Automorphisms of Stein tubes of the form $T^r M$ and complexification of the isometry group action on the tangent bundle

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AUTOMORPHISMS OF STEIN TUBES OF THE FORM T^rM AND COMPLEXIFICATION OF THE ISOMETRY GROUP ACTION ON THE TANGENT BUNDLE

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0. Introduction.

Assume that M is a real analytic manifold. Then every coordinate patch $U \subset \mathbb{R}^n$ can be thickened to obtain an open set $\mathbb{C}U \subset \mathbb{C}^n$. Since the coordinate changes of M are real analytic maps, they can be extended holomorphically to such enlarged domains (by taking power series expansions and by possible shrinking $\mathbb{C}U$ to get convergence) and thus they can be used as holomorphic transition functions for a complex manifold $\mathbb{C}M$.

If one does this procedure carefully, the obtained complex manifold will be Hausdorff. The complex conjugation of $\mathbb{C}^n$ induces a conjugation on $\mathbb{C}M$, i.e. an anti-holomorphic involution, whose fixed point set is precisely M. (See [Wh-Br].) This complexification process makes it possible to extend real analytic objects (functions, metrics, connections etc.) given on M to obtain holomorphic ones on the complexification.

This idea has been very fruitful in twistor theory for instance. H. Grauert also used this complexification in his famous proof about embeddability of abstract real analytic manifolds. (See [Gra].)

Despite of its naturality the above procedure also has some drawbacks. Namely it is not canonical, it really depends on the choices we have made along the construction procedure and $\mathbb{C}M$ is only unique as a germ of complex manifolds.

Recently another approach arose and has been studied in several papers. (See [Le-Sz], [Le2], [Gu-S], [St1 and 2] and [Sz2 and 3].) The idea is that with an extra piece of information, a Riemannian metric g on M, one is really able to define a canonical complex manifold X associated to (M, g). The underlying differentiable manifold structure of X is a certain disk bundle over M. More precisely, let r be a positive real number. Then T^rM will denote the set of vectors in the tangent bundle of M that have length less than r. We also allow r to be infinite, when T^rM will simply mean the tangent bundle of M. With the help of the metric, it is possible to define (if r is small enough, M is compact and the metric is also real analytic) a canonical (called *adapted*) complex manifold structure on T^rM. (See Section 3. for details.)

Many interesting properties of these structures were revealed in the above mentioned papers. Among others, the g-norm-square function is strictly plurisubharmonic on T^rM and thus it is a potential function for a Kahler metric κ_g. The restriction of this metric to the zero section in TM gives back the original metric g.

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Therefore the complex manifold (T^rM, κ_g) can be thought of as a canonical Kahler extension of (M, g).

The organisation of the paper is as follows. In Section 1. and 2. we recall some notations concerning the symplectic structure of the tangent bundle. In Section 3. we give a precise definition of the adapted complex structures together with listing some of their properties we are going to use later on in the paper. In Section 4. we explicitely calculate the metric κ_g in some special frame that was introduced in Section 2. With Section 5. we start our systematic study of the Kahler manifolds (T^rM, κ_g). Among others we prove the following result (Theorem 5.3).

Theorem A. *Let (M, g) and (N, h) be two n-dimensional compact Riemannian manifolds and $0 < r, s < \infty$. Assume that adapted complex structure exists on T^rM and T^sN. Denote by κ_g and κ_h the induced Kahler metrics. Suppose there exists an*

$$\Phi : (T^rM, \kappa_g) \longrightarrow (T^sN, \kappa_h),$$

biholomorphic isometry. Then $r = s$. Denote by f the restriction of Φ to M. Then f maps M isometrically onto N and the induced map f_* agrees with Φ on the whole tube.

Section 6. treats the automorphisms of tubes T^rM which have finite radius. In section 7. we prove a similar rigidity result (Theorem 7.1).

Theorem B. *Let (M, g) and (N, h) be compact Riemannian manifolds. Assume that adapted complex structure exists on TM and TN. Suppose that $H^1(M, \mathbb{R}) = 0$. Denote by κ_g and κ_h the induced Kahler metrics. Let*

$$\Phi : (TM, \kappa_g) \longrightarrow (TN, \kappa_h),$$

be a biholomorphic isometry. Then Φ maps M diffeomorphically onto N, the restriction map

$$f := \Phi|_M : (M, g) \longrightarrow (N, h),$$

is an isometry and $\Phi \equiv f_*$.

Section 8. treats the isometry group action on the tangent bundle of a Riemannian manifold. Our main result here is the following (Theorem 8.8).

Theorem C. *Let (M, g) be a compact Riemannian manifold that admits an adapted complex structure on its entire tangent bundle. Denote by G the unit component of the compact Lie group $Isom(M, g)$. Consider G as a transformation group, acting on TM by the induced action. This G-action extends to a group action of the complexified group $G_{\mathbb{C}}$, and the transformation map*

$$G_{\mathbb{C}} \times TM \longrightarrow TM$$

is holomorphic. The subgroup of $G_{\mathbb{C}}$ that consists of elements acting trivially on TM is discrete.

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1. Parallel vector fields.

Let us first recall a few notations concerning the symplectic structure of the tangent bundle of a Riemannian manifold. (See [Kl] as a general source of information.)

In this paper M^n will always denote a smooth $n-$dimensional manifold. The tangent bundle of M will be TM and $\pi : TM \to M$ will stand for the bundle projection map. If M^n is equipped with a Riemannian metric g, then TM will inherit a symplectic structure from the cotangent bundle of M as follows.

Define the canonical 1-form Θ on TM by

$$\langle \Theta, v\rangle := g(z, \pi_* v), \qquad v \in T_z M.$$

Then $\Omega := d\Theta$ is a symplectic form on TM. Denote by ρ the smooth function on TM which is $g-$length squared. For simplicity, from now on we will assume that (M, g) is complete.

The geodesic flow $\phi_t : TM \to TM$ is the Hamiltonian flow induced by the Hamiltonian ρ. Let $\gamma : \mathbb{R} \to M$ be a geodesic. The image of $T\mathbb{R} \setminus \mathbb{R}$ under the induced map $\gamma_* : T\mathbb{R} \to TM$ is a two dimensional surface. As γ runs through all the geodesics in M, these surfaces define a foliation of $TM \setminus M$. We call this the *Riemann foliation.*

For a $\gamma : \mathbb{R} \to M$ geodesic, a *parallel vector field* ξ *along* γ_* is a vector field along the map γ_* (i.e. a section of the pullback bundle $(\gamma_*)^*(TM)$), such that there exists a smooth family $\gamma_t : \mathbb{R} \to M$ of geodesics with $\gamma_0 = \gamma$ and

$$\left.\frac{d}{dt}\right|_{t=0} \gamma_{t*} = \xi.$$

If 0_σ is the zero vector in the tangent space $T_\sigma\mathbb{R}$, then $\xi(\sigma) := \xi(0_\sigma)$ is tangential to the zero section in TM. Indeed,

$$\xi(\sigma) = \left.\frac{d}{dt}\gamma_{t*}(0_\sigma)\right|_{t=0} = \left.\frac{d}{dt}\gamma_t(\sigma)\right|_{t=0},$$

is a Jacobi field along γ. Parallel vector fields can be thought of as canonical extensions of Jacobi fields on M to TM.

Since any point $z \in TM \setminus M$ determines a unique geodesic $\gamma : \mathbb{R} \to M$ such that $\dot\gamma(0) = z$, it follows that given a vector $0 \neq v \in T_0\mathbb{R}$ and a vector $\bar\xi \in T_{\gamma_* v}(TM)$, there exists a unique parallel vector field ξ along γ_* with $\xi(v) = \bar\xi$.

For a real number s define the map $N_s : TM \to TM$ by multiplying every vector in each fibre with s. For a non-zero s, N_s will be a diffeomorphism. Parallel vector fields along γ_* can be characterised by the following invariance property

$$N_{s*}\xi = \xi, \quad \phi_{s*}\xi = \xi \qquad s \in \mathbb{R}. \tag{1.1}$$

(See [Le-Sz, Proposition 6.1]). The relation of the symplectic form Ω and N_s, ϕ_s is also quite simple,

$$N_s^*\Omega = s\Omega, \quad \phi_s^*\Omega = \Omega \qquad s \in \mathbb{R}. \tag{1.2}$$

2. Symplectic frames.

The metric g determines the Levi-Cività connection on TM and thus a splitting of $T_z(TM)$, $(z \in TM)$ into vertical and horizontal subspaces. The vertical subspace is simply the fiber $T_{\pi(z)}M$ which canonically sits in $T_z(TM)$. The horizontal subspace is the kernel of some projection $T_z(TM) \to T_{\pi(z)}M$. The collection of these projections is the connection map $K : T(TM) \to TM$.

Denote by H_z the horizontal subspace at the point $z \in TM$. The tangent space of the zero section in TM at a point $0_m \in T_mM$ can (and with a little abuse of notation will be) identified with T_mM. With this identification in mind we obtain that for any $z \in TM$, the map

$$\pi_* : H_z \longrightarrow T_{\pi(z)}M$$

is an isomorphism of vector spaces. With the help of this isomorphism we can talk about the horizontal lift of a vector $v \in T_{\pi(z)}M$ into a vector $\bar{v} \in H_z(TM)$, meaning that $\bar{v}$ and v under π_* correspond. Vertical lift of $v \in T_{\pi(z)}M$ is given by the canonical imbedding of the fiber $T_{\pi(z)}M$, into $T_z(TM)$.

Recall that for a symplectic vector space (V^{2n}, ω), a 2n-tuple of vectors,

$$(u_1, \ldots, u_n, v_1, \ldots, v_n)$$

is called a *symplectic base*, if

$$\omega(u_j, u_k) = \omega(v_j, v_k) = 0,$$

and

$$\omega(u_j, v_k) = \delta_{jk},$$

for every $1 \leq j, k \leq n$.

A special type of frame of parallel vector fields along a leaf of the Riemann foliation plays a crutial role in what follows. Let $\gamma : \mathbb{R} \to M$ be a unit speed geodesic, and $\xi_1, \ldots, \xi_n, \eta_1 \ldots, \eta_n$ be parallel vector fields along the leaf L_γ defined by γ. We shall call $(\xi_1, \ldots, \xi_n, \eta_1 \ldots, \eta_n)$ a *symplectic frame*, if there exists a real number σ and a vector $v \in T_\sigma\mathbb{R}$, with $||v|| = 1$, and an orthonormal frame

$$v_1, \ldots, v_{n-1}, v_n = \dot{\gamma}(\sigma) \in T_{\gamma(\sigma)}M,$$

such that for any $1 \leq j \leq n$, $\xi_j(\gamma_* v)$ is the horizontal and $\eta_j(\gamma_* v)$ is the vertical lift of v_j.

This condition is equivalent to the following: the Jacobi fields $\xi_j|_{\mathbb{R}}$, $\eta_j|_{\mathbb{R}}$ have the initial condition ($'$ means covariant derivative along γ)

$$\xi_j(\sigma) = v_j, \qquad \xi_j'(\sigma) = 0, \quad 1 \leq j \leq n,$$

$$\eta_j(\sigma) = 0, \qquad \eta_j'(\sigma) = v_j, \quad 1 \le j \le n,$$

(see [Le-Sz] for instance). In particular the set S of those real numbers σ, where the n-tuple $\xi_1(\sigma), \ldots, \xi_n(\sigma) \in T_{\gamma(\sigma)}M$ is linearly dependent, is discrete. Moreover there exists a smooth matrix valued map $\varphi = (\varphi_{jk})$, defined on $\mathbb{R} \setminus S$, such that

$$\eta_k(\sigma) = \sum_j \varphi_{jk}(\sigma)\xi_j(\sigma), \qquad \sigma \in \mathbb{R} \setminus S, \quad 1 \le k \le n. \tag{2.1}$$

An explanation for the name "*symplectic frame*" is the following proposition.

Proposition 2.1. *Let (M^n, g) be a Riemanian manifold. Let γ be a unit speed geodesic and $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$ be a symplectic frame along the leaf L_γ (of the Riemann foliation) . Then for any $v \in (T\mathbb{R} \setminus \mathbb{R})$, the 2n-tuple*

$$\{\xi_j(\gamma_*(v)), \eta_j(\gamma_*(v))\}_{j=1}^n$$

forms a symplectic base of the symplectic vector space (ϵ denotes the sign of the real number $-\tau$)

$$\left(T_{\gamma_*(v)}(TM), \, (\epsilon/||\gamma_*(v)||_g)\Omega|_{\gamma_*(v)}\right).$$

Proof.

The orbit of a fixed point of the leaf $\gamma_*(T\mathbb{R} \setminus \mathbb{R})$, under repeated applications of N_s and ϕ_t is the whole leaf. Therefore, according to (1.1) and (1.2), it is enough to check our statement in one point q of the leaf. Thus we can assume that q has norm one, and that we have a Riemannian normal coordinate system around the point $\pi(q)$, which we take to be the origin.

With this choice we have

$$\xi_j(q) = \frac{\partial}{\partial q_j} \quad \text{and} \quad \eta_j(q) = \frac{\partial}{\partial p_j},$$

and

$$\Omega|_q = \sum_j dp_j \wedge dq_j.$$

This proves our claim. $\square$

3. Adapted complex structures.

When $M = \mathbb{R}$, there is a natural identification $T\mathbb{R} \cong \mathbb{C}$, given by

$$T_\sigma \mathbb{R} \ni \tau \frac{\partial}{\partial \sigma} \longleftrightarrow \sigma + i\tau \in \mathbb{C}, \tag{3.1}$$

and this equipes $T\mathbb{R}$ with a complex structure. In (3.1) σ denotes the coordinate on $\mathbb{R}$. This complex structure depends only on the metric (which we chose to be the standard one), in other words, an isometry of $\mathbb{R}$ induces a biholomorphic mapping on $T\mathbb{R}$. From now on we fix this complex structure.

Besides the full tangent bundle we shall also need the following type of tube domains. Let (M, g) be Riemannian and $0 < r \le \infty$. Let T^rM be defined by

$$T^rM = \{v \in TM \mid ||v||_g < r\}.$$

We will call r the *radius* of the tube T^rM. The main object of this paper is to study a certain complex structure on these tubes that is canonically associated to the metric.

Definition 3.1. *Let (M, g) be a complete Riemannian manifold. Let $0 < r \leq \infty$ be given. A smooth complex structure on the manifold T^rM will be called an **adapted complex structure** if for any geodesic $\gamma : \mathbb{R} \to M$, the map*

$$\gamma_* : \gamma_*^{-1}(T^rM) \longrightarrow T^rM$$

is holomorphic, where $\gamma_*^{-1}(T^rM) \subset T\mathbb{R}$ and $T\mathbb{R}$ is endowed with the complex structure as explained above.

Given the notion of adapted complex structures, natural questions arise about existence, uniqueness and regularity. Some of the answers to these questions can be found in [Gu-S], [Le-Sz], [Sz2] and [Le2]. From these papers one knows that the metric uniquely determines the adapted complex structure (assuming that it exists), and real analyticity of the metric is necessary for the existence. On the other hand for a compact, real analytic manifold (M, g) there always exists an $0 < r \leq \infty$, such that T^rM carries an adapted complex structure. (See [Sz2] or [Gu-S]).

It is immediate from its definition that once we have an adapted complex structure on T^rM for some $0 < r \leq \infty$, then the same complex structure will supply us an (and because of uniqueness, we could say the) adapted complex structure on T^sM for every $0 < s < r$. We shall call r the *critical radius* if adapted complex structure exists on T^rM but it does not exists on any other tube T^sM with radius s larger that r. If the adapted complex structure happens to exists on the entire tangent bundle, then we take ∞ to be the critical radius.

In what follows, it is sometimes important whether r is critical or not. Usually it is easier to treat the tubes with noncritical radius. The reason is that in this case T^rM is a relatively compact subdomain of a complex (in fact Stein) manifold with smooth, strictly pseudoconvex boundary.

We need some more notations concerning complex manifolds. Suppose X is a complex manifold with an almost complex tensor $J : TX \to TX$. The (1,0) resp. (0,1) tangent bundles, $T^{1,0}X$ resp. $T^{0,1}X$ are complex vector bundles ($T^{1,0}X$ being in fact a holomorphic bundle). They are both isomorphic to TX as real vector bundles, the isomorphism being given by

$$\begin{aligned} \lambda^{1,0} &: TX \ni \xi \longmapsto \xi^{1,0} = \frac{1}{2}(\xi - iJ\xi) \in T^{1,0}X, \\ \lambda^{0,1} &: TX \ni \xi \longmapsto \xi^{0,1} = \frac{1}{2}(\xi + iJ\xi) \in T^{0,1}X. \end{aligned} \tag{3.2}$$

In fact for every $p \in X$, the complex structure $J_p : T_pX \to T_pX$ defines a complex vector space structure on T_pM and the map $\lambda^{1,0}$, resp. $\lambda^{0,1}$ is $\mathbb{C}$ linear resp. antilinear.

For a complex manifold X we shall denote its group of biholomorphisms by $Aut(X)$. Since we are going to use it several times, we formulate precisely the above mentioned regularity result of Lempert.

Theorem 3.2. *(see [Le2, Theorem 1.5]) Let (M, g) be a Riemannian manifold. Assume that adapted complex structure exists on T^rM, for some positive r. Then*

M is a real analytic submanifold of the complex manifold T^rM and the the metric on M is real analytic.

We shall also need a result of Shiffman, that gives a very useful criterium to decide when a partially real-analytic function is real-analytic in all its variables.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^M$, $V \subset \mathbb{C}^N$ be domains, such that $\emptyset \neq A = V \cap \mathbb{R}^N$. Let $f : \Omega \times V \to \mathbb{C}$ be a function such that $f(x,.)$ is holomorphic on V for all $x \in \Omega$ and $f(.,w)$ is real analytic on Ω for all $x \in A$. Then f is real analytic on $\Omega \times V$.*

Proof. See [Sh, Theorem 1] for an even more general statement.

From now on we assume that the Riemannian manifold (M,g) induces an adapted complex structure on T^rM. Denote by $Isom(M,g)$ the group of isometries of (M,g). The properties we shall need later on are as follows.

(PROP.I) *The function on T^rM that associates to any vector its norm, is plurisubharmonic. Its square (denote it by ρ) is strictly plurisubharmonic and thus it induces a Kahler metric κ_g, which is defined by*

$$\kappa_g(V,W) = -i\partial\overline{\partial}\rho(JV \wedge \overline{W}), \qquad V,W \in T_z(TM) \otimes \mathbb{C}, \quad z \in T^rM.$$

(PROP.II) *The Kahler form of κ_g is Ω, the symplectic form of the tangent bundle.*

(PROP.III) *$M \subset T^rM$ is a κ_g-totally geodesic Lagrangean submanifold and $\kappa_g|_M = g$. If we denote by $dist_{\kappa_g}$ the distance with respect to the metric κ_g, then for any $p \in T^rM$ we get*

$$dist_{\kappa_g}(p,M) = dist_{\kappa_g}(p,\pi(p)) = ||p||_g.$$

(PROP.IV) *The map*

$$\begin{aligned} T^rM &\longrightarrow T^rM \\ v &\longmapsto -v. \end{aligned}$$

is antibiholomorphic.

(PROP.V) *For every $f \in Isom(M,g)$, $f_* : T^rM \to T^rM$ is a biholomorphism and thus*

$$Isom(M,g) \leq Aut(T^rM).$$

(PROP.VI) *Let $D_r := \{\zeta = \sigma + i\tau \in \mathbb{C} \mid |\tau| < r\}$. Assume that we are given a function $h : T^rM \to \mathbb{C}$, that is real analytic along the zero section, and for every unit speed geodesic γ, the composition map $h \circ \gamma_* : D_r \to \mathbb{C}$ is holomorphic. Then h is holomorphic.*

The proofs of $(PROP.I, \ldots, V)$ can be found in [Le-Sz].

Proof of (PROP. VI).

In fact this is implicitly contained in [Sz2]. The regularity result of Lempert (Theorem 3.2) implies that in fact g is real analytic. Then the proof of [Sz2, Proposition 3.2] shows that h must be holomorphic in an open neighbourhood of the zero section.

In order to prove that h is holomorphic everywhere, it suffices to show that h is real analytic on T^rM. We will use Theorem 3.3 above to achieve this.

Let p be a point of $T^rM\setminus M$ with norm one. Choose a small open neighbourhood U_p of p in the unit sphere bundle. We can assume that the Hamiltonian flow can be straighten out in U_p, i.e. there exists an $\epsilon > 0$ and a real-analytic diffeomorphism

$$\psi : (-\epsilon, \epsilon) \times \mathbb{B}_\epsilon \to U_p,$$

($\mathbb{B}_\epsilon \subset \mathbb{R}^{2n-2}$ being the open ϵ-ball), such that the curves

$$t \mapsto \psi(t, x)$$

are precisely the flow lines in U_p, for every $x \in \mathbb{B}_\epsilon$. Let

$$D_\epsilon = (-\epsilon, \epsilon) \times (0, r) \times \mathbb{B}_\epsilon.$$

Then the map

$$\begin{aligned} D_\epsilon &\longrightarrow T^rM \\ \Psi : (\sigma, \tau, x) &\longmapsto \tau\psi(t, x) \end{aligned}$$

is a real-analytic diffeomorphism onto its image. It is enough to show that the composition map $h \circ \Psi$ is real-analytic. But using the fact that h was holomorphic in a small neighbourhood of the zero section, we get that $h \circ \Psi$ is real-analytic in the region $|\sigma| < \epsilon, 0 < \tau < \epsilon, x \in \mathbb{B}_\epsilon$, and holomorphic in $\zeta = \sigma + i\tau$ for each $x \in \mathbb{B}_\epsilon$. Apply Theorem 3.3. □

In the last section we shall need to know about complexifications of Lie groups. We close this section by connecting this notion with our adapted complex structures.

Let G be a compact, connected Lie group. The *complexification* of G is a complex, connected Lie group $G_{\mathbb{C}}$, and a group monomorphism $\iota : G \to G_{\mathbb{C}}$, such that for any representation

$$\chi : G \longrightarrow GL(n, \mathbb{C}),$$

there exists a unique representation

$$\chi_{\mathbb{C}} : G_{\mathbb{C}} \longrightarrow GL(n, \mathbb{C}),$$

with $\chi_{\mathbb{C}}(\iota(a)) = \chi(a)$, for every element a in G. (See [Br-Di] for a detailed discussion.)

The group $G_{\mathbb{C}}$ can explicitely be constructed as follows. Take a faithful unitary representation of G, i.e. imbed it as a closed subgroup of $U(N)$, for some large N. Denote by $\mathfrak{g}$ the Lie algebra of G. The underlying manifold for $G_{\mathbb{C}}$ is just $G \times \mathfrak{g}$. The group structure and the complex structure can be defined by pulling them back with the embedding

$$\begin{aligned} G \times \mathfrak{g} &\longrightarrow GL(N, \mathbb{C}) \\ \Lambda : (a, Y) &\longmapsto a\exp(iY), \end{aligned} \tag{3.3}$$

where exp is the usual exponential of a matrix and product is ordinary matrix multiplication.

Proposition 3.4. *Let G be a compact, connected Lie group. Equipp G with a two-sided invariant Riemannian metric g. Then (G, g) admits an adapted complex structure on TG which can be canonically identified with $G_{\mathbb{C}}$.*

Proof.

G, being a Lie group, is parallelizable. Hence $TG = G \times \mathfrak{g}$. Because of homogeneity, to show that the complex structure on TG arising from (3.3) is adapted to g, it suffices to show that for every unit speed g-geodesic γ through the unit element, $\gamma_* : T\mathbb{R} \to TG$ is holomorphic. It is well known (see [Ab-Ma, Corollary 4.4.13] for instance) that such geodesics are precisely the 1-parameter subgroups of G. Let $X \in \mathfrak{g}$, and $\gamma(\sigma) := \exp(\sigma x)$. The induced map is then

$$\gamma_* : T\mathbb{R} \cong \mathbb{C} \ni \sigma + i\tau \longmapsto (\exp(\sigma X), \tau X).$$

Thus

$$\Lambda \circ \gamma_* : T\mathbb{R} \ni \zeta = \sigma + i\tau \longmapsto \exp(\sigma X)\exp(i\tau X) = \exp(\zeta X),$$

is holomorphic in ζ. Therefore the complex structure of $TG = G \times \mathfrak{g}$ is indeed adapted to g. $\square$

4. Calculating the metric κ_g.

In this section we are going to give explicit formulas for κ_g using symplectic frames. But first we need to recall some more notation.

Denote by $M^n_{\mathbb{C}}$ the set of $n \times n$ complex matrices. For a $Z \in M^n_{\mathbb{C}}$, we will use the symbol $Z^\top$ to denote the transpose (without conjugation) of the matrix Z.

Re Z resp. Im Z will mean the $n \times n$ real matrix, obtained by taking the real resp. imaginary part of every entry of Z. For a real matrix X, we will use the symbol $X > 0$ to denote that X is symmetric and positive definite.

The subset of $M^n_{\mathbb{C}}$,

$$\mathcal{H}^n = \{Z \in M^n_{\mathbb{C}} \mid Z = Z^\top, \quad \mathrm{Im}\, Z > 0\}$$

is called the *Siegel upper half plane*. In particular $\mathcal{H}^1$ is the ordinary upper half plane, that we also denote by $\mathbb{C}^+$.

Let (V, ω) be again a symplectic vector space. A complex structure $J : V \to V$ is said to *calibrate* the symplectic form ω, if the bilinear form $\omega(u, Jv)$, $u, v \in V$ is symmetric and positive definite. We will denote the set of calibrating complex structures on (V, ω) by $\mathcal{J}_\omega$.

Proposition 4.1. *Let (V^{2n}, ω) be a symplectic vector space. Then $\mathcal{J}_\omega$ can be identified with the Siegel upper half plane $\mathcal{H}^n$. This correspondence can be defined as follows. Fix a symplectic base $u_1, \dots, u_n, v_1, \dots, v_n$. If $J \in \mathcal{J}_\omega$, then the n-tuples $\{u_j\}_{j=1}^n$ and $\{v_j\}_{j=1}^n$ both provide a $\mathbb{C}$ basis of the complex vector space (V, J). Denote by $Z := (f_{kl})$ the transition matrix, i.e.*

$$v_k = \sum_l f_{lk} u_l, \qquad k = 1, \dots, n. \tag{4.1}$$

And vica versa, assume that $Z = \mathrm{Re}\, Z + i\mathrm{Im}\, Z \in \mathcal{H}^n$. Then declaring $\{u_j\}_{j=1}^n$ and $\{v_j\}_{j=1}^n$ to be a $\mathbb{C}$ basis with transition matrix Z (as in (4.1)), we define a complex structure $J_Z : V \to V$ which calibrates ω, and can be expressed as

$$J_Z u_k = \sum_{j=1}^n (\mathrm{Im}\, Z)_{jk}^{-1} [v_j - \sum_{l=1}^n (\mathrm{Re}\, Z_{lj}) u_l]. \tag{4.2}$$

The matrix of the symmetric, positive definite bilinear form $\omega(.,J.)$ in the base u_j, v_k is

$$\begin{pmatrix} [\mathrm{Im}\, Z]^{-1} & [\mathrm{Im}\, Z]^{-1}\mathrm{Re}\, Z \\ \mathrm{Re}\, Z[\mathrm{Im}\, Z]^{-1} & \mathrm{Re}\, Z[\mathrm{Im}\, Z]^{-1}\mathrm{Im}\, Z + \mathrm{Im}\, Z \end{pmatrix} \tag{4.3}$$

Proof. Left to the reader.

Proposition 4.2. *Let X and Y be complex manifolds and $\epsilon > 0$. Suppose we have a smooth map $f : (-\epsilon, \epsilon) \times X \to Y$ and for every fixed $-\epsilon < t < \epsilon$, the map $f_t(.) := f(t,.) : X \to Y$ is holomorphic. Let*

$$\xi = df_t/dt|_{t=0}.$$

This is a section of $f_0^* TY_0$. Then $\xi^{1,0}$ is a holomorphic section of $f_0^* T^{1,0}Y$.

Proof.. (cf. [Le-Sz] Prop. 5.1, p. 698) The statement is local, therefore we can assume $X = D_1 \subset \mathbb{C}^n$, $Y = D_2 \subset \mathbb{C}^m$ and $f : (-\epsilon, \epsilon) \times D_1 \to D_2$. We have to show that $df/dt|_{t=0}$ is holomorphic. But

$$\bar\partial_\zeta \left(\frac{df}{dt}(t,\zeta)|_{t=0} \right) = \frac{d}{dt} \left(\bar\partial_\zeta f(t,\zeta) \right) \Big|_{t=0} \equiv 0. \quad \square$$

Armed with the last two propositions, we are now ready to prove the main theorem of this section.

Theorem 4.3. *Let (M,g) be a Riemannian manifold and $0 < r \le \infty$. Assume that adapted complex structure exists on $T^r M$. Let γ be a unit speed geodesic and $(\xi_1, \dots \xi_n, \eta_1, \dots, \eta_n)$ be a symplectic frame along the leaf L_γ. Let $D_r := \{\zeta = \sigma + i\tau \in \mathbb{C} \mid |\tau| < r\}$. Denote by $S \subset \mathbb{R}$ the discrete set of points $\sigma \in \mathbb{R}$, for which the vectors $\xi_1(\sigma), \dots, \xi_n(\sigma) \in T_{\gamma(\sigma)}M$ are linearly dependent. (see (2.1)) Then there exist meromorphic maps $f_{jk} : D_r \to \mathbb{C} \cup \{\infty\}$, $1 \le j,k \le n$, which are holomorphic on $D_r \setminus S$ and*

$$F = (f_{jk}) : (D_r \setminus \mathbb{R}) \longrightarrow \mathcal{H}^n. \tag{4.4}$$

Let $T\mathbb{R} \cong \mathbb{C}$ as in (3.1). Then

$$\eta_k^{1,0}(\gamma_*(\zeta)) = \sum_{j=1}^n f_{jk}(\zeta) \xi_j^{1,0}(\gamma_*(\zeta)), \qquad \zeta \in D_r \setminus S, \tag{4.5}$$

and $k = 1, \ldots, n$. For $\sigma \in \mathbb{R} \setminus S$, $\varphi_{jk}(\sigma) = f_{jk}(\sigma)$, (see (2.1). Let $\zeta \in D_r \setminus \mathbb{R}$ and $p := \gamma_*(\zeta)$ and $J_p : T_p(TM) \to T_p(TM)$ be the adapted complex structure. Then

$$J_p\xi_k(p) = \sum_{l=1}^{n} (Im\ F)^{-1}_{lk}(\zeta) \left[\eta_l(p) - \sum_{j=1}^{n} (Re\ f_{jl})(\zeta)\xi_j(p) \right], \tag{4.6}$$

and for the κ_g Kahler metric (remember (PROP.I)) we obtain

$$\begin{aligned} \langle \xi_i(p), \xi_k(p) \rangle_{\kappa_g} &= ||p||_g (Im\ F)^{-1}_{ik}(\zeta), \\ \langle \xi_i(p), \eta_k(p) \rangle_{\kappa_g} &= ||p||_g \left[(Im\ F)^{-1} Re\ F \right]_{ik}(\zeta), \\ \langle \eta_i(p), \eta_k(p) \rangle_{\kappa_g} &= ||p||_g \left[Re\ F (Im\ F)^{-1} Re\ F + Im\ F \right]_{ik}(\zeta). \end{aligned} \tag{4.7}$$

Proof. (Compare with [Le-Sz].)

From (PROP.II) we know that the Kahler form of κ_g is Ω, the symplectic form of the tangent bundle. Thus for any $p \in T^rM \setminus M$, and $X, Y \in T_p(TM)$,

$$\langle X, Y \rangle_\kappa = \Omega(JX, Y) = ||p||_g \left[(-1/||p||_g)\Omega(X, JY) \right]. \tag{4.8}$$

This implies that the adapted complex structure $J_p : T_p(TM) \to T_p(TM)$ calibrates the symplectic form

$$(-1/||p||_g)\Omega_p.$$

Then Proposition 4.1 tells us that for any $\zeta \in D_r \setminus \mathbb{R}$,

$$\{\xi_j^{1,0}(\gamma_*(\zeta))\}_{j=1}^n \qquad \text{resp.} \qquad \{\eta_j^{1,0}(\gamma_*(\zeta))\}_{j=1}^n$$

are both $\mathbb{C}$-basis of the vector space $T^{1,0}_{\gamma_*(\zeta)}(TM)$. If $\sigma \in \mathbb{R} \setminus S$, then $\{\xi_j(\sigma)\}_{j=1}^n$ being an $\mathbb{R}$ basis of the vector space $T_{\gamma(\sigma)}$ (and $J_\sigma : T_\sigma(TM) \to T_\sigma(TM)$ being the canonical complex tensor), is also a $\mathbb{C}$ basis of $T^{1,0}_{\gamma(\sigma)}(TM)$. Therefore for any $\zeta \in D_r \setminus S$, there exists a matrix $F(\zeta) = (f_{jk}(\zeta)_{j,k=1}^n$ such that (4.5) holds. (3.2) and (2.1) gives the equality $(\varphi_{jk}(\sigma)) = (f_{jk}(\sigma)), \sigma \in \mathbb{R} \setminus S$. From Proposition 4.2 we know that the maps

$$\xi_j^{1,0}, \eta_j^{1,0} : D_r \longrightarrow T^{1,0}(TM), \qquad j, k = 1, \ldots, n$$

are all holomorphic. Hence F is holomorphic on $D_r \setminus S$ and meromorphic on D_r. Then (4.4) follows from Proposition 4.1, and (4.2) implies (4.6) and (4.3) together with (4.8) yields the expressions of the metric in (4.7). $\square$

Proposition 4.4.

(a) *Let (M_j, g_j), j=1,2 be Riemannian manifolds with adapted complex structure defined on $T^rM_j, 0 < r \leq \infty$. Then the adapted complex structure of $(M_1 \times M_2, g_1 \times g_2)$ is the product of the complex manifolds T^rM_1 and T^rM_2.*

(b) *Let (M, g) be a Riemannian manifold, and $0 < r \leq \infty$. Suppose adapted complex structure exists on T^rM. Assume also that N is a totally geodesic submanifold of (M, g). Then $T^rN \subset T^rM$ is a complex submanifold and its complex structure is adapted to $g|_N$.*

Proof. (a) is immediate from Definition 3.1 and the uniqueness of an adapted complex structure . (See [Le-Sz, Theorem 4.2]) For the case (b), use (4.6) and totally geodesity to show that for any $p \in T^rN$, $T_p(TN)$ is a J_p invariant subspace of $T_p(TM)$. This yields that T^rN is in fact a complex submanifold (obviously an adapted one). □

5. Holomorphic isometries of tubes with finite radius.

In this section we only deal with tubes T^rM, with $0 < r < \infty$. The case $r = \infty$ will be treated separately in Sections 7 and 8. This separation seems natural because the complex analytic properties of these two kind of complex manifolds are quite different.

Proposition 5.1 (Schwarz lemma). *Let (M^n, g) and (N^n, h) be compact Riemannian manifolds and $0 < r, s < \infty$. Assume that adapted complex structure exists on T^rM and T^sN. Let*

$$\Phi : T^rM \to T^sN$$

be a holomorphic map such that

$$\Phi(M) \subset N.$$

Then

$$r||\Phi(p)||_h \leq s||p||_g, \qquad p \in T^rM. \tag{5.1}$$

Proof:.

Let $u(p) := ||p||_g$ and $v(p) := ||\Phi(p)||_h$. From (PROP.I) we know that u and v are plurisubharmonic. Let η be a small but positive real number, fixed for the moment. Define c_η by

$$c_\eta := \max\{v(p) \mid u(p) = r - \eta\}.$$

For every positive ϵ the function

$$w_\epsilon := \frac{c_\eta}{r - \eta} u + \epsilon$$

takes the value $c_\eta + \epsilon$ on the set

$$S_{r-\eta} := \{p \in T^rM \mid u(p) = r - \eta\}$$

and hence

$$w_\epsilon|_{S_{r-\eta}} > v|_{S_{r-\eta}}.$$

For every small enough $\delta > 0$, $w_\epsilon > v$ on the set $\{u = \delta\}$. Both w_ϵ and v satisfies the Monge-Ampère equation on $T^rM \setminus M$. Applying the minimum principle of Bedford and Taylor (see [Be-Ta]) for the functions w_ϵ and v on the domain

$$D_{\delta\eta} := \{p \in T^rM \mid \delta < u(p) < r - \eta\},$$

we get

$$w_\epsilon|_{D_{\delta\eta}} \geq v|_{D_{\delta\eta}}.$$

Because the minimum of w_ϵ is ϵ and v goes to zero as we approach M, we obtain that

$$w_\epsilon \geq v, \qquad \text{on} \quad \{p \in T^rM \mid u(p) \leq r - \eta\}.$$

Let now ϵ go to zero. This yields

$$\frac{c_\eta}{r-\eta} u \geq v, \qquad \text{on} \quad \{p \in T^rM \mid u(p) \leq r - \eta\}. \tag{5.2}$$

Fix now a point p in T^rM. For every small enough η, $u(p) < r - \eta$. It follows then from (5.2) that

$$\frac{c_\eta}{r-\eta} u(p) \geq v(p).$$

Let now η go to zero. Then the denominator goes to r and c_η is bounded above by s. This gives (5.1). □

Theorem 5.2. *Let (M^n, g), (N^n, h) be compact Riemannian manifolds and $0 < r, s < \infty$. Assume that adapted complex structure exists on T^rM and T^sN. Let*

$$\Phi : T^rM \to T^sN$$

be a biholomorphism, such that

$$\Phi(M) \subset N.$$

Then

$$f := \Phi|_M : (M, sg) \to (N, rh)$$

is an onto isometry and $\Phi \equiv f_* : T^rM \to T^sN$.

Proof:.

The fact that Φ is a biholomorphism and that N is compact and connected gives that f is indeed onto. Denote by κ_g and κ_h the Kahler metrics on T^rM and T^sN, induced by the strictly plurisubharmonic Kahler potential function $|| \; ||_g^2$ and $|| \; ||_h^2$ accordingly (PROP.I). Applying our (5.1) Schwarz lemma for both Φ and its inverse, we obtain

$$r^2 ||\Phi(p)||_h^2 = s^2 ||p||_g^2, \qquad p \in T^rM. \tag{5.3}$$

It follows easily from its definition that rescaling the metric does not change the induced complex structure, i.e. for any $\lambda > 0$, g and λg have the same adapted complex structures defined on the same tube except the radius is measured with different scales. Thus

$$\Phi : (T^rM, \kappa_{sg}) \to (T^rN, \kappa_{rh})$$

is a biholomorphic isometry. This, together with (PROP.III) implies that

$$f : (M, sg) \longrightarrow (N, rh)$$

is indeed an isometry itself. Hence f_*, (see (PROP.V)) and Φ are both biholomorphic and agree on the maximal dimensional totally real (PROP.III) submanifold M. This implies that they must agree everywhere. □

Theorem 5.3. *Let (M^n, g), (N^n, h) be compact Riemannian manifolds and $0 < r, s < \infty$. Assume that adapted complex structure exists on T^rM and T^sN. Denote by κ_g and κ_h the induced Kahler metrics on T^rM and T^sN. Let*

$$\Phi : (T^rM, \kappa_g) \to (T^sN, \kappa_h) \tag{5.4}$$

be a biholomorphic isometry. Then $r = s$. Denote by f the restriction of Φ to M. Then f maps M isometrically onto N and $\Phi \equiv f_*$.

Proof:.

We can assume that $s \geq r$. Denote by ρ_1 and ρ_2 the norm-square functions on T^rM and T^sN accordingly. Now (5.4) yields

$$\partial\bar{\partial}\rho_1 = \Phi^*\partial\bar{\partial}\rho_2 = \partial\bar{\partial}(\rho_2 \circ \Phi). \tag{5.5}$$

Let

$$h := \rho_2 \circ \Phi - \rho_1 + r^2 - s^2.$$

It follows from (5.5) that h is a pluriharmonic function on T^rM.

Let $\gamma : \mathbb{R} \to M$ be an arbitrary unit speed geodesic, parametrized by arclength. Let $v := h \circ \gamma_*$. Notice that because γ_* is holomorphic, v is a harmonic function on the domain $D := \{\sigma + i\tau \mid \sigma \in \mathbb{R}, |\tau| < r\}$. From the definition of h it follows that v is bounded by $2(r^2 + s^2)$. Furthermore if $\zeta_n \in D$, $\zeta_n \to z_0 = \sigma_0 + i\tau_0$ with $|\tau_0| = r$, then $v(\zeta_n)$ must go to zero (Φ is a biholomorphism). This yields that v must vanish everywhere. This is true for every geodesic, thus h must also vanish identically. Hence we obtain

$$||\Phi(p)||_h^2 = ||p||^2 + s^2 - r^2, \qquad p \in T^rM. \tag{5.6}$$

Φ is biholomorphic, so we can take a point $p \in T^rM$ with $||\Phi(p)||_h = 0$. Since we assumed $s \geq r$, (5.6) implies $r = s$ and

$$||\Phi(p)||_h^2 = ||p||^2, \qquad p \in T^rM.$$

Apply Theorem 5.2. $\square$

Theorem 5.4. *Let (M^n, g), (N^n, h) be compact Riemannian manifolds and $0 < r_1, r_2 < \infty$. Assume that adapted complex structure exists on $T^{r_1}M$ and $T^{r_2}N$. Let $\Phi : T^{r_1}M \to T^{r_2}N$ be a biholomorphism, which preserves a level set of the norm functions, i.e. suppose that for $0 \leq s_j < r_j$, $j = 1, 2$, Φ maps the level set $\{|| \,||_g = s_1\}$ into $\{|| \,||_h = s_2\}$. Then*

$$r_1 s_2 = r_2 s_1 \tag{5.7}$$

and the map $f := \Phi|_M$ maps M diffeomorphically onto N,

$$f : (M, (r_2 - s_2)g) \to (N, (r_1 - s_1)h) \tag{5.8}$$

is an isometry and $\Phi \equiv f_*$.

Proof:.

Because of dimension reasons, $s_1 = 0$ implies $s_2 = 0$, and vica versa. This case was treated in Theorem 5.2. Hence we can assume that s_1 and s_2 are both positive. Denote by u the norm function on $T^{r_1}M$ and v on $T^{r_2}N$ accordingly. As in the proof of Theorem 5.2, we can conclude that Φ maps the level set $\{u = s_1\}$ diffeomorphically onto the level set $\{v = s_2\}$.

As in the proof of Proposition 5.1, let η be a small positive number, and let

$$c_\eta := \max_{u=r_1-\eta} v \circ \Phi.$$

Let η be so small that $r_1 - \eta > s_1$. Since M is connected, $T^{r_1}M \setminus \{\| \|_g = s_1\}$, resp. $T^{r_2}N \setminus \{\| \|_h = s_2\}$ has two components. One which containes the zero section, D_0^M and D_0^N, and one which does not, D_1^M and D_1^N. Since Φ is a diffeomorphism, only two cases could occur. First possibility is that under Φ, D_1^M and D_0^N correspond to each other. But D_0^N is homotopically equivalent to a compact n-dimensional manifold, namely N and D_1^M is homotopically equivalent to a compact (2n-1)-dimensional manifold, namely the unit sphere bundle of M. For instance looking at homology groups, it is clear that this case cannot occur. Hence, under the map Φ, D_0^M and D_0^N (resp. D_1^M and D_1^N) correspond to each other. Therefor we get $c_\eta \leq c_2$. Let $\tilde{v}$ be defined by

$$\tilde{v} := \frac{c_\eta - s_2}{r_1 - \eta - s_1} u + \frac{r_1 s_2 - s_2 \eta - c_\eta s_1}{r_1 - \eta - s_1}, \tag{5.9}$$

and let $D_\eta = \{s_1 < u < r_1 - \eta\}$. Then $\tilde{v}$ is plurisubharmonic and

$$\tilde{v}|_{\partial D_\eta} \geq v \circ \Phi|_{\partial D_\eta}.$$

Thus the minimum principle of Bedford and Taylor ([Be-Ta]) implies

$$\tilde{v}|_{D_\eta} \geq v \circ \Phi|_{D_\eta}. \tag{5.10}$$

Let now p be a fixed point in $T^{r_1}M$, with $s_1 < u(p) < r_1$. Then for every small enough positive η, $p \in D_\eta$. Hence (5.10) holds. Let η go to zero. Then c_η will go to r_2 and from (5.9) and (5.10) we obtain

$$(r_2 - s_2)u(p) + r_1 s_2 - r_2 s_1 \geq (r_1 - s_1)v(\Phi(p)).$$

Repeating this argument for the inverse of Φ, we obtain that in the domain

$$\{p \in T^{r_1}M \mid s_1 < u(p) < r_1\}$$

we have

$$(r_2 - s_2)u(p) + r_1 s_2 - r_2 s_1 = (r_1 - s_1)v(\Phi(p)). \tag{5.11}$$

Thus

$$\begin{aligned}\Phi : (\{p \in T^{r_1}M \mid s_1 < u(p) < r_1\}, \kappa_{(r_2-s_2)g}) \\ \longrightarrow (\{q \in T^{r_2}N \mid s_2 < v(q) < r_2\}, \kappa_{(r_1-s_1)h})\end{aligned}$$

is a biholomorphic isometry. Using the fact that analytic continuation of an isometry is an isometry (see for instance [Ko-No, Lemma 3, p.253]), we get

$$\Phi : (T^{r_1(r_2-s_2)}M, \kappa_{(r_2-s_2)g}) \longrightarrow (T^{r_2(r_1-s_1)}N, \kappa_{(r_1-s_1)h})$$

is a holomorphic isometry. (Here we measured the radii of the tubes in the new, rescaled metrics.) Then Theorem 5.3 implies (5.7) and (5.8). □

6. Biholomorphisms of tubes with finite radius.

Now that we completely described all the biholomorphic isometries of our tube, we would like to drop the condition isometry and want to study the biholomorphism group of T^rM, that we denote by $Aut(T^rM)$.

First let us recall a few definitions. Let X, Y be complex manifolds. Denote by $Hol(Y,X)$, the set of holomorphic maps. A sequence $\{f_n\} \subset Hol(Y,X)$ is called *compactly divergent* if for every pair of compact sets $K_1 \subset Y, K_2 \subset X$, there exists an index n_0, such that $f_n(K_1) \cap K_2 = \emptyset$ for $n \geq n_0$. A family $\mathcal{F} \subset Hol(Y,X)$ is called *normal*, if every sequence in $\mathcal{F}$ admits either a convergent subsequence or a compactly divergent one.

Denote by U the unit disk on the complex plane. A complex manifold X is called *taut* if $Hol(U,X)$ is normal.

Theorem 6.1. *(see [Ba, Theorem 2]) A complex manifold X is taut iff for any complex manifold Y, $Hol(Y,X)$ is normal.*

A complex manifold X is called *hyperbolic* (in the sense of Kobayashi) if the Kobayashi pseudometric is a genuine metric on X.

Theorem 6.2. *Let X be a complex manifold. Suppose X admits a bounded strictly plurisubharmonic exhaustion function. Then X is a taut and hyperbolic Stein manifold and $Aut(X)$ is a Lie group .*

Proof:. The fact that X is taut, is proved in [Si, Corollary 5]. Tautness implies hyperbolicity (see [Kie]). Hence according to a theorem of Kobayashi (see [Kob]), $Aut(X)$ is a Lie group. Steinnes follows from Grauert's theorem. □

Theorem 6.3. *Let X^n be a complex manifold which admits a bounded strictly plurisubharmonic exhaustion function. Suppose that the n-th homology group, $H_n(X,\mathbb{Z})$ is finitely generated and nonzero. Then X is a taut Stein manifold, $Aut(X)$ is a compact Lie group. Furthermore if $f : X \to X$ is a holomorphic map which induces an isomorphism of $H_n(X,\mathbb{Z})$ and f is injective, then $f \in Aut(X)$.*

Proof. The theorem is essentially contained in [Mo, Theorem 1], except that Mok works with manifolds with a stronger assumption than ours, namely he treats complex manifolds which are hyperbolic in the sense of Caratheodory. But the only place in his proof where he uses this stronger condition is to prove his Proposition 1.1. This can be bypassed by using Theorem 6.1 and 6.2 above. The rest of Mok's proof works in our situation as well. □

It is not possible to apply Mok's Theorem literally to our tubes, because they can be non-Caratheodory hyperbolic, as the following example shows.

Example 6.4.

Let (M,g) be a compact Riemannian manifold of constant sectional curvature minus 1/4. Then of course M will be a quotient of the real hyperbolic space H^n, $(M,g) \cong H^n/\Gamma$. (Here Γ is some discrete subgroup of the isometry group of H^n.) If we take the unit ball of $\mathbb{R}^n$ with the Cayley-Klein metric h, we get a model of H^n and a modification of the construction in [Le1] gives that the adapted complex structure of (H^n,h) on $T^\pi H$ is naturally biholomorphic to $D := (\mathbb{C}^n \backslash \mathbb{R}^n) \cup H^n$. The isometries of H^n act on $T^\pi H$ by biholomorphisms and the quotient map $(T^\pi H) \to$

$T^\pi H/\Gamma = T^\pi M$ is a holomorphic covering. ($T^\pi M$ also equipped with the adapted complex structure .)

Now if f is any bounded holomorphic function on $T^\pi M$, then pulling it back to $D = T^\pi H$, we get a bounded holomorphic function on D. If n is at least 2 then such a function must extend to the whole $\mathbb{C}^n$ to give a bounded holomorphic and thus constant function. Therefore even though $T^\pi M$ is Kobayashi hyperbolic, its Caratheodory pseudometric is identically zero. □

It was a problem, proposed by H.Wu, whether the existence of a bounded strictly plurisubharmonic exhaustion function implies not only hyperbolicity, but complete hyperbolicity. This was refuted by an example of Sibony (see [Si]). It seems to be an interesting question, which we cannot answer at the moment, whether this large class of complex manifolds namely our tube domains $T^r M$ are all complete hyperbolic or not.

Theorem 6.5. *Let (M, g) be a compact Riemannian manifold. Assume that adapted complex structure exists on $T^r M$ for some $0 < r < \infty$.*

- a) *Then $T^r M$ is a taut Stein manifold.*
  If M is orientable, or the universal cover is compact, then
- bi) *$Aut(T^r M)$ is a compact Lie group, and*
- bii) *for any $0 < s < S \leq r$, the complex manifolds $T^s M$ and $T^S M$ are not biholomorphic.*

Proof:.
From (PROP.I) we know that the norm square function on $T^r M$ is a bounded strictly plurisubharmonic exhaustion function. Thus a) follows from Theorem 6.2.

If M is orientable, then $H_n(T^r M, \mathbb{Z}) \cong H_n(M, \mathbb{Z}) \cong \mathbb{Z}$. Therefore bi) and bii) follows from Theorem 6.3. (When $S < r$ or the adapted complex structure extends to a definitely larger tube than $T^r M$, then we do not need to rely on Mok's theorem, it is enough to quote a much easier fact [Be2, Corollary 1.5].)

Now suppose that M is not necessarily orientable but the universal cover $\widetilde{M}$ is compact. We can lift the metric g to $\widetilde{M}$ to obtain $\tilde{g}$. Then $(\widetilde{M}, \tilde{g})$ will admit an adapted complex structure on $T^r\widetilde{M}$. The covering map $p : \widetilde{M} \to M$ induces a holomorphic covering $p_* : T^r\widetilde{M} \to T^r M$. Since $\widetilde{M}$ is simply connected, any element of $Aut(T^r M)$ can be lifted to give an element of $Aut(T^r\widetilde{M})$. Similarly any element of $Hol(T^S M, T^s M)$ can be lifted to an element of $Hol(T^S\widetilde{M}, T^s\widetilde{M})$ and lifts of biholomorphisms are biholomorphic maps. Applying what we have already proved for $\widetilde{M}$, we are done. □

Remark.
Suppose that the compact Riemannian manifold (M, g) admits an adapted complex structure on $T^r M$, $0 < r \leq \infty$. Assume the the Euler characteristic of M is negative. Then the Stein manifold $T^r M$ can never be affine algebraic. This follows from [To, Theorem 2/2], since after identifying the tangent bundle of M with its cotangent bundle, the Kahler form becomes the standard symplectic form and $T^{*r}M$ and T^*M can easily be seen to be orientable diffeomorphic. (Orientation is defined by the standard symplectic form.)

Proposition 6.6. *Let (M,g) be a compact Riemannian manifold, with an adapted complex structure on T^rM for some $0 < r < \infty$. The map*

$$\sigma : Aut(T^rM) \to Aut(T^rM), \qquad \sigma(\Phi)(p) := -\Phi(-p),$$

is an involutive automorphism of the group $Aut(T^rM)$. The fixedpoint set of σ is precisely the isometry group $Isom(M,g)$, and thus (denoting by index u the unit component of a group)

$$(Aut_u(T^rM), Isom_u(M,g), \sigma)$$

is a symmetric space (compact if M is orientable or the universal cover of M is compact).

Proof:.
From (PROP.IV) we know that the map

$$T^rM \to T^rM,$$
$$p \mapsto -p,$$

is an antiholomorphic diffeomorphism. Hence σ really maps into $Aut(T^rM)$. The fact that σ is an involutive automorphism is straightforward. Suppose that $\Phi \in Aut(T^rM)$ and $\sigma(\Phi) = \Phi$. This implies in particular that Φ preserves the zero section. The rest follows then from Theorem 5.2 and Theorem 6.5(bi). □

It is not clear, whether anything more, then they are compact, can be said in general about the groups $Aut(T^rM)$. It would be interesting to see examples, when $Aut_u(T^rM)$ is really larger than the group $Isom_uT^rM$. This would show that we are able to move the zero section by a biholomorphism and that the symmetric space defined in Proposition 6.6 is nontrivial. This perhaps would lead to new type of invariants of the manifold M.

On the other hand if it turns out that $Aut_u(T^rM)$ is always equal to $Isom_u(M,g)$, this would mean a certain rigidity property of our adapted complex structure . In special cases with large isometry groups we are indeed able to show that this is what really happens.

Theorem 6.7. *Let (M,g) be a compact Riemannian manifold. Suppose that a compact, connected Lie group G acts transitively on M by isometries. Then there exists an $0 < s \leq \infty$ such that adapted complex structure exists on T^sM. Let R be the critical radius (see Section 3.). Let $0 < r \leq R$, but finite. Suppose that the isotropy representation of the isotropy group of G has no common eigenvector. Assume furthermore that M is simply connected. Let Γ be any discrete subgroup of $Isom(M,g)$, such that $\widehat{M} := M/\Gamma$ is a manifold. Denote by g_Γ the projection of the metric g. Then $(\widehat{M}, g_\Gamma)$ admits an adapted complex structure on $T^r\widehat{M}$. Moreover*

$$Aut(T^rM) = Isom(M,g) \tag{6.1}$$

and

$$Aut(T^r\widehat{M}) = Isom(\widehat{M}, g_\Gamma) \tag{6.2}$$

Proof:.

Because G is transitive on M, the metric g is real analytic. Hence existence of an r with adapted complex structure on T^rM follows from [Sz2]. We can lift the elements of $Aut(T^r\widehat{M})$ to get elements of $Aut(T^rM)$. Hence it is enough to prove (6.1).

From Theorem 6.5(bi) we know that $Aut(T^rM)$ is a compact Lie group. Denote by ψ the average of the norm-square function, i.e. let

$$\psi(p) := \int_{\Phi\in Aut(T^rM)} ||\Phi(p)||^2 dvol,$$

where $dvol$ is the two sided invariant Haar measure on $Aut(T^rM)$. ψ is $Aut(T^rM)$ invariant and strictly plurisubharmonic . $0 \leq \psi < r^2$ and easy to see that ψ is an exhaustion function. Let

$$S := \{p \in T^rM \mid \psi(p) = \psi_{min}\}.$$

Then S is $Aut(T^rM)$ invariant and also totally real (see [Ha-We]) and thus its dimension is at most n. By our assumption on the isotropy group of G, any point q in $T^rM \setminus M$ must have an orbit, under the isotropy group action, of dimension at least one and therefore an orbit, under the group action G, of dimension at least $n+1$. Hence S must be a subset of M. But G is transitive on M. Thus S must be equal to M. That means M is $Aut(T^rM)$ invariant and so Theorem 5.2 finishes the proof of (6.1). □

Theorem 6.7 covers, among others, all the compact rank one symmetric spaces and thus generalizes a theorem of Bedford who proved (6.1) and (6.2) for the round spheres (see [Be1]).

7. Holomorphic isometries of the form $TM \to TN$.

In this and the next Section we will be working with Riemannian manifolds (M, g), which admit adapted complex structure on its entire tangent bundle. The arising complex manifold TM will never be Kobayashi-hyperbolic, unlike the tubes with finite radius, because any geodesic γ in M induces a nontrivial holomorphic map $\gamma_* : T\mathbb{R} \simeq \mathbb{C} \to TM$, by definition. This makes the situation harder, for we do not apriory know whether $Aut(TM)$ is a Lie group or not. In fact this group is not always finite dimensional, as the following example shows.

Example.

The tangent bundle TS^1 of the unit circle, equipped with the adapted complex structure induced by the standard metric on S^1, is nothing else than the punctured complex plane $\mathbb{C}^*$. Denote by $T^n = S^1 \times \cdots \times S^1$ the n-dimensional torus with the product metric. It follows from Proposition 4.4 that $T(T^n)$ with its adapted complex structure is biholomorphic to $\mathbb{C}^{*n} := \mathbb{C}^* \times \cdots \times \mathbb{C}^*$. Let know f be any holomorphic function on the complex plane. Then the map

$$\begin{gathered} \mathbb{C}^n \longrightarrow \mathbb{C}^n \\ \Phi : (z_1, z_2, z_3, \ldots, z_n) \longmapsto (e^{f(z_1,z_2)}z_1, e^{-f(z_1,z_2)}z_2, z_3, \ldots, z_n) \end{gathered}$$

is an element of $Aut(\mathbb{C}^{*n})$. In fact we could take any flat metric on T^n, the induced adapted complex structure on $T(T^n)$ would all be biholomorphic to each other and thus to $\mathbb{C}^{*n}$. This shows the infinite dimensionality of $Aut(T(T^n))$.

Now we would like to prove an analogue of Theorem 5.3.

Theorem 7.1. *Let (M, g) and (N, h) be compact Riemannian manifolds. Assume that adapted complex structure exists on TM and TN. Suppose that the first cohomology group of M with real coefficients vanishes. Denote by κ_g and κ_h the Kahler metrics on TM and TN, induced by the metrics. Let*

$$\Phi : (TM, \kappa_g) \to (TN, \kappa_h) \tag{7.1}$$

be a biholomorphic isometry. Then Φ maps M diffeomorphically onto N and for the restriction map, $f := \Phi|_M$,

$$f : (M, g) \to (N, h)$$

is an isometry and $\Phi \equiv f_*$.

Proof:.
Just as in the proof of Theorem 5.3, let ρ_1 and ρ_2 be the norm-square functions on TM and TN. (7.1) implies

$$\partial\bar{\partial}\rho_1 = \Phi^*\partial\bar{\partial}\rho_2 = \partial\bar{\partial}(\rho_2 \circ \Phi). \tag{7.2}$$

Let

$$h := \rho_2 \circ \Phi - \rho_1. \tag{7.3}$$

According to (7.2), h is a pluriharmonic function on TM.

We need two key observation to prove the theorem.

Proposition 7.2. *There exists a holomorphic function $\mathcal{H} : TM \to \mathbb{C}$ with imaginary part h.*

Proof of Proposition 7.2.
Let $\{U_\alpha\}$ be a covering of TM with κ_g metric-convex balls. Thus for any α and β, $U_\alpha \cap U_\beta$ is also convex and in particular connected. We can choose each U_α so small that it is contained in a local holomorphic coordinate patch and that on U_α there exists a holomorphic function $\mathcal{H}_\alpha : U_\alpha \to \mathbb{C}$, with imaginary part $h|_{U_\alpha}$. This holomorphic function is well defined, up to a real constant. Since every $U_\alpha \cap U_\beta$ is connected, for every α and β we obtain a real constant $c_{\alpha\beta}$, such that

$$c_{\alpha\beta} = (H_\alpha - H_\beta)|_{U_\alpha \cap U_\beta}.$$

The data $(\{U_\alpha\}, \{c_{\alpha\beta}\})$ give us on TM a Čech cocycle with real coefficients. Denote by $\check{H}$ the Čech and by H the singular cohomology group. Then by our assumption

$$\check{H}^1(TM, \mathbb{R}) \cong H^1(TM, \mathbb{R}) \cong H^1(M, \mathbb{R}) = 0.$$

It is well known that for any (paracompact, Hausdorff) X, the vanishing of $\check{H}^1(X, \mathbb{R})$ implies that the group $\check{H}^1(X, \{U_\alpha\}, \mathbb{R})$ also vanishes. Hence there exist real numbers c_α, with $c_\beta - c_\alpha = c_{\alpha\beta}$. Thus the function $\mathcal{H}$, defined by $c_\alpha + H_\alpha$ on the set U_α is actually a well defined holomorphic function on the entire tangent bundle with imaginary part h. □

We need one more ingredient to finish the proof of Theorem 7.1.

Lemma 7.3. *Let $\gamma : \mathbb{R} \to M$ be a unit speed geodesic. Then there exist a real number β_γ depending on γ and a universal constant $A \in \mathbb{R}$ such that for every complex number $z = \sigma + i\tau$,*

$$\rho_2 \circ \Phi(\gamma_* z) = \tau^2 + \beta_\gamma \tau + A \ = \rho_1(\gamma_*(z)) + \beta_\gamma \tau + A \tag{7.4}$$

Proof of Lemma 7.3.

Let x be an arbitrary element of N and $q \in T_x N$. Denote by $dist_{\kappa_g}$ and $dist_{\kappa_h}$ the distance function for the metric κ_g and κ_h accordingly. From (PROP.III) we know

$$dist_{\kappa_h}(q, x) = dist_{\kappa_h}(q, N) = ||q||_h. \tag{7.5}$$

Let now m be an arbitrary point of M and $p \in T_m M$. Denote by $x \in N$ the image of the point $\Phi(p)$ under the projection map $\pi : TN \to N$. (7.5) together with the triangle inequality and the fact that Φ is an isometry, implies

$$\begin{aligned} ||\Phi(p)||_h &= dist_{\kappa_h}(\Phi(p), x) \\ &\le dist_{\kappa_h}(\Phi(p), \Phi(m)) + dist_{\kappa_h}(\Phi(m), x) \\ &= dist_{\kappa_g}(p, m) + dist_{\kappa_h}(\Phi(m), x) \\ &\le ||p||_g + \max_{a \in M, b \in N} dist_{\kappa_h}(\Phi(a), b) = ||p||_g + C. \end{aligned} \tag{7.6}$$

Taking square of both sides of (7.6), we obtain

$$\rho_2 \circ \Phi(p) \le \rho_1(p) + 2||p||_g C + C^2. \tag{7.7}$$

Since h is pluriharmonic (see (7.2) and (7.3)) and γ_* is holomorphic, the function $v(z) := h(\gamma_*(z))$ is harmonic on the entire complex plane. The estimate (7:7) gives us an upper bound for the growth of v,

$$v(z = \sigma + i\tau) = h(\gamma_*(z)) = \rho_2(\Phi(\gamma_*(z)) - \rho_1(\gamma_* z) \le 2|\tau| C + C^2. \tag{7.8}$$

But harmonic functions on the complex plane with such growth condition can only be linear (see [Sa-Zy, (10.13), p. 335]). Thus there exist real constants β_γ and A_γ such that

$$v(z) = \beta_\gamma \tau + A_\gamma.$$

Notice that A_γ is the value what the function h takes along the curve γ. In particular h is a constant function along any geodesic in M. But geodesics intersect each other and thus h must be constant on M. This implies that A_γ does not depend on γ. $\square$

End of the proof of Theorem 7.1.

Let now γ be any unit speed geodesic in M. It follows from Proposition 7.2, (7.3) and (7.4) that the holomorphic function $\mathcal{H}(\gamma_*(z))$ must be of the form

$$\mathcal{H}(\gamma_*(z)) = \beta_\gamma z + iA + \tilde{A}_\gamma, \tag{7.9}$$

for some real number $\tilde{A}_\gamma$. By our assumption M is compact and hence the real part of $\mathcal{H}$ is bounded there. Together with (7.9) this implies that $\beta_\gamma = 0$.

Therefore (7.4) reads as

$$\rho_2(\Phi(\gamma_*(z))) = \rho_1(\gamma_*(z)) + A. \tag{7.10}$$

This is true for every unit speed geodesic. Thus

$$||\Phi(p)||_h^2 = ||p||_g^2 + A, \tag{7.11}$$

for every $p \in TM$.

Plugging a point of M into (7.11) we obtain that A must be nonnegative. On the other hand Φ is onto and thus for some $p \in TM$ the left side of (7.11) must vanish. This gives that $A = 0$ and therefore Φ maps M diffeomorphically onto N. Φ was an isometry, thus its restriction to the zero section, which we will call f, will also be an isometry. From (PROP.III) it follows then that $f : (M, g) \to (N, h)$ is also an isometry. The biholomorphisms Φ and f_* (see (PROP.V) agree on the maximal dimensional totally real subset M, hence they must agree everywhere. $\square$

Remarks.

If we drop the condition on the cohomology group of M in Theorem 7.1, then in general there can be other biholomorphic isometries besides the ones that come from isometries between M and N. For instance take (M, g) to be a flat torus T^n, i.e. a quotient of $\mathbb{R}^n$ with respect to a lattice Γ. The complex manifold $T(T^n)$ (equipped with the adapted complex structure) is just $\mathbb{C}^n/\Gamma$. In $\mathbb{C}^n$ any translation with a nonzero, purely imaginary vector is a holomorphic isometry, which descends to $\mathbb{C}^n/\Gamma$ and does not respect the zero section torus.

But in fact this is the worst what can happen. More precisely, assume that we are given two compact Riemannian manifolds (M, g) and (N, h). Suppose that both admits adapted complex structure on its entire tangent bundle. Let us given an

$$\Phi : (TM, \kappa_g) \longrightarrow (TN, \kappa_h)$$

biholomorphic isometry. From [Le-Sz] we know that the existence of a global adapted complex structure induces non-negative curvature. That means all the sectional curvatures of (M, g) and (N, h) are non-negative. Denote by $\widetilde{M}$ resp. $\widetilde{N}$ the universal covers, and by $\tilde{g}$ resp. $\tilde{h}$ the pull back metrics.

From [Ch-Gr] we know that $(\widetilde{M}, \tilde{g})$ and $(\widetilde{N}, \tilde{h})$ split isometrically, i.e.

$$(\widetilde{M}, \tilde{g}) \cong (M_0 \times \mathbb{R}^k, g_0 \times g_e), \qquad (\widetilde{N}, \tilde{h}) \cong (N_0 \times \mathbb{R}^l, h_0 \times g_e),$$

where (M_0, g_0) and (N_0, h_0) are compact simply connected Riemannian manifolds with global adapted complex structure and g_e denotes the Euclidean metric. Φ can be lifted to a

$$\widetilde{\Phi} : (TM_0 \times \mathbb{C}^k, , \kappa_{g_0} \times \kappa_{g_e}) \longrightarrow (TN_0 \times \mathbb{C}^l, \kappa_{h_0} \times \kappa_{g_e})$$

biholomorphic isometry. Therefore $k = l$ and there must exists an

$$\Phi_1 : (TM_0, \kappa_{g_0}) \longrightarrow (TN_0, \kappa_{h_0}),$$

and an

$$\Phi_2 : (\mathbb{C}^k, \kappa_{g_e}) \longrightarrow (\mathbb{C}^k, \kappa_{g_e})$$

biholomorphic isometries, such that

$$\widetilde{\Phi} = \Phi_1 \times \Phi_2.$$

According to Theorem 7.1, there must exists an

$$\tilde{f} : (M_0, g_0) \longrightarrow (N_0, h_0)$$

isometry with $\tilde{f}_* \equiv \Phi_1$. Φ_2 can of course be only a unitary action plus a translation with a vector in $\mathbb{C}^k$.

8. Complexifying the isometry group action.

In the previous section we obtained a reasonable clear picture how the biholomorphic isometries look like between the tangent bundles of two Riemannian manifolds. Now we would like to drop once again the extra condition isometry and wish to study the biholomorphism maps between TM and TN. But this task seems far too ambitious at such generality. At the moment we cannot even answer to such "simple" questions as the following.

From [Sz2] or [St1] one knows that the round metric on the n-dimensional sphere S^n induces its adapted complex structure on the entire tangent bundle and as a complex manifold TS^n is biholomorphic to the affine hyperquadric, Q^n

$$Q^n = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \mid z_1^2 + \cdots + z_{n+1}^2 = 1\}.$$

Question: what is the biholomorphism group of $Q^n \cong TS^n$? Even though we do not know the answer to this question but it is obvious that the complex orthogonal group $O(n+1, \mathbb{C})$ is a subgroup of $Aut(Q^n)$.

Notice that $O(n+1, \mathbb{C})$ is the complexification of the compact Lie group $O(n+1, \mathbb{R})$ which is the isometry group of S^n. This is the key observation and in this section we would like to show that this example is not special, but in fact it is the general case. First we need some preparatory lemmas and propositions.

Let us recall Fatou's classic theorem concerning representations of positive harmonic functions defined on the upper half plane $\mathbb{C}^+$.

Theorem 8.1. *(Fatou, see [Koo]) Let u be a positive harmonic function defined on $\mathbb{C}^+$. Then there exist a non-negative Borel measure μ on the real line and a non-negative real number α, such that $\int_{\mathbb{R}} 1/(1+t^2)d\mu(t)$ is finite and for any ζ in the upper half plane*

$$u(\zeta = \sigma + i\tau) = \alpha\tau + \frac{1}{\pi}\int_{\mathbb{R}} \frac{\tau}{|w - t|^2} d\mu(t). \tag{8.1}$$

Define the domain $D_\sqcup$ by

$$D_\sqcup := \{\zeta = \sigma + i\tau \in \mathbb{C} \mid |\sigma| < 1, 1 < \tau\}.$$

Lemma 8.2. *Let $\zeta = \sigma + i\tau \in D_{\sqcup}$ and t be an arbitrary real number. Then*

$$\frac{1}{3\tau(1+t^2)} < \frac{\tau}{\tau^2 + (\sigma - t)^2}, \tag{8.2}$$

$$\frac{3}{1+t^2} > \frac{(\tau - 1)^2}{[(\sigma - t)^2 + 1][(\sigma - t)^2 + \tau^2]}. \tag{8.3}$$

Proof. Elementary and left to the reader.

Lemma 8.3. *Let f be an $f : \mathbb{C}^+ \to \mathbb{C}^+$ holomorphic function, and $\zeta = \sigma + i\tau \in D_{\sqcup}$. Then there exists an α, $0 \le \alpha \le Im\ f(i)$, such that*

$$\begin{aligned} |Re\ f(\sigma + i\tau)| &\le |Re\ f(\sigma + i)| + \sqrt{3} Im\ f(i), \\ Im\ f(\sigma + i\tau) &\le Im\ f(\sigma + i) + \sqrt{3} Im\ f(i) + \alpha(\tau - 1), \\ Im\ f(\sigma + i\tau) &> \frac{\alpha(3\tau^2 - 1) + Im\ f(i)}{3\tau}. \end{aligned} \tag{8.4}$$

Proof. Applying (8.1) to the imaginary part of f, for any $\zeta = \sigma + i\tau \in \mathbb{C}^+$ we get (for some nonnegative α and nonnegative Borel measure μ)

$$\text{Im } f(\sigma + i\tau) = \alpha\tau + \frac{1}{\pi} \int_{\mathbb{R}} \frac{\tau}{(\sigma - t)^2 + \tau^2} d\mu(t). \tag{8.5}$$

In particular for $\zeta = i$, (8.5) gives

$$\text{Im } f(i) = \alpha + \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{1+t^2} d\mu(t). \tag{8.6}$$

Since μ is nonnegative, this yields that α is in the desired range.

Applying the estimate (8.2) to the integrand in (8.5) and using (8.6), we obtain

$$\begin{aligned} \text{Im } f(\sigma + i\tau) &> \alpha\tau + \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{3\tau(1+t^2)} d\mu(t) \\ &= \alpha\tau + \frac{\text{Im } f(i) - \alpha}{3\tau}. \end{aligned}$$

This proves the last inequality of (8.4). To prove the first two, we have to differentiate (8.5) with respect to ζ:

$$f'(\sigma + i\tau) = \alpha + \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{(\sigma - t + i\tau)^2} d\mu(t).$$

This yields, by changing the order of integration,

$$
\begin{aligned}
f(\sigma+i\tau) &= f(\sigma+i) + \int_{\sigma+i}^{\sigma+i\tau} f'(\zeta)d\zeta \\
&= f(\sigma+i) + \int_{\sigma+i}^{\sigma+i\tau} \left(\alpha + \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{(\zeta-t)^2}d\mu(t)\right) d\zeta \\
&= f(\sigma+i) + i\alpha(\tau-1) + \frac{1}{\pi}\int_{\mathbb{R}} \frac{i(\tau-1)}{(\sigma-t+i)(\sigma-t+i\tau)}d\mu(t).
\end{aligned} \tag{8.7}
$$

Using (8.3) and (8.6), we can estimate the integral in (8.7) by above,

$$
\left|\frac{1}{\pi}\int_{\mathbb{R}} \frac{i(\tau-1)}{[\sigma-t+i][\sigma-t+i\tau]}d\mu(t)\right| \le \frac{\sqrt{3}}{\pi}\int_{\mathbb{R}} \frac{1}{1+t^2}d\mu(t) \le \sqrt{3}\mathrm{Im}\ f(i).
$$

Applying this estimate and taking real and imaginary parts of (8.7), we obtain the first two inequalities of (8.4). ☐

Lemma 8.4. *Let $F : \mathbb{C}^+ \to \mathcal{H}^n$ be a holomorphic map. Let $w \in \mathbb{C}$ and $\zeta \in D_\sqcup$. Denote by$\langle .,.\rangle$ the ordinary hermitian scalar product on $\mathbb{C}^n$. Then*

$$
\begin{aligned}
|\langle Re\ F(\zeta)w,w\rangle| &< |\langle Re\ F(\sigma+i)w,w\rangle| + \sqrt{3}\langle Im\ F(i)w,w\rangle, \\
\langle Im\ F(\zeta)w,w\rangle &< \langle Im\ F(\sigma+i)w,w\rangle + (\tau+\sqrt{3}-1)\langle Im\ F(i)w,w\rangle, \\
\langle Im\ F(\zeta)w,w\rangle &> (\tau\langle Im\ F(i)w,w\rangle)/3.
\end{aligned} \tag{8.8}
$$

Suppose now that we have a continuous map

$$
F : \mathbb{C}^+ \times K \to \mathcal{H}^n,
$$

where K is a compact topological space. Assume that for every $x \in K$ the map

$$
F(.,x) : \mathbb{C}^+ \to \mathcal{H}^n
$$

is holomorphic. Then there exists a constant $A > 0$, such that for every $\zeta = \sigma+i\tau \in D_\sqcup$, and $x \in K$ we have ($\|\ \|$ denotes the matrix norm)

$$
\begin{aligned}
&\|Re\ F\|(\zeta,x) \le A, \\
&\|Im\ F\|(\zeta,x) \le A(\tau+1), \\
&\|(Im\ F)^{-1}\|(\zeta,x) < A\tau.
\end{aligned} \tag{8.9}
$$

Proof. To prove (8.8), we can assume that $w \neq 0$. Then the function

$$
f_w(\zeta) := \langle F(\zeta)w,w\rangle
$$

is holomorphic and the real resp. imaginary part of f_w is $\langle \mathrm{Re\ F}(\zeta)w,w\rangle$, resp. $\langle \mathrm{Im\ F}(\zeta)w,w\rangle$. Therefore f_w maps $\mathbb{C}^+$ into itself and we can apply Lemma 8.3 (and the fact that $0 \le \alpha \le \mathrm{Im}\ f_w(i)$ in the formulas in (8.4)) to obtain (8.8). By compactness argument, (8.9) follows immediately. ☐

Proposition 8.5. *Let (M, g) be a Riemannian manifold which admits an adapted complex structure on the entire tangent bundle. Let X be an element of the Lie algebra of the isometry group $Isom(M, g)$. Denote by $X_\#$ the induced infinitesimal vector field on TM (by the action of $Isom(M, g)$ on TM). Then $X_\#^{1,0}$ is holomorphic on TM.*

Proof. $Isom(M, g)$ acts on TM by biholomorphisms. (see (PROP.V)) Let

$$\begin{aligned} f : \mathbb{R} \times TM &\longrightarrow TM \\ (t, p) &\longmapsto (\exp tX)_* p. \end{aligned}$$

Then $X_\# = df/dt|_{t=0}$. Apply Proposition 4.2. □

Theorem 8.6. *Let (M, g) be a compact Riemannian manifold. Suppose that adapted complex structure exists on the entire tangent bundle. Let κ_g be the induced Kahler metric. Let X be an element of the Lie algebra of the isometry group $Isom(M, g)$. Denote by $X_\#$ the induced infinitesimal vector field on TM (by the action of $Isom(M, g)$ on TM). Then there exist positive constants A_X and B_X such that*

$$||X_\#(p)||_{\kappa_g} \leq A_X ||p||_g + B_X, \tag{8.10}$$

for every $p \in TM$.

Remark. Let (M^n, g) be the standard round sphere S^n. Let X_j be an orthonormal base of the Lie algebra $\mathfrak{so}(n+1)$. Then we can read off from [St2] that for $p \in TS^n$,

$$\sum_j ||X_{j\#}||^2(p) = O(||p||_g,), \quad \text{as} \quad ||p||_g \to \infty.$$

This shows that the estimate in (8.10) is not sharp, but it is enough to prove that the flow of the vector field $JX_\#$ is complete (See Corollary 8.7 below).

Proof of Theorem 8.6.

Let p be a point in TM with norm one. Let ϵ be a small, positive number. Denote by D_ϵ the unit ball in $\mathbb{R}^{2n-2}$. If ϵ is small enough, then we can choose a neighbourhood U_p of the point p in the unit sphere bundle of TM, such that U_p is diffeomorphic to $(-\epsilon, \epsilon) \times D_\epsilon$ (because of Theorem 3.2, we can assume that in fact we have a real analytic diffeomorphism) and under this diffeomorphism the curves (for every fixed point x in the euclidean ball D_ϵ)

$$t \mapsto (t, x) \in (-\epsilon, \epsilon) \times D_\epsilon$$

correspond to the trajectories of the geodesic flow. Denote the image of the euclidean ball $0 \times D_\epsilon$, under the above diffeomorphism, by $\mathcal{N}$. Thus $\mathcal{N}$ parametrizes the trajectories of the geodesic flow in the neighbourhood U_p.

Choose real analytic vector fields $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$ in a neighbourhhod of $\overline{U}_p$ (the neighbourhood is still in the unit sphere bundle, but the vector fields are not all tangential to it, they are only sections of $T(TM)$), such that they are invariant under the geodesic flow, and for every point $q \in \mathcal{N}$,

$$\xi_1(q), \ldots, \xi_n(q), \quad \text{and} \quad \eta_1(q), \ldots \eta_n(q)$$

are the horizontal and vertical lifts, of an orthonormal frame $v_1, \ldots, v_n \in T_{\pi(q)}M$, and $v_n = q$. For a fixed point q in the neighbourhood U_p, take all positive multiples of q, to get a half line in the tangent space $T_{\pi(q)}M$. If we do this for every point in U_p, the union of these half lines provides us a domain D_p in TM. Extend now the vector fields ξ_j, η_k to be defined on D_p and invariant with respect to the N_s actions, (see (1.1)) for every positive s, and call these extended fields with the same name.

Then for every $q \in \mathcal{N}$, the frame $\{\xi_j, \eta_j\}_{j=1}^n$ is a symplectic frame along the leaf L_q of the Riemann foliation, defined by q (see (1.1) and the discussion above (2.1)). Since $X_\#$ is coming from the isometry group action, the vector field $X_\#$ is parallel along every leaf of the Riemannian foliation. Moreover, according to Proposition 8.5, $X_\#^{1,0}$ is holomorphic. Therefore there exists smooth (in fact real analytic) functions

$$\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n : \mathcal{N} \to \mathbb{R}$$

(which we also consider to be defined on D_p, being constant along every leaf of the Riemann foliation) such that in D_p,

$$X_\# = \sum_{j=1}^{n} \alpha_j \xi_j + \sum_{k=1}^{n} \beta_k \eta_k. \tag{8.11}$$

For a point $q \in \mathcal{N}$, denote by γ_q the unit speed geodesic, with initial datum $\dot\gamma(0) = q$.

From (4.5) we obtain a map

$$F = (f_{jk}) : \mathbb{C}^+ \times \mathcal{N} \longrightarrow \mathcal{H}^n,$$

such that

$$\eta_k^{1,0}(\gamma_{q*}(\zeta)) = \sum_{j=1}^{n} f_{jk}(\zeta)\xi_j^{1,0}(\zeta),$$

and because of our choice, F is also real analytic in the subdomain

$$\{(\zeta = \sigma + i\tau, q) \in \mathbb{C}^+ \times \mathcal{N}, |\sigma| < \epsilon\}.$$

From (4.4) it follows that for every $x \in \mathcal{N}$, the map

$$F(.,q) : \mathbb{C}^+ \to \mathcal{H}^n$$

is holomorphic. According to Theorem 3.3, this implies that

$$F : \mathbb{C}^+ \times \mathcal{N} \longrightarrow \mathcal{H}^n$$

is real analytic. (Continuity would actually be enough for our purposes.) Let

$$\alpha := (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) : \mathcal{N} \to \mathbb{R}^n.$$

Then (4.7) and (8.11) yields for any $\zeta \in \mathbb{C}^+$, $p := \gamma_{q*}(\zeta)$

$$\begin{aligned}\langle X_\#, X_\# \rangle_{\kappa_g}(p) = \|p\|_g \{ &\langle (\mathrm{Im}\ F)^{-1}(\zeta, q)\alpha(q), \alpha(q)\rangle \\ &+ 2\langle ((\mathrm{Im}\ F)^{-1}\mathrm{Re}\ F)(\zeta, q)\alpha(q), \beta(q)\rangle \\ &+ \langle [(\mathrm{Re}\ F(\mathrm{Im}\ F)^{-1}\mathrm{Re}\ F)(\zeta, q) \\ &+ \mathrm{Im}\ F(\zeta, q)]\beta(q), \beta(q)\rangle \}.\end{aligned} \tag{8.12}$$

Using our estimate (8.9), for $K = \overline{U}_p$, we can find a positive constant $A = A_{\overline{U}_p}$, such that for any $\zeta = \sigma + i\tau \in D_\sqcup$, and $q \in \mathcal{N}$,

$$\begin{aligned}\langle (\mathrm{Im}\ F)^{-1}(\zeta, q)\alpha(q), \alpha(q)\rangle_{\kappa_g} &\le A(\sup_{\mathcal{N}} \|\alpha\|^2)\tau \\ |\langle [(\mathrm{Im}\ F)^{-1}\mathrm{Re}\ F](\zeta, q)\alpha(q), \beta(q)\rangle_{\kappa_g}| &\le A^2(\sup_{\mathcal{N}} \|\alpha\| \sup_{\mathcal{N}} \|\beta\|)\tau \\ |\langle [\mathrm{Re}\ F(\mathrm{Im}\ F)^{-1}\mathrm{Re}\ F + \mathrm{Im}\ F](\zeta, q)\beta(q), \beta(q)\rangle_{\kappa_g}| &\le (A^3\tau + A\tau + A)\sup_{\mathcal{N}} \|\beta\|^2.\end{aligned} \tag{8.13}$$

(By shrinking the neighbourhood U_p a bit, all the sumpremums will be finite.) (8.12) and (8.13) shows that for some $\tilde{A}$ and $\tilde{B}$, and arbitrary $z \in D_p$, with norm at least one,

$$\langle X_\#, X_\# \rangle_{\kappa_g}(z) \le \tilde{A}\|z\|_g + \tilde{B}. \tag{8.14}$$

Our manifold M is compact, and its unit sphere bundle is as well. Hence we can choose the constants in (8.14) such that the estimate (8.14) in fact holds for every point z of the tangent bundle. □

Corollary 8.7. *Let (M, g) be a compact Riemannian manifold which admits an adapted complex structure on the entire tangent bundle. Let X be an element of the Lie algebra of $Isom(M, g)$. Denote by $X_\#$ the induced infinitesimal vector field on TM. Then the flow of $JX_\#$ is complete.*

Proof.

For the sake of brevity denote again by ρ the norm-square function on TM and by κ_g the induced Kahler metric. From [Le-Sz, Prop.3.2] we know that

$$\|grad\rho\|_{\kappa_g} = \sqrt{\rho}.$$

Thus using the Cauchy-Schwarz inequality and Theorem 8.6 to estimate the quantity

$$(JX_\#)\rho(p) = \langle \mathrm{grad}\rho, JX_\# \rangle_{\kappa_g}(p),$$

we obtain that there exist positive constants C_X and D_X such that

$$|(JX_\#)\rho(p)| \le C_X\rho(p) + D_X.$$

Applying [Ab-Ma, Prop. 2.1.20], we are done. □

Theorem 8.8. *Let (M, g) be a compact Riemannian manifold that admits an adapted complex structure on its entire tangent bundle. Denote by G the unit component of the compact Lie group $Isom(M, g)$. Consider G as a transformation group acting on TM by the induced action. This G-action extends to a group action of the complexified group $G_{\mathbf{C}}$, and the map*

$$G_{\mathbf{C}} \times TM \longrightarrow TM$$

is holomorphic. The subgroup of $G_{\mathbf{C}}$ that consists of elements acting trivially on TM is discrete.

Remarks.

If G is a compact, connected Lie group that acts on a compact, complex manifold X by biholomorphisms, then this action always extends to an action of $G_{\mathbf{C}}$ (also by biholomorphisms). This was shown in [Gu-St, Theorem 4.4] The key fact they used was that for a compact, complex manifold X, $Aut(X)$ is a complex Lie group. (See [Bo-Mo1 and 2]).

When we drop the condition compactness, we are going to face two problems. Firstly, it may happen that even though $Aut(X)$ is still a Lie group, its Lie algebra is totally real. This happens for instance for any bounded domain in $\mathbb{C}^n$ (see [Kob, ChIII. Theorem 1.3, or Bo-Mo2]). This makes of course the complexification of the group action impossible.

Secondly it can happen that $Aut(X)$ is so large that it is not even a (finite dimensional) Lie group. This phenomenon does not apriory prevent us from complexifying a group action but certainly makes the situation harder.

In our setup this second possibility can occur as it was pointed out in the Example at the beginning of section 7. Therefore we have to choose a more cumbersome detour to get the same conclusion as Guillemin and Sternberg.

When G is an arbitrary compact, connected Lie group and H is a closed subgroup, we can form the homogeneous space G/H. We can equip this space with the so called normal metric, to obtain a Riemannian homogeneous space $(G/H, g)$. It was shown by M. Stenzel in [St1] that the complex homogeneous manifold $G_{\mathbf{C}}/H_{\mathbf{C}}$ can naturally be identified with $T(G/H)$, this latter equiped with the adapted complex structure of the normal metric. Theorem 8.8 can be considered as a generalisation of this situation.

Proof of Theorem 8.8.

Denote by $\mathcal{A}(TM)$ the complex vector space of holomorphic vector fields on the complex manifold TM, i.e. $\mathcal{A}(TM)$ consists of vector fields V such that $V^{1,0}$ is a holomorphic section of $T^{1,0}TM$. In fact $\mathcal{A}(TM)$ becomes a complex Lie algebra if we take the obvious complex multiplication and Lie bracket= minus the ordinary Lie bracket of vector fields. The integrability of the almost complex tensor assures that $\mathcal{A}(TM)$ is indeed a complex Lie algebra. The reason for the sign convention is to make things compatible with induced infinitesimal generators. (See below.) Denote by $\mathfrak{g}$ the Lie algebra of G. From Proposition 8.5 we know that for any $X \in \mathfrak{g}$, the induced infinitesimal generator $X_{\#}$ on TM belongs to $\mathcal{A}(TM)$. The map

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} + i\mathfrak{g} \longrightarrow \mathcal{A}(TM)$$

$$\delta : X + iY \longmapsto X_{\#} + JY_{\#}$$

is a $\mathbf{C}$ linear Lie-algebra monomorphism. $\mathbf{C}$ linearity is obvious. Lie-algebra homomorphism follows from [Ab-Ma, Proposition 4.1.26] because of our sign convention.

Corollary 8.7 tells us that every element of $\mathcal{L} := \delta(\mathfrak{g}_\mathbf{C})$ induces a one parameter group of diffeomorphisms of TM. It follows from Palais' work (see [Pa]), that there exists a unique, connected Lie group $\widehat{G}$, whose underlying group is a subgroup of the group of diffeomorphisms of TM, the Lie algebra of $\widehat{G}$ is $\mathcal{L}$, the map

$$\widehat{G} \times TM \longrightarrow TM$$

is differentiable (i.e. $\widehat{G}$ is a connected Lie transformation group on TM, each element of $\widehat{G}$ different from the identity acts nontrivially on TM), and $\widehat{G}$ extends the G action on TM. (Hence G can also be considered a Lie subgroup of $\widehat{G}$.)

Since $\mathcal{L}$ is a complex Lie-algebra, the corresponding group $\widehat{G}$ will be a complex Lie group. (Recall Proposition 3.4.) Let $\widetilde{G}$ be the universal covering group of G. Then $T\widetilde{G} \cong \widetilde{G}_\mathbf{C}$ will be the universal cover of $TG \cong G_\mathbf{C}$. By a classical theorem in Lie theory, there exists a unique homomorphism,

$$\Delta : \widetilde{G}_\mathbf{C} \longrightarrow \widehat{G},$$

with differential δ at the unit element.

Therefore Δ is a holomorphic covering map. Then (since $\widehat{G}$ extends the action of G)

$$\text{Ker } \Delta \supset K = \text{Ker } (\widetilde{G} \to G) = \text{Ker } (\widetilde{G}_\mathbf{C} \to G_\mathbf{C}).$$

Thus we get a holomorphic covering map,

$$\widetilde{\Delta} : G_\mathbf{C} = \widetilde{G}_\mathbf{C}/K \longrightarrow \widehat{G}.$$

Hence $G_\mathbf{C}$ indeed acts on TM and Ker $\widetilde{\Delta}$ is discrete. (only the elements of Ker $\widetilde{\Delta}$ act trivially on TM.)

Since the Lie-algebra of $\widehat{G}$ is $\mathcal{L} \subset \mathcal{A}(TM)$, all the elements of $\widehat{G}$ that belong to a 1-parameter subgroup, act by biholomorphisms on TM. But these elements generate the whole group, hence $\widehat{G}$, and then of course $G_\mathbf{C}$ as well, acts on TM by biholomorphisms. This implies that the transformation map

$$\beta^\mathbf{C} : G_\mathbf{C} \times TM \longrightarrow TM,$$

is holomorphic in the second variable. Since $\beta^\mathbf{C}$ is smooth, in order to prove that it is holomorphic in all its variables, it suffices to show that for any point $p \in TM$, the map

$$\beta^\mathbf{C}_p : G_\mathbf{C} \to TM, \qquad G_\mathbf{C} \ni a \mapsto \beta^\mathbf{C}(a,p),$$

is holomorphic. From Theorem 3.2 we know that the metric on M is real-analytic and therefore the restricted transformation map

$$\beta := \beta^\mathbf{C}\big|_G : G \times TM \longrightarrow TM,$$

is real-analytic and consequently $\beta_p^{\mathbf{C}}|_G$ as well. Since TM is a Stein manifold, we can think of $\beta_p^{\mathbf{C}}$ as a map going into $\mathbb{C}^N$ for some large N. Equipping G with a two-sided invariant metric h, from Proposition 3.4 we know that TG with the adapted complex structure of h is precisely $G_{\mathbf{C}}$. Hence, using (PROP.VI), it suffices to prove that for any unit-speed geodesic $\gamma : \mathbb{R} \to G$, the composition map $\beta_p^{\mathbf{C}} \circ \gamma_*$ is holomorphic. Since homogeneity, it suffices to check this for geodesics through the unit element, i.e. for 1-parameter subgroups of G. Let $X \in \mathfrak{g}$, and $\gamma(\sigma) = \exp(\sigma X)$. The induced map is (just like in the proof of Proposition 3.4)

$$\gamma_* : T\mathbf{R} \cong \mathbb{C} \ni \zeta = \sigma + i\tau \longmapsto \exp(\zeta X) \in G_{\mathbf{C}}.$$

Hence the composition map $\beta_p^{\mathbf{C}} \circ \gamma_*$ can also be written as a composition of the holomorphic maps,

$$\mathbb{C} \ni \sigma + i\tau \longmapsto \sigma X_\# + \tau J X_\# \in \mathcal{L},$$

and

$$\mathcal{L} \ni V \longmapsto \chi_V(1),$$

and thus itself is also holomorphic. (Here χ is the trajectory of the vector field $V \in \mathcal{L}$ with initial condition $\dot{\chi}(0) = V(p)$. This latter map is holomorphic since: solutions of an O.D.E. that depends holomorphically on some parameters, also depend holomorphically on the same parameters.) $\square$

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