Stability, complex-analyticity and constancy of pluriharmonic maps from compact Kaehler manifolds

Yoshihiro Ohnita[*)] and Seiichi Udagawa[**)]

*) Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str.26
5300 Bonn 3
Federal Republik of Germany
and
Department of Mathematics
Tokyo Metropolitan University
Fukasawa, Setagaya, Tokyo 158
Japan

**) Department of Mathematics
School of Medicine
Nihon University
Itabashi-ku, Tokyo 173
Japan

MPI/88-23

Stability, complex-analyticity and constancy of pluriharmonic maps from compact Kaehler manifolds

Yoshihiro Ohnita and Seiichi Udagawa

Dedicated to Professor S. Murakami for his sixtieth birthday

§0. Introduction.

Siu([18], [21]) studied the complex-analyticity of harmonic maps from compact Kaehler manifolds into compact Kaehler manifolds with strongly negative curvature tensor(see §1 for the definition) or compact quotients of irreducible symmetric bounded domains. In particular, he obtained the following results

<u>Theorem</u> A.

<u>Let</u> $f : M \to N$ <u>be a harmonic map from a compact Kaehler manifold into a compact Kaehler manifold with strongly negative curvature tensor. Then,</u> f <u>is holomorphic or anti-holomorphic if</u> $\operatorname{Max}_M \operatorname{rank}_{\mathbb{R}} df \geqq 4$.

<u>Theorem</u> B.

<u>Let</u> $f : M \to N$ <u>be a harmonic map from a compact Kaehler manifold into a compact quotient of an irreducible symmetric bounded domain</u> D. <u>Then,</u> f <u>is holomorphic or anti-holomorphic</u> <u>if</u> $\operatorname{Max}_M \operatorname{rank}_{\mathbb{R}} df \geqq 2p(D) + 1$, <u>where</u>

$p(D^{I_{mn}}) = (m-1)(n-1) + 1$, $p(D^{II_n}) = (1/2)(n-2)(n-3) + 1$, $p(D^{III_n}) = (1/2)n(n-1) + 1$, $p(D^{IV_n}) = 2$, $p(D^{V}) = 6$, $p(D^{VI}) = 11$.

The number $p(D)$ is important in this paper.

Definition 0.1.

Let N be an irreducible Hermitian symmetric space of compact or non-compact type. We define an integer $p(N)$ as follows

Type of N	N of compact type	$p(N)$	$\dim_{\mathbb{C}} N$
I_{mn}	$G_{m,n}(\mathbb{C}) = SU(m+n)/S(U(m)\times U(n))$	$(m-1)(n-1)+1$	mn
II_n	$SO(2n)/U(n)$	$(1/2)(n-2)(n-3)+1$	$(1/2)n(n-1)$
III_n	$Sp(n)/U(n)$	$(1/2)n(n-1)+1$	$(1/2)n(n+1)$
IV_n	$Q^n(\mathbb{C}) = SO(n+2)/SO(n)\times SO(2)$	2	n
V	$E_6/Spin(10)\cdot T$	6	16
VI	$E_7/E_6\cdot T$	11	27

Definition 0.2.

A map is called $\pm$-holomorphic if it is either holomorphic or anti-holomorphic.

Let $f : M \to N$ be a smooth map from a Kaehler manifold into a Riemannian manifold. Then, f is called pluriharmonic if $(1,1)$-part of the second fundamental form of f vanishes identically(see §1). Hence, a pluriharmonic map is a harmonic map, and if $\dim_{\mathbb{C}} M = 1$, a pluriharmonic map is nothing but a harmonic map. Obviously, a totally geodesic map is a pluriharmonic map. Moreover, a holomorphic(or anti-holomorphic) map is a pluriharmonic map if N is a Kaehler manifold. If N is a Kaehler manifold of which curvature tensor is strongly semi-negative in the sense of Siu[18], any harmonic map from compact Kaehler manifolds into N becomes a pluriharmonic map[20].

A harmonic map between compact Riemannian manifolds is called stable if the second variation of the energy is non-negative for every variation of the map. Lichnerowicz showed that any holomorphic(or anti-holomorphic) map between compact Kaehler manifolds is energy-minimizing in its homotopy class, hence stable(see [23]). From the second variation formula for the energy integral[23], it is known that any harmonic map into a Riemannian manifold of non-positive sectional curvature is stable.

From Theorem B and these points of views, it is natural to ask that "Is any stable harmonic map(or pluriharmonic map) from compact Kaehler manifolds into irreducible Hermitian symmetric spaces of compact type N $\pm$-holomorphic (with some assumptions of $p(N)$) ?".

There are some related results

Theorem C. ([2], [16]).

Any stable harmonic map from compact Riemann surface into irreducible Hermitian symmetric space of compact type is $\pm$-holomorphic.

We denote by $\mathbb{C}P^n$ an n-dimensional complex projective space with Fubini-Study metric.

<u>Theorem</u> D. ([24]).

<u>Any stable pluriharmonic map from compact Kaehler manifold into</u> $\mathbb{C}P^n$ <u>is</u> ±-<u>holomorphic.</u>

<u>Theorem</u> E. ([2]).

<u>Any stable harmonic map from</u> $\mathbb{C}P^n$ <u>into irreducible Hermitian symmetric space of compact type</u> N <u>is</u> ±-<u>holomorphic.</u>

The following result of the first author is used to prove Theorem E

<u>Theorem</u> F. ([14]).

<u>Any stable harmonic map from</u> $\mathbb{C}P^n$ <u>into any Riemannian manifold is pluriharmonic.</u>

<u>Remark</u> 0.3.

The case where $N = \mathbb{C}P^n$ in Theorem E is proved by the first author[14]. The special cases of Theorem C are treated in [3], [4] and [19](see also [27]).

We denote by $c_1(M)$ and $b_2(M)$ the first Chern class and the second Betti number of M, respectively.

Our main results are the following

<u>Theorem</u> 2.13.

<u>Let</u> M <u>be a compact Kaehler manifold with</u> $c_1(M) > 0$ <u>and</u> $b_2(M) = 1$. <u>Let</u> $f : M \to N$ <u>be a pluriharmonic map.</u> <u>Assume that</u> N <u>has positive curvature on totally isotropic 2-planes.</u> <u>Then, one of the following cases occurs</u>

i) f <u>is a constant map,</u>

ii) $\dim_{\mathbb{C}} M = 1$ <u>and</u> f <u>is a branched minimal immersion.</u>

For the definition of the "positive on totally isotropic 2-planes", see A1.

Theorem 3.5.

Let $f : M \to N$ be a pluriharmonic(resp. harmonic) map from an $m(\geqq 2)$-dimensional compact Kaehler manifold with $b_2(M) = 1$ into a Kaehler manifold with strongly positive(resp. negative) curvature tensor. Then, f is $\pm$-holomorphic.

Theorem 3.9.

Let M be an m-dimensional compact Kaehler manifold with $c_1(M) > 0$ and $b_2(M) = 1$ and let N be an irreducible Hermitian symmetric space of compact type. If $m \geqq p(N) + 1$, then any pluriharmonic map from M into N is $\pm$-holomorphic.

Theorem 3.18.

Let M be a compact Kaehler manifold with $c_1(M) > 0$ and $b_2(M) = 1$ and let N be a Kaehler manifold with $\dim_{\mathbb{C}} N = \dim_{\mathbb{C}} M$. Then, any pluriharmonic map from M into N is $\pm$-holomorphic.

Theorem 4.1.

Let M be an m-dimensional compact Kaehler manifold with $b_2(M) = 1$ and let N be an irreducible Hermitian symmetric space of compact type. If $m \geqq p(N)$, then any stable pluriharmonic map from M into N is $\pm$-holomorphic.

Theorem 4.3.

Let M be a compact Kaehler manifold with $c_1(M) > 0$ and $b_2(M) = 1$ and let N be an irreducible Hermitian symmetric space of compact type. Then, any stable pluriharmonic map from M into N is $\pm$-holomorphic.

Combining Theorem F and the above results, we have the various conclusions for the complex-analyticity and the constancy of stable harmonic map from $\mathbb{C}P^m$, which are stated in §5.

In §6, we show that there exist stable(resp. unstable) pluriharmonic, but non ±-holomorphic maps from compact Hermitian symmetric spaces M with $b_2(M) = 2$ (resp. $b_2(M) = 1$) and $\dim_{\mathbb{C}} M = p(N)$ into irreducible Hermitian symmetric space of compact type N except for the case where $N = \mathbb{C}P^n$.

In §7, we show that any stable totally geodesic isometric immersion between irreducible Hermitian symmetric spaces of compact type is ±-holomorphic.

In §8, we give a construction of non ±-holomorphic pluriharmonic maps into complex Grassmann manifolds using the method of Eells and Wood[7], in particular, we make pluriharmonic maps into complex Grassmann manifolds from holomorphic maps into complex projective spaces.

§1. Basic notations and definitions.

(A). Pluriharmonic maps.

Let $f : M \to N$ be a smooth map from a complex manifold into a Riemannian manifold. Let $TM^{\mathbb{C}}$ and $TN^{\mathbb{C}}$ be the complexifications of the tangent bundles of M and N, repectively. We have

$$(1.1)\quad TM^{\mathbb{C}} = TM^{1,0} + TM^{0,1} ,$$

where the fibre $T_x M^{1,0}$(resp. $T_x^{0,1}$) at $x \in M$ is the $\sqrt{-1}$(resp.

$-\sqrt{-1}$)-eigenspace of the complex structure tensor of M. We denote by T^*M the dual bundle of TM. The differential f_* of f extends by complex linearity to $f_* : TM^{\mathbb{C}} \to TN^{\mathbb{C}}$, which may be interpreted as a homomorphism from $TM^{\mathbb{C}}$ to $f^{-1}TN^{\mathbb{C}}$. We denote by df this homomorphism. We may define the bundle maps $\partial f : TM^{1,0} \to f^{-1}TN^{\mathbb{C}}$ and $\bar{\partial} f : TM^{0,1} \to f^{-1}TN^{\mathbb{C}}$. Let ∇^f be the pull-back connection on $f^{-1}TN$, which is extended by complex linearity to $f^{-1}TN^{\mathbb{C}}$. By this connection and $\bar{\partial}$-operator of M, we may define the $\bar{\partial}$-exterior derivative of $\partial f = (\partial f^A)$, which is an $f^{-1}TN^{\mathbb{C}}$-valued (1,1)-form on M and denoted by $\nabla''\partial f$, is defined by

$$(1.2)\quad \nabla''\partial f = \bar{\partial}\partial f^A + \Sigma\Gamma^A_{BC}\bar{\partial} f^B \wedge \partial f^C ,$$

where $\Gamma^A_{BC} = \Gamma^A_{BC} \circ f$ is the Christoffel symbol of N.

In the same way, we may define the ∂-exterior derivative $\nabla'\bar{\partial} f$ of $\bar{\partial} f$, which is also an $f^{-1}TN^{\mathbb{C}}$-valued (1,1)-form on M. Alternatively, $\nabla''\partial f$ is defined by

$$(1.3)\quad (\nabla''_{\bar{W}}\partial f)(Z) = \nabla^f_{\bar{W}}(\partial f(Z)) - \partial f(\bar{\partial}_{\bar{W}} Z) \quad \text{for any } Z, W \in C^\infty(TM^{1,0})$$

Then, f is called <u>pluriharmonic</u> if

(1.4) $\nabla''\partial f = 0$.

Now, assume that M is a Kaehler manifold. Set $\nabla = \nabla' + \nabla''$. ∇df is called the <u>second fundamental form</u> of f. It is known that

(1.5) $(\nabla_V df)(W) = (\nabla_W df)(V)$ for any $V, W \in C^\infty(TM^{\mathbb{C}})$,

that is, ∇df is a symmetric $\mathbb{C}$-bilinear form. Thus, f is called pluriharmonic if (1,1)-part of the second fundamental form of f vanishes identically. In terms of local complex coordinate system (z^i) of M, (1.4) is rewritten as

(1.6) $\nabla_j f_{\bar{i}} = 0$ for any $1 \leqq i, j \leqq m = \dim_{\mathbb{C}} M$,

where $\nabla_j = \nabla_{\partial/\partial z^j}$ and $f_{\bar{i}} = \partial f/\partial z^{\bar{i}}$. By (1.5), we have $\nabla_j f_{\bar{i}} = \nabla_{\bar{i}} f_j$ and $\nabla_j f_i = \nabla_i f_j$.

(B). Curvature conditions.

Let (M, g) be an m-dimensional Riemannian manifold. We denote by $\langle\ ,\ \rangle$ (real) inner product of tensor bundles of M induced by g and by $(\ ,\ )$ the complex extension of $\langle\ ,\ \rangle$. Define the Hermitian inner product $\langle\langle\ ,\ \rangle\rangle$ by

(1.7) $\langle\langle u, v\rangle\rangle = (u, \bar{v})$.

The curvature tensor R of (M, g) is defined by

(1.8) $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$, $X, Y \in TM$,

where ∇ is the Riemannian connection of (M, g).
The sectional curvature K is defined by

(1.9) $K = K(\sigma)$

$= \langle R(X, Y)Y, X\rangle$,

where $\sigma = \mathrm{span}_{\mathbb{R}}(X, Y)$ is a 2-dimensional subspace of T_pM, $p \in M$. We denote by Q the curvature operator of (M, g), which is defined by

(1.10) $\langle Q(X\wedge Y), Z\wedge W\rangle = \langle R(X, Y)W, Z\rangle$, $X,Y,Z,W \in T_pM$.

The complex extension of Q to $\wedge^2 TM^{\mathbb{C}}$ is also denoted by Q.

<u>Definition</u> 1.11.

Let σ be a complex 2-dimensional subspace of $T_pM^{\mathbb{C}}$, $p \in M$. Then, the <u>complex sectional curvature</u> for σ, denoted by $\tilde{K}(\sigma)$, is defined by

(1.12) $\tilde{K}(\sigma) = \langle\langle Q(Z\wedge \bar{W}), \overline{Z\wedge W}\rangle\rangle$,

where $\{Z, W\}$ is a unitary basis of σ.

<u>Remark</u> 1.13.

If σ is real, i.e., $\sigma = \bar{\sigma}$, $\tilde{K}(\sigma)$ is nothing but a sectional curvature $K(\sigma)$. If $Q > 0$ (resp. $\geqq 0$, < 0, $\leqq 0$), then $\tilde{K} > 0$ (resp. $\geqq 0$, < 0, $\leqq 0$).

<u>Definition</u> 1.14.

An element $Z \in T_pM^{\mathbb{C}}$ is called <u>isotropic</u> if $(Z, Z) = 0$. Let σ be a complex subspace of $T_pM^{\mathbb{C}}$. Then, σ is called <u>totally isotropic</u> if $(Z, Z) = 0$ for any $Z \in \sigma$.

<u>Definition</u> 1.15.

We say that (M, g) has positive (resp. non-negative, negative, non-positive) curvature on totally isotropic 2-planes if $\tilde{K}(\sigma) > 0$ (resp. $\geqq 0$, < 0, $\leqq 0$) for any totally isotropic 2-dimensional complex subspace σ of $T_pM^{\mathbb{C}}$ and any $p \in M$.

<u>Remark</u> 1.16.

By [12], the following are known

1) If M has positive curvature operator, then M has positive curvature on totally isotropic 2-planes.

2) If $(1/4)\delta < K \leqq \delta$, where δ is a positive function on M, then M has positive curvature on totally isotropic 2-planes.

<u>Example</u> 1.17.

Let (M, g) be a symmetric space of compact type(resp. non-compact type). Then, $Q \geqq 0$ (resp. $\leqq 0$). In particular, (M, g) has non-negative (resp. non-positive) complex sectional curvature.

Now, assume that (M, g) is a Kaehler manifold. We have the decomposition

$$\Lambda^2 TM^{\mathbb{C}} = \Lambda^{(2,0)}TM + \Lambda^{(1,1)}TM + \Lambda^{(0,2)}TM .$$

By the Kaehler identity of M, we have

$$Q|_{\Lambda^{(2,0)}TM} = Q|_{\Lambda^{(0,2)}TM} = 0 .$$

Set $Q^{(1,1)} = Q : \Lambda^{(1,1)}TM \to \Lambda^{(1,1)}TM$.

<u>Example</u> 1.18.

If $M = \mathbb{C}P^m$ (resp. $\mathbb{C}H^m$), then $Q^{(1,1)} > 0$ (resp. < 0), where $\mathbb{C}H^m$ is an m-dimensional complex hyperbolic space.

<u>Remark</u> 1.19.

If $\sigma = \mathrm{span}_{\mathbb{C}}\{Z, W\} \subset T_pM^{\mathbb{C}}$, where $\{Z, W\}$ is a unitary basis for σ, then

$$\begin{aligned}\tilde{K}(\sigma) &= \langle\langle Q^{(1,1)}(Z\wedge W), Z\wedge W\rangle\rangle \\ &= \langle\langle Q^{(1,1)}(Z\wedge W)^{(1,1)}, (Z\wedge W)^{(1,1)}\rangle\rangle \\ &= \langle\langle Q^{(1,1)}(Z^{(1,0)}\wedge W^{(0,1)} - W^{(1,0)}\wedge Z^{(0,1)}), \\ &\qquad Z^{(1,0)}\wedge W^{(0,1)} - W^{(1,0)}\wedge Z^{(0,1)}\rangle\rangle .\end{aligned}$$

where $Z^{(p,q)}$ is the (p,q)-part$(p,q = 0$ or $1)$ of the vector Z.

<u>Definition</u> 1.20. ([18]).

We say that (M, g) has <u>very strongly positive (resp. semi-positive, negative, semi-negative) curvature tensor if</u>

$Q^{(1,1)} > 0$ (resp. $\geqq 0$, < 0, $\leqq 0$) .

<u>Definition</u> 1.21. ([18]).

We say that (M, g) has <u>strongly positive (resp. semi-positive, negative, semi-negative) curvature tensor if</u>

$\langle\langle Q^{(1,1)}(\xi), \xi\rangle\rangle > 0$ (resp. $\geqq 0$, < 0, $\leqq 0$) for any

$\xi = Z_1 \wedge \bar{W}_1 + Z_2 \wedge \bar{W}_2 \neq 0$, $Z_1, Z_2, W_1, W_2 \in T_pM^{1,0}$ and any $p \in M$.

<u>Example</u> 1.22.

$\mathbb{C}P^m$ (resp. $\mathbb{C}H^m$) has very strongly positive (resp. negative) curvature tensor.

§2. Constancy of pluriharmonic maps into Riemannian manifolds.

Let (M, g, J) and (N, h) be a Kaehler manifold and a Riemannian manifold, respectively. Let $f : M \to N$ be a pluriharmonic map. We define a smooth section ω of $\overset{2}{\otimes} T^* M^{\mathbb{C}}$ by

(2.1) $\omega(X, Y) = (f^* h)(JX, Y)$ for any $X, Y \in T_x M^{\mathbb{C}}$, any $x \in M$.

Then, ω can be decomposed as $\omega = \omega^{(2,0)} + \omega^{(1,1)} + \omega^{(0,2)}$ according to the decomposition

$$\overset{2}{\otimes} T^* M^{\mathbb{C}} = \overset{2}{\otimes} T^* M^{1,0} + (T^* M^{1,0} \otimes T^* M^{0,1} + T^* M^{0,1} \otimes T^* M^{1,0}) + \overset{2}{\otimes} T^* M^{0,1}.$$

Note that $\omega^{(1,1)}$ is a real (1,1)-form on M given by

$$\omega^{(1,1)} = \sqrt{-1}\Sigma_{i,j} h(f_i, f_{\bar{j}}) dz^i \wedge dz^{\bar{j}} .$$

Lemma 2.2. ([14]).

$\omega^{(2,0)} = \overline{\omega^{(0,2)}}$ *is a holomorphic section of* $\otimes^2 T^* M^{1,0}$, *and* $\omega^{(1,1)}$ *is a nonnegative closed real* (1,1)-*form on* M.

Lemma 2.3.

Let M *be a compact Kaehler manifold with* $c_1(M) > 0$. *Then,* $\omega = \omega^{(1,1)}$.

Proof. The solution of Yau for Calabi conjecture[26] ensures an existence of a Kaehler metric on M with positive Ricci curvature. Using a formula of Bochner type with respect to this Kaehler metric, we can show a nonexistence of nonzero holomorphic sections of $\otimes^2 T^* M^{1,0}$. By Lemma 2.2, we get $\omega^{(2,0)} = \overline{\omega^{(0,2)}} = 0$. Thus, $\omega = \omega^{(1,1)}$. Q.E.D.

Proposition 2.4.

Let M *be an* m-*dimensional compact Kaehler manifold with* $b_2(M) = 1$ *and* $f : M \to N$ *be a pluriharmonic map.*

(1) *If* $\mathrm{rank}_{\mathbb{C}} \partial f < m$ *on* M, *then* f *is a constant map.*

(2) *If* $c_1(M) > 0$ *and* f *is nonconstant, then there exists an open subset* U *of* M *such that*

(i) $f : U \to N$ *is an immersion,*

(ii) (U, f^*h) <u>is a Kaehler manifold, and</u>

(iii) $f : (U, f^*h) \to N$ <u>is a pluriharmonic isometric immersion.</u>

Proof. Assume that ∂f is not injective at every point of M. Then, we have

$$(\omega^{(1,1)})^m = \underbrace{\omega^{(1,1)}\wedge \cdots \wedge\omega^{(1,1)}}_{m\text{-times}} = 0 \quad \text{on} \quad M,$$

Hence, $[\omega^{(1,1)}]^m = 0$ as an element of $H^*(M, \mathbb{R})$. Since $b_2(M) = 1$, we have $[\omega^{(1,1)}] = 0$ as an element of $H^2(M, \mathbb{R})$. Since ω is a closed real (1,1)-form cohomologous to 0, there exists a real smooth function ψ on M such that $\omega^{(1,1)} = \sqrt{-1}\partial\bar{\partial}\psi$. By the non-negativity of $\omega^{(1,1)}$, ψ is a subharmonic function on M, hence constant. Thus, we have $\omega^{(1,1)} = 0$, which implies that $\partial f = 0$. Therefore, f is a constant map. Then, we get (1). Next, we show (2). By Lemma 2.3, we have

$$h((\partial f)(T_xM^{1,0}), \overline{(\bar{\partial}f)(T_xM^{0,1})}) = 0 \quad \text{for any} \quad x \in M.$$

Therefore, $\mathrm{rank}_{\mathbb{R}} df = 2\mathrm{rank}_{\mathbb{C}}\partial f$ at each point of M. If $\mathrm{rank}_{\mathbb{R}} df < 2m$ on M, then $\mathrm{rank}_{\mathbb{C}}\partial f < m$ on M, and by (1), f is a constant map, which is a contradiction. Hence, $\mathrm{rank}_{\mathbb{R}} df = 2m$ on some open subset of M. By Lemma 2.2 and 2.3, we get (i), (ii) and (iii).

Q.E.D.

<u>Remark</u> 2.5.

If (N, h) is a real analytic Riemannian manifold, we can take the above open subset U of M so that U is dense in M and $M \setminus U$ is a real analytic subvariety of M.

For later use, we prepare the following lemma

Lemma 2.6.

If $f : M \to (N, h)$ *is a harmonic map from a Kaehler manifold into a Riemannian manifold, then, with respect to local unitary frame fields on* M, *we have*

$$(2.7)\quad \Sigma_{i,j}\nabla_{\bar{j}}\nabla_{\bar{i}}h(f_i, f_j) = \Sigma|\nabla_{\bar{j}}f_i|^2 - \Sigma h(f_i, {}_N R(f_{\bar{i}}, f_{\bar{j}})f_j) ,$$

where ${}_N R$ *is the curvature tensor of* N.

Proof. Since f is harmonic, we have

$$\Sigma_i\nabla_{\bar{i}}h(f_i, f_j) = \Sigma_i h(f_i, \nabla_{\bar{i}}f_j) \quad \text{for any } 1 \leqq j \leqq m = \dim_{\mathbb{C}}M.$$

Thus,

$$\Sigma_{i,j}\nabla_{\bar{j}}\nabla_{\bar{i}}h(f_i, f_j) = \Sigma|\nabla_{\bar{j}}f_i|^2 + \Sigma h(f_i, \nabla_{\bar{j}}\nabla_{\bar{i}}f_j) .$$

On the other hand, by the harmonicity of f and the Ricci identity, we get

$$\Sigma_j\nabla_{\bar{j}}\nabla_{\bar{i}}f_j = -\Sigma_N R(f_{\bar{i}}, f_{\bar{j}})f_j .$$

Thus, we have (2.7). Q.E.D.

Note that $\nabla_{\bar{k}}h(f_i, f_j) = 0$ for any $1 \leqq i,j,k \leqq m$ if f is a pluriharmonic map. Since $(f^*h)^{(2,0)} = 0$ if and only if $h(f_i, f_j) = 0$ for any $1 \leqq i,j \leqq m = \dim_{\mathbb{C}}M$, we have

Corollary 2.8.

Let $f : M \to (N, h)$ be a harmonic map from a Kaehler manifold into a Riemannian manifold with non-positive complex sectional curvature. If $(f^*h)^{(2,0)} = 0$, then f is pluriharmonic.

We recall the definitions of "isotropy", "total isotropy" and "positivity or negativity on totally isotropic 2-planes". The condition that N has positive (or negative) curvature on totally isotropic 2-planes is always satisfied if $\dim N \leqq 3$.

By Lemma 2.3, we have

Lemma 2.9.

Let M be a compact Kaehler manifold with $c_1(M) > 0$. Let $f : M \to (N, h)$ be a pluriharmonic map into a Riemannian manifold. Then, $df(T_xM^{1,0})$ is a totally isotropic subspace of $T_{f(x)}N^{\mathbb{C}}$ for any $x \in M$.

Proposition 2.10.

Let $f : M \to (N, h)$ be a pluriharmonic map from a Kaehler manifold into a Riemannian manifold. If N has positive(or negative) curvature operator, then

$$\operatorname{rank}_{\mathbb{C}} \partial f \leqq 1 \quad \text{on} \quad M.$$

Further if M *is compact,* $m = \dim_{\mathbb{C}} M \geqq 2$ *and* $b_2(M) = 1$, *then* f *is a constant map.*

Proof. By (2.7),

$$0 = \Sigma h(f_i, {}_N R(f_{\bar{i}}, f_{\bar{j}}) f_j)$$

$$= \Sigma h({}_N Q(f_{\bar{i}} \wedge f_{\bar{j}}), f_i \wedge f_j) ,$$

where ${}_N Q$ is the curvature operator of N. Since N has positive (or negative) curvature operator, we have

$$f_i \wedge f_j = 0 \quad \text{for any} \quad 1 \leqq i, j \leqq m .$$

Thus, $\mathrm{rank}_{\mathbb{C}} \partial f \leqq 1$. The last statement of Proposition 2.10 follows from Proposition 2.4 (1). Q.E.D.

Proposition 2.11.

Let M *be a compact Kaehler manifold with* $c_1(M) > 0$. *Let* $f : M \to N$ *be a pluriharmonic map. Assume that* N *has positive (or negative) curvature on totally isotropic 2-planes. Then,*

$$\mathrm{rank}_{\mathbb{C}} \partial f \leqq 1 \quad \text{on} \quad M.$$

Proof. By Lemma 2.9, $\mathrm{span}_{\mathbb{C}}\{f_i, f_j\}$ for $1 \leqq i \neq j \leqq m$ is a totally isotropic 2-planes. Then, the proof of Proposition 2.10 yields the conclusion. Q.E.D.

Remark 2.12. $\mathrm{rank}_{\mathbb{C}} \partial f \leqq 1$ implies $\mathrm{rank}_{\mathbb{R}} df \leqq 2$.

By Propositions 2.4, 2.11 and the well-known facts for harmonic maps of Riemann spheres(see [7], see also §3 in [25]), we obtain

Theorem 2.13.

Let M be a compact Kaehler manifold with $c_1(M) > 0$ and $b_2(M) = 1$. Let $f : M \to N$ be a pluriharmonic map. Assume that N has positive curvature on totally isotropic 2-planes. Then, one of the following cases occurs

(i) f is a constant map,

(ii) $\dim_{\mathbb{C}} M = 1$ and f is a branched minimal immersion.

§3. Complex-analyticity and constancy of pluriharmonic maps into Kaehler manifolds.

Let $f : M \to N$ be a smooth map between Kaehler manifolds. Let (z^i) and (w^α) be local complex coordinate systems for M and N, respectively, and put $f^\alpha = w^\alpha \circ f$. By §1, it is clear that f is pluriharmonic if and only if

$$(3.1) \quad \nabla_i f^{\bar{\alpha}}_{\bar{j}} = 0$$

for any $1 \leqq i,j \leqq m = \dim_{\mathbb{C}} M$, $1 \leqq \alpha \leqq n = \dim_{\mathbb{C}} N$.

It is also clear that $\nabla_{\bar{j}} f^{\alpha}_{\bar{i}} = \nabla_{\bar{i}} f^{\alpha}_{\bar{j}} = \overline{\nabla_i f^{\bar{\alpha}}_{j}}$.

In [24], modifying the proof of Theorem B, the following is obtained

Theorem G.

Let $f : M \to N$ be a pluriharmonic map from a Kaehler manifold into an irreducible Hermitian symmetric space of compact or non-compact type. Then, f is $\pm$-holomorphic if

$\underset{M}{\text{Max}}\ \text{rank}_{\mathbb{R}} df \geqq 2p(N) + 1.$

Let $f : M \to N$ be a harmonic map from a Kaehler manifold into a Kaehler manifold with the Kaehler metric h. Then, by Lemma 2.6, denoting by ${}_N R_{\bar{\alpha}\beta\gamma\bar{\delta}}$ the component of the curvature tensor of N, we get

$$(3.2)\quad \Sigma \nabla_{\bar{j}} \nabla_{\bar{i}} h(f_i, f_j)$$

$$= \Sigma |\nabla_{\bar{j}} f_i|^2 - \Sigma {}_N R_{\bar{\alpha}\beta\gamma\bar{\delta}} (f_{\bar{i}}^{\bar{\alpha}} f_{\bar{j}}^{\beta} - f_{\bar{j}}^{\bar{\alpha}} f_{\bar{i}}^{\beta})(f_{\bar{i}}^{\gamma} f_{\bar{j}}^{\bar{\delta}} - f_{\bar{j}}^{\gamma} f_{\bar{i}}^{\bar{\delta}}) ,$$

$$= \Sigma |\nabla_{\bar{j}} f_i|^2$$

$$- \Sigma h({}_N Q^{(1,1)}(f_i^{1,0} \wedge f_j^{0,1} - f_j^{1,0} \wedge f_i^{0,1}, \overline{f_i^{1,0} \wedge f_j^{0,1} - f_j^{1,0} \wedge f_i^{0,1}}),$$

where ${}_N Q^{(1,1)}$ is the curvature operator of N and $f_i^{1,0}$ (resp. $f_i^{0,1}$) is the (1,0) (resp. (0,1))-component of the vector f_i.

Using (3.2), we can prove Theorem G for the case where $N = \mathbb{C}P^n$ or $\mathbb{C}H^n$. In fact, the smarter equation(see §3 in [24]) can be applied. However, the constancy of the holomorphic sectional curvature of $\mathbb{C}P^n$ or $\mathbb{C}H^n$ is not essential, that is, we have

Proposition 3.3.

Let $f : M \to N$ be a pluriharmonic map from an m-dimensional Kaehler manifold into a Kaehler manifold with strongly positive or negative curvature tensor. Then, one of the following cases occurs

(i) f is $\pm$-holomorphic,

(ii) $\mathrm{rank}_{\mathbb{C}}\partial f \leq 1$ on M.

Proof. From the proof of Theorem G, it is enough to prove that if $\mathrm{rank}_{\mathbb{C}}\partial f \geq 2$ at some point $p \in M$, then f is $\pm$-holomorphic at p. Since f is pluriharmonic, by (3.2) and the strong positivity or negativity of the curvature tensor of N, we obtain

(3.4) $f_i^{1,0} \wedge f_j^{0,1} - f_j^{1,0} \wedge f_i^{0,1} = 0$ for any $1 \leq i,j \leq m$.

We may assume that $m \geq 2$. Suppose that f is non $\pm$-holomorphic at $p \in M$. At p, there exist the indices i and k such that $f_i^{1,0} \neq 0$, $f_k^{0,1} \neq 0$. Moreover, we may assume that $i \neq k$. Otherwise, we have $\mathrm{rank}_{\mathbb{C}}\partial f \leq 1$ at p. By (3.4), we have $f_k^{1,0} \neq 0$, $f_i^{0,1} \neq 0$. Thus, (3.4) implies that for any fixed index j $(1 \leq j \leq m)$,

$$f_j^{1,0} = c\, f_i^{1,0},$$

$$f_j^{0,1} = c\, f_i^{0,1} \qquad \text{for some } c \in \mathbb{C}.$$

Hence, $\mathrm{rank}_{\mathbb{C}}\partial f \leq 1$ at p. Q.E.D.

Theorem 3.5.

Let $f : M \to N$ be a pluriharmonic (resp. harmonic) map from $m(\geq 2)$-dimensional compact Kaehler manifold with $b_2(M) = 1$

into a Kaehler manifold with strongly positive(resp. negative) curvature tensor. Then, f is ±-holomorphic.

Proof. First, note that any harmonic map from compact Kaehler manifold into a Kaehler manifold with strongly seminegative curvature tensor becomes a pluriharmonic map(see [20]). By Proposition 3.3, f is ±-holomorphic or $\mathrm{rank}_{\mathbb{C}}\partial f \leqq 1$ on M. If $\mathrm{rank}_{\mathbb{C}}\partial f \leqq 1$ on M, by $m \geqq 2$ and Proposition 2.4, f is a constant map. Q.E.D.

Let (N, h) be an irreducible Hermitian symmetric space of noncompact type. Then, it is known [20] that N has strongly seminegative curvature tensor. Then, by Corollary (2.8) (or (3.2)) and Theorem G, we obtain

Theorem 3.6.

Let $f : M \to (N, h)$ be a smooth map from a Kaehler manifold into an irreducible Hermitian symmetric space of non-compact type. Assume that $\underset{M}{\mathrm{Max}}\ \mathrm{rank}_{\mathbb{R}} df \geqq 2p(N) + 1$. Then, the following conditions are mutually equivalent

(i) f is ±-holomorphic,

(ii) f is pluriharmonic,

(iii) f is harmonic and $(f^{*}h)^{(2,0)} = 0$.

Since an isometric immersion f satisfies $(f^{*}h)^{(2,0)} = 0$, we get

Corollary 3.7. ([24]).

Any isometric minimal immersion from a Kaehler manifold M into an irreducible Hermitian symmetric space N of non-compact type is ±-holomorphic if $\dim_{\mathbb{C}} M \geqq p(N) + 1$.

Remark 3.8.

The condition $(f^*h)^{(2,0)} = 0$ is satisfied if $c_1(M) > 0$ by Lemma 2.3. However, it is known that any harmonic map from a compact manifold with positive Ricci curvature into a manifold with non-positive sectional curvature must be constant.

Theorem G and Proposition 2.4 yields

Theorem 3.9.

Let M be a compact Kaehler manifold with $c_1(M) > 0$ and $b_2(M) = 1$. Let N be an irreducible Hermitian symmetric space of compact type. Then, any pluriharmonic map from M into N is $\pm$-holomorphic if $\dim_{\mathbb{C}} M \geq p(N) + 1$.

Lemma 3.10.

Let M be a compact Kaehler manifold with $c_1(M) > 0$ and let (N, h) be a Kaehler manifold. If $f : M \to N$ is a pluriharmonic map, then

$$(3.11) \quad \Sigma h_{\alpha\bar{\beta}} f^\alpha_i f^{\bar{\beta}}_j dz^i \otimes dz^j = 0 \ .$$

Proof. Define a smooth section $\zeta = (\zeta_{ij})$ of $\overset{2}{\otimes} T^* M^{1,0}$ by

$\zeta_{ij} = \Sigma h_{\alpha\bar{\beta}} f^\alpha_i f^{\bar{\beta}}_j$. Since f is pluriharmonic, we obtain

$$\begin{aligned}\nabla_{\bar{k}} \zeta_{ij} &= \Sigma h_{\alpha\bar{\beta}} ((\nabla_{\bar{k}} f^\alpha_i) f^{\bar{\beta}}_j + f^\alpha_i (\nabla_{\bar{k}} f^{\bar{\beta}}_j)) \\ &= 0 \ ,\end{aligned}$$

that is, ς is a holomorphic section of $\overset{2}{\otimes} T^* M^{1,0}$. By the same way as Lemma 2.3, we see that $\varsigma = 0$. Q.E.D.

Proposition 3.12.

Let M be as in Lemma 3.10. Let N be a Riemann surface. Then, any pluriharmonic map from M into N is $\pm$-holomorphic.

Proof. Since $\dim_{\mathbb{C}} N = 1$, by (3.11), we obtain

$$f_i^1 f_j^{\bar{1}} = 0 \quad \text{for any} \quad 1 \leqq i,j \leqq m,$$

which implies that f is $\pm$-holomorphic at each point of M. If f is $\pm$-holomorphic on some open subset of M, then Siu's unique continuation theorem[18] yields $\pm$-holomorphicity of f. Q.E.D.

Remark 3.13.

Proposition 3.12 for the case where $\dim_{\mathbb{C}} M = 1$ is due to Eells and Wood[6].

Lemma 3.14.

Let $f : M \to (N, h)$ be a pluriharmonic map from a compact Kaehler manifold into a Kaehler manifold. Then,

$$\omega_1 = \sqrt{-1}\Sigma h_{\alpha\bar{\beta}} f^{\alpha}_{i} f^{\bar{\beta}}_{\bar{j}} dz^{i} \wedge dz^{\bar{j}} \quad \text{and} \quad \omega_2 = \sqrt{-1}\Sigma h_{\alpha\bar{\beta}} f^{\bar{\beta}}_{i} f^{\alpha}_{\bar{j}} dz^{i} \wedge dz^{\bar{j}}$$

are nonnegative closed real (1,1)-forms on M.

Proof. Let $\bar{\partial}$ be the exterior differential operator which sends (p,q)-forms to (p,q+1)-forms. We have

$$(3.15) \quad (\bar{\partial}\omega_1)_{i\bar{j}\bar{k}} = -\nabla_{\bar{j}}\omega_{i\bar{k}} + \nabla_{\bar{k}}\omega_{i\bar{j}}$$

$$= -\sqrt{-1}\Sigma h_{\alpha\bar{\beta}}(f^{\alpha}_{i}(\nabla_{\bar{j}} f^{\bar{\beta}}_{\bar{k}}) - f^{\alpha}_{i}(\nabla_{\bar{k}} f^{\bar{\beta}}_{\bar{j}}))$$

by the plurifarmonicity of f.

$\bar{\partial}\omega_1$ is skew-symmetric with respect to the indices j and k. On the other hand, the right hand side of (3.15) is symmetric with respect to the indices j and k. Thus, $\bar{\partial}\omega_1 = 0$. Since ω_1 is real, we get $d\omega_1 = 0$. In the same way as the case of ω_1, we may treat ω_2. Q.E.D.

Proposition 3.16.

Let $f : M \to N$ be a pluriharmonic map from m-dimensional compact Kaehler manifold with $b_2(M) = 1$ into a Kaehler manifold. If $\mathrm{rank}_{\mathbb{C}}(\partial f^{\alpha}) < m$ (resp. $\mathrm{rank}_{\mathbb{C}}(\bar{\partial} f^{\alpha}) < m$) on M, then, f is anti-holomorphic(resp. holomorphic.).

Proof. Let $\omega_1 = \sqrt{-1}\Sigma h_{\alpha\bar{\beta}} f^{\alpha}_i \bar{f^{\beta}_j} dz^i \wedge d\bar{z}^j$. If $\mathrm{rank}_{\mathbb{C}}(\partial f^{\alpha}) < m$ on M, by Lemma 3.14 we see that $[\omega_1]^m = [\omega_1^m] = 0$ as an element of $H^*(M, \mathbb{R})$. In the same way as the proof of Proposition 2.4, we obtain $\omega_1 = 0$, which implies that f is anti-holomorphic. If we replace ω_1 by ω_2 of Lemma 3.14,
we get the holomorphicity of f. Q.E.D.

Corollary 3.17.

Let $f : M \to N$ be as in Proposition 3.16. If $\dim_{\mathbb{C}} N < \dim_{\mathbb{C}} M$, then f is a constant map.

Theorem 3.18.

Let $f : M \to N$ be a pluriharmonic map from a compact Kaehler manifold with $c_1(M) > 0$ and $b_2(M) = 1$ into a Kaehler manifold. Assume that $\dim_{\mathbb{C}} M = \dim_{\mathbb{C}} N$. Then, f is $\pm$-holomorphic.

Proof. Let $\dim_{\mathbb{C}} M = \dim_{\mathbb{C}} N = m$. By Lemma 3.10, we have

$$(3.19) \quad \Sigma_{\alpha} f^{\alpha}_i \bar{f}^{\alpha}_j = 0 \quad \text{for any } 1 \leqq i, j \leqq m,$$

where we have used unitary basis of N. If f is not anti-holomorphic, then, by Proposition 3.16, there exists an open subset U of M such that $\mathrm{rank}_{\mathbb{C}}(\partial f^{\alpha}) = m$ on U. Thus,

(f_i^α) is the non-singular m×m-matrix on U. Therefore, (3.19) implies that f is holomorphic on U. Then, Siu's unique continuation theorem yields the holomorphicity of f. Q.E.D.

A4. Complex-analyticity of stable pluriharmonic maps.

If a pluriharmonic map f is stable as a harmonic map, then we say that f is a stable pluriharmonic map. In this section, we investigate the complex-analyticity of stable pluriharmonic maps from certain compact Kaehler manifolds into an irreducible Hermitian symmetric spaces of compact type. First, we state the following

Theorem H. [24].

Let f : M → N be a stable pluriharmonic map from a compact Kaehler manifold into an irreducible Hermitian symmetric space of compact type. Assume that

$$\operatorname*{Max}_M(\operatorname{rank}_{\mathbb{C}}(\partial f^\alpha) + \operatorname{rank}_{\mathbb{C}}(\bar\partial f^\alpha)) \geqq p(N) + 1.$$

Then, f is ±-holomorphic.

It can be verified that the condition $\operatorname*{Max}_M \operatorname{rank}_{\mathbb{R}} df \geqq 2p(N) + 1$ implies the condition $\operatorname*{Max}_M(\operatorname{rank}_{\mathbb{C}}(\partial f^\alpha) + \operatorname{rank}_{\mathbb{C}}(\bar\partial f^\alpha)) \geqq p(N) + 1$ (see [24]). If $N = \mathbb{C}P^n$, then Theorem H is reduced to Theorem D. Theorem H is used to prove the following

Theorem 4.1.

Let M be an m-dimensional compact Kaehler manifold with $b_2(M) = 1$ and let N be an irreducible Hermitian symmetric space of compact type. If $m \geqq p(N)$, then any stable pluriharmonic map f from M into N is $\pm$-holomorphic.

Proof. Assume that f is non $\pm$-holomorphic. Then, by Theorem H we have

$$(4.2) \quad \mathrm{rank}_{\mathbb{C}}(\partial f^{\alpha}) + \mathrm{rank}_{\mathbb{C}}(\bar{\partial} f^{\alpha}) \leqq p(N) \quad \text{on} \quad M.$$

On the other hand, by Proposition 3.16, there exists an open subset U of M such that $\mathrm{rank}_{\mathbb{C}}(\partial f^{\alpha}) = m$ on U, which, together with (4.2), yields

$$\begin{aligned} p(N) &\geqq \mathrm{rank}_{\mathbb{C}}(\partial f^{\alpha}) + \mathrm{rank}_{\mathbb{C}}(\bar{\partial} f^{\alpha}) \\ &\geqq m \end{aligned}$$

on U.

Therefore, if $m \geqq p(N)$, f is holomorphic on U. Then, by Siu's unique continuation theorem, f is holomorphic, which is a contradiction. Q.E.D.

Theorem 4.3.

Let M be a compact Kaehler manifold with $c_1(M) > 0$ and $b_2(M) = 1$ and let N be an irreducible Hermitian symmetric space of compact type. Then, any stable pluriharmonic map from M into N is $\pm$-holomorphic.

To prove Theorem 4.3, we need the following lemma

<u>Lemma</u> 4.4. [16].

<u>If</u> $f : M \to N$ <u>is a stable harmonic map from a compact Kaehler manifold into an irreducible Hermitian symmetric space of compact type, then, with respect to unitary basis of</u> M, <u>we have</u>

(4.5) $\Sigma_i (f^{\alpha}_{i} f^{\beta}_{\bar i} + f^{\beta}_{i} f^{\alpha}_{\bar i}) = 0$ <u>for any</u> $1 \leqq \alpha, \beta \leqq n = \dim_{\mathbb{C}} N$.

Proof. This is an immediate consequence of Proposition 1 in [16](see (1.5) of p.387 in [16]). Q.E.D.

Proof of Theorem 4.3. We use unitary bases of M and N. By Lemma 3.10, we get

(4.6) $\Sigma_{\alpha} f^{\alpha}_{i} \overline{f^{\alpha}_{j}} = 0$ for any $1 \leqq i, j \leqq m = \dim_{\mathbb{C}} M$.

Since f is a stable pluriharmonic map, by Lemma 4.4, the equation (4.5) holds. The equation (4.5) and (4.6) yield

(4.7) $\Sigma_{\alpha,\beta,i} f^{\alpha}_{\bar i} \overline{f^{\alpha}_{\bar j}} f^{\beta}_{i} \overline{f^{\beta}_{\bar k}} = 0$ for any $1 \leqq j, k \leqq m$.

We put $\zeta_{ij} = \Sigma_{\alpha} f^{\alpha}_{\bar i} \overline{f^{\alpha}_{\bar j}}$. If f is not holomorphic, by Proposition 3.16, there exists an open subset U of M such that $\mathrm{rank}_{\mathbb{C}}(\bar\partial f^{\alpha}) = m$ on U. Therefore, $\zeta = (\zeta_{ij})$ is the non-singular $m \times m$-hermitian matrix at each point of U. Then, (4.7) implies that f is anti-holomorphic on U. This, together with Siu's unique continuation theorem, yields the anti-holomorphicity of f . Q.E.D.

<u>Remark</u> 4.8.

In Theorem 4.3, if $\dim_{\mathbb{C}} M = 1$, Theorem 4.3 is contained in Theorem C. If $N = \mathbb{C}P^n$, Theorem 4.3 is contained in Theorem D. If $N = Q^n (n \geqq 3)$(complex hyperquadric), Theorem 4.3 is contained in Theorem 4.1.

§5. Complex-analyticity and constancy of stable harmonic maps from $\mathbb{C}P^m$.

In this section, we state the immediate consquences of Theorem F and the results in §2 - §4.

<u>Theorem</u> 5.1.

<u>Let</u> $f : \mathbb{C}P^m \to N$ <u>be a stable harmonic map into a Riemannian manifold.</u> <u>Assume that</u> N <u>has positive curvature on totally isotropic 2-planes with</u> $\dim N \geqq 4$ <u>or</u> N <u>has positive Ricci curvature with</u> $\dim N = 3$. <u>Then,</u> f <u>is constant.</u>

Proof. If $m = 1$, Theorem 5.1 is due to Micallef and Moore[12]. If $m \geqq 2$, Theorem F and Theorem 2.13 yield the conclusion. Q.E.D.

We conjecture that "<u>If</u> $f : M \to N$ <u>is a stable harmonic map from a compact Riemannian manifold into a simply connected compact Riemannian manifold with</u> $1/4 < K \leqq 1$ <u>and</u> $\dim N \geqq 3$, <u>where</u> K <u>is the sectional curvature of</u> N, <u>then,</u> f <u>is constant.</u>". This is the harminic map version of Lawson-Simons' conjecture[11]. Refer to [9], [17] for other partial answers to this conjecture.

Theorem 5.2.

Let $f : \mathbb{C}P^m \to N$ be a stable harmonic map into a Kaehler manifold with strongly positive curvature tensor. Then, f is $\pm$-holomorphic.

Proof. The case where $m = 1$ is due to Siu and Yau[22]. If $m \geqq 2$, Theorem F and Theorem 3.5 yield the conclusion.

Q.E.D.

Problem. Let N be a Kaehler manifold with positive holomorphic bisectional curvature (or 1/2-pinched holomorphic sectional curvature H, i.e., $1/2 < H \leqq 1$). Then, is any stable harmonic map $f : \mathbb{C}P^m \to N$ $\pm$-holomorphic ? .

Theorem F, Corollary 3.17 and Theorem 3.18 yield

Theorem 5.3.

Let $f : \mathbb{C}P^m \to N$ be a stable harmonic map into a Kaehler manifold. Assume that $\dim_{\mathbb{C}} N = m$ (resp. $\dim_{\mathbb{C}} N < m$). Then, f is $\pm$-holomorphic (resp. constant).

Moreover, Theorem F and Theorem 4.3 yield

Theorem 5.4.

Any stable harmonic map from $\mathbb{C}P^m$ into an irreducible Hermitian symmetric space of compact type is $\pm$-holomorphic.

Theorem 5.4 is nothing but Theorem E.

§6. Examples of stable or unstable pluriharmonic but non ±-holomorphic maps.

Let $f : (M, g) \to (N, h)$ be an isometric immersion between Kaehler manifolds. Then, f is called <u>Kaehler immersion</u> if it is holomorphic. If we denote by ω_N the Kaehler form of N, then f is called <u>totally real</u> if $f^*\omega_N = 0$ on M. We show the following seven examples(c.f. [5]).

<u>Example</u> 1. $N = G_{2,m}(\mathbb{C})$, $p(N) = (m-1)(2-1) + 1 = m$.

Let $M = Q^m$. Then, $\dim_{\mathbb{C}} M = p(N)$. Let

$$h = g \circ f : Q^m \xrightarrow{f} SO(m+2)/S(O(2)\times O(m)) \xrightarrow{g} G_{2,m}(\mathbb{C}),$$

where f is a covering map and g is a totally real, totally geodesic isometric immersion. Then, h is non ±-holomorphic, pluriharmonic map. This example shows that Theorem 3.9 and Corollary 3 in [24] are best possible for $N = G_{2,m}(\mathbb{C})$. Moreover, h is unstable as a harmonic map by Theorem 4.3. Thus, the assumption of the stability in Theorem 4.1 can not be excluded for $N = G_{2,m}(\mathbb{C})$.

Let $f : M = M_1\times M_2 \to N$ be a totally geodesic Kaehler immersion from a reducible Kaehler manifold into a Kaehler manifold. Then, f is a stable pluriharmonic map. If we define a new Kaehler structure on M so that f is not ±-holomorphic, we obtain a non ±-holomorphic, stable pluriharmonic map because the stability and the total geodesicity of f depend on the Riemannian structures of M and N only. Now, the following is a list of totally geodesic Kaehler immersions f from reducible

Hermitian symmetric spaces M into an irreducible Hermitian symmetric spaces N of compact type except for $N = \mathbb{C}P^n$. In the following examples, $\dim_{\mathbb{C}} M = p(N)$ holds, hence the assumption of $b_2(M) = 1$ in Theorem 4.1 and Theorem 4.3 can not be excluded. Thus, Theorem H is also best possible.

<u>Example</u> 2. $N = Q^n$, $p(N) = 2$.

$$f : M = \mathbb{C}P^1 \times \mathbb{C}P^1 \to N .$$

Then, $\dim_{\mathbb{C}} M = p(N)$.

<u>Example</u> 3. $N = G_{p,q}(\mathbb{C})$, $p(N) = (p-1)(q-1) + 1$.

$$f : M = G_{k,k}(\mathbb{C}) \times G_{p-k,q-k}(\mathbb{C}) \to G_{p,q}(\mathbb{C}) \quad (0 < k \leqq p \leqq q),$$

$\dim_{\mathbb{C}} M = (p-k)(q-k) + k^2$. If $k = 1$, $\dim_{\mathbb{C}} M = p(N)$.

<u>Example</u> 4. $N = SO(2n)/U(n)$, $p(N) = (1/2)(n-2)(n-3) + 1$.

$$f : M = (SO(2k)/U(k)) \times (SO(2(n-k))/U(n-k)) \to SO(2n)/U(n) \quad (0 < k < n) ,$$

$\dim_{\mathbb{C}} M = (1/2)k(k-1) + (1/2)(n-k)(n-k-1)$. If $k = 2$, $\dim_{\mathbb{C}} M = p(N)$.

<u>Example</u> 5. $N = Sp(n)/U(n)$, $p(N) = (1/2)n(n-1) + 1$.

$$f : M = (Sp(k)/U(k)) \times (Sp(n-k)/U(n-k)) \to Sp(n)/U(n) \quad (0 < k < n) ,$$

$\dim_{\mathbb{C}} M = (1/2)k(k+1) + (1/2)(n-k)(n-k+1)$. If $k = 1$, $\dim_{\mathbb{C}} M = p(N)$.

Example 6. $N = E_6/Spin(10)\cdot T$, $p(N) = 6$.

$$f : M = \mathbb{C}P^1 \times \mathbb{C}P^5 \to E_6/Spin(10)\cdot T .$$

Then, $\dim_{\mathbb{C}} M = p(N)$.

Example 7. $N = E_7/E_6\cdot T$, $p(N) = 11$.

$$f : M = \mathbb{C}P^1 \times Q^{10} \to E_7/E_6\cdot T .$$

Then, $\dim_{\mathbb{C}} M = p(N)$.

These examples 2 ~ 7 satisfy $c_1(M) > 0$ and $b_2(M) = 2$.

A7. Totally geodesic isometric immersions between irreducible Hermitian symmetric spaces of compact type.

Let M and N be a compact Riemannian manifold and Riemannian manifold, respectively. Let $f : M \to N$ be a totally geodesic isometric immersion. We denote by Id_M the identity map of M. Then, we have

Proposition 7.1.

f is stable as a harmonic map if and only if f is stable as a minimal immersion and Id_M is stable as a harmonic map.

Proof. Any smooth section V of $C^\infty(f^{-1}TN)$ is represented as

$$V = df(V^T) + V^N , \qquad V^T \in C^\infty(TM), \quad V^N \in C^\infty(NM),$$

where NM is the normal bundle of f. We denote by L_f, L_{Id_M} and L_f^m Jacobi operators of harmonic map f, identity map as a harmonic map and minimal immersion f, respectively(for the Jacobi operators, see [11], [14]). Since f is totally geodesic isometric immersion, it is easy to verify that

$$L_f(V) = (df)L_{Id_M}(V^T) + L_f^m(V^N) \quad , \qquad V \in C^\infty(f^{-1}TN).$$

This and the total geodesicity of f yield

$$L_f = L_{Id_M} + L_f^m \, . \qquad\qquad \text{Q.E.D.}$$

If M is a compact Kaehler manifold, then Id_M is holomorphic, hence stable. Thus, we obtain

Proposition 7.2.

Let f : M → N be a totally geodesic isometric immersion of a compact Kaehler manifold into a Riemannian manifold. Then, f is stable as a harmonic map if and only if it is stable as a minimal immersion.

By Proposition 7.2 and Theorem 4.3, we obtain

Theorem 7.3.

Any stable totally geodesic isometric immersion between irreducible Hermitian symmetric spaces of compact type is ±-holomorphic.

Remark 7.4.

Let f : M → N be a totally geodesic map between Riemannian manifolds. If M is irreducible in the sense of de Rham, then f is homothetic. By Theorem 2 in [2], if f : M → N is a stable harmonic immersion from a compact Riemannian manifold into an irreducible Hermitian symmetric space of compact type, then M is even-dimensional and $\tilde{J} = df^{-1} \circ J_N \circ df$, where J_N is the complex structure tensor of N, is a complex structure and hermitian with respect to the given metric on M. Thus, if f is totally geodesic, we have $\nabla\tilde{J} = 0$, that is, <u>if</u> $f : M \to N$ <u>is a stable totally geodesic isometric immersion of a compact Riemannian manifold into an irreducible Hermitian symmetric space of compact type, then there exists a unique Kaehler structure on</u> M <u>such that</u> f <u>is holomorphic with respect to this Kaehler structure.</u>

A8. A construction of non ±-holomorphic, pluriharmonic maps into complex Grassmann manifolds

In this section, we give a method of manufacturing pluriharmonic maps into complex Grassmann manifolds from holomorphic maps into complex projective space with Fubini-Study metrics.

Let $\mathbb{C}P^n$ be an n-dimensional complex projective space with constant holomorphic sectional curvature c and $L \to \mathbb{C}P^n$ be the universal line bundle. L is naturally regarded as a holomorphic line subbundle of the trivial bundle $\underline{\mathbb{C}}^{n+1} = \mathbb{C}P^n \times \mathbb{C}^{n+1}$. We denote by L^* the dual bundle of L. Let $\langle\ ,\ \rangle$ denotes the standard Hermitian inner product of $\mathbb{C}^{n+1}$. We denote by $L^\perp$ the orthogonal complement of L in $\underline{\mathbb{C}}^{n+1}$ with respect to $\langle\ ,\ \rangle$. We have a natural exact sequence :

$$0 \longrightarrow L^* \otimes L \xrightarrow{\ i\ } L^* \otimes \underline{\mathbb{C}}^{n+1} \xrightarrow{\ j\ } L^* \otimes L^\perp \longrightarrow 0$$

where i denotes the inclusion and j denotes the Hermitian projection. We endow each bundle with the natural Hermitian connected structure induced from $\langle\ ,\ \rangle$. There is a linear isomorphism $h : T(\mathbb{C}P^n)^{1,0} \to L^* \otimes L^\perp$ preserving the connections such that $(c/2)g(Z, \bar{W}) = \langle h(Z), h(W)\rangle$ for $Z, W \in T(\mathbb{C}P^n)^{1,0}$, where g is the Kaehler metric of $\mathbb{C}P^n$.

Let $f : M \to \mathbb{C}P^n$ be a smooth map from a manifold M. Consider the exact sequence of the pull back bundles :

$$0 \longrightarrow f^{-1}(L^*\otimes L) \longrightarrow f^{-1}(L^*\otimes \underline{\mathbb{C}}^{n+1}) \longrightarrow f^{-1}(L^*\otimes L^{\perp}) \longrightarrow 0 .$$

Set $E = f^{-1}(L^*\otimes \underline{\mathbb{C}}^{n+1})$. The section $\phi_f = i(1) \in C^\infty(E)$ is called the universal lift of f(c.f. [7]), where 1 denotes the identity section of $f^{-1}(L^*\otimes L)$. Denote by D the induced connection of E. The curvature form R^E of the connection D is given by

$$(8.1) \quad R^E(X, Y) = (c/2)\sqrt{-1}(f^{-1}\omega)(X, Y)\cdot Id_E$$

for $X, Y \in T_pM$, where ω is a fundamental 2-form of $(\mathbb{C}P^n, g)$ defined by $\omega(u, v) = g(Ju, v)$. Then, we have

$$h((df)^{1,0}(X)) = D_X\phi_f \qquad \text{for} \quad X \in TM^{\mathbb{C}},$$

where $(df)^{1,0}$ denotes the (1,0)-component of df. Assume that M is a complex manifold. f is holomorphic(resp. anti-holomorphic) if and only if

$$(8.2) \quad D_{\bar{Z}}\phi_f = 0 \text{ (resp. } D_Z\phi_f = 0) \quad \text{for any} \quad Z \in TM^{1,0}.$$

Let $f : M \to \mathbb{C}P^n$ be a holomorphic map. We define the osculating spaces along the map f. At each $p \in M$, we define $O_p^k(f)$ ($k \in Z$, $k \geq 0$) by

$$O_p^0(f) = i(f^{-1}(L^*\otimes L))_p ,$$

$$O_p^k(f) = \mathrm{span}_{\mathbb{C}}\{D_{Z_1}\cdots D_{Z_i}\Phi_f \ ; \ Z_1, \cdots Z_i \in T_pM^{1,0}, \ 0 \leqq i \leqq k \}$$

$$\subset E_p .$$

Put $\tilde{O}_p^k(f) = \mathrm{Im}\{O_p^k(f)\} = \mathrm{span}_{\mathbb{C}}\{ V(\rho) \ ; \ V \in O_p^k(f), \ \rho \in (f^{-1}L)_p\}$

$$\subset \mathbb{C}^{n+1} .$$

Note that $\tilde{O}_p^0(f) = (f^{-1}L)_p$. Set $R_0 = M$ and $R_k = \{ p \in R_{k-1} ;$ $\dim \tilde{O}_p^k(f)$ is maximal $\}$ for $k \geqq 1$. Then, there exists uniquely a positive integer d such that $\tilde{O}_p^{d-1}(f) \subsetneq \tilde{O}_p^d(f)$ and $\tilde{O}_p^d(f) = \tilde{O}_p^{d+i}(f)$ for any $p \in R_d$ and any $i \geqq 1$. Such integer d is called the <u>osculating degree</u> of the holomorphic map f and denoted by $d = d(f)$. Note that $R_d = R_{d+1} = \cdots$. This open subset R_d of M will be denoted by $\bar{R}$ and called the set of <u>regular points</u> of M. Let $\tilde{O}^k(f)$ denote the complex vector subbundle of $\underline{\mathbb{C}}^{n+1}$ over R_k with the fibre $\tilde{O}_p^k(f)$. It is standard to check the following lemma

<u>Lemma</u> 8.3.

(i) $M \setminus R_k$ <u>is a complex analytic subvariety of</u> M .

(ii) If $\tilde{O}^k(f) = \tilde{O}^{k+1}(f)$ over R_k, then $\tilde{O}^k(f) = \tilde{O}^{k+i}(f)$ for each $i \geqq 0$.

(iii) $f^{-1}L = \tilde{O}^0(f) \subsetneq \tilde{O}^1(f) \subsetneq \cdots \subsetneq \tilde{O}^{d-1}(f) \subsetneq \tilde{O}^d(f)$ over $\bar{R}$.

(iv) $\tilde{O}^k(f)$ is a holomorphic vector subbundle of $\mathbb{C}^{n+1}$ over $\bar{R}$.

(v) If f is full, that is, the image of f is not contained in any proper projective subspace of $\mathbb{C}P^n$, then we have $\tilde{O}^d(f) = \mathbb{C}^{n+1}$ over $\bar{R}$.

Lemma 8.4.

For each $p \in \bar{R}$ and any $Z \in TM^{1,0}$,

(i) $\partial_Z C^\infty(\tilde{O}^k(f)) \subset C^\infty(\tilde{O}^{k+1}(f))$,

(ii) $\partial_{\bar{Z}} C^\infty(\tilde{O}^k(f)) \subset C^\infty(\tilde{O}^k(f))$.

Proof. Let $0 \leqq i \leqq k$ and $Z, Z_1, \cdots, Z_i$ be local smooth sections of $TM^{1,0}$ over $\bar{R}$. For any $\rho \in C^\infty(f^{-1}L)$,

$$\partial_Z((D_{Z_1} \cdots D_{Z_i}\Phi_f)(\rho)) = (D_Z D_{Z_1} \cdots D_{Z_i}\Phi_f)(\rho)$$

$$+ (D_{Z_1} \cdots D_{Z_i}\Phi_f)((\partial_Z \rho)^L)$$

is a local smooth section of $\tilde{O}^{i+1}(f) \subset \tilde{O}^{k+1}(f)$, where $(\cdot)^L$ denotes the Hermitian projection of $(\cdot)$ to the universal line bundle L. From this we get (i). By (8.1) and (8.2) we compute

$$\partial_{\bar{Z}}((D_{Z_1} \cdots D_{Z_i}\Phi_f)(\rho)) = (D_{\bar{Z}}D_{Z_1} \cdots D_{Z_i}\Phi_f)(\rho)$$

$$+ (D_{Z_1} \cdots D_{Z_i}\Phi_f)((\partial_{\bar{Z}}\rho)^L)$$

$$\equiv (D_{Z_1} \cdots D_{Z_i}D_{\bar{Z}}\Phi_f)(\rho) \quad \mathrm{mod} \quad \tilde{O}^i(f)$$

$$\equiv 0 \quad \mathrm{mod}\ \tilde{O}^i(f) .$$

From this we get (ii). Q.E.D.

Next we prepare twistor fibrations of complex flag manifolds onto a complex Grassmann manifold. Let $\mathbb{F} = \mathbb{F}(n_1, n_2 ; n+1)$ denote the complex flag manifold $U(n+1)/U(n_1)\times U(n_2-n_1)\times U(n+1-n_2)$, where $0 \leqq n_1 \leqq n_2 \leqq n+1$. A point in $\mathbb{F}$ may be viewed as a pair (V, W) of an n_1-dimensional complex subspace V of $\mathbb{C}^{n+1}$ and an n_2-dimensional complex subspace W of $\mathbb{C}^{n+1}$ satisfying $V \subset W$. There are three tautological subbundles T_1, T_2, T_3 of the trivial bundle $\mathbb{F}\times\mathbb{C}^{n+1}$ with the fibre at $b = (V, W) \in \mathbb{F}$ being given by $(T_1)_b = V$, $(T_2)_b = V^{\perp}\cap W$, $(T_3)_b = W^{\perp}$. It is well-known that the complexified tangent bundle $T\mathbb{F}^{\mathbb{C}}$ of $\mathbb{F}$ is naturally isomorphic to

$$\Sigma_{1\leqq i\neq j\leqq 3}\mathrm{Hom}(T_i, T_j) .$$

$GL(n+1, \mathbb{C})$ acts on $\mathbb{F}$ via

$$a(V, W) = (a(V), a(W)) \in F$$

for $a \in GL(n+1, \mathbb{C})$ and $(V, W) \in F$.
$PGL(n+1, \mathbb{C})$ acts effectively on F. F is expressed as a complex homogeneous space $PGL(n+1, \mathbb{C})/P$, where P is a parabolic subgroup of $PGL(n+1, \mathbb{C})$. We equip F with the complex structure via this expression as a complex homogeneous space. With respect to this complex structure, the (1,0)- and (0,1)-tangent spaces of F are given by

$$(8.5) \quad TF^{1,0} = Hom(T_1, T_2) \oplus Hom(T_1, T_3) \oplus Hom(T_2, T_3) ,$$

$$(8.6) \quad TF^{0,1} = Hom(T_2, T_1) \oplus Hom(T_3, T_1) \oplus Hom(T_3, T_2) .$$

We denote by $h_{(i,j)}$ the Hermitian metric on $Hom(T_i, T_j)$ induced from the flat Hermitian metric on $F \times \mathbb{C}^{n+1}$. For each pair $\xi = (\xi_1, \xi_2)$ of positive real numbers, the Hermitian metric

$$h_\xi = \xi_1 h_{(1,2)} + (\xi_1 + \xi_2) h_{(1,3)} + \xi_2 h_{(2,3)}$$

on $TF^{1,0}$ defines a Kaehler metric on F (c.f. [1]). With respect to this Kaehler metric, $U(n+1)$ acts isometrically on F. h_ξ is an Einstein-Kaehler metric if and only if $\xi = c(n_2, n+1 - n_1)$ for some $c > 0$. Let $G_r(\mathbb{C}^{n+1})$ denote a complex Grassmann manifold $U(n+1)/U(r)\times U(n+1-r)$ of r-dimensional complex subspaces of $\mathbb{C}^{n+1}$ and T denote a tautological subbundle of the trivial bundle $G_r(\mathbb{C}^{n+1})\times\mathbb{C}^{n+1}$. $G_r(\mathbb{C}^{n+1})$ has a natural symmetric Kaehler manifold structure such that the (1,0)-tangent spaces of $G_r(\mathbb{C}^{n+1})$ are given by

$$TG_r(\mathbb{C}^{n+1})^{1,0} = \mathrm{Hom}(T,\ T^{\perp})\ .$$

For a pair $(n_1,\ n_2)$ of integers with $0 \leqq n_1 < n_2 \leqq n+1$, define

$$\pi\ :\ F = F(n_1,\ n_2\ ;\ n+1) \longrightarrow G_r(\mathbb{C}^{n+1})$$

by $\pi(V,\ W) = V^{\perp}\cap W$ for $(V,\ W) \in F$, where $r = n_2 - n_1$. Then, we endow F with a homogeneous Kaehler metric ($\varepsilon_1 = \varepsilon_2$) such that π is a homogeneous Riemannian submersion and we have $\pi^{-1}T = T_2$. Let H_b (resp. V_b) be the horizontal(resp. vertical) subspace of T_bF with respect to the Riemannian submersion π. We have an orthogonal direct sum

$$TF^{\mathbb{C}} = H^{\mathbb{C}} \oplus V^{\mathbb{C}}\ ,$$

where

(8.7)

$$H^{\mathbb{C}} = (\mathrm{Hom}(T_1,\ T_2)\oplus \mathrm{Hom}(T_2,\ T_3))\oplus(\mathrm{Hom}(T_2,\ T_1)\oplus \mathrm{Hom}(T_3,T_2))$$

and

$$V^{\mathbb{C}} = \mathrm{Hom}(T_1, T_3) \oplus \mathrm{Hom}(T_3, T_1) \quad .$$

The action of $PGL(n+1, \mathbb{C})$ on F preserves the horizontal distribution H. Let M be a complex manifold and $f : M \to \mathbb{C}P^n$ be a full holomorphic map with the osculating degree d. Fix a pair (s, t) of integers with $-1 \leqq s < t \leqq d$. Define a smooth map $f_{s,t} : \bar{R} \to F = F(n(s), n(t) ; n+1)$ by

$$f_{s,t}(p) = (\tilde{O}^s_p(f), \tilde{O}^t_p(f)) \quad \text{for} \quad p \in \bar{R},$$

where $n(s) = \mathrm{rank}_{\mathbb{C}}\tilde{O}^s(f)$, $n(t) = \mathrm{rank}_{\mathbb{C}}\tilde{O}^t(f)$ and $\tilde{O}^{-1}(f) = \{0\}$. Note that $f^{-1}_{s,t}T_1 = \tilde{O}^s(f)$ and $f^{-1}_{s,t}(T_1 \oplus T_2) = \tilde{O}^t(f)$.

<u>Lemma</u> 8.8.

(i) $f_{s,t} : \bar{R} \to F$ <u>is a holomorphic map.</u>

(ii) $f_{s,t}$ <u>is horizontal with respect to the fibration</u> $\pi : F \to G_r(\mathbb{C}^{n+1})$, <u>where</u> $r = n(t) - n(s) \geqq 1$.

Proof. Set $\bar{f} = f_{s,t}$. Let Z be a local smooth section of $TM^{1,0}$ over $\bar{R}$. By the holomorphicity of f and Lemma 8.4,

$$(8.9) \quad \partial_Z C^\infty(\bar{f}^{-1}T_1) \subset C^\infty(\bar{f}^{-1}T_1) \oplus C^\infty(\bar{f}^{-1}T_2) \ ,$$

$$(8.10)\quad \partial_{\bar{Z}} C^{\infty}(\bar{f}^{-1}T_1) \subset C^{\infty}(\bar{f}^{-1}T_1) \ ,$$

$$(8.11)\quad \partial_{\bar{Z}} C^{\infty}(\bar{f}^{-1}(T_1 \oplus T_2)) \subset C^{\infty}(\bar{f}^{-1}(T_1 \oplus T_2)) \ .$$

From (8.11) we have

$$(8.12)\quad \partial_{Z} C^{\infty}(\bar{f}^{-1}T_3) \subset C^{\infty}(\bar{f}^{-1}T_3) \ .$$

Let $V_1 \in C^{\infty}(\bar{f}^{-1}T_1)$ and $V_2 \in C^{\infty}(\bar{f}^{-1}T_2)$. Differentiating $<V_1, V_2> = 0$ by $\partial_{\bar{Z}}$, we have $<\partial_{\bar{Z}}V_1, V_2> + <V_1, \partial_Z V_2> = 0$. By (8.10), we have $<\partial_{\bar{Z}}V_1, V_2> = 0$. Hence, we get $<V_1, \partial_Z V_2> = 0$. Thus,

$$(8.13)\quad \partial_{Z} C^{\infty}(\bar{f}^{-1}T_2) \subset C^{\infty}(\bar{f}^{-1}T_1^{\perp}) = C^{\infty}(\bar{f}^{-1}T_2) \oplus C^{\infty}(\bar{f}^{-1}T_3) \ .$$

By (8.9), (8.12) and (8.13) we see that $(d\bar{f})(Z)$ is a local smooth section of $\bar{f}^{-1}Hom(T_1, T_2) \oplus \bar{f}^{-1}Hom(T_2, T_3)$ over $\bar{R}$. From (8.5) and (8.7) we conclude that $\bar{f}$ is horizontal and holomorphic. Q.E.D.

Proposition 8.14.

Let F be a Kaehler manifold and $\pi : F \to N$ be a Riemannian submersion of F onto a Riemannian manifold N. If

$\tilde{\Phi} : M \to F$ <u>is a horizontal holomorphic map from a complex manifold</u> M, <u>then</u> $\Phi = \pi \circ \tilde{\Phi} : M \to N$ <u>is a pluriharmonic map.</u>

Proof. By a formula of a Riemannian submersion and the pluriharmonicity of $\tilde{\Phi}$, for any local smooth section Z, W of $TM^{1,0}$,

$$\begin{aligned}(\nabla''_{\bar{W}}\ \partial\Phi)(Z) &= \nabla^{\Phi}_{\bar{W}}(\partial\Phi(Z)) - \partial\Phi(\bar{\partial}_{\bar{W}}Z)\\ &= \nabla^{\Phi}_{\bar{W}}(d\pi(\partial\tilde{\Phi}(Z))) - \partial\Phi(\bar{\partial}_{\bar{W}}Z)\\ &= d\pi(\nabla^{\tilde{\Phi}}_{\bar{W}}(\partial\tilde{\Phi}(Z))) - d\pi(\partial\tilde{\Phi}(\bar{\partial}_{\bar{W}}Z))\\ &= d\pi((\nabla''_{\bar{W}}\partial\tilde{\Phi})(Z))\\ &= 0\ .\end{aligned}$$

Q.E.D.

Set $\Phi_{s,t} = \pi \circ f_{s,t}$.

<u>Theorem</u> 8.15.

(i) $\Phi_{s,t}$ <u>is a pluriharmonic map.</u>

(ii) $s = -1$ <u>if and only if</u> $\Phi_{s,t}$ <u>is holomorphic.</u>

(iii) $t = d$ <u>if and only if</u> $\Phi_{s,t}$ <u>is anti-holomorphic</u>.

Proof. (i) is due to Lemma 8.8 and Proposition 8.14. We show (ii) and (iii) :

(ii) $\Phi_{s,t}$ is holomorphic $\Longleftrightarrow (df_{s,t})(TM^{1,0}) \subset f_{s,t}^{-1}\mathrm{Hom}(T_2, T_3)$

$$\Longleftrightarrow \partial_Z C^\infty(f_{s,t}^{-1}T_1) \subset C^\infty(f_{s,t}^{-1}T_1) \quad \text{for any} \quad Z \in C^\infty(TM^{1,0})$$

$$\Longleftrightarrow \partial_Z C^\infty(\tilde{O}^s(f)) \subset C^\infty(\tilde{O}^s(f)) \quad \text{for any} \quad Z \in C^\infty(TM^{1,0})$$

$$\Longleftrightarrow \tilde{O}^s(f) = \tilde{O}^{s+1}(f) \Longleftrightarrow \tilde{O}^s(f) = \{0\} \Longleftrightarrow s = -1 .$$

(iii) $\Phi_{s,t}$ is anti-holomorphic

$$\Longleftrightarrow (df_{s,t})(TM^{1,0}) \subset f_{s,t}^{-1}\mathrm{Hom}(T_1, T_2)$$

$$\Longleftrightarrow \partial_Z C^\infty(f_{s,t}^{-1}T_2) \subset C^\infty(f_{s,t}^{-1}T_2) \quad \text{for any} \quad Z \in C^\infty(TM^{1,0})$$

$$\Longleftrightarrow \partial_Z C^\infty(\tilde{O}^t(f)) \subset C^\infty(\tilde{O}^t(f)) \quad \text{for any} \quad Z \in C^\infty(TM^{1,0})$$

$$\Longleftrightarrow \tilde{O}^t(f) = \tilde{O}^{t+1}(f) \Longleftrightarrow t = d .$$

Q.E.D.

<u>Theorem</u> 8.16.

<u>For each</u> $a \in PGL(n+1, \mathbb{C})$, $\Phi_{s,t}^a = \pi(\, a \cdot f_{s,t}) : M \to G_r(\mathbb{C}^{n+1})$ <u>is also pluriharmonic.</u>

Proof. Since $PGL(n+1, \mathbb{C})$ acts holomorphically on F and this action preserves the horizontal distribution on F, $a \circ f_{s,t}$ is also holomorphic and horizontal. Q.E.D.

<u>Remark</u> 8.17.

(1) Our argument is inspired by the work of Eells and Wood [7]. In case of $\dim_{\mathbb{C}} M = 1$ and $t = s+1$, this is a result of [7]. The maps $\Phi_{-1,1}$ and $\Phi_{0,1}$ were studied by Nishikawa[13] and Ishihara[10], respectively. Guest[8] investigated the harmonicity of $\Phi_{s,s+1}$ for a flag manifold $M = G/T$.

(2) The first author[15] constructed a series of harmonic maps from each compact homogeneous Kaehler manifold into a complex projective space, which are neither $\pm$-holomorphic nor totally real.

<u>Acknowledgements</u> : The first author was supported by the Max-Planck-Institut für Mathematik in Bonn during the preparation of this paper. He would like to thank the Max-Planck-Institut for the hospitality.

References.

[1] A. Borel and F. Hirzebruch, Characteristic classes and homogeneous spaces I, Amer. J. Math. 80(1958), 458-538.

[2] D. Burns, F. Burstall, P. de Bartolomeis and J. Rawnsley, Stability of harmonic maps of Kaehler manifolds, preprint.

[3] D. Burns and P. de Bartolomeis, Applications harmoniques stables dans P^n, to appear.

[4] F. Burstall, J. Rawnsley and S. Salamon, Stable harmonic 2-spheres in symmetric spaces, Bull. Amer. Math. Soc. 16(1987), 274-278.

[5] B. Y. Chen and T. Nagano, Totally geodesic submanifolds of symmetric spaces, II, Duke Math. J. 45(1978), 405-425.

[6] J. Eells and J. C. Wood, Restrictions on harmonic maps of surfaces, Topology 15(1976), 263-266.

[7] J. Eells and J. C. Wood, Harmonic maps from surfaces to complex projective spaces, Adv. Math. 49(1983), 217-263.

[8] M. A. Guest, Geometry of maps between generalized flag manifolds, J. Diff. Geom. 25(1987), 223-247.

[9] R. Howard, The nonexistence of stable submanifolds, varifolds and harmonic maps in sufficiently pinched simply connected Riemannian manifolds, Mich. Math. J. 32(1985), 321-334.

[10] T. Ishihara, The Gauss map and non holomorphic harmonic maps, Proc. Amer. Math. Soc. 89(1983), 661-665.

[11] H. B. Lawson and J. Simons, On stable currents and their application to global problems in real and complex geometry, Ann. of Math. 98(1973), 427-450.

[12] M. J. Micallef and J. D. Moore, Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two planes, to appear in Ann. of Math. .

[13] S. Nishikawa, Gauss map of Kaehler immersions, Tohoku Math. J. 27(1975), 453-460.

[14] Y. Ohnita, On pluriharmonicity of stable harmonic maps, J. London Math. Soc. 35(1987), 563-568.

[15] Y. Ohnita, Homogeneous harmonic maps into complex projective spaces, in preparation.

[16] Y. Ohnita and S. Udagawa, Stable harmonic maps from Riemann surfaces to compact Hermitian symmetric spaces, Tokyo J. Math. 10(1987), 385-390.

[17] T. Okayasu, Pinching and nonexistence of stable harmonic maps, preprint.

[18] Y. T. Siu, The complex analyticity of harmonic maps and the strong rigidity of compact Kaehler manifolds, Ann. of Math. 112(1980), 73-111.

[19] Y. T. Siu, Curvature characterization of hyperquadrics, Duke Math. J. 47(1980), 641-654.

[20] Y. T. Siu, Strong rigidity of compact quotients of exceptional bounded symmetric domains, Duke Math. J. 48(1981), 857-871.

[21] Y. T. Siu, Complex-analyticity of harmonic maps, vanishing and Lefschetz Theorems, J. Diff. Geom. 17(1982), 55-138.

[22] Y. T. Siu and S. T. Yau, Compact Kaehler manifolds of positive bisectional curvature, Invent. Math. 59(1980), 189-204.

[23] R. T. Smith, The second variation formula for harmonic mappings, Proc. Amer. Math. Soc. 47(1975), 229-236.

[24] S. Udagawa, Holomorphicity of certain stable harmonic maps and minimal immersions, to appear in Proc. London Math. Soc. .

[25] S. Udagawa, Pluriharmonic maps and minimal immersions of Kaehler manifolds, to appear in J. London Math. Soc. .

[26] S. T. Yau, On Calabi's conjecture and some results in algebraic geometry, Proc. Nat. Acad. Sci. USA 74(1977), 1798-1799.

[27] J. C. Zhong, The degree of strong nondegeneracy of the bisectional curvature of exceptional bounded symmetric domains, Several Complex Variables, Intern. Conf. Hangzhou, China 1981, (1984), 127-139.

Y. Ohnita
Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
5300 Bonn 3
Federal Republic of Germany
and
Department of Mathematics
Tokyo metropolitan University
Fukasawa, Setagaya-Ku, Tokyo 158
Japan

and

S. Udagawa
Department of Mathematics
School of Medicine
Nihon University
Itabashi-Ku, Tokyo 173
Japan