Birational geometry of Fano double spaces of index two

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We study birational geometry of Fano varieties, realized as double covers $\sigma: V \to \mathbb{P}^M$, $M \geq 5$, branched over generic hypersurfaces $W = W_{2(M-1)}$ of degree $2(M-1)$. We prove that the only structures of a rationally connected fiber space on V are the pencils-subsystems of the free linear system $|-\frac{1}{2}K_V|$. The groups of birational and biregular self-maps of the variety V coincide: $\mathrm{Bir}\, V = \mathrm{Aut}\, V$.

Bibliography: 41 title.

Introduction

0.1. Setting up of the problem and formulation of the main result. The integer $M\geq 4$ is fixed throughout the paper. The symbol $\mathbb{P}$ stands for the projective space $\mathbb{P}^M$ over an algebraically closed field of characteristic zero (in the first place, we mean $\mathbb{C}$). Let $W=W_{2(M-1)}\subset\mathbb{P}$ be a smooth hypersurface of degree $2(M-1)$. There exists a uniquely determined double cover

$$\sigma\colon V\to\mathbb{P},$$

branched over W. It can be explicitly defined as the hypersurface, given by the equation

$$x_{M+1}^2=f(x_0,\dots,x_M)$$

in the weighted projective space $\mathbb{P}^{M+1}(1,\dots,1,M-1)$, where $f(x_*)$ is the equation of the hypersurface W.

The variety V is a Fano variety of index two:

$$\operatorname{Pic}V=\mathbb{Z}H,$$

where H is the ample generator, $K_V=-2H$, the class H is the pull back via σ of a hyperplane in $\mathbb{P}$. On the variety V there are the following natural structures of a rationally connected fiber space: let $\alpha_P\colon\mathbb{P}\dashrightarrow\mathbb{P}^1$ be the linear projection from an arbitrary linear subspace P of codimension two, then the map

$$\pi_P=\alpha_P\circ\sigma\colon V\dashrightarrow\mathbb{P}^1$$

fibers V into $(M-1)$-dimensional Fano varieties of index one. Now let us formulate the main result.

Recall [1], that a (non-trivial) rationally connected fiber space is a surjective morphism $\lambda\colon Y\to S$ of projective varieties, where $\dim S\geq 1$ and the variety S and a fiber of general position $\lambda^{-1}(s)$, $s\in S$, are rationally connected (and the variety Y itself is automatically rationally connected by the theorem of Graber, Harris and Starr [2]).

Theorem 1. *Assume that $M\geq 5$ and the branch hypersurface $W\subset\mathbb{P}$ is sufficiently general. Let $\chi\colon V\dashrightarrow Y$ be a birational map onto the total space of the rationally connected fiber space $\lambda\colon Y\to S$. Then $S=\mathbb{P}^1$ and for some isomorphism $\beta\colon\mathbb{P}^1\to S$ and a subspace $P\subset\mathbb{P}$ of codimension two we get*

$$\lambda\circ\chi=\beta\circ\pi_P,$$

that is, the following diagram commutes:

$$\begin{array}{ccccc} & V & \stackrel{\chi}{\dashrightarrow} & W & \\ \pi_P & \downarrow & & \downarrow & \lambda\\ & \mathbb{P}^1 & \stackrel{\beta}{\to} & S. & \end{array}$$

Corollary 1. *For a generic double space V of dimension $\dim V \geq 5$ the following claims hold.*

(i) *On the variety V there are no structures of a rationally connected fiber space with the base of dimension ≥ 2. In particular, on V there are no structures of a conic bundle and del Pezzo fibration, and the variety V itself is non-rational.*

(ii) *Assume that there is a birational map $\chi\colon V \dashrightarrow Y$, where Y is a Fano variety of index $r \geq 2$ with factorial terminal singularities, such that $\operatorname{Pic} Y = \mathbb{Z}H_Y$, where $K_Y = -rH_Y$, and moreover, the linear system $|H_Y|$ is non-empty and base point free. Then $r = 2$ and the map χ is a biregular isomorphism.*

(iii) *The groups of birational and biregular self-maps of the variety V coincide:*

$$\operatorname{Bir} V = \operatorname{Aut} V = \mathbb{Z}/2\mathbb{Z}.$$

Proof of the corollary. The claim (i) and the equality $r = 2$ in (ii) are obvious (any linear subsystem of projective dimension $\leq r-1$ in the complete linear system $|H_Y|$ defines a structure of a rationally connected fiber space on Y). Furthermore, the χ-preimage of a generic divisor in the system $|H_Y|$ is by Theorem 1 a divisor in the linear system $|H|$, which completes the proof of the claim (ii). The part (iii) follows from (ii) in an obvious way. Q.E.D.

Tha aim of the present paper is to prove Theorem 1. As usual, its claim will be derived from another fact, a much more technical and less visual Theorem 2 on the thresholds of canonical adjunction of movable linear systems on the variety V. However, first of all, let us discuss the position of Theorem 1 in the context of known results on birational geometry of higher-dimensional Fano varieties.

0.2. From birationally rigid varieties to birationally non-rigid ones. Recall the principal definitions of the theory of birational rigidity. Let X be a smooth projective rationally connected variety. It satisfies the classical condition of termination of adjunction of the canonical class: for any effective divisor D the linear system $|D+mK_X|$ is empty for $m \gg 0$, since K_X is negative on every family of rational curves sweeping out X, whereas an effective divisor is non-negative on any such family. In order to fix the moment of termination precisely, let us consider the Picard group $A^1X = \operatorname{Pic} X$, set $A^1_{\mathbb{R}}X = A^1X \otimes \mathbb{R}$ and define the cones $A^1_+X \subset A^1_{\mathbb{R}}X$ of *pseudo-effective* classes and $A^1_{\rm mov}X \subset A^1_{\mathbb{R}}X$ of *movable* clases as the closed cones (with respect to the standard real topology of $A^1_{\mathbb{R}}X \cong \mathbb{R}^k$), generated by the classes of effective divisors and movable divisors (that is, divisors in the linear systems with no fixed components), respectively.

Definition 0.1. The *threshold of canonical adjunction* of a divisor D on the variety X is the number $c(D,X) = \sup\{\varepsilon \in \mathbb{Q}_+ | D + \varepsilon K_X \in A^1_+X\}$. If Σ is a non-empty linear system on X, then we set $c(\Sigma, X) = c(D,X)$, where $D \in \Sigma$ is an arbitrary divisor.

Example 0.1. (i) Let X be a smooth Fano variety, and assume that $\operatorname{Pic} X = \mathbb{Z}H_X$, where H_X is the ample generator and $K_X = -rH_X$, $r \geq 1$. For any effective divisor D we have $D \sim nH_X$ for some $n \geq 1$, so that

$$c(D,X) = \frac{n}{r}.$$

(ii) Let $\pi: X \to S$ be a rationally connected fiber space with $\dim X > \dim S \geq 1$, Δ an effective divisor on the base S. Obviously, $c(\pi^*\Delta, X) = 0$. If $\operatorname{Pic} X = \mathbb{Z}K_X \oplus \pi^* \operatorname{Pic} S$, that is, X/S is a standard Fano fiber space, and D is an effective divisor on X, which is not a pull back of a divisor on the base S, then $D \in |-mK_X + \pi^* R|$ for some divisor R on S, where $m \geq 1$. Obviously, $c(D, X) \leq m$, and moreover, if the divisor R is effective, then $c(D, X) = m$. Indeed, K_X is negative on the fibers of the morphism π (in particular, on the dense families of rational curves sweeping out fibers of π), whereas any divisor, pulled back from the base, is trivial on the fibers.

Example 0.2. Let Y be a Fano variety of index $r \geq 2$, described in part (ii) of Corollary 1. Consider a movable linear system Σ, spanned by the divisors $D_1, \ldots, D_r \in |H_Y|$ of general position. Obviously,

$$Q = D_1 \cap \ldots \cap D_r \subset Y$$

is an irreducible subvariety of codimension $r \geq 2$, whereas $\operatorname{Bs} \Sigma = Q$. Let $\varphi: Y^+ \to Y$ be the blow up of Q and Σ^+ the strict transform of the system Σ on Y^+. Obviously, the free system Σ^+ defines a morphism

$$\pi_Q: Y^+ \to \mathbb{P}^{r-1},$$

the fibers of which are Fano varieties and therefore are rationally connected. We get the equalities

$$c(\Sigma, Y) = \frac{1}{r} \quad \text{and} \quad c(\Sigma^+, Y^+) = 0,$$

because Σ^+ is pulled back from the base $\mathbb{P}^{r-1}$.

Definition 0.2. For a movable linear system Σ on a variety X define the *virtual threshold of canonical adjunction* by the formula

$$c_{\text{virt}}(\Sigma) = \inf_{X^\sharp \to X} \{c(\Sigma^\sharp, X^\sharp)\},$$

where the infimum is taken over all birational morphisms $X^\sharp \to X$, $X^\sharp$ is a smooth projective model of $\mathbb{C}(X)$, $\Sigma^\sharp$ is the strict transform of the system Σ on $X^\sharp$.

The virtual threshold is obviously a birational invariant of the pair (X, Σ): if $\chi: X \dashrightarrow X^+$ is a birational map, $\Sigma^+ = \chi_* \Sigma$ is the strict transform of the system Σ with respect to χ^{-1}, then we get $c_{\text{virt}}(\Sigma) = c_{\text{virt}}(\Sigma^+)$.

The following obvious claim shows how the information on the virtual threshold of canonical adjunction makes it possible to describe birational property of algebraic varieties.

Proposition 0.1. (i) *Assume that on the variety X there are no movable linear systems with the zero virtual threshold of canonical adjunction. Then on X there are no structures of a non-trivial fibration into varieties of negative Kodaira dimension, that is, there is no rational dominant map $\rho: X \dashrightarrow S$, $\dim S \geq 1$, the generic fiber of which has negative Kodaira dimension.*

(ii) *Let $\pi: X \to S$ be a rationally connected fiber space. Assume that every movable linear system Σ on X with the zero virtual threshold of canonical adjunction,*

$c_{\rm virt}(\Sigma)=0$, *is the pull back of a system on the base:* $\Sigma=\pi^*\Lambda$, *where* Λ *is some movable linear system on* S. *Then any birational map*

$$\begin{array}{rcccl} & X & \stackrel{\chi}{\dashrightarrow} & X^{\sharp} & \\ \pi & \downarrow & & \downarrow & \pi^{\sharp} \\ & S & & S^{\sharp}, & \end{array} \qquad (1)$$

where $\pi^{\sharp}\colon V^{\sharp}\to S^{\sharp}$ *is a fibration into varieties of negative Kodaira dimension, is fiber-wise, that is, there exists a rational dominant map* $\rho\colon S \dashrightarrow S^{\sharp}$, *making the diagram (1) commutative,* $\pi^{\sharp}\circ\chi=\rho\circ\pi$.

The only known way to compute the virtual thresholds of canonical adjunction is by reduction to the ordinary thresholds.

Definition 0.3. (i) The variety X is said to be *birationally superrigid*, if for any movable linear system Σ on X the equality

$$c_{\rm virt}(\Sigma)=c(\Sigma,X)$$

holds.

(ii) The variety X (respectively, the Fano fiber space X/S) is said to be *birationally rigid*, if for any movable linear system Σ on X there exists a birational self-map $\chi\in\operatorname{Bir}X$ (respectively, a fiber-wise birational self-map $\chi\in\operatorname{Bir}(X/S)$), providing the equality

$$c_{\rm virt}(\Sigma)=c(\chi_*\Sigma,X).$$

(iii) The variety X is said to be *almost birationally rigid*, if it has a model $\widetilde{X}$, which is a birationally rigid variety, that is, the condition (ii) is satisfied for some smooth projective variety $\widetilde{X}$, birational to X.

Example 0.3. (i) A smooth three-dimensional quartic $X=X_4\subset\mathbb{P}^4$ is birationally superrigid: this follows immediately from the arguments of [3].

(ii) A generic smooth hypersurface $X=X_d\subset\mathbb{P}^d$, $d\geq 5$, is a birationally superrigid variety [4]. A generic complete intersection $X_{d_1\cdot\ldots\cdot d_k}\subset\mathbb{P}^{M+k}$ of index one (that is, $d_1+\ldots+d_k=M+k$) and dimension $M\geq 4$ is birationally superrigid for $X\geq 2k+1$ [5]. For more examples, see [1].

(iii) Generic smooth complete intersections $X_{2\cdot 3}\subset\mathbb{P}^5$ of a cubic and a quadric hypersurfaces are birationally rigid, but not superrigid [6,7]. A generic complete intersection $X_{2\cdot 3}\subset\mathbb{P}^5$ with a non-degenerate quadratic singularity $o\in X_{2\cdot 3}$ is almost birationally rigid. For a birationally rigid model $\widetilde{X_{2\cdot 3}}$ one can take the blow up of the singular point [8].

Conjecture 0.1. *A smooth Fano complete intersection of index one and dimension* ≥ 4 *in a weighted projective space is birationally rigid, of dimension* ≥ 5 *birationally superrigid.*

It follows from Example 0.2 that the double space V, the main object of study in the present paper, is neither superrigid, nor rigid, nor almost rigid.

0.3. The start and scheme of the proof of Theorem 1. Similarly to birationally rigid varieties, Theorem 1 is based on some claim on the virtual threshold of canonical adjunction of a movable linear system on V. For an arbitrary linear subspace $P \subset \mathbb{P}$ of codimension two let V_P be the blow up of the subvariety $\sigma^{-1}(P) \subset V$ (it is irreducible by the conditions of general position, see Sec. 0.4). For a movable linear system Σ on V the symbol Σ_P stands for its strict transform on V_P.

Theorem 1 follows from a more technical fact.

Theorem 2. *Assume that $M \geq 5$ and for a movable linear system Σ the inequality*

$$c_{\rm virt}(\Sigma) < c(\Sigma, V) \tag{2}$$

holds. Then there exists a uniquely determined linear subspace $P \subset \mathbb{P}$ of codimension two, satisfying the inequality

$$\mathrm{mult}_{\sigma^{-1}(P)}\, \Sigma > c(\Sigma, V),$$

whereas for the strict transform Σ_P the equality

$$c_{\rm virt}(\Sigma) = c_{\rm virt}(\Sigma_P) = c(\Sigma_P, V_P)$$

holds.

Theorem 1 is derived from Theorem 2 in a few lines, see §1. Almost all paper is devoted to proving Theorem 2. Let us fix a movable linear system Σ, satisfying the inequality (2). Taking, if necessary, a symmetric power of Σ, we may assume that

$$\Sigma \subset |2nH| = |-nK_V|,$$

where $n \geq 1$ is a positive integer. The system Σ (and the integer n) are fixed throughout the paper, with the exception of a few technical sections (in the first place, §§4-5), where the notations are independent; we usually point this out but it is always clear from the context. Obviously,

$$c(\Sigma, V) = n.$$

Proposition 0.2 (the Noether-Fano inequality). *There exists a birational morphism $\varphi\colon \widetilde{V} \to V$ and an irreducible exceptional divisor $E \subset \widetilde{V}$, satisfying the estimate*

$$\mathrm{ord}_E\, \varphi^*\Sigma > na(E, V). \tag{3}$$

Proof is well known (see, for instance, [1]).

The divisor E (or the corresponding discrete valuation of the field of rational functions of the variety V) is called a *maximal singularity* of the linear system Σ. If φ is the blow up of an irreducible subvariety $B \subset V$ (and in that case $E = \varphi^{-1}(B)$), then the latter is called a *maximal subvariety* of the system Σ. In that case (3) is equivalent to the inequality

$$\mathrm{mult}_B\, \Sigma > n(\mathrm{codim}\, B - 1).$$

In any case the subvariety $B = \varphi(E)$ is called the *centre* of the maximal singularity E, see [1] for the language of maximal singularities.

An equivalent formulation of Proposition 0.2: the pair

$$\left(V, \frac{1}{n}\Sigma\right) \tag{4}$$

is not canonical and the prime divisor $E \subset \widetilde{V}$ is a non canonical singularity of this pair. If, instead of (3), the stronger inequality

$$\mathrm{ord}_E\, \varphi^*\Sigma > n(a(E)+1), \tag{5}$$

holds, then E is a *log maximal singularity*, the pair (4) is not log canonical and the estimate (5) is the log Noether-Fano inequality. The bigger part of the paper is devoted to proving the following fact.

Proposition 0.3. *There exists a unique linear subspace $P \subset \mathbb{P}$ of codimension two, such that the subvariety $\sigma^{-1}(P)$ is a maximal subvariety of the system Σ.*

Proposition 0.3 is proved in §§2-6 in the "negative" version: assuming that the system Σ has no maximal subvarieties of the form $\sigma^{-1}(P)$, where $P \subset \mathbb{P}$ is a linear subspace of codimension two, we exclude one by one all possibilities for a maximal singularity of the system Σ, thus coming to a contradiction with Proposition 0.2. The arguments of §2 exclude also the possibility that the system Σ has two maximal subvarieties, $\sigma^{-1}(P_1)$ and $\sigma^{-1}(P_2)$, where $P_1 \neq P_2$ are distinct linear subspaces of codimension two.

Now we blow up the maximal subvariety $\sigma^{-1}(P)$ and on the new (generally speaking, singular) variety complete the proof of Theorem 2, which directly implies Theorem 1. This part of our work, although it is the concluding one in the sense of the proof as a whole, is based on Proposition 0.3 only and is independent of the contents of §§2-6. So it is done in §1 (in the assumption that Proposition 0.3 holds).

§7 is devoted to proving the conditions of general position, which the double space V is supposed to satisfy. There is no doubt that these conditions are unnecessary, that is, that Theorems 1 and 2 are true for any smooth double space of index two. However, the conditions of general position are essentially used in the proof. Some of those conditions could have been at least relaxed, however, this would have made our proof, hard and long as it is, even more complicated. On the other hand, we do not use the conditions of general position at all or use them not to the full extent in those cases where it does not make the proof too complicated. Some types of maximal singularities are excluded under the only assumption that the variety V is smooth.

Note that in this paper we do a considerable part of work for the double spaces of dimension $M = 4$ (we exclude a majority of types of maximal singularities). For $M = 5$, in order to avoid the paper getting too long, we omit the proof for one of the cases when we exclude infinitely near maximal singularities (§6). For $M \geq 6$ we consider all possible cases.

0.4. Formulation of conditions of general position. As we mentioned above, the main result of the present paper is obtained under the assumption that the double space V is sufficiently general (that is, the branch divisor $W \subset \mathbb{P}$ is a sufficiently general hypersurface of degree $2(M-1)$). We will use several conditions of general position, the three principal ones of which are formulated below and proved in §7 (we show that a general hypersurface W satisfies these conditions indeed). Some other, less significant, conditions are given where they are used.

The first main condition deals with lines on V. As usual, a curve $C \subset V$ is called a *line*, if the equality $(C \cdot H) = 1$ holds. In particular, a line is a smooth irreducible rational curve. We have

Proposition 0.4. *On a generic variety V there are finitely many lines through any point.*

The second and third condition deal with linear subspaces (planes) in $\mathbb{P}$ of codimension two. Consider an arbitrary plane $P \subset \mathbb{P} = \mathbb{P}^M$ of codimension two. The intersection $P \cap W$, generally speaking, is singular:

$$p \in \operatorname{Sing} P \cap W$$

if and only if

$$P \subset T_p W.$$

It is well known that (without the assumption that the hypersurface W is generic) the set $\operatorname{Sing} P \cap W$ is at most one-dimensional (see, for instance, [7]).

The assumption that W is generic makes it possible to improve this claim.

Proposition 0.5. *For a generic hypersurface W and an arbitrary plane $P \subset \mathbb{P}$ of codimension two the set* $\operatorname{Sing} P \cap W$ *is finite (or empty). In particular, the closed set $R = \sigma^{-1}(P)$ is irreducible, that is, it is a subvariety, and the set of its singular points is at most finite.*

The third condition characterizes the singularities of the variety $\sigma^{-1}(P)$ and the singularities of its blow up on V. For a quadratic singular point, that is, a hypersurface singularity with a local equation

$$0 = w_2(u_1, \dots, u_N) + w_3(u_*) + \dots,$$

where $w_i(u_*)$ is a homogeneous polynomial of degree i, then we say that this point is of rank $\operatorname{rk} w_2$. When such a singularity is blown up, the exceptional divisor is the quadric $\{w_2 = 0\} \subset \mathbb{P}^{N-1}$ of rank $\operatorname{rk} w_2$.

Let V_P be the blow up of the (irreducible by Proposition 0.5) subvariety $\sigma^{-1}(P)$ on V.

Proposition 0.6. *For a generic hypersurface W and an arbitrary plane $P \subset \mathbb{P}$ of codimension two:*

(i) *for $M \geq 6$ every singular point of the variety V_P is an isolated quadratic point of rank ≥ 4,*

(ii) *for $M \geq 4$ every singular point of the variety V_P is an isolated quadratic point of rank ≥ 3,*

(iii) *for $M \geq 6$ every singular point of the variety $\sigma^{-1}(P)$ is an isolated quadratic point of rank ≥ 2.*

The properties, formulated in Propositions 0.4-0.6, will be assumed to take place (sometimes we remind about this in the course of our arguments).

0.5. Historical remarks. Up to this day, only very few papers were describing birational geometry of Fano varieties of index $r \geq 2$. (We should explain: here we mean dealing with the problems that give information about the birational type of a variety, such as the rationality problem, computation of the group of birational self-maps, description of the set of structures of rationally connected fiber spaces etc. There are a lot of papers where *particular* birational maps are constructed and studied, for instance, birational transformations of the projective space, but this is a completely different area.) Fano himself pioneered the study of varieties of index ≥ 2, he tried to describe the group of birational self-maps of the three-dimensional cubic [9]. This attempt, as it is clear now, could not be successful: the problem was too complicated for the methods of his time. After V.A.Iskovskikh and Yu.I.Manin in 1971 proved birational superrigidity (in the modern terminology) of three-dimensional quartics, it was natural to try to apply the new technique of the method of maximal singularities to varieties of higher index, and such an attempt was immediately made: in [10] certain auxiliary claims are formulated for varieties of arbitrary index $r \geq 1$ and some work is done on description of birational geometry of the Veronese double cone of dimension three (it is a Fano variety of index two). The paper [11] aimed at completing that work (in particular, at solving the rationality problem for that class of varieties). Unfortunately, the above mentioned paper [11], as it became clear later, was faulty, see [12], where the proof was completed 20 years later. However, the fact that there was a mistake in [11], was already clear enough in the mid-nineties: in order to study the Veronese double cone successfully, one should be able to exclude maximal singularities of movable linear systems on the pencils of del Pezzo surfaces (because there are such pencils on the double cone), whereas the technique of their exclusion was developed in [13]. It is impossible to solve this problem by the methods that were used in the eighties (the test class technique).

It is worth mentioning that the talk [14] at the ICM in Warsaw announced S.I.Khashin's description of the group of birational self-maps of the double space of index two and dimension three (corresponding to the value $M = 3$ in the notations of the present paper), however, this announcement was not confirmed later and no proof was produced. (Note that the three-dimensional space of index two is a much more difficult object of study than the Veronese double cone, so that the announcement, given in [14], looks somewhat naive.) For the modern techniques this variety seems already to be within reach, however, the problem is still very hard. For the up to date description of this problem, see [15,16].

Thus the Veronese double cone of dimension three up to this day was the only Fano variety of index two, birational geometry of which was completely studied. A series of remarkable results were obtained by means of other methods: non-rationality of the three-dimensional cubic was proved by Clemens and Griffiths in [17] (see also [18,19]), non-rationality of "very general" Fano hypersurfaces of arbitrary dimension and of index two and higher was proved by Kollár [20], non-rationality of double spaces $\mathbb{P}^3$ of index two follows from the fact that they admit no structures of a conic bundle, as A.S.Tikhomirov showed in [21-23]. These are just three examples; we do not give a complete list of those results, since the methods used in the above mentioned papers are very far from the method of maximal singularities that makes the basis of the present paper. It should be mentioned, though, that both the "transcendent method" (or the method of intermediate Jacobian, developed by Clemens and Griffiths) and Kollár's approach make it possible to obtain much less information about birational geometry of a given variety than the method of maximal singularities that gives its almost exhaustive description. In particular, only the method of maximal singularities describes all structures of a rationally connected fiber space (in the case when the work is completed).

The Veronese double cone of dimension three is, in a sense, an exceptional variety by its numerical characteristics. The double spaces of arbitrary dimension, considered in this paper, are already quite typical. Theorems 1 and 2 show that the behaviour which is natural to expect from higher-dimensional Fano varieties of index two, really takes place.

1 The structures of rationally connected fiber spaces

In this section we prove the main results of the paper, Theorems 1 and 2, assuming that Proposition 0.3 on the maximal subvariety of codimension two holds.

1.1. Fano fiber space over $\mathbb{P}^1$. According to Proposition 0.3, there exist a (unique) linear subspace $P \subset \mathbb{P}$ of codimension two, satisfying the estimate

$$\mathrm{mult}_R \Sigma > n,$$

where $R = \sigma^{-1}(P)$ is an irreducible variety with at most zero-dimensional singularities (Proposition 0.5). Let $\varphi: V^+ \to V$ be the blow up of the (possibly singular) subvariety $R = \sigma^{-1}(P)$, $E = \varphi^{-1}(R)$ the exceptional divisor.

Lemma 1.1. (i) *The variety V^+ is factorial and has at most finitely many isolated double points (not necessarily non-degenerate).*

(ii) *The linear projection $\pi_{\mathbb{P}}: \mathbb{P} \dashrightarrow \mathbb{P}^1$ from the plane P generates the regular projection*

$$\pi = \pi_{\mathbb{P}} \circ \sigma \circ \varphi: V^+ \to \mathbb{P}^1,$$

the general fiber of which $F_t = \pi^{-1}(t)$, $t \in \mathbb{P}^1$ is a non-singular Fano variety of index one, and finitely many fibers have isolated double points.

(iii) *The following equalities hold:*

$$\mathrm{Pic}\, V^+ = \mathbb{Z}H \oplus \mathbb{Z}E = \mathbb{Z}K^+ \oplus \mathbb{Z}F,$$

where $H = \varphi^* H$ for simplicity of notations, $K^+ = K_{V^+}$ is the canonical class of the variety V^+, F is the class of a fiber of the projection π, whereas

$$K^+ = -2H + E, \ F = H - E.$$

Proof. These claims follow directly from the definition of the blow up φ, Proposition 0.5 and the well known fact that an isolated hypersurface singularity of a variety of dimension ≥ 4 is factorial (see [24]).

Let Σ^+ be the strict transform of the system Σ on the blow up V^+ of the subvariety R.

Proposition 1.1. *The following equality holds:*

$$c_{\mathrm{virt}}(\Sigma^+) = c(\Sigma^+, V^+).$$

Proof of Theorem 2. This theorem is just the union of Proposition 0.3 and Proposition 1.1. Q.E.D.

Corollary 1.1. *Assume that $c_{\mathrm{virt}}(\Sigma^+) = 0$. Then the system Σ^+ is composed from the pencil $|H - R|$, that is,*

$$\Sigma^+ \subset |2nF|.$$

Proof of the corollary. Assume the converse:

$$\Sigma^+ \subset | - mK^+ + lF|,$$

where $m \geq 1$. By the part (iii) of Lemma 1.1,

$$m = 2n - \nu, \ l = 2\nu - 2n \geq 2,$$

so that for the threshold of canonical adjunction we get

$$c(\Sigma^+, V^+) = m.$$

Since $c_{\mathrm{virt}}(\Sigma^+) = 0$, by Proposition 1.1 we get $m = 0$, as we claimed. Q.E.D. for the corollary.

Proof of Theorem 1. For the linear system Σ we take the strict transform with respect to χ of any linear system of the form $\lambda^* \Lambda$, where Λ is a movable system on the base S. Applying Theorem 2 (or Corollary 1.1), we complete the proof.

1.2. Movable linear systems on the variety V. We start our proof of Proposition 1.1 with the well known step: assume that the inequality

$$c_{\mathrm{virt}}(\Sigma^+) < c(\Sigma^+, V^+) = m$$

holds. Then the pair

$$(V^+, \frac{1}{m}\Sigma^+) \qquad (6)$$

is not canonical, so that the linear system Σ^+ has a maximal singularity, that is, for some birational morphism $\psi: \widetilde{V} \to V^+$ and irreducible exceptional divisor $E^+ \subset \widetilde{V}$ the Noether-Fano inequality holds:

$$\nu_E(\Sigma^+) > ma(E^+, V^+).$$

Lemma 1.2. *The centre of maximal singularity E^+ is contained in some fiber $F_t = \pi^{-1}(t)$, that is,*

$$B = \pi \circ \psi(E^+) = t \in \mathbb{P}^1.$$

Proof. Assume the converse: $\pi \circ \psi(E^+) = \mathbb{P}^1$. Restricting the linear system Σ^+ onto the fiber of general position $F = F_s$, we get that the pair

$$(F, \frac{1}{m}\Sigma_F)$$

is not canonical, where $\Sigma_F \subset |-mK_F|$. However, F is a smooth double space of index one and it is well known [25], that this is impossible. Q.E.D. for the lemma.

For simplicity of notations, let $F = F_t$ be the fiber, containing the centre of singularity E^+.

Proposition 1.2. *The centre B is a singular point of the fiber F.*

Proof. Since the anticanonical degree of the divisor $D_F \in \Sigma_F$ is $2m$, and by genericity of the branch divisor the anticanonical degree of any subvariety of codimension one on F is at least 2, we get the inequality $\operatorname{codim}_F B \geq 2$, so that

$$\operatorname{codim}_{V^+} B \geq 3.$$

In the notations of Sec. 1.1 let $\Pi = \sigma \circ \varphi(F) \subset \mathbb{P}$ be the hyperplane, corresponding to the fiber F. It is easy to see that

$$\sigma_F = \sigma \circ \varphi: F \to \Pi = \mathbb{P}^{M-1}$$

is the double cover, branched over $W_\Pi = W \cap \Pi$: the blow up φ does not affect the divisors-elements of the pencil $|H - R|$. Now we need to consider two cases:

1) $\sigma_F(B) \not\subset W_\Pi$,

2) $\sigma_F(B) \subset W_\Pi$, but the generic point of the subvariety B is a non-singular point of the fiber F.

The case 1) is excluded by the arguments of [26]. Let $o \in B$ be a point of general position,

$$\lambda: F^\sharp \to F$$

its blow up, $E^\sharp=\lambda^{-1}(o)\subset F^\sharp$ the exceptional divisor, $E^\sharp\cong\mathbb{P}^{M-2}$. By inversion of adjunction for a general divisor $D\in\Sigma^+$ we get: the pair

$$(F,\frac{1}{m}D_F) \tag{7}$$

is not log canonical at B, so that by [26, Proposition 3] there is a hyperplane $\Lambda\subset E^\sharp$, satisfying the inequality

$$\operatorname{mult}_o D_F+\operatorname{mult}_\Lambda D^\sharp_F>2m,$$

where $D^\sharp_F$ is the strict transform of the divisor D_F on $F^\sharp$. Now the arguments of [26, Sec. 2.2] give a contradiction.

Consider the case 2). If $\dim B\geq 1$, then for a point $o\in B$ of general position the intersection of divisors

$$T_pW_\Pi\quad\text{and}\quad\sigma_F(D_F),$$

where $p=\sigma_F(o)$, is of codimension two (by the condition of general position, for any hyperplane $\Lambda\subset\Pi$ we get $\dim\operatorname{Sing}\Lambda\cap W=0$, so that the tangent hyperplanes T_pW_Π, $p\in B$, form a $\dim B$-dimensional family). In particular, the scheme-theoretic intersection

$$(\sigma_F^{-1}(T_pW_\Pi)\circ D_F)$$

is an effective cycle of codimension two on F, of H-degree $2m$ and of multiplicity at least

$$2\operatorname{mult}_o D_F>2m$$

at the point o, which is impossible.

Thus it remains to consider the case when $B=o$ is a smooth point on the ramification divisor of the morphism σ_F. Since the condition of non log canonicity of the pair (7) is linear in the divisor $D_F\in|-mK_F|$, one may assume that D_F is a prime divisor. Set $\Lambda=T_pW_\Pi$. If

$$D_F\neq\sigma_F^{-1}(\Lambda),$$

then we argue as above in the case $\dim B\geq 1$. Let us show that the equality $D_F=\sigma_F^{-1}(\Lambda)$ is impossible. It can be done by inspection of possible singularities of the intersection $W_\Pi\cap\Lambda$ for a hypersurface W of general position. We will give a simpler argument suggested by the anonymous referee of the paper [27], see also [28]. Namely, if the pair (7) is not log canonical for $D_F=\sigma_F^{-1}(\Lambda)$, then by [29] (or [30]), the pair

$$(\Pi,\Lambda+\frac{1}{2}W_\Pi)$$

is not log canonical, either, which, in its turn, implies that the pair

$$(\Lambda,\frac{1}{2}W_\Lambda)$$

is not log canonical, $W_\Lambda = (W \circ \Lambda) = W \cap \Lambda$. However, as we pointed out above, the restriction W_Λ has at most isolated double points as singularities. This contradiction proves Proposition 1.2.

Let $B = o$ be the centre of the maximal singularity E^+.

Proposition 1.3. *The point o is a singularity of the variety V^+.*

Proof. Assume the converse: the point $o \in V^+$ is non-singular. Since the pair (6) is not canonical, we get the inequality

$$\mathrm{mult}_o\, \Sigma^+ > m,$$

whence by Proposition 1.2 it follows that

$$\mathrm{mult}_o\, D_F > 2m$$

(since $o \in F$ is a singular point of the fiber). As we pointed out above, this is impossible, which proves the proposition.

1.3. The centre of the maximal singularity is a singular point of the variety V^+. We have shown above that the centre of the maximal singularity E^+ is a singular point $o \in V^+$, which we will assume from now on. Let

$$\lambda\colon V^\sharp \to V^+$$

be the blow up of the point o, $E^\sharp = \lambda^{-1}(o) \subset V^\sharp$ the exceptional divisor, which can be seen as a quadratic hypersurface in $\mathbb{P}^M$.

Recall (Proposition 0.6), that for $M \geq 6$ we may assume that for a generic hypersurface $W \subset \mathbb{P}$, arbitrary plane $P \subset \mathbb{P}$ of codimension two and any singularity $o \in V^+$ the quadric $E^\sharp$ is of rank at least 4.

Define the integer $\beta \in \mathbb{Z}_+$ by the formula

$$D^\sharp \sim \lambda^* D - \beta E^\sharp,$$

where $D \in \Sigma^+$ is a generic divisor, $D^\sharp$ its strict transform on $V^\sharp$. By Proposition 1.4, which we prove below, Proposition 0.6 implies the inequality

$$\beta > m.$$

Furthermore, the divisor

$$\lambda_F^* D_F - \beta E_F^\sharp$$

on the strict transform $F^\sharp \subset V^\sharp$ is effective (the symbols λ_F and $E_F^\sharp$ stand for the blow up of the point $o \in F$ and for the exceptional divisor $\lambda_F^{-1}(o)$, respectively). This implies the inequality

$$\mathrm{mult}_o\, D_F \geq 2\beta > 2m,$$

which is impossible. Proof of Proposition 1.1 for $M \geq 6$ is complete.

1.4. Maximal singularities over quadratic points. Consider the following local situation. Let $o \in X$ be a germ of a quadratic singularity, $\dim X \geq 3$. Let us blow up the point o:

$$\lambda\colon X^+ \to X,$$

and denote by the symbol E the exceptional divisor $\lambda^{-1}(o)$, which we consider as a quadric hypersurface

$$E \subset \mathbb{P}^{\dim X}$$

Let, furthermore, D be an effective $\mathbb{Q}$-Cartier divisor on the variety X, D^+ its strict transform on X^+. Assuming the exceptional quadric E to be irreducible, define the number $\beta \in \mathbb{Q}_+$ by the relation

$$D^+ \sim \lambda^* D - \beta E.$$

Proposition 1.4. *Assume that the rank of the quadric hypersurface E is at least 4 and the pair*

$$(X, D)$$

has the point o as an isolated centre of a non canonical singularity, that is, it is non canonical, but canonical outside the point o. Then the following inequality holds:

$$\beta > 1.$$

Proof. If $\dim X = 3$, then by assumption the point $o \in X$ is a non-degenerate quadratic singularity, and this fact is well known [31]. (If $\beta \leq 1$, then the pair (X^+, D^+) is non canonical, so that by inversion of adjunction the pair (E, D^+_E) is not log canonical, but $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ and D^+_E is an effective curve of bidegree (β, β), which is impossible [32].) If $\dim X \geq 4$, then, restricting D onto a generic hyperplane section $Y \ni o$ of the variety X with respect to some embedding $X \hookrightarrow \mathbb{P}^N$, and repeating this procedure $\dim X - 3$ times, we reduce the problem (by inversion of adjunction) to the already considered case $\dim X = 3$. Proof of the proposition is complete.

1.5. Double spaces of dimension five. Assume now that $M = \dim V = 5$. Let the singular point $o \in V^+$ be an isolated centre of non log canonical singularities of the pair $(V^+, \frac{1}{m}\Sigma^+)$. The fiber $F \ni o$ is a Cartier divisor, so that the point o is the centre of a non log canonical singularity of the pair

$$(F, \frac{1}{m} D_F), \tag{8}$$

where $D_F \in \Sigma^+|_F$ is a general divisor, $D_F \sim -mK_F$. By the arguments of Sec. 1.2, the point o is an isolated centre of non log canonical singularities of that pair. If the quadratic singularity $o \in F$ is of rank 4 or 5, we argue as for $M \geq 6$ and come to a contradiction. Since by the conditions of general position the rank of the quadratic point $o\in F$ is at least 3, we assume that it is equal to 3.

The variety F is realized as the double cover

$$\sigma_F\colon F\to\mathbb{P}^4,$$

generated by the morphism σ. By Proposition 1.8, proved below, the conditions of general position imply (see Proposition 1.7) that the pair (8) is log canonical for $m=1$. Now Proposition 1.1 for $M=5$ comes from the following claim.

Proposition 1.5. *For any effective Cartier divisor $D\sim -mK_F$ on F the pair*

$$(F,\frac{1}{m}D), \tag{9}$$

is log canonical.

Proof will be given by induction on $m\geq 2$ (as we mentioned above, for $m=1$ the claim of the proposition is true). It is sufficient to show that non log canonicity of the pair (9) implies non log canonicity of a similar pair with a smaller value of the parameter $m\geq 2$.

We may assume that the point o is an isolated centre of non log canonical singularities of the pair (9). For a generic surface $S\ni o$ (a section of the germ $o\in F$ by two generic hyperplanes with respect to some projective embedding) the pair

$$(S,\frac{1}{m}D_S),$$

where $D_S=D|_S$, is not log canonical at the point o by inversion of adjunction. On the other hand, the point o is an isolated centre of non log canonical singularities of that pair: in the opposite case on F there is a divisor T such that

$$D=aT+D_1,$$

where $a>m$ and D_1 is effective, which is impossible. The singularity $o\in S$ is a non-degenerate quadratic point.

Let

$$\psi\colon F^\sharp\to F\quad\text{and}\quad\bar\psi\colon\mathbb{P}^\sharp\to\mathbb{P}^4$$

be the blow ups of the points $o\in F$ and $p=\sigma_F(o)\in\mathbb{P}^4$. Denote the exceptional divisors of the blow ups ψ and $\bar\psi$ by the symbols $E^\sharp$ and $\bar E^\sharp$, respectively. Obviously, $\bar E^\sharp\cong\mathbb{P}^3$, and σ_F extends to a double cover

$$\sigma_\sharp\colon F^\sharp\to\mathbb{P}^\sharp,$$

which on the level of exceptional divisors gives a double cover

$$\sigma_E=\sigma_\sharp|_{E^\sharp}\colon E^\sharp\to\bar E^\sharp.$$

Set

$$\psi^*D=D^\sharp+\nu E^\sharp,$$

where $D^\sharp$ is the strict transform. If $\nu > m$, then, as above, we get $\mathrm{mult}_o D > 2m$, which is impossible. For this reason we assume that $\nu \leq m$. Applying Proposition 1.6, which is proved below, to the pair $(S, \frac{1}{m}D_S)$, we conclude that on the quadric $E^\sharp$ there is a plane $P \cong \mathbb{P}^2$, such that the centre of any non log canonical of the pair $(S, \frac{1}{m}D_S)$ on the strict transform $S^\sharp \subset F^\sharp$ is a point $P \cap S^\sharp$. Obviously, $\sigma_E(P)$ is a plane in $\bar{E}^\sharp \cong \mathbb{P}^3$.

Let $Q \subset \mathbb{P}^4$ be the only hyperplane, such that

$$Q^\sharp \cap \bar{E}^\sharp = \sigma_E(P)$$

(as always, $Q^\sharp \subset \mathbb{P}^\sharp$ is the strict transform). Set

$$\Pi = \sigma_F^{-1}(Q) \subset F.$$

The divisor Π is irreducible, and moreover,

$$\Pi^\sharp \cap E^\sharp \supset P.$$

Now write down

$$D = a\Pi + D^*,$$

where $a \in \mathbb{Z}_+$ and D^* is effective and does not contain Π as a component.

Lemma 1.3. *The inequality $a \geq 1$ holds. The pair*

$$(F, \frac{1}{m^*}D^*) \tag{10}$$

is not log canonical at the point o, where $m^* = m - a$.

Proof. By the conditions of general position, the pair (F, Π) is log canonical. Now by linearity we conclude that the pair (10) is not log canonical at the point o, and by the arguments of Sec. 1.2 this point is an isolated centre of non log canonical singularities. This proves the second claim of the lemma.

However, it is true for $a = 0$ in a trivial way: $m^* = m$ and $D^* = D$. Let us show that in fact $a \geq 1$. Indeed, by Proposition 1.6 (which is proven below), the inequality

$$\mathrm{mult}_P D^\sharp + 2\nu > 2m \tag{11}$$

holds. If $a = 0$, then Π is not contained in the support of the divisor D, so that the effective cycle

$$Y = (\Pi \circ D)$$

of codimension two on F is well defined. By (11) we get

$$\mathrm{mult}_o Y > 2m,$$

but at the same time $\deg Y = 2m$. Contradiction. Q.E.D. for the lemma.

Since for the pair (10), where $D^* \sim -m^* K_F$, the point $o \in F$ is an isolated centre of a non log canonical singularity and $m^* < m$, we apply the induction hypothesis and complete the proof of Proposition 1.5.

Now Proposition 1.1 is proven for $M=5$ as well.

1.6. Non log canonical singularities over a singular point of the surface. Let us consider the following local situation. Let $o\in S$ be a germ of a non-degenerate double point on a surface S (that is, a germ, analytically isomorphic to the germ $(0,0,0)\in\{x^2+y^2+z^2=0\}\subset\mathbb{C}^3$). Let

$$\varphi\colon S^+\to S$$

be the blow up of the double point o, $E=\varphi^{-1}(o)$ the exceptional conic. Assume that C is an effective 1-cycle on S, and for some positive m the pair

$$(S,\frac{1}{m}C) \tag{12}$$

is not log canonical at the point o, but log canonical outside this point. Define the number $\nu\in\mathbb{Z}_+$ by the relation

$$C^+\sim\varphi^*C-\nu E,$$

where C^+ is the strict transform of the 1-cycle C on S^+. Similar to Proposition 3 in [26], for the double point we have

Proposition 1.6. *There exists a point $q\in E$ such that*

$$2\nu+\operatorname{mult}_q C^+>2m. \tag{13}$$

Proof. Note that if the exceptional conic E makes itself a non log canonical singularity of the pair $(S,\frac{1}{m}C)$, then the inequality $\nu>m$ holds, that is, (13) holds for any point $q\in E$. If $\nu\leq m$, then the connectedness principle implies that the centre of any non log canonical singularity of the pair (12) is some uniquely defined (by the pair) point $q\in E$. We will prove that the inequality (13) holds for that point.

Let

$$\varphi_i\colon S_i\to S_{i-1}, \tag{14}$$

$i=1,\dots,N$, be the sequence of blow ups of the points which are the centres of a fixed non log canonical singularity of the (12). More precisely, let

$$\beta\colon\widetilde{S}\to S^+$$

be some birational morphism, $E^+\subset\widetilde{S}$ an irreducible exceptional curve, realizing the non log canonical singularity of the pair (12), that is, the log Noether-Fano inequality holds:

$$\nu_{E^+}(C)>m(a(E^+,S)+1).$$

By assumption, $\beta(E^+)$ is the point $q\in E$. Let us define a sequence of blow ups (14), setting

$$S_0=S,\ S_1=S^+,$$

φ_i blows up the point

$$x_{i-1} = \mathrm{centre}(E^+, S_{i-1}),$$

$i = 1, \dots, N$ (so that $x_0 = o$, $x_1 = q$), and the last exceptional curve

$$E_N = \varphi_N^{-1}(x_{N-1}) \subset S_N$$

realizes E^+. As usual, we denote the exceptional curves by the symbols

$$E_i = \varphi_i^{-1}(x_{i-1}) \subset S_i,$$

so that $E_1 = E$. Set

$$\nu_i = \mathrm{mult}_{x_{i-1}} C^{i-1} \in \mathbb{Z}_+,$$

where C^{i-1} is the strict transform of the cycle C on S_{i-1}, $i = 2, \dots, N$, and $\nu_1 = \nu$. The log Noether-Fano inequality now is re-written in the traditional form:

$$p_1\nu + \sum_{i=2}^{N} p_i\nu_i > m\left(\sum_{i=2}^{N} p_i + 1\right), \tag{15}$$

where $p_i = p_{Ni}$ is the number of paths from the vertex N to the vertex i in the graph of the sequence of blow ups (14). Note that in (15) in the right hand past there is no component with $i = 1$, because the discrepancy of E is zero. The multiplicities ν_i satisfy the system of linear inequalities

$$\nu_i \geq \sum_{j \to i} \nu_j \tag{16}$$

for $i = 2, \dots, N$ and, besides,

$$2\nu \geq \sum_{j \to 1} \nu_j. \tag{17}$$

Finally, $\nu_N \geq 0$. Non-negativity of the other numbers $\nu_1, \nu_2, \dots, \nu_{N-1}$ follows from (16,17). For simplicity let us denote by the symbol the $(15)^*$ *non-strict* log Noether-Fano inequality, that is, the inequality (15), in which the sign $>$ is replaced by $\geq$. Finally, by the symbol $\mathcal{L}$ denote the system of non-strict linear inequalities $(15)^*$, (16,17) and $\nu_N \geq 0$.

Let us show that if the set of real numbers

$$\nu_1, \dots, \nu_N$$

satisfies the system $\mathcal{L}$, then the estimate

$$2\nu_1 + \nu_2 \geq 2m \tag{18}$$

holds. This immediately implies the inequality (13).

Set $\Lambda \subset \mathbb{R}^N$ to be the convex subset defined by the system $\mathcal{L}$. Obviously, the linear function $2\nu_1 + \nu_2$ is bounded from below on Λ, and moreover, the infimum is a minimum, attained at some point

$$v = (\theta_1, \dots, \theta_N) \in \mathbb{R}^N.$$

We may assume that the point v is one of the vertices of the set Λ, that is, that N inequalities from the system $\mathcal{L}$ become equalities at that point.

Lemma 1.4. *Assume that $\theta_N = 0$. Then there exists $K \in \{2, \ldots, N-1\}$ such that the inequality*

$$\sum_{i=1}^{K} p_{Ki}\theta_i > m\left(\sum_{i=2}^{K} p_{Ki} + 1\right) \tag{19}$$

holds.

Arguing by induction on the length N of the resolution of the singularity E^+, we obtain the estimate (18) in the case $\theta_N = 0$.

Proof of the lemma 1.4. Since the curve

$$\bigcup_{i=1}^{N} E_i^N$$

is a normal crossing divisor on a non-singular surface, the vertex N is connected by arrows with one or two vertices: always

$$N \to N-1$$

and, possibly, $N \to L$ for some $L \leq N-2$. The first case is trivial; we will consider the second one (our arguments, with simplifications, prove the claim of the lemma in the first case as well). Setting $p_{ij} = 0$ for $i < j$, by definition of the incidence graph we get

$$p_{Ni} = p_{N-1,i} + p_{L,i}$$

for any $i \leq N-1$. Now, taking into account that $\theta_N = 0$, we can re-write the inequality $(15)^*$ in the following way:

$$\left(\sum_{i=1}^{N-1} p_{N-1,i}\theta_i - m\left(\sum_{i=2}^{N-1} p_{N-1,i} + 1\right)\right) +$$

$$+\left(\sum_{i=1}^{L} p_{Li}\theta_i - m\left(\sum_{i=2}^{L} p_{Li} + 1\right)\right) \geq 0,$$

which implies that the inequality (19) holds either for $K = N-1$, or for $K = L$ (or for both these values). Q.E.D. for the lemma.

Thus we may assume that $\theta_N > 0$ and therefore for the vector v the inequalities $(15)^*$, (16) and (17) are equalities. It follows that for $\theta = \theta_N$

$$\theta_i = p_i\theta$$

for $i = 2, \ldots, N$, and $\theta_1 = \frac{1}{2}p_1\theta$, so that θ can be found from the equation

$$\left(\frac{1}{2}p_1^2 + \sum_{i=2}^{N} p_i^2\right)\theta = \left(\sum_{i=2}^{N} p_i + 1\right)m.$$

The value of the linear function $2\nu_1 + \nu_2$ at the vector v is $(p_1 + p_2)\theta$, so that the inequality (18) comes from the following combinatorial fact.

Lemma 1.5. *The following inequality holds:*

$$(p_1 + p_2) \left(\sum_{i=2}^{N} p_i + 1 \right) \geq p_1^2 + 2 \sum_{i=2}^{N} p_i^2. \tag{20}$$

Proof will be given by induction on the number N of vertices in the incidence graph. If $N = 2$, then $p_1 = p_2 = 1$ and (20) holds. Furthermore, the inequality

$$(p_2 + p_3) \left(\sum_{i=3}^{N} p_i + 1 \right) \geq p_2^2 + 2 \sum_{i=3}^{N} p_i^2$$

holds by the induction hypothesis. In order to obtain (20), it is sufficient to show the estimate

$$(p_1 + p_2)p_2 + (p_1 - p_3) \left(\sum_{i=3}^{N} p_i + 1 \right) \geq p_1^2 + p_2^2. \tag{21}$$

In [33, Lemma 1.6] it was proved that the inequality

$$\sum_{i=3}^{N} p_i + 1 \geq p_1$$

holds, so that (21) follows from the estimate

$$(p_1 + p_2)p_2 + (p_1 - p_3)p_1 \geq p_1^2 + p_2^2,$$

which is obvious, since $p_2 \geq p_3$.

Q.E.D. for Lemma 1.5 and Proposition 1.6.

1.7. Additional conditions of general position for $M = 5$. Here we assume that $M = 5$. For an arbitrary point $p \in W$ set

$$T(p) = \sigma^{-1}(T_p W).$$

Let

$$\varphi \colon T^+(p) \to T(p)$$

be the blow up of the isolated double point $o = \sigma^{-1}(p)$ with the exceptional divisor $E(p)$, a three-dimensional quadric in $\mathbb{P}^4$. Set

$$Y_i = \{p \in W \,|\, \mathrm{rk}\, E(p) = i\} \subset W.$$

Proposition 1.7. *For a generic variety V we have $\mathrm{rk}\, E(p) \geq 3$, that is, $Y_1 = Y_2 = \emptyset$. Furthermore, $\dim Y_3 = 1$. For a point $p \in Y_3$ the singularities of the variety $T(p)$ are of the following form:*

1) for a point $p\in Y_3$ of general position on the line $L=\mathrm{Sing}\,E(p)$ there are three distinct singular points of the variety $T^+(p)$, which are non-degenerate quadratic points, and on $E(p)$ the variety $T^+(p)$ has no other singular points;

2) for a finite set of points $p\in Y_3$, which do not satisfy the condition 1), on the line $L=\mathrm{Sing}\,E(p)$ there are two distinct singular points p_1 and p_2 of the variety $T(p)$. On $E(p)$ the variety $T(p)$ has no other singular points. On of these points (say, p_1) is a non-degenerate quadratic singularity. The point p_2 is an isolated quadratic point of rank 4. Its blow up

$$\varphi_\sharp\colon T^\sharp(p)\to T^+(p)$$

has a unique singular point p_3 on the exceptional divisor $E^\sharp=\varphi_\sharp^{-1}(p_2)$, which is the vertex of the cone $E^\sharp$, and moreover, $p_3\in T^\sharp(p)$ is a non-degerate quadratic point.

Proof: an easy dimension count for the local equation

$$y^2=q_2(z_1,z_2,z_3,z_4)+q_3(z_*)+\dots$$

of the variety $T(p)$ at the point o. For a point $p\in Y_3$ singularities of the variety $T^+(p)$ correspond to the zeros of the polynomial

$$q_3(z_*)|_L$$

on the vertex line L of the quadric $E(p)$. If the three roots are distinct, we get the case 1). If one of the roots is a double root, we get the case 2), where the point $p_2\in L$ corresponds to the double root. Simple calculations are left to the reader. Q.E.D.

Proposition 1.8. *For a generic variety V, an arbitrary point $p\in Y_3$ and an arbitrary hyperplane $R\subset T_pW$, $R\ni p$, the pair*

$$(T(p),\Pi=\sigma^{-1}(R))$$

is canonical at the point o.

Proof. Since

$$\mathrm{mult}_o\,\Pi=a(E,T)=2,$$

where $E=E(p)$, $T=T(p)$ for simplicity of notations, it is sufficient to prove that the pair

$$(T^+,\Pi^+),$$

is canonical, where $\Pi^+\subset T^+$ is the strict transform of the divisor Π.

Let

$$y^2=q_2^*(z_1,z_2,z_3)+q_3^*(z_*)+\dots$$

be the local equation of the variety Π. If $\mathrm{rk}\,q_2^*=3$, then $o\in\Pi$ is an ordinary double point and there is nothing to prove. If $\mathrm{rk}\,q_2^*=2$, then it is easy to check that the singularities of Π over the point o are resolved by a sequence of $k\le 6$ blow ups of isolated quadratic points of rank ≥ 3. In that case it is also obvious that the pair (T, Π) is canonical.

Now assume that $\operatorname{rk} q_2^* = 1$. It imposes 3 independent conditions on Π (more precisely, on the polynomial $f|_R$), so that there is a 4-dimensional family of subspaces $R \subset \mathbb{P}$, for which Π satisfies this property. Let

$$E_\Pi = \Pi^+ \cap E$$

be the exceptional quadric of rank 2 in $\mathbb{P}^3$, that is, a pair of planes, $L = \operatorname{Sing} E_\Pi$ the line of their intersections.

If

$$q_3^*|_L \equiv 0$$

(this imposes 4 additional independent conditions on Π, so that there are only finitely many such pairs), then let

$$\varphi_L : \Pi_L \to \Pi^+$$

be the blow up of the line L. It is easy to check that Π_L has finitely many isolated double points, resolved by one blow up. It is now obvious that the pair (T, Π) is canonical, taking into account that

$$\operatorname{mult}_L \Pi^+ = a(L, T^+) = 2.$$

Assume that $q_3^*|_L \not\equiv 0$, that is, at a general point of the line L the variety Π^+ is non-singular. It is sufficient to check that the pair (T^+, Π^+) is canonical at the singular points of the variety Π^+ on the line L. The explicit computations in local coordinates (they are elementary but tiresome and we omit them) show that singularities of the divisor Π^+ are resolved by a sequence of blow ups of isolated quadratic points (of rank ≥ 2), which implies canonicity of the pair (T^+, Π^+).

Assume finally that $q_2^* \equiv 0$. This imposes 6 independent conditions on Π, so that there is a one-dimensional family of such subvarieties on V. The quadric E_Π is the double plane $2\Lambda = \{y = 0\}$, however, the arguments of general position imply that $\operatorname{mult}_\Lambda \Pi^+ = 1$ and, moreover,

$$C = \operatorname{Sing} \Pi^+ = \{q_3^*|_\Lambda = 0\}$$

(we mean the singularities over the point o) is an irreducible cubic curve, and moreover, if it is singular, then the only singular point of that curve lies outside the line $L = \operatorname{Sing} E(p)$. We get

$$\operatorname{mult}_C \Pi^+ = a(C, T^+) = 2.$$

Blowing up of the cubic C gives the variety Π_C, which is non-singular over a generic point of C. It is easy to check that for any point $p \in C \backslash L$ the pair (T^+, Π^+) is canonical (even terminal) at that point, either. Finally, the variety Π_C has only isolated quadratic points of rank ≥ 3, so that it is easy to check that the pair (T^+, Π^+) is canonical also over the points $q \in C \cap L$. Note that for the blow up

$$\varphi_q : T_q \to T^+$$

of a point $q\in C\cap L$, with $E_q=\varphi_q^{-1}(q)$ the exceptional divisor, we get

$$\mathrm{mult}_q\,\Pi^+=a(E_q,T^+)=2,$$

E_q is a quadric of rank ≥ 4, so that E_q realizes one more canonical, but not terminal singularity of the pair (T,Π). This completes our proof of Proposition 1.8.

Remark 1.1. Since the multiplicities of singular points and subvarieties are equal to 2, the proof of Proposition 1.8 reduces to checking that if the strict transform of the divisor Π has a curve of singular points then the strict transform T is non-singular at the generic point of this curve, and that the singularities of the strict transform of Π are at most one-dimensional.

2 Exclusion of maximal subvarieties of codimension two

In this section, we start to prove Proposition 0.3: we show that, except for the preimage $\sigma^{-1}(P)$, where $P\subset\mathbb{P}$ is a linear subspace of codimension two, no subvariety of codimension two can make a maximal subvariety of the system Σ.

2.1. Set up of the problem. The following claim is true.

Proposition 2.1. *If an irreducible subvariety $B\subset V$ of codimension two is maximal for the movable linear system $\Sigma\subset|2nH|$, that is, the inequality $\mathrm{mult}_B\,\Sigma>n$ holds, then $B=\sigma^{-1}(\bar{B})$, where $\bar{B}\subset\mathbb{P}$ is a linear subspace of codimension two.*

Proof. The self-intersection $Z=(D_1\circ D_2)$, $D_i\in\Sigma$, of the linear system Σ is of H-degree $8n^2$ and contains the subvariety B with multiplicity strictly higher than n^2. Therefore, $\deg B\leq 7$. It is necessary to show that only one of these possibilities realize: $\deg B=2$, and moreover, $\bar{B}=\sigma(B)$ is a $(M-2)$-plane in $\mathbb{P}$, that is, the double cover $\sigma^{-1}(\bar{B})\to\bar{B}$ is irreducible.

Note that for $\dim V=M\geq 5$ we have $A^2V=\mathbb{Z}H^2$, so that only three possibilities occur, $B\sim H^2$ or $2H^2$ or $3H^2$. In particular, $\deg B\in\{2,4,6\}$. However, we exclude below maximal subvarieties of codimension two for $M=4$, either.

Let us exclude, first of all, the case $\deg B=1$. Here $M=4$, so that $\bar{B}\subset\mathbb{P}=\mathbb{P}^4$ is a 2-plane, and moreover, the double cover $\sigma^{-1}(\bar{B})\to\bar{B}$ is reducible. Therefore, the curve $\bar{B}\cap W$ is everywhere non-reduced (it is a cubic curve with multiplicity two). This is impossible by generality of the hypersurface W, see Sec. 0.4.

If $B=\sigma^{-1}(\bar{B})$, then $\deg B\in\{2,4,6\}$. Assume that $\deg B\in\{4,6\}$, that is, $\deg\bar{B}\in\{2,3\}$. Let us show that these cases do not realize. Indeed, let $L\subset\mathbb{P}$ be a generic secant line of the subvariety $\bar{B}\subset\mathbb{P}$. By generality, the curve $C=\sigma^{-1}(L)$ is non-singular and irreducible, and such curves sweep out at least a divisor on V, so that $C\not\subset\mathrm{Bs}\,\Sigma$. For a general divisor $D\in\Sigma$ we get $C\not\subset D$ and $(C\cdot D)=4n$. On the other hand, let $p_1\neq p_2$ be the points of intersection $L\cap\bar{B}$, then (by generality)

$\sigma^{-1}(p_i)=\{p_{i1},p_{i2}\}$, $i=1,2$, where p_{ij} are four distinct points on B. For this reason,

$$4n=(C\cdot D)\geq\sum_{i,j}(C\cdot D)_{p_{ij}}>4n.$$

Contradiction.

Thus if $B=\sigma^{-1}(\bar{B})$, then $\bar{B}\subset\mathbb{P}$ is a $(M-2)$-plane, which is exactly what we need.

Starting from this moment, we assume that $\sigma^{-1}\bar{B}=B\cup B'$ breaks into two irreducible components and

$$\deg B=\deg\bar{B}\in\{2,3,4,5,6,7\}.$$

We show below that none of these cases realizes. Let us describe first of all the main technical tools that will be used for their exclusion.

2.2. Conics on the variety $\bar{B}$. Let $C\subset\bar{B}$ be an irreducible conics, $P=<C>$ its linear span (a 2-plane). Assume that $C\not\subset W$, the curve $W\cap P$ is reduced and the two finite sets

$$C\cap W\quad\text{and}\quad\mathrm{Sing}(W\cap P)$$

are disjoint. Set $S=\sigma^{-1}(P)$, this is an irreducible surface with a finite set of singular points $\sigma^{-1}(\mathrm{Sing}(W\cap P))$. Let C_+ and C_- be components of the curve $\sigma^{-1}(C)=C_+\cup C_-$, where $C_+\subset B$, $C_-\subset B'$.

Lemma 2.1. *The surface S is contained in the base set* $\mathrm{Bs}\,\Sigma$.

Proof. Assume the converse. Then for a general divisor $D\in\Sigma$ we get $S\not\subset D$, so that $(D\circ S)$ is an effective curve on S, containing C_+ with some multiplicity

$$\nu_+\geq\mathrm{mult}_B\,\Sigma>n$$

and C_- with some multiplicity $\nu_-\in\mathbb{Z}_+$. Let $H_S=H\,|\,_S$ be the class of a hyperplane section of S. By what we have said,

$$((2nH_S-\nu_+C_+-\nu_-C_-)\cdot C_\pm)\geq 0. \tag{22}$$

Note that by assumption the curves $C_\pm$ do not contain singular points of the surface S, so that the local intersection numbers $(C_+\cdot C_-)_x$ are equal to $\frac{1}{2}(C\cdot W)_{\sigma(x)}$ and therefore

$$(C_+\cdot C_-)=\frac{1}{2}(C\cdot W)=2(M-1).$$

Furthermore, $C_++C_-\sim 2H_S$, whence we obtain

$$(C_+^2)=(C_-^2)=2(3-M).$$

Therefore, the inequalities (22) take the form of linear inequalities

$$\begin{array}{c}4n+2(M-3)\nu_+-2(M-1)\nu_-\geq 0,\\ 4n-2(M-1)\nu_++2(M-3)\nu_-\geq 0,\end{array} \tag{23}$$

whence we get $\nu_{\pm} \leq n$. Contradiction. Q.E.D. for the lemma.

Corollary 2.1. *The following inequality holds:* $\deg B \geq 4$.

Proof. We have to exclude two cases: $\deg B = 2$ and $\deg B = 3$. First assume that $\deg B = 2$. Applying Lemma 2.1 to the irreducible conic $C = \bar{B} \cap P$, where $P \subset < \bar{B} >$ is a generic 2-plane in the linear span of $\bar{B}$, we get that $\sigma^{-1}(P) \subset \mathrm{Bs}\,\Sigma$. Therefore,

$$\sigma^{-1}(< \bar{B} >) \subset \mathrm{Bs}\,\Sigma,$$

which is impossible, since $< \bar{B} >$ is a divisor in $\mathbb{P}$. If $\deg B = 3$, the arguments are similar: the variety $\bar{B}$ is swept out by conics, and moreover a generic conic $C \subset \bar{B}$ satisfies the assumptions of Lemma 2.1. The linear spans $P =< C >$ of those conics sweep out at least a divisor in $\mathbb{P}$, which again contradicts the fact that the linear system Σ is movable. Q.E.D. for the corollary.

2.3. The secant lines of the variety $\bar{B}$. Let $C \subset \bar{B}$ be an irreducible curve, not contained in W. Let $x \in \mathbb{P}$ be a point, satisfying the following conditions of general position:

- $x \notin C$,
- for any point $p \in C \cap W$ the line $L =< x, p >$, connecting the points x and p, intersects the hypersurface W transversally at the point p and is not a secant line of the curve C, that is, $C \cap L = \{p\}$.

Consider the cone $\Delta = \Delta(x, C)$ with the vertex at the point x and the base C. Set $S = \sigma^{-1}(\Delta)$, it is an irreducible surface. Let C_+ and C_- once again be the components of the curve $\sigma^{-1}(C) = C_+ \cup C_-$, where $C_+ \subset B$, $C_- \subset B'$. By the assumptions above, all the points of intersection of the curves C_+ and C_- are smooth points of the surface S. Obviously,

$$(C_+ \cdot C_-) = \frac{1}{2}(C \cdot W) = (M-1)\deg C.$$

Furthermore, it is well known [1], that on the cone Δ the curve C is numerically equivalent to the hyperplane section. Thus on the surface S

$$C_+ + C_- \equiv H_S = \sigma^* H_\Delta,$$

where H_Δ is the hyperplane section of the cone Δ. From here we get that

$$(C_+^2) = (C_-^2) = -(M-2)\deg C.$$

The restriction $\Sigma_S = \Sigma\,|\,_S$ of the system Σ on S is a non-empty linear system of curves, containing $C_\pm$ with the multiplicity $\nu_\pm$, respectively, where $\nu_+ \geq \mathrm{mult}_B\,\Sigma > n$. Therefore,

$$((2nH_S - \nu_+ C_+ - \nu_- C_-) \cdot C_\pm) \geq 0,$$

which yields the system of linear inequalities

$$\begin{aligned} 2n-(M-1)\nu_+ +(M-2)\nu_- \geq 0, \\ 2n+(M-2)\nu_+ -(M-1)\nu_- \geq 0. \end{aligned} \tag{24}$$

From here we immediately get

Proposition 2.2. *The following estimate holds*

$$\mathrm{mult}_{B'}\Sigma > \frac{M-3}{M-2}n \geq \frac{n}{2}. \tag{25}$$

Proof. For a general choice of the vertex x of the cone Δ we obtain $\nu_+ = \mathrm{mult}_B\Sigma$, $\nu_- = \mathrm{mult}_{B'}\Sigma$, whereas the inequality

$$\nu_- > \frac{M-3}{M-2}n$$

follows directly from (24). Q.E.D. for the proposition.

Note that the estimate (25) is the stronger, the higher is M. Proposition 2.2 makes it possible to exclude the case $\deg B = 7$ straightaway.

Proposition 2.3. *The case* $\deg B = 7$ *does not realize.*

Proof. Assume the converse: $\deg B = 7$. Let $D_1, D_2 \in \Sigma$ be general divisors, $Z = (D_1 \circ D_2)$ the self-intersection of the system Z. We obtain the inequality

$$8n^2 = \deg Z \geq 7((\mathrm{mult}_B\Sigma)^2 + (\mathrm{mult}_{B'}\Sigma)^2) > 7\cdot\frac{5}{4}n^2,$$

which is impossible. Contradiction. Q.E.D. for the proposition.

Note that, repeating this argument word for word, we exclude the case $\deg B = 6$ for $M \geq 5$: for the multiplicity of the subvariety B' Proposition 2.2 gives the estimate $\mathrm{mult}_{B'}\Sigma > 2n/3$, so that

$$8n^2 > 6\cdot(1+\frac{4}{9})n^2 = \frac{26}{3}n^2,$$

which is impossible once again.

2.4. Three-secant lines of the variety $\bar{B}$. Thus it remains to exclude three cases: $\deg B = 4, 5, 6$, whereas in the two latter cases $\dim V = M = 4$. We will need another simple construction. Let $L \subset \mathbb{P}$ be a 3-secant line of the variety $\bar{B}$, that is, a line that intersects $\bar{B}$ at (at least) three points outside W.

Proposition 2.4. *If the curve $\sigma^{-1}(L) = C$ is irreducible, then $C \subset \mathrm{Bs}\,\Sigma$. If $C = C_+ \cup C_-$ is reducible, then at least one of the components $C_\pm$ is contained in* $\mathrm{Bs}\,\Sigma$.

Proof. Let $D \in \Sigma$ be an arbitrary divisor. The curve C intersects D at at least 6 points. The total multiplicity of D at those points is at least

$$3(\mathrm{mult}_B\Sigma + \mathrm{mult}_{B'}\Sigma) > \frac{9}{2}n,$$

whereas $C \cdot D = 4n$. Therefore $L \subset \sigma(D)$, which is what we need. Q.E.D. for the proposition.

Therefore, if the subvariety $\bar{B} \subset \mathbb{P}$ has sufficiently many 3-secant lines (more precisely, if they sweep out at least a divisor on $\mathbb{P}$), then the subvariety $B \subset V$ can not be maximal since the linear system Σ is movable.

Remark 2.1. The claim of Proposition 2.4 (and its proof) remain true if the line L intersects $\bar{B}$ at two distinct points outside W, and in one of them, say, $\bar{x} \in L \cap \bar{B}$, is tangent to $\bar{B}$. In that case the curve $C = \sigma^{-1}(L)$ is tangent to B and B' at the points x, x', respectively, where $\sigma^{-1}(\bar{x}) = \{x, x'\}$, $x \in B$, $x' \in B'$, and it is easy to see that the local intersection numbers satisfy the inequalities

$$(C \cdot D)_x \geq 2 \operatorname{mult}_B \Sigma, \quad (C \cdot D)_{x'} \geq 2 \operatorname{mult}_{B'} \Sigma,$$

which makes it possible to argue in word for word the same way as in the case of three distinct points. In the sequel, when speaking about 3-secant lines, we will include the limit case of tangency without special reservations.

As a first application of the construction of Proposition 2.4 we exclude the case $\deg B = 4$ (the dimension $M \geq 4$ is arbitrary).

Proposition 2.5. *The case* $\deg B = 4$ *does not take place.*

Proof. Assume the converse. Let $P \subset \mathbb{P}$ be a generic 3-plane. For the irreducible curve $B_P = \bar{B} \cap P$ in $\mathbb{P}^3$ the four cases are possible:

1) $B_P \subset R$ is a plane curve, $R = \mathbb{P}^2$ is a plane in P;

2) $B_P = Q_1 \cap Q_2$ is a smooth elliptic curve, the intersection of quadrics Q_1 and Q_2;

3) B_P is a smooth rational curve;

4) B_P has a double point.

The case 1) does not realize, because any line $L \subset R$ is a 4-secant line. Proposition 2.4 implies that the entire surface $\sigma^{-1}(R)$ is contained in the base set $\operatorname{Bs}\Sigma$. This is impossible, since P is a generic 3-plane.

In the case 2) we come to a contradiction in exactly the same way as in the proof of Lemma 2.1. Namely, let Q be a generic quadric, containing the curve B_P. On the surface Q we get $B_P \sim 2H_Q$, where H_Q is the plane section. Set

$$S = \sigma^{-1}(Q), \quad \sigma^{-1}(B_P) = C_+ \cup C_-, \quad C_+ \subset B, \quad C_- \subset B',$$

so that on S we have $C_+ + C_- \sim 2H_S$, where $H_S = H\,|\,_S$ is the class of a hyperplane section. Now we argue in exactly the same way as in Lemma 2.1 and obtain the inequalities (22), which give the system of linear inequalities (23). This contradiction excludes the case 2). Note that of key importance (as in Lemma 2.1) is the fact that the curve B_P is equivalent to *two* hyperplane sections of the surface Q. In a general case, a curve can be embedded into a surface as a hyperplane section (with multiplicity one), which gives just some estimate for the multiplicity of the second component B', but does not allow to get a contradiction in one step.

Consider the case 3). Let $x \in B_P$ be a point of general position, $\pi_x \colon B_P \to \mathbb{P}^2$ the projection from the point x. The image $\pi_x(B_P) \subset \mathbb{P}^2$ is a rational cubic curve with a double point. Therefore, the curve B_P has a 3-secant line, passing through the point x. Since P is a 3-plane and $x \in B_P$ is a general point, we apply Proposition 2.4 and obtain a contradiction.

Consider the case 4). The curve B_P has a unique double point. This implies that the variety $\bar{B}$ contains a $(M-3)$-plane Π of double points. Let $L \subset \Pi$ be a generic line, $\Lambda \supset L$ a generic 3-plane, containing L. Now the curve $B_\Lambda = B \cap \Lambda$ is a quartic in $\mathbb{P}^3$, containing the line L with multiplicity 2. Therefore,

$$B_\Lambda = C_\Lambda + 2L,$$

where C_Λ is a (in the general case irreducible) conic. The variety $\bar{B}$ is swept out by the conics C_Λ. Now we apply Lemma 2.1 and obtain a contradiction.

Q.E.D. for Proposition 2.5.

2.5. Exclusion of the cases $\deg B = 5$ **and** 6**.** Recall that we may assume that $\dim V = M = 4$ (although our arguments work in arbitrary dimension). Let $P \subset \mathbb{P}$ be a generic hyperplane (that is, a 3-plane), $B_P = \bar{B} \cap P$ an irreducible curve. We may assume that the linear span of the curve B_P is $P = \mathbb{P}^3$ (otherwise we argue as in the case 1) for $\deg B = 4$). Besides, the curve B_P does not contain singular points of multiplicity ≥ 3 if $\deg B = 5$ and of multiplicity ≥ 4 if $\deg B = 6$ (otherwise we argue as in the case 4) for $\deg B = 4$).

Assume that $\deg B = 5$. It is easy to check that there is 3-secant line through a generic point $x \in B_P$. Indeed, if the curve B_P is smooth, then the projection from the point x realizes B_P as a plane quartic $Q \subset \mathbb{P}^2$, which can not be smooth: if the curve Q were smooth, by Riemann-Roch we would have got

$$h^0(l_Q + x) - h^1(l_Q + x) = 5 + 1 - 3 = 3,$$

where $l_Q = L \cap Q$ is the section of Q by a line $L \subset \mathbb{P}^2$. Furthermore,

$$h^1(l_Q + x) = h^0(-x) = 0,$$

whence $h^0(l_Q + x) = 3$, but at the same time $l_Q + x$ is a plane section of the smooth curve $B_P \subset \mathbb{P}^3$ and for that reason $h^0(l_Q + x) \geq 4$. Contradiction. Therefore, the quartic Q is singular and B_P has a 3-secant line, passing through the point x. Now Proposition 2.4 gives a contradiction.

Therefore, the curve B_P has $\delta \geq 1$ double points. Let $p \in \operatorname{Sing} B_P$ be a double point. The projection $\pi_p \colon B_P \to \mathbb{P}^2$ from the point p realizes B_P as a plane cubic with $\geq (\delta - 1) \geq 0$ double points, that is, a curve of genus $\leq 2 - \delta$. On the other hand, the projection $\pi_x \colon B_P \to \mathbb{P}^2$ from a generic point $x \in B_P$ realizes B_P as a plane quartic with $\delta^* \geq \delta$ double points, that is, a curve of genus $3 - \delta^*$. Therefore, we get the inequality $\delta^* \geq \delta + 1$, that is, there is a 3-secant line $L \subset P$ through the point x, that *does not contain* the double points of the curve B_P. Now we can apply Proposition 2.4 and obtain a contradiction. This excludes the case $\deg B = 5$.

Assume that $\deg B = 6$. The fact that the curve B_P can not be smooth is proved as the similar fact for $\deg B = 5$. Assume that the point $p \in B_P$ is of multiplicity 3. Comparing the curves

$$\pi_p(B_P) \subset \mathbb{P}^2 \qquad \pi_x(B_P) \subset \mathbb{P}^2,$$

where x is a generic point, we exclude this case by the same arguments as in the case of a curve of degree 5 with singularities. Thus we may assume that the curve B_P contains $\delta \geq 1$ double points and does not contain points of higher multiplicity.

Now we argue in word for word the same way as for $\deg B = 5$: we compare the curve $\pi_p(B_P) \subset \mathbb{P}^2$ of degree 4 with $\geq (\delta - 1) \geq 0$ double points ($p \in \operatorname{Sing} B_P$ is one of the singular points) with the curve $\pi_x(B_P) \subset \mathbb{P}^2$ of degree 5 with $\delta^* \geq \delta$ double points. We get that B_P has a 3-secant line, passing through the point $x \in B_P$ of general position and not containing singular points of the curve B_P. (If the curve $\pi_p(B_P) \subset \mathbb{P}^2$ is a conic, that is, $\deg \pi_p = 2$, then B_P is contained in a quadric cone with the vertex p, and moreover, we may assume that B_P has no other double points. In that case the genus of the curve B_P is easy to compute and we can show that there exists a 3-secant line through a point of general position.) Applying Proposition 2.4, we get a contradiction. The case $\deg B = 6$ is excluded.

This completes the proof of Proposition 2.1.

3 Exclusion of maximal singularities with the centre of codimension three

In this section we continue the proof of Proposition 0.3: we prove that the linear system Σ has no maximal singularities, the centre of which is a subvariety of codimension three on V.

3.1. Set up of the problem. Exclusion of centres of degree ≥ 2. Recall that the linear system Σ has a maximal singularity, the centre of which is an irreducible subvariety $B \subset V$. In Sec. 2 we proved that if Σ has no maximal subvariety of the form $\sigma^{-1}(P)$, where $P \subset \mathbb{P}$ is a linear subspace of codimension two, then Σ has no maximal subvarieties of codimension two at all. Therefore we may assume that $\operatorname{codim} B \geq 3$.

Proposition 3.1. *The subvariety B is of codimension ≥ 4.*

Proof. Assume the converse. By Proposition 2.1 then $\operatorname{codim} B = 3$. The case $\deg B = 1$ is excluded below in Sec. 3.2 and 3.3. Therefore we may assume that $\deg B \geq 2$.

Note that the morphism

$$\sigma|_B \colon B \to \sigma(B) = \bar{B}$$

is birational. Indeed, let $Z = (D_1 \circ D_2)$ be the self-intersection of the system Σ, then

$$\operatorname{mult}_B Z > 4n^2,$$

whence it follows that if $\sigma|_B$ is a double cover, then

$$\mathrm{mult}_{\bar{B}}\,\sigma_* Z > 8n^2,$$

however, $\sigma_* Z$ is an effective cycle of codimension two on $\mathbb{P}$ of degree $8n^2$. We get a contradiction.

Therefore, $\deg \bar{B} = \deg_H B \geq 2$ (since the branch divisor W does not contain linear subspaces of codimension three). Let $p, q \in \bar{B}$ be points of general position, $L \subset \mathbb{P}$ the line, connecting these points, $\Pi \supset L$ a generic (two-dimensional) plane, $\Lambda = \sigma^{-1}(\Pi)$ an irreducible surface on V. If $L \not\subset \mathrm{Supp}\,\sigma_* Z$, then the intersection $\Pi \cap \mathrm{Supp}\,\sigma_* Z$, and therefore, also the intersection $\Lambda \cap \mathrm{Supp}\, Z$, is zero-dimensional, so that we get

$$8n^2 = (\Lambda \cdot Z) \geq \sum_{x \in \sigma^{-1}(L) \cap B} (\Lambda \cdot Z)_x \geq$$

$$\geq \sum_{x \in \sigma^{-1}(L) \cap B} \mathrm{mult}_x\, Z > 8n^2,$$

a contradiction. Therefore, $L \subset \mathrm{Supp}\,\sigma_* Z$.

Let $Q \subset \mathbb{P}$ be the irreducible subvariety, swept out by all secant lines of the variety $\bar{B}$. By what we have proved, $\mathrm{codim}\, Q = 2$, so that Q is a subspace of codimension two and $\bar{B} \subset Q$ is some hypersurface.

Now let us write down

$$Z = a\sigma^{-1}(Q) + Z^{\sharp},$$

where $Z^{\sharp}$ does not contain the subvariety $\sigma^{-1}(Q)$ as a component and $a \geq 1$. The cycle Z satisfies the linear inequality

$$2\,\mathrm{mult}_B\, Z > \deg Z.$$

It is easy to see that any effective cycle of codimension two, satisfying this inequality, contains the subvariety $\sigma^{-1}(Q)$ as a component: as above,

$$\deg Z = (\Lambda \cdot Z) \geq \sum_{x \in \sigma^{-1}(L) \cap B} \mathrm{mult}_B\, Z > \deg Z$$

for every secant line L of the variety $\bar{B}$, which is not contained in the support of the cycle $\sigma_* Z$ (and a generic plane $\Pi \supset L$). However,

$$\mathrm{mult}_B\, \sigma^{-1}(Q) = 1 \text{ and } \deg \sigma^{-1}(Q) = 2$$

(recall that for a general hypersurface W the intersection $Q \cap W$ has at most zero-dimensional singularity, so that $\sigma^{-1}(Q)$ is an irreducible set), whence it follows that the cycle $Z^{\sharp}$ satisfies the inequality

$$2\,\mathrm{mult}_B\, Z^{\sharp} > \deg Z^{\sharp}$$

and therefore contains the subvariety $\sigma^{-1}(Q)$ as a component. Contradiction.

This excludes the case $\deg B \geq 2$.

3.2. Exclusion of infinitely near singularities with $\deg B = 1$. Starting from this moment and up to the end of the section we assume that $\deg B = 1$. By the conditions of general position this case can realize for the double spaces of dimension 4 only. Let X be the σ-preimage of a generic 3-plane in $\mathbb{P}$ (in particular, intersecting $\bar{B}$ at exactly one point). Then $\sigma_X: X \to \mathbb{P}^3$ is a double cover branched over a smooth hypersurface $W_X \subset \mathbb{P}^3$ of degree $2m_X \geq 8$, $o = X \cap B$ a point lying outside the ramification divisor:

$$p = \sigma_X(o) \notin W_X,$$

where H_X is the pull back via σ_X of the class of a plane in $\mathbb{P}^3$. To simplify the notations, we write H instead of H_X. By Proposition 0.4 we may assume that on X there are no lines passing through the point o, that is, for any line $L \subset \mathbb{P}^3$, $L \ni p$, the curve $\sigma_X^{-1}(L)$ is irreducible.

By the symbol Σ_X we denote the restriction of the system Σ onto X. The movable linear system $\Sigma_X \subset |2nH|$ has a maximal singularity with the centre at the point o, that is, for the pair $(X, \frac{1}{n}\Sigma_X)$ the point o is a centre of a non canonical singularity. Assume that the inequality

$$\nu = \operatorname{mult}_o \Sigma_X \leq 2n$$

holds, that is, the point o itself is not maximal (see Lemma 3.2 which is proved below). Let us blow up this point:

$$\varphi: \widetilde{X} \to X,$$

$E = \varphi^{-1}(o) \cong \mathbb{P}^2$ is the exceptional divisor.

Proposition 3.2. *The centre of the maximal singularity on $\widetilde{X}$ is a line in $E \cong \mathbb{P}^2$.*

Proof. If the centre of the maximal singularity is a curve $C \subset E$ of degree $d_C \geq 1$, then the inequality

$$\nu > nd_C$$

holds, whence by the assumptions above we get $d_C = 1$, that is, C is a line. Therefore, it is sufficient to exclude the case when the centre of the singularity is a point $y \in E$. Let us assume that this is the case and show that this assumption leads to a contradiction.

Lemma 3.1. *For any irreducible curve $C \subset X$ the inequality*

$$\operatorname{mult}_o C + \operatorname{mult}_y \widetilde{C} \leq \deg C = (C \cdot H), \tag{26}$$

holds, where $\widetilde{C} \subset \widetilde{X}$ is the strict transform.

Proof. Let

$$\bar{\varphi}: \widetilde{\mathbb{P}} \to \mathbb{P}^3$$

be the blow up of the point $p = \sigma_X(o)$, $\bar{E} = \bar{\varphi}^{-1}(o)$ the exceptional divisor. The morphism σ_X induces an isomorphism

$$\sigma_E \colon E \to \bar{E}.$$

Set $\bar{y} = \sigma_E(y) \in \bar{E}$. For any plane $P \ni p$ such, that its strict transform $\widetilde{P} \subset \widetilde{\mathbb{P}}$ contains the point $\bar{y}$, its inverse image $H = \sigma_X^{-1}(P)$ contains the point o and $\widetilde{H} \ni y$. Let us denote by the symbol

$$|H - o - y|$$

the linear subsystem of the system H, defined by that condition. Obviously,

$$\mathrm{Bs}\,|H - o - y| = \sigma^{-1}(L),$$

where $L \ni p$ is the line in $\mathbb{P}^3$ with the tangent direction $\bar{y}$ at the point p. Let $C \subset X$ be an irreducible curve, $C \ni p$.

Recall that by assumption there are no lines through the point o, that is, the curve $\sigma^{-1}(L)$ is irreducible. We get

$$\mathrm{mult}_o\, \sigma^{-1}(L) = \mathrm{mult}_y\, \widetilde{\sigma^{-1}(L)} = 1$$

and $(H \cdot \sigma^{-1}(L)) = 2$, so that for the curve $\sigma^{-1}(L)$ the inequality (26) holds.

Assume that $C \neq \sigma^{-1}(L)$. For a generic divisor $R \in |H - o - y|$ we have

$$(C \cdot R) = (C \cdot H) \geq (C \cdot R)_o \geq$$

$$\geq \mathrm{mult}_o\, C + (\widetilde{C} \cdot \widetilde{R})_y \geq \mathrm{mult}_o\, C + \mathrm{mult}_y\, \widetilde{C},$$

which is what we need ($\widetilde{R}$ is the strict transform of R on $\widetilde{X}$). Q.E.D. for the lemma.

Now we complete the proof of Proposition 3.2 by word for word the same arguments as the proof of the $8n^2$-inequality (Lemma 4.2). Indeed, let

$$\varphi_{i,i-1} \colon X_i \to X_{i-1},$$

$i = 1, \ldots, N$, be the resolution of the maximal singularity, that is, $\varphi_{i,i-1}$ blows up its centre B_{i-1} on X_{i-1}, $E_i = \varphi_{i,i-1}^{-1}(B_{i-1})$ is the exceptional divisor. For $i = 1, \ldots, L$ the centres of the blow ups are points, for $i = L+1, \ldots, N$ they are curves, and moreover, it follows from the inequality $\nu \leq 2n$ that all these curves are smooth and rational: $B_L \subset E_L \cong \mathbb{P}^2$ is a line, $B_i \subset E_i$ is a section of the ruled surface $E_i \to B_{i-1}$ for $i = L+1, \ldots, N-1$. By the same inequality $\nu \leq 2n$ we have $N \geq L+1$ and

$$B_L \not\subset E_{L-1}^L,$$

that is, $L+1 \nrightarrow L-1$ in the oriented graph of the sequence of blow ups $\varphi_{i,i-1}$. Finally, by assumption $L \geq 2$: more precisely, $B_0 = o$ and $B_1 = y \in E_1$. Now repeating the proof of Lemma 4.2 word for word, we get the inequality

$$\mathrm{mult}_o\, Z + \mathrm{mult}_y\, \widetilde{Z} > 8n^2$$

for the self-intersection $Z=(D_1\circ D_2)$ of the movable linear system Σ. However, Z is an effective 1-cycle of degree $\deg Z=(Z\cdot H)=8n^2$. We obtained a contradiction with Lemma 3.1 which completes the proof of Proposition 3.2.

3.3. Exclusion of the last case: preliminary constructions. In order to complete the proof of Proposition 3.1, it remains to exclude the situation described in Proposition 3.2. We assume that $M\geq 4$.

Let $L\subset\mathbb{P}^4$ be the line generating a line on V, that is, $\sigma^{-1}(L)=C_+\cup C_-$, where $C=C_+$ and C_- are smooth rational curves. Let

$$\varphi\colon\widetilde{V}\to V\quad\text{and}\quad\varphi_{\mathbb{P}}\colon\widetilde{\mathbb{P}}\to\mathbb{P}^4$$

be the blow ups of the curve C and the line L, respectively, with the exceptional divisors

$$E=\varphi^{-1}(C)\subset\widetilde{V}\quad\text{and}\quad E_{\mathbb{P}}=\varphi_{\mathbb{P}}^{-1}(L)\subset\widetilde{\mathbb{P}}.$$

The morphism σ induces a rational map

$$\sigma_E\colon E\dashrightarrow E_{\mathbb{P}},$$

which is a birational isomorphism, mapping $E\backslash\varphi^{-1}(C\cap\sigma^{-1}(W))$ isomorphically onto $E_{\mathbb{P}}\backslash\sigma_{\mathbb{P}}^{-1}(L\cap W)$. In particular, for any irreducible surface $S\subset E$, covering C, its image

$$\sigma_E(S)\subset E_{\mathbb{P}}\cong L\times\mathbb{P}^2$$

is well defined.

Proposition 3.3. *The movable linear system $\Sigma\subset|2nH|$ can not have a maximal singularity, the centre of which on V is the curve C, and on $\widetilde{V}$ it is some surface $S\subset E$.*

Proof. Assume the converse: such a maximal singularity exists.

Lemma 3.2. *The following inequality holds:*

$$\mathrm{mult}_C\,\Sigma\leq 2n,$$

that is, the curve C itself is not a maximal subvariety of the system Σ.

Proof. Let $P\subset\mathbb{P}^4$ be a generic plane, containing the line L. It is easy to see that the intersection $P\cap W$ is a non-singular curve, so that $Q=\sigma^{-1}(P)$ is a non-singular $K3$ surface. The restriction $\Sigma|_Q=\Sigma_Q$ is a linear system of curves that has, generally speaking, two fixed components, C_+ and C_-, of multiplicity ν_+ and ν_-, respectively. Therefore, on Q the inequalities

$$((2nH_Q-\nu_+C_+-\nu_-C_-)\cdot C_\pm)\geq 0$$

hold, where $H_Q=H|_Q$, or, explicitly,

$$\begin{gathered}2n+2\nu_+-3\nu_-\geq 0,\\ 2n-3\nu_++2\nu_-\geq 0\end{gathered}$$

(since $(C_\pm^2) = -2$ and $(C_+ \cdot C_-) = 3$). Multiplying the first inequality by 2, the second one by 3 and putting them together, we obtain

$$10n - 5\nu_+ \geq 0,$$

which is what we need. Q.E.D. for the lemma.

Corollary 3.1. *For a point $x \in L$ we have:*

$$S \cap \sigma^{-1}(x)$$

is a line in $\sigma^{-1}(x) \cong \mathbb{P}^2$. The graph of the resolution of the maximal singularity is a chain.

Proof: the inequality

$$\mathrm{mult}_C \Sigma \geq n \deg(S \cap \sigma^{-1}(x))$$

holds, which implies the first claim. The second is obvious.

Let us continue our proof of Proposition 3.3.

The surface $\sigma_E(S)$ in $E_{\mathbb{P}} \cong L \times \mathbb{P}^2 = \mathbb{P}^1 \times \mathbb{P}^2$ is of bidegree $(d, 1)$.

3.4. The hard case $d = 0$.

Proposition 3.4. *The case $d = 0$ is impossible.*

Proof. This is the hardest case and to consider it, we have to inspect quite a few possible cases.

First of all, note, that there exists a unique hyperplane $\Pi \subset \mathbb{P}^4$, cutting out S on $E_{\mathbb{P}}$:

$$S = \widetilde{\Pi} \cap E_{\mathbb{P}},$$

where $\widetilde{\Pi} \subset \widetilde{\mathbb{P}}$ is the strict transform. Since the linear system Σ is movable, its restriction

$$\Sigma_\Pi = \Sigma|_{\sigma^{-1}(\Pi)}$$

is a non-empty linear system (possibly with fixed components). Now let us consider a generic plane $P \supset L$, $P \subset \Pi$, and argue in exactly the same way as in the proof of Lemma 2.1: we restrict an effective divisor in the system Σ_Π onto the surface $Q = \sigma^{-1}(P)$ and show that the effective curve Σ_Q obtained in this way cannot contain the curve C with a multiplicity strictly higher than $2n$.

Unfortunately, we can not argue in word for word the same way as in the proof of Lemma 2.1, since the surface Q, generally speaking, has singular points. We need to resolve the singularities, and for that purpose, in its turn, to list the possible cases for a generic branch divisor W. The singularities can appear because the hyperplane Π (which is uniquely determined by the system Σ) may turn out to be the tangent hyperplane to W at one or more points of intersection of the line L and W.

By the symbol $T_x W$ for a point $x \in W$ we denote the hyperplane in $\mathbb{P}^4$, which is tangent to W at the point x.

For the scheme-theoretic intersection $(L \circ W)$ there are three possible cases:

1. $(L \circ W) = 2x_1 + 2x_2 + 2x_3$, where x_1, x_2, x_3 are distinct points on the line L,
2. $(L \circ W) = 4x_1 + 2x_2$, where $x_1 \neq x_2$ are two distinct point,
3. $(L \circ W) = 6x$, $x \in L$.

The first case takes place for a line of general position (a three-dimensional family). The case 2 takes place for a two-dimensional, the case 3 for a one-dimensional family of lines $C \subset V$.

Furthermore, the hyperplane Π is tangent to the divisor W at the points x, y if and only if

$$\Pi = T_x W = T_y W.$$

Therefore we can detailise the cases 1 and 2 in the following way:

1.1. The three hyperplanes $T_{x_i} W$ are distinct (the case of general position),

1.2. $T_{x_1} W = T_{x_2} W \neq T_{x_3} W$, this case takes place for a one-dimensional family of lines, since the coincidence of two tangent hyperplanes imposes two independent conditions on the line C,

1.3. $T_{x_1} W = T_{x_2} W = T_{x_3} W$, this case does not take place for a general divisor W, however we consider it to make the picture complete,

2.1. $T_{x_1} W \neq T_{x_2} W$ are distinct hyperplanes,

2.2. $T_{x_1} W = T_{x_2} W$, this case takes place for a finite number of lines.

The strategy of the further arguments is as follows. Assuming that Π is tangent to W at at least one point (otherwise we repeat the proof of Lemma 2.1 without modifications), we resolve singularities of the surface $Q = \sigma^{-1}(P)$ for a general plane P, where $L \subset P \subset \Pi$. Let T_i, $i \in I$, be the set of irreducible exceptional curves (they are (-2)-curves on a $K3$-surface $\widetilde{Q}$). Let $\widetilde{C_\pm}$ be strict transforms of the curves $C_\pm$ on $\widetilde{Q}$, $\widetilde{\Sigma}_Q$ the strict transform of the linear system Σ_Q on $\widetilde{Q}$. For some no-negative integers $a_i \in \mathbb{Z}_+$, $i \in I$, we get

$$\Sigma_Q \subset |2nH_Q - \sum_{i \in I} a_i T_i|,$$

where $H_Q = H|_Q$ (the pull back of this class on $\widetilde{Q}$ we denote by the same symbol). The linear system $\widetilde{\Sigma}_Q$ contains the curves $C_\pm$ as fixed components of multiplicities $\nu_\pm$, whereas at least one of these two multiplicities by construction is strictly higher than $2n$. We assume that $\nu_+ > 2n$. Therefore, the $\sharp I + 2$ linear inequalities hold:

$$((2nH_Q - \nu_+ \widetilde{C}_+ - \nu_- \widetilde{C}_- - \sum_{i \in I} a_i T_i) \cdot R) \geq 0,$$

where $R \in \{\widetilde{C}_+, \widetilde{C}_-\} \cup \{T_i | i \in I\}$. In each of the possible cases this system of linear inequalities gives the estimate $\nu_+ \leq 2n$, contradicting the initial assumption. This would complete the proof of Proposition 3.4.

Let us realize the program that was described. For that purpose, we list all possible types of singularities of the surface Q for a *generic* plane $P \subset \Pi$, $P \supset L$ (in the assumption that the divisor W is generic). For an arbitrary plane there are many more cases, but we do not need them. The types of singularities, listed below, are obtained by elementary computations in the affine coordinates z_1, z_2, z_3, z_4 on $\mathbb{P}^4$, in which the hyperplane Π is given by the equation $z_4 = 0$, and the line L is given by the system of equations $z_1 = z_2 = 0$, so that the plane P is given by the equation $z_1 + \beta z_2 = 0$, where $\beta \in \mathbb{C}$ is some number. Direct coordinate computations show that if singularities of the surface Q are worse than in the cases listed below, for at least one line $C \subset V$, then W is not a hypersurface of general position. The computations are absolutely elementary and we omit them.

Here is the list of possible types of singularities.

Type A. One ordinary double point, on $\widetilde{Q}$ there is one exceptional curve E. The multiplication table:

	$\widetilde{C}_+$	$\widetilde{C}_-$	E
$\widetilde{C}_+$	-2	2	1
$\widetilde{C}_-$	2	-2	1
E	1	1	-2

This type takes place in the cases 1.1, 2.1 and 3.

Type B. One degenerate double point, resolved by one blow up, on $\widetilde{Q}$ there are two exceptional lines E_+ and E_-. The multiplication table:

	$\widetilde{C}_+$	$\widetilde{C}_-$	E_+	E_-
$\widetilde{C}_+$	-2	2	1	0
$\widetilde{C}_-$	2	-2	0	1
E_+	1	0	-2	1
E_-	0	1	1	-2

This type takes place in the cases 1.1, 2.1.

Type C. A degenerate double point on Q, the exceptional divisor of its blow up is a pair of lines, the point of their intersection is an ordinary double point of the surface (resolved by one blow up). On $\widetilde{Q}$ there are three exceptional curves E_+, E_-, E with the multiplication table

	$\widetilde{C}_+$	$\widetilde{C}_-$	E_+	E_-	E
$\widetilde{C}_+$	-2	2	1	0	0
$\widetilde{C}_-$	2	-2	0	1	0
E_+	1	0	-2	0	1
E_-	0	1	0	-2	1
E	0	0	1	1	-2

This type takes place in the case 1.1.

Type D. The surface Q has two ordinary double points (resolved by one blow up). On $\widetilde{Q}$ there are two exceptional curves E_1 and E_2. The multiplication table:

	$\widetilde{C}_+$	$\widetilde{C}_-$	E_1	E_2
$\widetilde{C}_+$	-2	1	1	1
$\widetilde{C}_-$	1	-2	1	1
E_1	1	1	-2	0
E_2	1	1	0	-2

This type takes place in the cases 1.2 and 2.2.

These four types complete the list of possible singularities of the surface Q for a variety V of general position. However the condition of general position is not essential. The author considered examples of more complicated singularities and our method in all cases gives a proof of Proposition 3.4. As an illustration, in addition to the types A-D, we will give two more examples (they do not take place on a variety of general position).

Type E. Two singular points: a non-degenerate one and a degenerate one. Both are resolved by one blow up. On $\widetilde{Q}$ there are three exceptional curves: E (corresponds to the non-denerate point) and $E_\pm$ (they correspond to the exceptional lines on the blow up of the degenerate point). The multiplication table:

	$\widetilde{C}_+$	$\widetilde{C}_-$	E_+	E_-	E
$\widetilde{C}_+$	-2	1	1	0	1
$\widetilde{C}_-$	1	-2	0	1	1
E_+	1	0	-2	1	0
E_-	0	1	1	-2	0
E	1	1	0	0	-2

This type takes place on a variety of non-general position in the case 1.2.

Type F. On the surface Q there are three non-degenerate double points. On $\widetilde{Q}$ there are three exceptional curves E_1, E_2, E_3. The multiplication table:

	$\widetilde{C}_+$	$\widetilde{C}_-$	E_1	E_2	E_3
$\widetilde{C}_+$	-2	0	1	1	1
$\widetilde{C}_-$	0	-2	1	1	1
E_1	1	1	-2	0	0
E_2	1	1	0	-2	0
E_3	1	1	0	0	-2

This type takes place on a variety of non-general position in the case 1.3.

It remains to realize the program described above for each type of singularities. We will consider two examples, A and C, in the other cases the computations are similar. After that we explain the essence of the computations.

Consider the type A. Let ν_+, ν_- and α be the multiplicities of the curves $\widetilde{C}_+$, $\widetilde{C}_-$ and E in the linear system Σ_Q, pulled back on $\widetilde{Q}$. Multiplying the class

$$2nH_Q - \nu_+\widetilde{C}_+ - \nu_-\widetilde{C}_- - \alpha E$$

by $\widetilde{C}_+$, $\widetilde{C}_-$ and E, we obtain a system of linear inequalities:

$$\begin{array}{rl} 2n & +2\nu_+ - 2\nu_- - \alpha \geq 0, \\ 2n & -2\nu_+ + 2\nu_- - \alpha \geq 0, \\ & \quad -\nu_+ - \nu_- + 2\alpha \geq 0. \end{array}$$

Adding to the first and second inequalities one half of the third one, we get the system

$$\begin{array}{l} 4n + 3\nu_+ - 5\nu_- \geq 0, \\ 4n - 5\nu_+ + 3\nu_- \geq 0. \end{array}$$

Multiplying the first inequality by 3, the second one by 5 and putting them together, we get

$$32n - 16\nu_+ \geq 0,$$

which is precisely what we need.

Let us consider the type C. Denoting the multiplicities of the components $\widetilde{C}_+$, $\widetilde{C}_-$, E_+, E_-, E by the symbols ν_+, ν_-, α_+, α_-, α, respectively, multiply the effective class

$$2nH_Q - \nu_+\widetilde{C}_+ - \nu_-\widetilde{C}_- - \alpha_+E_+ - \alpha_-E_- - \alpha E$$

by $\widetilde{C}_+$, $\widetilde{C}_-$, E_+, E_-, E and obtain the system of inequalities

$$\begin{array}{cccccccl} 2n & +2\nu_+ & -2\nu_- & -\alpha_+ & & & \geq 0 \\ 2n & -2\nu_+ & +2\nu_- & & -\alpha_- & & \geq 0 \\ & -\nu_+ & & +2\alpha_+ & & -\alpha & \geq 0 \\ & & -\nu_- & & +2\alpha_- & -\alpha & \geq 0 \\ & & & -\alpha_+ & -\alpha_- & +2\alpha & \geq 0. \end{array}$$

By means of the fifth inequality eliminate α in the third and fourth inequalities, which take the form

$$\begin{array}{ccccl} -\nu_+ & & +\frac{3}{2}\alpha_+ & -\frac{1}{2}\alpha_- & \geq 0, \\ & -\nu_- & -\frac{1}{2}\alpha_+ & +\frac{3}{2}\alpha_- & \geq 0. \end{array}$$

Multiplying one of these inequalities by $\frac{3}{2}$, another one by $\frac{1}{2}$ and putting them together, we obtain the inequalities

$$\begin{array}{l} -3\nu_+ - \nu_- + 4\alpha_+ \geq 0, \\ -\nu_+ - 3\nu_- + 4\alpha_- \geq 0, \end{array}$$

which make it possible to eliminate α_+, α_- in the first two inequalities and obtain the system of inequalities

$$\begin{array}{l} 8n + 5\nu_+ - 9\nu_- \geq 0, \\ 8n - 9\nu_+ + 5\nu_- \geq 0, \end{array}$$

whence, similar to the case A, we get

$$112n - 56\nu_+ \geq 0,$$

which is precisely what we need.

The other types of singularities are considered in a similar way. Now let us explain, why all the types listed above lead to the inequality $2n \geq \nu_+$. Let us consider the space

$$\mathcal{L} = \mathbb{R}[\widetilde{C}_+] \oplus \mathbb{R}[\widetilde{C}_-] \oplus \mathcal{E},$$

where

$$\mathcal{E} = \bigoplus_{i=1}^{k} T_i,$$

where $\{T_i | i = 1, \dots, k\}$ is the set of irreducible exceptional curves on $\widetilde{Q}$. On $\mathcal{L}$ there is a natural bilinear form, generated by intersection of curves. Set

$$\Theta = \|(T_i \cdot T_j)\|_{1 \leq i,j \leq k}$$

to be the negative definite matrix of the intersection form on $\mathcal{E}$. Set $\Theta^{-1} = \|\lambda_{ij}\|$ to be the inverse matrix. It is easy to check that in each of the cases A-F (and in all other cases of singularities of non-general position, studied by the author) the matrix Θ satisfies the following condition:

all coefficients $\lambda_{i,j}$, $1 \leq i,j \leq k$, of the inverse matrix Θ^{-1} are negative.

Let $\mathcal{E}_+ = \{a_i T_i \,|\, a_i \in \mathbb{R}_+\}$ be the positive coordinate cone. Since the matrix Θ is non-degenerate, there exist uniquely determined vectors $e_\pm \in \mathcal{E}$, such that

$$R_\pm = \widetilde{C}_\pm + e_\pm \in \mathcal{E}^\perp.$$

Lemma 3.3. $e_\pm \in \mathcal{E}_+$.

Proof. This follows immediately from the inequalities $(C_\pm \cdot T_i) \geq 0$ and the properties of the matrix Θ ($\lambda_{ij} < 0$).

Since the intersection form on $\mathcal{L}$ is non-degenerate, the class H_Q can be considered as an element of the space $\mathcal{L}$, $H_Q \in \mathcal{E}^\perp$.

Lemma 3.4. *The following inequality holds:* $H_Q = R_+ + R_-$.

Proof. H_Q is the class of a Cartier divisor on Q, which consists of the curves C_+ and C_-. Therefore, for some $e \in \mathcal{E}$ we get the equality

$$H_Q = \widetilde{C}_+ + \widetilde{C}_- + e.$$

Since $H_Q \in \mathcal{E}^\perp$, the fact that the quadratic intersection form on $\mathcal{L}$ is non-degenerate, implies the claim of the lemma.

Obviously, $(R_\pm \cdot H_Q) = 1$.

It is easy to check that $(R_\pm^2) = -a < 0$, so that the intersection form on the two-dimensional space

$$\mathcal{R} = \mathbb{R}[R_+] \oplus \mathbb{R}[R_-]$$

is given by the matrix

$$\begin{pmatrix} -a & 1+a \\ 1+a & -a \end{pmatrix},$$

the inverse matrix for which is

$$\frac{1}{1+2a}\left(\begin{array}{cc} a & 1+a \\ 1+a & a \end{array}\right).$$

By what was said above, non-negativity of the intersections of the class

$$2nH_Q-\nu_+\widetilde{C}_+-\nu_-\widetilde{C}_--\sum_{i=1}^{k}b_iT_i$$

(where $b_i\in\mathbb{Z}_+$) with the classes $\widetilde{C}_+$, $\widetilde{C}_-$, T_i implies non-negativity of the intersections of the class

$$\beta_+R_++\beta_-R_-=2nH_Q-\nu_+R_+-\nu_-R_-$$

with the classes R_+ and R_-, which implies that $\beta_\pm\geq 0$. However,

$$\beta_\pm=2n-\nu_\pm,$$

which is what we need.

This general argument works for any type of singularities of the surface Q, satisfying the two properties: the elements λ_{ij} of the matrix Θ^{-1} are all negative and $(R^2_\pm)=-a<0$. These properties are to be checked directly (for instance, for the type C the matrix Θ^{-1} is

$$\left(\begin{array}{ccc} -\frac{3}{4} & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -1 \end{array}\right),$$

as we need). It seems, however, that this is a consequence of some general fact.

Proof of Proposition 3.4 is complete.

Let us get back to the proof of Proposition 3.3.

3.5. The case $d\geq 1$: end of the proof of Proposition 3.3. We assume that $d\geq 1$. Let

$$\varphi_{i,i-1}\colon V_i\to V_{i-1},$$

$i=1,\dots,k+1$, be the resolution of the maximal singularity. Here $V_0=V$,

$$\varphi_{1,0}\colon V_1\to V$$

is the blow up of the curve C, that is, $\widetilde{V}\cong V_1$, and $B_1\subset E_1=E$ is the surface S. Setting, as usual,

$$\nu_i=\mathrm{mult}_{B_{i-1}}\Sigma^{i-1},$$

write down the Noether-Fano inequality:

$$\nu_1+\ldots+\nu_{k+1}>(k+2)n.$$

Consider a generic plane $P \supset L$ and a non-singular surface $Q = \sigma^{-1}(P)$. By the inequality $d \geq 1$ the section $\widetilde{P} \cap E_{\mathbb{P}}$ intersects transversally the surface $\sigma_E(S)$ at at least one point of general position, and the surface $\widetilde{P}$ intersects transversally $\sigma_E(S)$ at a point of general position. Therefore, the surface $Q^1 = \widetilde{Q}$ intersects transversally the surface S at at least one point of general position, say

$$x \in S \cap Q^1,$$

and we may assume that $x \notin C^1_-$.

Furthermore, by generality of P the restriction $\Sigma^1|_{Q^1}$ can have only one fixed component, the curve C^1_-. Set

$$\nu_- = \mathrm{mult}_{C^1_-}(\Sigma^1|_{Q^1}) = \mathrm{mult}_{C_-}(\Sigma|_Q) = \mathrm{mult}_{C_-}\Sigma.$$

Since the intersection $S \cap Q^1$ is transversal, the surfaces $B_2, \dots, B_k$ generate infinitely near base points

$$x_i \in B_i \cap Q^i$$

of the linear system of curves $\Sigma^1|_Q$ on the non-singular surface $Q^1 = Q$, lying over the point $x = x_1$. Since the point x lies outside the fixed component C_-, the self-intersection of the movable part of the linear system $\Sigma^1|_Q$ is not less than

$$\nu_2^2 + \ldots + \nu_{k+1}^2.$$

This gives the inequality

$$f(\nu_-, \nu_1, \dots, \nu_{k+1}) = (2nH_Q - \nu_1 C_+ - \nu_- C_-)^2 - \sum_{i=2}^{k+1} \nu_i^2 =$$

$$= 8n^2 - 4n\nu_1 - 4n\nu_- - 2\nu_1^2 - 2\nu_-^2 + 6\nu_1\nu_- - \sum_{i=2}^{k+1} \nu_i^2 \geq 0$$

(see the proof of Lemma 2.1 for the intersection numbers). Besides, as we have shown in the proof of Lemma 3.2, the inequality

$$\nu_- \geq \frac{3\nu_1 - 2n}{2} > \frac{n}{2}$$

holds. Now let us estimate the function $f(\nu_-, \nu_1, \dots, \nu_{k+1})$ from above. Let us replace the Noether-Fano inequality by the equality

$$\nu_1 + \nu_2 + \ldots + \nu_{k+1} = (k+2)n, \qquad (27)$$

which can only increase the value $f(\cdot)$. Furthermore, let us fix ν_- and ν_1 and consider f as a function of $\nu_2, \dots, \nu_{k+1}$ under the constraint (27). Obviously its maximum is attained at

$$\nu_2 = \ldots = \nu_{k+1} = \frac{(k+2)n - \nu_1}{k}.$$

On the other hand, the maximum of f as a function of one argument ν_- is attained, as it is easy to check, at

$$\nu_- = \frac{3\nu_1 - 2n}{2}.$$

Substituting these values of $\nu_2, \dots, \nu_{k+1}, \nu_-$ into $f(\cdot)$, we obtain the following expression:

$$\frac{1}{k}[-(k^2+4)n^2 + (4-2k)n\nu_1].$$

For $k \geq 2$ it is obviously negative. Let $k = 1$, then we have

$$-5n^2 + 2n\nu_1.$$

We have shown above that $\nu_1 \leq 2n$. Therefore, for any $k \geq 1$ we obtain the estimate

$$f(\nu_-, \nu_1, \dots, \nu_{k+1}) < -kn^2.$$

We get a contradiction which completes the proof of Proposition 3.3.

Q.E.D. for Proposition 3.1.

4 A local inequality for the self-intersection of a movable system

We give a proof of the so called $8n^2$-inequality for the self-intersection of a movable linear system, correcting the mistake in the papers [34-36]. The notations of this section are independent of the rest of the paper.

4.1. Set up of the problem and start of the proof. Let $o \in X$ be a germ of a smooth variety of dimension $\dim X \geq 4$. Let Σ be a movable linear system on X, and the effective cycle

$$Z = (D_1 \circ D_2),$$

where $D_1, D_2 \in \Sigma$ are generic divisors, its self-intersection. Blow up the point o:

$$\varphi\colon X^+ \to X,$$

$E = \varphi^{-1}(o) \cong \mathbb{P}^{\dim X - 1}$ is the exceptional divisor. The strict transform of the system Σ and the cycle Z on X^+ we denote by the symbols Σ^+ and Z^+, respectively.

Proposition 4.1 ($8n^2$-inequality). *Assume that the pair*

$$(X, \frac{1}{n}\Sigma)$$

is not canonical, but canonical outside the point o, where n is some positive number. There exists a linear subspace $P \subset E$ of codimension two (with respect to E), such that the inequality

$$\mathrm{mult}_o Z + \mathrm{mult}_P Z^+ > 8n^2$$

holds.

An equivalent claim, but formulated in a rather cumbersome way, was several times published by Cheltsov [34-36], however his proof is essentially faulty (see [37]).

Proof. The first part of our arguments follows [34,36]. Note that if $\operatorname{mult}_o Z > 8n^2$, then for P we may take any subspace of codimension two in E. However, if $\operatorname{mult}_o Z \leq 8n^2$, then the subspace P is uniquely determined: it follows easily from the connectedness principle of Shokurov and Kollár [29,38].

Restricting Σ onto a germ of a generic smooth subvariety, containing the point o, we may assume that $\dim X = 4$. Moreover, we may assume that $\nu = \operatorname{mult}_o \Sigma \leq 2\sqrt{2}n < 3n$, since otherwise

$$\operatorname{mult}_o Z \geq \nu^2 > 8n^2$$

and there is nothing to prove.

Lemma 4.1. *The pair*

$$(X^+, \frac{1}{n}\Sigma^+ + \frac{(\nu - 2n)}{n}E) \tag{28}$$

is not log canonical, and the centre of any of its non log canonical singularities is contained in the exceptional divisor E.

Proof. Let $\lambda\colon \widetilde{X} \to X$ be a resolution of singularities of the pair $(X, \frac{1}{n}\Sigma)$ and $E^* \subset \widetilde{X}$ a prime exceptional divisor, realizing a non-canonical singularity of that pair. Then $\lambda(E^*) = o$ and the Noether-Fano inequality holds:

$$\nu_{E^*}(\Sigma) > na(E^*).$$

For a generic divisor $D \in \Sigma$ we get $\varphi^* D = D^+ + \nu E$, so that

$$\nu_{E^*}(\Sigma) = \nu_{E^*}(\Sigma^+) + \nu \cdot \nu_{E^*}(E)$$

and

$$a(E^*, X) = a(E^*, X^+) + 3\nu_{E^*}(E).$$

From here we get

$$\nu_{E^*}\left(\frac{1}{n}\Sigma^+ + \frac{\nu - 2n}{n}E\right) = \nu_{E^*}\left(\frac{1}{n}\Sigma\right) - 2\nu_{E^*}(E) >$$

$$> a(E^*, X^+) + \nu_{E^*}(E) \geq a(E^*, X^+) + 1,$$

which proves the lemma.

Let $R \ni o$ be a generic three-dimensional germ, $R^+ \subset X^+$ its strict transform on the blow up of the point o. For a small $\varepsilon > 0$ the pair

$$\left(X^+, \frac{1}{1+\varepsilon}\frac{1}{n}\Sigma^+ + \frac{\nu - 2n}{n}E + R^+\right)$$

still satisfies the connectedness principle (with respect to the morphism $\varphi\colon X^+ \to X$), so that the set of centres of non log canonical singularities of that pair is connected. Since R^+ is a non log canonical singularity itself, we obtain, that there is a non log canonical singularity of the pair (28), the centre of which on X^+ is of positive dimension, since it intersects R^+.

Let $Y \subset E$ be a centre of a non log canonical singularity of the pair (28) that has the maximal dimension.

If $\dim Y = 2$, then consider a generic two-dimensional germ S, intersecting Y transversally at a point of general position. The restriction of the pair (28) onto S is not log canonical at that point, so that, applying Proposition 4.2, which is proven below, we see that

$$\mathrm{mult}_Y(D_1^+ \circ D_2^+) > 4\left(3 - \frac{\nu}{n}\right) n^2,$$

so that

$$\begin{aligned} \mathrm{mult}_o Z \geq \nu^2 + \mathrm{mult}_Y(D_1^+ \circ D_2^+) \deg Y > \\ > (\nu - 2n)^2 + 8n^2, \end{aligned}$$

which is what we need.

If $\dim Y = 1$, then, since the pair

$$\left(R^+, \frac{1}{1+\varepsilon}\frac{1}{n}\Sigma_R^+ + \frac{\nu - 2n}{n} E_R\right), \tag{29}$$

where $\Sigma_R^+ = \Sigma^+|_{R^+}$ and $E_R = E|_{R^+}$, satisfies the condition of the connectedness principle and R^+ intersects Y at $\deg Y$ distinct points, we conclude that $Y \subset E$ is a line in $\mathbb{P}^3$.

Now we need to distinguish between the following two cases: when $\nu \geq 2n$ and when $\nu < 2n$. The methods of proving the $8n^2$-inequality in these two cases are absolutely different. Consider first the case $\nu \geq 2n$.

Let us choose as $R \ni o$ a generic three-dimensional germ, satisfying the condition $R^+ \supset Y$. Since the pair (28) is effective (recall that $\nu \geq 2n$), one may apply inversion of adjunction [29, Chapter 17] and conclude that the pair (29) is not log canonical at Y.

Applying now to the pair (29) (where $R^+ \supset Y$) Proposition 4.2 in the same way as it was done for $\dim Y = 2$, we obtain the inequality

$$\mathrm{mult}_Y(D_1^+|_{R^+} \circ D_2^+|_{R^+}) > 4\left(3 - \frac{\nu}{n}\right) n^2.$$

On the left in brackets we have the self-intersection of the movable system Σ_R^+, which breaks into two natural components:

$$(D_1^+|_{R^+} \circ D_2^+|_{R^+}) = Z_R^+ + Z_R^{(1)},$$

where Z_R^+ is the strict transform of the cycle $Z_R = Z|_R$ on R^+ and the support of the cycle $Z_R^{(1)}$ is contained in E_R. The line Y is a component of the effective 1-cycle $Z_R^{(1)}$.

On the other hand, for the self-intersection of the movable linear system Σ^+ we get

$$(D_1^+ \circ D_2^+) = Z^+ + Z_1,$$

where the support of the cycle Z_1 is contained in E. From the genericity of R it follows that outside the line Y the cycles $Z_R^{(1)}$ and $Z_1|_{R^+}$ coincide, whereas for Y we get the equality

$$\mathrm{mult}_Y\, Z_R^{(1)} = \mathrm{mult}_Y\, Z^+ + \mathrm{mult}_Y\, Z_1.$$

However, $\mathrm{mult}_Y\, Z_1 \leq \deg Z_1$, so that

$$\begin{gathered}\mathrm{mult}_o\, Z + \mathrm{mult}_Y\, Z^+ = \\ = \nu^2 + \deg Z_1 + \mathrm{mult}_Y\, Z^+ \geq \\ \geq \nu^2 + \mathrm{mult}_Y\, Z_R^{(1)} > 8n^2,\end{gathered}$$

which is what we need. This completes the case $\nu \geq 2n$.

Note that the key point in this argument is that the pair (28) is effective. For $\nu < 2n$ inversion of adjunction can not be applied (as it was done in [39]). The additional arguments in [34-36], proving inversion of adjunction specially for this pair for $\nu < 2n$, are faulty, see [37].

4.2. The case $\nu < 2n$. Consider again the pair (29) for a generic germ $R \ni o$. Let $y = Y \cap R^+$ be the point of (transversal) intersection of the line Y and the variety R^+. Since $a(E_R, R) = 2$, the non log canonicity of the pair (29) at the point y implies the non log canonicity of the pair

$$\left(R, \frac{1}{n}\Sigma_R\right)$$

at the point o, whereas the centre of some non log canonical (that is, log maximal) singularity on R^+ is a point y.

Now the $8n^2$-inequality comes from the following fact.

Lemma 4.2. *The following inequality holds:*

$$\mathrm{mult}_o\, Z_R + \mathrm{mult}_y\, Z_R^+ > 8n^2,$$

where Z_R is the self-intersection of a movable linear system Σ_R and Z_R^+ is its strict transform on R^+.

Proof. Consider the resolution of the maximal singularity of the system Σ_R, the centre of which on R^+ is the point y:

$$\begin{array}{ccc} R_i & \stackrel{\psi_i}{\to} & R_{i-1} \\ \cup & & \cup \\ E_i & & B_{i-1}, \end{array}$$

where B_{i-1} is the centre of the singularity on R_{i-1}, $R_0 = R$, $R_1 = R^+$, $E_i = \psi_i^{-1}(B_{i-1})$ is the exceptional divisor, $B_0 = o$, $B_1 = y \in E_1$, $i = 1, \dots, N$, where the first L blow ups correspond to points, for $i \geq L+1$ curves are blown up. Since

$$\mathrm{mult}_o \Sigma_R = \mathrm{mult}_o \Sigma < 2n,$$

we get $L < N$, $B_L \subset E_L \cong \mathbb{P}^2$ is a line and for $i \geq L+1$

$$\deg[\psi_i|_{B_i} \colon B_i \to B_{i-1}] = 1,$$

that is, $B_i \subset E_i$ is a section of the ruled surface E_i.

Consider the graph of the sequence of blow ups ψ_i.

Lemma 4.3. *The vertices $L+1$ and $L-1$ are not connected by an arrow:*

$$L+1 \nrightarrow L-1.$$

Proof. Assume the converse: $L+1 \to L-1$. This means that

$$B_L = E_L \cap E_{L-1}^L$$

is the exceptional line on the surface E_{L-1}^L and the map

$$E_{L-1}^{L+1} \to E_{L-1}^L$$

is an isomorphism. As usual, set

$$\nu_i = \mathrm{mult}_{B_{i-1}} \Sigma_R^{i-1},$$

$i = 1, \dots, N$. Let us restrict the movable linear system Σ_R^{L+1} onto the surface E_{L-1}^{L+1} (that is, onto the plane $E_{L-1} \cong \mathbb{P}^2$ with the blown up point B_{L-1}). We obtain a non-empty (but, of course, not necessarily movable) linear system, which is a subsystem of the complete linear system

$$\left| \nu_{L-1}(-E_{L-1}|_{E_{L-1}}) - (\nu_L + \nu_{L+1})B_L \right|.$$

Since $(-E_{L-1}|_{E_{L-1}})$ is the class of a line on the plane E_{L-1}, this implies that

$$\nu_{L-1} \geq \nu_L + \nu_{L+1} > 2n,$$

so that the more so $\nu_1 = \nu > 2n$. A contradiction. Q.E.D. for the lemma.

Similarly, one can deduce from the inequality $\nu_1 \leq 2n$ that the upper part of the graph of the sequence of blow ups ψ_i, that is, the part, corresponding to the vertices

$$L+1, \dots, N,$$

is a chain. In other words, the section B_i of the ruled surface E_i, $i \geq L+1$, is different from the section $E_i \cap E_{i-1}^i$. Now set, as usual,

$$m_i = \mathrm{mult}_{B_{i-1}} (Z_R)^{i-1},$$

$i = 1, \ldots, L$, so that, in particular,

$$m_1 = \text{mult}_o Z_R \quad \text{and} \quad m_2 = \text{mult}_y Z_R^+.$$

For $1 \leq i \leq L-1$ set $p_i \geq 1$ to be the number of paths in the graph of the sequence of blow ups ψ_i from the vertex L to the vertex i, and

$$p_N = p_{N-1} = \ldots = p_L = 1$$

by definition. By the technique of counting multiplicities, see [1,40] and §5 below, we get the inequality

$$\sum_{i=1}^{L} p_i m_i \geq \sum_{i=1}^{N} p_i \nu_i^2. \tag{30}$$

Besides, the inequality

$$\sum_{i=1}^{N} p_i \nu_i > n \left(2 \sum_{i=1}^{L} p_i + \sum_{i=L+1}^{N} p_i \right) \tag{31}$$

holds. (In fact, a somewhat stronger inequality holds, the *log* Noether-Fano inequality, but we do not need that.) The inequality (31) follows from the Noether-Fano inequality for the maximal singularity E_N by the assumption that $\nu_1 \leq 2n$.

Indeed, let p_i^* be the number of paths in the graph of the sequence of blow ups $\{\psi_i\}$, going from the vertex N to the vertex i. By what we have proved,

$$p_N^* = p_{N-1}^* = \ldots = p_L^* = p_{L-1}^* = 1$$

and for $i \leq L$ we have $p_i \leq p_i^*$, since p_i is the number of paths from the vertex N to the vertex i, *passing through the vertex* L. The Noether-Fano inequality can be written down in the form

$$\sum_{i=L+1}^{N} p_i^* (\nu_i - n) > \sum_{i=1}^{L} p_i^* (2n - \nu_i),$$

where in the right hand side each summand is non-negative, since $2n \geq \nu_i$ for all $i = 1, \ldots, N$. It follows that

$$\sum_{i=L+1}^{N} p_i (\nu_i - n) > \sum_{i=1}^{L} p_i (2n - \nu_i)$$

(recall that $p_i = p_i^*$ for $i = L+1, \ldots, N$, so that the left hand side of the inequality remains unchanged whereas the right hand side can only get smaller), as we claimed. From the estimates (30, 31) one can get in the standard way [1,40] the inequality

$$\sum_{i=1}^{L} p_i m_i > \frac{(2\Sigma_0 + \Sigma_1)^2}{\Sigma_0 + \Sigma_1} n^2,$$

where $\Sigma_0 = \sum\limits_{i=1}^{L} p_i$ and $\Sigma_1 = \sum\limits_{i=L+1}^{N} p_i = N - L$. Taking into account that for $i \geq 2$ we get

$$m_i \leq m_2$$

and the obvious inequality $(2\Sigma_0 + \Sigma_1)^2 > 4\Sigma_0(\Sigma_0 + \Sigma_1)$, we obtain the following estimate

$$p_1 m_1 + (\Sigma_0 - p_1) m_2 > 4n^2 \Sigma_0.$$

Now assume that the claim of the lemma is false:

$$m_1 + m_2 \leq 8n^2.$$

Lemma 4.4. *The following inequality holds:* $\Sigma_0 \geq 2p_1$.

Proof. By construction, p_1 is the number of paths from the vertex L to the vertex 1. Marking in each path the last but one vertex, we get

$$p_1 = \sum_{L \geq i \to 1} p_i,$$

so that $p_1 \leq \Sigma_0 - p_1$, which is what we need. Q.E.D. for the lemma.

Now, taking into account that $m_2 \leq m_1$, we obtain

$$p_1 m_1 + (\Sigma_0 - p_1) m_2 = p_1(m_1 + m_2) + (\Sigma_0 - 2p_1) m_2 \leq$$

$$\leq 8 p_1 n^2 + (\Sigma_0 - 2p_1) \cdot 4n^2 = 4n^2 \Sigma_0.$$

This is a contradiction. Q.E.D. for Lemma 4.2.

Proof of Proposition 4.1 is complete.

Remark 4.1. Let us explain the key point in the proof of $8n^2$-inequality for $\nu_1 \leq 2n$. As it follows from the technique of counting multiplicities, the graph of the sequence of blow ups $\{\psi_i\}$ can be modified in such a way that all applications still hold, namely, one can delete all the arrows going from the vertices

$$L+1, \ldots, N$$

of the upper part of the graph to the vertices

$$1, \ldots, L-1$$

of the lower part (and both the Noether-Fano inequality and the estimate for the multiplicities of the self-intersection of the linear system are intact). The graph, modified in this way, satisfies the property of Lemma 4.4.

4.3. A local inequality for a surface. Let $o \in X$ be a germ of a smooth surface, $C \ni o$ a smooth curve and Σ a movable linear system on X. Let, furthermore,

$Z = (D_1 \circ D_2)$ be the self-intersection of the linear system Σ, that is, an effective 0-cycle. Since the situation is local, we may assume that the support of the cycle Z is the point o, that is,

$$\deg Z = (D_1 \cdot D_2)_o.$$

Proposition 4.2. *Assume that for some real number $a < 1$ the pair*

$$\left(X, \frac{1}{n}\Sigma + aC\right) \tag{32}$$

is not log canonical (that is, for a general divisor $D \in \Sigma$ the pair $(X, \frac{1}{n}D + aC)$) is not log canonical, where $n > 0$ is a positive number. Then the estimate holds

$$\deg Z > 4(1-a)n^2. \tag{33}$$

Proof. The original argument see in [31]. We will show that the inequality (33) follows directly from some well known facts on the infinitely near singularities of curves on a non-singular surface [32,33]. Assume that the sequence of blow ups

$$\varphi_{i,i-1} \colon X_i \to X_{i-1},$$

$i = 1, \dots, N$, where $X_0 = X$, resolves the non log canonical singularity of the pair (32). We use the standard notations and conventions: the centre of the blow up $\varphi_{i,i-1}$ is the point $x_{i-1} \in X_{i-1}$, its exceptional line is

$$E_i = \varphi_{i,i-1}^{-1}(x_{i-1}) \subset X_i,$$

the first point to be blown up is $o = x_0$, the blown up points x_i lie over each other: $x_i \in E_i$. The last exceptional line E_N realizes the non log canonical singularity of the pair (32), that is the log Noether-Fano inequality holds:

$$\sum_{i=1}^{N} \nu_i p_i + an \sum_{x_{i-1} \in C^{i-1}} p_i > n\left(\sum_{i=1}^{N} p_i + 1\right), \tag{34}$$

where $\nu_i = \operatorname{mult}_{x_{i-1}} \Sigma^{i-1}$, the symbols Σ^i and C^i stand for the strict transforms on X_i and p_i is the number of paths in the graph Γ of the constructed sequence of blow ups from the vertex E_N to the vertex E_i, see [1,40]. Assume that

$$x_{i-1} \in C^{i-1}$$

for $i = 1, \dots, k \leq N$, then the inequality (34) takes the form

$$\sum_{i=1}^{N} \nu_i p_i > n\left(\sum_{i=1}^{k} (1-a) p_i + \sum_{i=k+1}^{N} p_i + 1\right). \tag{35}$$

Lemma 4.5. *The following inequality holds:*

$$\deg Z \geq \sum_{i=1}^{N} \nu_i^2. \tag{36}$$

Proof: this is obvious.

Lemma 4.6. *For each $i \in \{1, \dots, N-1\}$ the estimate*

$$\nu_i \geq \sum_{j \to i} \nu_j \tag{37}$$

holds.

Proof. This is a very well known property of multiplicities of curves at infinitely near points on a non-singular surface.

Lemma 4.7. *The following estimate is true:*

$$\sum_{i=1}^{N} \nu_i^2 > \frac{\Delta^2}{q} n^2,$$

where

$$\Delta = 1 + (1-a)\sum_{i=1}^{k} p_i + \sum_{i=k+1}^{N} p_i$$

and $q = \sum\limits_{i=1}^{N} p_i^2$ (so that $n\Delta$ is the right-hand side of the inequality (35)).

Proof. The minimum of the quadratic form in the right-hand side of the inequality (36) under the restrictions (37) and

$$\sum_{i=1}^{N} \nu_i p_i = \Delta n \tag{38}$$

is attained at $\nu_i = p_i \theta$, where $\theta = \frac{\Delta n}{q}$ is computed from (38). Q.E.D. for the lemma.

Now the claim of Proposition 4.2 follows from a purely combinatorial fact about the graph Γ, which we will now prove.

Lemma 4.8. *Assume that the starting segment of the graph Γ with the vertices $1, \dots, k$ is a chain. Then the estimate*

$$\Delta^2 \geq 4(1-a)q \tag{39}$$

holds.

Proof will be given by induction on the number N of vertices of the graph Γ. If $N = 1$, then the inequality (39) holds in a trivial way:

$$(2-a)^2 \geq 4(1-a).$$

Consider the inequality (39) as a claim on the non-negativity of a quadratic function of the argument a:

$$a^2\left(\sum_{i=1}^{k}p_i\right)^2+2a\left(2q-\left(\sum_{i=1}^{k}p_i\right)\left(\sum_{i=1}^{N}p_i+1\right)\right)+\left(\left(\sum_{i=1}^{N}p_i+1\right)^2-4q\right)\geq 0$$

on the interval $a\leq 1$. Since for $a\to\pm\infty$ this function is positive, it is sufficient to check that its minimum is non-negative. Elementary computations show that, up to an inessential positive factor, this minimum is given by the formula

$$\left(\sum_{i=1}^{k}p_i\right)\left(\sum_{i=k+1}^{N}p_i+1\right)-\sum_{i=1}^{N}p_i^2. \tag{40}$$

Non-negativity of the latter expression we will prove by induction on the number of vertices N. Recall that the only assumption, restricting the choice of the number $k\geq 1$, is that there are no arrows $i\to j$ for $i\geq j+2$ and $i\leq k$.

Consider first the case $k=1$. Assume that $l\geq 1$ vertices are connected by arrows with 1, that is,

$$2\to 1,\ldots,l+1\to 1,\quad \text{but}\quad l+2\nrightarrow 1.$$

In this case $p_1=p_2+\ldots+p_{l+1}$ and the subgraph of the graph Γ with the vertices $\{2,\ldots,l+1\}$ either consists of one vertex or is a chain. The expression (40) transforms to the formula

$$\left(\sum_{i=2}^{l+1}p_i\right)\left(\sum_{i=l+2}^{N}p_i+1\right)-\sum_{i=2}^{N}p_i^2,$$

so that one can apply the induction hypothesis to the subgraph with the vertices $\{2,\ldots,N\}$. This completes the case $k=1$.

Now let $k\geq 2$. The following key fact is true.

Lemma 4.9. *The following inequality holds:*

$$p_i\leq \sum_{j=i+2}^{N}p_i+1. \tag{41}$$

Proof: this is Lemma 1.6 in [33].

By the lemma that we have just proved, we get the inequality

$$p_1=p_2=\ldots=p_{k-1}\leq \sum_{i=k+1}^{N}p_i+1.$$

For this reason, for $k\geq 2$ the expression (40) is bounded from below by the number

$$\left(\sum_{i=2}^{k}p_i\right)\left(\sum_{i=k+1}^{N}p_i+1\right)-\sum_{i=2}^{N}p_i^2.$$

Now, applying the induction hypothesis to the subgraph with the vertices $\{2,\dots,N\}$ we complete the proof of Lemma 4.8 and Proposition 4.2.

5 The technique of counting multiplicities

In this section we give a stronger version of the technique of counting multiplicities for the self-intersection of a movable linear system. We obtain the result that, together with the $8n^2$-inequality, forms the technical basis for the exclusion of maximal singularities, the centre of which is of codimension ≥ 4. The notations in this section are independent of other parts of this paper.

5.1. Set up of the problem. Let $o \in X$ be a germ of a smooth three-dimensional variety, $\varphi\colon \widetilde{X} \to X$ a birational morphism, $E \subset \widetilde{X}$ an irreducible exceptional divisor over the point o, that is, $\varphi(E) = o$. Consider the resolution [1,40] of the discrete valuation ν_E, that is, the sequence of blow ups

$$\varphi_{i,i-1} : X_i \to X_{i-1},$$

$i = 1,\dots,N$, where $X_0 = X, \varphi_{i,i-1}$ blows up an irreducible subvariety $B_{i-1} \subset X_{i-1}$ (a point or a curve),

$$E_i = \varphi_{i,i-1}^{-1}(B_{i-1}) \subset X_i$$

is the exceptional divisor, where B_i is uniquely defined by the conditions $B_0 = o$ and for $i = 1,\dots,N-1$

$$B_i = \mathrm{centre}(E, X_i),$$

and, finally, the geometric discrete valuations

$$\nu_E \quad \text{and} \quad \nu_{E_N}$$

of the field of rational functions of the variety X coincide. Geometrically this means that the birational map

$$\varphi_{N,0}^{-1} \circ \varphi\colon \widetilde{X} \dashrightarrow X_N$$

is biregular at the generic point of the divisor E and maps E onto E_N (see the details in [1,40]). Here

$$\varphi_{N,0} = \varphi_{1,0} \circ \ldots \circ \varphi_{N,N-1}\colon X_N \to X_0,$$

and more generally set for $i > j$

$$\varphi_{i,j} = \varphi_{j+1,j} \circ \ldots \circ \varphi_{i,i-1}\colon X_i \to X_j.$$

The strict transform of an irreducible subvariety (by linearity, also of an effective cycle) $Y \subset X_j$ on X_i we denote, as usual, by adding the upper index i: we write Y^i.

Assume that for $i = 1,\dots,L \leq N$ the centres B_{i-1} of blow ups are points, for $i \geq L+1$ they are curves. Let Γ be the graph of the constructed resolution, that is, an oriented graph with the vertices $1, \ldots, N$, where an oriented edge (arrow) joins i and j for $i > j$ (notation: $i \to j$), if and only if

$$B_{i-1} \subset E_j^{i-1}.$$

In particular, by construction always $i+1 \to i$.

Let us describe the obvious combinatorial properties of the graph Γ.

Lemma 5.1. *Let $i < j < k$ be three distinct vertices. If $k \to i$, then $j \to i$.*

Proof. By definition, $k \to i$ means that $B_{k-1} \subset E_i^{k-1}$. By construction of the resolution of singularities, we get

$$\varphi_{k-1,j-1}(B_{k-1}) = B_{j-1}$$

(the centres of blow ups with higher numbers cover the centres of previous blow ups) and, besides, obviously

$$\varphi_{k-1,j-1}(E_i^{k-1}) = E_i^{j-1}.$$

From here the lemma follows directly. Q.E.D.

Definition 5.1. We say that the vertex i of the graph Γ is of *class* $e \geq 1$ (notation: $\varepsilon(i) = e$), if precisely e arrows come out of it, that is,

$$\sharp\{j | i \to j\} = e.$$

We say, furthermore, that the graph Γ is of class $e \geq 1$, if for each vertex i we have $\varepsilon(i) \leq e$.

For instance, a graph of class 1 is a chain:

$$1 \longleftarrow 2 \longleftarrow \ldots \longleftarrow N$$

(no other arrows but $i+1 \to i$). The graph of a sequence of blow ups of points on a non-singular surface is of class 2.

Lemma 5.2. *The graph of the resolution of the valuation ν_E is of class 3. If for some vertex i we have $\varepsilon(i) = 3$, then $i \leq L$, that is, B_{i-1} is a point.*

Proof. By definition,

$$B_{i-1} \subset E(i) = \bigcup_{\{j|i \to j\}} E_j^{i-1},$$

and moreover, at the general point B_{i-1} is a smooth variety and $E(i)$ is a normal crossings divisor, each component of which contains B_{i-1}. Q.E.D. for the lemma.

Remark 5.1. Word for word the same arguments show that in the case of arbitrary dimension $\dim X$ the graph of the resolution of any valuation is of class at most $\dim X$ and if

$$\sharp\{j | i \to j\} = a,$$

then $\operatorname{codim} B_{i-1} \geq a$.

Let Σ be a germ of a movable (that is, free from fixed components) linear system on X, Σ^i its strict transform on X_i,

$$\nu_i = \mathrm{mult}_{B_{i-1}} \Sigma^{i-1},$$

so that for a general divisor $D \in \Sigma$ we have

$$D^i = D - \sum_{j=1}^{i} \nu_j E_j,$$

where we write D instead of $\varphi^*_{i,0} D$ and similarly for the exceptional divisors E_i.

Consider a pair of general divisors $D_1, D_2 \in \Sigma$ and construct the *self-intersections* of linear systems Σ^i, which are (not uniquely determined) effective 1-cycles

$$Z_i = (D^i_1 \circ D^i_2)$$

on X_i. These cycles admit the natural decomposition

$$\begin{array}{lcl} Z_1 & = & Z^1_0 + Z_{1,1}, \\ Z_2 & = & Z^2_1 + Z_{2,2} = Z^2_0 + Z_{1,2} + Z_{2,2}, \\ & \ldots & \\ Z_i & = & Z^i_{i-1} + Z_{i,i} = Z^i_0 + Z_{1,i} + \ldots + Z_{i,i}, \end{array}$$

where $Z_{a,i} = (Z_{a,i-1})^i = \ldots = Z^i_{a,a}$, $i = 1, \ldots, L$.

Definition 5.2. A function $a\colon \{1, \ldots, L\} \to \mathbb{Z}_+$ is said to be *compatible with the graph structure* Γ, if the inequalities

$$a(i) \geq \sum_{j \to i} a(j)$$

hold.

By construction, $B_L \subset E_L \cong \mathbb{P}^2$ is a plane curve. Set $\beta_L = \deg B_L$ to be its degree in $\mathbb{P}^2$ and for an arbitrary $i \geq L+1$

$$\beta_i = \beta_L \deg[\varphi_{i,L}|_{B_i}\colon B_i \to B_L]. \tag{42}$$

The main computational tool of the theory of birational rigidity is the folowing local fact.

Proposition 5.1. *For any function $a(\cdot)$, compatible with the graph structure, the inequality*

$$\sum_{i=1}^{L} a(i) m_i \geq \sum_{i=1}^{L} a(i) \nu_i^2 + a(L) \sum_{i=L+1}^{N} \beta_i \nu_i^2 \tag{43}$$

holds, where $m_i = \mathrm{mult}_{B_{i-1}} Z^{i-1}_0$, $i = 1, \ldots L$.

Proof is given in [1,40].

The aim of this section is to prove a stronger estimate that includes (43) as a particular case.

Definition 5.3. A vertex $i \in \{4, \ldots, L\}$ of the graph Γ is said to be *complex*, if precisely three arrows come out of this vertex, that is, for three distinct vertices $i_1 < i_2 < i_3$ we have

$$i \to i_1,\ i \to i_2,\ i \to i_3.$$

Note that in the notations of Definition 5.3, by Lemma 5.1 we always have

$$i_2 \to i_1, \quad i_3 \to i_1, \quad \text{and} \quad i_3 \to i_2.$$

If in the graph Γ there are no complex vertices, then it is of class ≤ 2.

Definition 5.4. A *simplification* of the graph Γ is the oriented graph Γ^* of class ≤ 2 with the set of vertices

$$1, \ldots, L,$$

the arrows in which join the vertices in accordance with the following rule: if

$$\sharp\{j | i \to j\} \leq 2,$$

then $i \to j$ in Γ^* if and only if $i \to j$ in Γ; however, if i is a complex vertex, then in the notations of Definition 5.3 in Γ^* there are two arrows coming out ofthe vertex i,

$$i \to i_2 \quad \text{and} \quad i \to i_3,$$

that is, the arrow $i \to i_1$ is deleted.

Thus the graph Γ^* is obtained from Γ by means of deleting some arrows (and the vertices that correspond to the blow ups of curves).

Proposition 5.2. *For any function $a\colon \{1, \ldots, L\} \to \mathbb{Z}_+$, compatible with the structure of the graph Γ^*, the inequality (43) holds.*

5.2. Proof of the improved inequality. If the function $a(\cdot)$ is compatible with the structure of Γ, then, the more so, it is compatible with Γ^*, so that Proposition 5.2 implies Proposition 5.1. Following the general scheme of the proof of the inequality (43) in [1,40], set for $i = 1, \ldots, L$

$$d_i = \deg Z_{i,i},$$

where $Z_{i,i} \subset E_i \cong \mathbb{P}^2$ is a plane curve and for $i < j \leq L$

$$m_{i,j} = \mathrm{mult}_{B_{j-1}} Z_{i,j-1}.$$

We obtain the following system of equalities

$$\begin{array}{rcl} \nu_1^2 + d_1 & = & m_1, \\ \nu_2^2 + d_2 & = & m_2 + m_{1,2}, \\ & \cdots & \\ \nu_i^2 + d_i & = & m_i + m_{1,i} + \ldots + m_{i-1,i}, \\ & \cdots & \\ \nu_L^2 + d_L & = & m_L + m_{1,L} + \ldots + m_{L-1,L}. \end{array} \tag{44}$$

Besides, we get the obvious inequality

$$d_L \geq \sum_{i=L+1}^{N} \beta_i \nu_i^2,$$

where the numbers β_i are defined by the formula (42). Let us multiply the i-th equality in (44) by $a(i)$ and put together the resulting equalities. In the left hand side we get

$$\sum_{i=1}^{L} a(i)\nu_i^2 + \sum_{i=1}^{L} a(i) d_i.$$

In the right hand side we get

$$\sum_{i=1}^{L} a(i) m_i + \sum_{i=1}^{L-1} \left(\sum_{j=i+1}^{L} a(j) m_{i,j} \right).$$

Thus Proposition 5.2 follows immediately from the following claim.

Lemma 5.3. *For any $i = 1, \ldots, L-1$ the inequality*

$$a(i) d_i \geq \sum_{j=i+1}^{L} a(j) m_{i,j} \tag{45}$$

holds.

Proof. Up to this moment our argument repeated word for word the corresponding arguments in [1,40]. If the function $a(\cdot)$ is compatible with the structure of the graph Γ (and not Γ^*), then, taking into account that the inequality $m_{i,j} > 0$ is possible only if $j \to i$, and that always

$$d_i \geq m_{i,j},$$

we obtain (45) from the inequality $a(i) \geq \sum\limits_{j \to i} a(j)$ (this proves Proposition 5.1). However, this is not sufficient for the proof of Proposition 5.2, since the fucntion $a(\cdot)$ is compatible with the structure of the graph Γ^* only, and the latter has, generally speaking, less arrows than Γ does.

To prove (45), recall, first of all, that the integer-valued functions

$$d_i = \deg Z_{i,i} \qquad m_{i,j} = \mathrm{mult}_{B_{j-1}} Z_{i,j-1}$$

are linear functions of effective 1-cycles $Z_{i,i}$ on the exceptional plane $E_i \cong \mathbb{P}^2$. Since the inequality (45) is also linear, the claim of Lemma 5.3 follows from a simpler fact.

Lemma 5.4. *For any irreducible curve $C \subset E_i$ the following inequality holds:*

$$a(i) \deg C \geq \sum_{j=i+1}^{L} a(j)\, \mathrm{mult}_{B_{j-1}} C^{j-1}. \tag{46}$$

Proof. Set

$$d = \deg C,\ \mu_j = \mathrm{mult}_{B_{j-1}} C^{j-1},\ j = i+1, \ldots, L.$$

As we pointed out above, if $\mu_j > 0$, then $j \to i$, so that it is necessary to prove the inequality

$$a(i)d \geq \sum_{j \to i} a(j)\mu_j.$$

This inequality is a claim about singularities of plane curves. Let us consider two cases:

1) $C \subset E_i$ is a line in $E_i \cong \mathbb{P}^2$,

2) C is a curve of degree $d \geq 2$.

In the case 1) define the integer $k \geq 1$ by the condition

$$\{j | B_{j-1} \in C^{j-1}\} = \{i+1, \ldots, i+k\}. \tag{47}$$

In order to distinguish between the arrows in the graphs Γ and Γ^*, we write $a \stackrel{*}{\to} b$, if the vertices a, b are joined by an arrow in Γ^*, leaving the usual arrow for Γ.

The following fact is of key importance.

Lemma 5.5. *For each e, $1 \leq e \leq k$, we have:*

$$i + e \stackrel{*}{\to} i.$$

Proof. Assume that $(i+e)$ is a complex vertex of the graph Γ (otherwise $i + e \stackrel{*}{\to} i$ by definition). Recall that the simplification procedure removes, of the three arrows, coming out of $i+e$, the one that goes to the *lowest* vertex. Therefore, we may assume that $e \geq 3$. However, the points

$$B_i, B_{i+1}, B_{i+2}, \ldots, B_{i+k-1}$$

lie on the strict transform of the smooth curve C, and therefore the subgraph

$$i+1 \longleftarrow i+2 \longleftarrow i+3 \longleftarrow \ldots \longleftarrow 1+k$$

is a chain, that is, between the vertices $i+1, \ldots, i+k$ in the original graph Γ there are no arrows, except for the consecutive. In any case there are two arrows coming out of the vertex $i+e$:

$$i+e \to i+e-1 \quad \text{and} \quad i+e \to i.$$

What has been said implies that if a third arrow comes out the vertex $i+e$, that is, $i+e \to j$, then inevitably

$$j \leq i-1.$$

It is this arrow that the simplification procedure deletes. Therefore, the arrow $i+e \to i$ will not be deleted. Q.E.D. for the lemma.

Let is come back to the case 1). The inequality (46) takes the form of the estimate

$$a(i) \geq \sum_{j=i+1}^{i+k} a(j). \tag{48}$$

By the lemma we have just proved, (48) is true, because the function $a(\cdot)$ is compatible with the structure of the graph Γ^*. Q.E.D. for Lemma 5.4 in the case 1).

Let us consider the case 2). Again let us define $k \geq 1$ by the condition (47). If $k=1$, then there is nothing to prove, since

$$i+1 \stackrel{*}{\to} i$$

and $\mu_j \leq d$ for any j. For this reason we assume that $k \geq 2$.

Lemma 5.6. *The inequality $d \geq \mu_1 + \mu_2$ holds.*

Proof. Let $L \subset E_i$ be the line, passing through the point B_i in the direction of the infinitely near point $B_{i+1} \in E_i^{i+1}$. By assumption, $C \neq L$. Then for the intersection number on the surface E_i^{i+2} we get

$$0 \leq (C^{i+2} \cdot L^{i+2}) = d - \mu_1 - \mu_2,$$

which is what we need. Q.E.D. for the lemma.

Let Γ_C be the subgraph of the graph Γ with the vertices $\{i+1, \dots, i+k\}$ (and the same arrows), Γ_C^* the subgraph of the graph Γ^* with the same set of vertices. By definition of the simplification procedure, the arrow

$$i+a \to i+b, \quad a > b \leq 1,$$

can not be deleted, because in Γ there is an arrow that goes to a lower vertex:

$$i+a \to i. \tag{49}$$

This implies, that $\Gamma_C = \Gamma_C^*$. Furthermore, the arrow (49) is deleted by the simplification procedure if and only if $i+a$ is a complex vertex and the arrow (49) is the lowest. In the language of the graph Γ_C this means that the vertex $i+a$ is of class 2 (as a vertex of that graph). The following claim is obvious.

Lemma 5.7. *The integer-valued function*

$$i+a \longmapsto \mu_{i+a} \in \mathbb{Z}_+$$

is compatible with the structure of the graph Γ_C.

Proof: this is a well known property of multiplicities of a curve at infinitely near points (on a non-singular surface).

By what has been said, the proof of the inequality (46) in the case 2) is reduced to the following combinatorial fact. Let Δ be a subgraph of class 2 with the set of vertices $\{1,\dots,k\}$ and $\varepsilon_\Delta(\cdot) \in \{0,1,2\}$ the function of the class of a vertex ($\varepsilon_\Delta(1)=0$). Let $\mu(\cdot)$ and $a(\cdot)$ be $\mathbb{Z}_+$-valued functions, compatible with the structure of the graph Δ.

Lemma 5.8. *The following inequality holds:*

$$(\mu(1)+\mu(2)) \sum_{\varepsilon_\Delta(j)\le 1} a(j) \ge \sum_{j=1}^{k} \mu(j)a(j). \tag{50}$$

Proof is given by induction on the number of vertices $k \ge 2$. If $k=2$, then $\varepsilon_\Delta(1)=0$, $\varepsilon_\Delta(2)=1$ and the inequality (50) is of the form

$$(\mu(1)+\mu(2))(a(1)+a(2)) \ge \mu(1)a(1)+\mu(2)a(2),$$

so that there is nothing to prove.

Assume that $k \ge 3$ and the vertices 3 and 1 are not joined by an arrow: $3 \not\to 1$, that is, $\varepsilon_\Delta(3)=1$. In that case we apply the induction hypothesis to the graph Δ_1 with the vertices $\{2,\dots,k\}$ and the same arrows as in Δ: for that graph the inequality (50) takes the form

$$(\mu(2)+\mu(3)) \sum_{\varepsilon_\Delta(j)=1} a(j) \ge \sum_{j=2}^{k} \mu(j)a(j),$$

whence, taking into account the inequality $\mu(1) \ge \mu(2) \ge \mu(3)$, we obtain the required inequality (50) for Δ.

Assume that $k \ge 3$ and there are arrows

$$3 \to 1, \dots, 2+l \to 1,$$

where $l \ge 1$. In that case let us write down the left hand side of (50) as

$$(\mu(1)+\mu(2)) \left(a(1)+a(2)+ \sum_{\substack{j \ge l+3,\\ \varepsilon_\Delta(j)=1}} a(j) \right),$$

and the right hand side of (50) as

$$\mu(1)a(1)+\mu(2)a(2)+\sum_{j=3}^{l+2}\mu(j)a(j)+\sum_{j=l+3}^{k}\mu(j)a(j).$$

Since the functions $\mu(\cdot)$ and $a(\cdot)$ are compatible with the structure of the graph, we get the inequality

$$\mu(2)a(1) \ge \mu(2)(a(2)+a(3)+\ldots+a(l+2))$$

and the symmetric inequality for $\mu(1)a(2)$. Applying the induction hypothesis to the subgraph Δ_{l+3} with the vertices $\{l+3,\dots,k\}$, we complete the proof of the lemma.

Q.E.D. for Proposition 5.2.

5.3. Counting multiplicities of the self-intersection for a non log canonical singularity. Let $o \in X$ be a non-singular three-dimensional germ, Σ a movable linear system, such that

$$\operatorname{mult}_o \Sigma \leq 2n,$$

but the point o is an isolated centre of non log canonical singularities of the pair

$$\left(X, \frac{1}{n}\Sigma\right), \tag{51}$$

where $n > 0$ is some number. Let $Z = (D_1 \circ D_2)$ be the self-intersection of the system Σ (an effective 1-cycle on X). Let

$$\varphi_{i,i-1}\colon X_i \to X_{i-1},$$

$i = 1,\dots,K$, be the sequence of blow ups of the centres of some non log canonical singularity of the pair (51), $X_0 = X$, $\varphi_{i,i-1}$ blows up an irreducible subvariety $B_{i-1} \subset X_{i-1}$, a point or a curve, $B_0 = o$,

$$E_i = \varphi^{-1}_{i,i-1}(B_{i-1}) \subset X_i$$

is the exceptional divisor, finally, $E_K \subset X_K$ realizes a non log canonical singularity of the pair (51), that is, the log Noether-Fano inequality

$$\sum_{i=1}^{K} p_{Ki}\nu_i > n\left(\sum_{i=1}^{K} p_{Ki}\delta_i + 1\right) \tag{52}$$

holds, where $\delta_i = \operatorname{codim} B_{i-1} - 1$, p_{K_i} is the number of paths from E_K to E_i and, as usual,

$$\nu_i = \operatorname{mult}_{B_{i-1}} \Sigma^{i-1} \leq 2n,$$

Σ^i is the strict transform of the movable system Σ on X_i. Set

$$\{1,\dots,L\} = \{j \mid \dim B_{j-1} = 0\}.$$

Since by assumption $\nu_i \leq 2n$, (52) implies that $K \geq L+1$, that is, among the centres of blow ups there is at least one curve.

Denote by the symbol Γ the oriented graph of the sequence of blow ups $\varphi_{i,i-1}$, $i = 1,\dots,K$, and by the symbol Γ^* its simplification, a graph of class ≤ 2. The number of paths from the vertex i to the vertex j, $i > j$, in the graph Γ^* we denote by the symbol p^*_{ij}. By definition, $p^*_{ii} = 1$. Set also

$$m_i = \operatorname{mult}_{B_{i-1}} Z^{i-1},$$

$i = 1, \ldots, L$. The following fact improves the classical $4n^2$-inequality for the case of a non *log* canonical singularity of the pair (51).

Proposition 5.3. *The following estimate holds:*

$$\sum_{i=1}^{L} p^*_{Li} m_i > 4n^2 \left(\sum_{i=1}^{L} p^*_{Li} + 1 \right).$$

Proof. Set

$$N = \min \left\{ e \,\middle|\, \sum_{i=1}^{e} p_{ei}(\nu_i - \delta_i n) > 0 \right\},$$

that is, E_N is the non canonical singularity of the pair (51) with the minimal number. In particular, the pair (51) is canonical at $E_1, \ldots, E_{N-1}$. It follows easily from the inequality $\nu_1 \leq 2n$ (see §4), that the segment of the graph Γ with the vertices

$$L-1, L, L+1, \ldots, N$$

is a chain. Define the number a, $0 \leq a \leq 1$, by the equality

$$a = \frac{1}{n} \sum_{i=1}^{N-1} p_{Ni}(\delta_i n - \nu_i)$$

(the numbers of paths p_{N_i} and $p_{N-1,i}$ coincide by what was said above). If $a = 1$, then it is easy to see that

$$\nu_1 = \ldots = \nu_N = 2n,$$

whence the claim of the proposition follows directly. For that reason, we assume that $a < 1$.

First assume that $N = K$. In that case the technique of counting multiplicities together with the inequality (52) gives:

$$\sum_{i=1}^{L} p^*_{Li} m_i > \frac{(2\Sigma^*_0 + \Sigma^*_1 + 1)^2}{\Sigma^*_0 + \Sigma^*_1} n^2 = 4(\Sigma^*_0 + 1)n^2 + \frac{(\Sigma^*_1 - 1)^2}{\Sigma^*_0 + \Sigma^*_1} n^2,$$

where

$$\Sigma^*_{2-\alpha} = \sum_{\delta_i = \alpha} p^*_{Ni}, \quad \alpha = 1, 2,$$

which was required (note that the log Noether-Fano inequality can get only stronger when we replace p_{Ki} by p^*_{Ki}, since those coefficients are changed only for $i = 1, \ldots, L-1$ and $\nu_i \leq 2n$ for these values of i, this is a well known fact).

Thus for $N = K$ Proposition 5.3 holds.

Assume that $K \geq N + 1$.

Lemma 5.9. *The pair*

$$(X_{N-1}, \frac{1}{n}\Sigma^{N-1} - aE_{N-1})$$

is not log canonical at the curve $B_{N-1}\subset E_{N-1}$.

Proof. Taking into account the standard properties of the numbers p_{ij}, rewrite the inequality (52) in the form

$$\left(\sum_{\alpha\to N-1} p_{K\alpha}\right)\left(\sum_{i=1}^{N-1} p_{Ni}(\nu_i-\delta_i n)\right)+\sum_{i=N}^{K} p_{Ki}(\nu_i-\delta_i n)>n$$

(obviously, for $i\leq N-1$ the equality $p_{Ni}=p_{N-1,i}$ holds). Now the claim of the lemma follows directly from the definition of the number a and the fact that $N\to N-1$. Q.E.D. for the lemma.

Now setting

$$\Sigma_0^*=\sum_{i=1}^{L} p^*_{Li},\quad \Sigma_1^*=\sum_{i=L+1}^{N-1} p^*_{Ni},$$

by Propositions 4.2 and 5.2 we get

$$\sum_{i=1}^{L} p^*_{Li}m_i>\sum_{i=1}^{N-1} p^*_{Ni}\nu_i^2+4(1+a)n^2,$$

whence, taking into account the Noether-Fano inequality, we get

$$\sum_{i=1}^{L} p^*_{Li}m_i>\left[\frac{(2\Sigma_0^*+\Sigma_1^*-a)^2}{\Sigma_0^*+\Sigma_1^*}+4(1+a)\right]n^2=4(\Sigma_0^*+1)n^2+\frac{(\Sigma_1^*+a)^2}{\Sigma_0^*+\Sigma_1^*}n^2. \quad (53)$$

Q.E.D. for Proposition 5.3.

6 Exclusion of infinitely near maximal singularities

In this section we complete the proof of Proposition 0.3: under the assumption that the system Σ has no maximal subvariety of the form $\sigma^{-1}(P)$, where $P\subset\mathbb{P}$ is a linear subspace of codimension two, we prove that Σ has no maximal singularities with the centre B of codimension ≥ 4.

6.1. The centre of the singularity is not contained in the ramification divisor. By Propositions 2.1 and 3.1 we may assume that the centre B of the maximal singularity of the linear system Σ is of codimension ≥ 4 (and the same is true for any other maximal singularity). Let $P\subset\mathbb{P}$ be a generic linear subspace of dimension $\operatorname{codim} B$ and $o\in\sigma^{-1}(P)\cap B$ is some point. Set $V_P=\sigma^{-1}(P)$.

Consider first the case when $\sigma(B)\not\subset W$. In this case $p=\sigma(o)\notin W$. The variety V_P is smooth,

$$\sigma_P=\sigma|_{V_P}\colon V_P\to P$$

is the double cover, branched over $W_P = W \cap P$,

$$\operatorname{Pic} V_P = \mathbb{Z} H_P,$$

where $H_P = H|_{V_P}$. Let Σ_P be the restriction of the system Σ onto V_P. This is a movable linear system and the pair

$$(V_P, \frac{1}{n}\Sigma_P)$$

is not log canonical, where the point o is an isolated centre of a non log canonical singularity. Let

$$\varphi\colon V_P^+ \to V_P$$

be the blow up of the point o, $E = \varphi^{-1}(o)$ the exceptional divisor. Set

$$Z_P = (D_1 \circ D_2)$$

to be the self-intersection of the linear system Σ_P and Z_P^+ its strict transform on V_P^+. By Proposition 4.1, for some plane $\Pi \subset E$ of codimension two the inequality

$$\operatorname{mult}_o Z_P + \operatorname{mult}_\Pi Z_P^+ > 8n^2 \tag{54}$$

holds. Now let

$$\varphi_P\colon P^+ \to P$$

be the blow up of the point $p = \sigma(o)$ and $E_P = \varphi_P^{-1}(p)$ the exceptional divisor, E identifies naturally with E_P. Let $\Lambda \subset P$ be the unique plane of codimension two, containing the point p and cutting out Π on $E_P = E$:

$$\Lambda^+ \cap E_P = \Pi,$$

where $\Lambda^+ \subset P^+$ is the strict transform. The subvariety $Q = \sigma_P^{-1}(\Lambda) \subset V_P$ is irreducible, of codimension two (with respect to V_P), and moreover,

$$\deg Q = 2, \quad \operatorname{mult}_o Q = \operatorname{mult}_\Pi Q^+ = 1, \tag{55}$$

where $Q^+ \subset V_P^+$ is the strict transform. Since the cycle Z_P satisfies the inequality

$$\operatorname{mult}_o Z_P + \operatorname{mult}_\Pi Z_P^+ > \deg Z_P = 8n^2,$$

writing down

$$Z_P = aQ + Z_P^\sharp,$$

where $a \in \mathbb{Z}_+$ and $Z_P^\sharp$ does not contain Q as a component, we obtain

$$\operatorname{mult}_o Z_P^\sharp + \operatorname{mult}_\Pi (Z_P^\sharp)^+ > \deg Z_P^\sharp,$$

$(Z_P^\sharp)^+$ is the strict transform. Finally, let R be the σ-preimage of a generic hyperplane in P, containing the point p and cutting out Π on $E_P = E$, that is,

$$\sigma(R)^+ \cap E_P = \Pi.$$

The divisor R contains no component of the effective cycle $Z_P^\sharp$, so that for the scheme-theoretic intersection

$$Z_R^\sharp = (Z_P^\sharp \circ R)$$

we obtain the inequality

$$\mathrm{mult}_o\, Z_R^\sharp \geq \mathrm{mult}_o\, Z_P^\sharp + \mathrm{mult}_\Pi (Z_P^\sharp)^+ > \deg Z_R^\sharp,$$

which is impossible. This contradiction proves Proposition 0.3 in the case when $\sigma(B) \not\subset W$.

6.2. The centre of the singularity is contained in the ramification divisor: the simple case. Consider, finally, the last case when $\sigma(B) \subset W$. Once again, we work on the variety $V_P = \sigma^{-1}(P)$, the linear system Σ_P is movable and the point o is an isolated centre of log maximal singularity of this linear system. Set $p = \sigma(o)$. For the blow ups

$$\varphi\colon V_P^+ \to V_P \quad \text{and} \quad \varphi\colon P^+ \to P$$

of the points o and p, respectively, with the exceptional divisors

$$E = \varphi^{-1}(o) \quad \text{and} \quad E_P = \varphi_P^{-1}(p),$$

the double cover $\sigma_P\colon V_P \to P$ does not extend to an isomorphism of the exceptional divisors E and E_P. Let $T_P = T_p W_P$ be the tangent hyperplane to the branch divisor $W_P = W \cap P$ at the point p, $\mathbb{T}_P$ the corresponding hyperplane in E_P. It is easy to see that there exist a hyperplane $\mathbb{T} \subset E$ and a point $\xi \in E \backslash \mathbb{T}$ such that σ_P induces an isomorphism of $\mathbb{T}$ and $\mathbb{T}_P$ and the rational map

$$\sigma_E\colon E \dashrightarrow E_P$$

is the composition of the projection $\mathrm{pr}_\xi\colon E \dashrightarrow \mathbb{T}$ from the point ξ and the isomorphism $\mathbb{T} \cong \mathbb{T}_P$. In particular, $\sigma_E(E) = \mathbb{T}_P$ (all these facts are easy to check in suitable local coordinates $z_1, \dots, z_k$ at the point p on P, in which V_P is given by the local equation $y^2 = z_1$).

Let, as above, Z_P and Z_P^+ be the self-intersection of the linear system Σ_P and its strict transform on V_P^+, respectively. Let $\Pi \subset E$ be the plane of codimension two, satisfying the inequality (54),

$$\Pi_P = \sigma_E(\Pi) \subset \mathbb{T}_P \subset E_P$$

the image of the plane Π. Obviously, Π_P is a linear subspace in $\mathbb{T}_P$ of codimension 1 or 2. In the latter case for a generic hyperplane $R \ni p$, $R \subset P$, such that $R^+ \supset \Pi_P$, we get: none of the components of the effective cycle $(\sigma_P)_* Z_P$ of codimension two is not contained in R. By the inequality (54) we get

$$\mathrm{mult}_o(\sigma^{-1}(R) \circ Z_P) \geq \mathrm{mult}_o\, Z_P + \mathrm{mult}_\Pi\, Z_P^+ >$$

$$> 8n^2 = \deg Z_P = \deg(\sigma^{-1}(R) \circ Z_P),$$

which is impossible. Therefore, Π_P is a hyperplane in $\mathbb{T}_P$. Let $\Lambda \subset P$ be the unique plane of codimension two, such that $\Lambda \ni p$ and $\Lambda^+ \cap E_P = \Pi_P$. The subvariety $Q = \sigma_P^{-1}(\Lambda)$ is irreducible. If Q^+ does not contain Π, then we get

$$\mathrm{mult}_o\, Q = 2, \quad \mathrm{mult}_\Pi\, Q^+ = 0,$$

so that $\mathrm{mult}_o\, Q + \mathrm{mult}_\Pi\, Q^+ = \deg Q$ and we may argue as above. Recall that by the conditions of general position (and taking into account the genericity of the subspace P), the point $o \in Q$ is an isolated quadratic singularity. If the rank of that singularity is at least 3, then the quadric $Q^+ \cap E$ is irreducible (and reduced), so that $\Pi \not\subset Q^+$. An easy dimension count (similar to the one carried out below in Sec. 7.3) shows that for $M \geq 8$ for a sufficiently general hypersurface W, a general subspace P and a general subspace $\Lambda \subset P$, $\Lambda \ni p$ of codimension two in P, where $p \in P \cap W$ is an arbitrary point, the rank of the corresponding quadratic singularity $o \in Q$ is at least 3.

However, for $M \leq 7$ the possibility $\Pi \subset Q^+$ takes place for any hypersurface W. Now we can not argue in the same way as in the case when the point o does not lie on the ramification divisor: let $\Lambda \subset P$ be the unique plane of codimension two, such that $\Lambda \ni p$ and $\Lambda^+ \cap E_P = \Pi_P$. The subvariety $Q = \sigma_P^{-1}(\Lambda)$ is irreducible, however,

$$\mathrm{mult}_o\, Q = 2, \quad \mathrm{mult}_\Pi\, Q^+ = 1,$$

so that the arguments, similar to the case when $\sigma(o) \notin W$, do not work. To exclude this case, we need the improved technique of counting multiplicities (§5). We complete the proof in all details for $M \geq 6$.

6.3. The centre of the singularity is contained in the ramification divisor: the hard case. The following claim is true.

Lemma 6.1. *We have the inequality* $\nu = \mathrm{mult}_o \Sigma \leq 2n$.

Proof. Consider the divisor $T = \sigma^{-1}(T_p W)$, where $T_p W \subset \mathbb{P}$ is the tangent hyperplane. The system Σ is movable, so that the effective cycle $(D \circ T)$ is well defined, where $D \in \Sigma$ is a general divisor. Now we get

$$2\nu \leq \mathrm{mult}_o(D \circ T) \leq \deg(D \circ T) = \deg D = 4n,$$

as we claimed.

Let $\Delta \subset P$, $\Delta \ni p$, be a generic 3-plane, so that $V_\Delta = \sigma^{-1}(\Delta)$ is a smooth variety, $\sigma_\Delta \colon V_\Delta \to \Delta$ a double cover, Σ_Δ the restriction of the system Σ onto V_Δ. The pair

$$(V_\Delta, \frac{1}{n}\Sigma_\Delta) \tag{56}$$

is not log canonical at the point o, and, moreover, o is an isolated centre of non log canonical singularities of this pair.

Set $C = Q \cap V_\Delta = \sigma^{-1}(L)$, where $L = \Delta \cap \Lambda$ is the line, passing through the point p and tangent to W at that point (for the definition of the plane Λ, see above in the end of Sec. 6.2). Set also

$$y = \Pi \cap V_\Delta^+,$$

where V_Δ^+ is the strict transform of V_Δ on V^+, that is,

$$\varphi_\Delta = \varphi|_{V_\Delta^+}\colon V_\Delta^+ \to V_\Delta$$

is the blow up of the point o with the exceptional plane E_Δ, $y = \Pi \cap E_\Delta$. There is a non log canonical singularity of the pair (56), the centre of which on V_Δ^+ is the point y.

Consider the self-intersection

$$Z_\Delta = (D_1 \circ D_2) = Z|_{V_\Delta}$$

of the movable linear system Σ_Δ and write down

$$Z_\Delta = bC + Z_1,$$

where $b \in \mathbb{Z}_+$, the effective 1-cycle Z_1 does not contain the curve C as a component and for this reason satisfies the inequality

$$\mathrm{mult}_o Z_1 + \mathrm{mult}_y Z_1^+ \leq \deg Z_1 = 8n^2 - 2b. \tag{57}$$

Assume now that at the point o the curve C has two distinct branches:

$$C^+ \cap E_\Delta = \{y, y^*\},$$

where $y, y^* \in E_\Delta$ are distinct points. This assumption is justified for $M \geq 6$ by the conditions of general position, since for any subspace $\mathcal{U} \subset \mathbb{P}$ of codimension two, $\mathcal{U} \ni p$, the quadratic point $o \in \sigma^{-1}(\mathcal{U})$ is of rank at least two, see Proposition 0.6, (iii), so that the same is true for the quadratic point $o \in \sigma^{-1}(L) = C$, either, since $L = \mathcal{U} \cap \Delta$, where Δ is a generic 3-plane, containing the point p. For $M = 5$ we must consider also the case when $o \in C$ is a simple cuspidal singularity, see below.

In the notations of Sec. 5.3 let

$$\varphi_{i,i-1}\colon X_i \to X_{i-1},$$

$i = 1, \dots, N$, $X_0 = V_\Delta$, be the resolution of the non log canonical singularity, the centre of which on $X_1 = V_\Delta^+$ is the point $B_1 = y$. Set

$$\{1, \dots, k\} = \{i \,|\, 1 \leq i \leq L, B_{i-1} \in C^{i-1}\}.$$

By the assumption about the branches of the curve C, we get $k \geq 2$, and, moreover, the subgraph with the vertices $1, \dots, k$ is a chain.

Note that $b \geq 1$: otherwise the inequality

$$\mathrm{mult}_o\, Z_\Delta + \mathrm{mult}_y\, Z_\Delta^+ \leq \deg Z_\Delta = 8n^2$$

holds and one can argue in word for word the same way as for $\sigma(o) \notin W$.

We have

$$p^*_{L1} = \ldots = p^*_{L,k-1}.$$

Set, furthermore,

$$\mu_i = \mathrm{mult}_{B_{i-1}}\, Z_1^{i-1}$$

for $i = 1, \ldots, L$. By the inequality (57), we get the estimate

$$\begin{array}{c} \sum\limits_{i=1}^{L} p^*_{Li} m_i = b \left(\sum\limits_{i=1}^{k} p^*_{Li} + p^*_{L1} \right) + \sum\limits_{i=1}^{L} p^*_{Li}\mu_i \leq \\ \leq b \left(\sum\limits_{i=1}^{k} p^*_{Li} + p^*_{L1} \right) + \frac{1}{2} \deg Z_1 \sum\limits_{i=1}^{L} p^*_{Li}. \end{array} \tag{58}$$

Lemma 6.2. *The following estimate holds*

$$p^*_{L1} = \ldots = p^*_{L,k-1} \leq 1 + \sum_{i=k+1}^{L} p^*_{Li}$$

Proof. For $L \geq i > j \geq 1$ denote by the symbol $\mathcal{P}^*_{i,j}$ the set of paths in the graph Γ^* going from the vertex i to the vertex j. We say that an arrow $i \stackrel{*}{\to} j$ is a *jump*, if $i \geq j+2$. A path $\pi \in \mathcal{P}^*_{i,j}$ is said to be *simple*, if it contains no jumps, that is, it goes subsequently through all vertices

$$i \stackrel{*}{\to} i-1 \stackrel{*}{\to} \ldots \stackrel{*}{\to} a \stackrel{*}{\to} a-1 \stackrel{*}{\to} \ldots \stackrel{*}{\to} j.$$

It is obvious that each set $\mathcal{P}^*_{i,j}$ contains precisely one simple path. Denote it by the symbol σ_{ij}.

If the set $\mathcal{P}^*_{i,j} \setminus \{\sigma_{ij}\}$ is non-empty, then every path $\pi \in \mathcal{P}^*_{i,j} \setminus \{\sigma_{ij}\}$ has at least one jump. Let

$$l(\pi) \stackrel{*}{\to} q(\pi)$$

be the *last* jump in the path π (that is, the jump from the vertex $l(\pi)$ with the *smallest* number). Since the graph Γ^* is of class ≤ 2, the jump $a \stackrel{*}{\to} b$ is uniquely determined by the vertex a (since there are precisely two arrows coming out of the vertex a, and one of them is the arrow $a \stackrel{*}{\to} a-1$). Therefore, the path π is uniquely determined after the vertex $l(\pi)$: first, there is the uniquely determined jump to the vertex $q(\pi)$, and after that vertex (if $q(\pi) \neq j$) the path π is simple.

Denote by the symbol $\mathcal{J}^*_{i,j} \subset \{j+2, \ldots, i\}$ the set of such indices l, where $j+2 \leq l \leq i$, that from the vertex l there is a jump $l \stackrel{*}{\to} a \geq j$. It follows from what was said that, associating to every path $\pi \in \mathcal{P}^*_{i,j} \setminus \{\sigma_{ij}\}$ the vertex $l(\pi)$ of the last jump, we get a one-to-one correspondence between the sets

$$\mathcal{P}^*_{i,j} \setminus \{\sigma_{ij}\} \quad \text{and} \quad \coprod_{l \in \mathcal{J}^*_{i,j}} \mathcal{P}^*_{i,l}$$

(where the symbol $\coprod$ means the disjoint union). Therefore,

$$p^*_{ij} = 1 + \sum_{l \in \mathcal{J}^*_{i,j}} p^*_{il} \leq 1 + \sum_{l \geq j+2} p^*_{il},$$

since for $l \in \mathcal{J}^*_{i,j}$ in any case $j + 2 \leq l$. Q.E.D. for the lemma.

Note that the argument above gives a new proof of Lemma 4.9 and, therefore, of the $4n^2$-inequality (see [33]).

By Lemma 6.2, the right hand side of the inequality (58) is bounded from above by the number

$$b + (b + \frac{1}{2} \deg Z_1) \sum_{i=1}^{L} p^*_{Li} = b + 4n^2 \sum_{i=1}^{L} p^*_{Li}.$$

Now Proposition 5.3 implies the estimate

$$b > 4n^2,$$

which is impossible. This contradiction completes the proof of Proposition 0.3 for $M \geq 6$.

For $M = 5$ to complete the proof it remains to consider the case when the curve C has a simple cuspidal singularity at the point o, so that C^+ is tangent to the exceptional divisor E at the point y (the tangency is simple). If $L = 2$, then the previous arguments work. If $L \geq 3$ and $B_2 \in C^2$, then the vertices 3 and 1 are joined by an arrow, $3 \to 1$, so that

$$p^*_{L1} \geq p^*_{L2} + p^*_{L3}$$

and the input of the first vertex of the graph into the sum $\sum\limits_{i=1}^{L} p^*_{Li} m_i$ is not compensated by the number

$$b \left(1 + \sum_{i=k+1}^{L} p^*_{Li} \right),$$

as above. However, the previous estimates can be improved in the following way. Set

$$\mu = \mathrm{mult}_C\, \Sigma \leq n$$

(if $\mu > n$, then the system Σ has a maximal subvariety of codimension two, which is what we need). Let us restrict Σ onto the σ-preimage S of a generic 2-plane in $\mathbb{P}$, containing the line L. Obviously, $\Sigma_S = \Sigma|_S$ is a non-empty linear system of curves with a single fixed component C of multiplicity μ. For a generic curve $G \in \Sigma_S$ we get the inequality

$$((G-\mu C)\cdot C) \geq 2\,\mathrm{mult}_o(G-\mu C) + \sum_{i=2}^{k} \mathrm{mult}_{B_{i-1}}(G-\mu C),$$

where

$$\{2,\dots,k\} = \{i \mid B_{i-1} \in C^{i-1}\},$$

$k \geq 3$. There is a standard technique (used in [41, §8], also in [12,15,32]), which makes it possible to derive from this estimate and the Noether-Fano inequality, that the "upper" sum Σ_1^* is high compared with the "lower" one Σ_0^*, whence a considerable improvement of the quadratic inequality (53) is obtained. That improved inequality is already sufficient to exclude the cuspidal case. We omit the details; they will be given in another paper. This completes the proof of Proposition 0.3 for $M = 5$.

7 Double spaces of general position

In this section we prove Propositions 0.4-0.6.

7.1. Lines on the variety V. Let us prove Proposition 0.4. The non-trivial part of that claim is that through every point there are at most finitely many lines; the fact that any (not necessarily generic) double space of index two is swept out by lines, is almost obvious. It is easy to see that the image $L = \sigma(C)$ of a line $C \subset V$ on $\mathbb{P}$ is a line in the usual sense and

$$\sigma|_C \colon C \to L \subset \mathbb{P}$$

is an isomorphism. Thus there are two possible cases: either $L \not\subset W$, so that $\sigma^{-1}(L) = C \cup C^*$ is a part of smooth rational curves (permuted by the Galois involution of the double cover σ), or $L \subset W$ is contained entirely in the branch divisor, that is, $\sigma^{-1}(L) = C$. The converse is also true: if a line $L \subset \mathbb{P}$ is such that the curve $\sigma^{-1}(L)$ is reducible or $L \subset W$, then $\sigma^{-1}(L)$ consists of two or one lines on V, respectively. An easy dimension count shows that on a generic hypersurface in $\mathbb{P}$ of degree $2(M-1)$ there are no lines, so that the second option does not take place. Furthermore, the double cover $\sigma^{-1}(L) \to L$ is reducible if and only if the divisor $W \mid_L$ on $L = \mathbb{P}^1$ is divisible by 2, that is,

$$\frac{1}{2}(W \mid_L) \in \mathrm{Div}\, L$$

is an integral divisor. Thus Proposition 0.4 follows immediately from the following fact.

Proposition 7.1. *For a generic smooth hypersurface $W \subset \mathbb{P}$ of degree $2(M-1)$ through every point $x \in \mathbb{P}$ there are finitely many lines L such that $W \mid_L \in 2\,\mathrm{Div}\, L$.*

Proof. Let us denote by the symbol $\mathcal{P}_k(\mathbb{P}^l)$ the space of homogeneous polynomials of degree k on the projective space $\mathbb{P}^l$ (that is, $H^0(\mathbb{P}^l, \mathcal{O}_{\mathbb{P}^l}(k))$), considered as an affine algebraic variety of dimension $\binom{k+l}{l}$. Let

$$\begin{array}{rccc} \mathrm{sq}\colon & \mathcal{P}_k(\mathbb{P}^l) & \to & \mathcal{P}_{2k}(\mathbb{P}^l), \\ \mathrm{sq}\colon & f & \mapsto & f^2 \end{array}$$

be the map of taking the square. Its image

$$\mathrm{sq}(\mathcal{P}_k(\mathbb{P}^l)) \subset \mathcal{P}_{2k}(\mathbb{P}^l)$$

will be denoted by the symbol $[\mathcal{P}_k(\mathbb{P}^l)]^2$. Consider the space of pairs

$$\Pi = \mathbb{P} \times \mathcal{P}_{2(M-1)}(\mathbb{P})$$

and set $\Pi(x) = \{x\} \times \mathcal{P}_{2(M-1)}(\mathbb{P})$ for an arbitrary point $x \in \mathbb{P}$. Set

$$Y(x) \subset \Pi(x)$$

to be the closed algebraic subset of pairs (x, F), $F \in \mathcal{P}_{2(M-1)}(\mathbb{P})$, defined by the condition

(+) the set of lines $L \subset \mathbb{P}$, $L \ni x$, for which $F\,|\,_L \in [\mathcal{P}_{M-1}(L)]^2$, is of positive dimension.

It is easy to see that the closure

$$\overline{\bigcup_{x \in \mathbb{P}} Y(x)} \subset \Pi$$

is a closed algebraic subset of dimension $\leq M + \dim Y(x)$. Therefore, Proposition 7.1, in its turn, is implied by the following fact.

Proposition 7.2. *The codimension of the closed set $Y(x)$ in $\Pi(x) \cong \mathcal{P}_{2(M-1)}(\mathbb{P})$ is at least $M+1$.*

In fact, as we will see from the proof, a much stronger estimate for the codimension of the set $Y(x)$ holds. In particular, the claim of Proposition 7.1 remains true for double spaces of index two with elementary singularities (quadratic points). However, we do not need it here.

Proof of Proposition 7.2. Let $z_1, \ldots, z_M$ be a system of affine coordinates on $\mathbb{P}$ with the origin at the point $x = (0, \ldots, 0)$. We write the polynomial $F \in \mathcal{P}_{2(M-1)}(\mathbb{P})$ in the form

$$F = q_0 + q_1(z_1, \ldots, z_M) + \ldots + q_{2(M-1)}(z_1, \ldots, z_M),$$

where $q_i(z_*)$ is a homogeneous polynomial of degree i. The line $L \ni x$ corresponds to a set of homogeneous coordinates

$$(a_1 : \ldots : a_M) \in \mathbb{P}^{M-1},$$

$L = \{t(a_1,\dots,a_M)\,|\,t \in \mathbb{C}\}$. Obviously, $F|_L \in [\mathcal{P}_{M-1}(L)]^2$ if and only if the polynomial

$$q_0 + tq_1(a_*) + \ldots + t^{2(M-1)}q_{2(M-1)}(a_*) \in \mathbb{C}[t]$$

is a full square in $\mathbb{C}[t]$.

For each $k = 0,1,\dots,2(M-1)$ we define the set $Y_k(x) \subset \Pi(x)$ by the condition

$+_k$ there exists an irreducible closed subset $Z \subset \mathbb{P}^{M-1}$ of positive dimension such that for a general line $L \in Z$ we have $F_L \in [\mathcal{P}_{M-1}(L)]^2$, and moreover, $F|_L$ has a zero of order k at the point $x \in L$.

(Recall that we identify the points $\mathbb{P}^{M-1}$ with the lines in $\mathbb{P}$, passing through the point x.) Obviously, for an odd $k \notin 2\mathbb{Z}$ we have $Y_k = \emptyset$ and

$$Y(x) = \bigcap_{i=0}^{M-1} Y_{2i}(x).$$

Therefore, it is sufficient to prove the estimate of Proposition 7.2 for each of the (constructive) sets $Y_{2i}(x)$, $i = 0,\dots,M-1$. Consider first the set $Y_0(X)$ (corresponding to the lines on V, passing through the point outside the branch divisor). For $F \in Y_0(x)$ we have $q_0 \neq 0$ and we may assume that $q_0 = 1$.

Lemma 7.1. *For any $m \geq 1$ there exists a set of quasi-homogeneous polynomials*

$$A_{m,i}(s_1,\dots,s_m) \in \mathbb{Q}[s_1,\dots,s_m]$$

of degree $\deg A_{m,i} = i \in \{m+1,\dots,2m\}$, where the weight of the variable s_j is $\mathrm{wt}(s_j) = j$, such that the polynomial

$$1 + b_1 t + \ldots + b_{2m}t^{2m}$$

is a full square in $\mathbb{Q}[t]$ for $b_1,\dots,b_{2m} \in \mathbb{C}$ if and only if the following system of equalities is satisfied:

$$b_i = A_{m,i}(b_1,\dots,b_m),$$

$i = m+1,\dots,2m$.

Proof. Consider the equality

$$1 + s_1 t + \ldots + s_{2m}t^{2m} = (1 + r_1 t + \ldots + r_m t^m)^2.$$

Equating the coefficients at the same powers of t, we find r_i as polynomials in $s_1,\dots,s_i$ for $i \leq m$ (with coefficients in $\mathbb{Z}[\frac{1}{2}]$). The equality of coefficients at $t^{m+1},\dots,t^{2m}$, gives the required system of equations. Q.E.D.

By the lemma, the restriction of F onto the line $\{t(a_*)\}$ is a full square if and only if the system of equations

$$q_i(a_*) = A_{M-1,i}(q_1(a_*),\dots,q_{M-1}(a_*)), \tag{59}$$

$i = M,\dots,2(M-1)$, is satisfied. This is a system of $(M-1)$ polynomial homogeneous equations in $(a_1 : \ldots : a_M)$ of degrees $M,\dots,$ $2(M-1)$, respectively (which, in particular, implies immediately that through every point $x \in V$ there are at least two lines, and in the case of general position through $x \in V$ there are

$$2 \cdot (M \cdot (M+1) \cdot \ldots \cdot 2(M-1)) = 2\frac{(2M-2)!}{(M-1)!}$$

lines). As can be seen from (59), the coefficients of the right hand side depend polynomially on the coefficients of the polynomials $q_1, \ldots, q_{M-1}$. Therefore, $Y_0(x)$ consists of such polynomials $F \in \mathcal{P}_{2(M-1)}(\mathbb{P})$, for which the system (59) defines an algebraic set of positive dimension.

Now the codimension $\operatorname{codim} Y_0(x)$ can be estimated by the method of [4]. Assuming the polynomials $q_1, \ldots, q_{M-1}$ to be fixed, we obtain the equality

$$\operatorname{codim} Y_0(x) = \operatorname{codim} Y_0^*(x),$$

where the closed set $Y_0^* \subset \mathcal{P}_M(\mathbb{P}^{M-1}) \times \ldots \times \mathcal{P}_{2(M-1)}(\mathbb{P}^{M-1})$ (which consists of sets $q^*_M, \ldots, q^*_{2(M-1)}$ of homogeneous polynomials of the corresponding degrees) is defined by the condition: the system of equations

$$q^*_M = \ldots = q^*_{2(M-1)} = 0$$

has a positive-dimensional set of solutions. Repeating the proof of Lemma 3.3 in [1] (this argument is well known, it was applied and published many times), define the subsets $Y^*_{0,j} \subset Y^*_0$ for $j = M, \ldots, 2(M-1)$, fixing the first "incorrect" codimension:

$$Y^*_{0,j} = \{(q^*_M, \ldots, q^*_{2(M-1)}) | \operatorname{codim}\{q^*_M = \ldots = q^*_j = 0\} = j - M\},$$

so that

$$Y^*_0 = \bigvee_{j=M}^{2(M-1)} Y^*_{0,j},$$

where the symbol $\bigvee$ stands for a disjoint union (for instance, $Y^*_{0,M}$ consists of the sets (q^*_*) with $q^*_M \equiv 0$). For $(q^*_M, \ldots, q^*_{2(M-1)}) \in Y^*_{0,j}$ there is an irreducible component

$$B \subset \{q^*_M = \ldots = q^*_{j-1} = 0\}$$

of codimension precisely $j - M$, on which q^*_j vanishes identically. Let

$$\pi \colon B \to \mathbb{P}^{\dim B} \subset \mathbb{P}^{M-1}$$

be a generic linear projection onto a generic $\dim B$-dimensional plane. Since the π-pull back of a non-zero homogeneous polynomial on $\mathbb{P}^{\dim B}$ does not vanish on B identically, we get the estimate

$$\operatorname{codim} Y^*_{0,j} \geq \dim \mathcal{P}_j(\mathbb{P}^{2M-1-j}) = \binom{2M-1}{j}$$

(since $\dim B = 2M-1-j$, $j \in \{M,\dots,2M-2\}$), so that

$$\operatorname{codim} Y_0^* \geq \min \left\{ \binom{2M-1}{j} \;\middle|\; j = M,\dots,2M-2 \right\} = 2M-1.$$

Thus $\operatorname{codim} Y_0(x) \geq 2M-1 \geq M+1$, as required.

Let us consider now the problem of estimating the codimension of the set $Y_k(x)$, $k = 2e \geq 2$. For $F \in Y_k(x)$ there exists a set $Z_F \subset \mathbb{P}^{M-1}$ of positive dimension, on which identically vanish the polynomials

$$q_0, q_1(z_*), \dots, q_{k-1}(z_*)$$

and for a point of general position $(a_1 : \dots : a_M) \in Z_F$ the polynomial

$$t^k q_k(a_*) + \dots + t^{2(M-1)} q_{2(M-1)}(a_*) \in \mathbb{C}[t]$$

is a full square, and moreover, $q_k(a_*) \neq 0$. Applying Lemma 7.1, we obtain the system of equalities

$$\frac{q_i(a_*)}{q_k(a_*)} = A_{M-e-1,i}\left(\frac{q_{k+1}(a_*)}{q_k(a_*)}, \dots, \frac{q_{M+e-1}(a_*)}{q_k(a_*)} \right),$$

$i = M-e, \dots, 2(M-e-1)$, or, after multiplying by $q_k(a_*)^i$,

$$q_i(a_*) q_k(a_*)^{i-1} = A^+_{M-e-1,i}(q_k(a_*), q_{k+1}(a_*), \dots, q_{M+e-1}(a_*)),$$

$i = M-e, \dots, 2(M-e-1)$, where $A^+(\cdot)$ is the appropriately modified polynomial. We obtain a system of $M+e-1$ homogeneous equations on $\mathbb{P}^{M-1}$, and the set $Y_k(x)$ consists of those polynomials F, for which that system has a set of solutions of positive dimension, on which q_k does not vanish identically. Here q_0 is a non-zero constant by definition of the set $Y_k(x)$ for $k = 2e \geq 2$.

Now we argue as above: we assume the polynomials $q_k, \dots, q_{M+e-1}$ to be fixed, so that the codimension of the set $Y_k(x)$ is the codimension of the subset

$$Y_k^* \subset \mathbb{C} \times \mathcal{P}_1(\mathbb{P}^{M-1}) \times \dots \times \mathcal{P}_{k-1}(\mathbb{P}^{M-1}) \times \mathcal{P}_{M+e}(\mathbb{P}^{M-1}) \times \dots \times \mathcal{P}_{2(M-1)}(\mathbb{P}^{M-1}),$$

defined by the condition: the sequence

$$(q_0, q_1, \dots, q_{k-1}, q_{M+e}, \dots, q_{2(M-1)}) \in Y_k^*$$

if and only if $q_0 = 0$ and the system of equations

$$q_1 = \dots = q_{k-1} = q_{M+e} = \dots = q_{2(M-1)} = 0$$

defines an algebraic set that has a component of positive dimension, on which $q_k \not\equiv 0$. Now we may forget about the latter condition.

Now we estimate the codimension of the set Y_k^* by the method of [4] (see [1]) in precisely the same way, as it was done above for $k = 0$: we fix the first "incorrect" codimension, when the next polynomial q_i in the sequence that was written out above vanishes on an irreducible component, defined by the previous equations (which is of "correct" codimension). We get the worst estimate at the first step: the condition $q_1 \equiv 0$ (together with the condition $q_0 = 0$) gives the codimension

$$\operatorname{codim} Y_2^* = M + 1,$$

and this estimate is optimal. Indeed, $q_1 \equiv 0$ means that the branch divisor is singular at the point x, and then through this point there is a one-dimensional family of lines. In all other cases the estimate for $\operatorname{codim} Y_k^*$ is considerably stronger (we omit the elementary computations). Q.E.D. for Proposition 0.4.

7.2. Isolated singular points. The proof of Proposition 0.5 is elementary and we just point out its main steps. Assume that for some subspace $P \subset \mathbb{P}$ of codimension two the intersection $P \cap W$ has a whole curve C of singular points. It is convenient to consider the pair $p \in C$, where p is an arbitrary point, so that in any case

$$P \subset T_p W.$$

There is a $(2M-3)$-dimensional family of pairs $(p, P \ni p)$, satisfying this condition. It is sufficient to show that the number of independent condition, which are imposed on the (non-homogeneous) polynomial

$$f|_P = f(z_1, \dots, z_{M-2})$$

of degree $2M - 2$ by the condition that the hypersurface $\{f|_P = 0\} = W \cap P$ contains a curve C of singular points, passing through $p = (0, \dots, 0)$, is at least $2M - 2$. There are three possible cases:

— C is a line,

— C is a plane curve, $C \subset \Lambda \subset P$, where Λ is some 2-plane,

— the linear span of the curve C is a k-plane, where $k \geq 3$.

In the first case one can compute the number of independent conditions precisely, this is an elementary exercise.

In the second case the plane curve $\{f|_\Lambda = 0\}$ has an irreducible component C of degree ≥ 2 and multiplicity ≥ 2, which gives an estimate from below for the number of independent conditions (which is essentially stronger than we need).

In the third case we choose on the curve C $3(M-1)$ points in general position (neither three lie on a line and neither four lie in the same plane). It is easy to check (considering hypersurfaces that are unions of singular quadrics), that being singular at these points imposes on f independent conditions, which completes the proof of Proposition 0.5. (In fact, the codimension of the set of hypersurfaces with a whole curve of singular points is much higher, but we do not need that.) The details are left to the reader.

7.3. The rank of quadratic singularities. Let us prove Proposition 0.6. We will show the claim (i). The claims (ii) and (iii) are proved in a similar way. It is easy to check that the planes $P \subset \mathbb{P}$ of codimension two that are tangent to W at at least one point (that is, $\operatorname{Sing} P \cap W \neq \emptyset$) form a $(2M-3)$-dimensional family. So it is sufficient to prove the following fact.

Let $P \subset \mathbb{P}$ be a fixed plane of codimension two, $p \in P$ a fixed point,

$$\mathcal{W} = \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2M-2)))$$

the space of hypersurfaces of degree $2M-2$. Let us define the subset $\mathcal{W}_P \subset \mathcal{W}$ by the conditions:

- a hypersurface $W \in \mathcal{W}_P$ is non-singular at the point p,
- the tangent hyperplane T_pW contains P,
- the point p is an isolated singular point of the intersection $P \cap W$.

For $W \in \mathcal{W}_P$ set $\sigma_W \colon V_W \to \mathbb{P}$ to be the double cover, branched over W, $o = \sigma_W^{-1}(p)$ is a singular point, $R = \sigma_W^{-1}(P)$,

$$\varphi \colon V_W^+ \to V_W$$

the blow up of the subvariety R. On the variety V_W^+ there is a unique singular point $o^+ \in \varphi^{-1}(o)$. Let us define the closed subset $Y \subset W_P$ by the condition that for $W \in Y$ the rank of the quadratic singularity $o^+ \in V_W^+$ is at most 3. Now Proposition 0.6, (i) follows immediately from the estimate

$$\operatorname{codim}(Y \subset W_P) \geq 2M-2 \tag{60}$$

for $M \geq 6$.

The proof of the inequality (60) is obtained by simple local computations which we will just describe. Let $(z_1, \ldots, z_M)$ be affine coordinates at the point p, where the plane P is defined by the system of equations $z_1 = z_2 = 0$, and the tangent hyperplane to W is $z_1 = 0$. The local equation of the double cover V_W at the point $o = \sigma_W^{-1}(p)$ is of the form

$$y^2 = z_1 + q_2(z_1, \ldots, z_M) + q_3(z_*) + \ldots\ ,$$

and the local equation of the blow up V_W^+ at the point o^+ is of the form

$$u^2 = u_1u_2 + q_2(0, u_2, \ldots, u_M) + \ldots\ .$$

This implies that the condition that the rank of the quadratic point o^+ is at most $M+1-k$, imposes

$$\frac{k(k+1)}{2}$$

independent conditions on the coefficients of the equation of the hypersurface W. If $M+1-k \leq 3$, then we get at least

$$\frac{(M-2)(M-1)}{2}$$

independent conditions, which for $M \geq 6$ is strictly higher than $2M-3$. Q.E.D. for Proposition 0.6, (i). The claims (ii) and (iii) are shown in a similar way.

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