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Abstract. For a complex reductive Lie group G Tits defined an extension W_G^T of the corresponding Weyl group W_G. The extended group is supplied with an embedding into the normalizer $N_G(H)$ of the maximal torus $H \subset G$ such that W_G^T together with H generate $N_G(H)$. We give an interpretation of the Tits classical construction in terms of the maximal split real form $G(\mathbb{R}) \subset G(\mathbb{C})$, leading to a simple topological description of W_G^T. We also propose a different extension W_G^U of the Weyl group W_G associated with the compact real form $U \subset G(\mathbb{C})$. This results in a presentation of the normalizer of maximal torus of the group extension $U \ltimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ by the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We also describe explicitly the adjoint action of W_G^T and W_G^U on the Lie algebra of G.

1 Introduction

In the standard approach to classification of complex semisimple Lie groups the problem is reduced to an equivalent problem of classification of root data. In other words the root data, i.e. the system of roots and coroots describing maximal tori $H \subset G$ and the induced adjoint action of $\mathfrak{h} = \mathrm{Lie}(H)$ on $\mathfrak{g} = \mathrm{Lie}(G)$, defines the corresponding semisimple Lie group up to isomorphism. Curtis, Wiederhold and Williams [CWW] demonstrate that for classification of compact connected semisimple Lie groups G it is enough to classify the normalizers $N_G(H)$ of maximal tori $H \subset G$. The normalizer provides information on the action of the Weyl group $W_G := N_G(H)/H$ on H but this is not enough for classification as one needs the precise structure of the extension of W_G by H. Thus for the classification problem one might replace an involved non-commutative object, semisimple Lie group by a finite group extended by an abelian Lie group. The deep reason for this equivalence is not clear. One perspective is to look at $N_G(H)$ as a kind of degeneration of G [CWW]. An apparently related but more conceptual approach is based on attempts to look at $N_G(H)$ as the Lie group G defined over some non-standard number field (closely akin to mysterious field $\mathbb{F}_1$ "with one element" introduced by Tits [T3] probably with regard to this subject). In this way the equivalence of the classification problems for compact semisimple Lie groups and normalizers looks like a manifestation of a general principle (due to C. Chevalley [C]) that classification of semisimple algebraic groups should not essentially depend on the nature of the base local algebraically closed field.

The above reasoning suggests a more detailed study of group extension structure on $N_G(H)$. The important fact is that this extension does not split in general [D], [T1], [T2], [CWW], [AH] so to have a universal description of $N_G(H)$ one should look for a section of the projection $N_G(H) \to W_G$ realized by a minimal extension of W_G. Such construction was proposed by Demazure [D] and Tits [T1], [T2] and may be naturally formulated in terms of the Tits extension W_G^T of the Weyl group W_G by $\mathbb{Z}_2^{\mathrm{rank}(\mathfrak{g})}$. This construction allows an explicit presentation of $N_G(H)$ by generators and relations.

Although the Tits construction is known for a long time there seems no simple natural explanation for its precise form even in the case of the complex reductive group (for recent discussions Tits groups see e.g. [N], [DW], [AH]). This paper is an attempt to understand the Tits construction better. After reminding the general results on normalizers of maximal tori in Section 2 we reconsider the Tits construction in Section 3. We stress that the Tits group construction is defined for maximally split form $G(\mathbb{R}) \subset G(\mathbb{C})$ of complex semisimple group $G(\mathbb{C})$. This allows us to present in Proposition 3.1 a simple purely topological description of the Tits extension of the Weyl group W_G (our considerations appear to be very close to the final section of [BT]). Taking into account the relevance of the real structure for Tits description of maximal tori normalizers, we consider the opposite case of the real structure on $G(\mathbb{C})$ leading to maximal compact subgroup $U \subset G(\mathbb{C})$. It turns out that in this case there exists an analog of the Tits construction that takes into account the action of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. The main result of this paper is Theorem 4.1 in Section 4 describing the structure of the maximal tori normalizers of compact connected semi-simple Lie groups. In Section 5 we calculate explicitly the adjoint action of the Tits group and of its unitary analog on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. This action, in contrast with the adjoint action on $\mathfrak{h} \subset \mathfrak{g}$, depends on the lift of W_G into G. Finally in Section 6 we provide details of the proof of Theorem 4.1.

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2 Normalizers of maximal tori and Weyl groups

We start with recalling the standard facts on normalizers of maximal tori and the associated Weyl groups. Let $G(\mathbb{C})$ be a complex semisimple Lie group, $H \subset G(\mathbb{C})$ be a maximal torus and $N_G(H)$ be its normalizer in $G(\mathbb{C})$. Then there is the following exact sequence

$$1 \longrightarrow H \longrightarrow N_G(H) \stackrel{p}{\longrightarrow} W_G \longrightarrow 1, \tag{2.1}$$

where p is the projection on the finite group $W_G := N_G(H)/H$, the Weyl group of $G(\mathbb{C})$. The Weyl group W_G does not actually depend on the choice of H and thus produce an invariant of $G(\mathbb{C})$. Let $\mathfrak{g} := \mathrm{Lie}(G)$ and let I be the set of vertexes of the Dynkin diagram associated to

$G(\mathbb{C})$, where $|I| = \mathrm{rank}(\mathfrak{g})$. Let $(\Delta, \Delta^\vee)$ be the root-coroot system corresponding to $G(\mathbb{C})$, $\{\alpha_i,\, i \in I\}$ be a set of positive simple roots and $\{\alpha_i^\vee,\, i \in I\}$ be the corresponding set of positive simple coroots. Let $A = \|a_{ij}\|$, $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$ be the Cartan matrix of $(\Delta,\, \Delta^\vee)$. The Weyl group W_G has the simple description in terms of generators and relations. Precisely, W_G is generated by simple root reflections $\{s_i,\, i \in I\}$ subjected to

$$s_i^2 = 1, \tag{2.2}$$

$$\underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}}, \qquad i \neq j \in I, \tag{2.3}$$

where $m_{ij} = 2, 3, 4, 6$ for $a_{ij}a_{ji} = 0, 1, 2, 3$, respectively. Equivalently these relations may be written in the Coxeter form:

$$s_i^2 = 1, \qquad (s_i s_j)^{m_{ij}} = 1\,, \qquad i \neq j \in I\,. \tag{2.4}$$

The exact sequence (2.1) defines the canonical action of W_G on H. The corresponding action on the Lie algebra $\mathfrak{h} = \mathrm{Lie}(H)$ and on its dual is as follows

$$\begin{aligned} s_i(h_j) &= h_j - \langle \alpha_i, \alpha_j^\vee \rangle h_i \;=\; h_j - a_{ji} h_i\,, \\ s_i(\alpha_j) &= \alpha_j - \langle \alpha_j, \alpha_i^\vee \rangle \alpha_i \;=\; \alpha_j - a_{ij}\alpha_i\,. \end{aligned} \tag{2.5}$$

Unfortunately the exact sequence (2.1) does not split in general, i.e. $N_G(H)$ is not necessary isomorphic to a semi-direct product of W_G and H. A peculiar situation in this regard is described by the following result due to [CWW], [AH].

Theorem 2.1 *Assume G is a simple complex Lie group and let $Z(G)$ be the center of G. Then modulo low rank isomorphisms of classical groups, the exact sequence* (2.1) *splits in the following cases, and not otherwise:*

- *Type A_ℓ such that $|Z(G)|$ is odd;*
- *Type B_ℓ for the adjoint form;*
- *Type D_ℓ, for all forms except $Spin(2\ell)$;*
- *Type G_2.*

Thus to have an explicit description of the normalizer $N_G(H)$ one should look for a minimal section of the projection map p in (2.1). In the following Section we provide the construction of the resulting extension of the Weyl group by a finite group. Let us note that for a normal finite subgroup $G^0 \subset G$ one has: if (2.1) splits for G then it splits for G/G^0. In the following for simplicity we consider only the case of simply-connected complex groups.

3 Tits extension of Weyl group

To describe the extension (2.1) in terms of generators and relations Tits proposed the following extension W_G^T of the Weyl group W_G by a discrete group [T1], [T2] (closely related results were obtained by Demazure [D]).

Definition 3.1 *Let $A = \|a_{ij}\|$ be the Cartan matrix corresponding to a semi-simple Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and let $m_{ij} = 2, 3, 4, 6$ for $a_{ij}a_{ji} = 0, 1, 2, 3$, respectively. The Tits group W_G^T is an extension of the Weyl group W_G by an abelian group $\mathbb{Z}_2^{|I|}$ generated by $\{\tau_i,\, \theta_i,\, i \in I\}$ subjected to the following relations:*

$$(\tau_i)^2 \,=\, \theta_i\,, \qquad \theta_i\theta_j = \theta_j\theta_i\,, \qquad \theta_i^2 \,=\, 1\,, \tag{3.1}$$

$$\tau_i\theta_j \,=\, \theta_i^{-a_{ji}}\theta_j\tau_i\,, \tag{3.2}$$

$$\underbrace{\tau_i\tau_j\tau_i\cdots}_{m_{ij}} \,=\, \underbrace{\tau_j\tau_i\tau_j\cdots}_{m_{ij}}\,, \qquad i \neq j\,, \tag{3.3}$$

where the abelian subgroup is generated by $\{\theta_i,\, i \in I\}$.

Let $\{h_i,\, e_i,\, f_i\, : i \in I\}$ be the Chevalley-Serre generators of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{C}))$, satisfying the standard relations

$$[h_i,\, e_j] \,=\, a_{ij}e_j\,, \qquad [h_i,\, f_j] \,=\, -a_{ij}f_j\,, \qquad [e_i,\, f_j] \,=\, \delta_{ij}h_j\,, \tag{3.4}$$

$$\mathrm{ad}_{e_i}^{1-a_{ij}}(e_j) = 0, \qquad \mathrm{ad}_{f_i}^{1-a_{ij}}(f_j) = 0, \tag{3.5}$$

where $A = \|a_{ij}\|$ is the Cartan matrix i.e. $a_{ij} = \langle \alpha_i^\vee,\, \alpha_j\rangle$.

According to [BT] (see also [T1]) there exists a subset $\{\zeta_i,\, i \in I\} \subset H$ of canonical elements of order two satisfying the following relations

$$s_i(\zeta_j) = \zeta_j\, \zeta_i^{-a_{ji}}, \qquad i, j \in I\,, \tag{3.6}$$

where s_i, $i \in I$ are generators of the Weyl group W_G (2.2), (2.3).

Theorem 3.1 (Demazure-Tits) *Let W_G^T be the Tits group associated with the complex semi-simple Lie groups $G(\mathbb{C})$, then the map*

$$\tau_i \longmapsto \dot{s}_i := e^{f_i}\, e^{-e_i}\, e^{f_i}, \qquad \theta_i \longmapsto \zeta_i\,, \qquad i \in I\,, \tag{3.7}$$

defines a section of p in (2.1) by embedding the Tits group W_G^T into $N_G(H)$. In particular, the normalizer group $N_G(H)$ is generated by H and by the image of the Tits group under (3.7), so that the following relations hold:

$$\dot{s}_i h \dot{s}_i^{-1} \,=\, s_i(h)\,, \qquad \forall h \in \mathfrak{h} = \mathrm{Lie}(H)\,, \qquad i \in I. \tag{3.8}$$

Example 3.1 *In the standard faithful two-dimensional representation* $\phi : SL_2(\mathbb{C}) \to \mathrm{End}(\mathbb{C}^2)$ *given by* (6.3) *we have*

$$\phi(\dot{s}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \phi(\dot{s})^2 = \phi(\zeta) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.9}$$

The appearance of the Tits extension W_G^T as a minimal section seems unmotivated. However the construction of W_G^T may be elucidated by considering the maximally split real form $G(\mathbb{R}) \subset G(\mathbb{C})$ of $G(\mathbb{C})$. For the maximal split real form $G(\mathbb{R}) \subset G(\mathbb{C})$ there is an analog of (2.1)

$$1 \longrightarrow H(\mathbb{R}) \longrightarrow N_{G(\mathbb{R})}(H(\mathbb{R})) \stackrel{p}{\longrightarrow} W_G \longrightarrow 1, \tag{3.10}$$

with the real form maximal torus given by the intersection

$$H(\mathbb{R}) = H \cap G(\mathbb{R}), \tag{3.11}$$

of the complex maximal torus with maximally split real subgroup. Thus a section of (3.10) provides a section of (2.1). The group $H(\mathbb{R})$ allows the product decomposition

$$H(\mathbb{R}) = MA, \qquad M := H(\mathbb{R}) \cap K, \tag{3.12}$$

where $K \subset G(\mathbb{R})$ is a maximal compact subgroup of $G(\mathbb{R})$, M is isomorphic to the group $\mathbb{Z}_2^{|I|}$ and A is an abelian connected exponential group $A = \exp(\mathfrak{a})$. Therefore $H(\mathbb{R})$ is not connected and consists of $2^{|I|}$ components. Hence the group M may be identified with the discrete group of connected components of $H(\mathbb{R})$

$$M = \pi_0(H(\mathbb{R})). \tag{3.13}$$

Considering the groups of connected components of the topological groups entering (3.10) we obtain the induced exact sequence

$$1 \longrightarrow \pi_0(H(\mathbb{R})) \longrightarrow \pi_0(N_{G(\mathbb{R})}(H(\mathbb{R}))) \longrightarrow W_G \longrightarrow 1. \tag{3.14}$$

Explicitly the groups of connected components may be identified with the quotients by the connected normal subgroup A

$$1 \longrightarrow M \longrightarrow N_{G(\mathbb{R})}(H(\mathbb{R}))/A \stackrel{p}{\longrightarrow} W_G \longrightarrow 1\,, \tag{3.15}$$

and we have the exact sequence

$$1 \longrightarrow A \longrightarrow N_{G(\mathbb{R})}(H(\mathbb{R})) \longrightarrow \pi_0(N_{G(\mathbb{R})}(H(\mathbb{R}))) \longrightarrow 1. \tag{3.16}$$

Lemma 3.1 *The exact sequence* (3.16) *splits and thus* $\pi_0(N_{G(\mathbb{R})}(H(\mathbb{R})))$ *allows an embedding into* $N_{G(\mathbb{R})}(H(\mathbb{R}))$.

Proof. The extension (3.16) is an instance of extensions of $\pi_0(N_{G(\mathbb{R})}(H(\mathbb{R})))$ by A. Such extensions are classified by the group $H^2(\pi_0(N_{G(\mathbb{R})}(H(\mathbb{R}))), A)$. The triviality of this group follows from the fact that A is an exponential group and the second cohomology of any finite group with coefficients in a free module is trivial. Thus the extension (3.16) is necessarily trivial and therefor there exists the required embedding. □

Proposition 3.1 *The following isomorphism holds*

$$\pi_0(N_{G(\mathbb{R})}(H(\mathbb{R}))) \simeq W_G^T. \tag{3.17}$$

Proof. Let us take into account that the images $\dot{s}_i$, ζ_i, $i \in I$ of Tits generators belong to the maximally split real subgroup $G(\mathbb{R}) \subset G(\mathbb{C})$. Then by Theorem 3.1 the normalizer group $N_{G(\mathbb{R})}(H(\mathbb{R}))$ is generated by $H(\mathbb{R})$ and the image of W_G^T under the homomorphism (3.7) is given by the semidirect product $H(\mathbb{R}) \rtimes W_G^T$ over M. Considering the connected components we arrive at (3.17). □

Example 3.2 *For maximal split form $SL_2(\mathbb{R}) \subset SL_2(\mathbb{C})$ we have*

$$H(\mathbb{R}) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \in \mathbb{R}^* \right\}, \qquad H(\mathbb{R}) = MA\,, \tag{3.18}$$

$$A = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \in \mathbb{R}_+ \right\}, \qquad M = \{\pm \mathrm{Id}\}\ . \tag{3.19}$$

Elements $g \in N_{SL_2(\mathbb{R})}(H(\mathbb{R}))$ are defined by the condition that for each $\lambda \in \mathbb{R}^*$ there exists a $\tilde{\lambda} \in \mathbb{R}^*$ such that

$$g \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} \tilde{\lambda} & 0 \\ 0 & \tilde{\lambda}^{-1} \end{pmatrix} g \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1. \tag{3.20}$$

It is easy to check directly that the normalizer group $N_{SL_2(\mathbb{R})}(H(\mathbb{R}))$ is a union of two components

$$N_{SL_2}(H(\mathbb{R})) = N_1 \sqcup N_s, \tag{3.21}$$

where N_1 is a set of diagonal elements ($c = b = 0, ad = 1 \neq 0$) and N_s is the set of anti-diagonal ($a = d = 0, cb = -1$) elements. Each of these groups splits further into two connected components

$$N_1 = N_1^+ \sqcup N_1^-, \qquad N_s = N_s^+ \sqcup N_s^-, \tag{3.22}$$

depending on the sign of the non-zero elements in the last row.

The group $\pi_0(N_{SL_2(\mathbb{R})}(H(\mathbb{R})))$ consists of four elements corresponding to the classes of $N_1^{\pm}$, $N_s^{\pm}$ and is isomorphic to the quotient of $N_{SL_2(\mathbb{R})}(H(\mathbb{R}))$ by $A \simeq \mathbb{R}_+$. It is useful to pick the following parameterization of the connected components

$$N_1^+ = A, \qquad N_s^+ = \dot{s}A, \qquad N_1^- = \theta A, \qquad N_s^- = \theta \dot{s} A\,, \tag{3.23}$$

where

$$\dot{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \theta = \dot{s}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \theta\dot{s} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.24}$$

It is easy to check directly that the group $\pi_0(N_{SL_2(\mathbb{R})}(H(\mathbb{R})))$ generated by classes $N^{\pm}_{1,s}$ coincides with the finite group generated by $\dot{s}$ in accordance with (3.17).

4 Weyl group extensions for compact real forms

As we have demonstrated in the previous Section the Tits group extension W_G^T appears quite naturally if we consider the totally split real subgroup $G(\mathbb{R}) \subset G(\mathbb{C})$. This motivates to look for analogs of the Tits construction associated with other real forms of $G(\mathbb{C})$. Here we consider the connected compact real form $U \subset G(\mathbb{C})$ of the Lie group $G(\mathbb{C})$

$$U = \{g \in G(\mathbb{C}) |\, g^\dagger g = 1\}, \tag{4.1}$$

where $g \to g^\dagger$ is the composition of the Cartan involution and complex conjugation. Let us extend U by considering the semidirect product

$$U^\Gamma := (U \ltimes \Gamma) \subset G^\Gamma := G(\mathbb{C}) \ltimes \Gamma. \tag{4.2}$$

Here $\Gamma := \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}_2$ is the Galois group of $\mathbb{R}$ generated by γ, $\gamma^2 = 1$ so that γ acts by complex conjugation:

$$\gamma\,\lambda\,\gamma^{-1} = \bar{\lambda}, \qquad \forall \lambda \in \mathbb{C}. \tag{4.3}$$

In the following we chose the generators e_i, f_i, h_i, $i \in I$ to be real and thus commuting with γ.

Definition 4.1 *Let W_G^U be a group generated by $\{\sigma_i,\, \bar{\sigma}_i,\, \eta_i;\, i \in I\}$ subjected to*

$$\begin{gathered} \sigma_i^2 \,=\, \bar{\sigma}_i^2 \,=\, 1\,, \qquad \sigma_i\bar{\sigma}_i \,=\, \bar{\sigma}_i\sigma_i = \eta_i\,, \qquad i \in I\,, \\ \sigma_i\eta_j \,=\, \eta_j\eta_i^{-a_{ji}}\sigma_i\,, \qquad \bar{\sigma}_i\eta_j \,=\, \eta_j\eta_i^{-a_{ji}}\bar{\sigma}_i\,, \qquad i \neq j \in I\,, \quad a_{ij} \neq 0\,, \end{gathered} \tag{4.4}$$

$$\underbrace{\sigma_i\sigma_j\cdots}_{m_{ij}} = \underbrace{\bar{\sigma}_j\bar{\sigma}_i\cdots}_{m_{ij}}, \qquad i \neq j \in I\,, \tag{4.5}$$

where in (4.5) $m_{ij} = 2, 3, 4, 6$ *for* $a_{ij}a_{ji} = 0, 1, 2, 3$.

The group W_G^U has outer automorphism:

$$\gamma : \sigma_i \longleftrightarrow \bar{\sigma}_i\,, \tag{4.6}$$

which we will consider below as an extension of complex conjugation γ onto W_G^U. Now the group $W_G^U \ltimes \Gamma$ can be presented via generators $\{\sigma_i,\, i \in I\}$ and γ and relations (4.4), (4.5) with $\bar{\sigma}_i := \gamma\sigma_i\gamma$.

Lemma 4.1 *(i) The elements $\eta_i = \sigma_i\bar{\sigma}_i$ are real of order two and pairwise commute:*

$$\eta_i^2 = 1, \qquad \eta_i\eta_j = \eta_j\eta_i\,, \qquad i,j \in I\,. \tag{4.7}$$

(ii) For $a_{ij} = 0$ the following relations hold:

$$\sigma_i\sigma_j\bar{\sigma}_j = \sigma_j\bar{\sigma}_j\sigma_i, \qquad \bar{\sigma}_i\sigma_j\bar{\sigma}_j = \sigma_j\bar{\sigma}_j\bar{\sigma}_i\,, \tag{4.8}$$

completing the set of relations (4.4) *for all allowed values of a_{ij}.*

Proof. (i) For the first relation in (4.7) we have

$$(\sigma_i\bar{\sigma}_i)^2 = \sigma_i\bar{\sigma}_i\sigma_i\bar{\sigma}_i = \sigma_i\bar{\sigma}_i\bar{\sigma}_i\sigma_i = 1. \tag{4.9}$$

The second relation in (4.7) follows from the set of identities:

$$\sigma_i\bar{\sigma}_i\sigma_j\bar{\sigma}_j = \sigma_i\,\sigma_j\bar{\sigma}_j(\sigma_i\bar{\sigma}_i)^{-a_{ij}}\bar{\sigma}_i \;=\; \sigma_j\bar{\sigma}_j(\sigma_i\bar{\sigma}_i)^{-2a_{ij}}\sigma_i\bar{\sigma}_i = \sigma_j\bar{\sigma}_j\sigma_i\bar{\sigma}_i. \tag{4.10}$$

(ii) For (4.8) we have

$$\sigma_i\,\sigma_j\bar{\sigma}_j = \bar{\sigma}_j\bar{\sigma}_i\bar{\sigma}_j = \bar{\sigma}_j\sigma_j\sigma_i, \tag{4.11}$$

where have used the basic relation (4.5) twice

$$\sigma_i\sigma_j = \bar{\sigma}_j\bar{\sigma}_i\,, \qquad \bar{\sigma}_i\bar{\sigma}_j = \sigma_j\sigma_i\,, \qquad a_{ij} = a_{ji} = 0. \tag{4.12}$$

This completes our proof. □

Lemma 4.2 *For any $i,j \in I$ such that $a_{ij} = -1,\,-3$ the following holds:*

$$\underbrace{\sigma_i\sigma_j\cdots\sigma_i}_{m_{ij}} = \underbrace{\bar{\sigma}_i\bar{\sigma}_j\cdots\bar{\sigma}_i}_{m_{ij}}\,, \tag{4.13}$$

and thus

$$\underbrace{\sigma_i\sigma_j\cdots\sigma_i}_{m_{ij}} = \underbrace{\sigma_j\sigma_i\cdots\sigma_j}_{m_{ij}}\,. \tag{4.14}$$

Proof. Note that for $a_{ij} = -1, -3$ we have

$$\begin{aligned}\bar{\sigma}_i\bar{\sigma}_j\bar{\sigma}_i\sigma_i\sigma_j\sigma_i \;&=\; \bar{\sigma}_i\bar{\sigma}_j\,\eta_i\,\sigma_j\sigma_i \;=\; \bar{\sigma}_i(\bar{\sigma}_j\sigma_j)\,\eta_i\eta_j^{-a_{ij}}\,\sigma_i \;=\; \bar{\sigma}_i\,\eta_i\eta_j^{1-a_{ij}}\,\sigma_i\\ &=\; \bar{\sigma}_i\sigma_i\eta_i^{-1} \;=\; \eta_i\eta_i^{-1} \;=\; 1\,,\end{aligned} \tag{4.15}$$

which entails

$$\bar{\sigma}_i\bar{\sigma}_j\bar{\sigma}_i = \sigma_i\sigma_j\sigma_i\,. \tag{4.16}$$

The assertion then follows from (4.16). □

Let us note that there exists a slightly different but equivalent definition of the extended Weyl group W_G^U which directly follows from Definition 4.1, Lemma 4.1 and Lemma 4.2.

Corollary 4.1 *The group W_G^U is defined by the following set $\{\sigma_i,\, \bar{\sigma}_i;\, i \in I\}$ of generators subjected to*

$$\sigma_i^2 = \bar{\sigma}_i^2 = 1\,, \qquad \sigma_i\bar{\sigma}_i = \bar{\sigma}_i\sigma_i\,, \qquad i \in I\,;$$
$$\sigma_i\sigma_j = \bar{\sigma}_j\bar{\sigma}_i, \qquad i,j \in I\,, \quad a_{ij} = a_{ji} = 0; \tag{4.17}$$
$$\sigma_j\sigma_i\sigma_j = \bar{\sigma}_j\bar{\sigma}_i\bar{\sigma}_j = \sigma_i\sigma_j\sigma_i = \bar{\sigma}_i\bar{\sigma}_j\bar{\sigma}_i, \qquad i,j \in I\,, \quad a_{ij} = a_{ji} = -1\,,$$

$$\sigma_i\sigma_j\sigma_i\sigma_j = \sigma_i\bar{\sigma}_j\sigma_i\bar{\sigma}_j = \bar{\sigma}_j\bar{\sigma}_i\bar{\sigma}_j\bar{\sigma}_i = \sigma_j\bar{\sigma}_i\sigma_j\bar{\sigma}_i\,, \qquad i,j \in I\,, \quad a_{ji} = -2\,, \tag{4.18}$$

$$\begin{aligned} \sigma_i\sigma_j\sigma_i\sigma_j\sigma_i\sigma_j &= \bar{\sigma}_i\bar{\sigma}_j\bar{\sigma}_i\sigma_j\sigma_i\sigma_j = \bar{\sigma}_i\bar{\sigma}_j\bar{\sigma}_i\bar{\sigma}_j\bar{\sigma}_i\bar{\sigma}_j = \\ = \bar{\sigma}_j\bar{\sigma}_i\bar{\sigma}_j\bar{\sigma}_i\bar{\sigma}_j\bar{\sigma}_i &= \bar{\sigma}_j\bar{\sigma}_i\bar{\sigma}_j\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j\sigma_i\sigma_j\sigma_i\,, \qquad i,j \in I\,, \quad a_{ji} = -3\,. \end{aligned} \tag{4.19}$$

Proposition 4.1 *The group W_G^U is an extension*

$$1 \longrightarrow \mathfrak{R} \longrightarrow W_G^U \longrightarrow W_G \longrightarrow 1\,, \tag{4.20}$$

of W_G by the commutative group $\mathfrak{R} = \mathbb{Z}_2^{|I|}$ identified with the subgroup of W_G^U generated by elements

$$\eta_i := \sigma_i\bar{\sigma}_i\,, \qquad i \in I\,. \tag{4.21}$$

Proof. Due to the following relations

$$\sigma_i = \eta_i\bar{\sigma}_i\,, \qquad \sigma_i\,\eta_j = \eta_j\,\eta_i^{-a_{ji}}\,\sigma_i, \tag{4.22}$$

each element of W_G^U may be represented as a product of some η's times the product of some σ's. Taking the quotient over the relations $\eta_i = 1$ we recover the defining relations (2.2), (2.3) of W_G. □

The $\mathbb{Z}_2^{|I|}$-extension (4.20) basically arises due to the following simple fact. Given a complex invertable operator a with real square satisfying the relation $a\bar{a} = 1$ it defines an element a^2 of order two

$$(a^2)^2 = (\bar{a}^2)^2 = \mathrm{id}. \tag{4.23}$$

In the considered case we have $\sigma_i = a_i\gamma$ and the required properties of a_i follows from the first line in (4.4).

The following analog of Theorem 3.1 is the main result of this Note (the proof is given in Section 6).

Theorem 4.1 *Let $U \subset G(\mathbb{C})$ be a maximal compact subgroup of the complex semi-simple Lie group. Let (Δ, W_G) be a root system for the corresponding Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ with* *the Cartan matrix* $A = \|a_{ij}\|$. *Let* γ *be the generator of the Galois group* $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ *of the field* $\mathbb{R}$ *of real numbers. Then the following map*

$$\begin{gathered}\sigma_i \longmapsto \varsigma_i := e^{\imath\pi(e_i+f_i)/2}\,\gamma, \qquad \bar{\sigma}_i \longmapsto \bar{\varsigma}_i := e^{-\imath\pi(e_i+f_i)/2}\,\gamma, \\ \eta_i \longmapsto \xi_i := e^{\imath\pi h_i}\,, \qquad i \in I,\end{gathered} \tag{4.24}$$

defines a homomorphim $W_G^U \longrightarrow U^\Gamma$ *where* U^Γ *is defined by* (4.2). *The elements* ς_i, $i \in I$ *and* γ *together with the maximal torus* H *generate* $N_G(H) \ltimes \Gamma$.

Let us stress that there is a clear analogy between the two cases. On the one hand we have $\varsigma_i \in U\gamma \subset G(\mathbb{C})\gamma$, U being a maximal compact subgroup of $G(\mathbb{C})$ and on the other hand we have $\dot{s}_i \in K \subset G(\mathbb{R})$, K being the maximal compact subgroup of $G(\mathbb{R})$. The last statement follows form the relation

$$\dot{s}_i^T \dot{s}_i = 1, \qquad i \in I, \tag{4.25}$$

where index T denotes the standard Cartan involution interchanging e_i and f_i for each $i \in I$. The elements $\eta_i = \sigma_i\bar{\sigma}_i$ are real

$$\gamma(\sigma_i\bar{\sigma}_i)\gamma = \bar{\sigma}_i\sigma_i = \sigma_i\bar{\sigma}_i. \tag{4.26}$$

and pairwise commute. Moreover ς_i and $\bar{\varsigma}_i$ are in $U\gamma$ and thus their products are in U. From this we may infer that the images of θ_i and η_i, $i \in I$ in (3.7) and (4.24) both belong to the same triple intersection

$$M = H \cap U \cap G(\mathbb{R}) = H(\mathbb{R}) \cap G(\mathbb{R}). \tag{4.27}$$

In a sense W_G^U looks like a complex analog of the real discrete group W_G^T where the relation $\tau_i^2 = \theta_i$ is replaced by $\sigma_i\bar{\sigma}_i = \eta_i$.

5 Adjoint action of the extended Weyl groups

While the action of W_G on the maximal commutative subalgebra $\mathfrak{h} = \mathrm{Lie}(H)$ is defined canonically (2.5) and does not depend on a lift of W_G into $N_G(H)$ its action on the whole Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ does depend on the lift. Above we have considered two extensions of the Weyl group W_G together with their homomorphisms into the corresponding Lie group. Here we describe their induced adjoint actions on $\mathfrak{g}$.

Proposition 5.1 *The adjoint action of the Tits group* W_G^T *on the Lie algebra* $\mathfrak{g} = \mathrm{Lie}(G)$ *via homomorphism* (3.7) *is given by*

$$\dot{s}_i\, e_i\, \dot{s}_i^{-1} \;=\; -f_i, \qquad \dot{s}_i\, f_i\, \dot{s}_i^{-1} \;=\; -e_i\,, \tag{5.1}$$

$$\dot{s}_i\, e_j\, \dot{s}_i^{-1} = e_j, \qquad \dot{s}_i\, f_j\, \dot{s}_i^{-1} \;=\; f_j, \qquad a_{ij} = 0\,, \tag{5.2}$$

$$
\begin{aligned}
\dot{s}_i\, e_j\, \dot{s}_i^{-1} \;&=\; \frac{(-1)^{|a_{ij}|}}{|a_{ij}|!}\, \underbrace{[e_i, [\ldots [e_i}_{|a_{ij}|},\, e_j]\ldots]]\,, \\
\dot{s}_i\, f_j\, \dot{s}_i^{-1} \;&=\; \frac{1}{|a_{ij}|!}\, \underbrace{[f_i, [\ldots [f_i}_{|a_{ij}|},\, f_j]\ldots]], \qquad i \neq j\,.
\end{aligned}
\tag{5.3}
$$

Proof. Relations (5.1) are actually relations for $\mathfrak{sl}_2$ Lie subalgebras generated by (e_i, h_i, f_i) and may easily be checked using for example the standard faithful representation (6.3). Relations (5.2) trivially follow from the Lie algebra relations (3.4). Thus we need to prove (5.3). Let us define $\dot{s}_i(a) := \dot{s}_i\, a\, \dot{s}_i^{-1}$. For the conjugated generators we have

$$
[h_k, \dot{s}_i(e_j)] = \dot{s}_i([h_{s_i(k)}, e_j]) = \langle s_i(\alpha_k^\vee), \alpha_j\rangle\, \dot{s}_i(e_j) = (a_{kj} - a_{ki}a_{ij})\dot{s}_i(e_j)\,, \tag{5.4}
$$

$$
[h_k, \dot{s}_i(f_j)] = \dot{s}_i([h_{s_i(k)}, f_j]) = -\langle s_i(\alpha_k^\vee), \alpha_j\rangle\, \dot{s}_i(f_j) = -(a_{kj} - a_{ki}a_{ij})\dot{s}_i(f_j)\,. \tag{5.5}
$$

These relations fix the r.h.s. of (5.3) up to coefficients. Let us calculate the coefficients by taking into account only the terms of the right weights. We have

$$
\dot{s}_i(e_j) = e^{f_i}e^{-e_i}e^{f_i}\, e_j\, e^{-f_i}e^{e_i}e^{-f_i} = \frac{(-1)^{|a_{ij}|}}{|a_{ij}|!}\, \left(\mathrm{ad}^{|a_{ij}|}_{e^{f_i}e_ie^{-f_i}}(e_j)\right) + \cdots\,, \tag{5.6}
$$

where we have used the Serre relations (3.5) and denote by $\cdots$ the terms of the "wrong" weight. Taking into account

$$
e^{f_i}e_ie^{-f_i} = e_i + \cdots\,, \tag{5.7}
$$

we obtain the first relation in (5.3). The second relation is obtained quite similarly using the isomorphism (for a proof see Lemme 6.1)

$$
e^{f_i}e^{-e_i}e^{f_i} = e^{-e_i}e^{f_i}e^{-e_i}. \tag{5.8}
$$

In this case we have

$$
\dot{s}_i(f_j) = e^{-e_i}e^{f_i}e^{-e_i}\, f_j\, e^{e_i}e^{-f_i}e^{e_i} = \frac{1}{|a_{ij}|!}\, \left(\mathrm{ad}^{|a_{ij}|}_{e^{-e_i}f_ie^{e_i}}(f_j)\right) + \cdots\,. \tag{5.9}
$$

Taking into account

$$
e^{-e_i}f_ie^{e_i} = f_i + \cdots\,, \tag{5.10}
$$

we obtain the second relation in (5.3). □

Let us stress that there is a simple way to get rid of sign factors in (5.1) and (5.3). Define a new set of generators $\tilde{e}_i = -e_i$, $\tilde{f}_i = f_i$. Then we have

$$
\dot{s}_i\, \tilde{e}_i\, \dot{s}_i^{-1} \;=\; \tilde{f}_i\,, \qquad \dot{s}_i\, \tilde{f}_i\, \dot{s}_i^{-1} \;=\; \tilde{e}_i\,, \tag{5.11}
$$

$$
\dot{s}_i\, \tilde{e}_j\, \dot{s}_i^{-1} = \tilde{e}_j\,, \qquad \dot{s}_i\, \tilde{f}_j\, \dot{s}_i^{-1} \;=\; \tilde{f}_j\,, \qquad a_{ij} = 0\,, \tag{5.12}
$$

$$\dot{s}_i\, \tilde{e}_j\, \dot{s}_i^{-1} \;=\; \frac{1}{|a_{ij}|!}\, \big[\underbrace{\tilde{e}_i, [\ldots [\tilde{e}_i}_{|a_{ij}|},\, \tilde{e}_j]\ldots]\big]\,,$$

$$\dot{s}_i\, \tilde{f}_j\, \dot{s}_i^{-1} \;=\; \frac{1}{|a_{ij}|!}\, \big[\underbrace{\tilde{f}_i, [\ldots [\tilde{f}_i}_{|a_{ij}|},\, \tilde{f}_j]\ldots]\big], \qquad i \neq j\,. \tag{5.13}$$

Now we describe the action on $\mathfrak{g}$ of the Weyl group extension W_G^U introduced in Section 4. It is convenient to express it in terms of purely imaginary generators $\imath e_i$, $\imath f_i$, $i \in I$.

Proposition 5.2 *The elements of the group* W_G^U *act on the Lie algebra* $\mathfrak{g} = \mathrm{Lie}(G)$ *via homomorphism* (4.24) *as follows*

$$\varsigma_i\, (\imath e_i)\, \varsigma_i^{-1} \;=-\, \imath f_i\,, \qquad \varsigma_i\, (\imath f_i)\, \varsigma_i^{-1} \;=-\, \imath e_i\,, \tag{5.14}$$

and

$$\varsigma_i\, (\imath e_j)\, \varsigma_i^{-1} = -\imath e_j, \qquad \varsigma_i\, (\imath f_j)\, \varsigma_i^{-1} \;=\; -\imath f_j, \qquad a_{ij} = 0\,, \tag{5.15}$$

$$\varsigma_i\, (\imath e_j)\, \varsigma_i^{-1} \;=\; -\,\frac{1}{|a_{ij}|!}\, \big[\underbrace{\imath e_i, [\ldots [\imath e_i}_{|a_{ij}|},\, \imath e_j]\ldots]\big],$$

$$\varsigma_i\, (\imath f_j)\, \varsigma_i^{-1} \;=\; -\,\frac{1}{|a_{ij}|!}\, \big[\underbrace{\imath f_i, [\ldots [\imath f_i}_{|a_{ij}|},\, \imath f_j]\ldots]\big], \qquad i \neq j\,. \tag{5.16}$$

Proof. Taking into account (3.4) we have

$$e^{\imath\pi t\, \mathrm{ad}_{h_i}}(e_j) = e_j\, e^{\imath\pi t a_{ij}}\,, \qquad e^{\imath\pi t\, \mathrm{ad}_{h_i}}(f_j) = f_j\, e^{-\imath\pi t a_{ij}}\,. \tag{5.17}$$

Using the representation

$$\varsigma_i = \dot{s}_i e^{\imath\pi h_i/2}\gamma, \tag{5.18}$$

and Proposition 3.1 we obtain (5.16) and (5.14). □

6 Proof of Theorem 4.1

We start the proof by establishing a precise relation of the generators (4.24) to the Tits generators.

Lemma 6.1 *The following identities hold*

$$\dot{s}_i := e^{f_i}\, e^{-e_i}\, e^{f_i} = e^{-e_i}\, e^{f_i}\, e^{-e_i} = e^{\imath\pi h_i/4}\, e^{\imath\pi(e_i+f_i)/2}\, e^{-\imath\pi h_i/4},$$

$$\dot{s}_i^2 = e^{\imath\pi h_i}, \tag{6.1}$$

and thus the generators defined by (4.24) *may be represented as follows*

$$\varsigma_i = e^{-\imath\pi h_i/4}\, \dot{s}_i\, e^{\imath\pi h_i/4}\gamma. \tag{6.2}$$

Proof. The identities (6.1) follow from the corresponding relations for $SL_2 \subset G$. Thus to prove (6.1) we might use the standard two-dimensional faithful representation $\phi : SL_2 \to \mathrm{End}(\mathbb{C}^2)$

$$\phi(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \phi(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \phi(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.3}$$

Direct calculations show that

$$\phi\left(e^f\, e^{-e}\, e^f\right) = \phi\left(e^{-e}\, e^f\, e^{-e}\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{6.4}$$

$$\phi\left(e^{-e}\, e^f\, e^{-e}\, e^{-e}\, e^f\, e^{-e}\right) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \phi\left(e^{\imath\pi h}\right), \tag{6.5}$$

$$\phi\left(e^{-\imath\pi h/4} e^f\, e^{-e}\, e^f e^{\imath\pi h/4}\right) = \phi\left(e^{\imath\pi(e+f)/2}\right) = \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix}. \tag{6.6}$$

Then (6.1) follows from the faithfulness of ϕ. □

The proof of Theorem 4.1 is provided by a series of Lemmas below.

Lemma 6.2 *The following relations hold*

$$\varsigma_i^2 = 1, \qquad i \in I\,. \tag{6.7}$$

Proof. Direct calculation gives

$$\varsigma_i^2 = e^{-\imath\pi h_i/4}\dot{s}_i e^{\imath\pi h_i/4}\,\gamma e^{-\imath\pi h_i/4}\dot{s}_i e^{\imath\pi h_i/4}\,\gamma = e^{-\imath\pi h_i/4}\dot{s}_i e^{\imath\pi h_i/4}\, e^{\imath\pi h_i/4}\dot{s}_i e^{-\imath\pi h_i/4} = \tag{6.8}$$

$$e^{-\imath\pi h_i/4}\dot{s}_i e^{\imath\pi h_i/2}\,\dot{s}_i e^{-\imath\pi h_i/4} = e^{-\imath\pi h_i/4} e^{-\imath\pi h_i/2}\, e^{-\imath\pi h_i/4}\,\dot{s}_i^2 = e^{-\imath\pi h_i}\cdot e^{\imath\pi h_i} = 1\,. \tag{6.9}$$

□

Lemma 6.3 *For any $i, j \in I$ such that $a_{ij} = a_{ji} = 0$ the following relations hold*

$$(\varsigma_i\varsigma_j) = \gamma\,(\varsigma_j\varsigma_i)\gamma\,. \tag{6.10}$$

Proof. We have

$$\varsigma_i\varsigma_j = e^{-\imath\pi h_i/4}\dot{s}_i e^{\imath\pi h_i/4}\,\gamma\, e^{-\imath\pi h_j/4}\dot{s}_j e^{\imath\pi h_j/4}\,\gamma = e^{-\imath\pi h_i/4}\dot{s}_i e^{\imath\pi(h_i+h_j)/4}\,\dot{s}_j e^{-\imath\pi h_j/4} \tag{6.11}$$

$$= e^{-\imath\pi h_i/4} e^{\imath\pi(-h_i+h_j)/4}\,\dot{s}_i e^{\imath\pi h_j/4}\,\dot{s}_j = e^{-\imath\pi h_i/4} e^{\imath\pi(-h_i+h_j)/4}\, e^{\imath\pi h_j/4}\,\dot{s}_i\dot{s}_j = e^{\imath\pi(h_j-h_i)/2}\,\dot{s}_i\dot{s}_j,$$

On the other hand

$$\varsigma_j\varsigma_i = e^{\imath\pi(h_i-h_j)/2}\,\dot{s}_j\dot{s}_i = e^{\imath\pi(h_i-h_j)/2}\,\dot{s}_i\dot{s}_j = e^{\imath\pi(h_i-h_j)/2}\,e^{-\imath\pi(h_j-h_i)/2}\,\varsigma_i\varsigma_j = e^{\imath\pi(h_i-h_j)}\,\varsigma_i\varsigma_j,$$

and thus

$$\varsigma_i\varsigma_j = e^{\imath\pi(h_j+h_i)}\,\varsigma_j\varsigma_i = \varsigma_j\varsigma_i\,e^{\imath\pi(h_j+h_i)}, \tag{6.12}$$

where we have used the fact that in U the following identity holds

$$e^{2\pi\imath h_i} = 1\,, \qquad i \in I. \tag{6.13}$$

It is easy to check that

$$\gamma\varsigma_i\gamma = e^{\imath\pi h_i}\varsigma_i, \qquad \gamma\varsigma_i\varsigma_j\gamma = e^{\imath\pi(h_i+h_j)}\varsigma_i\varsigma_j. \tag{6.14}$$

Now (6.10) follows from (6.12) and (6.14). □

Lemma 6.4 *For any $i,j \in I$ such that $a_{ij} = a_{ji} = -1$ the following relations hold*

$$\varsigma_i\varsigma_j\varsigma_i = \gamma\varsigma_j\varsigma_i\varsigma_j\gamma. \tag{6.15}$$

Proof. We have

$$\varsigma_i\varsigma_j\varsigma_i = e^{-\imath\pi h_i/4}\dot{s}_i e^{\imath\pi h_i/4}\,\gamma e^{-\imath\pi h_j/4}\dot{s}_j e^{\imath\pi h_j/4}\,\gamma e^{-\imath\pi h_i/4}\dot{s}_i e^{\imath\pi h_i/4}\,\gamma \tag{6.16}$$

$$= e^{-\imath\pi h_i/4}\dot{s}_i e^{\imath\pi h_i/4}\,e^{\imath\pi h_j/4}\dot{s}_j e^{-\imath\pi h_j/4}\,e^{-\imath\pi h_i/4}\dot{s}_i e^{\imath\pi h_i/4}\,\gamma. \tag{6.17}$$

Using explicit form of the Weyl group action

$$s_i(h_j) = h_j + h_i \;=\; s_j(h_i), \qquad a_{ij} = a_{ji} = -1, \tag{6.18}$$

we obtain

$$\varsigma_i\varsigma_j\varsigma_i = e^{-\imath\pi h_i/4}\dot{s}_i e^{\imath\pi(h_i+h_j)/4}\,\dot{s}_j e^{-\imath\pi(h_j+h_i)/4}\,\dot{s}_i e^{\imath\pi h_i/4}\,\gamma = \tag{6.19}$$

$$= e^{-\imath\pi h_i/4}e^{\imath\pi(-h_i+h_j+h_i)/4}\dot{s}_i\,\dot{s}_j e^{-\imath\pi(h_j+2h_i)/4}\,\dot{s}_i\gamma = \tag{6.20}$$

$$= e^{\imath\pi(h_j-h_i)/4}e^{-\imath\pi(-(h_j+h_i)+2(-h_i+h_j+h_i))/4}\dot{s}_i\dot{s}_j\dot{s}_i\gamma = \dot{s}_i\dot{s}_j\dot{s}_i\gamma. \tag{6.21}$$

The required identity follows from the identity $\dot{s}_i\dot{s}_j\dot{s}_i = \dot{s}_j\dot{s}_i\dot{s}_j$ (consequence of (3.3) and (3.7) for $a_{ij} = a_{ji} = -1$) and the relation

$$\gamma\varsigma_j\varsigma_i\varsigma_j\gamma = \gamma\dot{s}_j\dot{s}_i\dot{s}_j = \dot{s}_j\dot{s}_i\dot{s}_j\gamma = \varsigma_j\varsigma_i\varsigma_j. \tag{6.22}$$

□

Lemma 6.5 *For any $i,j \in I$ such that $a_{ji} = -2$ the following relations hold*

$$(\varsigma_i\varsigma_j)^2 = \gamma(\varsigma_j\varsigma_i)^2\gamma\,. \tag{6.23}$$

Proof. We have

$$\varsigma_i\varsigma_j = e^{-\imath\pi h_i/4}\dot{s}_i e^{\imath\pi h_i/4}\,\gamma\, e^{-\imath\pi h_j/4}\dot{s}_j e^{\imath\pi h_j/4}\,\gamma = e^{-\imath\pi h_i/4}\dot{s}_i e^{\imath\pi(h_i+h_j)/4}\,\dot{s}_j e^{-\imath\pi h_j/4} \tag{6.24}$$

$$= e^{-\imath\pi h_i/4}e^{\imath\pi(-h_i+(h_j+2h_i))/4}\,\dot{s}_i e^{\imath\pi h_j/4}\,\dot{s}_j = e^{\imath\pi(h_j+h_i)/2}\,\dot{s}_i\dot{s}_j, \tag{6.25}$$

where we take into account

$$\begin{aligned} s_i(h_j) \;&=\; h_j - \langle\alpha_i,\,\alpha_j^\vee\rangle h_i \;=\; h_j + 2h_i\,,\\ s_j(h_i) \;&=\; h_i - \langle\alpha_j,\,\alpha_i^\vee\rangle h_j \;=\; h_i + h_j\,. \end{aligned} \tag{6.26}$$

Thus we have

$$\varsigma_i\varsigma_j\varsigma_i\varsigma_j = e^{\imath\pi(h_j+h_i)/2}\,\dot{s}_i\dot{s}_j e^{\imath\pi(h_j+h_i)/2}\,\dot{s}_i\dot{s}_j = e^{\imath\pi(h_j+h_i)/2}\,\dot{s}_i\, e^{\imath\pi(-h_j+(h_i+h_j))/2}\,\dot{s}_j\dot{s}_i\dot{s}_j =$$

$$= e^{\imath\pi(h_i+h_j)/2}\,\dot{s}_i\, e^{\imath\pi h_i/2}\,\dot{s}_j\dot{s}_i\dot{s}_j = e^{\imath\pi(h_j+h_i)/2}\, e^{-\imath\pi h_i/2}\,\dot{s}_i\dot{s}_j\dot{s}_i\dot{s}_j = e^{\imath\pi h_j/2}\,\dot{s}_i\dot{s}_j\dot{s}_i\dot{s}_j. \tag{6.27}$$

Similarly we have

$$\varsigma_j\varsigma_i = e^{-\imath\pi h_j/4}\dot{s}_j e^{\imath\pi h_j/4}\,\gamma\, e^{-\imath\pi h_i/4}\dot{s}_i e^{\imath\pi h_i/4}\,\gamma = e^{-\imath\pi h_j/4}\dot{s}_j e^{\imath\pi(h_j+h_i)/4}\,\dot{s}_i e^{-\imath\pi h_i/4} \tag{6.28}$$

$$= e^{-\imath\pi h_j/4}e^{\imath\pi(-h_j+(h_i+h_j))}\,\dot{s}_j e^{\imath\pi h_i/4}\,\dot{s}_i = e^{\imath\pi h_i/2}\,\dot{s}_j\dot{s}_i\,, \tag{6.29}$$

where we use the Weyl group relations (6.26). Thus we have

$$\varsigma_j\varsigma_i\varsigma_j\varsigma_i = e^{\imath\pi h_i/2}\,\dot{s}_j\dot{s}_i e^{\imath\pi h_i/2}\,\dot{s}_j\dot{s}_i = e^{\imath\pi h_i/2}\,\dot{s}_j\, e^{-\imath\pi h_i/2}\,\dot{s}_i\dot{s}_j\dot{s}_i = \tag{6.30}$$

$$= e^{\imath\pi h_i/2}\, e^{-\imath\pi(h_i+h_j)/2}\,\dot{s}_j\dot{s}_i\dot{s}_i\dot{s}_j = e^{-\imath\pi h_j/2}\,\dot{s}_j\dot{s}_i\dot{s}_j\dot{s}_i = e^{\imath\pi h_j/2}\,\dot{s}_j\dot{s}_i\dot{s}_j\dot{s}_i. \tag{6.31}$$

Thus we prove the relation using $(\dot{s}_i\dot{s}_j)^2 = (\dot{s}_j\dot{s}_i)^2$. $\square$

Lemma 6.6 *For any $i,j \in I$ such that $a_{ji} = -3$ the following relations hold*

$$(\varsigma_i\varsigma_j)^3 = \gamma(\varsigma_j\varsigma_i)^3\gamma\,. \tag{6.32}$$

Proof. We have

$$\varsigma_i\varsigma_j = e^{-\imath\pi h_i/4}\dot{s}_i e^{\imath\pi h_i/4}\,\gamma\, e^{-\imath\pi h_j/4}\dot{s}_j e^{\imath\pi h_j/4}\,\gamma = e^{-\imath\pi h_i/4}\dot{s}_i e^{\imath\pi(h_i+h_j)/4}\,\dot{s}_j e^{-\imath\pi h_j/4} \tag{6.33}$$

$$= e^{-\imath\pi h_i/4}e^{\imath\pi(-h_i+(h_j+3h_i))/4}\,\dot{s}_i e^{\imath\pi h_j/4}\,\dot{s}_j = e^{\imath\pi(h_j+2h_i)/2}\,\dot{s}_i\dot{s}_j\,, \tag{6.34}$$

where we take into account the following relations

$$
\begin{aligned}
s_i(h_j) \,&=\, h_j - \langle \alpha_i,\, \alpha_j^\vee \rangle h_i \,=\, h_j + 3h_i\,, \\
s_j(h_i) \,&=\, h_i - \langle \alpha_j,\, \alpha_i^\vee \rangle h_j \,=\, h_i + h_j\,.
\end{aligned}
\tag{6.35}
$$

Thus we have

$$
\begin{aligned}
(\varsigma_i\varsigma_j)^3 \,&=\, e^{\imath\pi(h_j+2h_i)/2}\,\dot{s}_i\dot{s}_j e^{\imath\pi(h_j+2h_i)/2}\,\dot{s}_i\dot{s}_j e^{\imath\pi(h_j+2h_i)/2}\,\dot{s}_i\dot{s}_j \\
&=\, e^{\imath\pi(h_j+2h_i)/2}\, e^{\imath\pi\, s_is_j(h_j+2h_i)/2}\,(\dot{s}_i\dot{s}_j)^2 e^{\imath\pi(h_j+2h_i)/2}\,\dot{s}_i\dot{s}_j \\
&=\, e^{\imath\pi(h_j+2h_i)/2}\, e^{\imath\pi\, s_is_j(h_j+2h_i)/2}\, e^{\imath\pi\,(s_is_j)^2(h_j+2h_i)/2}\,(\dot{s}_i\dot{s}_j)^3\,,
\end{aligned}
\tag{6.36}
$$

which due to

$$
s_is_j(2h_i+h_j) \,=\, h_i+h_j\,, \qquad (s_is_j)^2(2h_i+h_j) \,=\, -h_i\,,
\tag{6.37}
$$

results into

$$
(\varsigma_i\varsigma_j)^3 \,=\, e^{\imath\pi(h_j+2h_i)/2}\, e^{\imath\pi(h_i+h_j)/2}\, e^{-\imath\pi h_i/2}\,(\dot{s}_i\dot{s}_j)^3 = e^{\imath\pi(h_i+h_j)}\,(\dot{s}_i\dot{s}_j)^3\,.
$$

Similarly, we find out

$$
\varsigma_j\varsigma_i \,=\, e^{-\pi\imath h_j/2}\dot{s}_j\gamma\, e^{-\pi\imath h_i/2}\dot{s}_i\gamma \,=\, e^{-\pi\imath h_j/2}\dot{s}_j\, e^{\pi\imath h_i/2}\dot{s}_i = e^{\pi\imath h_i/2}\,\dot{s}_j\dot{s}_i\,,
\tag{6.38}
$$

where we applied (6.35). Then we derive

$$
\begin{aligned}
(\varsigma_j\varsigma_i)^3 \,&=\, e^{\pi\imath h_i/2}\,\dot{s}_j\dot{s}_i e^{\pi\imath h_i/2}\,\dot{s}_j\dot{s}_i e^{\pi\imath h_i/2}\,\dot{s}_j\dot{s}_i \\
=\, e^{\pi\imath h_i/2}\, e^{\pi\imath s_js_i(h_i)/2}\,(\dot{s}_j\dot{s}_i)^2 e^{\pi\imath h_i/2}\,\dot{s}_j\dot{s}_i \,&=\, e^{\pi\imath h_i/2}\, e^{\pi\imath s_js_i(h_i)/2}\, e^{\pi\imath(s_js_i)^2(h_i)/2}\,(\dot{s}_j\dot{s}_i)^3\,,
\end{aligned}
\tag{6.39}
$$

which due to

$$
s_js_i(h_i) \,=\, -h_i-h_j \qquad (s_js_i)^2(h_i) \,=\, -2h_i-h_j\,,
\tag{6.40}
$$

results into

$$
(\varsigma_j\varsigma_i)^3 \,=\, e^{\pi\imath h_i/2}\, e^{-\pi\imath(h_i+h_j)/2}\, e^{-\pi\imath(2h_i+h_j)/2}\,(\dot{s}_j\dot{s}_i)^3 \,=\, e^{-\pi\imath(h_i+h_j)}\,(\dot{s}_j\dot{s}_i)^3\,,
\tag{6.41}
$$

and therefore

$$
\gamma(\varsigma_j\varsigma_i)^3\,\gamma \,=\, e^{\pi\imath(h_i+h_j)}\,(\dot{s}_j\dot{s}_i)^3\,.
\tag{6.42}
$$

This completes the proof. □

Combining the previous Lemmas we obtain the proof of Theorem 4.1.

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