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THE INDEX OF REPRESENTATIONS ASSOCIATED WITH STABILISERS

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ABSTRACT. Let Q be an algebraic group with Lie algebra $\mathfrak q$ and V a finite-dimensional Q-module. The index of V, denoted $\operatorname{ind}(\mathfrak q, V)$, is the minimal codimension of the Q-orbits in the dual space V^*. By Vinberg's inequality, $\operatorname{ind}(\mathfrak q, V^*) \leqslant \operatorname{ind}(\mathfrak q_v, (V/\mathfrak q{\cdot}v)^*)$ for any $v \in V$. In this article, we study conditions that guarantee us the equality. In case of reductive group actions, we show that it suffices to test only nilpotent elements in V. It was recently proved by J.-Y. Charbonnel that the equality for indices holds for the adjoint representation of a semisimple group. Another proof for the classical series was given by the second author. One of our goals is to understand what is going on in the case of isotropy representations of symmetric spaces.

INTRODUCTION

The ground field $\Bbbk$ is algebraically closed and of characteristic zero. For any finite-dimensional representation $\rho : \mathfrak q \to \mathfrak{gl}(V)$ of a Lie algebra $\mathfrak q$, one can define a non-negative integer which is called the *index* of (the $\mathfrak q$-module) V. Namely, if V^* is the dual $\mathfrak q$-module, then

$$\operatorname{ind}(\mathfrak q, V) = \dim V - \max_{\xi \in V^*}(\dim \mathfrak q{\cdot}\xi)$$

Here $\mathfrak q{\cdot}\xi = \{s{\cdot}\xi \mid s \in \mathfrak q\}$ and $s{\cdot}\xi$ is a shorthand for $\rho^*(s)\xi$. This definition goes back to RAÏS [11]. Let $\mathfrak q_v$ denote the stationary subalgebra of $v \in V$. For any $v \in V$, we can form the $\mathfrak q_v$-module $V/\mathfrak q{\cdot}v$. It was noticed by VINBERG that one always have the inequality

$$\operatorname{ind}(\mathfrak q, V^*) \leqslant \operatorname{ind}(\mathfrak q_v, (V/\mathfrak q{\cdot}v)^*)\ . \tag{0.1}$$

The goal of this paper is to study conditions that guarantee us the equality. If V is the coadjoint representations of $\mathfrak q$, then the above index is equal to the index of $\mathfrak q$ in the sense of Dixmier. Here Vinberg's inequality reads

$$\operatorname{ind}\mathfrak q \leqslant \operatorname{ind}\mathfrak q_\xi \quad \text{for any } \ \xi \in \mathfrak q^*\ .$$

It is not always true that $\operatorname{ind}\mathfrak q = \operatorname{ind}\mathfrak q_\xi$, see Example 1.1 below. However, it was conjectured by ELASHVILI that if $\mathfrak q = \mathfrak g$ is semisimple, then this equality always holds. It easily seen that it suffices to prove the equality $\operatorname{ind}\mathfrak g = \operatorname{ind}\mathfrak g_\xi$ only for the nilpotent elements $\xi \in \mathfrak g \simeq \mathfrak g^*$. The conjecture was recently proved by CHARBONNEL [3]. A proof for the classical Lie algebras, with weaker assumptions on the ground field, was found independently by the second author [14].

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One can consider two types of problems connected with Eq. (0.1). *First*, to find properties of v that guarantee the equality of the indices. *Second*, to describe representations such that (0.1) turns into equality for each $v \in V$.

We begin with pointing out two simple sufficient conditions. If either $\mathfrak{q}_v$ is reductive or $\dim \mathfrak{q}_v{\cdot}v$ is maximal, then Eq. (0.1) turns into equality. Let Q be a connected algebraic group with Lie algebra $\mathfrak{q}$. Given a representation $\rho : Q \to GL(V)$ (or $(Q:V)$ for short), we say that $(Q:V)$ has *good index behaviour* (GIB), if $\operatorname{ind}(\mathfrak{q}, V^*) = \operatorname{ind}(\mathfrak{q}_v, (V/\mathfrak{q}{\cdot}v)^*)$ for each $v \in V$. We prove that most of sufficiently large reducible representations have GIB. Namely, if V is any (finite-dimensional rational) Q-module, then mV has GIB for any $m \geqslant \dim V$. Another result of this sort asserts that if V is a Q-module having GIB and there is $\xi \in V^*$ such that $\mathfrak{q}_\xi = 0$, then $V \oplus W$ has GIB for any Q-module W. It is also easily seen that any representation of an algebraic torus has GIB.

Then we restrict ourselves to the case of reductive Lie algebras. Here one can use the rich machinery and various tools of Invariant Theory. Let G be a connected reductive group with Lie algebra $\mathfrak{g}$. Given a representation $\rho : G \to GL(V)$ (or $(G:V)$ for short), we say that $(G:V)$ has *good nilpotent index behaviour* (GNIB), if the equality $\operatorname{ind}(\mathfrak{g}, V^*) = \operatorname{ind}(\mathfrak{g}_v, (V/\mathfrak{g}{\cdot}v)^*)$ holds for any <u>nilpotent</u> element $v \in V$. Using Luna's slice theorem, we prove that GIB is equivalent to that GNIB holds for any slice representation of $(G:V)$. Furthermore, we prove that if $(G:V)$ is *observable* (i.e., the number of nilpotent orbits is finite), then GNIB implies GIB. As is well-known, the adjoint representation of G is observable.

A related class of representations, with nice invariant-theoretic properties, consists of the isotropy representations of symmetric pairs. Since these representations are observable, it suffices to consider the property of having GNIB for them. Let (G, G_0) be a symmetric pair with the associated $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and the isotropy representation $(G_0 : \mathfrak{g}_1)$. Abusing notation, we will say that (G, G_0) has GNIB whenever the isotropy representation has. A down-to-earth description of GNIB in the context of isotropy representations is as follows. Let $e \in \mathfrak{g}_1$ be a nilpotent element, and $\mathfrak{g}_e = \mathfrak{g}_{e,0} \oplus \mathfrak{g}_{e,1}$. Then GNIB property for e means that the codimension of generic $G_{e,0}$-orbits in $(\mathfrak{g}_{e,1})^*$ equals the codimension of generic G_0-orbits in $\mathfrak{g}_1$, that is, the rank of the symmetric variety G/G_0. (By Vinberg's inequality, the first codimension cannot be less than the second one.) It turns out that the analogue of the Elashvili conjecture (= Charbonnel's theorem) does not always holds here, so that it is of interest to explicitly describe the isotropy representations having GNIB.

In Sections 3-5, we prove, using explicit matrix models, that the symmetric pairs (SL_n, SO_n), (SL_{2n}, Sp_{2n}), (Sp_{2n}, GL_n), and (SO_{2n}, GL_n) have GNIB. It is also shown that each symmetric pair of rank 1 has GNIB, see Section 7. On the other hand, we present a method of constructing isotropy representations without GNIB, which makes use of even nilpotent orbits of height 4. Combining this method with the slice method, we are able to prove that most of the remaining isotropy representations do not have GNIB, see Section 6. As a result of our analysis and explicit calculations for small rank cases, we get a complete answer for the isotropy representations related to the classical simple Lie algebras. The answer for $\mathfrak{sl}_n$ is given below.

Theorem 0.1. *Let (SL_n, G_0) be a symmetric pair. Then it has GNIB if and only if $\mathfrak{g}_0$ belong to the following list: (i) $\mathfrak{so}_n$, (ii) $\mathfrak{sp}_{2m}$ for $n = 2m$, (iii) $\mathfrak{sl}_m \times \mathfrak{sl}_{n-m} \times \mathfrak{t}_1$ with $m = 1, 2$, (iv) $\mathfrak{sl}_3 \times \mathfrak{sl}_3 \times \mathfrak{t}_1$ for $n = 6$. [Here $\mathfrak{t}_1$ stands for the Lie algebra of a one-dimensional torus.]*

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1. The index of a representation

Let $\mathfrak{q}$ be a Lie algebra and $\rho : \mathfrak{q} \to \mathfrak{gl}(V)$ a finite-dimensional representation of $\mathfrak{q}$, i.e., V is a $\mathfrak{q}$-module. Abusing notation, we write $s{\cdot}v$ in place of $\rho(s)v$, if $s \in \mathfrak{q}$ and $v \in V$. An element $v \in V$ is called *regular* or $\mathfrak{q}$-*regular* whenever its stationary subalgebra $\mathfrak{q}_v = \{s \in \mathfrak{q} \mid s{\cdot}v = 0\}$ has minimal dimension. Because the function $v \mapsto \dim \mathfrak{q}_v$ ($v \in V$) is upper semicontinuous, the set of all $\mathfrak{q}$-regular elements is open and dense in V.

Definition 1. The nonnegative integer

$$\dim V - \max_{\xi \in V^*}(\dim \mathfrak{q}{\cdot}\xi) = \dim V - \dim \mathfrak{q} + \min_{\xi \in V^*}(\dim \mathfrak{q}_\xi)$$

is called the *index* of (the $\mathfrak{q}$-module) V. It will be denoted by $\operatorname{ind}(\mathfrak{q}, V)$.

Notice that in order to define the index of V we used elements of the dual $\mathfrak{q}$-module V^*. This really makes a difference, since $\operatorname{ind}(\mathfrak{q}, V)$ is not necessarily equal to $\operatorname{ind}(\mathfrak{q}, V^*)$ unless $\mathfrak{q}$ is reductive.

In case $\mathfrak{q}$ is an algebraic Lie algebra, a more geometric description is available. Let Q be an algebraic group with Lie algebra $\mathfrak{q}$. Then $\operatorname{ind} \mathfrak{q} = \dim \mathfrak{q} - \max_{\xi \in \mathfrak{q}^*} \dim Q{\cdot}\xi$. By the Rosenlicht theorem [12], this number is also equal to $\operatorname{trdeg} \Bbbk(V^*)^Q$. Below, we always assume that $\mathfrak{q}$ is algebraic, and consider Q whenever it is convenient.

If $v \in V$, then $\mathfrak{q}{\cdot}v$ is a $\mathfrak{q}_v$-submodule of V. Geometrically, it is the tangent space of the orbit $Q{\cdot}v$ at v. Then $V_v := V/\mathfrak{q}{\cdot}v$ is a $\mathfrak{q}_v$-module as well. By Vinberg's Lemma (see [9, 1.6]), we have

$$\max_{x \in V} \dim(Q{\cdot}x) \geqslant \max_{\eta \in V_v} \dim(Q_v{\cdot}\eta) + \dim(Q{\cdot}v) \tag{1.1}$$

for any $v \in V$. It can be rewritten in equivalent forms:

$$\operatorname{trdeg} \Bbbk(V)^Q \leqslant \operatorname{trdeg} \Bbbk(V/\mathfrak{q}{\cdot}v)^{Q_v} \quad \text{or} \tag{1.2}$$

$$\min_{x \in V} \dim(Q_x) \leqslant \min_{\eta \in V_v} \dim((Q_v)_\eta) \quad \text{or} \tag{1.3}$$

$$\operatorname{ind}(\mathfrak{q}, V^*) \leqslant \operatorname{ind}(\mathfrak{q}_v, (V/\mathfrak{q}{\cdot}v)^*)\ . \tag{1.4}$$

It is then natural to look for conditions that guarantee us the equality This article is devoted to several aspects of the following problem

Problem. *When does the equality hold in Eq. (1.1)–(1.4) ?*

Every Lie algebra has a distinguished representation, namely, the adjoint one. The index of the adjoint representation of $\mathfrak{q}$ is called simply *the index of* $\mathfrak{q}$, denoted $\operatorname{ind}\mathfrak{q}$. That is, $\operatorname{ind}(\mathfrak{q},\mathfrak{q}) = \operatorname{ind}\mathfrak{q}$. Let us take $V = \mathfrak{q}^*$. Then $\mathfrak{q}^*/\mathfrak{q}{\cdot}\xi \simeq (\mathfrak{q}_\xi)^*$ for any $\xi \in \mathfrak{q}^*$. Therefore inequality (1.4) in this situation reads

$$\operatorname{ind}\mathfrak{q} \leqslant \operatorname{ind}\mathfrak{q}_\xi \quad \text{for any } \xi \in \mathfrak{q}^*. \tag{1.5}$$

The coadjoint representation has some interesting features. For instance, the Q-orbits in $\mathfrak{q}^*$ are symplectic manifolds. Hence $\operatorname{ind}\mathfrak{q}_\xi - \operatorname{ind}\mathfrak{q}$ is even for any $\xi \in \mathfrak{q}^*$. However, even in this situation the inequality (1.5) and hence (1.4) can be strict.

Example 1.1. Let $\mathfrak{q}$ be a Borel subalgebra of $\mathfrak{gl}_4$. It is well known that $\operatorname{ind}\mathfrak{q} = 2$, see e.g. [8, 4.9]. But there is a point $\xi \in \mathfrak{q}^*$ such that $\mathfrak{q}_\xi$ is a 4-dimensional commutative subalgebra, i.e., $\operatorname{ind}\mathfrak{q}_\xi = 4$. If $\mathfrak{q}$ is represented as the space of all upper-triangular matrices, then $\mathfrak{q}^* \simeq \mathfrak{gl}_4/[\mathfrak{q},\mathfrak{q}]$ can be identified with the space of all lower-triangular matrices. Then we take ξ to be the following matrix $\xi = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

Since the equality in Eq. (1.1)–(1.4) does not always holds, one has to impose some constraints on V and Q. We begin with the following simple assertion, which is well known to the experts.

Proposition 1.1. *Suppose Q_v is reductive. Then* $\operatorname{ind}(\mathfrak{q}, V^*) = \operatorname{ind}(\mathfrak{q}_v, (V/\mathfrak{q}{\cdot}v)^*)$.

Proof. In this case the Q_v-module V is completely reducible, so that there is a Q_v-stable complement of $\mathfrak{q}{\cdot}v$, say N_v. Let us form the associated fibre bundle $Z_v := Q *_{Q_v} N_v$. Recall that it is the (geometric) quotient of $Q \times N_v$ by the Q_v-action defined by $Q_v \times Q \times N_v \to Q \times N_v$, $(s,q,n) \mapsto (qs^{-1}, s{\cdot}n)$. The image of $(q,n) \in Q \times N_v$ in Z_v is denoted by $q * n$. Consider the natural Q-equivariant morphism $\psi : Z_v \to V$, $\psi(q*n) = q{\cdot}(v+n)$. By construction, ψ is étale in $e * v \in Z_v$. It follows that the maximal dimensions of Q-orbits in V and Z_v are the same, i.e., $\operatorname{trdeg}\Bbbk(Z_v)^Q = \operatorname{trdeg}\Bbbk(V)^Q$. It remains to observe that

$$\max_{z\in Z_v} \dim(Q{\cdot}z) = \max_{\eta\in N_v} \dim(Q_v{\cdot}\eta) + \dim(Q{\cdot}v)\ ,$$

which is a standard property of associated fibre bundles. □

For the sake of completeness, we mention the following obvious consequence of (1.1).

Proposition 1.2. *If the dimension of $Q{\cdot}v$ is maximal, then the action $(Q_v : V/\mathfrak{q}{\cdot}v)$ is trivial and the equality holds in (1.1).*

Definition 2. We say that the representation $(Q:V)$ has *good index behaviour* (*GIB*, for short), if the equality

$$\operatorname{ind}(\mathfrak{q},V^*) = \operatorname{ind}(\mathfrak{q}_v,(V/\mathfrak{q}{\cdot}v)^*) \tag{1.6}$$

holds for every $v\in V$. That is, inequality (1.1) or (1.4) always turns into equality. Another way is to say that $(Q:V)$ has GIB if and only if the function $v\mapsto \operatorname{trdeg}\Bbbk(V/\mathfrak{q}{\cdot}v)^{Q_v} = \operatorname{ind}(\mathfrak{q}_v,(V/\mathfrak{q}{\cdot}v)^*)$ is constant on V.

As an immediate consequence of Proposition 1.1, we obtain

Proposition 1.3. *Let Q be an algebraic torus. Then any Q-module has GIB.*

For an arbitrary Q, it is not easy to prove that V has (or has not) GIB. However, sufficiently "large" reducible representations always have GIB.

Theorem 1.4. *Let $\rho: Q\to GL(V)$ be an arbitrary linear representation and $\dim V=n$. Then the representation $(Q:mV^*)$ has GIB for any $m\geqslant n$. In this case,* $\operatorname{ind}(\mathfrak{q},mV^*) = nm-\dim\mathfrak{q}$.

Proof. Our plan is to prove first the assertions for $\mathfrak{q}=\mathfrak{gl}(V)$, and then deduce from this the general case.

1) Assume that $\mathfrak{q}=\mathfrak{gl}(V)$. It is clear that the generic stabiliser for $(\mathfrak{gl}(V):mV)$ is trivial for $m\geqslant n$, whence the equality for the index.

Let $\tilde v=(v_1,\dots,v_m)$ be an arbitrary element of mV. The rank of $\tilde v$, denoted $\operatorname{rk}\tilde v$, is the dimension of the linear span of the components v_i. If $\operatorname{rk}\tilde v = r\leqslant n$, then without loss of generality one may assume that $\tilde v=(v_1,\dots,v_r,0,\dots,0)$, where the vectors $v_1,\dots,v_r$ form the part of the standard basis for V. (Use the action of GL_m that permutes the coordinates of $\tilde v$.) Then $GL(V)_{\tilde v} = \begin{pmatrix} I_r & * \\ 0 & * \end{pmatrix}$ and $mV/\mathfrak{gl}(V){\cdot}\tilde v \simeq (m-r)V$. It is easily seen that $GL(V)_{\tilde v}$ has an orbit with trivial stabiliser in $mV/\mathfrak{gl}(V){\cdot}\tilde v$. This means that $n(m-n)=\operatorname{trdeg}\Bbbk(mV)^{GL(V)} = \operatorname{trdeg}\Bbbk(mV/\mathfrak{gl}(V){\cdot}\tilde v)^{GL(V)_{\tilde v}}$ for any $\tilde v$, as required.

2) If $Q\subset GL(V)$ is arbitrary and $\tilde v$ is as above, then $mV/\mathfrak{q}{\cdot}\tilde v\supset (m-r)V$ and $\mathfrak{q}_{\tilde v}\subset \mathfrak{gl}(V)_{\tilde v}$. Hence $Q_{\tilde v}$ also has an orbit in $mV/\mathfrak{q}{\cdot}\tilde v$ with trivial stabiliser. □

Theorem 1.5. *Let V be a Q-module having GIB such that* $\operatorname{ind}(\mathfrak{q},V^*)=\dim V-\dim\mathfrak{q}$. *Then for any Q-module W, $(Q:W\oplus V)$ has GIB and* $\operatorname{ind}(\mathfrak{q},V^*\oplus W^*) = \dim V+\dim W-\dim\mathfrak{q}$.

Proof. The assumption of having GIB and the equality for $\operatorname{ind}(\mathfrak{q},V^*)$ mean that for any $v\in V$ there is v_0 such that $\dim(\mathfrak{q}_v)_{\overline{v_0}}=0$, where $\overline{v_0}$ stands for the image of v_0 in $V/\mathfrak{q}{\cdot}v$. Our aim is to establish the similar property for $W\oplus V$. Let $w+v\in W\oplus V$ be an arbitrary vector. Then

$$(\mathfrak{q}_{v+w})_{\overline{(0,v_0)}}\subset(\mathfrak{q}_v)_{\overline{(0,v_0)}}\subset(\mathfrak{q}_v)_{\overline{v_0}}=\{0\},$$

where $\overline{(0,v_0)}$ the image of v_0 in $(W\oplus V)/\mathfrak{q}{\cdot}(w+v)$. Therefore

$$\operatorname{ind}(\mathfrak{q}_{v+w},(W\oplus V)/\mathfrak{q}{\cdot}(w+v)^*) = \dim(W\oplus V)-\dim\mathfrak{q}{\cdot}(w+v)-\dim\mathfrak{q}_{w+v} = \dim(W\oplus V)-\dim\mathfrak{q}.$$

Thus the function $v+w\mapsto \operatorname{ind}(\mathfrak{q}_{v+w},(W\oplus V)/\mathfrak{q}{\cdot}(w+v)^*)$ is constant, and we are done. □

Combining the above theorems, we obtain

Corollary 1.6. *If V_1, V_2 are arbitrary Q-modules and $m \geqslant \dim V_1$, then $mV_1 \oplus V_2$ has GIB.*

2. Representations of reductive groups having GIB and GNIB

Let G be a reductive algebraic group, and let $\rho : G \to GL(V)$ be a finite-dimensional rational representation of G. Recall that $v \in V$ is called *nilpotent*, if the closure of the orbit $G{\cdot}v$ contains the origin, i.e., $\overline{G{\cdot}v} \ni 0$. The set of all nilpotent elements is called the *nullcone* and is denoted by $\mathfrak{N}(V)$. Whenever we wish to stress that the nullcone depends on the group, we write $\mathfrak{N}_G(V)$. A vector v is said to be *semisimple*, if $\overline{G{\cdot}v} = G{\cdot}v$. If v is semisimple, then G_v is reductive, and therefore the tangent space $\mathfrak{g}{\cdot}v \subset V$ has a G_v-stable complement, say N_v. The natural representation $(G_v : N_v)$ is called the *slice representation* (associated with v). We also say that it is a slice representation of $(G{:}V)$. Notice that the initial representation itself can be regarded as the slice representation associated with $0 \in V$. In this general situation, there is an analogue of the *Jordan decomposition*, which is well known for the elements of $\mathfrak{g}$. That is, for any $v \in V$ there are a semisimple element v_s and a (nilpotent) element v_n such that

- $v = v_s + v_n$;
- $G_v \subset G_{v_s}$;
- v_n is nilpotent with respect to G_{v_s}, i.e., $\overline{G_{v_s}{\cdot}v_n} \ni 0$.

This readily follows from Luna's slice theorem [6]. Below, we recall how such a decomposition is being constructed. But, unlike the case of the adjoint representation, a decomposition with the above properties is not unique.
As usual, $V/\!/G := \operatorname{Spec} \Bbbk[V]^G$ is the *categorical quotient* and $\pi : V \to V/\!/G$ is the *quotient mapping*. Recall that $\mathfrak{N}_G(V) = \pi^{-1}\pi(0)$.

Definition 3. We say that the representation $(G : V)$ has *good nilpotent index behaviour* (*GNIB*, for short), if equality (1.6) holds for each *nilpotent* element $v \in V$.

First, we demonstrate that there are irreducible representations of reductive groups having no GNIB and thereby no GIB.

Example 2.1. Let $G = SL_2 \times SL_2$ and $V = R_3 \otimes R_1$. Here R_d stands for the simple SL_2-module of dimension $d+1$. Hence V is a simple G-module of dimension 8. Let us show that V has no GNIB. A generic stabiliser for this representation is finite, hence $\operatorname{ind}(\mathfrak{g}, V^*) = \dim V - \dim \mathfrak{g} = 2$. As usual, we regard R_d as the space of binary forms of degree d. Let (x^3, x^2y, xy^2, y^3) be a basis for R_3 and (u, z) a basis for R_1. Take $v = (x^3 + y^3) \otimes u$. It is easily seen that v is nilpotent. A direct computation shows that the identity component of G_v is 1-dimensional and unipotent. However, the $\mathfrak{g}_v$-module $V/\mathfrak{g}{\cdot}v$ is trivial (and 3-dimensional). Hence equality (1.6) does not hold for v.

Our next goal is to understand a relationship between GIB and GNIB. Clearly, if a representation has GIB, then it has GNIB as well. As for the converse, we have the following general criterion.

Theorem 2.1. *The representation $(G{:}V)$ has GIB if and only if every slice representation of $(G{:}V)$ has GNIB.*

Proof. Actually, we prove a more precise statement. Namely, suppose $v \in V$ is semisimple. Then equality (1.6) is satisfied for every $y \in \pi^{-1}(\pi(v))$ if and only if the slice representation $(G_v{:}N_v)$ has GNIB.

1. "If" part. By Luna's slice theorem, $\pi^{-1}(\pi(v)) \simeq G *_{G_v} \mathfrak{N}(N_v)$. Therefore we may assume that $y = v + x$, where $x \in \mathfrak{N}(N_v)$. This expression is just a Jordan decomposition for y, in the sense described above. By assumption, we know that for any $x \in \mathfrak{N}(N_v)$ the following holds:

$$\dim G_v{\cdot}x + \max_{\xi \in N_v/\mathfrak{g}_v\cdot x} \dim (G_v)_x{\cdot}\xi = \max_{z \in N_v} \dim G_v{\cdot}z \ . \tag{2.1}$$

Notice that $(G_v)_x = G_{v+x} = G_y$, since $y = v + x$ is a Jordan decomposition. We want to show that

$$\dim G{\cdot}y + \max_{\eta \in V/\mathfrak{g}\cdot y} \dim (G_y){\cdot}\eta = \max_{z \in V} \dim G{\cdot}z \ . \tag{2.2}$$

Again, since $y = v + x$ is a Jordan decomposition, we have $\dim G{\cdot}y = \dim G{\cdot}v + \dim G_v{\cdot}x$. The following assertion is one of the many consequences of Luna's slice theorem.

Lemma 2.2. *The G_y-modules $N_v/\mathfrak{g}_v{\cdot}x$ and $V/\mathfrak{g}{\cdot}y$ are isomorphic.*

Proof. First, we notice that both N_v and $\mathfrak{g}_v{\cdot}x$ are G_y-modules, since $G_y = G_v \cap G_x$. Hence the first quotient is also a G_y-module. Consider the G-equivariant morphism

$$\psi : G *_{G_v} N_v \longrightarrow V \ .$$

Recall that if $g{*}n \in G *_{G_v} N_v$ is an arbitrary point, then $\psi(g{*}n) := g{\cdot}(v + n)$. Hence $\psi(1{*}x) = y$. Set $\tilde{y} = 1{*}x$. It follows from the slice theorem that $G_{\tilde{y}} = G_y$ and

$$T_{\tilde{y}}(G *_{G_v} N_v)/T_{\tilde{y}}(G{\cdot}\tilde{y}) \to T_y V/T_v(G{\cdot}y) = V/\mathfrak{g}{\cdot}y$$

is a G_y-equivariant bijection. It remains to observe that the left-hand side is isomorphic to $N_v/\mathfrak{g}_v{\cdot}x$. □

Thus, it follows from Lemma 2.2 and the previous argument that the left-hand side of (2.2) can be transformed as follows

$$\begin{aligned}
&\dim G{\cdot}y + \max_{\eta \in V/\mathfrak{g}\cdot y} \dim G_y{\cdot}\eta = \\
&\qquad = \dim G{\cdot}v + \Big(\dim G_v{\cdot}x + \max_{\xi \in N_v/\mathfrak{g}_v\cdot x} \dim G_y{\cdot}\xi\Big) \overset{(2.1)}{=} \\
&\qquad\qquad \dim G{\cdot}v + \max_{z \in N_v} \dim G_v{\cdot}z \overset{Prop.\,1.1}{=} \max_{z \in V} \dim G{\cdot}z \ ,
\end{aligned}$$

which completes the proof of the "if" part.

2. "Only if" part. Notice that the previous argument can be reversed. □

In the light of the previous theorem, it is natural to ask the following natural

Question. Is it true that "GNIB" implies "GIB" for any representation of a reductive group?

We can give a partial answer to this question. Recall that a representation $(G:V)$ is said to be *observable* if the number of nilpotent orbits is finite. This implies that each fibre of π consists of finitely many orbits, see e.g. [5].

Theorem 2.3. *Suppose $(G:V)$ is observable. Then GNIB implies GIB.*

Proof. Assume that this is not the case, i.e., $(G:V)$ has GNIB but there is $v\in V$ such that $\overline{G{\cdot}v}\not\ni 0$ and Eq. (1.6) is not satisfied for v. We use the method of associated cones developed in [1, § 3]. The variety $\overline{\Bbbk^*(G{\cdot}v)}\cap\mathfrak{N}(V)$ is the *associated cone* of $G{\cdot}v$, denoted $\mathcal{C}(G{\cdot}v)$. It can be reducible, but each irreducible component is of dimension $\dim G{\cdot}v$. Let $G{\cdot}u$ be the orbit that is dense in an irreducible component of $\mathcal{C}(G{\cdot}v)$. Here we use the hypothesis that $(G:V)$ is observable. There is a morphism $\tau:\Bbbk\to\overline{\Bbbk^*(G{\cdot}v)}$ such that $\tau(\Bbbk\setminus\{0\})\subset\Bbbk^*(G{\cdot}v)$ and $\tau(0)=u$. Since $\dim G{\cdot}u=\dim G{\cdot}v$, this implies that

$$\lim_{t\to 0}\mathfrak{g}_{\tau(t)}=\mathfrak{g}_u \ \text{ and } \ \lim_{t\to 0}\mathfrak{g}{\cdot}\tau(t)=\mathfrak{g}{\cdot}u. \tag{2.3}$$

These two limits are taken in the suitable Grassmannians. By the assumption, we have $\operatorname{ind}(\mathfrak{g}_{\tau(t)},(V/\mathfrak{g}{\cdot}\tau(t))^*)<\operatorname{ind}(\mathfrak{g},V^*)$ for any $t\neq 0$. In other words,

$$\max_{\eta\in V/\mathfrak{g}\cdot\tau(t)}\dim G_{\tau(t)}{\cdot}\eta<\max_{z\in V}\dim G{\cdot}z-\dim G{\cdot}v\ .$$

We claim that $\max_{\zeta\in V/\mathfrak{g}\cdot u}\dim G_u{\cdot}\zeta\leqslant\max_{\eta\in V/\mathfrak{g}\cdot\tau(t)}\dim G_{\tau(t)}{\cdot}\eta$. This follows from the upper semi-continuity of dimensions of orbits and Eq. (2.3). The inequality obtained means that Eq. (1.6) is not satisfied for the nilpotent element u. Hence $(G:V)$ has no GNIB, which contradicts the initial assumption. This completes the proof. □

We do not know of whether the statement of Theorem 2.3 remains true for arbitrary representations of G.

Now, we turn to considering the adjoint representation of a reductive group G. Here the condition of having GIB means that inequality (1.5) is, in fact, equality. Because now $\mathfrak{g}\simeq\mathfrak{g}^*$, one may deal with centralisers of elements in $\mathfrak{g}$. As above, we write $\mathfrak{g}_e$ for the centraliser of $e\in\mathfrak{g}$. The following fundamental result was conjectured by Elashvili at the end of 1980's and is recently proved by Charbonnel [3].

Theorem 2.4. *(Charbonnel) The adjoint representation of a reductive group G has GNIB. In other words, if $e\in\mathfrak{g}$ is a nilpotent element, then* $\operatorname{ind}\mathfrak{g}_e=\operatorname{rk}\mathfrak{g}$.

In [14], this theorem is independently proved for the classical Lie algebras. Some partial results for "small" orbits were obtained earlier in [9] and [10].

A remarkable fact is that, for the adjoint representation, each slice representation is again the adjoint representation (of a centraliser). Hence Theorems 2.1 and 2.4 readily imply that the adjoint representation has GIB. Another way to deduce GIB is to refer to Theorems 2.3 and 2.4, and the fact that the adjoint representation is observable.

From the invariant-theoretic point of view, adjoint representations have the best possible properties. Isotropy representations of symmetric spaces form a class with close properties. So, it is natural to inquire whether these representations have GIB and GNIB. Recall the necessary setup.

Let σ be an involutory automorphism of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the direct sum of σ-eigenspaces. Here $\mathfrak{g}_0$ is a reductive subalgebra and $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module. Write G_0 for the connected subgroup of G with Lie algebra $\mathfrak{g}_0$. With this notation, our object of study is $(G_0{:}\mathfrak{g}_1)$, the *isotropy representation* of the symmetric pair (G, G_0). By [4], these representations are observable, so that Theorem 2.3 applies. Therefore we will not distinguish the properties GIB and GNIB in the context of isotropy representations of symmetric pairs. In the rest of the paper, we deal with the following

Problem. For which involutions σ the representation $(G_0{:}\mathfrak{g}_1)$ has GNIB ?

For future use, we record the following result.

Lemma 2.5. *Let $\mathfrak{q} = \mathfrak{q}_0 \oplus \mathfrak{q}_1$ be an arbitrary $\mathbb{Z}_2$-graded Lie algebra and $\mathfrak{q}^* = \mathfrak{q}_0^* \oplus \mathfrak{q}_1^*$ the corresponding decomposition of the dual space. For any $\xi \in \mathfrak{q}_1^*$ the stationary subalgebra $\mathfrak{q}_\xi$ posesses the induced $\mathbb{Z}_2$-grading and* $\dim \mathfrak{q}_0 - \dim \mathfrak{q}_1 = \dim \mathfrak{q}_{\xi,0} - \dim \mathfrak{q}_{\xi,1}$.

Proof. This claim is well known if $\mathfrak{q}$ is reductive and one identifies $\mathfrak{q}$ and $\mathfrak{q}^*$, see [4, Prop. 5]. The general proof is essentially the same. □

Let us give an interpretation of GNIB for the isotropy representations, which is helpful in practical applications. It is known that $x \in \mathfrak{g}_1$ is nilpotent in the sense of the above definition (i.e., as an element of the G_0-module $\mathfrak{g}_1$) if and only if it is nilpotent as an element of $\mathfrak{g}$. Formally, $\mathfrak{N}_{G_0}(\mathfrak{g}_1) = \mathfrak{N}_G(\mathfrak{g}) \cap \mathfrak{g}_1$. If $e \in \mathfrak{N}(\mathfrak{g}_1)$, and $\mathfrak{g}_e = \mathfrak{g}_{e,0} \oplus \mathfrak{g}_{e,1}$ is the induced $\mathbb{Z}_2$-grading, then $\mathfrak{g}_{e,0}$ is precisely the stationary subalgebra of e in $\mathfrak{g}_0$. Now, inequality (1.4) reads

$$\mathrm{ind}\,(\mathfrak{g}_0, (\mathfrak{g}_1)^*) \leqslant \mathrm{ind}\,(\mathfrak{g}_{e,0}, (\mathfrak{g}_1/[\mathfrak{g}_0, e])^*) \ .$$

Using a G-invariant inner product on $\mathfrak{g}$, one easily shows that $\mathfrak{g}_1/[\mathfrak{g}_0, e] \simeq (\mathfrak{g}_{e,1})^*$. Recall also that $\mathfrak{g}_1$ is an orthogonal G_0-module, i.e., $G_0 \to SO(\mathfrak{g}_1)$. The number $\mathrm{ind}\,(\mathfrak{g}_0, (\mathfrak{g}_1)^*) = \mathrm{ind}\,(\mathfrak{g}_0, \mathfrak{g}_1)$ equals the Krull dimension of the invariant ring $\Bbbk[\mathfrak{g}_1]^{G_0}$, which in turn is equal to the rank of G/G_0 (in the sense of the theory of symmetric varieties). Thus, we obtain

Proposition 2.6.

1. *For any $e \in \mathfrak{N}(\mathfrak{g}_1)$, we have* $\mathrm{rk}\,(G/G_0) = \mathrm{ind}\,(\mathfrak{g}_0, \mathfrak{g}_1) \leqslant \mathrm{ind}\,(\mathfrak{g}_{e,0}, \mathfrak{g}_{e,1})$.
2. *The following conditions are equivalent:*

(i) *The isotropy representation* $(G_0{:}\mathfrak{g}_1)$ *has GNIB;*
(ii) *for any* $e \in \mathfrak{N}(\mathfrak{g}_1)$ *we have* $\mathrm{rk}\,(G/G_0) = \mathrm{ind}\,(\mathfrak{g}_{e,0}, \mathfrak{g}_{e,1})$*;*
(iii) *for any* $e \in \mathfrak{N}(\mathfrak{g}_1)$ *there is an* $\alpha \in \mathfrak{g}_e^*$ *such that* $\alpha(\mathfrak{g}_{e,0}) = 0$ *and* $\dim(\mathfrak{g}_{e,1})_\alpha = \mathrm{rk}\,(G/G_0)$.

Proof. Part 1 and the equivalence of (i) and (ii) follow from the previous discussion. To prove the equivalence of (ii) and (iii), we note that if $\alpha(\mathfrak{g}_{e,0}) = 0$, then α can be regarded as an element of $(\mathfrak{g}_{e,1})^*$. Then

$$\mathrm{codim}\,\mathfrak{g}_{e,0}{\cdot}\alpha = \dim\mathfrak{g}_{e,1} - \dim\mathfrak{g}_{e,0} + \dim(\mathfrak{g}_{e,0})_\alpha \overset{\textit{Lemma } 2.5}{=} \dim(\mathfrak{g}_{e,1})_\alpha.$$

Hence, $\mathrm{ind}\,(\mathfrak{g}_{e,0}, \mathfrak{g}_{e,1}) = \min\dim(\mathfrak{g}_{e,1})_\alpha$, where minimum is taken over all $\alpha \in \mathfrak{g}_e^*$ such that $\alpha(\mathfrak{g}_{e,0}) = 0$. $\square$

Below, we show that there are isotropy representations with and without GNIB.

3. Isotropy representations for the outer involutions of $\mathfrak{gl}(V)$

Let V be a finite-dimensional vector space over $\Bbbk$. If σ is an outer involution of $\mathfrak{sl}(V)$, then $\mathfrak{g}_0$ is isomorphic to either $\mathfrak{sp}(V)$ or $\mathfrak{so}(V)$. Of course, the first case is only possible if $\dim V$ is even. It will technically be easier to deal with $\mathfrak{g} = \mathfrak{gl}(V)$ and assume that the centre of $\mathfrak{gl}(V)$ lies in $\mathfrak{g}_1$. Then the $Sp(V)$-module $\mathfrak{g}_1$ is isomorphic to $\wedge^2 V$ and the $SO(V)$-module $\mathfrak{g}_1$ is isomorphic to $\mathcal{S}^2 V$. The goal of this section is to prove that the isotropy representations $(Sp(V) : \wedge^2 V)$ and $(SO(V) : \mathcal{S}^2 V)$ have GIB.

Recall the necessary set-up. Let $(\ ,\ )$ be a non-degenerate symmetric or skew-symmetric form on V; that is, $(v,w) = \varepsilon(w,v)$, where $v,w \in V$ and $\varepsilon = +1$ or -1. Let J denote the matrix of $(\,,)$ with respect to some basis of V. Then $(v,w) = v^t J w$, where v,w are regarded as column vectors and the symbol $(\ )^t$ stands for the transpose. Since $J^t = \pm J$, the mapping $A \to \sigma(A) := -J^{-1}A^tJ$ is an involution of $\mathfrak{gl}(V)$. Let $\mathfrak{gl}(V) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the corresponding $\mathbb{Z}_2$-grading. Here $\mathfrak{g}_0$ consists of the linear transformations preserving the form $(\,,)$, i.e., satisfying the property $(vA,w) = -(v,Aw)$ for all $v,w \in V$. The elements of $\mathfrak{g}_1$ multiply the form $(\ ,\ )$ by -1, i.e.,

$$(Av,w) = (v,Aw) \quad \text{for all } A \in \mathfrak{g}_1 \text{ and } v,w \in V. \tag{3.1}$$

Recall standard facts concerning nilpotent elements in $\mathfrak{g} = \mathfrak{gl}(V)$, mainly in order to fix the notation.

Let $e \in \mathfrak{g}$ be a nilpotent element and $m = \dim\mathrm{Ker}\,(e)$. By the theory of Jordan normal form, there are vectors $w_1,\dots,w_m \in V$ and non-negative integers $d_1,\dots,d_m$ such that $e^{d_i+1}{\cdot}w_i = 0$ and $\{e^s{\cdot}w_i \mid 1 \leqslant i \leqslant m,\ 0 \leqslant s \leqslant d_i\}$ is a basis for V. Set $V_i = \langle w_i, e{\cdot}w_i, \dots, e^{d_i}{\cdot}w_i\rangle$ and $W = \langle w_1,\dots,w_m\rangle$. Then $V = \oplus_{i=1}^m V_i$ and $V = W \oplus \mathrm{Im}\,(e)$. The spaces $\{V_i\}$ are called the Jordan (or *cyclic*) spaces of the nilpotent element e.

Suppose $\varphi \in \mathfrak{g}_e$. Because $\varphi(e^s{\cdot}w_i) = e^s{\cdot}\varphi(w_i)$, the linear map φ is determined by its values on W. In other words, if

$$\varphi(w_i) = \sum_{j,s} c_i^{j,s}(e^s{\cdot}w_j), \text{ where } c_i^{j,s} \in \Bbbk\,,$$

then φ is determined by the coefficients $c_i^{j,s} = c_i^{j,s}(\varphi)$. In what follows, we will only indicate the values of φ on the cyclic vectors $\{w_i\}$.

A basis for $\mathfrak{g}_e$ consists of the maps $\{\xi_i^{j,s}\}$ given by

$$\xi_i^{j,s} : \begin{cases} w_i & \mapsto & e^s{\cdot}w_j \\ w_t & \mapsto & 0 \qquad \text{if } t \neq i \end{cases}, \quad \text{where} \quad 1 \leqslant i,j \leqslant m \ \text{ and } \ \max\{d_j - d_i, 0\} \leqslant s \leqslant d_j \ .$$

Lemma 3.1. *In the above setting, suppose that $e \in \mathfrak{N}(\mathfrak{g}_1)$. Then the cyclic vectors $\{w_i\}$ and thereby the spaces $\{V_i\}$ can be chosen such that the following conditions are satisfied:*

- (i) *If $\varepsilon = -1$, then the set $\{1, 2, \dots, m\}$ can be partitioned in pairs (i, i^*) such that $d_i = d_{i*}$ and w_i is orthogonal to all basis vectors $e^s{\cdot}w_j$ except $e^{d_i}{\cdot}w_{i^*}$. (*Here *$i \neq i^*$.)*
- (ii) *If $\varepsilon = 1$, then $(V_i, V_j) = 0$ for $i \neq j$ and the restriction of $(\ ,\ )$ to each V_i is non-degenerate.*

Proof. We argue by induction on $m = \dim \mathrm{Ker}\,(e)$.

It follows from Eq. (3.1) that $\mathrm{Ker}\,(e^i)$ and $\mathrm{Im}\,(e^i)$ are orthogonal with respect to $(\ ,\ )$. In particular, $\mathrm{Ker}\,(e)$ is orthogonal to $\mathrm{Im}\,(e)$, and $(\ ,\ )$ induces a non-degenerate pairing between W and $\mathrm{Ker}\,(e)$. Suppose $d_1 = \min_i\{d_i\}$. There is a vector $e^{d_i}{\cdot}w_i \in \mathrm{Ker}\,(e)$ for some i such that $(w_1, e^{d_i}w_i) \neq 0$. Then $d_i \leqslant d_1$, hence $d_i = d_1$ in view of the minimality of d_1.
The rest of the argument splits.

(i) In the symplectic case ($\varepsilon = -1$), we have

$$(w_1, e^{d_1}{\cdot}w_1) = (e^{d_1}{\cdot}w_1, w_1) = -(w_1, e^{d_1}{\cdot}w_1) = 0.$$

Hence, $i \neq 1$. It is easily verified that the restriction of $(\ ,\ )$ to either V_1 or V_i is zero, while the restriction to $V_1 \oplus V_i$ is non-degenerate. Therefore, we may take $1^* = i$. Then all other w_j can be chosen in $(V_1 \oplus V_i)^\perp$, the e-invariant orthogonal complement to $V_1 \oplus V_i$.

(ii) Consider the orthogonal case ($\varepsilon = 1$). If $i = 1$, then the restriction of $(\ ,\ )$ to V_1 is non-degenerate and we may choose the remaining cyclic vectors in $V_1^\perp$. If $i \neq 1$ and $(\ ,\ )$ is degenerate on both V_1 and V_i, then we make the following modification of w_1 and w_i. Our assumption implies that $(w_1, e^{d_1}{\cdot}w_1) = 0$ and $(w_i, e^{d_1}{\cdot}w_i) = 0$. Set $w_1' := w_1 + w_i$ and $w_i' := w_1 - w_i$. Then $(w_1', e^{d_1}{\cdot}w_1') = 2(w_1, e^{d_1}{\cdot}w_i) \neq 0$ and $(w_i', e^{d_1}{\cdot}w_i') = -2(w_1, e^{d_1}{\cdot}w_i) \neq 0$. This means that the restriction of $(\ ,\ )$ to the cyclic space generated by either w_1' or w_i' is non-degenerate. □

Theorem 3.2. *The representation $(SO(V) : \mathcal{S}^2V)$ has GNIB.*

Proof. Here $\mathrm{rk}\,(G/G_0) = \dim V$. Let $e \in \mathfrak{N}(\mathfrak{g}_1)$. Recall that σ induces the decomposition $\mathfrak{g}_e = \mathfrak{g}_{e,0} \oplus \mathfrak{g}_{e,1}$. We choose the cyclic vectors for e as described in Lemma 3.1(ii). Define $\alpha \in (\mathfrak{g}_e)^*$ by

$$\alpha(\varphi) = \sum_{i=1}^{m} a_i c_i^{i,d_i},$$

where $c_i^{j,s}$ are the coefficients of φ and $\{a_i\}$ are pairwise different non-zero numbers. Then $(\mathfrak{g}_e)_\alpha$ consists of all maps in $\mathfrak{g}_e$ preserving the Jordan spaces V_i [14, Sect. 2], i.e.,

$$(\mathfrak{g}_e)_\alpha = \langle \xi_i^{i,s} \mid 1 \leqslant i \leqslant m,\ 0 \leqslant s \leqslant d_i \rangle\ ,$$

where $\langle \dots \rangle$ denotes the $\Bbbk$-linear span. Hence $\dim(\mathfrak{g}_e)_\alpha = \sum_i (d_i+1) = \dim V$. We claim that $\alpha(\mathfrak{g}_{e,0}) = 0$. Indeed, assume the converse, i.e., $\varphi = \sum c_i^{j,s}\xi_i^{j,s} \in \mathfrak{g}_{e,0}$ and $\alpha(\varphi) \neq 0$. This means that $c_i^{i,d_i} \neq 0$ for some i. Then

$$(\varphi(w_i), w_i) = (w_i, \varphi(w_i)) = c_i^{i,d_i}(w_i, e^{d_i}{\cdot}w_i) \neq 0,$$

which in view of Eq.(3.1) contradicts the fact that $\varphi \in \mathfrak{g}_{e,0}$. Hence $\alpha \in (\mathfrak{g}_{e,1})^* \subset (\mathfrak{g}_e)^*$. By Lemma 2.5, we have

$$\dim(\mathfrak{g}_{e,1})_\alpha - \dim(\mathfrak{g}_{e,0})_\alpha = \dim \mathfrak{g}_{e,1} - \dim \mathfrak{g}_{e,0} = \dim \mathfrak{g}_1 - \dim \mathfrak{g}_0 = \dim V\ .$$

On the other hand,

$$\dim(\mathfrak{g}_{e,1})_\alpha + \dim(\mathfrak{g}_{e,0})_\alpha = \dim(\mathfrak{g}_e)_\alpha = \dim V.$$

Hence $\dim(\mathfrak{g}_{e,1})_\alpha = \dim V = \mathrm{rk}\, G/G_0$. Thus, $(SO(V) : \mathcal{S}^2 V)$ has GNIB in view of Proposition 2.6(2). □

Theorem 3.3. *The representation $(Sp(V) : \wedge^2 V)$ has GNIB.*

Proof. Put $\dim V = 2n$. It is well known that $\mathrm{rk}\,(G/G_0) = \dim V/2 = n$.
Let $e \in \mathfrak{N}(\mathfrak{g}_1)$. We choose the cyclic vectors for e as described in Lemma 3.1(i). By Eq. (3.1), we have $(w_i, e^{d_i}{\cdot}w_{i^*}) = (e^{d_i}{\cdot}w_i, w_{i^*}) = -(w_{i^*}, e^{d_i}{\cdot}w_i)$. This implies that $\xi_i^{i,d_i} + \xi_{i^*}^{i^*,d_i} \in \mathfrak{g}_{e,1}$ and $\xi_i^{i,d_i} - \xi_{i^*}^{i^*,d_i} \in \mathfrak{g}_{e,0}$ for each i.

Define $\alpha \in (\mathfrak{g}_e)^*$ by

$$\alpha(\varphi) = \sum_{i=1}^{m} a_i c_i^{i,d_i},$$

where $c_i^{j,s}$ are the coefficients of φ and $\{a_i\}$ are non-zero numbers such that $a_i = a_j$ if and only if $i = j^*$. A direct computation shows that $(\mathfrak{g}_e)_\alpha$ consists of all elements of $\mathfrak{g}_e$ preserving the subspaces $V_i \oplus V_{i^*}$ for each pair (i, i^*), cf. [14, Sect. 2]. More concretely,

$$(\mathfrak{g}_e)_\alpha = \langle \xi_i^{i,s} \mid 1 \leqslant i \leqslant m,\ 0 \leqslant s \leqslant d_i \rangle \oplus \langle \xi_i^{i^*,s} \mid 1 \leqslant i \leqslant m,\ 0 \leqslant s \leqslant d_i \rangle\ .$$

Hence, $\dim(\mathfrak{g}_e)_\alpha = 2\dim V = 4n$. As in the proof of Theorem 3.2, one can show that $\alpha|_{\mathfrak{g}_{e,0}} = 0$. Therefore α can be regarded as an element of $(\mathfrak{g}_{e,1})^* \subset (\mathfrak{g}_e)^*$. Using Lemma 2.5, we obtain

$$\dim(\mathfrak{g}_{e,0})_\alpha - \dim(\mathfrak{g}_{e,1})_\alpha = \dim \mathfrak{g}_{e,0} - \dim \mathfrak{g}_{e,1} = \dim \mathfrak{g}_0 - \dim \mathfrak{g}_1 = 2n\ .$$

It follows that $\dim(\mathfrak{g}_{e,0})_\alpha = 3n$. and $\dim(\mathfrak{g}_{e,1})_\alpha = n$. By Proposition 2.6(2), we conclude that $(Sp(V) : \wedge^2 V)$ has GNIB. □

In Section 6, we show that most of the isotropy representations associated with inner involutions of $\mathfrak{gl}(V)$ do not have GNIB.

4. THE ISOTROPY REPRESENTATION OF $(\mathfrak{sp}_{2n}, \mathfrak{gl}_n)$

In this section, $\dim V = 2n$, $\mathfrak{g} = \mathfrak{sp}(V)$, and $(\,,\,)$ is a $\mathfrak{g}$-invariant skew-symmetric form on V. Let σ be an involution of $\mathfrak{g}$ such that $\mathfrak{g}_0 \simeq \mathfrak{gl}_n$. This can explicitly be described as follows. Let $V = V_+ \oplus V_-$ be a Lagrangian decomposition of V. Then G_0 can be taken as the subgroup of $G = Sp(V)$ preserving this decomposition. Here $G_0 \simeq GL(V_+)$, $V_- \simeq (V_+)^*$ as G_0-module, and the G_0-module $\mathfrak{g}_1$ is isomorphic to $\mathcal{S}^2V_+ \oplus (\mathcal{S}^2V_+)^*$.

Keep the notation introduced in the previous section. In particular, V_i, $i = 1, \ldots, m$, are the Jordan spaces of $e \in \mathfrak{N}(\mathfrak{gl}(V))$, $\dim V_i = d_i + 1$, and $w_i \in V_i$ is a cyclic vector.

Lemma 4.1. *Let $e \in \mathfrak{N}(\mathfrak{g}_1)$. Then the cyclic vectors $\{w_i\}_{i=1}^m$ and hence the $\{V_i\}$'s can be chosen such that the following properties are satisfied:*

- (i) *there is an involution $i \mapsto i^*$ on the set $\{1, \ldots, m\}$ such that*
  - $d_i = d_{i^*}$,
  - *$i = i^*$ if and only if d_i is odd,*
  - *$(V_i, V_j) = 0$ if $i \neq j, j^*$;*
- (ii) $\sigma(w_i) = \pm w_i$.

The proof is left to the reader (cf. the proof of Lemma 3.1). Actually, part (i) is a standard property of nilpotent orbits in $\mathfrak{sp}(V)$. Then part (ii) says that in the presence of the involution σ the cyclic vectors for $e \in \mathfrak{N}(\mathfrak{g}_1)$ can be chosen to be σ-eigenvectors.

Theorem 4.2. *The representation $(GL(V_+) : \mathcal{S}^2V_+ \oplus (\mathcal{S}^2V_+)^*)$ has GNIB.*

Proof. Recall that $\mathfrak{sp}(V)$ is a symmetric subalgebra of $\widetilde{\mathfrak{g}} := \mathfrak{gl}(V)$. Let $\widetilde{\mathfrak{g}} = \mathfrak{sp}(V) \oplus \widetilde{\mathfrak{g}}_1$ be the corresponding $\mathbb{Z}_2$-grading. Then we have a hierarchy of involutions:

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \ \text{ and } \ \tilde{\mathfrak{g}}_0 = \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

Let $e \in \mathfrak{N}(\mathfrak{g}_1)$. In this case, we have $\mathrm{rk}\,(G/G_0) = n$. Hence, by Proposition 2.6 our goal is to find an element $\alpha \in (\mathfrak{g}_{e,1})^*$ such that $\dim(\mathfrak{g}_{e,1})_\alpha = n$. Let $\widetilde{\mathfrak{g}}_e$ and $\widetilde{\mathfrak{g}}_{e,1}$ denote the centraliser of e in $\widetilde{\mathfrak{g}}$ and $\widetilde{\mathfrak{g}}_1$, respectively. In view of the above hierarchy, we have

$$\tilde{\mathfrak{g}}_e = \mathfrak{g}_e \oplus \tilde{\mathfrak{g}}_{e,1} = \mathfrak{g}_{e,0} \oplus \mathfrak{g}_{e,1} \oplus \tilde{\mathfrak{g}}_{e,1}.$$

Choose the cyclic vectors for e as prescribed by Lemma 4.1. We normalise these vectors such that $(w_i, e^{d_i}{\cdot}w_i) = 1$ if $i = i^*$; and $(w_i, e^{d_i}{\cdot}w_{i^*}) = -(w_{i^*}, e^{d_i}{\cdot}w_i) = \pm 1$ if $i \neq i^*$. Then $\mathfrak{g}_e$ has a basis $\xi_i^{j,d_j-s} + \varepsilon(i,j,s)\xi_{j^*}^{i^*,d_i-s}$, where $\varepsilon(i,j,s) = \pm 1$ depending on i, j and s; and $\xi_i^{j,d_j-s} - \varepsilon(i,j,s)\xi_{j^*}^{i^*,d_i-s}$ form a basis for $\widetilde{\mathfrak{g}}_{e,1}$.

Define $\alpha \in (\tilde{\mathfrak{g}}_e)^*$ by

$$\alpha(\varphi) = \left(\sum_{i,\ i=i^*} a_i c_i^{i,d_i} \right) + \sum_{(i,i^*),\ i \neq i^*} a_i (c_i^{i^*,d_i} + c_{i^*}^{i,d_i}),$$

where $c_i^{j,s}$ are coefficients of $\varphi \in \widetilde{\mathfrak{g}}_e$, $a_i = a_{i^*}$, and $a_i \neq \pm a_j$ if $i \neq j, j^*$. The stationary subalgebra $(\widetilde{\mathfrak{g}}_e)_\alpha$ consists of all maps preserving cyclic spaces generated by w_i for $i = i^*$ and

$w_i + w_{i^*}$, $w_i - w_{i^*}$ for $i \neq i^*$. More precisely,

$$(\widetilde{\mathfrak{g}}_e)_\alpha = \langle \xi_i^{i,s} \mid 1 \leqslant i \leqslant m,\ i = i^*,\ 0 \leqslant s \leqslant d_i \rangle \oplus \\ \oplus \langle \xi_i^{i,s} + \xi_{i^*}^{i^*,s},\ \xi_i^{i^*,s} + \xi_{i^*}^{i,s} \mid 1 \leqslant i \leqslant m,\ i \neq i^*,\ 0 \leqslant s \leqslant d_i \rangle\ .$$

First, we show that $\alpha(\widetilde{\mathfrak{g}}_{e,1}) = 0$. Assume that $\alpha(\xi_i^{j,d_j-s} - \varepsilon(i,j,s)\xi_{j^*}^{i^*,d_i-s}) \neq 0$ for some $\xi_i^{j,d_j-s} - \varepsilon(i,j,s)\xi_{j^*}^{i^*,d_i-s} \in \widetilde{\mathfrak{g}}_{e,1}$. Then $j = i^*$, $s = 0$ and $\varepsilon(i,i^*,0) = -1$. But $\xi_i^{i^*,d_i} \in \mathfrak{g}$ for all i, i.e., $\varepsilon(i,i^*,0) = 1$. Thus $\alpha(\widetilde{\mathfrak{g}}_{e,1}) = 0$ and, hence, $(\mathfrak{g}_e)_\alpha = \mathfrak{g} \cap (\widetilde{\mathfrak{g}}_e)_\alpha$.

Suppose $i \neq i^*$. Then $\xi_i^{i,s} + \xi_{i^*}^{i^*,s} \in \mathfrak{g}$ if and only if s is odd; and $\xi_i^{i^*,s} + \xi_{i^*}^{i,s} \in \mathfrak{g}$ if and only if s is even. Suppose now that $i = i^*$. Then $\xi_i^{i,s} \in \mathfrak{g}$ if and only if s is odd. Summing up, we get $\dim(\mathfrak{g}_e)_\alpha = \frac{1}{2}(\sum\limits_{i=i^*} d_i) + \sum\limits_{(i,i^*),\ i\neq i^*} d_i = n$.

Next, we show that $\alpha(\mathfrak{g}_{e,0}) = 0$. Since $\sigma(e^s{\cdot}w_i) = (-1)^s e^s{\cdot}\sigma(w_i) = \pm e^s{\cdot}w_i$, all vectors $\{e^s{\cdot}w_i\}$ are eigenvectors of σ. Hence, $\sigma(\xi_i^{j,s}) = \pm\xi_i^{j,s}$. Suppose $i \neq i^*$ and $\sigma(w_i) = w_i$. Then $\sigma(e^{d_i}{\cdot}w_i) = e^{d_i}w_i$ and, since $(e^{d_i}{\cdot}w_i, w_{i^*}) \neq 0$, we get $\sigma(w_{i^*}) = -w_{i^*}$, $\sigma(e^{d_i}{\cdot}w_{i^*}) = -e^{d_i}{\cdot}w_{i^*}$. Thus $\xi_i^{i^*,d_i}, \xi_{i^*}^{i,d_i} \in \mathfrak{g}_{e,1}$. In case $i = i^*$, d_i is odd, $\sigma(\xi_i^{i,d_i}) = -\xi_i^{i,d_i}$, and $\xi_i^{i,d_i} \in \mathfrak{g}_{e,1}$. Suppose $\varphi = (\sum c_i^{j,s}\xi_i^{j,s}) \in \widetilde{\mathfrak{g}}_{e,1}$. Then all coefficients $c_i^{i^*,d_i}$ of φ equal zero. In particular, $\alpha(\varphi) = 0$. Thus $\alpha(\mathfrak{g}_{e,0}) = 0$ and indeed α is a point of $\mathfrak{g}_{e,1}^*$. Finally, notice that $\dim(\mathfrak{g}_{e,1})_\alpha \leqslant \dim(\mathfrak{g}_e)_\alpha = n$. Hence $\dim(\mathfrak{g}_{e,1})_\alpha = n$, and we are done. □

In Section 6, we show that most of the isotropy representations associated with other involutions of $\mathfrak{sp}(V)$ do not have GNIB.

5. The isotropy representations of $(\mathfrak{so}_{2n}, \mathfrak{gl}_n)$

In this section, $\dim V = 2n$, $\mathfrak{g} = \mathfrak{so}(V)$, and $(\ ,\ )$ is a $\mathfrak{g}$-invariant symmetric form on V. Let σ be an involution of $\mathfrak{g}$ such that $\mathfrak{g}_0 \simeq \mathfrak{gl}_n$. This can explicitly be described as follows. Let $V = V_+ \oplus V_-$ be a Lagrangian decomposition of V. Then G_0 can be taken as the subgroup of $G = SO(V)$ preserving this decomposition. Here $G_0 \simeq GL(V_+)$, $V_- \simeq (V_+)^*$ as G_0-module, and the G_0-module $\mathfrak{g}_1$ is isomorphic to $\wedge^2 V_+ \oplus (\wedge^2 V_+)^*$.

Keep the notation introduced in Section 3. In particular, V_i, $i = 1, \ldots, m$, are the Jordan spaces of $e \in \mathfrak{N}(\mathfrak{gl}(V))$, $\dim V_i = d_i + 1$, and $w_i \in V_i$ is a cyclic vector.

Lemma 5.1. *Let $e \in \mathfrak{N}(\mathfrak{g}_1)$. Then the cyclic vectors $\{w_i\}$ and hence the spaces $\{V_i\}$ can be chosen such that the following properties are satisfied:*

- (i) *there is an involution $i \mapsto i^*$ on the set $\{1, \ldots, m\}$ such that*
  - *$i^* \neq i$ for each i;*
  - *$d_i = d_{i^*}$;*
  - *$(V_i, V_j) = 0$ if $i \neq j^*$. In particular, $(V_i, V_i) = 0$.*
- (ii) *$\sigma(w_i) = \pm w_i$. More precisely, if d_i is even, then the signs for $\sigma(w_i)$ and $\sigma(w_{i^*})$ are the same; if d_i is odd, then the signs are opposite.*

The proof is left to the reader (cf. the proof of Lemma 3.1).

Theorem 5.2. *The representation $(GL(V_+) : \wedge^2 V_+ \oplus (\wedge^2 V_+)^*)$ has GNIB.*

Proof. In the argument below, we omit routine but tedious calculations of stabilisers and verifications that some functions $\alpha \in (\mathfrak{g}_e)^*$ actually belong to $\mathfrak{g}_{e,1}^*$. All this is similar to computations already presented in Sections 3 and 4.

Recall that $\mathfrak{so}(V)$ is a symmetric subalgebra of $\widetilde{\mathfrak{g}} := \mathfrak{gl}(V)$. We follow the notation similar to that used in the proof of Theorem 4.2. In particular, $\widetilde{\mathfrak{g}} = \mathfrak{so}(V) \oplus \widetilde{\mathfrak{g}}_1$ is a $\mathbb{Z}_2$-grading, and there is again a hierarchy of two involutions.

Let $e \in \mathfrak{N}(\mathfrak{g}_1)$. In this case, $\mathrm{rk}\,(G/G_0) = [n/2]$ and, by Proposition 2.6, our goal is to find an element $\alpha \in (\mathfrak{g}_{e,1})^*$ such that $\dim(\mathfrak{g}_{e,1})_\alpha = [n/2]$. Choose the cyclic vectors for e as prescribed by Lemma 5.1. We normalise these vectors such that $(w_i, e^{d_i}{\cdot}w_{i^*}) = -(w_{i^*}, e^{d_i}{\cdot}w_i) = \pm 1$. Then $\mathfrak{g}_e$ has a basis $\xi_i^{j,d_j-s} + \varepsilon(i,j,s)\xi_{j^*}^{i^*,d_i-s}$, where $\varepsilon(i,j,s) = \pm 1$ depending on i, j and s; and $\xi_i^{j,d_j-s} - \varepsilon(i,j,s)\xi_{j^*}^{i^*,d_i-s}$ form a basis for $\widetilde{\mathfrak{g}}_{e,1}$.

We argue by induction on m. Notice that by Lemma 5.1 m is even.

- Suppose first that $m = 2$. Then $d_1 = d_2$ and $(V_i, V_i) = 0$. Abusing notation, we write $\sigma(v)/v$ for the sign in the formula $\sigma(v) = \pm v$. By Lemma 5.1(ii), we have $\sigma(w_1)/w_1 = \sigma(w_2)/w_2$ if d_1 is odd, and $\sigma(w_1)/w_1 = -\sigma(w_2)/w_2$ if d_1 is even. The algebra $\mathfrak{g}_e$ has a basis

$$\{\xi_1^{1,s} + (-1)^{s+1}\xi_2^{2,s} \mid s = 0, \ldots, d_1\} \cup \{\xi_i^{i,d_1-s} \mid i = 1,2;\, 0 \leqslant s \leqslant d_1,\, s \text{ is odd}\}\,.$$

Here $\sigma(\xi_1^{1,s} + (-1)^{s+1}\xi_2^{2,s}) = (-1)^s(\xi_1^{1,s} + (-1)^{s+1}\xi_2^{2,s})$ and $\sigma(\xi_i^{i,d_1-s}) = \xi_i^{i,d_1-s}$. Therefore $\dim \mathfrak{g}_{e,1} = d_1 = [n/2]$. Since $\dim(\mathfrak{g}_{e,1})_\alpha$ cannot be less than $\mathrm{rk}\,(G/G_0)$, we obtain $\dim(\mathfrak{g}_{e,1})_\alpha = [n/2]$ for any α, as required.

- Assume that $m \geqslant 4$ and the statement holds for all $m_0 < m$. In the induction step, we use the following simple fact. Suppose there is $\alpha \in \mathfrak{g}_{e,1}^*$ such that $\mathrm{ind}\,((\mathfrak{g}_{e,0})_\alpha, (\mathfrak{g}_{e,1})_\alpha) = [n/2]$. Then

$$[n/2] \leqslant \mathrm{ind}\,(\mathfrak{g}_{e,0}, \mathfrak{g}_{e,1}) \leqslant \mathrm{ind}\,((\mathfrak{g}_{e,0})_\alpha, (\mathfrak{g}_{e,1})_\alpha) = [n/2].$$

Hence, $\mathrm{ind}\,(\mathfrak{g}_{e,0}, \mathfrak{g}_{e,1}) = [n/2]$.

Choose an ordering of cyclic spaces such that $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_m$. Without loss of generality, we may assume that $i^* = i+1$ if i is odd. Then there are four possibilities:

(1) d_1 is odd;

(2) d_1 is even and there is some $k \in \{3, \ldots, m-2\}$ such that d_k is also even;

(3) d_1, d_2, d_{m-1}, d_m are even and all other d_i are odd;

(4) d_1, d_2 are even and all other d_i are odd.

Consider all these possibilities in turn. In cases **(1)** and **(2)** we argue by induction, whereas in cases **(3)** and **(4)** we explicitly indicate a generic point in $(\mathfrak{g}_{e,1})^*$.

(1) Set $\mathfrak{f}_1 = \mathfrak{so}(V_1 \oplus V_2)$ and $\mathfrak{f}_2 = \mathfrak{so}(V_3 \oplus \cdots \oplus V_m)$. Then $e = e_1 + e_2$, where $e_i \in \mathfrak{f}_i$. Define $\alpha \in (\mathfrak{g}_{e,1})^*$ by the formula $\alpha(\varphi) = c_1^{1,d_1} + c_2^{2,d_2}$, where c_1^{1,d_1}, c_2^{2,d_2} are coefficients of $\varphi \in \mathfrak{g}_e$. Then $(\mathfrak{g}_e)_\alpha = (\mathfrak{f}_1)_{e_1} \oplus (\mathfrak{f}_2)_{e_2}$. By the inductive hypothesis, $\mathrm{ind}\,((\mathfrak{g}_{e,0})_\alpha, (\mathfrak{g}_{e,1})_\alpha) = [n/2]$.

(2) Let $k > 2$ be the first (odd) number such that d_k is even. We may assume that $\sigma(w_1) = w_1$ and $\sigma(w_k) = w_k$, while $\sigma(w_2) = -w_2$ and $\sigma(w_{k+1}) = -w_{k+1}$. Define $\beta \in (\mathfrak{g}_{e,1})^*$ by the formula

$$\beta(\varphi) = \left(\sum_{i=3}^{k-1} a_i c_i^{i,d_i} \right) + b_1 (c_1^{k+1,d_k} - c_k^{2,d_2}) + b_2 (c_2^{k,d_k} - c_{k+1}^{1,d_1}),$$

where $a_i = a_j$ if and only if $i = j^*$ and $b_1 \neq \pm b_2$. One can show that $(\mathfrak{g}_e)_\beta = \mathfrak{h} \oplus (\mathfrak{f}_2)_{e_2}$, where $\mathfrak{h}$ is a subalgebra of $\mathfrak{f}_1 = \mathfrak{so}(V_1 \oplus \cdots \oplus V_{k+1})$, $\mathfrak{f}_2 = \mathfrak{so}(V_{k+2} \oplus \cdots \oplus V_m)$, $e = e_1 + e_2$, $e_1 \in \mathfrak{h}$, and $e_2 \in \mathfrak{f}_2$. By the inductive hypothesis $\operatorname{ind}((\mathfrak{f}_2)_{e_2,0}, (\mathfrak{f}_2)_{e_2,1}) = \left[\frac{\operatorname{rk} \mathfrak{f}_2}{2}\right]$.

Let $\mathfrak{p}$ be an $(\mathfrak{f}_1 \oplus \mathfrak{f}_2)$-invariant complement of $\mathfrak{f}_1 \oplus \mathfrak{f}_2$ in $\mathfrak{g}$. Then there is a σ-invariant decomposition $\mathfrak{g}_e = (\mathfrak{f}_1 \oplus \mathfrak{f}_2)_e \oplus \mathfrak{p}_e$. If $\alpha \in (\mathfrak{f}_1)_e^*$, then $(\mathfrak{g}_e)_\alpha = ((\mathfrak{f}_1)_e)_\alpha \oplus (\mathfrak{f}_2)_e \oplus \operatorname{Ker} \hat{\alpha}$, where $\operatorname{Ker} \hat{\alpha} \subset \mathfrak{p}_e$ is the kernel of the symplectic form $\hat{\alpha}$ defined by $\hat{\alpha}(\xi, \eta) = \alpha([\xi, \eta])$. Since $\operatorname{Ker} \widehat{\beta} = 0$, this is also true for generic points $\alpha \in (\mathfrak{f}_1)_e^*$ such that $\alpha(\mathfrak{g}_{e,0}) = 0$. Therefore, we can find a point $\alpha \in (\mathfrak{f}_1)_e^*$ such that $\alpha(\mathfrak{g}_{e,0}) = 0$ and $(\mathfrak{g}_e)_\alpha = \mathfrak{h} \oplus (\mathfrak{f}_2)_e$, where $\operatorname{ind}(\mathfrak{h}_0, \mathfrak{h}_1) = \left[\frac{\operatorname{rk} \mathfrak{f}_1}{2}\right]$. For that point α we get

$$\operatorname{ind}((\mathfrak{g}_{e,0})_\alpha, (\mathfrak{g}_{e,1})_\alpha) = \operatorname{ind}(\mathfrak{h}_0, \mathfrak{h}_1) + \operatorname{ind}((\mathfrak{f}_2)_{e_2,0}, (\mathfrak{f}_2)_{e_2,1}) = \left[\frac{\operatorname{rk} \mathfrak{f}_1}{2}\right] + \left[\frac{\operatorname{rk} \mathfrak{f}_2}{2}\right] = \left[\frac{\operatorname{rk} \mathfrak{g}}{2}\right].$$

(3) & **(4)** We may assume that $\sigma(w_1) = w_1$ and $(w_1, e^{d_1} \cdot w_2) = 1$. In case **(3)**, we also assume that $(w_{m-1}, e^{d_m} \cdot w_m) = 1$ and $\sigma(w_m) = -w_m$. Set $t = m-2$ in case **(3)** and $t = m$ in case **(4)**. Take a point $\alpha \in (\mathfrak{g}_{e,1})^*$ such that $\alpha(\varphi) = b(c_1^{1,d_1-1} + c_2^{2,d_2-1}) + \sum\limits_{i=3}^{t} a_i c_i^{i,d_i}$, where $a_i = a_j$ if and only if $i = j^*$ and each $a_i \neq b$. For each i odd, we set $\mathfrak{h}_i := \mathfrak{so}(V_i \oplus V_{i+1}) \cap \mathfrak{g}_e$. Then there exist numbers $\varepsilon(1,i), \varepsilon(2,i) \in \{+1, -1\}$, depending on i, such that

$$(\mathfrak{g}_e)_\alpha = \left(\bigoplus_{i \text{ odd}} \mathfrak{h}_i \right) \oplus \left\langle \xi_1^{i,d_i} + \varepsilon(1,i) \xi_{i^*}^{2,d_1}, \xi_2^{i,d_i} + \varepsilon(2,i) \xi_{i^*}^{1,d_1} \mid i = 3, 4, \ldots, m-1, m \right\rangle.$$

The second summand, denoted by $\mathfrak{a}$, is a commutative ideal of $(\mathfrak{g}_e)_\alpha$. Since $\sigma(w_1) = w_1$ and $\sigma(w_2) = -w_2$, one of the vectors $\xi_1^{i,d_i} + \varepsilon(1,i) \xi_{i^*}^{2,d_1}$, $\xi_2^{i,d_i} + \varepsilon(2,i) \xi_{i^*}^{1,d_1}$ lies in $\mathfrak{g}_0$ and another in $\mathfrak{g}_1$ for each pair $\{i, i^*\} \neq \{1,2\}$.

Each $\mathfrak{h}_i$ has a Levi decomposition $\mathfrak{h}_i = \mathfrak{l}_i \oplus \mathfrak{n}_i$, where $\mathfrak{l}_i := \mathfrak{h}_i \cap \mathfrak{gl}(\Bbbk w_i \oplus \Bbbk w_{i^*})$ is reductive and $\mathfrak{n}_i$ is the nilpotent radical. If d_i is even, then $\mathfrak{l}_i \cong \mathfrak{so}_2$; and if d_i is odd, then $\mathfrak{l}_i \cong \mathfrak{sl}_2$. In any case, $\mathfrak{l}_i \subset \mathfrak{g}_0$. Moreover, we have $[\mathfrak{n}_i, \mathfrak{a}] = 0$ for each odd i, and $[\mathfrak{l}_i, \xi_1^{j,d_j} + \varepsilon(1,j) \xi_{j^*}^{2,d_1}] = [\mathfrak{l}_i, \xi_2^{j,d_j} + \varepsilon(2,j) \xi_{j^*}^{1,d_1}] = 0$ if $i \neq 1, j, j^*$.

Set $\mathfrak{r} := (\mathfrak{g}_e)_\alpha$. We claim that $\operatorname{ind}(\mathfrak{r}_0, \mathfrak{r}_1) = [n/2]$. Define $\beta \in \mathfrak{r}^*$ by the following rule:
If $x \in \bigoplus\limits_i \mathfrak{h}_i$, then $\beta(x) = 0$; if x is one of the vectors $\xi_1^{i,d_i} + \varepsilon(1,i) \xi_{i^*}^{2,d_1}, \xi_2^{i,d_i} + \varepsilon(2,i) \xi_{i^*}^{1,d_1}$, then $\beta(x) = 1$ if $x \in \mathfrak{r}_1$ and $\beta(x) = 0$ if $x \in \mathfrak{r}_0$. In particular $\beta(\mathfrak{r}_0) = 0$, i.e., $\beta \in (\mathfrak{r}_1)^*$. Then

$$(\mathfrak{r}_1)_\beta = \bigoplus_{i \text{ odd}} (\mathfrak{h}_i \cap \mathfrak{g}_{e,1}) \oplus \Bbbk \eta_0,$$

where $\eta_0 = 0$ in case **(4)**, and $\eta_0 = \xi_1^{m,d_m} - \xi_{m-1}^{2,d_1} + \xi_2^{m-1,d_i} - \xi_m^{1,d_1}$ in case **(3)**. For each $\mathfrak{h}_i$, we have $\dim(\mathfrak{h}_i \cap \mathfrak{g}_{e,1}) = \left[\frac{d_i+1}{2}\right]$. Therefore, in both cases **(3)** and **(4)** we obtain $\dim(\mathfrak{r}_1)_\beta = [n/2]$, as required. □

6. Isotropy representations without GNIB

Here we describe a method for finding nilpotent orbit in isotropy representations without equality in (1.6). There is an obvious method of constructing $\mathbb{Z}_2$-graded Lie algebras: take any $\mathbb{Z}$-grading and then glue it modulo 2. This will be applied in the following form. Given $e \in \mathfrak{N}(\mathfrak{g})$, take an $\mathfrak{sl}_2$-triple containing e, say $\{e,h,f\}$. Consider the $\mathbb{Z}$-grading of $\mathfrak{g}$ that is determined by h:

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i), \text{ where } \ \mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h,x] = ix\} \ .$$

Here $e \in \mathfrak{g}(2)$. Suppose that e is *even*, i.e., $\mathfrak{g}(i) = 0$ if i is odd. Gluing modulo 2 means that we define $\mathfrak{g}_0 = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(4i)$ and $\mathfrak{g}_1 = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(4i+2)$. Then $e \in \mathfrak{g}_1$ and it is sometimes possible to prove that, for this nilpotent element, Equality (1.6) does not hold.

Our point of departure is an even nilpotent element e of height 4 (the latter means that $(\mathrm{ad}\, e)^5 = 0$). Then the corresponding $\mathbb{Z}$-grading is $\mathfrak{g} = \bigoplus_{i=-2}^{2} \mathfrak{g}(2i)$. The centraliser of e lies in the non-negative part of this grading, i.e., $\mathfrak{g}_e = \mathfrak{g}(0)_e \oplus \mathfrak{g}(2)_e \oplus \mathfrak{g}(4)$. Therefore,

$$\mathfrak{g}_{e,0} = \mathfrak{g}(0)_e \oplus \mathfrak{g}(4) \quad \text{and} \quad \mathfrak{g}_{e,1} = \mathfrak{g}(2)_e \ .$$

Here $\dim \mathfrak{g}(2)_e = \dim \mathfrak{g}(2) - \dim \mathfrak{g}(4)$ and $\dim \mathfrak{g}(0)_e = \dim \mathfrak{g}(0) - \dim \mathfrak{g}(2)$ and hence

$$\dim \mathfrak{g}_1 - \dim \mathfrak{g}_0 = 2\dim \mathfrak{g}(2) - 2\dim \mathfrak{g}(4) - \dim \mathfrak{g}(0) = \dim \mathfrak{g}(2)_e - \dim \mathfrak{g}(0)_e - \dim \mathfrak{g}(4) \ .$$

We wish to compare $\mathrm{ind}\,(\mathfrak{g}_0, \mathfrak{g}_1)$ and $\mathrm{ind}\,(\mathfrak{g}_{e,0}, \mathfrak{g}_{e,1})$. Let S denote the identity component of a generic stabiliser for $(G_0 : \mathfrak{g}_1)$. Then

$$\mathrm{ind}\,(\mathfrak{g}_0, \mathfrak{g}_1) = \dim \mathfrak{g}_1 - \dim \mathfrak{g}_0 + \dim S = \dim \mathfrak{g}(2)_e - \dim \mathfrak{g}(0)_e - \dim \mathfrak{g}(4) + \dim S \ .$$

In our situation, $\mathfrak{g}(4)$ acts trivially on $\mathfrak{g}(2)_e$ and hence on $\mathfrak{g}(2)_e^*$. Hence the action $(G_{e,0} : (\mathfrak{g}_{e,1})^*)$ essentially reduces to a *reductive* group action $(G(0)_e : \mathfrak{g}(2)_e^*)$. Let $S^{\{e\}}$ denote the identity component of a generic stabiliser for the representation $(G(0)_e : \mathfrak{g}(2)_e)$. Then

$$\mathrm{ind}\,(\mathfrak{g}_{e,0}, \mathfrak{g}_{e,1}) = \mathrm{ind}\,(\mathfrak{g}(0)_e, \mathfrak{g}(2)_e) = \dim \mathfrak{g}(2)_e - \dim \mathfrak{g}(0)_e + \dim S^{\{e\}} \ .$$

Hence

$$\delta := \mathrm{ind}\,(\mathfrak{g}_{e,0}, \mathfrak{g}_{e,1}) - \mathrm{ind}\,(\mathfrak{g}_0, \mathfrak{g}_1) = \dim \mathfrak{g}(4) + \dim S^{\{e\}} - \dim S \ , \tag{6.1}$$

and, as was shown in Proposition 2.6, this quantity is non-negative. The stabilisers S are well-known. (Actually, they can be directly read off from the Satake diagram of the involution in question.) Some work is only needed for computing $\dim S^{\{e\}}$.

Remark 6.1. The involutions obtained in this way are always inner.

Below, we provide a series of examples covered by the previous scheme.

Example 6.1. Suppose $\mathfrak{g}$ is a simple Lie algebra such that the highest root is a fundamental weight. Take the weighted Dynkin diagram of the minimal nilpotent orbit. Then twice this diagram is again a weighted Dynkin diagram. This new diagram determines an even nilpotent orbit (element) of height 4. In this situation, $\dim\mathfrak{g}(4) = 1$ and $\mathfrak{g}_0 = \mathfrak{g}(0)' \oplus \mathfrak{sl}_2$. Then straightforward calculations show that $S^{\{e\}} = S$. Hence the quantity in (6.1) is equal to 1. This yields the following list of symmetric pairs without GNIB:

$(\mathbf{E}_8, \mathbf{E}_7 \times \mathbf{A}_1)$; $(\mathbf{E}_7, \mathbf{D}_6 \times \mathbf{A}_1)$; $(\mathbf{E}_6, \mathbf{A}_5 \times \mathbf{A}_1)$; $(\mathbf{F}_4, \mathbf{C}_3 \times \mathbf{A}_1)$; $(\mathbf{G}_2, \mathbf{A}_1 \times \mathbf{A}_1)$;
$(\mathfrak{so}_n, \mathfrak{so}_{n-4} \times \mathfrak{so}_4)$, $n \geqslant 7$;

Remark 6.2. If $\mathfrak{g}$ is of type $\mathbf{G}_2$, then this procedure leads to Example 2.1.

Example 6.2. Let e be a nilpotent element in $\mathfrak{gl}_{3k+l}$ corresponding to the partition $(3^k, 1^l)$. Then e is even and of height 4, and the related symmetric pair is $(\mathfrak{gl}_{3k+l}, \mathfrak{gl}_{2k} \times \mathfrak{gl}_{k+l})$. We have the following data for the dimension of graded pieces for the $\mathbb{Z}$-grading:

i	0	2	4
$\dim\mathfrak{g}(i)$	$2k^2 + (k+l)^2$	$2k(k+l)$	k^2
$\dim\mathfrak{g}(i)_e$	$k^2 + l^2$	$k^2 + 2kl$	k^2

To compute $S^{\{e\}}$, we notice that $\mathfrak{g}(0)_e \simeq \mathfrak{gl}_k \times \mathfrak{gl}_l = \mathfrak{gl}(V_1) \times \mathfrak{gl}(V_2)$ and the $\mathfrak{g}(0)_e$-module $\mathfrak{g}(2)_e$ is isomorphic to $(V_1 \otimes V_1^*) \oplus (V_1 \otimes V_2) \oplus (V_1 \otimes V_2)^*$. Suppose $k \geqslant l$. Then $S^{\{e\}} = T_1 \times GL_{l-k}$, where T_j stands for a j-dimensional torus. The group S is isomorphic to $T_{2k} \times GL_{l-k}$, Hence the quantity δ in (6.1) is equal to $k^2 - 2k + 1$, which is positive for $k \geqslant 2$. The same type of argument shows that $\delta = 1$ if $k = 2$ and $l = 1$. In particular, this means that the symmetric pair $(\mathfrak{gl}_n, \mathfrak{gl}_4 \times \mathfrak{gl}_{n-4})$ has no GNIB for any $n \geqslant 7$.

Example 6.3. Now $\mathfrak{g} = \mathfrak{sp}(V)$. Let e be a nilpotent element in $\mathfrak{sp}_{6k+2l}$ corresponding to the partition $(3^{2k}, 1^{2l})$. Then e is even and of height 4, and the related symmetric pair is $(\mathfrak{sp}_{6k+2l}, \mathfrak{sp}_{4k} \times \mathfrak{sp}_{2k+2l})$. Here $\mathfrak{g}(0)_e = \mathfrak{sp}_{2k} \times \mathfrak{sp}_{2l} = \mathfrak{sp}(V_1) \times \mathfrak{sp}(V_2)$ and $\mathfrak{g}(2)_e$ is isomorphic to $\wedge^2 V_1 \oplus (V_1 \otimes V_2)$. If $k \leqslant l$, then $S^{\{e\}} = \begin{cases} Sp_{2(l-k)}, & \text{if } k \geqslant 3 \\ SL_2 \times Sp_{2(l-k)}, & \text{if } k \leqslant 2 \end{cases}$. Also, $S = (SL_2)^{2k} \times Sp_{2(l-k)}$. Since $\dim\mathfrak{g}(4) = k(2k+1)$, we see that δ is positive for $k \geqslant 2$. Similarly to the previous example, one also verifies that δ is positive for $k = 2, l = 1$. In particular, the symmetric pair $(\mathfrak{sp}_{2n}, \mathfrak{sp}_8 \times \mathfrak{sp}_{2n-8})$ has no GNIB for any $n \geqslant 7$.

Example 6.4. Let $\mathfrak{g}$ be of type $\mathbf{E}_7$ and e a nilpotent element with weighted Dynkin diagram $\begin{smallmatrix} 0\text{-}0\text{-}0\text{-}0\text{-}0\text{-}0 \\ 2 \end{smallmatrix}$. Then $\dim\mathfrak{g}(0) = 49$, $\dim\mathfrak{g}(2) = 35$, and $\dim\mathfrak{g}(4) = 7$. Therefore $\dim\mathfrak{g}_0 = 63$ and $\dim\mathfrak{g}_1 = 70$. Hence the related involution is of maximal rank. (Here $\mathfrak{g}_0 \simeq \mathfrak{gl}(V)$ with $\dim V = 7$ and $\mathfrak{g}_1 \simeq \wedge^4 V$). Therefore the group S is trivial. This already means that $\dim\mathfrak{g}(4) + \dim S^{\{e\}} - \dim S \geqslant 7$.

Example 6.5. Let $\mathfrak{g}$ be of type $\mathbf{E}_8$ and e a nilpotent element with weighted Dynkin diagram $\begin{smallmatrix} 0\text{-}0\text{-}0\text{-}0\text{-}0\text{-}0\text{-}2 \\ 0 \end{smallmatrix}$. Then $\dim\mathfrak{g}(0) = 92$, $\dim\mathfrak{g}(2) = 64$, and $\dim\mathfrak{g}(4) = 14$. Therefore $\dim\mathfrak{g}_0 = 120$

and $\dim \mathfrak{g}_1 = 128$. Hence the related involution is of maximal rank. (Here $\mathfrak{g}_0 \simeq \mathfrak{so}_{16}$ and $\mathfrak{g}_1$ is a half-spinor representation of $\mathfrak{so}_{16}$). Therefore the group S is trivial. This already means that $\dim \mathfrak{g}(4) + \dim S^{\{e\}} - \dim S \geqslant 14$.

One can find more isotropy representations without GNIB using the above examples and the slice method. The following assertion readily follows from Theorems 2.1 and 2.3.

(6.6) *Suppose* $(G : V)$ *is observable and* $(L : W)$ *is a slice representation of* $(G : V)$*. If* $(L : W)$ *has no GNIB, then so does* $(G : V)$.

Example 6.7. Consider the symmetric pair $(\mathbf{E}_6, \mathfrak{sp}_8)$. The corresponding $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is of maximal rank and the Sp_8-module $\mathfrak{g}_1$ is isomorphic to the 4-th fundamental representation. Since the rank is maximal, for any Levi subalgebra $\mathfrak{l} \subset \mathbf{E}_6$, there is $x \in \mathfrak{g}_1$ whose centraliser in $\mathfrak{g}$ is conjugate to $\mathfrak{l}$. The induced $\mathbb{Z}_2$-grading of $\mathfrak{z}(x) \simeq \mathfrak{l}$ is also of maximal rank. In particular, taking x such that the semisimple part of $\mathfrak{z}(x)$ is of type $\mathbf{D}_4$, we obtain, up to the centre of $\mathfrak{z}(x)$, the symmetric pair $(\mathfrak{so}_8, \mathfrak{so}_4 \times \mathfrak{so}_4)$, which has no GNIB. Hence the symmetric pair $(\mathbf{E}_6, \mathfrak{sp}_8)$ has no GNIB, too.
The similar argument also works for the involutions in Examples 6.4, 6.5.

Theorem 6.3. *For the following symmetric pairs, the isotropy representation has no GNIB:*

(i) $(\mathfrak{gl}_n, \mathfrak{gl}_m \times \mathfrak{gl}_{n-m})$ *with* $4 \leqslant m \leqslant n-m$;
(ii) $(\mathfrak{so}_n, \mathfrak{so}_m \times \mathfrak{so}_{n-m})$ *with* $4 \leqslant m \leqslant n-m$;
(iii) $(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2m} \times \mathfrak{sp}_{2n-2m})$ *with* $4 \leqslant m \leqslant n-m$.

The cases with $m = 3$ *and* $n - m = 4$ *also yield the isotropy representations without GNIB.*

Proof. (i) It is easily seen that $(G_0 : \mathfrak{g}_1)$ has a slice representation which is isomorphic to the isotropy representation of the pair $(\mathfrak{gl}_{n-2}, \mathfrak{gl}_{m-1} \times \mathfrak{gl}_{n-m+1})$. Iterating this procedure yields the isotropy representation of the pair $(\mathfrak{gl}_{n-2m+8}, \mathfrak{gl}_4 \times \mathfrak{gl}_{n-2m+4})$. The latter has no GNIB by Example 6.2. Then one applies assertion 6.6.
(ii),(iii). Here one uses the similar reductions, with '$\mathfrak{so}$' and '$\mathfrak{sp}$' in place of '$\mathfrak{gl}$', and Examples 6.1 and 6.3. □

Making use of a direct computation, we strengthen the assertion of Theorem 6.3(i).

Theorem 6.4. *The following symmetric pairs* $(\mathfrak{g}, \mathfrak{g}_0)$ *have no GNIB:*

(i) $(\mathfrak{gl}_n, \mathfrak{gl}_3 \oplus \mathfrak{gl}_{n-3})$, $n \geqslant 7$,
(ii) $(\mathfrak{so}_n, \mathfrak{so}_3 \oplus \mathfrak{so}_{n-3})$, $n \geqslant 7$,
(iii) $(\mathfrak{sp}_{2n}, \mathfrak{sp}_6 \oplus \mathfrak{sp}_{2n-6})$, $n \geqslant 7$.

Proof. For all these symmetric pairs, we have $\mathrm{rk}\,(G/G_0) = 3$, and the case of $n = 7$ is covered by Theorem 6.3. We show that for $n > 7$ there is a reduction to $n = 7$.

(i) Let $\mathfrak{h} = \mathfrak{gl}_7 \subset \mathfrak{gl}_n$ be a regular σ-invariant subalgebra such that $\mathfrak{h}^\sigma = \mathfrak{gl}_3 \oplus \mathfrak{gl}_4 \subset \mathfrak{gl}_3 \oplus \mathfrak{gl}_{n-3}$. By Example 6.2, the nilpotent H-orbit with partition $(3,3,1)$ meets $\mathfrak{h}_1$ and yields an H_0-orbit without GNIB. Let $e \in \mathfrak{h}_1$ be an element in this orbit. Using the embedding $\mathfrak{h}_1 \subset$

$\mathfrak{g}_1$, we may regard e as element of $\mathfrak{g}_1$. Then the corresponding partition is $(3,3,1^{n-6})$. We are going to prove that $\mathrm{ind}(\mathfrak{g}_{e,0},\mathfrak{g}_{e,1}) \geqslant \mathrm{ind}(\mathfrak{h}_{e,0},\mathfrak{h}_{e,1})$. An explicit model of e is as follows. Let $w_1, w_2, \ldots, w_{n-4}$ be cyclic vectors for e, where $e^3{\cdot}w_i = 0$ for $i = 1,2$ and $e{\cdot}w_i = 0$ for $i \geqslant 3$. Let $\Bbbk^n = \Bbbk^3 \oplus \Bbbk^{n-3}$ be the $\mathfrak{g}_0$-stable decomposition corresponding to σ. Then we assume that $w_3 \in \Bbbk^3$ and and all other w_i's lie in $\Bbbk^{n-3}$. (Hence $\Bbbk^3 = \langle e{\cdot}w_1, e{\cdot}w_2, w_3\rangle$.) Now, all information for e can be presented rather explicitly. We have $\dim\mathfrak{g}_{e,0} = (n-4)^2+5$, $\dim\mathfrak{h}_{e,0} = 9$, $\dim\mathfrak{h}_{e,1} = 8$. More precisely, $\mathfrak{g}_{e,0} = \mathfrak{h}_{e,0} \oplus \mathfrak{gl}_{n-7} \oplus \mathfrak{a}$, where $[\mathfrak{gl}_{n-7},\mathfrak{h}] = 0$ and

$$\mathfrak{a} = \langle \xi_1^{t,0}, \xi_2^{t,0}, \xi_t^{1,2}, \xi_t^{2,2} \mid 4 \leqslant t \leqslant n-4\rangle.$$

This means in particular that $\mathfrak{a} = \{0\}$ if $n = 7$. Next,

$$\mathfrak{h}_{e,1} = \left\langle \xi_i^{j,1}, \xi_i^{3,0}, \xi_3^{i,2} \mid i,j \in \{1,2\} \right\rangle.$$

It is easily seen that $[\mathfrak{a},\mathfrak{h}_{e,1}] = 0$. For instance, $[\xi_t^{1,2}, \xi_1^{i,1}] = -\xi_t^{i,3} = 0$, since $e^3 \cdot w_i = 0$. Because also $[\mathfrak{gl}_{n-4},\mathfrak{h}_{e,1}] = 0$, we get $[\mathfrak{g}_{e,0},\mathfrak{h}_{e,1}] = [\mathfrak{h}_{e,0},\mathfrak{h}_{e,1}] \subset \mathfrak{h}_{e,1}$. It remains to observe that for each $\alpha \in (\mathfrak{g}_{e,1})^*$ we have

$$\dim(\mathfrak{g}_{e,1})_\alpha \geqslant \dim(\mathfrak{h}_{e,1})_\alpha = \dim(\mathfrak{h}_{e,1})_{\widetilde{\alpha}} \geqslant 4 > \mathrm{rk}\,(G/G_0),$$

where $\widetilde{\alpha} \in (\mathfrak{h}_{e,1})^*$ is the restriction of α.

(ii) The previous argument goes through *mutatis mutandis* in the orthogonal case. We just consider the nilpotent element in $\mathfrak{so}_7$ with partition $(3,3,1)$. Using the natural embedding $\mathfrak{so}_7 \subset \mathfrak{so}_n$, $n > 7$, we obtain the nilpotent orbit with partition $(3,3,1^{n-6})$.

(iii) This case is similar to part (i) but in a different fashion. Here we start with a nilpotent element in $\mathfrak{sp}_{14}$ with partition $(3,3,3,3,1,1)$, and then embed $\mathfrak{sp}_{14}$ in $\mathfrak{sp}_{2n}$, $n > 7$. □

7. More affirmative results and open problems

We conclude with some more examples of isotropy representations having GNIB and state several questions.

Proposition 7.1. *If (G, G_0) is a symmetric pair of rank 1, then $(G_0 : \mathfrak{g}_1)$ has GNIB.*

Proof. The symmetric pairs of rank 1 are the following:

$$(\mathfrak{gl}_n, \mathfrak{gl}_{n-1} \times \mathfrak{gl}_1),\ (\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2} \times \mathfrak{sp}_2),\ (\mathfrak{so}_n, \mathfrak{so}_{n-1}),\ (\mathbf{F}_4, \mathfrak{so}_9).$$

The number of nonzero nilpotent G_0-orbits in $\mathfrak{N}(\mathfrak{g}_1)$ equals 3, 2, 1, 2, respectively. By Proposition 1.2, the orbit of maximal dimension is always "good", so that it remains to test the minimal orbit(s). This is done by hand.

We give some details for the last case. Here the isotropy representation is the spinor (16-dimensional) representation of $\mathfrak{g}_0 = \mathfrak{so}_9$. The weights are $\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$. For simplicity, weights will be represented by the set of 4 signs. For instance, the lowest weight is $(----)$. Let $v \in \mathfrak{g}_1$ be a lowest weight vector. Then $\dim\mathfrak{g}_0{\cdot}v = 11$ and $(\mathfrak{g}_0)_v$ is a semi-direct product of $\mathfrak{sl}_4$ and a nilpotent radical. As $\mathfrak{sl}_4$-module, the 5-dimensional space

$\mathfrak{g}_1/\mathfrak{g}_0{\cdot}v$ can be identified with the subspace W of $\mathfrak{g}_1$ whose weights are $(-+++), (+-++), (++-+), (+++-), (++++)$. Hence W is the sum of the trivial and 4-dimensional $\mathfrak{sl}_4$-modules. This shows that already SL_4, the reductive part of $(G_0)_v$, has an orbit of codimension one in $\mathfrak{g}_1/\mathfrak{g}_0{\cdot}v$. □

Example 7.1. The symmetric pair $(\mathbf{E}_6, \mathbf{F}_4)$ has rank two. However, its isotropy representation has only two nonzero nilpotent orbits. Here again one can easily check that the minimal orbit O_{min} satisfies GNIB-condition, i.e., Equality (1.6) is satisfied for $v \in O_{min}$. Hence this isotropy representation has GNIB.

Remark 7.2. Using explicit description of nilpotent G_0-orbits, one can honestly verify that the symmetric pairs $(\mathfrak{gl}_n, \mathfrak{gl}_2 \oplus \mathfrak{gl}_{n-2})$ and $(\mathfrak{gl}_6, \mathfrak{gl}_3 \oplus \mathfrak{gl}_3)$ have GNIB. Together with results of Sections 3 and 6, this completes the problem of classifying the isotropy representations of $\mathfrak{gl}_n$ with and without GNIB.

Remark 7.3. It is also not hard to verify that the pairs $(\mathfrak{so}_n, \mathfrak{so}_2 \times \mathfrak{so}_{n-2})$ and $(\mathfrak{sp}_{2n}, \mathfrak{sp}_4 \times \mathfrak{sp}_{2n-4})$ have GNIB. Furthermore, both pairs $(\mathfrak{so}_6, \mathfrak{so}_3 \times \mathfrak{so}_3)$ and $(\mathfrak{sp}_{12}, \mathfrak{sp}_6 \times \mathfrak{sp}_6)$ have GNIB. Together with results of Sections 4–6, this completes the problem of classifying the isotropy representations of $\mathfrak{so}_n$ and $\mathfrak{sp}_{2n}$ with and without GNIB.

Taking into account all symmetric pairs considered so far, one may notice that there remain only two unmentioned symmetric pairs: $(\mathbf{E}_6, \mathfrak{so}_{10} \times \mathfrak{t}_1)$ and $(\mathbf{E}_7, \mathbf{E}_6 \times \mathfrak{t}_1)$. Their ranks are 2 and 3, respectively. It is likely that the first of them has GNIB, but we have no assumption for the second case. We hope to consider these remaining cases in a subsequent article.

There are many interesting open questions on GIB and GNIB. Here are some of them.

(Q1) We have shown in Corollary 1.6 that sufficiently large reducible representations have GIB. However, no a priori results is known for irreducible representations of simple algebraic groups. We conjecture that for any semisimple G there are finitely many irreducible representations without GNIB.

(Q2) Let V be a simple G-module and $v \in V$ a highest weight vector. Is it true that Equality (1.6) holds for v?

(Q3) Suppose G has a dense orbit in V, i.e., $\Bbbk(V)^G = \Bbbk$. Is it true that V has GNIB ?

(Q4) Let V be a spherical G-module. Is it true that V has GNIB ? (It is a special case of (Q3).)

In connection with the last question, we mention that most of spherical modules are obtained by the following construction. Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra whose nilpotent radical, $\mathfrak{p}^u$, is Abelian. Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{p}^u$ be a Levi decomposition. Then $\mathfrak{p}^u$ is a spherical L-module. Using the theory developed in [7], one can prove that $\mathfrak{p}^u$ has GNIB. The point here is that, for any $v \in \mathfrak{p}^u$, already the reductive part of L_v has an open orbit in $\mathfrak{p}^u/\mathfrak{l}{\cdot}v$.

Finally, we recall that most of the observable representations of reductive groups are associated with automorphisms of finite order of simple Lie algebras, i.e., the corresponding linear groups are Θ-groups in the sense of Vinberg [13]. This is a generalisation of the situation considered in this paper. It is therefore natural to investigate when these Θ-representations have GNIB.

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