2) is the number of its real (resp. complex) places, D is the absolute value of the discriminant, R the regulator, w the number of roots of unity in k.

The main feature of (1.6) is that $N_{\mathbf{P}^n(k)}(\omega^{-1}; H)$ grows *asymptotically linearly in* H, whatever the dimension n and the ground field k are. This became possible only because we have chosen local norms $|.|_v$ as Haar multipliers. Therefore the height function (1.1) is non-invariant with respect to ground field extensions: if we replace k by $k' \supset k$, $h(x)$ becomes $h'(x) = h(x)^{[k':k]}$ so that $\mathbf{P}^n(k)$ does not contribute to the main term of the asymptotic formula for $N_{\mathbf{P}^n(k')}(\omega^{-1}; H)$: essentially, we count only "new points".

Schanuel proved (1.7) by reducing the problem to that of counting lattice points in a large domain. The volume of the domain furnishes the leading term, and if the boundary is not too bad, we get an asymptotic formula. We will now sketch an alternate approach via zeta functions.

1.3. Zetas. Consider the following abstract setting. Let U be a finite or countable set, $h_{\mathbf{L}} : U \to \mathbf{R}_+$ a counting function (this means that $N_U(\mathbf{L}; H)$ defined by (1.4) is finite for all H). Assume moreover that $N_U(\mathbf{L}; H) = O(H^c)$ for some $c > 0$. Put

$$Z_U(\mathbf{L}; s) = \sum_{x \in U} h_{\mathbf{L}}(x)^{-s}. \tag{1.8}$$

The better we understand the analytical properties of $Z_U(\mathbf{L}; s)$, the more precise information about $N_U(\mathbf{L}; H)$ we can obtain. We will distinguish here four levels of precision.

Level 0: Convergence abscisse. Put

$$\beta = \beta_U(\mathbf{L}) = \inf \{\sigma \mid Z_U(\mathbf{L}; s) \text{ converges for } \mathrm{Re}(s) > \sigma\}. \tag{1.9}$$

This is well defined and invariant if one replaces h by $\exp(O(1))h$. In particular, if $h_{\mathbf{L}}$ is a Weil or Arakelov ample height, β depends only on the isomorphism class of the relevant ample sheaf L.

It gives the following information about $N_U(\mathbf{L}; H)$:

$$\beta_U(\mathbf{L}) = \begin{cases} -\infty \text{ if } U \text{ is finite;} \\ \limsup \frac{\log \; N_U(\mathbf{L};H)}{\log H} \geq 0 \text{ otherwise.} \end{cases} \tag{1.10}$$

In other words, if $\beta \geq 0$, we have for all $\varepsilon > 0$:

$$N_U(\mathbf{L}; H) = \begin{cases} O(H^{\beta+\varepsilon}), \\ \Omega(H^{\beta-\varepsilon}). \end{cases} \tag{1.11}$$

Level 1: a Tauberian situation. Assume that $\beta = \beta_U(\mathbf{L}) \geq 0$, and for some $t = t_U(\mathbf{L}) \geq 0$ we have

$$Z_U(\mathbf{L}; s) = (s-\beta)^{-t} G(s), \tag{1.12}$$

where $G(\beta) \neq 0$, and $G(s)$ is holomorphic in a neighborhood of $\mathrm{Re}(s) \geq \beta$. In this case

$$N_U(\mathbf{L}; H) = \frac{G(\beta)}{\Gamma(t)} \frac{H^\beta}{\beta} (\log\ H)^{t-1}(1+o(1)). \tag{1.13}$$

In particular, assume that $U = U_1 \times \cdots \times U_m$, $h_{\mathbf{L}}(u_1, \dots, u_m) = h_{\mathbf{L}_1}(u_1) \dots h_{\mathbf{L}_m}(u_m)$. Put $\beta_i = \beta_{U_i}(L_i), t_i = t_{U_i}(\mathbf{L}_i)$ whenever they are defined, and $\tilde{\beta} = \max_i\ (\beta_i), J = \{i \mid \beta = \beta_i\}$. Using the zeta-description of these numbers, one readily sees that

$$\beta_U(\mathbf{L}) = \tilde{\beta}, \quad t_U(\mathbf{L}) = \sum_{i \in J} t_i. \tag{1.14}$$

Formula of the type (1.13) is valid for $(U, h_{\mathbf{L}})$ if the Tauberian condition is assumed only for $U_i, h_{\mathbf{L}_i}$ with $i \in J$.

Level 2: analytic continuation to a larger halfplane. Instead of axiomatizing the situation, I will only remind the contour deformation technique. Let us start with the formula valid for $\beta' > \beta$:

$$N_U(\mathbf{L}; H) = \int_{\beta'-i\infty}^{\beta'+i\infty} \frac{H^s}{s} Z_U(\mathbf{L}; s) ds. \tag{1.15}$$

In favorable case, one can integrate instead along a vertical line $\mathrm{Re}(s) = \gamma < \beta$ adding the contribution of poles $Z_U(\mathbf{L}; s)$ for $\gamma < \mathrm{Re}(s) < \beta'$. This contribution constitutes the leading term of the asymptotics; it will be of the type $cH^\beta P(\log H)$ where P is a polynomial if $Z_U(\mathbf{L}; s)$ has a pole at $s = \beta$ as its only singularity in $\gamma < \mathrm{Re}(s) < \beta'$. The integral over $\mathrm{Re}(s) = \gamma$ will grow slower, possibly as $O(H^{\beta-\epsilon})$, if Z_U has no more poles in $\mathrm{Re}(s) > \gamma$, and can be appropriately majorized.

To accomplish the necessary estimates, one has sometimes to first replace $N_U(\mathbf{L}; s)$ by an appropriate average, and the r.h.s. of (1.15) by something like $\int_{\beta'-i\infty}^{\beta'+i\infty} \frac{H^s}{s(s+1)} Z_U(\mathbf{L}; s)$ which converges better.

Level 3: explicit formulas. If one has a well-behaved meromorphic continuation of $Z_U(\mathbf{L}; s)$ to the whole complex plane, one can sometimes push β' to $-\infty$ in (1.15) and obtain a precise formula for $N_U(\mathbf{L}; H)$ as a series over all poles of $Z_U(\mathbf{L}; s)$.

1.4. A generalization of Schanuel's theorem. The behavior of the height zeta-function (1.8) is well understood only for two clases of projective manifolds: a) Abelian varieties; b) homogeneous Fano manifolds.

If $U = V(k), V$ is an Abelian variety, L is an ample symmetric sheaf on V, one can use Néron-Tate's height $\hat{h}_L$ to count points. Denote by W the image of $V(k)$ in $V(k) \otimes \mathbf{R}$, and let t be the order of $V(k)_{\mathrm{tors}}$. Then $\hat{h}_L(x) = \exp(q(x \bmod V(k)_{\mathrm{tors}})$ where q is a positive quadratic form on $V(k) \otimes \mathbf{R}$ so that our zeta is a theta-function:

$$Z_U(L; s) = t \sum_{y \in W} \exp(-q(y)s). \tag{1.16}$$

Hence, if $r := \text{rk } V(k) > 0$, we have $\beta > 0$, and

$$N_{V(k)}(L; H) = c \log^{r/2} H(1 + o(1)). \tag{1.17}$$

Notice that the convergence abscisse $\text{Re}(s) = 0$ is also the natural boundary for $Z_U(L; s)$. For abelian varieties, $K_V = 0$ so that (1.17) matches our naive expectation (0.2).

Let us turn now to homogeneous Fano varieties.

1.4.1. Theorem. *Every homogeneous Fano variety V is isomorphic to a generalized flag space $P \setminus G$ where G is a semi-simple linear algebraic group, and P is a k-rational conjugacy class of parabolic subgroups.*

If $V(k) \neq \emptyset$, we can take P to be a parabolic subgroup defined over k.

For a proof, see Demazure [10].

Flag spaces $P \setminus G$ admit a distinguished class of heights which can be defined in terms of Arakelov metrics invariant with respect to maximal compact subgroups of the adelic group of G. For such heights, the zeta function of $V = P \setminus G$ becomes essentially one of the Langlands–Eisenstein series. Their deep theory developed by Langlands allows one to use the technique of contour integration of the Level 3 above, and prove the following theorem, generalizing 1.2:

1.4.2. Theorem. *If V is a homogeneous Fano variety with $V(k) \neq \emptyset$, then for a distinguished anticanonical height we have*

$$N_V(-K_V; H) = Hp(\log H)(1 + H^{-\varepsilon}) \tag{1.18}$$

where $\varepsilon > 0$, and p is a polynomial of degree rk $\text{Pic}(V) - 1$.

For a proof, see [13]. In particular, $\beta_V(-K_V) = 1$.

This theorem can be extended to the distinguished heights corresponding to other invertible L. It must be stressed however that, even for projective line, there are natural situations when the relevant heights are not distinguished. This happens on accumulating Fano subvarieties, when a height is induced from the ambient space: see the next section. In the homogeneous case,the asymptotic is of the same form. A very interesting question of characterization the leading term coefficient directly in terms of the anticanonical height was recently attacked by E. Peyre.

The simplest variety for which the analytic properties of Z_U beyond the convergence abscisse are unknown is the affine Del Pezzo surface of degree 5 over $\mathbf{Q}$ which can be obtained by blowing up four rational points on $\mathbf{P}^2$ and then deleting all 10 exceptional curves. One reason for this may be a wrong choice of the function itself. The mirror conjecture on the curve count on, say, three-dimensional quintics, furnishes analytic continuation for a geometric version of the height zeta where the contribution of the curve x is $(\log h(x))^3 \frac{h(x)^{-s}}{1-h(x)^{-s}}$ rather than our simple-minded $h(x)^{-s}$. It would be quite important to guess a version of $Z_U(L; s)$ with good analytic properties.

1.5. Circle method. We will now briefly explain a classical approach to counting points which is efficient for Fano hypersurfaces and complete intersections (mostly over $\mathbf{Q}$) with many variables.

Let X be a finite set, $F : X \to \mathbf{Z}$ a function, $e(\alpha) = e^{2\pi i\alpha}$. Put

$$S(\alpha) = S_{(X,F)}(\alpha) = \sum_{x\in X} e(\alpha F(x)). \tag{1.19}$$

Then

$$\text{card } \{x \in X \mid F(x) = 0\} = \int_0^1 S(\alpha)d\alpha. \tag{1.20}$$

A useful version of this formula refers to the case of a vector function $F = (F_1, \ldots, F_r) : X \to \mathbf{Z}^r$. Then $\alpha = (\alpha_1, \ldots, \alpha_r)$ varies in a unit cube, $\alpha F(x) = \sum \alpha_i F_i(x), S(\alpha)$ is again defined by (1.19), and

$$\text{card } \{x \in X \mid F(x) = 0\} = \int_0^1 \cdots \int_0^1 S(\alpha)d\alpha_1 \ldots d\alpha_r. \tag{1.21}$$

The circle method, when it works, gives a justification to the following heuristic principle:

1.5.1. Circle principle. *Under favorable circumstances, there exists a finite set of rational points* $\alpha^i = \{\frac{a_1^{(i)}}{q_1^{(i)}}, \ldots, \frac{a_r^{(i)}}{q_r^i}\}$ *and small cubes* $I^{(i)}$ *centered at these points ("major arcs") such that*

$$\int_0^1 \cdots \int_0^1 d\alpha_1 \ldots d\alpha_r = \sum_i \int_{I^{(i)}} S(\alpha)d\alpha_1 \ldots d\alpha_r + \{\text{a small remainder term}\}.$$

To get some feeling of why it might be true, and what it implies, let us look at the case $r = 1$. First of all, the values of $S(\alpha)$ at rational points are related to the distribution of values of $F(x)$ modulo integers:

$$S(0) = \text{card}(X); S(\frac{1}{2}) = \text{card } \{x \mid F(x) \text{ even}\} - \text{card } \{x \mid F(x) \text{ odd}\};$$

$$S(\frac{a}{q}) = \sum_{p \mod q} e^{2\pi i a p/q} \text{card } \{x \mid F(x) \equiv p \mod q\}.$$

And if $X = [1, \ldots, N]$ with large N, $F(x) = x^2$, then $S(\frac{a}{q})$ is approximately $\frac{N}{q} \times \{\text{a Gauss sum}\}$ decreasing as $\frac{N}{\sqrt{q}}$ for large $q \ll N$.

Hence we may expect that $S(\alpha)$ is relatively small (in comparison with the number N of its summands) outside of a neighborhood of the set of rational points with denominators bounded in terms of N.

In the classical additive problems with large number of summands k, the remainder term can be effectively damped as $k \to \infty$, because

$$S_{(X^k, F_1+\cdots+F_k)}(\alpha) = S_{(X,F)}(\alpha)^k, \quad F_i = F \circ \text{pr}_i. \tag{1.22}$$

For example, in Waring's problem of degree n with k summands,

$$(X, F) = ([0, \ldots, [M^{1/n}]], x_1^n + \cdots + x_k^n - M)$$

so that

$$\text{card}\ \{(x_i) \mid \sum_{i=1}^{k} x_i^n = M\} = \int_0^1 e^{-2\pi i\alpha M}(\sum_{x=0}^{[M^{1/n}]} e^{2\pi\alpha x^n})^k d\alpha.$$

Below we review some results of W. Schmidt [25] who applied the circle method to the intersections of hypersurfaces in a projetive space over $\mathbf{Q}$. In fact, he worked with the corresponding affine cone, but this only changes the coefficient in the asymptotic formula.

1.5.2. The setting. Consider a finite system of r forms in s variables of degrees ≥ 2:$F = \{F_1, \ldots, F_r\}$, with integral coefficients. Let V be the variety $\{F_i = 0\}$ in the affine space. Let r_d be the number of forms of degree d, and $r = \sum_i r_i$. W. Schmidt proved an asymptottic formula of the type (0.2) in the cases when "the number of variables is large, and the forms are not too degenerate". Both conditions are used as a refined substitute for the classical damping effect (1.22). Let us state them more precisely.

A. Many variables. The basic bound is written in terms of the number

$$v(r_2, \ldots, r_k) = \max\ \{s \mid \text{for some F and some prime p},\ F(\mathbf{Q}_p) = \emptyset\}.$$

In other words, $s > v(r_2, \ldots, r_k)$, implies p-adic solvability for all p and all F with a given vector degree.

B. Degeneracy. The degeneracy is measured in terms of the tensor rank, well known in the computational complexity theory. Specifically, for one form F put

$$h(F) = \min\ \{h \mid \text{there exist non} - \text{constant forms } A_1, B_1, \ldots, A_h, B_h \in \mathbf{Q}[x_1, \ldots, x_s]$$

$$\text{such that } F = A_1B_1 + \cdots + A_hB_h\}.$$

For a system of forms of the same degree $F = \{F_i\}$, put

$$h(F) = \min\ \{h(\sum c_iF_i) \mid c_i \in \mathbf{Q}\}.$$

Finally, for a general system of forms put $h_d = h(\text{degree } d \text{ part of } F)$.

1.5.3. Theorem. *Assume that*
a). $h_d \geq 2^{4^d} d! r_d k v(r_2, \ldots, r_k)$.
b). $\dim V(\mathbf{R}) \geq s - \sum_{i=2}^{k} r_i$.
Then the number of integral points of V in $\{|x_i| \leq H\}$ is

$$\mu H^{s-\sum r_i}(1 + O(H^{-\varepsilon})),\ \varepsilon > 0,$$

where the constant $\mu > 0$ is a product of local densities.

Turning to the base of the cone V, we again see the linear growth rate with respect to an anticanonical height, at least when this base is only mildly singular so that the anticanonical sheaf exists and is given by the same formula as for the smooth complete intersections.

§2. Algebro-geometric methods

2.1. Accumulating subvarieties. The analytic methods described in §1 work efficiently only for those Fano varieties which are either homogeneous or complete intersections with many variables (or, more invariantly, of large index). Moreover, their success seems to be connected with the fact that the rational points are uniformly distributed with respect to a natural Tamagawa measure.

Algebro-geometric data suggest that generally we may not expect such a uniformity, and that rational points tend to concentrate upon proper subvarieties. Below we will discuss several ways to make this idea precise. Let U be a quasiprojective variety over a number field k.

a. *Zariski topology.* Denote by V the closure of $U(k)$ in Zariski topology. If a compactification of U is a curve of genus > 1, then V is a proper subvariety of U. This fancy way to state Faltings' theorem leads to the generalized Mordell conjecture: we expect that V is a proper subvariety of U whenever U is birationally equivalent to a variety of general type. Roughly speaking, this means that the description of $U(k)$ can be divided into two subproblems: to understand the distribution of rational points on varieties with $K \leq 0$, and to understand the distribution of such subvarieties in varieties of general type.

This pattern is characteristic for all definitions of accumulation.

b. *Hausdorff topology.* Let $k = \mathbf{Q}$. B. Mazur recently suggested that $U(\mathbf{Q})$ may be Hausdorff dense in the space of $\mathbf{R}$-points of its Zariski closure V. If this is universally true, it implies that $\mathbf{Z}$ cannot be a $\mathbf{Q}$-Diophantine subset subset of $\mathbf{Q}$ so that not all $\mathbf{Q}$-enumerable subsets are $\mathbf{Q}$-Diophantine. (Recall that $E \subset \mathbf{Q}^n$ is $\mathbf{Q}$-Diophantine if it is a projection of $U(\mathbf{Q}) \subset \mathbf{Q}^{n+m}$ for some affine U defined over $\mathbf{Q}$).In particular, Matiyasevich's strategy of proving the algorithmic undecidability of Diophantine equations over $\mathbf{Z}$ would not work for $\mathbf{Q}$.

c. *Measure theory.* Again for simplicity working over $\mathbf{Q}$ consider the limit

$$\mu = \lim_{N \to \infty} N^{-1} \sum_{i=1}^{N} \delta(x_i)$$

of the averaged delta-distributions over rational points $x_i \in U(\mathbf{Q})$ ordered, say, by increasing height. If such a limit exists, the support of μ provides a notion of accumulating subset which may be finer than the topological closure.

d. *Point count according to the polynomial growth rate.* The following notion was suggested in [4]: choose a height function h_L on (a projective closure of) U and call a Zariski closed subset $V \subset U$ accumulating w.r.t. h_L if

$$\beta_U(L) = \beta_V(L) > \beta_{U \setminus V}(L),$$

where the growth order β is defined by (1.9) or equivalently (1.10). One easily sees that there exists a unique minimal accumulating subset V_1; putting $U_1 = U \setminus V_1$ and applying the same reasoning to U_1 etc, one gets a sequence of Zariski open subsets

$$U_0 = U \supset U_1 \supset U_2 \supset \dots \tag{2.1}$$

such that $U_i \setminus U_{i+1}$ is the minimal h_L-accumulating subset in U_i. A description of (2.1) and of the corresponding growth order sequence

$$\beta_{U_0}(L) > \beta_{U_1}(L) > \beta_{U_2}(L) > \dots \tag{2.2}$$

is the natural first goal in understanding $U(k)$, which can be best attacked by algebro-geometric means.

We will now report on the results of [18], [19] concerning mostly Fano varieties, in particular surfaces and threefolds.

2.2. Invariant α and reductions. Let V be a projective manifold (we can also allow mild singularities). Denote by N^1_{eff} (resp. N^1_{ample}) the closure of the cone generated by effective (resp. ample) classes in $NS(V) \otimes \mathbf{R}$ where NS is the Néron-Severi group. For an invertible sheaf L, put

$$\alpha(L) = \inf\{p/q \quad | \quad p, q \in \mathbf{Z}, q > 0, p[L] + qK_V \in N^1_{eff}\}.$$

If V is Fano and L is ample then $\alpha(L) > 0$. The following two results allow us to reduce in certain cases the calculation of $\beta_U(L)$ to that of $\beta_U(-K_V)$, if $\alpha(L)$ is considered as a computable geometric invariant.

2.2.1. Theorem on the upper bound. *a). For every $\varepsilon > 0$, there exists a dense Zariski open subset $U(\varepsilon) \subset V$ such that for all $U \subset U(\varepsilon)$ we have*

$$\beta_U(L) \le \alpha(L)\beta_U(-K_V) + \varepsilon. \tag{2.3}$$

b). If in addition $\alpha(L)$ is rational (and positive), there exists a dense open subset $U \subset V$ such that for all $U' \subset U$ we have

$$\beta_{U'}(L) \le \alpha(L)\beta_U(-K_V). \tag{2.4}$$

Proof. a). Take p/q very close to $\alpha(L)$ such that $p[L] + qK_V$ is effective. Then $p/q = \alpha(L) + \eta$ with small $\eta > 0$. Denote by $U(p,q)$ the complement to the support of base points and fixed components of $|pL + qK_V|$. For all $x \in U(p,q)(k)$, we have $h_{pL+qK}(x) \ge c' > 0$ i.e. $h_L(x) > ch_{-K}^{q/p}(x)$, so that

$$\beta_{U(p,q)}(L) \le \frac{p}{q}\beta_{U(p,q)}(-K_V) = (\alpha(L) + \eta)\beta_{U(p,q)}(-K_V).$$

b). If $\alpha = p/q$, we can put $U = U(p,q)$.

Remark. This Theorem shows that it is important to know whether $\alpha(L)$ is rational for all ample L on Fano manifolds. This is true for surfaces in view of the Mori polyhedrality theorem and the convex duality of N^1_{eff} and N^1_{ample}. For threefolds, V. V. Batyrev showed that it is a (rather non-trivial) consequence of Mori's technique. In higher dimensions, this is an open problem.

2.2.2. Theorem on the lower bound. *Given an ample L on a Fano manifold V, assume that*

$$\alpha(L)[L] + K_V \in \partial N^1_{ample} \cap \partial N^1_{eff}. \tag{2.5}$$

Then $\alpha(L)$ is rational. Assume in addition that $\alpha(L)[L]+K_V := l$ belongs to exactly one face of ∂N^1_{ample} of codimension one. Then the contraction morphism associated to this face has a fiber F which is a non-singular Fano variety of dimension ≥ 1, and we have for any $U \supset V$:

$$\beta_U(L) \geq \alpha(L)\beta_{U\cap F}(-K_F). \tag{2.6}$$

The condition (2.5) is a strong one. However, if it is not satisfied for L, one can sometimes ameliorate the situation by an appropriate birational modification of V.

Whenever both inequalities (2.4) and (2.6) hold, we can get the best possible result $\beta_U(L) = \alpha(L)$ in the case when $\beta_U(-K) = 1$ for appropriate open subsets of subsets of V and F. We have already noticed in §1 that analytic methods when applicable give exactly this result. We will show below that this also seems to be a tendency for surfaces and threefolds, but only after deleting the accumulating subvarieties.

The following results heavily depend upon classification theorems. Geometric classification is done over a closed ground field; we generally dispose of subtler problems by passing to a finite extension of the ground field.

2.3. Del Pezzo surfaces. Fano manifolds of dimension two are called the del Pezzo surfaces. They split into ten deformation families. Two of them are homogeneous ($\mathbf{P}^2$ and $\mathbf{P}^1 \times \mathbf{P}^1$) so that point count on them reduces to the Schanuel's theorem. Family $\{V_a\}, 1 \leq a \leq 8$, consists of surfaces that can be obtained by blowing up a points on $\mathbf{P}^2$ in a sufficiently general position. We call a surface V_a *split* (over k), if these a points can be chosen k−rational.

Every surface V_a contains a finite number of exceptional curves ("lines"); they are all k−rational if V_a is split. Denote by U_a the complement to these lines, and put $\Lambda_a = V_a \setminus U_a$. The following Theorem is proved in [18]:

2.3.1. Theorem. *Let V_a be split. Then*
a). $\beta_{\Lambda_a}(-K_V) = 2$.
b). We have the following estimates for $\beta_{U_A}(-K_V) := \beta_a$.
For $k = \mathbf{Q}$: $\beta_1 = \cdots = \beta_4 = 1; \beta_5 \leq 5/4; \beta_6 \leq 5/3$.
For general k: $\beta_1 = \cdots = \beta_3 = 1; \beta_4 \leq 6/5; \beta_5 \leq 3/2$.

The results for $a = 5$ and $a = 6$ have especially direct Diophantine interpretation, since V_5 is an intersection of two quadrics in $\mathbf{P}^4$, and V_6 is a cubic in $\mathbf{P}^3$. We see that if all lines on these surfaces are rational they are accumulating, and, for $k = \mathbf{Q}$, the remainder term $N_{U_a}(-K, H)$ is $O(H^{5/4+\epsilon})$ (resp. $O(H^{5/3+\epsilon})$).

A proof of Theorem 2.3.1 given in [18] consists of two parts. The cases $a \leq 4$ are treated directly, by representing V_a as a blow-up of $\mathbf{P}^2$, comparing height on V_a with height on $\mathbf{P}^2$, and using explicit number-theoretical properties of the height. The remaining cases are treated via an inductive reasoning which shows that $\beta_{a+1} \leq \frac{9-a}{8-a}\beta_a$.

2.4. Fano threefolds. This case was treated in [19] where the following linear lower bound was established:

2.4.1. Theorem. *For any Fano threefold V over a number field k and any Zariski open dense subset $U \subset V$, there exists a finite extension k' of k such that if k'' contains k', then $N_{U\otimes k''}(_K, H) > cH$ for some $c > 0$ and large H. In particular, $\beta_{U\otimes k''} \geq 1$.*

The proof is based upon a description of all 104 deformation families of Fano threefolds obtained by Fano, Iskovskih, Shokurov, Mori, and Mukai. Studying this description, one can derive the following:

2.4.2. Main Lemma. *Every Fano threefold over a closure of the ground field becomes isomorphic to a member of at least one of the following families:*

a). A generalized flag space $P \setminus G$.

b). A Fano threefold covered by rational curves C with $(K_V.C) \leq 2$.

c). A blow-up of varieties of the previous two groups.

Group a) is treated via Eisenstein series. For the group b), it suffices to count points on a single rational curve invoking the Schanuel theorem. Finally, a blow up diminishes the anticanonical height in the complement of the exceptional set and increases the number of such points of bounded height.

2.5. Length of arithmetical stratification. We conjecture that for Fano manifolds, the length of the sequence (2.1) of the complements to accumulating subsets is always finite. However, it can be arbitrarily long:

2.5.1. Proposition. *For every $n \geq 1$, there exists a Fano manifold W of dimension $2n$ over $\mathbf{Q}$ and an ample invertible sheaf L on it such that the sequence (2.1) for (W, L) is of length $\geq 27n + 1$.*

Proof. For $n = 1$, take for W a split del Pezzo surface V_6. Representing it as a blow-up of six rational points on $\mathbf{P}^2$, denote by Λ the inverse image of $\mathcal{O}_{\mathbf{P}^2}(1)$, and by $l_1, \dots, l_{27}$ the exceptional classes, of which $l_1, \dots, l_6$ are represented by inverse images of blown up points. Choose a large positive integer N and small positive integers $\varepsilon_1, \dots, \varepsilon_6$. Take for L a class approximately proportional to $-K_V$: $L = 3N\Lambda - (N - \varepsilon_1)l_1 - \cdots - (N - \varepsilon_6)l_6$. Choose the parameters (N, ε_i) in such a way that $(l_i, L) \neq (l_j, L)$ for all $i \neq j; 1 \leq i, j \leq 27; (l_i, L) < \frac{6}{5}N$.

Theorem 2.3.1 then shows that the 27 lines will be consecutive accumulating subvarieties, with the growth orders $\frac{2}{(L,l_i)}$, and the complement to them will have $\beta < \frac{5}{3N}$, so that the total length is at least 28.

For $n \geq 2$, take n pairs (V_i, L_i) of this type. Arrange parameters $(N_i, \varepsilon_1^i, \dots, \varepsilon_6^i)$ in such a way that the spectra of the growth orders for various (V_i, L_i) do not intersect. Then put $W = V_1 \times \cdots \times V_n, L = pr_1^*(L_1) \otimes \cdots \otimes pr_n^*(L_n)$. From (1.14) one easily sees that the spectrum of the growth orders will have length at least $27n + 1$ (one can even get $28n - 1$).

2.5.2. Conjecture. *If V is a manifold with $K_V = 0$ on which there exist rational curves of arbitrarily high degree defined over a fixed number field, then* *the arithmetical stratification with respect to any ample sheaf L is infinite, and the consecutive growth orders tend to zero.*

The first non-trivial case of this conjecture is furnished by certain quartic surfaces, and more general $K3-$surfaces. In this case, the accumulating subvarieties must consist of unions of rational curves of consecutive $L-$degrees.

However, the problem of understanding rational curves on $K3-$surfaces is difficult, in particular because it is "unstable": even the rank of the Picard group depends on the moduli. It is expected that some stabilization occurs starting with tree-dimensional Calabi-Yau manifolds. We will devote the next Chapter to the highly speculative and fascinating picture whose contours were discovered by physicists.

CHAPTER II

COUNTING RATIONAL CURVES

§3. Calabi-Yau manifolds and mirror conjecture

3.1. Classification of manifolds with $K_V = 0$. In this Chapter, we discuss some conjectural identities involving, on the one hand, characteristic series for the numbers of rational curves of all degrees on certain manifolds V with $K_V = 0$, and on the other hand, hypergeometric functions expressing periods of "mirror dual" manifolds W in appropriate local coordinates. From the physical viewpoint, such identities mean that certain correlation functions of a string propagating on V coincide with other correlation functions of a string propagating on W; the passage from V to W involves also a Lagrangian change ("A– and B– models" of Witten [26]).

Recent physical literature contains a wealth of generalizations of these identities involving curves of arbitrary genus on varieties with $K_V \leq 0$. However, no single case of these conjectures has been rigorously proved. Therefore we have decided to concentrate upon the simplest case, that of Calabi-Yau threefolds.

In the framework of Kähler geometry, they can be introduced by means of the following classification theorem. Let us call a Kähler manifold V *irreducible* if no finite unramified cover of V can be represented as a non-trivial direct product.

3.1.1. Theorem. *For any compact Kähler manifold V with $K_V = 0$, there exists a finite unramified cover V' and its decomposition into irreducible factors*

$$V' \cong \prod_i T_i \times \prod_j S_j \times \prod_k C_k$$

such that

a). T_i are Kähler tori.

b). S_j are complex symplectic manifolds, (i.e. they admit everywhere non-degenerate closed holomorphic 2-form), but not tori.

c). C_k are neither tori, nor symplectic.

Irreducible Kähler manifolds of the type C_k can be called *Calabi-Yau*; in the physical literature this name is sometimes applied to any manifold with $K_V = 0$. The smallest dimension of a complex torus is 1, of a symplectic manifold 2 (any symplectic surface is a $K3$–surface); strictly Calabi-Yau manifolds occur first in dimension three. Classification of Calabi-Yau threefolds is a wide open problem; one does not know even whether they belong to a finite number of deformation families. Most of known examples are constructed as anticanonical hypersurfaces of Fano varieties W, or more generally, as "anticanonical complete intersections": $V = \cap_i D_i$, $\sum_i D_i \in |-K_W|$.

Every Kähler manifold belongs to the realm of three geometries: Riemannian, symplectic, and complex (or algebraic). Theorem 3.1.1 is basically a Riemannian statement (de Rham theorem on the holonomy groups). The curve count, seemingly a pure complex problem, at present can be properly approached only from the symplectic direction revealing its "quasi-topological" nature.

In this report we will concentrate upon algebro-geometric aspects of this vast and complex picture.

3.2. The structure of the mirror conjecture. Consider a Calabi-Yau threefold V and a complete local deformation family $W_z, z \in Z$ of Calabi-Yau threefolds. We will say that V and W_z are mirror related if a certain characteristic function F counting maps $\varphi : \mathbf{P}^1 \to V$ coincides with another function G describing the variation of the periods of W_z. The function G depends on $h^{11}(V)$ complex arguments which reflect the degree of φ with respect to a basis of $\mathrm{Pic}(V) = H^{11}(V, \mathbf{C}) \cap H^2(V, \mathbf{Z})$. The function G depends on $h^{12}(W_z)$ arguments because this is the dimension of Z. Hence we must have $h^{11}(V) = h^{12}(W_z)$.

To make this all precise, we start with the notion of pre-mirror data.

3.2.1. Definition. *The premirror data consist of the following objects:*
i). A pair (V, W_z) as above.
ii). A local isomorphism $q : Z = Z(W) \subset \mathrm{Mod}(W) \to U = U(V) \subset \mathrm{Pic}(V) \otimes \mathbf{C}$ where $\mathrm{Mod}(W)$ is the moduli space of W.
iii). A local trivialization: $\omega : \mathcal{L}_W \to \mathcal{O}_Z$ where $\mathcal{L}_W$ is the invertible sheaf on Z whose geometric fiber at $z \in Z$ is $H^0(W_z, \Omega^3_{W_z})$.
We assume in addition that $\mathrm{Pic}(W_z)$ is canonically trivialized over Z, and that U is contained in the tube domain $\mathrm{Pic}(V) \otimes \mathbf{R} + iK$ where K is the cone spanned by ample (or Kähler) classes.

3.2.2. Counting curves on V and function F. Given premirror data 3.2.1, we proceed as follows.

The holomorphic tangent sheaf $\mathcal{T}_U$ to $U(V)$ is canonically trivialized because U is a domain in the complex vector space $\mathrm{Pic}(V) \otimes \mathbf{C} = H^2(V, \mathbf{C}) : \mathcal{T}_U = \mathrm{Pic}(V) \otimes \mathcal{O}_U$. We define the $\mathcal{O}_U$-linear map

$$F : S^3(\mathcal{T}_U) \to \mathcal{O}_U \tag{3.1}$$

by

$$F(H, E_1 \otimes E_2 \otimes E_3) = \langle E_1 E_2 E_3 \rangle + \sum_C \frac{e^{2\pi i \langle C, H \rangle}}{1 - e^{2\pi i \langle C, H \rangle}} \langle C, E_1 \rangle \langle C, E_2 \rangle \langle C, E_3 \rangle. \tag{3.2}$$

Here $H \in U$; $E_i \in \mathrm{Pic}(V)$ are interpreted as vector fields on U; $\langle , \rangle$ means the intersection index, or cup-product; finally, C runs over rational curves in V.

However, the sum in the r.h.s. of (3.2) can be understood literally only if all rational curves in V are isolated and have the normal sheaf $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Otherwise the local contributions of rational curves can be formally defined by a general position argument involving a deformation of the complex structure of V which makes it non-integrable. More generally, this argument leads to the introduction of the so called Gromov-Witten invariants and quantum cohomology rings. Although these notions belong to the most significant geometric discoveries made by quantum field theorists, we have to omit their discussion because of the lack of mathematically rigorous treatment.

3.2.3. Calculating periods of W and function G. For the local family $\pi: W \to Z$, we have denoted by $\mathcal{L}$ the sheaf $R\pi_*\Omega^3_{W/Z}$ of holomorphic volume forms on the fibers of π. We will now define an $\mathcal{O}_Z$-linear map

$$G: \; S^3(\mathcal{T}_Z) \to \mathcal{L}^{-2} \tag{3.3}$$

as a symbol map of a Picard-Fuchs operator, or infinitesimal variation of Hodge structure.

Specifically, consider the exact sequence

$$0 \to \mathcal{T}_{W/Z} \to \mathcal{T}_W \to \pi^*(\mathcal{T}_Z) \to 0.$$

Its boundary map is the Kodaira-Spencer morphism

$$\mathcal{T}_Z \to R^1\pi_*\mathcal{T}_{W/Z} \tag{3.4}$$

which is an isomorphism if Z is a versal deformation.

The convolution map $i: \; \mathcal{T}_{W/Z} \times \Omega^p_{W/Z} \to \Omega^{p-1}_{W/Z}$ induces a pairing

$$R^1\pi_*(i): \; R^1\pi_*\mathcal{T}_{W/Z} \otimes_{\mathcal{O}_Z} R^q\pi_*\Omega^p_{W/Z} \to R^{q+1}\pi_*\Omega^{p-1}_{W/Z},$$

or a $\mathcal{O}_Z$-map

$$R^1\pi_*\mathcal{T}_{W/Z} \to End^{(-1,1)}(\oplus_{p,q} R^q\pi_*(\Omega^p_{W/Z})).$$

Iterating this map three times we get

$$(R^1\pi_*\mathcal{T}_{W/Z})^{\otimes 3} \to \mathrm{Hom}\;(\pi_*(\Omega^3_{W/Z}), R^3\pi_*((O_W)). \tag{3.5}$$

Actually, this map is symmetric because according to Ph. Griffiths it is the symbol map of the Gauss-Manin connection extended to the differential operators of order 3. Using the relative Serre duality, one can identify the r.h.s. of (3.5) with $\mathcal{L}^{-2}$. Finally, composing (3.5) with the Kodaira-Spencer map $S^3(\mathcal{T}_Z) \to S^3(R^1\pi_*\mathcal{T}_{W/Z})$, we obtain the function G in (3.5).

3.2.4. Definition. *The premirror data 3.2.1 are called mirror data if, after the identification of $U(V)$ and $Z(W)$ via q and trivialization of $\mathcal{L}^{-2}$ via ω, F and G coincide.*

3.3. Example. For V a generic quintic hypersurface, the relevant mirror data were given in the ground-breaking paper by Ph. Candelas, X. de la Ossa, P. Green, and L. Parkes [9]. In this case, $h^{11}(V) = 1$, and Z is a neighborhood of zero in $\mathbf{C}$, with complex coordinate z. Evaluating (3.2) on the positive generator H of Pic(V) (hyperplane section) multiplied by t in upper plane, and on $E_1 = E_2 = E_3 = H$ they get a function $F(q), q = e^{2\pi i t}$ of the form

$$F(q) = 5 + \sum_{k=1}^{\infty} n_k k^3 \frac{e^{2\pi i k t}}{1 - e^{2\pi i k t}} \tag{3.6}$$

where n_k is the number of rational curves of degree k (with appropriate multiplicities).

The mirror map $z \mapsto q(z) := e^{2\pi i t(z)}$ is calculated to be

$$t(z) = -\frac{5}{2\pi i}\log(5z^{-1/5}) + \frac{\sum_{N=0}^{\infty}\frac{(5N)!}{(N!)^5}\lambda(N)5^{-5N}z^N}{\sum_{N=0}^{\infty}\frac{(5N)!}{(N!)^5}5^{-5N}z^N}, \tag{3.7}$$

$$\lambda(0) = 0,\ \lambda(N) = -\sum_{m=N+1}^{5N}\frac{1}{m}.$$

Put

$$f_0(z) = \sum_{N=0}^{\infty}\frac{(5N)!}{(N!)^5}5^{-5N}z^N.$$

The function $G(z)$ is

$$G(z) = 5f_0(z)^{-2}(1-z)^{-1}(2\pi i z\frac{dt(z)}{dz})^{-3}. \tag{3.8}$$

Finally, the mirror identity states that

$$F(q(z)) \equiv G(z) \tag{3.9}$$

in a neighborhood of zero.

This identity says that two cubic differentials $F(q)(dq/q)^3$ and

$$\frac{5}{(1-z)f_0(z)^2}\left(\frac{dz}{z}\right)^3$$

are one and the same differential written in different local coordinates $q(z)$ and z respectively. This reminds one a Schwarz deritivative related to the linear differential operators of the second order and projective connections. In fact, this analogy can be made quite precise. The relevant differential operator annihilates $f_0(z)$: for $D = z\dfrac{d}{dz}$ it can be written as

$$L = D^4 - 5^{-4}(5D+1)(5D+2)(5D+3)(5D+4),$$

and $\log(5^5 q(z)/z)$ is a quotient of two solutions of the equation $Lf = 0$.

In the remaining part of this report, we will explain Batyrev's construction of toric premirror data.

§4. Toric mirrors

4.1. Convex geometry. Let M, N be a pair of free abelian groups of finite rank $r = d+1$ endowed with a pairing $\langle , \rangle : M \times N \to \mathbf{Z}$ making them dual to each other.

In $M_{\mathbf{R}} = M \otimes \mathbf{R}$, $N_{\mathbf{R}} = N \otimes \mathbf{R}$ consider a pair of convex compact closed polyhedra $\diamondsuit_M \subset M_{\mathbf{R}}$, $\diamondsuit_N \subset N_{\mathbf{R}}$. Each of them is an intersection of a finite set of closed halfspaces.

4.1.1. Definition. *a). $\diamondsuit_M, \diamondsuit_N$ are dual, if*

$$\diamondsuit_M = \{m \in M_{\mathbf{R}} | \langle m, n\rangle \geq -1 \text{ for all } n \in \diamondsuit_N\},$$

$$\diamondsuit_N = \{n \in N_{\mathbf{R}} | \langle m, n\rangle \geq -1 \text{ for all } m \in \diamondsuit_M\}. \tag{4.1}$$

b). $(\diamondsuit_M, \diamondsuit_N)$ form a mirror pair if they are dual and have integral vertices.

If we start with any convex compact closed polyhedron $\diamondsuit_N$ and define $\diamondsuit_M$ by the first line of (4.1), it will also be such a polyhedron, and the second condition will be satisfied automatically. Duality of $(\diamondsuit_M, \diamondsuit_N)$ induces an inclusion reversing isomorphism between the posets of faces of $\diamondsuit_M$ and $\diamondsuit_N$.

If in addition $\diamondsuit_N$ has integral vertices, then codimension one faces of $\diamondsuit_M$ are defined by equations of the type $\langle m, n_i\rangle = -1$, $n_i \in N$, but vertices of $\diamondsuit_M$ need not be integral. This is an additional (and restrictive) condition. It can be expressed via point count in $(a\diamondsuit_N) \cap N$. Specifically, there exists a polynomial of degree $r = \dim N_{\mathbf{R}}$, $l(a)$, such that card $(a\diamondsuit_N \cap N) = l(a)$ for all integral positive a.

T. Hibi proved that $\diamondsuit_M$ has integral vertices iff $l(-a-1) = (-1)^r l(a)$ for all a.

V. Batyrev calls members of mirror pairs *reflexive polyhedra.*

4.1.2. Lemma. *If $(\diamondsuit_M, \diamondsuit_n)$ form a mirror pair, they contain origin which is their only interior point.*

Proof. From (4.1), it is obvious that $0 \in \diamondsuit_M, 0 \in \diamondsuit_N$, and that 0 does not lie on the boundary.

In order to see that, say, $\diamondsuit_M$ does not contain any more integral interior points, represent $\diamondsuit_M$ as a union of cones $\sigma(E) = \cap_{t \in [0,1]} tE$ where E runs over all codimension one faces of $\diamondsuit_M$.

Any interior point $m_0 \in \diamondsuit_M$ belongs to some $t_0 E, 0 < t_0 < 1$. If m_0 lies in the face $\langle m, n_E\rangle = -1, n_E \in N$, we have $\langle m_0, n_E\rangle = -t_0$. If m_0 is integral, we must have $t_0 = 0$, that is $m_0 = 0$.

4.1.3. Classification results. For every r, there exists only a finite number of reflexive polyhedra, but they are completely enumerated only for $r = 1$ and 2. There are 16 of them for $r = 2$, hundreds for $r = 3$, and thousands for $r = 4$.

Here is one example for general r: put $M = \mathbf{Z}^r, e_i =$ the i-th coordinate vector,

$$\diamondsuit_M = \text{convex envelope of } \{e_1, \ldots, e_r, -(e_1 + \cdots + e_r)\}. \tag{4.2}$$

For $N = \mathbf{Z}^r$ and standard pairing we can easily check that

$$\diamondsuit_N = (-1, \ldots, -1) + \text{convex envelope of } \{(r+1)e_1, \ldots, (r+1)e_r, 0\}. \tag{4.3}$$

4.2. Affine toric mirrors. Given a pair of dual lattices M, N as in 4.1, we can construct a pair of tori. Writing elements of M (resp. N) multiplicatively as x^m (resp. y^n) we put

$$T(N) = \text{Spec } \mathbf{C}[x^M],\ T(M) = \text{Spec } \mathbf{C}[y^N].$$

For $G_m := \text{Spec } [t, t^{-1}]$ we have the following canonical identifications:

$$N = \text{Hom } (G_m, T(N)),\ M = \text{Hom } (T(N), G_m)$$

and similarly for $T(M)$.

Given in addition a mirror pair of polyhedra $(\Diamond_M, \Diamond_N)$, we put

$$v_M = \partial\Diamond_M \cap M = \Diamond_M \cap M \setminus \{0\}. \tag{4.4}$$

and similarly for v_N.

4.2.1. Definition. *The following two families of affine hypersurfaces in the tori $T(M), T(N)$ are called affine mirrors of each other:*

$$V(\Diamond_M) = V_N : 1 - \sum_{m \in v_M} a_m x^m = 0 \text{ (in } T(N)), \tag{4.5}$$

$$V(\Diamond_N) = V_M : 1 - \sum_{n \in v_N} b_n y^n = 0 \text{ (in } T(M)). \tag{4.6}$$

Notice that 1 in (4.5), (4.6) is actually x^0, resp. y^0, corresponding to $0 \in \Diamond_M, \Diamond_N$.

A word about our notation. Eventually we will construct toric premirror data as in 3.2.1, where V will be a partial compactification of the family V_N and W that of family V_M. We try to furnish the principal relevant objects by indices M, resp. N, in such a way that an object covariantly depended on its index. So $T(N)$ covariantly depends on its lattice of one-parametric subgroups N, and V_N is a family of hypersurfaces in $T(N)$, etc.

4.2.2. Example. In the notation of 4.1.3, put $x^{e_i} = x_i$ in M and $y^{e_i} = y_i$ in N. Then:

$$V_N : \qquad 1 - \frac{a}{x_1 \ldots x_r} + \sum_{i=1}^{r} a_i x_i = 0, \tag{4.7}$$

$$V_M : \qquad 1 - \frac{1}{y_1 \ldots y_r} \sum_{\nu} y_1^{\nu_1} \ldots y_r^{\nu_r} = 0, \tag{4.8}$$

$$\nu = (\nu_1, \ldots, \nu_r) \neq (1, \ldots, 1);\ 0 \leq \sum_i \nu_i \leq r + 1,\ \nu_i \geq 0.$$

If we compactify $T(M)$ to a projective space by introducing homogeneous coordinates $y_i = Y_i / Y_0$, (4.8) becomes the complete linear system of hypersurfaces of degree $r + 1$ in $\mathbf{P}^r$:

$$\overline{V}_M : \sum_{\mu} B_\mu Y_0^{\mu_0} \ldots Y_r^{\mu_r} = 0, \sum_i \mu_i = r + 1, \mu_i \geq 0. \tag{4.9}$$

For $r \geq 4$, they are Calabi-Yau manifolds outside the discriminantal locus defined by a universal polynomial in coefficients B_μ : $D(B_\mu) = 0$. For $r = 3$ (resp. $r = 2$), they are quartic $K3$-surfaces and cubic plane curves respectively. We have $h^{11} = 1$ for $\overline{V}_M$. On the other hand, (4.7) is actually a one–parametric family since a_i's can be made constant by rescaling x_i's. After some variable change in (4.7) and a suitable compactification, we obtain in this way for $r = 4$ the quintic mirrors of 3.4.

In order to discuss in a more systematic way compactifications both in the toric spaces $T(M), T(N)$ and the coefficient spaces a_m, b_n we will briefly recall some constructions of toric geometry.

4.3. Toric (partial) compactifications. Let L be a lattice of finite rank, $\sigma \subset L_{\mathbf{R}}$ a closed convex cone with vertex in origin. We will be working only with cones finitely generated by a family of elements of L. Put $\sigma^t = \{l^* \in L^*_{\mathbf{R}} | \langle l^*, l\rangle \geq 0 \text{ for all } l \in \sigma\}$, and

$$A_\sigma = \mathrm{Spec}\ (\oplus_{l \in \sigma^t} \mathbf{C} x^l).$$

The affine variety A_σ contains $T(L)$, i.e. $\sigma^t \cap L^*$ generates L^* as a group, iff σ is *strictly convex* that is, does not contain a non-trivial subspace. The natural action of $T(L)$ upon itself extends to the action $T(L) \times A_\sigma \to A_\sigma$. So A_σ is a partial toric compactification of $T(L)$.

A more general construction of of compactifications is obtained if one glues together A_σ's for an appropriate family of cones. Such families are called *fans*. For us, a fan Δ in $L_{\mathbf{R}}$ is a finite family of strictly convex cones, containing all faces of all its elements and such that the intersection of any two cones is a face of each of them. We put

$$\mathbf{P}(\Delta) = \coprod_{\sigma \in \Delta} A_\sigma / (\text{natural equivalence relation}).$$

When $|\Delta| := \cup_{\sigma \in \Delta} \sigma = L_{\mathbf{R}}$, $\mathbf{P}(\Delta)$ is a complete toric variety which can be considered as a natural generalization of projective space.

4.4. Compactifying members of affine families V_N, V_M. For a reflexive polyhedron $\Diamond_M$, denote by $F(\Diamond_M)$ the set of $\Diamond_M$*-compatible* fans Δ_M in $M_{\mathbf{R}}$, i.e. fans satisfying the following conditions:

4.4.1. Definition. *Δ_M is $\Diamond_M$-compatible if*

a). Every 1-cone of Δ_M is generated by some $m \in v_M$. and every $m \in v_M$ generates some 1-cone of Δ_M.

b). Δ_M is simplicial, i.e. every d–dimensional cone of Δ_M is generated by d 1-cones.

c). Δ_M is projective, i.e. there exists a strictly convex function $\eta : M_{\mathbf{R}} \to \mathbf{R}$ linear on every cone of Δ_M.

The property b) implies that $\mathbf{P}(\Delta_M)$ has only abelian quotient singularities. In c), η is called strictly convex (w.r.t. Δ_M) if it is convex, and every maximal subset of $M_{\mathbf{R}}$ on which it is linear is a cone of maximal dimension of Δ_M. The property c) implies that $\mathbf{P}(\Delta_M)$ is a projective variety.

The set $F(\Diamond_M)$ is obviously finite. Less obvious but true is that it is non-empty (condition c) can be satisfied).

4.4.2. Definition. *Given a mirror pair* $(\Diamond_M, \Diamond_N)$, *a pair of fans* $\Delta_M \in F(\Diamond_M)$, $\Delta_N \in F(\Diamond_N)$, *the Calabi-Yau families of the corresponding toric premirror data consist of fiber compactified families* $\overline{V}_N \subset \mathbf{P}(\Delta_N) = \overline{T(N)}$, $\overline{V}_M \subset \mathbf{P}(\Delta_M) = \overline{T(M)}$.

Remark. Since $\mathbf{P}(\Delta_M), \mathbf{P}(\Delta_N)$ have only abelian quotient singularities, its (anti)canonical divisor is $\mathbf{Q}$–Cartier. Families $\overline{V}_N, \overline{V}_M$ are precisely anticanonical systems of divisors. For $r = 4$ ($d = 3$), their generic members are nonsingular Calabi–Yau manifolds; for $d \geq 4$ they are generalized Calabi–Yau varieties with mild singularities.

4.5. Secondary lattices and tori. The equations (4.5) (resp. (4.6)) show that points of v_M (resp. v_N) define some one–parametric deformations of hypersurfaces $\overline{V}_N$ (resp. $\overline{V}_M$) represented by coefficients $a_m, m \in v_M$ (resp. $b_n, n \in v_N$).

On the other hand, according to 4.4.1 a), these points correspond bijectively to 1-cones of Δ_M (resp. Δ_N) that is, to the irreducible divisors D_m at infinity of Pic $\mathbf{P}(\Delta_M)$ (resp. Pic $\mathbf{P}(\Delta_N)$) which in turn define one-parametric subgroups in Pic $\mathbf{P}(\Delta_M)$ (resp. Pic $\mathbf{P}(\Delta_N)$) and by restriction, on members of $\overline{V}_M$ (resp. $\overline{V}_N$).

This is the first approximation to the second part of the premirror data where we need spaces parametrizing simultaneously members of $\overline{V}_N$ and elements of Pic $\mathbf{P}(\Delta_N) \otimes \mathbf{C}$, and vice versa.

To get the second approximation, we want to take into account that a_m, $m \in v_M$, can never parametrize $\overline{V}_N$ effectively because the whole linear system is acted upon by $T(N)$. Similarly, rays in Pic $\mathbf{P}(\Delta_M) \otimes \mathbf{C}$ generated by D_m, $m \in v_M$, cannot be linearly independent because divisors of monomials reduce to zero in Pic.

In order to proceed systematically, we have to construct new pairs of lattices and tori.

4.5.1. Secondary lattices. Denote by $\mathbf{Z}[v_M]$ the free abelian group generated by v_M, and similarly for v_N. Let $\mathrm{Rel}(v_M)$ be the kernel of the natural homomorphism $\mathbf{Z}[v_M] \to M : \sum_{m \in v_M} c_m[m] \mapsto \sum c_m m$, and similarly for N. The image of this homomorphism $\tilde{M} \subset M$ is a lattice of finite index in M, and similarly we define $\tilde{N} \subset N$. Thus we have exact sequences

$$0 \to \mathrm{Rel}(v_M) \to \mathbf{Z}[v_M] \to \tilde{M} \to 0, \tag{4.10}$$

$$0 \to \mathrm{Rel}(v_N) \to \mathbf{Z}[v_N] \to \tilde{N} \to 0. \tag{4.11}$$

Denote by L_N (resp. L_M) the lattice dual to $\mathrm{Rel}(v_M)$ (resp. $\mathrm{Rel}(v_N)$). Since (4.10) and (4.11) split, the dual sequences are exact. Identifying $\mathbf{Z}(v_M)^*$ (resp. $\mathbf{Z}(v_N)^*$) with space of functions $\mathbf{Z}^{v_M}$ (resp. $\mathbf{Z}^{v_N}$) and putting $\tilde{M}^* = N'$, $\tilde{N}^* = M'$, we get exact sequences

$$0 \to N' \to \mathbf{Z}^{v_M} \to L_N \to 0, \tag{4.12}$$

$$0 \to M' \to \mathbf{Z}^{v_N} \to L_M \to 0. \tag{4.13}$$

Clearly, $N \subset N' \subset N_{\mathbf{Q}}$, $M \subset M' \subset M_{\mathbf{Q}}$. The embedding $N \to \mathbf{Z}^{v_M}$ is just the restriction to v_M of N as the group of functions on M, and similarly for N.

4.5.2. Positive cones. Denote by $\mathrm{Rel}_{\geq 0}(v_N)$ the semigroup of relations with non-negative coefficients, and by $\mathrm{Rel}^{\mathbf{R}}_{\geq 0}(v_N)$ the respective cone in $\mathbf{R}[v_N]$. Denote by $\varepsilon_M \subset L_M \otimes \mathbf{R}$ the image of $\mathbf{R}^{v_N}_{\geq 0}$ in $L_M \otimes \mathbf{R}$. Spaces $\mathrm{Rel}(v_L) \otimes \mathbf{R}$ and $L_M \otimes \mathbf{R}$ are dual. Using the standard facts of convex duality, one sees that

$$\mathrm{Rel}^{\mathbf{R}}_{\geq 0}(v_N) = \varepsilon^t_M,\ \mathrm{Rel}^{\mathbf{R}}_{\geq 0}(v_M) = \varepsilon^t_N.$$

We will now construct tori $T(L_N), T(L_M)$ and show that they naturally parametrize simultaneously pre-mirror pairs (moduli space/complexified Picard group), or at least some subspaces of the latter, when toric linear systems do not form locally versal families. Then we will use cones $\varepsilon_M, \varepsilon_N$ in order to construct their partial compactifications crucial for understanding the mirror map.

4.6. Theorem. *There exist two natural maps*

$$T(L_N)(\mathbf{C}) \to Mod(\overline{V}_N), \tag{4.14}$$

$$T(L_N)(\mathbf{C}) \to \mathrm{Pic}(\overline{V}_M) \otimes \mathbf{C} \tag{4.15}$$

and similarly with (M, N) reversed. (The second map is multivalued: see (4.16) below).

Proof. a). By definition,

$$T(L_N) = \mathrm{Spec}\ [L^*_N] = \mathrm{Spec}\ \mathbf{C}[\mathrm{Rel}(v_M)].$$

Writing $\mathrm{Rel}(v_M)$ multiplicatively, we identify it with the group of monomials $\prod_{m \in v_M} a_m^{c_m}$ such that $\sum c_m m = 0, c_m \in \mathbf{Z}$.

For a point $\xi \in T(N)(\mathbf{C})$, put $\xi^m = x^m(\xi) \in \mathbf{C}^*$. The natural action of $T(N): x^m \mapsto \xi^m x^m,\ a_m \mapsto \xi^{-m} a_m$ leaves (4.5) invariant, and $\mathbf{C}[\mathrm{Rel}(v_M)]$ can be identified with the span of $T(N)$–invariant monomials in a_m. Hence $\mathbf{C}$–points of $T(L_N)$ bijectively correspond to the $T(N)$–orbits of hypersurfaces in $\overline{V}_N$ defined by equations with all $a_m \neq 0$. This defines (4.14).

More algebraically, we have an affine hypersurface (4.5) in $T(\mathbf{Z}^{v_M}) \times T(N)$ which is invariant with respect to the described $T(N)$–action. The affine quotient gives a hypersurface in $T(\mathbf{Z}^{v_M}) \times T(N)/T(N)$, which can be identified with $T(L_N) \times T(N')$ by choosing a splitting of (4.12). There is a natural isogeny $T(N) \to T(N')$ which allows one to lift this hypersurface back to $T(L_N) \times T(N)$.

b). For an arbitrary torus $T(L)$, we have a natural identification $L \otimes \mathbf{C} = Lie\ T(L)(\mathbf{C})$ which defines the exponential map $\exp: L \otimes \mathbf{C} \to T(L)(\mathbf{C})$. We can explicitly define an inverse map

$$\log: T(L)(\mathbf{C}) \to L \otimes \mathbf{R} + il \otimes \mathbf{R}/2\pi i L$$

whose real part is

$$L^* \ni m \mapsto \log\ |x^m(\eta)| \in \mathbf{R},\ \eta \in T(L)(\mathbf{C}),$$

and imaginary part is

$$L^* \ni m \mapsto i\arg\, x^m(\eta) \in \mathbf{R}/(2\pi i\mathbf{Z}).$$

On the other hand, (4.12) up to isogeny coincides with

$$0 \to \mathrm{Div}^0_\infty(\mathbf{P}(\Delta_M)) \to \mathrm{Div}_\infty(\mathbf{P}(\Delta_M)) \to \mathrm{Pic}(\mathbf{P}(\Delta_M)) \to 0$$

so that we have a natural isomorphism

$$L_N \otimes \mathbf{R} = \mathrm{Pic}(\mathbf{P}(\Delta_M)) \otimes \mathbf{R}$$

whereas $L_N \subset \mathrm{Pic}(\mathbf{P}(\Delta_M)) \otimes \mathbf{R}$ is a lattice commensurable with $\mathrm{Pic}(\mathbf{P}(\Delta_M))$ (and coinciding with it if v_M generates exactly M over $\mathbf{Z}$ as one sees from (4.10), (4.12)). So finally we get, combining with res : $\mathrm{Pic}(\mathbf{P}(\Delta_M)) \to \mathrm{Pic}(\overline{V}_M)$:

$$\begin{aligned} \mathrm{res} \circ \log : \ T(L_N)(\mathbf{C}) &\to \mathrm{Pic}(\mathbf{P}(\Delta_M)) \otimes \mathbf{R} \oplus \mathrm{Pic}(\mathbf{P}(\Delta_M)) \otimes \mathbf{R}i/2\pi i L_N \\ &\to \mathrm{Pic}(\overline{V}_M) \oplus \mathrm{Pic}(\overline{V}_M) \otimes \mathbf{R}i/2\pi i\ \mathrm{res}(L_N). \end{aligned} \tag{4.16}$$

This is our multivalued map (4.15).

4.7. Partial compactification. The cone $\varepsilon_N \subset L_N \otimes \mathbf{R}$ dual to $\varepsilon^t_N = \mathrm{Rel}_{\geq 0}(v_M) \otimes \mathbf{R}$ defines the affine toric variety $A_{\varepsilon_N} \supset T(N)$ whose function ring is just the span of $T(N)$-invariant monomials $\prod a_m^{c_m}$ with $c_m \geq 0$. Hence it contains in particular the point $a_m = 0$ for all m which defines the maximally degenerate anticanonical hypersurfaces in $\mathbf{P}(\Delta_N)$, the sum of all divisors at infinity.

We will use this degeneration below in order to trivialize the bundle of holomorphic volume forms on fibers of V_N by choosing a form with period 1 along a specific invariant cycle in the neighborhood of the degenerate hypersurface.

Now we proceed to refine the compactification A_{ε_N} by taking into account various possible choices of $\Delta_M \in F(\diamondsuit_M)$.

For the proof of the following result, see Oda–Park [23]. Consider the cone of convex functions on $M \otimes \mathbf{R}$ linear on all cones of Δ_M. Restrict them on $\mathbf{R}^{v_M}$ and then consider the image of the resulting cone in $L_N \otimes \mathbf{R}$. Denote this image $\varepsilon(\Delta_M) \subset L_N \otimes \mathbf{R}$.

4.8. Proposition. *a). $\varepsilon(\Delta_M)$ is a closed convex finite polyhedral cone in $L_N \otimes \mathbf{R}$. Under the identification $L_N \otimes \mathbf{R} = \mathrm{Pic}(\mathbf{P}(\Delta_M))_\mathbf{R}$ it coincides with the closure of the ample cone of* $\mathrm{Pic}(\mathbf{P}(\Delta_M))$.

b). All cones $\varepsilon(\Delta_M)$ for $\Delta_M \in F(\diamondsuit_M)$ and their faces form a finite convex polyhedral fan $f(\diamondsuit_M)$ with support ε_N; the cones $\varepsilon(\Delta_M)$ themselves are all cones of maximal dimension of this fan.

In this way we get the following diagram of spaces:

$$\mathbf{P}(f(\diamondsuit_M)) = \cup_{\Delta_M \in F(\diamondsuit_M)} A_{\varepsilon(\Delta_M)} \to A_{\varepsilon_N} = A(L_N).$$

The closed point $p_{\varepsilon_N} \in A_{\varepsilon_N}$ corresponding to $a_m = 0$, $m \in v_M$, is covered by the closed points

$$p_{\varepsilon(\Delta_M)} \in A_{\varepsilon(\Delta_M)},\ \Delta_M \in F(\diamondsuit_M).$$

Of course, the similar picture of partial compactifications of $T(L_M)$ takes place in the mirror setting. We now look at (parts of) $T(L_M)(\mathbf{C})$ as a space parametrizing (parts of) $\mathrm{Pic}(\mathbf{P}(\Delta_N)) \otimes \mathbf{C}$ for various $\Delta_N \in F(\Diamond_M)$ and therefore furnishing the arguments of the function F counting rational curves on the members of various compactified families $\overline{V}_N = \overline{V}_N(\delta_N)$. From this vantage point, the cones $\varepsilon(\Delta_N)$ correspond to various convergence domains of the same function which in its G–avatar depends on the moduli of $\overline{V}_M$ and does not see any difference between various compactifications Δ_N.

We will now make this more precise.

4.9. Curve counting function. We want to define an analog of the function F (see 3.3) in our situation.

We will choose a fan $\Delta_N \in F(\Diamond_N)$ and count rational curves C on a hypersurface $V \in |-K_{\mathbf{P}}(\Delta_N)|$, or more precisely, parametrized rational curves which are non-constant maps $\varphi_C :\ \mathbf{P}^1 \to V$.

Every such curve defines a $\mathbf{Z}$–valued function on $\mathrm{Pic}(\mathbf{P}(\Delta_N)) :\ \mathcal{L} \mapsto \deg \varphi_C^*(\mathcal{L})$. Hence we get a $\mathbf{Z}$–valued function on L_M which we denote, together with its extension to $L_M \otimes \mathbf{C}$, by l_C. It is non-negative on the ample cone of $\mathrm{Pic}(\mathbf{P}(\Delta_N))_{\mathbf{R}}$ that is on $\varepsilon(\Delta_N) \subset L_M \otimes \mathbf{R}$. Instead of logarithm, consider the function

$$t = \frac{1}{2\pi i}\log :\ T(L_M)(\mathbf{C}) \to L_M \otimes \mathbf{R}/L_M + iL_M \otimes \mathbf{R}.$$

Put

$$q_C :\ T(L_M)(\mathbf{C}) \to \mathbf{C}^*,\ q_C = e^{2\pi i \langle l_C, t\rangle}.$$

Define also

$$U(\Delta_N) = t^{-1}(L_M \otimes \mathbf{R}/L_M + i\varepsilon(\Delta_N)) \subset T(L_M)(\mathbf{C}).$$

The positivity property above implies the following fact:

$$|q_C(\xi)| < 1 \text{ for all } \varphi_C \text{ and all } \xi \in U(\Delta_N) \subset T(L_M)(\mathbf{C}),$$

$$|q_C(\xi)| \to 0 \text{ as } \mathrm{Im}(t(\xi)) \to \infty \text{ in } \varepsilon(\Delta_N).$$

Consider now the holomorphic tangent vector *bundle* $\mathbf{T}T(L_M)(\mathbf{C})$. It can be canonically trivialized by invariant vector fields. Restricting upon $U(\Delta_N)$ we get

$$\mathbf{T}U(\Delta_N) \cong U(\Delta_N) \times L_M \otimes \mathbf{C}.$$

Finally we define (now assuming $\dim(\mathbf{P}(\delta_N)) = 4$:

$$F_{\Delta_N} :\ S^3(\mathbf{T}U(\Delta_N)) = U(\Delta_N) \times S^3(L_M \otimes \mathbf{C}) \to U(\Delta_N) \times \mathbf{C},$$

$$F_{\Delta_N}(\xi; E_1, E_2, E_3) = (\xi; \langle E_1 E_2 E_3\rangle + \sum_C \frac{q_C(\xi)}{1 - q_C(\xi)} \langle l_C, E_1\rangle\langle l_C, E_2\rangle\langle l_C, E_3\rangle).$$

We remind to the reader that algebro-geometric aspects of summing over C's are far from being firmly established: see [15], [16], [1].

Consider now the open embedding

$$U(\Delta_N) \subset T(L_M)(\mathbf{C}) \subset A_{\varepsilon(\Delta_N)}(\mathbf{C}) \ni p_{\varepsilon(\Delta_N)}.$$

The closure $\overline{U(\Delta_N)}$ of $U(\Delta_N)$ in $A_{\varepsilon(\Delta_N)}(\mathbf{C})$ contains the maximal degeneracy point $p_{\varepsilon(\Delta_N)}$, and all q_C extend to this point and vanish there so that

$$F_{\Delta_N}(p_{\varepsilon(\Delta_N)}; E_1, E_2, E_3) = \langle E_1 E_2 E_3 \rangle.$$

We expect that F_{Δ_N} is meromorphic in the interior of $U(\Delta_N)$.
Let us put now

$$U(\Diamond_N) = t^{-1}(L_M \otimes \mathbf{R}/L_M + i\varepsilon_M) = \cup_{\Delta_N \in F(\Diamond_N)} U(\Delta_N).$$

4.9.1. Question. *Does there exist a meromorphic function F_Δ on $S^3(\mathbf{T})$ whose restriction on $U(\Diamond_N)$ coincides with F_{Δ_N}?*

If the answer to this question is positive, this means that counting curves on a set of flops of anticanonical toric hypersurfaces reduces to choosing various branches of the same analytic characteristic function.

4.10. Periods of the mirror family. We now want to define the function G on a part Z of $T(L_M)(\mathbf{C})$ considered as a moduli space for (compactified) hypersurfaces in $T(M)$.

We will assume that there exists a fan $\Delta_M \in F(\Diamond_M)$ such that the generic member of $\overline{V}_M = |-K_{\mathbf{P}(\Delta_M)}|$ is smooth. For $d = 3$ $(r = 4)$ any Δ_M will do.

For Z we will take $\widetilde{U}(\Delta_M) = T(L_M) \backslash D(\Delta_M)$ where $D(\Delta_M)$ is the discriminantal divisor of non-smooth anticanonical hypersurfaces.

In this way we get as in 3.3 ($W = \overline{V}_M$):

$$G: \ S^3(\mathcal{T}_{\widetilde{U}(\Delta_M)}) \to \mathcal{L}^{-2},$$

where $\mathcal{L}$ is the sheaf of holomorphic volume forms.

4.10.1. Trivialization of $\mathcal{L}$. To make it, we must choose a section ω of $\pi_* \Omega^d_{W/Z}$; it suffices to define it up to sign. Following D. Morrison [21], [22] we suggest to do it by choosing an appropriate invariant cycle γ in the local system of homology groups $H_d(V_{M,a}, \mathbf{Z})$, $a \in T(L_M) \setminus D(\Delta_M) = \widetilde{U}(\Delta_M)$. A complete understanding of the situation requires a description of the relevant modular group representation

$$\pi_1(\widetilde{U}(\Delta_M), a) \to \mathrm{Aut}(H_d(V_{M,a}, \mathbf{Z}))$$

which we lack at the moment. However the following prescription fits all the examples.

a). Invariant cycle. Consider a $(d+1)$-dimensional topological torus $\gamma_T = (S^1)^{d+1} \subset T(M)(\mathbf{C})$ given by $|x^n| = 1$ for all $n \in N$. Denote by $\widetilde{U} \subset \widetilde{U}(\Delta_M)$ the set of points $a = \{a_n | n \in v_N\}$ in $\widetilde{U}(\Delta_M)$ for which $\sum_{n \in \widetilde{U}} |a_n| < 1$. This means that

$$\gamma_T \cap V_{M,a} = \emptyset \text{ for } a \in \widetilde{U},$$

so that
$$[\gamma_T] \in H_{d+1}(\mathbf{P}(\Delta_M), V_{M,a}; \mathbf{Z}).$$

If d is odd (e.g. $d = 3$) we have a surjective map
$$\partial : \ H_{d+1}(\mathbf{P}(\Delta_M), V_{M,a}) \to H_d(V_{M,a}).$$

Denote by γ_a the image of $[\gamma_T]$ in $H_d(V_{M,a}, \mathbf{Z})$. By construction, it is monodromy invariant over at least $\widetilde{U} \subset T(L_M)(\mathbf{C})$. Recall that geometrically ∂ can be described as follows. Take a small tubular neighborhood $\tau(V_{M,a})$ in $\mathbf{P}(\Delta_M)$, then $\tau(V_{M,a}) \setminus V_{M,a}$ restricts to an S^1–fibration $\sigma(V_{M,a}) \subset \tau(V_{M,a})$ over $V_{M,a}$. For a cycle γ in $V_{M,a}$, take its inverse image γ' in $\sigma(V_{M,a})$. Then $\partial(\gamma') = \gamma$.

b). Residue map. Denote by $\Omega^{d+1}(\log\ V_{M,a})$ the sheaf of meromorphic forms $\omega_{\mathbf{P}}$ on $\mathbf{P}(\Delta_M)$ with pole of order ≤ 1 on $V_{M,a}$. There exists a well defined map
$$\mathrm{res}: \ H^0(\mathbf{P}(\Delta_M), \Omega^{d+1}(\log\ V_{M,a})) \to H^0(V_{M,a}, \Omega^d_{V_{M,a}})$$
for which
$$\frac{1}{2\pi i}\int_\gamma \omega_{\mathbf{P}} = \int_{\partial\gamma} \mathrm{res}(\omega_{\mathbf{P}}).$$

c). Trivialization of $\mathcal{L}$. Choose $\omega_{\mathbf{P},a}$ in such a way that
$$\int_{\gamma_T} \omega_{\mathbf{P},a} = 2\pi i, \text{ i. e. } \int_{\gamma_a} \mathrm{res}(\omega_{\mathbf{P},a}) = 1$$
and trivialize $\mathcal{L}$ by choosing $\omega_a = \mathrm{res}(\omega_{\mathbf{P},a})$ as a unit section. Changing orientation of γ_T results in changing the sign of ω_a.

d). An explicit calculation of $\omega_{\mathbf{P},a}$. On the affine chart $T(L_N) \times T(M)$ with coordinates $(a_n, x_i^{n_i})$ where $n \in v_N$, $n_1, \dots n_{d+1}$ is a basis of N, we can put
$$\omega_{\mathbf{P},a} = (1 - \sum_{n\in v_N} a_n x^n)^{-1} x_1^{-1} dx_1 \wedge \dots \wedge x_{d+1}^{-1} dx_{d+1}.$$

For $a \in \widetilde{U}$, we can expand this and easily calculate:
$$\frac{1}{(2\pi i)^{d+1}}\int_{\gamma_T} \omega_{\mathbf{P},a} = 1 + \sum_{l\in \mathrm{Rel}_{\geq}(v_N)} (\sum_{n\in v_N} l(n))! \prod_{n\in v_N} a_n^{l(n)}/l(n)! := \Omega(a)$$
so that finally
$$\omega_a = \frac{1}{(2\pi i)^d \Omega(a)} \mathrm{res}(\omega_{\mathbf{P},a}).$$

4.11. Concluding remarks. We have now completed the construction of the toric pre-mirror data. This construction has however two drawbacks.

The first is that $T(L_M)$ (resp. $T(L_N)$)) not always parametrize the whole Mod (resp. Pic) spaces. This is however true when Aut $\mathbf{P}(\Delta_M)$has $T(M)$ as its connected component. And in general we can hope that partial toric pre-mirrors constructed here extend to complete mirror data.

The second is that we lack a general definition of the mirror maps q. The identity map of $T(L_N)$ (resp. $T(L_M)$) certainly is not the correct one; as examples suggest, it is "tangent" to the correct one.

Educated guesses about q in various situations were made in [22], [5], [9].

Addendum

(July 1994)

This report was written about a year ago. This version is only slightly revised and corrected.

Here is a list of some new results related to the questions discussed in the paper.

Counting points. E. Peyre [24] formulated a fairly precise conjecture about the constant c in (1.5) for anticanonical heights. He defined a Tamagawa measure that depends on a choice of the anticanonical height; the relevant Tamagawa number is the main ingredient of his constant. He has verified his prediction for certain small blow ups. He has also checked that it agrees with previous calculations for generalized flag varieties and the singular series for complete intersections furnished by the circle method. One remaining indeterminacy concerns the contribution of the Brauer group and/or more general obstructions of local–to–global type.

P. Salberger (paper in preparation) has shown that $\mathbf{P}^2$ with four blown up points over $\mathbf{Q}$ and deleted exceptional curves has $O(H(\log H)^4)$ points of height $\leq H$. His method is a refinement of that in [18]. A very careful strategy of estimates allows him to save one logarithm; unfortunately, it falls short of giving an asymptotic formula.

V. V. Batyrev and Yu. Tschinkel (paper in preparation) established the expected analytic properties of the height zeta function of toric varieties, at least for anisotropic tori. They developed a generalization of the Tate method which proved to be very efficient for studying this problem. In particular their constant has the same general structure as Peyre's one, with clearly visible contribution from the local–to–global obstructions.

Counting curves. An axiomatic treatment of the so called Gromov–Witten classes which is the mathematical basis of curve counting is given in

M. Kontsevich, Yu. Manin. *Gromov–Witten classes, quantum cohomology, and enumerative geometry.* Preprint MPI, 1994 (to appear in Comm. Math. Phys.)

This paper also contains a detailed discussion of the Fano case, which we omitted here concentrating on the Calabi–Yau varieties.

The existence theorems for Gromov–Witten classes in the context of symplectic geometry are proved in

Y. Ruan, G. Tian. *Mathematical theory of quantum cohomology.* Preprint, 1994.

See also

A. Givental, B. Kim. *Quantum cohomology of flag manifolds and Toda lattices.* Preprint hep–th/9312096

M. Kontsevich developed a very promising algebro–geometric approach to the curve counting and derived precise formulas in

M. Kontsevich. *Enumeration of rational curves via torus actions.* Preprint MPI, 1994.

References

[1] Aspinwall P., Morrison D. *Topological field theory and rational curves.* Comm. Math. Phys., 151 (1993), 245-262.

[2] Batyrev V.V. *Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties.* Essen University preprint, 1992.

[3] Batyrev V.V. *Variation of mixed Hodge structures of affine hypersurfaces in algebraic tori.* Essen University preprint, 1992.

[4] Batyrev V.V., Manin Yu.I. *Sur le nombre des points rationnels de hauteur bornée des variétés algébriques.* Math. Ann., Bd. 286 (1990), 27–43.

[5] Batyrev V.V., van Straten D. *Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties.* Preprint Essen University, 1993.

[6] Batyrev V.V. *Quantum cohomology rings of toric manifolds.* Preprint MSRI, 1993.

[7] Bershadsky M., Cecotti S., Ooguri H., Vafa C. *Holomorphic anomalies in topological field theories.* Preprint HUTP-93/A008.

[8] Bershadsky M., Cecotti S., Ooguri H., Vafa C. *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes.* Preprint HUTP-93/A025.

[9] Candelas P., de la Ossa X., Green P.S., Parkes L. *A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory.* Nucl. Phys ., B359 (1991), 21–74.

[10] Demazure M. *Automorphismes et déformations des variétés de Borel.* Inv. Math., 39 (1977), 179–186.

[11] Ellingsrud G., Stromme S.A. *The number of twisted cubic curves on the general quintic threefold.* In: [12], 181–240.

[12] *Essays on mirror manifolds.* Ed. by Sh.-T. Yau. International Press, Hong Kong, 1992.

[13] Franke J., Manin Yu.I., Tschinkel Yu. *Rational points of bounded height on Fano varieties.* Inv. Math. 95 (1989), 421–435.

[14] Katz Sh. *On the finiteness of rational curves on quintic threefolds.* Comp. Math., 60 (1986), 151–162.

[15] Katz Sh. *Rational curves on Calabi–Yau threefolds.* In: [12], 168–180.

[16] Kontsevich M. *A_∞-algebras in mirror symmetry.* Talk at the Bonn Arbeitstagung, 1993.

[17] Libgober A., Teitelbaum J. *Lines on Calabi–Yau complete intersections, mirror symmetry, and Picard–Fuchs equations.* Duke MJ, Int. Math. Res. Notices, 1(1993), 29.

[18] Manin Yu.I., Tschinkel Yu. *Points of bounded height on del Pezzo surfaces.* Comp. Math., 85 (1993), 315–332.

[19] Manin Yu.I. *Notes on the arithmetic of Fano threefolds.* Comp. Math., 85 (1993), 37–55.

[20] Mori S. *Threefolds whose canonical bundles are not numerically effective.* Ann. of Math., 116 (1982), 133–176.

[21] Morrison D. *Mirror symmetry and rational curves on quintic 3-folds: A guide for mathematicians.* Duke University preprint, 1991.

[22] Morrison D. *Picard-Fuchs equations and mirror maps for hypersurfaces.* In: [12], 241–264.

[23] Oda T., Park H.S. *Linear Gale transform and Gelfand-Kapranov-Zelevinsky decomposition.* Tôhoku Math. Journ., 43 (1991), 375–399.

[24] Peyre E. *Hauteurs et mesures de Tamagawa sur les variétés de Fano.* Max-Planck-Inst. Preprint 1993.

[25] Schmidt W.M. *The density of integer points on homogeneous varieties.* Acta Math., 154 (1985), 243–296.

[26] Witten E. *Mirror manifolds and topological field theory.* In: [12], 265–278.