MATHEMATICAL INSTANTON BUNDLES ON $\mathbb{P}^{2n+1}$

by

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0. Introduction

A mathematical instanton bundle with quantum number k on $\mathbb{P}^3$ is by definition a holomorphic rank-2 bundle E with Chern polynomial $c_t(E) = (\frac{1}{1-t^2})^k$ which has natural cohomology $H^q(E(l))$ in the range $-3 \leq l \leq 0$. These bundles are stable, hence trivial on generic lines and have a symplectic structure which is unique up to multiplication with scalars [13].

Via the Penrose transformation a certain subset of these bundles corresponds to self-dual solutions of the $SU(2)$ Yang-Mills equations on S^4 [1].

Recently the Penrose transformation has been generalized by Salamon [12].

Salamon constructs for every quaternionic manifold M a twistor bundle

$$\pi : Z \longrightarrow M$$

over M, such that the total space Z is a complex manifold [12]. If $M = \mathbb{P}^n_{\mathbb{H}}$ is the n-dimensional projective space over the quaternions $\mathbb{H}$ this twistor bundle is the well-known fibration

$$\pi : \mathbb{P}^{2n+1}_{\mathbb{C}} \longrightarrow \mathbb{P}^n_{\mathbb{H}} .$$

Identifying $\mathbb{C}^{2n+2}$ with ${}_{\mathbb{C}}\mathbb{H}^{n+1}$ one obtains a real structure σ on $\mathbb{P}^{2n+1}_{\mathbb{C}}$ such that the real lines are precisely the fibres of π.

The special case $M = \mathbb{P}^1_{\mathbb{H}} = S^4$ is just the usual Penrose transformation.

This makes it reasonable to try to construct holomorphic 2n-bundles on $\mathbb{P}_{\mathbb{C}}^{2n+1}$ using the fibration over $\mathbb{P}_{\mathbb{H}}^{n}$.
Salamon proves the following result [12]:
Let

$$A_i : \mathbb{H}^k \longrightarrow \mathbb{H}^{n+k} \qquad , \ i=0,1,\ldots,n \ ,$$

be $\mathbb{H}$-linear mappings. For every $q = (q_o,\ldots,q_n) \in \mathbb{H}^{n+1}$ define

$$A(q) = \sum_{i=0}^{n} A_i \, q_i \ .$$

Assume, that for all $q \in \mathbb{H}^{n+1} \setminus \{0\}$ we have

$$(*) \quad \overline{A(q)}^{t} \ A(q) \ \in GL(k, \mathbb{R}) \ .$$

Then there exists a holomorphic 2n-bundle E on $\mathbb{P}_{\mathbb{C}}^{2n+1}$ with Chern polynomial $c_t(E) = (\frac{1}{1-t^2})^k$, which is trivial on the real lines $\pi^{-1}([q])$ and has a symplectic structure.

Two special cases of this construction are well-known. For $k = 1$, $n \geq 1$ one gets the so called Nullcorrelation bundles [9]. For $n = 1$, $k \geq 1$ this construction gives all mathematical instanton bundles, which come from physics [13].
Unfortunately for $k > 1$, $n > 1$ the condition (*) is hard to check. Therefore it is not clear, that those bundles E really do exist.

This remark was the starting point for our paper.

Let E be an algebraic rank-2n bundle on $\mathbb{P}_{C}^{2n+1}$.

We say, E is a mathematical instanton bundle with quantum number k , if E is simple, has Chern polynomial $c_t(E) = (\frac{1}{1-t^2})^k$ and natural cohomology $H^q(E(l))$ in the range $-2n-1 \leq l \leq 0$. Furthermore we require that E has a symplectic structure and trivial splitting type.

We shall prove, that the set $MI_{\mathbb{P}^{2n+1}}(k)$ of isomorphism classes of mathematical instanton bundles with quantum number k on $\mathbb{P}_{\mathbb{C}}^{2n+1}$ can be identified with a quotient

$$MI_{\mathbb{P}^{2n+1}}(k) = SK(2n+2k) \;/\; GL(k,\mathbb{C}) ,$$

where $SK(2n+2k)$ denotes the variety of non-degenerate simple symmetric Kronecker modules of rank $2n+2k$ (Definition 1.2). From geometric invariant theory it follows that the set $MI^{s}_{\mathbb{P}^{2n+1}}(k)$ of stable bundles in $MI_{\mathbb{P}^{2n+1}}(k)$ carries the structure of a quasi-projective variety (theorem 1.13).
In the second part of the paper we prove that for all $k \geq 1$, $n \geq 1$ the moduli spaces $MI_{\mathbb{P}^{2n+1}}(k)$ are non-empty, giving an explicit construction of an appropriate Kronecker module.

1. Properties of Mathematical Instanton Bundles on $\mathbb{P}^{2n+1}$

We use the notation of [9] with some minor changes.
Let V be a complex vector space of dimension $2n+2$, $n \geq 1$, $\mathbb{P} = \mathbb{P}(V^*)$ the associated projective space of lines in V.
An algebraic vector bundle E on $\mathbb{P}$ has a symplectic structure, if there is an isomorphism

$$\varphi : E \longrightarrow E^*$$

with $\varphi^* = -\varphi$.
If E is simple, a symplectic structure is unique up to multiplication with scalars.
We say that E has natural cohomology in the range $r_1 \leq l \leq r_2$, if for every l in that range at most one of the cohomology groups $H^q(E(l))$ is non zero [6].

Definition 1.1. An algebraic rank-2n bundle E on $\mathbb{P}$ is a mathematical instanton bundle with quantum number $k \geq 1$ if it has the following properties:

(i) the Chern polynomial of E is $c_t(E) = \left(\frac{1}{1-t^2}\right)^k$

(ii) E has natural cohomology in the range $-2n-1 \leq l \leq 0$

(iii) E has trivial splitting type

(IV) E is simple

(v) E has a symplectic structure

Remark. (i)-(iv) are open properties but not (v) except in the case $n = 1$.

We denote the set of isomorphism classes of these bundles by $MI_{\mathbb{P}^{2n+1}}(k)$.

Let H be a complex vector space of dimension k,

$$\alpha : \Lambda^2 V \longrightarrow L(H,H^*)$$

a linear map. We define the adjoint

$$\hat{\alpha} : V \otimes H \longrightarrow V^* \otimes H^*$$

of α by

$$\hat{\alpha}\ (v_1 \otimes h_1)\ (v_2 \otimes h_2) = \alpha\ (v_1 \wedge v_2)(h_1)(h_2).$$

For every $v \in V$ let

$$v^{**} : V^* \otimes H^* \longrightarrow H^*$$

be the evaluation mapping associated to v.

<u>Definition 1.2.</u> A <u>Kronecker module</u> on H is a linear map

$$\alpha : \Lambda^2 V \longrightarrow L(H,H^*)$$

with the following properties

(i) $\hat{\alpha}\ (v \otimes -) : H \longrightarrow V^* \otimes H^*$ is injective for all $v \in V \setminus \{0\}$.

(ii) $v^{**} \circ \hat{\alpha} : V^* \otimes H^* \longrightarrow H^*$ is surjective for all $v \in V \setminus \{0\}$.

The <u>rank</u> of the Kronecker module α is the rank of the linear map $\hat{\alpha}$.

A Kronecker module α is <u>symmetric</u> if the image of α lies in the subspace $S^2H^* \subset L(H,H^*)$ of the symmetric bilinear forms on H , i.e. if $\hat{\alpha}$ is symplectic.

If for almost all $v_1, v_2 \in V$ the bilinear form $\alpha(v_1 \wedge v_2)$ is non-degenerate we call the Kronecker module α <u>non-degenerate</u>.

A Kronecker module α is <u>simple</u>, if for each pair $\varphi_1, \varphi_2 \in \operatorname{End} H$ with $\varphi_2^* \alpha = \alpha \varphi_1$ it follows that $\varphi_1 = \varphi_2 = \lambda \, id_H$.

A Kronecker module α is called <u>irreducible</u> (cf. [7],[11]) if the following condition holds.

If $U \subset H$, $U' \subset H^*$ are linear subspaces, such that $U' \neq 0$, $U' \neq H^*$ and $\alpha(v_1 \wedge v_2)(U) \subset U'$ for $v_1, v_2 \in V$, than $\dim U < \dim U'$.

<u>Remark.</u> Property (i) is equivalent to (ii) for symmetric Kronecker modules.

We want to associate to every mathematical instanton bundle with quantum number k on $\mathbb{P} = \mathbb{P}^{2n+1}$ a non-degenerate simple symmetric Kronecker module of rank $2n+2k$.

<u>Lemma 1.3.</u> Let E be a rank-2n vector bundle with Chern polynomial $c_t(E) = \left(\frac{1}{1-t^2}\right)^k$ on $\mathbb{P}$.

If E has natural cohomology in the range $-2n-1 \leq l \leq 0$, E is the cohomology bundle of a monad

$$(1) \quad 0 \longrightarrow H^{2n}(E(-2n-1)) \otimes \mathcal{O}(-1) \longrightarrow H^1(E(-1)) \otimes \Omega(1) \longrightarrow H^1(E) \otimes \mathcal{O} \longrightarrow 0 .$$

Proof. From the Riemann-Roch formula we find the Hilbert polynomial of E

$$\chi\,(E\,(l)) = 2n \binom{l+2n+1}{2n+1} - k \binom{l+2n}{2n-1} .$$

The proof follows now from the Beilinson spectral sequence [9]

$$E_1^{pq} = H^q(E(p)) \otimes \Omega^{-p}(-p) \Rightarrow E^{p+q} = \begin{cases} E & \text{for} \quad p+q = 0 \\ 0 & \text{for} \quad p+q \neq 0 \end{cases}$$

On $\mathbb{P} = \mathbb{P}(V^*)$ we have the Euler sequence

$$0 \longrightarrow \Omega(1) \longrightarrow V^* \otimes \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

Tensoring this sequence with $H^1(E(-1))$ and combining it with (1) we get the following commutative diagram

$$(2) \quad \begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{2n}(E(-2n-1)) \otimes \mathcal{O}(-1) & \xrightarrow{a} & H^1(E(-1)) \otimes \Omega(1) & \xrightarrow{b} & H^1(E) \otimes \mathcal{O} & \longrightarrow & 0 \\
 & & \downarrow a' & & \downarrow & & \downarrow = & & \\
0 & \longrightarrow & H^1(E(-1) \otimes \Omega(1)) \otimes \mathcal{O} & \longrightarrow & H^1(E(-1)) \otimes V^* \otimes \mathcal{O} & \xrightarrow{v} & H^1(E) \otimes \mathcal{O} & \longrightarrow & 0 \\
 & & \downarrow b' & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(E(-1)) \otimes \mathcal{O}(1) & \xrightarrow{=} & H^1(E(-1)) \otimes \mathcal{O}(1) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The first row and the first column are monads with the same cohomology E . The remaining rows and columns are exact.

Corollary 1.4. Let E be a bundle as in 1.3. Then E is the cohomology of a monad

$$0 \longrightarrow H^{2n}(E(-2n-1)) \otimes \mathcal{O}(-1) \xrightarrow{a'} H^1(E(-1) \otimes \Omega(1)) \otimes \mathcal{O} \xrightarrow{b'} H^1(E(-1)) \otimes \mathcal{O}(1) \longrightarrow 0 .$$

Define

$$H(E) = H^{2n}(E(-2n-1)) ,$$

$$K(E) = H^1(E(-1) \otimes \Omega(1)).$$

Lemma 1.5. Let E be a bundle as in 1.3. A symplectic structure $\varphi : E \longrightarrow E^*$ induces a symplectic structure $q : K(E) \longrightarrow K(E)^*$ on $K(E)$ such that E is the cohomology bundle of a self-dual monad

$$(3) \quad 0 \longrightarrow H(E) \otimes \mathcal{O}(-1) \xrightarrow{a'} K(E) \otimes \mathcal{O} \xrightarrow{a'^*q} H(E)^* \otimes \mathcal{O}(1) \longrightarrow 0 .$$

Proof. From [9] it follows that the morphisms of E to E* correspond to morphisms of the associated monads. So φ induces the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H(E) \otimes \mathcal{O}(-1) & \xrightarrow{a'} & K(E) \otimes \mathcal{O} & \xrightarrow{b'} & H^1(E(-1)) \otimes \mathcal{O}(1) & \longrightarrow & 0 \\ & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \\ 0 & \longrightarrow & H^1(E(-1))^* \otimes \mathcal{O}(-1) & \xrightarrow[b'^*]{} & K(E)^* \otimes \mathcal{O} & \xrightarrow[a'^*]{} & H(E)^* \otimes \mathcal{O}(1) & \longrightarrow & 0 \end{array}$$

Now $q = \varphi_2$ is the induced symplectic structure, and with φ_3 as an identification (Serre duality associated to the given symplectic structure φ) we get $b' = a'^* q$.

Now let E be a rank-2n bundle on $\mathbb{P}$ with the properties (i) (ii) and (v) of definition 1.1.
With respect to some symplectic structure $\varphi : E \longrightarrow E^*$ we get a canonical identification $H^1(E(-1)) \xrightarrow{\sim} H(E)^*$. The morphism a in the monad (1) can then be written as

$$a = a_{E,\varphi} : H(E) \otimes \mathcal{O}(-1) \longrightarrow H(E)^* \otimes \Omega(1) .$$

a is represented by a linear map

$$\hat{\alpha} = \hat{\alpha}_{E,\varphi} : V \otimes H(E) \longrightarrow V^* \otimes H(E)^* ,$$

which is the adjoint of a linear map

$$\alpha = \alpha_{E,\varphi} : \Lambda^2 V \longrightarrow L(H(E), H(E)^*) .$$

claim: $\hat{\alpha}$ is symplectic.
proof. From (2) and (3) we get the following commutative diagram

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 & & H(E)\otimes\mathcal{O}(-1) & \xrightarrow{a} & H(E)^*\otimes\Omega(1) \\
 & \swarrow^{\pi^*} & \downarrow{\scriptstyle a'} & & \downarrow \\
V\otimes H(E)\otimes\mathcal{O} & \xrightarrow{\alpha_2} & K(E)\otimes\mathcal{O} & \xrightarrow{\alpha_1} & V^*\otimes H(E)^*\otimes\mathcal{O} \\
 & & \downarrow{\scriptstyle a'^*q} & & \downarrow{\scriptstyle \pi} \\
 & & H(E)^*\otimes\mathcal{O}(1) & = & H(E)^*\otimes\mathcal{O}(1) \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}$$

It follows $\pi\alpha_1 = a'^*q = (\alpha_2\pi^*)^*q = \pi\,\alpha_2^* q$ and therefore $\alpha_1 = \alpha_2^* q$.

Now by definition $\hat{\alpha}$ is equal to $\alpha_1\alpha_2$ and thus we have

$$\hat{\alpha}^* = (\alpha_2^* q\,\alpha_2)^* = \alpha_2^*\, q^*\alpha_2 = -\hat{\alpha}\ .$$

So we can consider α as a map $\Lambda^2 V \longrightarrow S^2 H(E)^*$.

We then obtain the following "symplectic" commutative diagram with exact columns

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & H(E)\otimes\mathcal{O}(-1) & \xrightarrow{a} & H(E)^*\otimes\Omega(1) & \xrightarrow{b} & H^1(E)\otimes\mathcal{O} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow{=} & & \\
(4) \quad 0 & \longrightarrow & H^1(E)^*\otimes\mathcal{O} & \xrightarrow{-\beta^*} & V\otimes H(E)\otimes\mathcal{O} & \xrightarrow{\hat{\alpha}} & V^*\otimes H(E)^*\otimes\mathcal{O} & \xrightarrow{\beta} & H^1(E)\otimes\mathcal{O} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(E)^*\otimes\mathcal{O} & \xrightarrow{-b^*} & H(E)\otimes T(-1) & \xrightarrow{-a^*} & H(E)^*\otimes\mathcal{O}(1) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}$$

The second row of this diagram is also exact.

<u>Proposition 1.6.</u> Let E be a rank-$2n$ bundle on $\mathbb{P}$ with Chern polynomial $c_t(E) = \left(\frac{1}{1-t^2}\right)^k$ and natural cohomolgy in the range $-2n-1 \le l \le 0$, $\varphi : E \longrightarrow E^*$ a symplectic structure on E . Then the associated map

$$\alpha = \alpha_{E,\varphi} : \Lambda^2 V \longrightarrow S^2 H(E)^*$$

is a symmetric Kronecker module of rank $2n+2k$. Furthermore we have:

(i) α is simple if and only if E is simple

(ii) α is non-degenerate if and only if E has trivial splitting type.

Proof. Since a has to be injective on fibres we see from (4) that α is a symmetric Kronecker module. The rank of $\hat{\alpha}$ is $\dim(V \otimes H(E)) - h^1(E) = 2n+2k$.

(i) follows immediately from

<u>Lemma 1.7.</u> Let H_i', H_i be complex vector spaces , $i = 1,2,3,$ and

$$M = 0 \longrightarrow H_1 \otimes \mathcal{O}(-1) \xrightarrow{a} H_2 \otimes \Omega(1) \xrightarrow{b} H_3 \otimes \mathcal{O} \longrightarrow 0 ,$$

$$M' = 0 \longrightarrow H_1' \otimes \mathcal{O}(-1) \xrightarrow{a'} H_2' \otimes \Omega(1) \xrightarrow{b'} H_3' \otimes \mathcal{O} \longrightarrow 0$$

monads. Let $H^\bullet = \mathrm{Hom}^\bullet(M,M')$ be the following complex. H^i is the complex vector space of all homomorphisms $M \longrightarrow M'$ of degree i ; the differentials $d^i : H^i \longrightarrow H^{i+1}$ are defined by

$$d^0(x,y,z) = (a'x - ya ,\ b'y - zb) ,$$

$$d^1(x,y) = b'x + ya .$$

Then there exist canonical isomorphisms

$$\mathrm{Ext}^q(E,E') \xrightarrow{\sim} H^q(H^\bullet) \quad \text{for} \quad q \geq 0 ,$$

where $E = \ker b / \operatorname{im} a$, $E' = \ker b' / \operatorname{im} a'$.

Especially we have

$$\mathrm{Hom}(E,E') \simeq \ker d^0 = \{\text{homomorphisms of complexes } M \to M'\}.$$

Proof. [10].

(ii) follows from the following more precise result (cf.[9] II.4.2.3) .

Lemma 1.8. Let E, α be as in proposition 1.6. If $L \subset \mathbb{P}$ is the line defined by $v_1, v_2 \in V$, $v_1 \wedge v_2 \neq 0$, then the restriction E_L of E to L is trivial if and only if the symmetric bilinear form $\alpha(v_1 \wedge v_2)$ on $H(E)$ is non-degenerate, i.e. $\mathrm{rk}\,\alpha(v_1 \wedge v_2) = k$.

Proof. Let $W \subset V$ be the subspace generated by v_1 and v_2, $\alpha(v_1 \wedge v_2)$ can be considered as linear map

$$\alpha_W = \alpha(v_1 \wedge v_2) \;:\; \Lambda^2 W \longrightarrow S^2 H(E)^*$$

with adjoint

$$(\alpha_W)^\wedge : W \otimes H(E) \longrightarrow W^* \otimes H(E)^* .$$

Restricting the monad (a,b) in (4) to L and combining with the exact sequence

$$0 \longrightarrow (V/W)^* \otimes \mathcal{O}_L \longrightarrow \Omega(1)_L \longrightarrow \Omega_L(1) \longrightarrow 0$$

we get the following short exact sequence of complexes of vector bundles on L .

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H'\otimes\mathcal{O}_L(-1) & \xrightarrow{a_L'} & H(E)^*\otimes(V/W)^*\otimes\mathcal{O}_L & \xrightarrow{b_L'} & H^1(E)\otimes\mathcal{O}_L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H(E)\otimes\mathcal{O}_L(-1) & \xrightarrow{a_L} & H(E)^*\otimes\Omega(1)_L & \xrightarrow{b_L} & H^1(E)\otimes\mathcal{O}_L & \longrightarrow & 0 \\
 & & \downarrow & \searrow^{\tilde{a}_L} & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H''\otimes\mathcal{O}_L(-1) & \longrightarrow & H(E)^*\otimes\Omega_L(1) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

From this we obtain the long exact cohomology sequence

$$0 \longrightarrow \ker b_L' / \mathrm{im}\, a_L' \longrightarrow E_L \longrightarrow \mathrm{coker}\, \tilde{a}_L \longrightarrow \mathrm{coker}\, b_L' \longrightarrow 0$$

One easily sees that E_L is trivial if and only if $\tilde{a}_L$ is surjective. But $\tilde{a}_L$ is nothing else than the map associated to $(a_W)^{\wedge}$ and so $\tilde{a}_L$ is surjective iff α_W is non-degenerate. This completes the proof of the lemma and of the proposition.

<u>Remark.</u> The above proof also shows that $E_L \cong \mathcal{O}_L(1) \oplus \mathcal{O}_L^{\oplus 2n-2} \oplus \mathcal{O}_L(-1)$ if and only if $\mathrm{rk}\,\alpha(v_1 \wedge v_2) = k-1$.

Now let H, W be fixed complex vector spaces of dimension k , $2n(k-1)$ respectively.

<u>Proposition 1.9.</u> Let $\alpha : \Lambda^2 V \longrightarrow S^2 H^*$ be a simple symmetric Kronecker module of rank $2n+2k$ on H . Then α defines a monad $M(\alpha)$

$$(5) \qquad 0 \longrightarrow H \otimes \mathcal{O}(-1) \xrightarrow{a} H^* \otimes \Omega(1) \xrightarrow{b} W \otimes \mathcal{O} \longrightarrow 0$$

whose cohomology bundle $E(\alpha)$ is simple, has Chern polynomial $c_t(E(\alpha)) = \left(\frac{1}{1-t^2}\right)^k$ and natural cohomology in the range $-2n-1 \leq l \leq 0$.

Furthermore α induces a symplectic structure $\varphi : E(\alpha) \longrightarrow E(\alpha)^*$ on $E(\alpha)$ such that $\hat{\alpha} = g^* \hat{\alpha}_{E,\varphi}\, g$ with a suitable isomorphism $g : H \xrightarrow{\sim} H(E)$.

Proof. The first part of the proposition is clear. From (5) we get a commutative diagram analogous to (4). The corresponding connecting homomorphism

$$\partial : E(\alpha)^* = H^1(M(\alpha)^*) \longrightarrow H^2(M(\alpha)) = E(\alpha)$$

gives us a symplectic structure $\varphi = \partial^{-1}$.

Now the identity $id : E \longrightarrow E$ induces isomorphisms $g_1 : H \longrightarrow H(E)$, $g_2 : H^* \longrightarrow H(E)^*$ such that

$$\hat{\alpha}_{E,\varphi}\, g_1 = g_2\, \hat{\alpha} \ .$$

Since $\hat{\alpha}_{E,\varphi}$ and $\hat{\alpha}$ are symplectic we get $\hat{\alpha}(g_2^* g_1) = g_1^* \hat{\alpha}_{E,\varphi}\, g_1$ and thus $(g_2^* g_1)^* \hat{\alpha} = \hat{\alpha}(g_2^* g_1)$.

By assumption α is simple and so we have

$g_2^* g_1 = \lambda^2 \, id_H$ for some $\lambda \in \mathbb{C} \setminus \{0\}$.

Taking $g = \frac{1}{\lambda} g_1$ we are done.

Now let $SK_V(H) \subset L(\Lambda^2 V, S^2H^*)$ denote the set of all non-degenerate simple symmetric Kronecker modules of rank $2n+2k$.
We consider the natural action

$$\alpha^g = g^* \alpha g$$

of $GL(H)$ on $L(\Lambda^2 V, S^2H^*)$, where $g^* \alpha g$ is defined by

$$g^* \alpha g \; (v_1 \wedge v_2) = g^* \circ \alpha \, (v_1 \wedge v_2) \circ g \quad .$$

$SK_V(H)$ is $GL(H)$ - invariant.

<u>Proposition 1.10.</u> The map $\alpha \longmapsto E(\alpha)$ induces a bijection

$$\phi : SK_V(H) \, / \, GL(H) \longrightarrow MI_{\mathbb{P}}(k) .$$

Proof. If E is a mathematical instanton bundle, $\varphi : E \to E^*$ a symplectic structure on E , $g : H \to H(E)$ an isomorphism, then $\hat{\alpha} = g^* \hat{\alpha}_{E,\varphi} g$ defines a Kronecker module $\alpha \in SK_V(H)$ with $E(\alpha) \cong E$, thus ϕ is surjective. The injectivity of ϕ follows by the same argument as at the end of the proof of proposition 1.9.

Now we want to show, that the set $MI^s_{\mathbb{P}^{2n+1}}(k)$ of isomorphism classes of <u>stable</u> mathematical instanton bundles with quantum number k carries the structure of a quasi-projective variety.

Let $P = \mathbb{P}(L(\Lambda^2 V, S^2 H^*)^*)$ be the projective space of lines in $L(\Lambda^2 V, S^2 H^*)$.

We consider the closed subspace

$$X \subset P$$

consisting of all points $[\alpha] \in P$ which satisfy the rank condition

$$\mathrm{rk}\,\hat{\alpha} \leq 2n+2k$$

X is $SL(H)$-invariant under the natural action of $SL(H)$ on P. Let $X^s (X^{ss})$ be the open set of (semi-)stable points in X with respect to $SL(H)$ in the sense of Mumford [8]. Then the quotient $X^{ss}/SL(H)$ exists and is a projective variety. $X^s/SL(H)$ is an open subspace of $X^{ss}/SL(H)$ [8] .
In order to show that $MI^s_{\mathbb{P}^{2n+1}}(k)$ is an open subset of $X^s/SL(H)$ we need the following two lemmata.

Lemma 1.11. Let $\alpha \in SK_V(H)$ be a Kronecker module, $E = E(\alpha)$ the associated instanton bundle. If E is stable, then α is irreducible.

Proof. First we recall that E is stable if there doesn't exist any subsheaf $F \subset E$ with $0 < \mathrm{rk}\, F < \mathrm{rg}\, E$ and $c_1(F) \geq 0$. The proof is now essentially the same as the proof of Le Potier [11] and so we omit it.

<u>Lemma 1.12.</u> A Kronecker module $\alpha \in SK_V(H)$ is irreducible if and only if the point $[\alpha] \in X$ is stable with respect to $SL(H)$.

Proof. Again we omit the proof since the proof of Hulek[7] generalizes without difficulty to our case.

Now let $SK_V^s(H) \subset SK_V(H)$ be the set of Kronecker modules belonging to stable bundles, $\mathbb{P}(SK_V^2(H))$ the corresponding $SL(H)$-invariant open subset of X .
From Lemma 1.11 and Lemma 1.12 we know that $\mathbb{P}(SK_V^s(H)) \subset X^s$ and we get

<u>Theorem 1.13.</u> The map $\alpha \longmapsto E(\alpha)$ induces a bijection

$$\psi : \mathbb{P}(SK_V^s(H) / SL(H) \longrightarrow MI^s_{\mathbb{P}^{2n+1}}(k) .$$

ψ induces the structure of a quasi-projective variety on $MI^s_{\mathbb{P}^{2n+1}}(k)$. With this structure $MI^s_{\mathbb{P}^{2n+1}}(k)$ is a coarse moduli space for stable mathematical instanton bundles with quantum number k on $\mathbb{P}^{2n+1}$. $\overline{MI}^s_{\mathbb{P}^{2n+1}}(k) = X^{ss}/SL(H)$ is a natural compactification of $MI^s_{\mathbb{P}^{2n+1}}(k)$.

Let $\mathbb{G} = Grass_2(V)$ be the grassmannian of lines in $\mathbb{P}$,

$$\mathbb{G} = \{[v_1 \wedge v_2] \mid v_1, v_2 \in V,\ v_1 \wedge v_2 \neq 0\} \subset \mathbb{P}(\Lambda^2 V^*).$$

Let α be an element of $SK_V(H)$. With α we associate a theta-characteristic on $\mathbb{G}$.

We have the canonical inclusion

$$H \otimes \mathcal{O}_{\mathbb{C}}(-1) \longrightarrow H \otimes \Lambda^2 V \otimes \mathcal{O}_{\mathbb{C}} .$$

The composition with

$$\alpha : \ H \otimes \Lambda^2 V \otimes \mathcal{O}_{\mathbb{C}} \longrightarrow H^* \otimes \mathcal{O}_{\mathbb{C}}$$

defines a morphism

$$\theta_\alpha : H \otimes \mathcal{O}_{\mathbb{C}}(-1) \longrightarrow H^* \otimes \mathcal{O}_{\mathbb{C}}$$

Since α is non-degenerate θ_α is a monomorphism and

$$\theta(\alpha) = \operatorname{coker} \theta_\alpha(-1)$$

is a sheaf on $\mathbb{C}$ with support on the set $S_{E(\alpha)}$ of jumping lines of $E(\alpha)$.

We call $\theta(\alpha)$ the theta-characteristic associated to α .

Lemma 1.14. Let $\alpha, \alpha' \in SK_V(H)$ be Kronecker modules with associated theta-charakteristics θ, θ' .

θ and θ' are isomorphic if and only if α and α' lie in the same $GL(H)$-orbit.

Proof. From $\alpha' = g^* \alpha g$ we get the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H\otimes\mathcal{O}_{\mathbb{C}}(1) & \longrightarrow & H^*\otimes\mathcal{O}_{\mathbb{C}} & \longrightarrow & \theta(1) & \longrightarrow & 0 \\ & & \downarrow g^{-1} & & \downarrow g^* & & \downarrow \cong & & \\ 0 & \longrightarrow & H\otimes\mathcal{O}_{\mathbb{C}}(-1) & \longrightarrow & H^*\otimes\mathcal{O}_{\mathbb{C}} & \longrightarrow & \theta^1(1) & \longrightarrow & 0 . \end{array}$$

Conversely an isomorphism $\psi : \theta \longrightarrow \theta'$ induces isomorphisms $g_2 : H \longrightarrow H$, $g_1 : H^* \longrightarrow H^*$ such that $\theta_{\alpha'} g_2 = g_1 \theta_\alpha$ and thus $\hat{\alpha}' g_2 = g_1 \hat{\alpha}$.

Since α' is simple we get $\alpha' = g^* \alpha g$ if we put $g = \frac{1}{\lambda} g_1^*$ with a suitable scalar λ .

<u>Remark.</u> $S_{E(\alpha)} \subset \mathbb{C}$ is a hypersurface of degree k with equation $\det \alpha(v_1 \wedge v_2) = 0$.

Since $E(\alpha)$ always has jumping lines of higher order [4] the sheaf $\theta(\alpha)$ can't be invertible on $S_{E(\alpha)}$.

From proposition 1.10 we see that we can define a theta-characteristic θ_E for every mathematical instanton bundle E .

θ_E determines E up to isomorphism.

2. Existence of Mathematical Instanton Bundles on $\mathbb{P}^{2n+1}$

The purpose of this section is to show that the sets $MI_{\mathbb{P}^{2n+1}}(k)$ are non-empty for all $k \geq 1,\ n \geq 1$.

Proposition 1.10 shows, that it is sufficient to construct a non-degenerate simple symmetric Kronecker module α of rank $2n+2k$.

By definition α is a linear map

$$\alpha : \Lambda^2 V \longrightarrow S^2 H^*$$

We choose a basis in H and represent α by a $k \times k$-matrix A with entries in $\Lambda^2 V^*$,

$$A = (A_{ij})_{i,j=1,\dots,k} \quad , \quad A_{ij} \in \Lambda^2 V^* \ .$$

First we have to express the properties of α in terms of A .
Identifying $\Lambda^2 V^*$ with the space of symplectic linear maps

$$\Lambda^2 V^* = \{\varphi \in L(V,V^*) \ | \varphi^* = -\varphi\}$$

we define for every $v \in V^*$ a vector

$$A_i(v) = \begin{pmatrix} A_{i1}(v) \\ \vdots \\ A_{ik}(v) \end{pmatrix} \in V^{*\oplus k}$$

We then get

<u>Lemma 2.1.</u> Let $\alpha : \Lambda^2 V \longrightarrow L(H,H^*)$ be a linear map, $A = (A_{ij})$ a matrix, which represents α with respect to

a basis of H . α is a symmetric Kronecker module of rank $2n+2k$ if and only if A has the following properties

(i) $A_{ij} = A_{ji} \ \forall i,j$

(ii) For all $v \in V \setminus \{0\}$ we have in $\Lambda^k(V^{*\oplus k})$

$$A_1(v) \wedge \ldots \wedge A_k(v) \neq 0 \ .$$

(iii) $\operatorname{rk} A = 2n+2k$, where we consider A as a linear map

$$A : V^{\oplus k} \longrightarrow V^{*\oplus k}$$

α is non-degenerate iff the following holds

(iv) $\operatorname{rk}(A_{ij}(v_1 \wedge v_2)) = k$ for almost all $v_1, v_2 \in V$.

α is simple iff A has the property

(v) $AX = YA$ for complex $k \times k$-matrices X, Y implies $X = Y = \lambda I_k$.

Now let A be a matrix with the properties (i)-(iii) . Then A defines a monad

$$0 \longrightarrow \mathcal{O}(-1)^{\oplus k} \xrightarrow{a} \Omega(1)^{\oplus k} \longrightarrow \mathcal{O}^{\oplus m} \longrightarrow 0 \tag{6}$$

where $m = 2n(k-1)$.

The morphism a is given by A and b is given by a $m \times k$ - matrix

$$B = (v_{ij}) \begin{array}{l} i = 1, \ldots, m \\ j = 1, \ldots, k \end{array} \ , \quad v_{ij} \in V \ ,$$

with entries in V .

Lemma 2.2. A matrix $B = (v_{ij})$ defines an epimorphism $b : \Omega(1)^{\oplus k} \longrightarrow \mathcal{O}^{\oplus m}$ if and only if for all $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m \setminus \{0\}$ we have

$$\text{(vi)} \qquad \left(\sum_{\mu=1}^{m} \lambda_\mu v_{\mu i} \right) \wedge \left(\sum_{\mu=1}^{m} \lambda_\mu v_{\mu j} \right) \neq 0$$

for at least on pair $1 \leq i, j \leq k$.

Proof. b is an epimorphism if and only if b^* is injective in each fibre. This is condition (vi).

Proposition 2.3. Let $k \geq 2$. Choose a basis $\{e_1, \ldots, e_{n+1}, f_1, \ldots, f_{n+1}\}$ for V . We define

$$e = \begin{pmatrix} e_1 \\ \cdot \\ \cdot \\ \cdot \\ e_n \end{pmatrix}, \quad e' = \begin{pmatrix} e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_{n+1} \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{pmatrix}, \quad f' = \begin{pmatrix} f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_{n+1} \end{pmatrix} \text{and}$$

$$(7) \qquad B = \begin{pmatrix} e & e' & & & & \\ f & f' & & & & \\ & e & e' & & & \\ & f & f' & & & \\ & & & \ddots & & \\ & & & & e & e' \\ & & & & f & f' \end{pmatrix}$$

Then B defines an epimorphism $b : \Omega(1)^{\oplus k} \longrightarrow \mathcal{O}^{\oplus m}$.

Proof. We have to verify the condition (vi).

Let

$$\lambda_i = (\lambda_1^i, \ldots, \lambda_n^i) \in \mathbb{C}^n, \ \mu_i = (\mu_1^i, \ldots, \mu_n^i) \in \mathbb{C}^n, \ i = 1, \ldots, k-1,$$

$$x = (\lambda_1, \mu_1, \ldots, \lambda_{k-1}, \mu_{k-1}) \in \mathbb{C}^m .$$

If B_i denotes the i^{th} column of B we must show that

$$x B_i \wedge x B_j = 0 \in \Lambda^2 V \quad \forall\, i < j$$

implies $x = 0$.

Define $\lambda_j^i = 0$, $\mu_j^i = 0$ if $j \leq 0$ or $j \geq n+1$ or $i \geq k$.

Then we compute

$$x B_i = \sum_{\nu=1}^{n+1} (\lambda_\nu^{i-1} + \lambda_\nu^i)\, e_\nu + \sum_{\nu=1}^{n+1} (\mu_\nu^{i-1} + \mu_\nu^i)\, f_\nu .$$

Assume now $x B_i \wedge x B_j = 0 \ \forall i, j$.

We show by induction, that then $\lambda_i = 0$.

This is true for $i \leq 0$ by definition. For the induction step we assume $\lambda_\nu^{i-1} = 0$ for all ν and show $\lambda_\nu^i = 0$ using descending induction on ν. If $\lambda_{\nu+1}^i$ vanishes the coefficient of $e_\nu \wedge e_{\nu+j}$ in $x B_i \wedge x B_{i+j}$ $(j = 1, \ldots, n+1-\nu)$ is

$$\lambda_\nu^i (\lambda_{\nu+j-1}^{i+j-1} + \lambda_{\nu+j}^{i+j})$$

These coefficients vanish. We form the alternating sum and get

$$(\lambda_\nu^i)^2 = \sum_{j \geq 1} (-1)^{j+1} \lambda_\nu^i (\lambda_{\nu+j-1}^{i+j-1} + \lambda_{\nu+j}^{i+j}) = 0 .$$

This proves the proposition.

If we can find a matrix $A = (A_{ij}) \in (\Lambda^2 V^*)^{k\times k}$ which has the three properties

(i) $A_{ij} = A_{ji}$,

(ii) $A_1(v) \wedge \ldots \wedge A_k(v) \neq 0$ for $v \in V \setminus \{0\}$,

(iii)' $BA = 0$,

A will define a symmetric Kronecker module of rank $2n+2k$. Consider the vector space P_B of matrices $A \in (\Lambda^2 V^*)^{k\times k}$ with (i) and (iii)' . It is easy to define a basis for this vector space.

<u>Proposition 2.4.</u> Let $z = (z_1, \ldots, z_{2n+2k-1}) \in \mathbb{C}^{2n+2k-1}$ and

$$A_j'(z) = \begin{pmatrix} z_{2k-j} & z_{2k-j+1} & \cdots & z_{2k-j+n} \\ z_{2k-j+1} & & \cdot^{\cdot^{\cdot}} & \vdots \\ \vdots & \cdot^{\cdot^{\cdot}} & & \vdots \\ z_{2k-j+n} & & \cdots & z_{2k-j+2n} \end{pmatrix} ,$$

$$A_j(z) = (-1)^j \begin{pmatrix} 0 & -A_j'(z) \\ A_j'(z) & 0 \end{pmatrix} ,$$

$$A(z) = \begin{pmatrix} A_1(z) & A_2(z) & \dots & A_k(z) \\ A_2(z) & & \cdot^{\cdot^{\cdot}} & \\ \vdots & \cdot^{\cdot^{\cdot}} & & \vdots \\ A_k(z) & & \dots & A_{2k-1}(z) \end{pmatrix}$$

The map $z \longmapsto A(z)$ is an isomorphism $\mathbb{C}^{2n+2k-1} \longrightarrow P_B$.

Proof. Identifying A_{ij} with a skew-symmetric $(2n+2) \times (2n+2)$ - matrix condition (iii)' means: the ν^{th} column of A_{ij} equals the $(\nu+1)^{th}$ column of $-A_{i+1,j}$ for $\nu = 1,\dots,n$ and $\nu = n+2,\dots, 2n+1$.

The proof is now straightforward.

Now we use this isomorphism to define A .

If $\{ \varepsilon_1,\dots, \varepsilon_{2n+2k-1} \}$ is the standard basis of $\mathbb{C}^{2n+2k-1}$ we define

(8) $$A = \begin{cases} A(\varepsilon_{2k-1} + \varepsilon_{3k+n-1}) & \text{for } k \leq n \\ A(\varepsilon_{2k-1}) & \text{for } k = n+1 \\ A(\varepsilon_{k-n-1} + \varepsilon_{2k-1}) & \text{for } k > n+1 \end{cases}$$

Proposition 2.5. The matrix A defined in (8) has the property (ii) .

Proof. It is sufficient to prove that the equation

$$\begin{pmatrix} A_1'\lambda \\ \cdot \\ \cdot \\ \cdot \\ A_k'\lambda \end{pmatrix} \wedge \ldots \wedge \begin{pmatrix} A_k'\lambda \\ \cdot \\ \cdot \\ \cdot \\ A_{2k-1}'\lambda \end{pmatrix} = 0 \quad , \lambda \in \mathbb{C}^{n+1}$$

has only the trivial solution $\lambda = 0$.

This is equivalent to the following claim:

If all k-minors of the $k(n+1) \times k$ - matrix

$$A'\lambda = \begin{pmatrix} A_1'\lambda & A_2'\lambda & \ldots & A_k'\lambda \\ A_2'\lambda & & & \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ A_k'\lambda & & \ldots & A_{2k-1}'\lambda \end{pmatrix}$$

vanish, it follows, that $\lambda = 0$.

To prove this claim, one has to consider the two cases $k \geq n+1$, $k > n+1$ separately. Writing out the matrices $A'\lambda$ in each of these two cases it is only a matter of patience to check the claim.

We can now use the matrix A in (8) to construct an algebraic rank - 2n bundle E_A on $\mathbb{P}^{2n+1}$ with Chern polynomial $c_t(E_A) = (\frac{1}{1-t^2})^k$. E_A has a symplectic structure and natural cohomology in the range $-2n-1 \leq l \leq 0$. It remains to verify, that E_A is simple and trivial on generic lines, i.e. that A has the properties (v) and (iv) in lemma 2.1. (v) can be checked directly. To prove (iv), it is sufficient to find some special vectors $v_1 = \Sigma\, a_i\, e_i$, $v_2 = \Sigma\, b_i\, f_i$ such that the $k \times k$ - matrix

$$A(v_1 \wedge v_2) = \begin{pmatrix} -b^t A'_1 a & b^t A'_2 a & \dots & \pm b^t A'_k a \\ b^t A'_2 a & & ⋰ & \\ \vdots & ⋰ & & \vdots \\ \pm b^t A'_k a & & \dots & -b^t A'_{2k-1} a \end{pmatrix}$$

is non-degenerate.

For example if $k \leq n+1$ we get

$$A(e_1 \wedge f_k) = \pm \begin{pmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{pmatrix}$$

The case $k > n+1$ is similar.

This was the final step in proving.

<u>Theorem 2.6.</u> For every $k \geq 1$, $n \geq 1$ there exist mathematical instanton bundles with quantum number k on $\mathbb{P}^{2n+1}$.

References

[1] Atiyah, M.F., Ward, R.S.: Instantons and algebraic geometry. Comm. Math.Phys. 55, 117-124 (1977).

[2] Barth, W.: Irreducibility of the space of mathematical instanton bundles with rank 2 and $c_2=4$. Math.Ann. 258, 81-106 (1981).

[3] Barth, W.: Moduli of vector bundles on the projective plane. Invent.math. 42, 63-91 (1977).

[4] Brun, J., Hirschowitz, A.: Variété des droites santenses du fibré instanton général. Compos.Math. 53, 325-336 (1984).

[5] Hartshorne, R.: Stable vector bundles and instantons. Comm. Math.Phys. 59, 1-15(1978).

[6] Hartshorne, R., Hirschowitz, A.: Cohomology of a general instanton bundle. Ann.scient. Éc.Norm.Sup. 4^{e} série, t. 15, 365-390(1982).

[7] Hulek, K.: On the classification of stable rank-r vector bundles over the projective plane. Vector bundles and differential equations. Proceedings. Nice 1979. Progress in Math. 7, Birkhäuser Boston (1980).

[8] Mumford, D.: Geometric invariant theory. Springer (1965).

[9] Okonek, C., Schneider, M., Spindler H.: Vector bundles on complex projective spaces. Progress in Math. 3. Birkhäuser Boston (1980).

[10] Okonek, C., Spindler, H.: Stabile reflexive Garben vom Rang 3 auf $\mathbb{P}^3$ mit kleinen Chernklassen. Math.Ann. 264, 91-118 (1983).

[11] Le Potier.: Sur l'espace de modules des fibres de Yang et Mills. Sém. E.N.S. 1979-1982. Mathématique et Physique. Progress in Math.37, Birkhäuser Boston (1983).

[12] Salamon, S.M.: Quaternionic structures and twistor spaces. In: Global Riemannian geometry. Proceedings. Ellis Horwood Limited, Chichester (1984).

[13] Verdier, J.L.: Instantons. In: Les équations de Yang et Mills, Sém. E.N.S. 1977-1978, Astérisque 71-72 (1980).