Supplement to the paper "Scalar curvature of a metric with unit volume"

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In the above mentioned paper [4], the author showed an inequality concerning the Yamabe number $\mu(M)$, which is defined as $\mu(M) = \sup_C \inf_{g \in C} \int_M R_g dv_g / (\int_M dv_g)^{(n-2)/n}$, where the supremum is taken over all conformal classes C of Riemannian metrics of a compact n-manifold M, and R_g denotes the scalar curvature of the metric g. The purpose of this note is to give a generalization of it.

Theorem. (i) *If M_1 and M_2 are compact manifolds of dimension $n \geq 3$, then*

$$\mu(M_1 \# M_2) \geq \begin{cases} -(|\mu(M_1)|^{n/2} + |\mu(M_2)|^{n/2})^{2/n} & \text{if } \mu(M_1) \leq 0 \text{ and } \mu(M_2) \leq 0; \\ \min\{\mu(M_1), \mu(M_2)\} & \text{otherwise.} \end{cases}$$

(ii) *If M is an S^{n-1} bundle over S^1 with $n \geq 3$, then* $\mu(M) = \mu(S^n) = n(n-1)\mathrm{Vol}(S^n(1))^{2/n}$.

For example we can see $\mu(S^1 \times S^{n-1} \# S^1 \times S^{n-1}) = \mu(S^n)$ if $n \geq 3$, and so on. Also we get as a corollary that $\mu(M_1 \# M_2) > 0$ if $\mu(M_1) > 0$ and $\mu(M_2) > 0$. This corollary is originally due to Schoen-Yau [6] and Gromov-Lawson [3]. However our proof is different from theirs and has the advantage of giving a good estimate on the Yamabe number.

§1. Preliminaries.

For a conformal class C of Riemannian metrics on a compact n-manifold M we set $\mu(M,C) = \inf_{g \in C} \int_M R_g dv_g / (\int_M dv_g)^{(n-2)/n}$. Therefore $\mu(M) = \sup_C \mu(M,C)$. Take two n-manifolds with conformal structures, say (M_1,C_1) and (M_2,C_2). Then we write $(M,C) = (M_1,C_1) \amalg (M_2,C_2)$ if M is the disjoint union of M_1 and M_2, and $C_i = \{g|_{M_i};\ g \in C\}$ for i = 1, 2.

<u>Lemma 1.1</u>. $\mu((M_1,C_1) \amalg (M_2,C_2))$

$$= \begin{cases} \mu(M_1,C_1) + \mu(M_2,C_2) & \underline{\text{if}}\ n = 2; \\ -(|\mu(M_1,C_1)|^{n/2} + |\mu(M_2,C_2)|^{n/2})^{2/n} & \underline{\text{if}}\ \mu(M_1,C_1) \leqq 0 \\ & \underline{\text{and}}\ \mu(M_2,C_2) \leqq 0; \\ \min\{\mu(M_1,C_1), \mu(M_2,C_2)\} & \underline{\text{otherwise}}. \end{cases}$$

<u>Proof</u>. A straightforward computation. ▯

<u>Corollary 1.2</u>. $\mu(M_1 \amalg M_2)$

$$= \begin{cases} \mu(M_1) + \mu(M_2) & \underline{\text{if}}\ n = 2; \\ -(|\mu(M_1)|^{n/2} + |\mu(M_2)|^{n/2})^{2/n} & \underline{\text{if}}\ \mu(M_1) \leqq 0\ \underline{\text{and}}\ \mu(M_2) \leqq 0; \\ \min\{\mu(M_1), \mu(M_2)\} & \underline{\text{otherwise}}. \end{cases}$$

Combining this with a theorem of Aubin [1; p. 13], we get

<u>Corollary 1.4</u>. <u>If</u> $\tilde{M}$ <u>is a</u> k-<u>fold covering of</u> M <u>then</u>

$$\mu(\tilde{M}) \geqq \mu(\underbrace{M \amalg \ldots \amalg M}_{k\text{-times}}).$$

Here we cited the Aubin's result in the following form: $\mu(\tilde{M},\tilde{C}) > \mu(M,C)$ for any conformal class C and its lift $\tilde{C}$ if $k \geqq 2$, dim $M \geqq 3$ and $\mu(M,C) > 0$. This fact also yields that $\mu(S^1 x S^{n-1}) > \mu(S^1 x S^{n-1},C)$ for any C if $n \geqq 3$, which we can see also from our theorem (ii) because it is known [5] that $\mu(S^1 x S^{n-1},C) < \mu(S^n)$ for any C.

§2. Proof of Theorem.

Let M be a compact manifold of dimension $n \geqq 3$, and p_1 and p_2 two points of M. We take off two small balls around p_1 and p_2, and then attach a handle instead, the handle being topologically the product of a line segment and S^{n-1}. The new manifold obtained in this way will be denoted by $\bar{M}$. For example if $M = S^n$ then $\bar{M}$ is an S^{n-1} bundle over S^1 i.e., $S^1 x S^{n-1}$ or the generalized Klein bottle. And if $M = M_1 \amalg M_2$ and p_1 and p_2 are taken from M_1 and M_2 respectively, then $\bar{M} = M_1 \# M_2$. Therefore in order to prove Theorem it suffices to show $\mu(\bar{M}) \geqq \mu(M)$ because of Corollary 1.2 and the fact that $\mu(M) \leqq \mu(S^n)$ for all compact n-manifold M.

Now the proof proceeds as follows. Let ε be an arbitral positive number, which will be fixed throughout. First, we take a conformal class C of M such that

$$(2.1) \qquad \mu(M,C) > \mu(M) - \varepsilon.$$

Lemma 2.1. We may assume C is conformally flat around the points p_1 and p_2.

Proof. Pick a representative metric $g \in C$, then $\mu(M,C)$ is rewritten as

$$(2.2) \qquad \mu(M,C) = \inf_{f>0} \frac{4\frac{n-1}{n-2}\int_M |df|^2 dv_g + \int_M R_g f^2 dv_g}{(\int_M f^{\frac{2n}{n-2}} dv_g)^{\frac{n-2}{n}}} .$$

We shall denote the right side of (2.2) by $\mu(g)$. That is, $\mu(M,C) = \mu(g)$ for $g \in C$. From the expression (2.2), it is not hard to see that μ is a continuous function of the space of all smooth Riemannian metrics of M with respect to the topology that two metrics are close if theirselves and their scalar curvature functions are close to each other respectively in C^0.

On the other hand, by Lemma 3.10 of [4], we can choose another metric g', which may not be in C, such that g' is conformally flat around p_1 and p_2, and that g' and $R_{g'}$ are sufficiently close to g and R_g respectively in C^0.

Thus if we take the conformal class C' of g' instead of C, the proof is done. $\square$

So let us assume C is conformally flat around p_1 and p_2. In particular, there are a function $\lambda \in C^\infty(M \setminus \{p_1, p_2\})$ and $g \in C$ such that $\tilde{g} = e^\lambda g$ is a complete metric of $M \setminus \{p_1, p_2\}$ and that each of two ends is isometric to the half infinite cylinder $[0,\infty) \times S^{n-1}(1)$. For convenience, we write

$$(2.3) \qquad (M \setminus \{p_1, p_2\}, \tilde{g}) = [0,\infty) \times S^{n-1}(1) \cup (\tilde{M}, \tilde{g}) \cup [0,\infty) \times S^{n-1}(1),$$

where $\tilde{M}$ is the complement of the two cylinders. We can glue

$(\tilde{M},\tilde{g})$ and $[0,\ell]\times S^{n-1}(1)$, the product of the interval of length ℓ with the unit (n-1)-sphere, along their boundaries to get a smooth Riemannian manifold $(\bar{M},g_\ell)$, where $\bar{M}$ is as mentioned at the beginning of this section.

(2.4) $\quad (\bar{M},g_\ell) = (\tilde{M},\tilde{g}) \cup [0,\ell]\times S^{n-1}(1).$

Let C_ℓ denote the conformal class to which g_ℓ belongs, and we have as before

$$\mu(\bar{M},C_\ell) = \inf_{f>0} \frac{4\frac{n-1}{n-2}\int_{\bar{M}} |df|^2 dv_{g_\ell} + \int_{\bar{M}} R_{g_\ell} f^2 dv_{g_\ell}}{(\int_{\bar{M}} f^{\frac{2n}{n-2}} dv_{g_\ell})^{\frac{n-2}{n}}} .$$

So, take a positive function $f_\ell \in C^\infty(\bar{M})$ such that

(2.5) $\quad 4\frac{n-1}{n-2}\int_{\bar{M}} |df_\ell|^2 dv_g + \int_{\bar{M}} R_{g_\ell} f_\ell^2 dv_{g_\ell} < \mu(\bar{M},C_\ell) + \varepsilon \leq \mu(\bar{M}) + \varepsilon,$

and

(2.6) $\quad \int_{\bar{M}} f_\ell^{\frac{2n}{n-2}} dv_{g_\ell} = 1.$

<u>Lemma 2.2</u>. <u>There is a section, say</u> $\{t_\ell\}\times S^{n-1}$, <u>in the cylindrical part of</u> $\bar{M}$ <u>(cf. (2.4)) such that</u>

$$\int_{\{t\}\times S^{n-1}} (|df_\ell|^2 + f_\ell^2) dv_{S^{n-1}} < \frac{A}{\ell},$$

<u>where A is a constant independent of</u> ℓ.

<u>Proof</u>. Put $A_1 = -\min\{0, \min_{x\in\tilde{M}} R_{\tilde{g}}(x)\} \operatorname{Vol}(\tilde{M},\tilde{g})^{2/n}$. Then, using Hölder's inequality we get from (2.5) that

$$4\frac{n-1}{n-2}\int_{[0,\ell]\times S^{n-1}} |df_\ell|^2 dv_{g_\ell} + (n-1)(n-2)\int_{[0,\ell]\times S^{n-1}} f_\ell^{\,2} dv_{g_\ell}$$
$$< \mu(\bar{M}) + \varepsilon + A_1.$$

Therefore there is a $t_\ell \in [0,\ell]$ such that

$$4\frac{n-1}{n-2}\int_{\{t_\ell\}\times S^{n-1}} |df_\ell|^2 dv_{S^{n-1}} + (n-1)(n-2)\int_{\{t_\ell\}\times S^{n-1}} f_\ell^{\,2} dv_{S^{n-1}}$$
$$< (\mu(\bar{M}) + \varepsilon + A_1)/\ell,$$

which proves the assertion with $A = (n-2)(\mu(\bar{M}) + \varepsilon + A_1)/(n-1)$. $\square$

Now we cut off $\bar{M}$ on the section $\{t_\ell\}\times S^{n-1}$, and attach two half-infinite cylinders to it, so $(M\setminus\{p_1,p_2\},\tilde{g})$ reappears. But this time we describe it as follows

$$(2.7)\quad (M\setminus\{p_1,p_2\},\tilde{g}) = [0,\infty)\times S^{n-1}(1) \cup (\bar{M}\setminus\{t_\ell\}\times S^{n-1}, g_\ell) \cup [0,\infty)\times S^{n-1}(1).$$

We think of the function f_ℓ as defined on $\bar{M}\setminus\{t_\ell\}\times S^{n-1}$, and extend it to the whole space $M\setminus\{p_1,p_2\}$ as follows: Let F_ℓ be Lipshitz function of $\bar{M}\setminus\{p_1,p_2\}$ such that

$$F_\ell = f_\ell \qquad \text{on } \bar{M}\setminus\{t_\ell\}\times S^{n-1}$$

and

$$F_\ell(t,x) = \begin{cases} (1-t)\tilde{f}_\ell(x) & \text{for } (t,x)\in[0,1]\times S^{n-1}; \\ 0 & \text{for } (t,x)\in[1,\infty)\times S^{n-1}, \end{cases}$$

where $\tilde{f}_\ell = f_\ell|_{\{t_\ell\}\times S^{n-1}} \in C^\infty(S^{n-1})$. Now it is easy to see from (2.5) and Lemma 2.2 that

$$(2.8)\quad 4\frac{n-1}{n-2}\int_{M\setminus\{p_1,p_2\}}|dF_\ell|^2 dv_{\tilde g} + \int_{M\setminus\{p_1,p_2\}} R_{\tilde g} F_\ell{}^2 dv_{\tilde g}$$

$$= 4\frac{n-1}{n-2}\int_{\bar M}|df_\ell|^2 dv_{g_\ell} + \int_{\bar M} R_{g_\ell} f_\ell{}^2 dv_{g_\ell}$$

$$+ \frac{8(n-1)}{3(n-2)}\int_{S^{n-1}}|d\tilde f_\ell|^2 dv_{S^{n-1}} + \frac{2(n-1)(n^2-4n+16)}{3(n-2)}\int_{S^{n-1}}\tilde f_\ell{}^2 dv_{S^{n-1}}$$

$$< \mu(\bar M) + \varepsilon + \frac{B}{\ell},$$

where B is a constant independent of ℓ. Obviously from (2.6) we get

$$(2.9)\quad \int_{M\setminus\{p_1,p_2\}} F_\ell^{\frac{2n}{n-2}} dv > 1.$$

Therefore, we have

$$(2.10)\quad \inf \frac{4\frac{n-1}{n-2}\int_{M\setminus\{p_1,p_2\}}|dF|^2 dv_{\tilde g} + \int_{M\setminus\{p_1,p_2\}} R_{\tilde g} F^2 dv_{\tilde g}}{\left(\int_{M\setminus\{p_1,p_2\}} F^{\frac{2n}{n-2}} dv_{\tilde g}\right)^{\frac{n-2}{n}}}$$

$$\leq \mu(\bar M) + \varepsilon,$$

where the infimum is taken over all nonnegative Lipshitz functions F with compact support. It follows from the choice of the metric $\tilde g$ that the left side of (2.10) is equal to $\mu(M,C)$. Since ε can be chosen arbitrarily, we conclude $\mu(M) \leq \mu(\bar M)$, which completes the proof. □

Remark 1. The argument in [4; §4] works well to prove $\mu(M_1 \# M_2) \geq -((\mu(M_1)_-)^{n/2} + (\mu(M_2)_-)^{n/2})^{2/n}$, where $_-$ means the negative part, i.e., $a_- = \max\{-a,0\}$. Actually this gives a simpler proof than the above but cannot cover the case when both $\mu(M_1)$ and $\mu(M_2)$ are positive, and hence the second part of Theorem.

Remark 2. The part (ii) of Theorem can be proved in another way too. Here we shall show it briefly. For simplicity we assume $M = S^1 \times S^{n-1}$, $n \geq 3$. Put $g_\ell = dt^2 + g_0$, where dt^2 is the metric of S^1 with length$(S^1, dt^2) = \ell$, and g_0 is the standard metric of S^{n-1}. Solving the Yamabe problem for g_ℓ, we get a positive function f_ℓ such that $\mathrm{Vol}(f_\ell g_\ell) = 1$ and the scalar curvature of $f_\ell g_\ell$ is a constant equal to $\mu(S^1 x S^{n-1}, C_\ell)$, C_ℓ being the conformal class of g_ℓ. According to a theorem of Gidas, Ni and Nirenberg [2; Theorem 4], it turns out that the function f_ℓ depends only on the parameter t of S^1. So the problem is reduced to an ODE, and then by a routine argument we can see $\mu(S^1 x S^{n-1}, C_\ell) \to \mu(S^n)$ as $\ell \to \infty$.

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