ARITHMETIC HEIGHT ON THE MODULI SPACE OF CALABI–YAU MANIFOLDS

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#1. INTRODUCTION.

At the end of his address to the Congress of the Mathematicians in Nice I. R. Shafarevich wrote: "En conclusion mentionnons le probléme extrêment interessant de la generalisation de ces considèrations aux varietètes algèbriques de dimension quelconque et de classe canonique nulle." The articles [T1], [T2], [T3], [T4] and the present one are attempts to answer this Problem.

We know that Kähler manifolds with canonical class zero and finite fundamental groups are devided into two classes namely Hyper-Kählerian manifolds and Calabi-Yau manifolds. In both cases we can define the so called Teichmüller space, namelly all complex structures on a fixed manifold modulo the action of the group of diffeomorphisms isotopic to the identity. Like in the case of polarized abellian varieties and in the case of polarized K3 surfaces we proved that the Teichmüller spaces of polarized Hyper-Kählerian manifolds are symmetric domains. More precisely it is $SO(2,b_2-1)/SO(2)xSO(b_2-1)$, like in the case of polarized K3 surfaces. (See [T3] and [T4].) In case of the so called Calabi-Yau manifolds the Teichmüller space $\mathfrak{T}(M_o)$ turned out to be a Stein manifold on which the mapping class group Γ acts discretely. In this paper we proved that each component of the moduli space of a Calabi-Yau manifold M_o is a quasi-projective manifold defined over Spec$\mathbb{Z}$. More over we constructed an analogue of the "modular" height function on $\mathfrak{T}(M_o)/\Gamma$ with a "logarithmic growth". We will make some comparison on conjectural leval with the deep Falting's results about abellian varieties. (See [F&W].) For definition of the height function and its properties see [L].

From the point of view of deformation theory manifolds with canonical class zero are much simpler then manifolds with negative canonical class. On the other hand from point of view of arithmetic it seems that algebraic manifolds defined over Spec$\mathbb{Z}$ with canonical class zero are much more interesting then manifolds with negative canonical class. At the end of this introduction we state some conjectures about the arithmetics of manifodls with canonical class zero and defined over Spec$\mathbb{Z}$.

1.1. Definition.

Suppose that M_o is a compact Kähler manifolds such that a) $H^o(M_o,\Omega^i)=0$ for $0<i<n=\dim_{\mathbb{C}}M_o=n>2$ b) there exists a unique up to a constant holomorphic n-form $\omega_o(n,0)$ on M_o which has no zeroes. Then we will call M_o a Calabi-Yau manifold.

1.2. Review of the results in [T2].

1.2.1. Definition. Let $I(M_O)$ be the set of all integrable complex structures on M_O. Let

$$\mathfrak{T}(M_O):=I(M_O)/\mathrm{Diff}_O^+(M_O)$$

where $\mathrm{Diff}_O^+(M_O)$ is the group of the diffeomorphisms of M_O isotopic to the identity preserving the orientation of M_O, then we call $\mathfrak{T}(M_O)$ the Teichmüller space of M_O.

In [T2] we proved the following Theorem:

THEOREM 2.1.(See [T2].)

Each component of $\mathfrak{T}(M_O)$ is a non-singular complex manifold and $\dim_{\mathbf{C}}\mathfrak{T}(M_O)=\dim_{\mathbf{C}}H^1(M_O,\Theta_O)$.

In [T1] we showed that the Weil-Petersson metric which was introduced by Koiso can be *defined in the following way: We know that the tangent space* $T_{o,\mathfrak{T}(M_O)}$ *of* $\mathfrak{T}(M_O)$ *at a point* $o\in\mathfrak{T}(M_O)$ *that corresponds to* M_O *can be identified with* $H^1(M_O,\Theta_O)$. *Let* μ_1 *and* $\mu_2\in H^1(M_O,\Theta_O)$, *then the Weil-Petersson metric can be defined in the following way:*

$$<\mu_1,\mu_2>:=\int_{M_O}[\mu_1\perp\omega_O(n,0)]\wedge\overline{[\mu_2\perp\omega_O(n,0)]}$$

where $\mu_i\perp\omega_O(n,0)$ is a contraction of tensors. Notice that by the contraction with $\omega_O(n,0)$ we get an isomorphisim:

$$i\colon H^1(M_O,\Theta_O)\to H^{n-1}(M_O,\Omega^1)$$

$[\mu_i\perp\omega_O(n,0)]$ denote the class of cohomology of the form $\mu_i\perp\omega_O(n,0)$ in $H^n(M_O,\mathbf{C})$, where $\perp$ means contraction of tensors.

One of the main Theorem proved in [T2] is:

THEOREM 4.1.(See [T2].)

The Weil-Petersson metric is a complete metric on each of the component of $\mathfrak{T}(M_O)$.

Using THEOREM 4.1. we proved the following Theorem:

THEOREM 7.1.(See [T2].)

Each component of $\mathfrak{T}(M_O)$ is Stein and contractible manifold.

1.3. Statement of the main results in this article.

1.3.1. Definition. $\Gamma:=\mathrm{Diff}^+(M_O)/\mathrm{Diff}_O^+(M_O)$

In {T2} it is proved that Γ acts dicretely on each of the components of $\mathfrak{T}(M_O)$. From now on we will fix one of the components of $\mathfrak{T}(M_O)$.

CONJECTURE.

One can hope that the number of the components of $\mathfrak{T}(M_O)$ is equal to the number of Calabi-Yau manifilds that are birationaly equivelent to M_O but are not isomorphic to M_O.

1.3.2. Definition.

$$\mathfrak{M}(M_O):=\mathfrak{T}(M_O)/\Gamma$$

From a result of H. Cartan it follows that $\mathfrak{M}(M_O)$ is a complex analytic space. In #2 of this article we are going to prove the following Theorem:

THEOREM 2. $\mathfrak{M}(M_O)$ is a quasi-projective variety.

The proof of THEOREM 2. is based on the following result of Mok:

THEOREM (See [Mok].)

Suppose that X is a Stein manifold on which

A) there exists a Kähler metric (g) such that: the holomorphic sectional curvature $\mathcal{H}(g)$ of g is such that $-\infty<c_1<\mathcal{H}(g)\leq c_2<0$

B) On X a group Γ acts discretely and the fundamental domain of Γ has a finite volume with respect to g. Then X/Γ is a quasi-projective variety.

In order to prove THEOREM 2.1. we need to chek conditions A) and B) of the theorem proved by Mok. Condition A) follows from the explicit formulas of the curvature operator of the Weil-Petersson metric obtained in [T1]. Condition B) follows from the Global Torelli Theorem for Calabi-Yau manifolds proved in [T2] and the following deep result of Dennis Sullivan:

THEOREM (See [Su].)

Suppose that X is a compact Kähler manifold of complex dimension ≥ 3, then

$$\Gamma:=\mathrm{Diff}^{+}(M_O)/\mathrm{Diff}_O^{+}(M_O)$$

is an arithmetic group.

In #3. we construct an ample line bundle $\overline{\mathfrak{L}}$ over $\overline{\mathfrak{M}}(M_O)$, where $\overline{\mathfrak{M}}(M_O)$ is some comactification of $\mathfrak{M}(M_O)$. Let us describe this construction. We know from [T1] that there exists an universal family of marked Calabi-Yau manifolds $\mathfrak{S}\rightarrow\mathfrak{T}(M_O)$. Let

$$L:=\pi_*\omega_{\mathfrak{S}/\mathfrak{T}(M_O)}.$$

1.3.3. Definition.

On L we have a natural metric, namely from the definition of L it follows that the local sections of L are holomorphic families of holomorphic n forms on the Calabi-Yau manifolds parametrized by $\mathfrak{T}(M_0)$, i.e. $\omega_t(n,0)$, then

$$H(\omega_t(n,0)):=\|\omega_t(n,0)\|^2:=(-1)^{\frac{n(n-1)}{2}}(\sqrt{-1})^n\int_{M_0}\omega_t(n,0)\wedge\overline{\omega_t(n,0)}$$

1.3.4. Remark.

Notice that directly from the definitions of Γ, L and H it follows that L and H are invariant under Γ.

1.3.5. Definition.

From 1.3.4. it follows that we can define on $\mathfrak{M}(M_0):=\mathfrak{T}(M_0)/\Gamma$ $\mathcal{L}:=\tau_*L$, where

$$\tau:\mathfrak{T}(M_0)\rightarrow\mathfrak{M}(M_0)$$

Since H is invariant under the action of Γ it follows that H induced an Hemitian metric on $\mathcal{L}$. We will denote this metric again by H.

In order to formulate the next THEOREM we need the following definition:

1.3.6. Definition.

Let X be a normal complex space, $Y\subset X$ a closed analytic subser such that $U=X-Y$ is dense in X. If $\underline{E}$ is a vector bundle on X and < , > a hermitian metric on E/U, this metric has logarithmic singularities along Y if the following holds: For $y\in Y$, there exists a neighborhood V of y in X, holomorphic functions $f_1,\ldots,f_k$ on V with Y as common set of zeroes, and sections $e_1,\ldots,e_r$ of $\underline{E}$ over U which form a basis of $\underline{E}/U$, such that for some constants c_1, $c_2>0$,

$$|<e_i,e_j>|(z)\leq c_1|\log(\max(|f_i(z)|)|^{c_2}$$

$$|\det<e_i,e_j>|(z)^{-1}\leq c_1|\log(\max(|f_i(z)|)|^{c_2}$$

for $z\in U\cap V$.

1.3.7. REMARKS:

a) The extension $\underline{E}$ of $\underline{E}/U$ is uniquely determined by this property, since local section of $\underline{E}/U$ is holomorphic on X if and only if its norm grows at most logarithmically near Y.

b) The definition is essentially independent of the choice of the e_i and f_j. (See [F] and [M].)

THEOREM 3.

a) The hermitian scalar product H on $\mathcal{L}$ has logarithmic singularities.

b) Let $\overline{\mathcal{L}}$ be the extension of $\mathcal{L}$ defined as in the remark 1.3.7., then $\overline{\mathcal{L}}$ is an ample bundle on $\mathfrak{M}(M_O)$.

1.4. SOME REMARKS.

A natural generalization of a K3 surface in higher dimensions are the manifolds with canonical class zero and finite fundamental group are Kähler manifolds with canonical class zero. There are two types of such manifolds, namely Calabi-Yau manifolds and Hyper-Kählerian. We know that the moduli space of marked polarized Hyper-Kählerian manifolds is isomorphic to $SO(2,b_2-1)/SO(2)xSO(b_2-1)$, where $b_2:=\dim_{\mathbf{C}} H^2(X,\mathbf{C})$. So one should consider as a natural generalization of K3 surfaces the Hyper-Kählerian manifolds.

A) CONJECTURES ABOUT CALABI-YAU MANIFOLDS.

It is easy to see that $\mathfrak{M}(M_O)$ is defined over Spec $\mathbf{Z}$. So H defines an analogue of the Falting's height h on the moduli of polarized abelian varieties. So one can hope that the analogue of Falting's Theorem (Shafarevich's conjecture) will hold for Calabi-Yau manifolds defined over $\mathbf{Z}$, namely that the set $S(M; p_1,..,p_k)$, where M is a Calabi-Yau manifold defined over Spec$\mathbf{Z}$ and M has "bad reduction" over the prime numbers $p_1,..,p_k$, then S is a finite set. This conjecture should hold also for Hyper-Kählerian manifolds. The functional analogue of Falting's Theorem (Shafarevich's conjecture) is not true for Hyper-Kahlerian manifolds. (See [F] and [S&Z].) There are some evidence that the functional analogue of Falting's Theorem (Shafarevich's conjecture) is true for Calabi-Yau manifolds.

B) CONJECTURES ABOUT HYPER-KÄHLERIAN MANIFOLDS.

1.4.1. Definition.

Let X be a compact Kähler manifold such that $H^1(X,\mathcal{O}_X)=0$ and let on X there exists a unique up to a constant holomorphic two-form $\omega_X(2,0)$ which is non-degenerate at each $x\in X$. Then we will call X a hyper-Kählerian manifold.

1.4.2. In [T3] it was proved that the moduli space of all marked Hyper-Kählerian manifolds with a fixed polarization is isomorphic to $SO(2,b_2-1)/SO(2)xSO(b_2-1)$, where a marked Hyper-Kählerian manifolds with a fixed polarization means a triple $(X;\delta_1,..,\delta_{b_2}; L)$, where $\delta_1,..,\delta_{b_2}$is a basis of $H_2(X,\mathbf{Z})$ and $L\in H^{1,1}(X,\mathbf{R})\cap H^2(X,\mathbf{Z})$ is an imaginary part of a Kähler metric on X.

1.4.3. Definition.

Let $(X;\delta_1,..,\delta_{b_2};\ L)$ and $(Y;\delta_1,..,\delta_{b_2};\ L)$ be two marked polarized hyper-Kählerian manifolds and suppose that under the period map they correspond to points x and y in $SO(2,b_2-1)/SO(2)xSO(b_2-1)$. We will say that X is isogenous to Y if there exists a matrix $A\in SO(2,b_2-1;\mathbf{Q})$ such that Ax=y. Indeed A defines a homeomorphism

$$A_*: H_2(X,\mathbf{Z})\rightarrow H_2(Y,\mathbf{Z})$$

A very intersting question posed by Shafarevich and A. Weil is the following one: If A_* is induced by algebraic correspondence. (See [Sh] and [Weil].)

1.4.4. Conjectures.

a) Suppose that X and Y are defined over $\mathbf{Z}$ and suppose that L(X,s)=L(Y,s), then X is isogenous Y, L(X,s) is the L-function of X.

b) $S(X;p_1,..,p_k)$ is a finite set, where X is a hyper-Kählerian manifold defined over Spec$\mathbf{Z}$, $p_1,..,p_k$ are fixed prime numbers over which X mod p is singular.(The functional analogue of Falting's Theorem (Shafarevich's conjecture) is not true for Hyper-Kahlerian manifolds. (See [F] and [S&Z].)

Remarks.

(i) This conjecture is an analogue to one of many famous results of Faltings. See [F].

(ii) Independently another proof of the quasiprojectivity of $\mathfrak{X}(M_O)$ was given by Viehweg. (See [V1], [V2] and [V3].)

Aknowlegments.

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#2. PROOF OF THEOREM 2.1.

THEOREM 2. $\mathfrak{M}(M_O)$ is a quasi-projective variety, where $\mathfrak{M}(M_O)$ is defined in #1.3.

PROOF OF 2.1.:

We need to chek A) the holomorphic sectional curvature $\mathcal{K}(g)$ of the Weil-Petersson metric (g) has the following property:

$$-\infty<c_2<\mathcal{K}(g)\leq c_1<0$$

It was proved in [T1] and [Ti] that $\mathcal{K}(g)\leq c_1<0$, so we need to prove:

LEMMA A. $-\infty<c_2<\mathcal{K}(g)$.

LEMMA B. Γ acts on $\mathfrak{T}(M_O)$ discretely and has a finite volume with respect to g.

As it was pointed out in the Introduction THEOREM 2 follows directly from a result of Mok. (See [Mok].)

PROOF OF LEMMA A:

The proof of PROPOSITION A is based on some facts stated and proved in [T1]&[T2]. Let me remind these facts. First we will fix a polarization on M_O, i.e. we are going to consider a pair (M,L), where M is a Calabi-Yau manifold, $L\in H^{1,1}(M,\mathbb{Z})$ and L corresponds to an ample divisor on M. From the fundamental result of Yau, namely the solution of Calabi conjecture, we get that for each $t\in\mathfrak{T}(M_O)$ which corresponds to a Calabi-Yau manifold L determines a Ricci flat metric g(t) such that Im(g(t))=L. Let us fix $o\in\mathfrak{T}(M_O)$ that corresponds to M_O and let $g_{\alpha,\overline{\beta}}(0)$ be the Yau metric, i.e. the Ricci flat metric that corresponds to L. Let $\overline{\partial}^*$ be the conjugate oprator of $\overline{\partial}$ with respect to $g_{\alpha,\overline{\beta}}(0)$ on the space $\Gamma(M_O,\Theta\otimes\Omega_O^{0,1})$. Let

$$\left\{\mathcal{U}_j,z_j^1,\ldots,z_j^n\right\}$$

be a finite covering of M_O where each open set is with fixed coordinate system.

PROPOSITION A.1.

For any $\gamma\in\mathbf{H}^1(M_O,\Theta_O)$ where $\gamma|_{\mathcal{U}_j}=\sum_{\alpha,\beta}(\gamma_j)_{\overline{\beta}}^{\alpha}\,\overline{dz_j^{\beta}}\otimes\frac{\partial}{\partial z_j^{\alpha}}$

such that $||\gamma||^2=1$ there exixts a constant C>0 such that for all functions $(\gamma_j)_{\overline{\beta}}^{\alpha}$ we have:

$$\sum_j|(\gamma_j)_{\overline{\beta}}^{\alpha}|\leq C$$

REMARK. The constant depends on the covering $\left\{\mathcal{U}_i\right\}$ of M_O and the choice of $\left\{dz_j^i\right\}$. We will use later a new basis of $\Omega_O^{1,0}|_{\mathcal{U}_j}$, namely $\left\{\Theta_O^i\right\}$.

PROOF of Proposition A.1.:

This Proposition follows immediately from the compactness of M_O and the conditions $\gamma \in \mathbf{H}^1(M_O,\Theta_O)$ such that $\|\gamma\|^2=1$.

Q.E.D.

PROPOSITION A.2. $|\mathcal{H}(g)| \leq C^2$, where C is defined in Proposition A.1.

PROOF of Proposition A.2.:

Let γ_k for k=1,..,N be an orthonormal basis of $\mathbb{H}^1(M_O,\Theta_O)$. Let

$$\text{(A.2.1.)} \qquad \phi:=\phi(t^1,..,t^N):=\sum_{k=1}^{N} \gamma_k t^k+\tfrac{1}{2}\overline{\partial}^* G[\phi(t^1,..,t^N),\phi(t^1,..,t^N)] \in \Gamma(M_O,\Omega_O^{0,1} \otimes \Theta_O)$$

where G is the Green operator with respect to the Yau metric $g_{\alpha,\overline{\beta}}(0)$. It was proved in [T1] that we have for $\phi(t^1,..,t^N)$

$$\text{(A.2.2.)} \qquad \overline{\partial}\phi(t^1,..,t^N))=\tfrac{1}{2}[\phi(t^1,..,t^N)),\phi(t^1,..,t^N)]$$

i.e. $\phi:=\phi(t^1,..,t^N)$ defines integrable complex structures. More over in [T2] it was proved that $\phi(t^1,..,t^N)$ is defined for all $(t^1,..,t^N) \in \mathbf{C}^N$. Let $(z_j^1,...,z_j^n)$ be a local coordinate system in $\mathcal{U}_j$ such that

$$\text{(A.2.3.)} \qquad \omega_O(n,0)|_{\mathcal{U}_j}=dz_j^1 \wedge ... \wedge dz_j^n$$

and $\omega_O(n,0)$ is a holomorphic n-form on M_O such that

$$\text{(A.2.4.)} \qquad \omega_O(n,0) \wedge \overline{\omega_O(n,0)} = \mathrm{vol}(g_{\alpha,\overline{\beta}}(0))$$

((A.2.4.) was proved in [T2].) We can view $\phi(t^1,..,t^N) \in \Gamma(M_O,\mathrm{Hom}(\Omega_O^{1,0},\Omega_O^{0,1}))$. We define

$$\text{(A.2.5.)} \qquad A_t(dz_j^k):=dz_j^k+\phi(dz_j^k), \text{ where } \phi \text{ is defined in (A.2.1.)}$$

In [T1] the following lemma was proved:

(A.2.6.) LEMMA. (See [T1]&[T2].)

$\omega_t(n,0):=(A_t(dz_j^1))\wedge..\wedge(A_t(dz_j^n))$ is a globally defined holomorphic n-form for each $t\in\mathfrak{T}(M_o)$ on M_t.

(A.2.7.) LEMMA. (See [T2].)

For each $t\in\mathfrak{T}(M_o)$ which corresponds to a Calabi-Yau manifold there exist forms of type (1,0) $\left\{\Theta_t^i\right\}$, i=1,..,n such that

$$\text{(i)} \qquad <\Theta_t^i,\Theta_t^j>=c(t)\delta_{i\bar{j}}$$

where < , > is the scalar product defined by the Calabi-Yau metric $(g_{\alpha\bar{\beta}}(t))$ on M_t and c(t) is a finction on $\mathfrak{T}(M_o)$ such that c(0)=1 and c(t)>0. Moreover we have

$$\text{(ii)} \qquad \Theta_t^1\wedge..\wedge\Theta_t^n=\omega_t(n,0)$$

(A.2.8.) LEMMA. (See [T2].)

Let us define B_t in the following way $B_t(\Theta_o^i)=\Theta_t^i$ & $\overline{B_t(\Theta_o^i)}=\overline{\Theta_t^i}$ for i=1,..,n, then $B_t=A_t$ as linear operators.

Clearly we have $A_t(\Theta_o^i)=A_t(A_{t_o}^{-1}(\Theta_o^i))$. Next we define $A_{t-t_o}:=A_t\cdot A_{t_o}^{-1}$.

In the basis $\left\{..,\Theta_t^i,..,\overline{\Theta_t^i},..\right\}$ the matrix of A_{t-t_o} has the following assymptotic expension:

$$A_{t-t_o}(\Theta_t^i)=\Theta_{t_o}^i+\sum_{k=1}^{N}(t^k-t_o^k)\gamma_{t_o,k}(\Theta_{t_o}^i)+0(\sum_{k=1}^{N}(t^k-t_o^k)^2)$$

where $\left\{\gamma_{t_o,k}\right\}\in\mathbf{H}^1(M_{t_o},\Theta_{t_o})$, more over all matrices $\gamma_{t_o,k}$ are identical to the matrices of $\gamma_k\in\mathbf{H}^1(M_o,\Theta_o)$ written in the basis $\left\{\Theta_o^1,..,\Theta_o^n,\overline{\Theta_o^1},..,\overline{\Theta_o^n}\right\}$

(A.2.9.) REMARKS.

I.) Let

$$\text{(A.2.9.1)} \qquad \gamma_{t,k}:=\sum_{\alpha,\beta}(\gamma_{t,k})_{\bar{\beta}}^{\alpha}\ \overline{\Theta_t^{\beta}}\otimes(\Theta_t^{\alpha})^*, \text{ where}(\Theta_t^{\alpha})^* \text{ is the dual of } \Theta_t^{\alpha}.$$

So the matrix of $\gamma_{t,k}$ in the basis $\left\{\Theta_t^1,..,\Theta_t^n,\overline{\Theta_t^1},..,\overline{\Theta_t^n}\right\}$ will be $\left((\gamma_{t,k})_{\bar{\beta}}^{\alpha}\right)$. Since for $\left\{\Theta_t^i\right\}$ we have

(A.2.9.2) $<\Theta^i_t,\Theta^j_t>=c(t)\delta_{i\bar{j}}\quad c(t)>0$

we get that

(A.2.9.3) $(\gamma_{t,k})^{\alpha}_{\bar{\beta}}=(\gamma_{t,k})_{\bar{\alpha},\bar{\beta}}$, where $\gamma_{\bar{\alpha},\bar{\beta}}:=\sum_{\mu=1}^{n} g_{\mu,\bar{\alpha}}\gamma^{\mu}_{\bar{\beta}}$

II) Siu proved in [Siu] the following results: If $\gamma_{t,k}$ is a harmonic tensor with respect to the Ricci flat metric then we have

(A.2.9.4) $(\gamma_{t,k})_{\bar{\alpha},\bar{\beta}}=(\gamma_{t,k})_{\bar{\beta},\bar{\alpha}}$ (See also [N].)

From (A.2.9.4.) Siu deduce the following result: Let $\mathfrak{X}\rightarrow\mathfrak{U}$ be the Kuranishi family of manifolds with a Ricci flat metric defined by ϕ which fulfills

(A.2.1.) $\phi:=\phi(t^1,..,t^N):=\sum_{k=1}^{N}\gamma_k t^k+\frac{1}{2}\bar{\partial}^*G[\phi(t^1,..,t^N),\phi(t^1,..,t^N)]\in\Gamma(M_o,\Omega^{0,1}_o\otimes\Theta_o)$

(It was proved in [T1] that automatically ϕ fulfills

(A.2.2.) $\bar{\partial}\phi(t^1,..,t^N))=\frac{1}{2}[\phi(t^1,..,t^N)),\phi(t^1,..,t^N)])$

where $\gamma_k\in\mathbf{H}^1(M_o,\Theta_o)$ for all k, i.e. γ_k are harmonic tensors with respect to a Ricci flat metric. $(g_{\alpha,\bar{\beta}}(0))$. Suppose that on $\mathfrak{X}\rightarrow\mathfrak{U}$ we have a family of Ricci flat metrics $(g_{\alpha,\bar{\beta}}(t))$ such that the class of the cohomology $[\mathrm{Im}(g_{\alpha,\bar{\beta}}(t))]=L$, i.e is fixed, then

(A.2.9.5.) $\frac{d}{dt}(\mathrm{Im}(g_{\alpha,\bar{\beta}}(t))|_{t=o}=0$ (See also [N].)

From this result and from the fact that for each $t\in\mathfrak{T}(M_o)$ we have that $\gamma_{t,k}$ are harmonic tensors with respect to the Ricci flat metrics $(g_{\alpha,\bar{\beta}}(t))$ it follows that

(A.2.9.6.) $\frac{d}{dt}(\mathrm{Im}(g_{\alpha,\bar{\beta}}(t))=0$

which means that for all $t\in\mathfrak{T}(M_o)$ $\mathrm{Im}(g_{\alpha,\bar{\beta}}(t))$ define one and the same symplectic structure on M_o, i.e.

(A.2.9.7.) $\mathrm{Im}(g_{\alpha,\bar{\beta}}(t))=\mathrm{const}$

From (A.2.9.7.) it follows that:

(A.2.9.8.) $$\frac{d}{dt}(\mathrm{vol}(g_{\alpha,\overline{\beta}}(t))=0$$

so

(A.2.9.9.) $$\mathrm{vol}(g_{\alpha,\overline{\beta}}(t))=\Theta_t^1\wedge..\wedge\Theta_t^n\wedge\overline{\Theta_t^1\wedge..\wedge\Theta_t^n}=\mathrm{vol}(g_{\alpha,\overline{\beta}}(0))=\Theta_o^1\wedge..\wedge\Theta_o^n\wedge\overline{\Theta_o^1\wedge..\wedge\Theta_o^n}$$

<u>We are going to summerize the main facts that we are going to use:</u>

<u>FACT 1.</u>

Suppose that $\nu_o\in\mathbf{H}^1(M_o,\Theta_o)$ and

$$\nu_o:=\sum_{i,j}\nu_{o\overline{j}}^{\;i}\overline{\Theta_o^j}\otimes(\Theta_o^i)^*$$

Then

$$\nu_{t_o}:=\sum_{i,j}\nu_{o\,\overline{j}}^{\;i}\overline{A_{t_o}(\Theta_o^j)}\otimes(A_{t_o}(\Theta_o^i))^*$$

is a harmonic form with respect to the Calabi-Yau metric $g_{\alpha,\overline{\beta}}(t_o)$.

<u>FACT 2.</u>

We have the following identity:

$$\left(\wedge^2\nu_o\perp(\Theta_o^1\wedge..\wedge\Theta_o^n)\right)\wedge\left(\overline{\left(\wedge^2\nu_o\perp(\Theta_o^1\wedge..\wedge\Theta_o^n)\right)}=\right.$$

$$=\frac{1}{\|\omega_t(n,0)\|^2}\left(\wedge^2\nu_{t_o}\perp(A_{t_o}(\Theta_o^1)\wedge..\wedge A_{t_o}(\Theta_o^n)\right)\wedge\left(\overline{\left(\wedge^2\nu_o\perp(A_{t_o}(\Theta_o^1)\wedge..\wedge A_{t_o}(\Theta_o^n)\right)}\right)$$

<u>FACT 3.</u>

The forms $\wedge^2\nu_o\perp(A_{t_o}(\Theta_o^1)\wedge..\wedge A_{t_o}(\Theta_o^n)$ are primitive with respect to the Calabi-Yau metrics $g_{\alpha,\overline{\beta}}(t_o)$ for all $t_o\in\mathfrak{X}(M_o)$.

<u>PROOF OF FACT 3:</u>

We know that $\nu_{t_o}\in\mathbb{H}^1(M_{t_o},\Theta_{t_o})$. Let

$$\nu_{t_o}(t):=\nu_{t_o}t+\tfrac{1}{2}G_{t_o}\overline{\partial}_t^*[\nu_{t_o}(t),\nu_{t_o}(t)]$$

We know that $\nu_{t_o}(t)$ automatically fulfills the inegrability condition:

$$\overline{\partial}_{t_o}\nu_{t_o}(t):=\frac{1}{2}[\nu_{t_o}(t),\nu_{t_o}(t)]$$

Let

$$\nu_{t_o}(t):=\nu_{t_o}t+\nu_{t_o}^2t^2+....$$

On the other hand in [T] it was proved that:

$$\omega_t(n,0):=\Big((\Theta_{t_o}^1+\nu_{t_o}(t)(\Theta_{t_o}^1))\wedge..\wedge(\Theta_{t_o}^n+\nu_{t_o}(t)(\Theta_{t_o}^n))\Big)$$

for each t is a holomorphic n-form on M_t defined by $\nu_{t_o}(t)$. It is easy to see that:

(*) $\omega_t(n,0):=\omega_{t_o}(n,0)+t(\nu_{t_o}\perp\omega_{t_o}(n,0))+t^2(\nu_{t_o}^2\perp\omega_{t_o}(n,0)-(\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0))+)0(t^3)$
(See [T1].)

Clearly $\omega_t(n,0)$ is a primitive form with respect to $\mathrm{Im}(g_{\alpha,\overline{\beta}}(t))$ for each t. This means that:

$$\omega_t(n,0)\perp(\mathrm{Im}(g_{\alpha,\overline{\beta}}(t))^*=0$$

From $\mathrm{Im}(g_{\alpha,\overline{\beta}}(t))=\mathrm{Im}(g_{\alpha,\overline{\beta}}(t_o))$ for each t and the definition of a primitive form and from (*) we get :

$$(\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0))\perp(\mathrm{Im}(g_{\alpha,\overline{\beta}}(t_o))^*=0$$

i.e. $(\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0))$ is a primitive form for each t_o.

Q.E.D.

<u>After this prelimenary work we are ready to prove our estimate:</u>

<u>PROOF OF$|\mathcal{K}(g)|\leq C^2$, where C is defined in Proposition A.1.:</u>

The proof is based on the computation of the curvature tensor of the Weil-Petersson metric done in [T1], namely we proved that the holomorphic sectional curvature at a point $t_o\in\mathfrak{T}(M_o)$ that corresponds to a Calabi-Yau manifold M_{t_o} in the direction $\nu_{t_o}\in T_{t_o,\mathfrak{T}(M_{t_o})}:=\mathbf{H}^1(M_{t_o},\Theta_{t_o})$ is equal

$$(A.I.)\ 8(-1)^{\frac{n(n-1)}{2}}(\sqrt{-1})^{n-1}\frac{1}{\|\omega_{t_o}(n,0)\|^2}\int_{M_{t_o}}[\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0)]\wedge\overline{[\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0)]}\ -1$$

where $\|\nu_{t_o}\|^2=1$ and $[\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0)]$ means the class of cohomology of the form $\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0)$.
(Let me remind that we can view ν as a section of $\Gamma(M_{t_o},Hom(\Omega^{1,0}_{t_o}\Omega^{0,1}_{t_o}))$ so $\wedge^2\nu_{t_o}$ makes sence. Here all norms are with respect to the Calabi-Yau metric $g_{\alpha,\overline{\beta}}(t_o)$.)

Clearly we are going use formula (A.I.), Fact1, Fact2 and Fact3 in order to prove our estimate. From (A.1.) and Fact 2 we get

$$(A.II.)\qquad 8(-1)^{\frac{n(n-1)}{2}}(\sqrt{-1})^{n-1}\frac{1}{\|\omega_{t_o}(n,0)\|^2}\int_{M_{t_o}}\left(\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0)\right)\wedge\left(\overline{\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0)}\right)=$$

$$=8(-1)^{\frac{n(n-1)}{2}}(\sqrt{-1})^{n-1}\frac{1}{\|\omega_{t_1}(n,0)\|^2}\int_{M_{t_1}}\left(\wedge^2\nu_{t_1}\perp\omega_{t_1}(n,0)\right)\wedge\left(\overline{\wedge^2\nu_{t_1}\perp\omega_{t_1}(n,0)}\right)$$

for any t_o and $t_1\in\mathfrak{T}(M_o)$. From the definition of $\wedge^2\nu_{t_1}$ and the constant C defined in PROPOSITION A.1. we get that

$$(A.III.)\qquad \left|8(-1)^{\frac{n(n-1)}{2}}(\sqrt{-1})^{n-1}\frac{1}{\|\omega_{t_o}(n,0)\|^2}\int_{M_{t_o}}\left(\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0)\right)\wedge\left(\overline{\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0)}\right)\right|\leq C^2$$

From Fact 3, i.e. that $\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0)$ is a primitive form we get that:

$$(A.VI.)\qquad \left|\frac{8}{\|\omega_{t_o}(n,0)\|^2}\int_{M_{t_o}}(\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0))\wedge(\overline{\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0)})\right|=\|\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0)\|^2$$

From the Hodge decomposition of $\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0)$, i.e.

$$(A.V.)\qquad \wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0)=\mathbb{H}(\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0))+\overline{\partial}_{t_o}\mu_{t_o}$$

we get that

(A.VI.) $\|8\mathbf{H}(\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0))\|^2+\|8\bar{\partial}_{t_o}\mu_{t_o}\|^2=$

$$|8(-1)^{\frac{n(n-1)}{2}}(\sqrt{-1})^{n-1}\frac{1}{\|\omega_{t_o}(n,0)\|^2}\int_{M_{t_o}}\left(\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0)\right)\wedge\overline{\left(\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0)\right)}|^2$$

On the other hand we have

(A.VII.) $$|8(-1)^{\frac{n(n-1)}{2}}(\sqrt{-1})^{n-1}\frac{1}{\|\omega_{t_o}(n,0)\|^2}\int_{M_{t_o}}[\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0)]\wedge\overline{[\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0)]}|^2=$$

$\|8\mathbf{H}(\wedge^2\nu_{t_o}\perp\omega_{t_o}(n,0))\|^2$ (The norm $\|\ \|$ is with respect to Calabi-Yau metric.)

So from (A.I.), (A.III), (A.VI) & (A.VII.) we get that

$|\mathcal{H}(g))\leq M$.

Q.E.D.

PROOF OF LEMMA B:

Let me remind to the reader the definition of $\Gamma:=\mathrm{Diff}^+(M_o)/\mathrm{Diff}_o^+(M_o)$. This group has a natural representation in $H^n(M_o,\mathbb{Z})$ and this representation preserve the cup-product. In [T2] it was proved that Γ acts on $\mathfrak{T}(M_o)$ discretely. Let me denote by Γ_o the subgroup of $\mathrm{Aut}(H^n(M_o,\mathbb{Z}))$ which preserve the cup-product. Let G_o be the the idenity component of the group of the automorphisms of $H^n(M_o,\mathbf{R})$ which preserve the cup product. From a Theorem of Borel and Harish-Chandra we know that $\mathrm{vol}(G_o/\Gamma_o)<\infty$ with respect to the Haar measure.(See [B&H]. From the theorem of Sullivan about the arithmeticity of Γ it follows that $\mathrm{vol}(G_o/\Gamma)<\infty$. (See [Su].) From the global Torelli Theorem for Calabi-Yau manifolds (See [T2]), i.e. $\mathfrak{T}(M_o)\subset G_o/K$, where K is a compact subgroup of G_o, the fact that the Weil-Petersson metric is a restriction of a left invariant metric on G_o/K and $\mathrm{vol}(G_o/\Gamma)<\infty$ we get that:

$$\mathrm{vol}(\mathfrak{T}(M_o)/\Gamma)<\infty$$

with respect to the Weil-Petersson metric.

Q.E.D.

From LEMMA A, LEMMA B and the THEOREM OF MOK it follows that $\mathfrak{T}(M_o)/\Gamma$ is quasiprojective manifold.

#3. THE ARITHMETIC HEIGHT ON $\mathfrak{T}(M_0)$.

THEOREM 3.

a) The hermitian scalar product H on $\mathcal{L}$ has logarithmic singularities.

b) Let $\overline{\mathcal{L}}$ be the extension of $\mathcal{L}$ defined as in the remark 1.3.7., then $\overline{\mathcal{L}}$ is an ample bundle on $\mathfrak{M}(M_0)$.

($\mathcal{L}$, $\overline{\mathcal{L}}$ and H are defined in the introduction.)

PROOF OF THEOREM 3.A.:

We have proved that the Weil-Petersson metric g on $\mathfrak{T}(M_0)$ has a bounded away from zero and $-\infty$ holomorphic sectional curvature, i.e.

(3.A.1.) $$-\infty<c_2<\mathcal{K}(g)\leq c_1<0$$

On the other hand it was proved in [T1] and [Ti] that

(3.A.2.) $$\partial\overline{\partial}\log H=g$$

Let $\overline{\mathfrak{T}}(M_0)$ be a projective variety such that $D_\infty:=\overline{\mathfrak{T}}(M_0)\setminus\mathfrak{T}(M_0)$ is a divisor with normal crossings. Such compactification of $\mathfrak{T}(M_0)$ exists since $\mathfrak{T}(M_0)$ is a quasi-projective variety and by the famous theorem of Hironaka. (See [H].) Let $t\in D_\infty$ and let Δ^N be a polycylinder(Δ=unit disk and $N=\dim\overline{\mathfrak{T}}(M_0)$) such that $\Delta^N\subset\overline{\mathfrak{T}}(M_0)$. Since D_∞ is is a divisor with normal crossings we get that

(3.A.3.) $$\Delta^N\cap\overline{\mathfrak{T}}(M_0)\setminus\mathfrak{T}(M_0)=\begin{Bmatrix}\text{union of coordinate hyperplanes}\\ z_1=0,..,z_k=0\end{Bmatrix}$$

Hence:

(3.A.4.) $$\Delta^N\cap\mathfrak{T}(M_0=(\Delta^*)^k x\Delta^{N-k}$$

From (3.A.1.), (3.A.2) ,(3.A.3.) and the fact that g is a complete metric in $\mathfrak{T}(M_0)$ we get that g behaves like Poincare metric on $(\Delta^*)^k x\Delta^{N-k}$, i.e. there exist constants c_3 and c_4 such that:

(3.A.5.) $$c_3\Big(\sum_{i=1}^{k}\frac{|dz_i|}{|z_i|^2(\log|z_i|)^2}+\sum_{i=k+1}^{N}|dz_i|\Big)\leq g\leq c_4\Big(\sum_{i=1}^{k}\frac{|dz_i|}{|z_i|^2(\log|z_i|)^2}+\sum_{i=k+1}^{N}|dz_i|\Big)$$

which means that if u is a Tangent vector and P is the Poincare metric, then

$$c_3P(u,u)\leq g(u,u)\leq c_4P(u,u)$$

Let me remind that the Poincare metric on the unit disc is given by

$$(3.A.6.) \qquad \frac{|dz_i|}{|z_i|^2 (\log|z_i|)^2}$$

From here we get immediately that g has logarithmic singularities.

Q.E.D.

PROOF OF THEOREM 3.B.:

From the results of [M] and THEOREM 3.A. it follows that we can prolonged uniquely $\mathcal{L}$ to $\overline{\mathcal{L}}$ on some compactification $\overline{\mathfrak{M}(M_O)}$ of $\mathfrak{M}(M_O)$. Again in [M] it was proved that $c_1(\overline{\mathcal{L}},H)$ as a form is correctly defined as a current. In [T1] it was proved that

$$\sqrt{-1}\partial\overline{\partial}\log H=g$$

where g is the Weil-Petrsson metric. So $c_1(\overline{\mathcal{L}},H)$ is a positive current. From here and Moishezon-Nakai criterium for ampleness we conclute that $\overline{\mathcal{L}}$ is an ample divisor on the projective variety $\overline{\mathfrak{M}(M_O)}$.

Q.E.D.

REMARK A.

We need to prove that $\mathfrak{M}(M_O)$ is defined over Spec$\mathbb{Z}$. For this we need to use Geometric Invariant theory. (See [F&M].) Indeed formally $\mathfrak{M}(M_O):=S/\mathbf{PGL}(N_O)$ where S is the component of the Hilbert scheme of all non-singular Calabi-Yau manifolds in $\mathbf{PC}^{N_1}$ that contain M_O. From general theory of Grothendieck we know that S is defined over Spec$\mathbb{Z}$. On the other hand $\mathbf{PGL}(N_1+1)$ is an open subset of Proj $\mathbb{Z}[a_{OO},...,a_{N_1N_1}]$. Clearly $\mathcal{L}$ is a $\mathbb{PGL}(N_1+1)$ linearized ample sheaf on S. From the arguments of Mumford in Proposition 7.4. on page 135 of [F&M] we get that $\mathfrak{M}(M_O)$ is defined over Spec$\mathbb{Z}$.

REMARK B.

Since $\overline{\mathcal{L}}$ is an ample bundle and H(t) is an Hermitian metric on $\overline{\mathcal{L}}$ with logarithmic singularities, then H(t) defines a nice height function h(t) on $\mathfrak{M}(M_O)$. For the properties of the height function with logarithmic singularities see [F&W]. Namely Faltings proved the following theorem:

THEOREM. (See [F].)

Suppose that $\mathbf{Q}\subset K$ is a finite extension and h(x) is a height function with logarithmic singularities on a quasi-projective manifold $X(K)\setminus Y(K)$, then the number of points $x\in X(K)\setminus Y(K)$ with $h(x)\leq c$ is finite.

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