Dedicated to Professor Friedrich Hirzebruch

TOPOLOGY OF SYMPLECTOMORPHISMGROUPS AND PSEUDOHOLOMORPHIC CURVES

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Abstract. In this note we study the topology of symplectomorphismgroups using pseudoholomorphic curves and symplectic bundles. As application we show the non-triviality of certain homotopy groups of some symplectomorphismgroups.

1. Introduction.

Given a symplectic manifold (M^{2n}, ω), one of the basic mathematical objects associated to (M^{2n}, ω) is its automorphismgroup $Sym(M, \omega)$. There are many important questions related to this object: how this automorphismgroup acts on M and its natural geometrical objects (e.g. the loop space, the tangent bundle of M, the space of compatible almost complex structures)?, how to describe representations of this automorphismgroup into other (finite or infinite dimensional) space, how to compute the algebraical cohomology group of $Sym(M, \omega)$ and etc. On the other hand, since the group $Sym(M, \omega)$ can be equipped with a natural smooth (C^∞) topology, we would like also to know the homotopy type of this automorphismgroup. This note is devoted to the topological properties of the group $Sym(M, \omega)$. Namely we are interested in the following questions:

1) How big is the rank of the homotopy group $\pi_i(Sym_{smooth}(M, \omega) \otimes \mathbf{Q})$?

2) What are the characteristic classes of $Sym_{smooth}(M, \omega)$, that is, the cohomology ring of the classifying space $BSym_{smooth}(M, \omega)$.

Like diffeomorphismgroups, the group symplectomorphisms of a compact symplectic manifold has infinite dimension. One naturally wishes to apply the methods and ideas in the study of topology of diffeomorphismgroups to the symplectic case. We know three approachs in

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the study of topology of diffeomorphism groups: the geometro- homotopical approach (using gluing, bordism, cocordance,...), the geometro-algebraical approach (construction of concrete cycles or cocycles by invariant differential forms, by the imbedding of certain compact Lie groups, or by considering the action of the group of symplectomorphisms on the associated geometrical objects,..), and the geometro-analytical approach. Our note is influenced by the seminal Gromov's paper [Gr] and inspired by the talk of John Lott [L 1] on the third approach.

This note consists of three parts. In the section 2 we recall the definition of symplectic bundle and extend the notion of stable holomorphic map to a family version. In section 3 we show that the basic properties of the moduli space of stable maps also hold in the parametrized (family) version. As an immediate consequence we construct parametrized Gromov-Witten invariants for symplectic bundles. In section 4 we apply this construction to problems 1, 2 mentioned above. We associate to an element in $\pi_i(Sym(M,\omega))$ a symplectic bundle over S^{i+1} which is the sum of two trivial symplectic bundles over disk D^{i+1} glued along the boundary ∂D^{i+1} by this element in $\pi_i(Sym(M,\omega))$. By computing a parametrized Gromov-Witten invariant for a symplectic bundle over S^2 we reinterpret a result by Gromov [G, Theorem 2.4.C_2] on the existence of an element of infinite order in the symplectomorphism group of $S^2 \times S^2$, see Theorem 4.2. We also prove the following theorem on higher homotopy group of the group of symplectomorphisms of a symplectic manifold of dimension greater than 4.

Theorem 4.4 . *Let (M,ω) be a symplectic manifold $(S^2 \times S^2 \times N^{2k}, \omega_1 \oplus \omega_2 \oplus \omega_3)$ and let A_i denote the image of i-th sphere S^2 in M. Suppose that $\omega_1(A_1) > \omega_2(A_2)$. Then we have $rk(\pi_1(Sym(M,\omega)) \otimes \mathbf{Q}) \geq 1$ and $rk(\pi_3(Sym(M,\omega)) \otimes \mathbf{Q}) \geq 2$.*

In the section 4 we also construct characteristic classes of the group $Sym(M,\omega)$ by dualizing the construction of Gromov-Witten invariants. We compare this construction with a recent construction of the characteristic classes by Kontsevich.

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results and related (not yet published) results by Abbreu, McDuff and Seidel.

2. Symplectic bundles and vertical stable holomorphic maps

2.1. Symplectic bundles and their fiberwise compatible almost complex structures. We recall that a fibration $F \to E \stackrel{j}{\to} B$ is called **a symplectic fibration** if the fiber F is diffeomorphic to a symplectic manifold (F, ω) and the transition function takes its value in the group $Sym(F, \omega)$. Let us denote by $\mathcal{J}(B, F)$ the associated bundle over B whose fiber is the space $\mathcal{J}(F)$ of smooth compatible almost complex structures on (F, ω) which is compatible to ω. Since the fiber $\mathcal{J}(F)$ is contractible, the space of sections $J(E) : B \to \mathcal{J}(B, F)$ is also contractible, in particular non-empty.

In what follows we are interested in defining invariants which detect the non-triviality of symplectic bundles. Associated to any symplectic bundle $F \to E \stackrel{j}{\to} B$ there is a (co)homological bundle $H_*^{(*)}(F) \to E_*^{(*)} \stackrel{j_*^*}{\to} B$, (where, in what follows, the coefficients of cohomology groups are in $\mathbf{R}$ or $\mathbf{Z}$, and the coefficients of homology groups are in $\mathbf{Z}$). Clearly all the invariants of the associated (co)homological bundles are also invariants of symplectic bundles. In particular if a symplectic bundle is trivial then the associated (co)homological bundle is trivial. We call a bundle with the trivial associated homological bundle **homological simple**. We also observe that, for a symplectic bundle E there is always a section $s^{[\omega]} : B \to E^*$ which takes a constant value $[\omega] \in H^2(F, \mathbf{R})$ and there is also a section $s^{c_1} : B \to E^*$ which takes a constant value $c_1(F, \omega) \in H^2(F, \mathbf{Z})$.

2.2. Vertical stable holomorphic maps. First we introduce a notion of a stable vertical map (cf. [LT]). Let $(\Sigma, \vec{x})$ be a semi-stable (or prestable) curve i.e. a connected union of Riemann surfaces with possible singularity of ordinary double points and with the set of marked points $\vec{x}$ outside the singularity set (se [Ko1] and [KM], [BM])for a more precise definition). A map u from a semi-stable curve $(\Sigma, \vec{x})$ to E is called a vertical stable map if $j(u) = pt \in B$ and u is a stable map (i.e. the components of Σ which is mapped to a point must be stable).

Let $J(E)$ be a vertical compatible almost complex structure. A map $u : (\Sigma, \vec{x}) \to E$ is called a vertical stable $J(E)$-holomorphic map if the composition map $j \circ u$ sends $(\Sigma, \vec{x})$ to a point $pt \in B$ and u is a stable holomorphic map from $(\Sigma, \vec{z})$ to $j^{-1}(pt) = (F, J(E)_{|F})$.

The equivalence between two stable maps is defined via the biholomorphism between their domains which make the diagram involving the mappings commutative. (cf[Ko1], [FO], [LT].)

To define the moduli space of vertical stable holomorphic maps we assume first, for the sake of simplicity and a later application, that the fibration is homologically simple (e.g. if the base B is simply connected .[1]) In this case for a class $A \in H_2(F, \mathbf{Z})$ there is also a constant section $s_A : B \to E_*$ whose value is A. Then we denote by $C\mathcal{M}_{g,m}(E, J(E), s_A)$ the moduli space of vertical stable maps of $(\Sigma_g, \vec{x})$, where $|\vec{x}| = m$ and the image u realizes the homology class $s_A(j(u)) \in H_2(F, \mathbf{Z})$:

$$C\mathcal{M}_{g,m}(E, J(E), s_A) = \{(b \in B, u \in C\mathcal{M}_{g,m}(F = j^{-1}(b), J(E)_{|F}, A))\}.$$

If E is not homological simple, we still can define a notion of a moduli space of vertical stable holomorphic maps by considering a multi-valued section $s_A : B \to E_*$ which is a locally constant and takes a value A at a reference fiber $j_*^{-1}(b_0) \subset E_*$. We note that the pairing $< s^{[\omega]}(b), s_A(b) >$ as well as the pairing $< s^{c_1}(b), s_A(b) >$ are constant functions on B. Using the Gromov compactness theorem it is easy to see that there is only finite number of values of s_A in each fiber $H_2(F = j^{-1}(b), \mathbf{Z})$ such that there is a J-holomorphic curve realizing class $s_A(b)$ for some $J \in \mathcal{J}(F)$. (Thus the compactness of the moduli space of vertical stable holomorphic maps can be obtained as in the case of homological simple symplectic bundles). The moduli space of vertical stable $J(E)$-holomorphic curves is a subspace of the space $Map_{g,m}(E, j, s_A)$ of equivalence classes of stable maps of genus g with k marked points and realizing class s_A in each fiber F.

2.3. Examples of symplectic fibrations. There are several ways to construct symplectic fibrations.

The first way is the **induction method**. Suppose that a group G acts symplectically on a symplectic manifold (M, ω). Then we can associate to each principal G-bundle E over B a symplectic bundle $E \times_G (M, \omega)$. This symplectic bundle is non-trivial if and only if the image of the homotopy class $\lambda \in [B, BG]$ defining the G-bundle is non-trivial in $[B, BSymp(M, \omega)]$.

The second way is the **pull-back method**. Suppose we are given a symplectic bundle E over a base E. Then any map f from B' to B pulls the bundle E back to a symplectic bundle E' ove B'.

The third way is the **reduction method**. We begin with a differentiable fibration $F \to E \to B$ with F being a symplectic manifold and

[1] Symplectic fibration over S^1 is classified by the group $\pi_0(Sym(M, \omega))$. Using Floer homology we can detect non-trivial elements in this group [Li].

ask if this fibration also admits a structure of a symplectic fibration. Of course if the inclusion of $Sym(F,\omega)$ to $Diff^+(M)$ is a homotopy equivalence then it does (e.g. if dim $M = 2$). Otherwise we can use the following criterium in [McD-S 1].

Lemma 2.1. [McD-S1, p. 193]*Let* $\pi : E \to B$ *be a locally trivial fibration with connected base and* $\omega \in \Omega^2(E)$ *be a symplectic form such that the fibers are all symplectic submanifolds of* E. *Then* $\pi : E \to E$ *admits the structure of a symplectic fibration which is compatible with* ω.

The fourth way to construct symplectic fibrations is the **gluing method**. Suppose that we are given two symplectic bundles E_1 and E_2 over bases B_1 and B_2 respectively. Suppose that the restriction of E_1 over the boundary ∂B_1 is isomorphic to the restriction of E_2 to the boundary ∂B_2. Then we can glue the bundle E_1 with E_2 along the boundary $(E_i)_{|\partial B_i}$. In particular when the restriction of E_i to ∂B_i is trivial then the glued bundle is defined uniquely by a map $\partial B_1 \to Sym(F,\omega)$. If B_i is closed we can define the operation of (direct) fiber connected sum by throwing away from E_1 and E_2 the restriction of E_i over a small disk B_i and identifying $E_i|\ \partial B_i$ with the help of trivial map $\partial B_i \mapsto Id \in Symp(F,\omega)$.

Remark 2.2. Each element $g \in \pi_k(Sym(M,\omega))$ defines a symplectic bundle E with the fiber (M,ω) over a sphere S^{k+1} by gluing two trivial symplectic bundles $(M \times D^{k+1})$ along the boundary $M \times S^k$ by a representative of g. (We can also think in terms of classfying spaces and homotopy exact sequences.) Conversely any symplectic bundle over S^{k+1} is defined by such a method. Clearly (symplectic) bundles over S^k is homologically trivial if $k \geq 2$.

3. Parametrized Gromov-Witten invariants.

3.1. Geometric picture. We consider the case $3g-3+m \geq 0$ and we assume that the base B is compact and oriented. Let $C\mathcal{M}_{g,m}$ denote the Deligne-Mumford moduli space of stable curves, and $E^{\otimes m}$ - the Whitney sum of m copies of E. As in the non-parametrized case (see [KM, 2.4] and [F-O], [LT]) there is a projection-evaluation map

$$\Pi = pr \times ev_{g,m,s_A} : C\mathcal{M}_{g,m}(E, J(E), s_A) \to C\mathcal{M}_{g,m} \times E^{\otimes m}, \qquad (3.1)$$

where pr associates each vertical stable holomorphic map u to its stabilized domain (stabilization means consecutive contraction of nonstable components, which must be a $\mathbf{C}P^1$ without proper singular point) - an element in $C\mathcal{M}_{g,m}$. The evaluation map ev associates each stable holomorphic map u to the evaluation of u at m marked points. Denote by

$\Pi_*(C\mathcal{M}_{g,m}(E, J(E), s_A))$ the induced class in $H_*(C\mathcal{M}_{g,m} \times E^{\otimes m}, \mathbf{Q})$. The map Π induces a map in cohomologies

$$I^E_{g,m,s_A} : H^*(E^{\otimes m}, \mathbf{Q}) \to H^{*+\mu}(C\mathcal{M}_{g,m}, \mathbf{Q}) \qquad (3.2))$$

by

$$I^E_{g,m,s_A}(\gamma) = PD(\gamma \backslash \Pi_*(C\mathcal{M}_{g,m}(E, J(E), s_A))) \qquad (3.2.1)$$

The parametrized Gromov-Witten invariants, as in a non-parameterized case, are the collection of maps I^E_{g,m,s_A} defined in (3.2.1). The grading μ equals, as in non-parametrized case, $< 2c_1(F), A > +2(g-1)\dim F$.

Remark. As in the non-parametrized case we can also interpret the the parametrized Gromov-Witten invariants of homologically trivial symplectic bundles in term of "counting holomorphic curves". First we consider the case of Gromov-Witten invariants of codimension 0 (which are most important in our applications) i.e. those elements $I_{g,m,s_A}(\gamma)$ of maximal degree in $H^*(C\mathcal{M}_{g,m}, \mathbf{Q})$. It is known (by using intersection theory cf.[KM]) that the definition of codimension zero G-W classes is equivalent to the definition of counting holomorphic curves passing through certain cycles in our fiber bundle E (or $E^{\otimes n}$). Namely

$$< I^E_{g,m,s_A}(\gamma), [C\mathcal{M}_{g,m}] >= \int_{C\mathcal{M}_{g,m}} PD(\gamma \backslash \Pi_*(C\mathcal{M}_{g,m,s_A}(E)) = \int_{\Pi_*(C\mathcal{M}_{g,m,s_A}(E))} j_1^*(\gamma), \qquad (3.3)$$

where j_1 denotes the projection map to the factor $C\mathcal{M}_{g,m}$. We represent γ as (the sum of) an element $PD(i_*\Gamma_1) \wedge \cdots \wedge PD(i_*\Gamma_m) \wedge j^*(PD(\Gamma_B))$, where $\Gamma_i, i = 1, m$, is a cycle in the fiber F and Γ_B is a cycle in the base B. Then the RHS of (3.3) can be expressed as the number of the vertical stable holomorphic curves of genus g and m marked points which realizes fiberwise the class s_A and satisfies the following properties: the image of the i-th marked point is lying on (a fixed representative of) the cycle $\Gamma_i \times B$ of the fiber as well the projection of the marked points ia lying on (fixed representative of) cycle Γ_B.

The case of Gromov-Witten invariant of positive codimension can be treated analogously (see e.g. [R-T] for the non-parametrized case with a weakly monotone symplectic structure). Unfortunately we don't have a good geometric way to describe the group $H_*(E, \mathbf{Z})$ if the bundle E is not homologically trivial.

3.2 Technical (analytical) tool. To define rigorously the parametrized Gromov-Witten invariants for all compact symplectic bundles we need to show that the moduli space $(C\mathcal{M}_{g,m}(E, J(E), s_A))$ is compact; it carries a fundamental class, and moreover, the map I^E_{g,m,s_A} does not depend on the choice of $J(E)$. The compactness is automatically satisfied because we have the compactness fiberwise and we assume that the base is compact. To show that the moduli space carries a fundamental class and our definition does not depend on the choice of $J(E)$ we use the same technique as in non-parametrized case. (We also note that to construct the Gromov-Witten invariant in non-parametrized case we need to construct a bordism between the moduli spaces correspondent to different almost complex structures J_1 and J_2. This bordism is a version of the moduli space of parametrized vertical stable pseudoholomorphic curves considered in the present note). For the sake of completeness we shall review a needed technical construction here. This construction is due to Fukaya and Ono [F-O] and Li and Tian [L-T]. They developed a theory of Kuranishi structures (or differently speaking, a theory of the Euler class of generalized Fredholm orbibundles) to construct Gromov-Witten invariants for compact symplectic manifolds. (Their construction is in spirit close to a construction of Gromov-Witten invariants in algebraic geometry). First we observe that the moduli space of vertical stable holomorphic maps is the zero set of the Cauchy-Riemmann operator defined over the orbibundle $Map_{g,m}(E, j, s_A)$. The main observation now is that the zero set of the Cauchy-Riemmann operator may be not smooth but the Euler class of the Cauchy-Riemann operator (a section of a certain orbibundle) is well defined over $\mathbf{Q}$. To show this we perturb the (vertical) Riemann-Cauchy operator with the help of a multi-section of the related Banach bundle such that the perturbed (now a multi-)section is transversal to the zero section of the associated Banch bundle. Hence follows, that the moduli space of the (vertical) stable holomorphic maps defines a fundamental class over $\mathbf{Q}$. (If the symplectic manifold is weakly-monotone then it suffices to take a single perturbation and henceforth the moduli space defines a fundamental class over $\mathbf{Z}$.)

This construction consists of the following steps

Step 1: Set-up. Our original configuration space $Map_{g,m}(E, j, s_A)$ is the space of vertical stable maps u from semi-stable curves $(\Sigma, x_1, \cdots, x_k)$ to E such that the image of u realizes the class s_A (cf. [LT]). Clearly $Map_{g,m}(E, j, s_A)$ is a fibration of Banach orbifold (see [LT]) over B, therefore it possesses also a structure of a Banach orbifold.

Next we define a bundle $\mathcal{E}$ over $\bar{\mathcal{F}}(E, s_A, g, k)$ whose fiber over point (b, u) consists of $(0,1)$-forms over the domain Σ_u of map u taking values in $u^*(TF(j^{-1}(b)))$.

Clearly the moduli space of vertical stable holomorphic maps is the zero set of the Cauchy-Riemann section $\bar{\partial}_{J(E)}$ of the bundle $\mathcal{E}$:

$$\bar{\partial}_{J(E)}(x, u)_{|y} = \bar{\partial}_J u_{|y}$$

for all $y \in \Sigma_u$.

Step 2. We assume that our space is equipped with a sufficient good Sobolev norm or a C^l-norm. In order to show that our moduli space of vertical stable holomorphic curves carries a fundamental class it suffices to show that the bundle $\mathcal{E}$ with its Cauchy-Riemmann section $\bar{\partial}_{J(E)}$ is a generalized Fredholm orbifold bundle (see Theorem 1.2 in [LT], or that is to say that the moduli space possesses a Kuranishi structure, see [FO]). Thus the following Proposition (cf. Proposition 2.2 [L-T]) gives us the desired fundamental class.

Proposition 3.2.1. *(i) The section $\bar{\partial}_{J(E)}$ gives rise to a generalized Fredholm orbifold bundle with the natural orientation and of index $2c_1(s_A) + 2k + (2n-6)(1-g)$.*

(ii) The homotopy class of $\bar{\partial}_{J(E)}$ does not depends on the choice of $J(E)$.

Proof. (i) (We follow the argument in [LT]). We need to construct a covering $U_\delta(u_b, \Sigma, \vec{x})$ of the zero section of $\bar{\partial}_{J(E)}$, a smooth approximation E_i over a local uniformization $\tilde{U}_\delta(u_b, \Sigma, \vec{x})$ of our chosen covering, and a smooth (equivariant) resolution (F_i, ψ_i) for the smooth approximation $(\tilde{U}_\delta, E_i)$.

We construct a covering U_δ of the zero set as in [LT]. At a point $(u_b, \Sigma, \vec{x})$ a neighborhood

$$U_\delta(u_b, \Sigma, \vec{x}) = Map_\delta(D_\delta(b), W_0, K),$$

is defined via parameters: $D_\delta(b)$ - a small neighborhood of b in B, W_0 - a neighborhood of the stabilization $Stab(\Sigma, \vec{x})$ of $(\Sigma, \vec{x})$ in $C\mathcal{M}_{g,m}$ and K - a compact subset in the universal family $\tilde{U}$ of curves over the local uniformization $\tilde{W}$ of W, where W- a small neighborhood of the pair of a curve $(\Sigma, \vec{x}, \vec{z})$ and point b in the product of $C\mathcal{M}_{g,m+l}$ with $D_\delta(b)$, where $\vec{z}$ are l marked points added to bubbling components.

More precisely δ and $D_\delta(b)$ are chosen so small such that Lemma 3.1 in [L-T] holds.

Next we construct a smooth approximation E_i as follows. Let S be a finitely dimensional subspace in $\Gamma^{0,1}_{l-1}(\tilde{\mathcal{U}}, TF \times D_\delta(b))$ $(l \geq 2)$. Let E_S be a bundle (of finite rank) over $Map_\delta(D_\delta(b), W, K)$ whose fiber is S.

The following Lemma is just a copy of Proposition 3.4 [LT].

Lemma 3.2.2. *Suppose that the restriction $S_{|u}$ to a holomorphic stable map u is transversal to L_u. Then by shrinking W if necessary, if δ is sufficient small and K is sufficient large, the preimage $(\bar{\partial})_{J_E})^{-1}(E_S)$ is a smooth submanifolds, which contains u, in $Map_\delta(D_\delta(b), W, K)$ and* *of dimension* $2c_1(V)(A) + 2(n-3)(1-g) + 2k + \dim S$. *Moreover,* $E_S \mapsto (\bar\partial_{J_E}(E_S)$ *is a smooth bundle.*

This Lemma shows us that $(E_S, \tilde U(u_b, \Sigma, \vec x)$ is a smooth approximation of the local uniformization of our covering $U_\delta(u_b, \Sigma, \vec x)$.

At the final step we construct a resolution (F_i, ψ_i) inductively as in [LT] using the compactness of $\bar\partial^{-1}_{J(E)}(0)$.

The part (ii) can be proved in the same way.

3.3. Properties of parameterized Gromov-Witten invariants

The following theorem is obvious because our construction of parametrized Gromov-Witten invariants is preserved under symplectic bundle isomorphisms.

Theorem 3.3.1. *The parametrized Gromov-Witten invariant are invariants of symplectic bundles.*

Remark 3.3.2. The bordism type invariants of the moduli space of pseudo holomorphic curves may have more informations than the Gromov-Witten invariants. Such an examples of finer Gromov-invariants of symplectic manifolds can be found in [McD 3].

In order to distinguish symplectic bundles by using Gromov-Witten invariants we need to compute those for trivial symplectic bundles $E = B \times F$. As we have seen, it is convenient to switch time to times from cohomology classes to homology classes by using Poincare duality. As in (the Remark in) 3.1 we denote cohomology classes by small greek letters and their Poincare dual by capital Greek letter. Thus the map I^E_{g,m,s_A} has also arguments $(\Gamma_1, \cdots, \Gamma_n)$. By Kuenneth's formula the algebra $H^*(B \times (F)^m, \mathbf{Q})$ equals $(H^*(F, \mathbf{Q}))^m \otimes H^*(B, \mathbf{Q})$. Let $\alpha_i \in H^*(F, \mathbf{Q})$ and $\beta \in H^*(B, \mathbf{Q})$.

Proposition 3.3.3. *The Gromov-Witten invariants of a trivial symplectic fibration equals*

$$I^E_{g,m,S_A}(\alpha_1 \otimes \cdots \otimes \alpha_m \otimes \beta) = I^M_{g,m,A}(i^*(\alpha_1) \otimes \cdots \otimes i^*(\alpha_m)) \int_B \beta. \quad (3.4)$$

Proof. We choose a vertical compatible almost complex structure $J(E)$ as a fiber-wise constant compatible almost complex structure $J \in \mathcal{J}(F)$. Clearly the moduli space of vertical stable maps $C\mathcal{M}_{g,m}(E, B \times J, s_A)$ is the direct product $B \times C\mathcal{M}_{g,m}(J, A)$. Denote by Π^{pt} the mapping in (3.1) with base $B = pt$. By (3.2.1) the LHS of (3.4) equals

$$PD(\alpha_1 \otimes \cdots \alpha_m \otimes \beta \backslash \Pi_*[B \times C\mathcal{M}_{g,m}(J, A)]) = PD(\alpha_1 \otimes \cdots \otimes \beta \backslash ([B] \times \Pi^{pt}_*[C\mathcal{M}_{g,m}(F, A)]) \quad (3.5)$$

Clearly the RHS of (3.4) equals the RHS of(3.5). ☐

Now let us compute the Gromov-Witten invariants of a pull-back symplectic bundle. Let us consider the case when the base B_1 is k-covering of the base B_2 and a symplectic bundle E_1 over B_1 is induced by a symplectic bundle E_1 by the covering map. Clearly a section s_A in the associated bundle over B_2 can also be lifted to a section s_A of the associated bundle over B_1. We get immediately the following

Proposition 3.3.4. We have

$$I^{E_1}_{g,m,s_A} = kI^{E_2}_{g,m,s_A}.$$

At this moment we would like to introduce a class of parametrized Gromov-Witten invariants. We call an invariant I^E_{g,m,s_A} a **natural parmetrized Gromov-Witten invariant** if the moduli space $C\mathcal{M}_{g,m}(F,A)$ is empty (e.g. by dimension reason). Now we shall compute natural parametrized Gromov-Witten invariants for a direct fiber connected sum of two symplectic homological trivial bundles.. To get a good expression of natural parametrized Gromov-Witten invariants we use again the geometric way in terms of "fiberwise counting $J(E)$-holomorphic curves" as $I^E_{g,m,s_A}(\Gamma_1, \cdots\cdots, \Gamma_m, e_B^{dim\, B})$.

Proposition 3.3.5.*Let $E = E_1 \# E_2$ is a fiber connected sum of symplectic bundles E_1 and E_2. Then we have the following formula for natural parametried Gromov-Witten-Invariants.*

$$I^E = I^{E_1} + I^{E_2}.$$

Proof of Proposition 3.3.5. By a homotopy argument we see that there is a fiberwise compatible almost complex structure $J(E_i)$ such that its restriction to $D(\varepsilon) \times M$ equals the product of $D(\varepsilon)$ and a compatible almost complex structure J on M. In the procedure of a construction of a smooth resolution for $\bar\partial_{J_{(E_i)}}$ we can use a smooth resolution for the zero set of $\bar\partial_{J(E_i)}$ restricted to $D(\varepsilon) \times M$, in particular we can use a smooth resolution for $\bar\partial_J$. Now in the next step to construct a fundamental class for the moduli space of vertical holomorphic maps we use a perturbation which extends a regular perturbation for $\bar\partial_J$ over $D(\varepsilon)\times M$. The naturality of the Gromov-Witten invariants implies that the moduli space of vertical stable holomorphic curves on E equals the union of the 2 corresponding moduli spaces. Hence follows the statement. □

4. Homotopy groups of symplectomorphismgroups.

In this section we combine Remark 2.2, Propositions 3.3.3, 3.3.4, 3.3.5 and other ideas to estimate the rank of homotopy groups of symplectomorphismgroups. The following Proposition is an immediate consequence of Remark 2.2 and Proposition 3.3.5.

Proposition 4.1. *Natural parametrized Gromov-Witten invariants define elements in* $Hom(\pi_i(BSym(M,\omega), H^*(C\mathcal{M}_{g,m}, \mathbf{Q}))$.

As an application of Proposition 4.1 (or Proposition 3.3.4) we reinterpret a result by Gromov [Gr, Theorem 2.4.C_2].

Let $(M,\omega) = (S^2 \times S^2, \omega_1 \oplus \omega_2)$ be a product of symplectic manifolds. We denote by A_i the generators of $H_2(M, \mathbf{Z})$ realizing by the i-th sphere S^2.

Theorem 4.2. *If* $\omega_1(A_1) > \omega_2(A_2)$, *then there is a* $S^2 \times S^2$*-symplectic bundle over* S^2 *such that a natural parametrized Gromov-Witten invariant is non-trivial. In particular the rank of* $\pi_1(Sym(S^2 \times S^2, \omega))$ *is at least 1.*

Proof. According to Proposition 4.1 it suffices to find a symplectic bundle E over S^2 with fiber $(S^2 \times S^2, \omega_1 \oplus \omega_2)$ and a natural parametrized Gromov-Witten invariant whose value on E is non-trivial. We shall construct the bundle E by finding its associated element $g \in Symp(S^2 \times S^2, \omega_1 \oplus \omega_2)$. The existence of such element g was shown by Gromov [G], and in what follows we shall give a complete proof of this existence. First we need the following lemma (compare with [G, 2.4.B_3']).

Lemma 4.3. *The subspace* $\mathcal{J}_0$ *of compatible almost complex structures* J *on* $S^2 \times S^2$ *for which there exists a* J*-holomorphic sphere in a class* $A_1 - lA_2$, $1 \le l$, *is a non-empty closed subset of codimension 2 in* $\mathcal{J}(S^2 \times S^2, \omega)$. *For each* $J \in (\mathcal{J}(S^2 \times S^2, \omega) \setminus \mathcal{J}_0)$ *and for each point in* $S^2 \times S^2$ *there is a unique* J*-holomorphic sphere realizing class* A_i *and passing through* x.

Proof. The subspace $\mathcal{J}_0$ is non-empty because the sphere $(x, -x)$ is symplectic. To prove the closedness of $\mathcal{J}_0$ we first notice that the energy of a holomorphic sphere in class $A_1 - lA_2$ is less than $\omega_1(A_1)$. Thus we can apply the Gromov compactness argument to see the following fact. If at an almost complex structure J_0 which is a limit point of a sequence of almost complex structures in $\mathcal{J}_0$ a holomorphic sphere $A_1 - lA_2$ bubbles, then there most be J_0-holomorphic spheres of type $k_iA_1 - l_iA_2$ such that $\sum k_i = 1$ and $\sum l_i = l$. Since there is always a J_0-holomorphic sphere in class A_2, taking into account of positivity of intersection in dimension 4 we conclude that the bubbling spheres must be of type $A_1 + l_i A_2$ and $l_j A_2$ such that $\sum l_i + l_j = -l$. Since l_j must be positive we conclude that there must be a bubble of type $A_1 - l' A_2$, $l' \geq 1$, that proves the closedness of $\mathcal{J}_0$.

The codimension of $\mathcal{J}_0$ is at least 2 by the same argument as in [MD 1] or [LO 1,2]. (Namely for a fixed l we consider the universal moduli space of the pairs (J, $J-$holomorphic sphere in class $A_1 - lA_2$). The Fredholm index of the projection of this moduli space on the first factor is equal $4 + 2(1-l) - 6 \leq -2$.) To prove that the codimension is precisely 2 we use the uniqueness of J-holomorphic sphere in class $A_1 - A_2$ if it exists. (cf. with the argument in [HLS]. It follows that the kernel of the projection of the moduli space on the first factor equals zero.)

Now let us prove the last statement. Since A_2 is a minimal class in sense of Gromov for each $J \in \mathcal{J}(S^2 \times S^2, \omega)$ there is a unique J-holomorphic sphere realizing A_2 and passing through x. Finally to show the existence of J-holomorphic sphere in class A_1 for $J \in (\mathcal{J} \setminus \mathcal{J}_0)$ we use again the bubbling analysis. Indeed if a bubble happens there must be a J_0-holomorphic sphere in a class $A_1 - lA_2$ for $1 \leq l$. $\square$

Now let us find an element $g \in Sym(S^2 \times S^2, \omega_1 \oplus \omega_2)$ by studying the action of the group of symplectomorphisms on $\mathcal{J} \setminus \mathcal{J}_0$. Since $\mathcal{J}_0$ is a closed subset of codimension 2 we can choose a small disk D^2 in $\mathcal{J}(S^2 \times S^2)$ such that this disk intersects at $\mathcal{J}_0$ at exactly one interior point. By a result of Gromov (see also Appendix 1, Proposition A.1) there is a loop g_t in the group $Sym(S^2 \times S^2, x)$ such that the image $g_t(J_0)$ is homotopic to loop ∂D^2. Let D^2_* be a disk in $\mathcal{J}(S^2 \times S^2)$ with boundary $g_t(J_0)$ which intersects with $\mathcal{J}_1$ at exactly one point (this disk D^2_* can be chosen as the sum of the disk D^2 and a homotopy cylinder between ∂D^2 and $g_t(J_0)$).

Now we construct our bundle E by gluing two trivial $S^2 \times S^2$ bundles over a disk B^2 using this loop g_t. We denote by e^0 the cohomology class dual to the fundamental (homology) class of the bundle $E^{\otimes 3}$. We claim that the parametrized Gromov-Witten invariant $I_{0,3,A_1 - A_2}$ is natural and its value $I^E_{0,3,A_1-A_2}(e^0)$ is 1. The naturality follows from the fact that, $c_1(A_1 - A_2) = 0$ and hence the moduli space $C\mathcal{M}_{0,3}(J, A_1 - A_2)$ is empty for generic J. To compute this invariant for our bundle E explicitly we construct a generic fiberwise compatible almost complex structure $J(E)$ on E as follows. (Note that we in the case of weakly monotone symplectic manifolds and we are working with Gromov-Witten invariants of genus 0, therefore it suffices to perturb J to get the fundamental class of the correspondent moduli space). We observe that the standard product complex structure J_0 on $S^2 \times S^2$ is a $(A_1 - A_2)$-regular. Then we take $J(E)$ being the gluing of the product structure $J_0 \times B^2$ and D^2_* along the boundary ∂D^2 (on which they agree with the gluing map g_t). By a perturbation of $J(E)$ around the singular point (which does not change the intersection number of the perturbed disk D^2_* with the singular set $\mathcal{J}_0$) we can assume that this almost complex structure $J(E)$ is $(A_1 - A_2)$ regular. By the construction the parametrized moduli space $C\mathcal{M}_{0,3,A_1-A_2}(E, J(E))$ consists of one point, corresponding to the singular value in $\mathcal{J}_0$. According to (3.2.1) the value $I^E_{g,m,A_1-A_2}(e^0) = 1$. By Proposition 4.2 the non-trivial natural parametrized Gromov-Witten invariant defines a non-trivial element in $Hom(\pi_2(BSym(M,\omega), \mathbf{Q})$. Hence $\pi_1(Sym(M)) = \pi_2(BSym(M))$ contains an element of infinite order. (We can also use Proposition 3.3.4 to conclude that $g^n \neq 0$ for all n). □

Now we are going to prove the following

Theorem 4.4. *Let (M, ω) be a symplectic manifold $(S^2 \times S^2 \times N^{2k}, \omega_1 \oplus \omega_2 \oplus \omega_3)$ where (N^{2k}, ω_3) is a compact symplectic manifold. Let A_i denote the image of the i-th sphere S^2 in M. Suppose that $\omega_1(A_1) > \omega_2(A_2)$. Then we have $rk(\pi_1(Sym(M,\omega)) \otimes \mathbf{Q}) \geq 1$ and $rk(\pi_3(Sym(M,\omega)) \otimes \mathbf{Q}) \geq 2$.*

Proof. To prove the first statement we shall construct a symplectic bundle E with fiber (M,ω) over S^2 and compute a natural parametrized Gromov-Witten invariant of E. Namely we consider the moduli space of vertical holomorphic mappings $f : S^2 \to E$ whose image realizes class $A := A_1 - A_2$.

The dimension computation shows us the moduli space of vertical stable holomorphic curves in class A on a (M,ω)-bundle over S^2 has dimension $2k$. Thus we can "count" the number of vertical pseudo-holomorphic spheres in class A which intersect the cycle Γ representing by $S^2 \times S^2$. We claim that this number of the the trivial bundle $S^2 \times (M,\omega)$ equals zero. More precisely the corresponding Gromov-Witten invariant is natural. To show it we consider a vertical almost complex structure J^{pr} on the trivial bundle $(M,\omega) \times S^2$ such that on each fiber (M,ω) we have $J^{pr} = J_0 \times J^N$, where J_0 is the standard product complex structure on $S^2 \times S^2$ and J^N is an almost complex structure on $N^{2k}, \omega_3)$. Clearly any J^{pr}-sphere is also a J_0-sphere. Hence the moduli space of J^{pr}-sphere in class $A_1 - A_2$ is empty.

Now we construct a symplectic (M,ω)-bundle E over S^2 by gluing two trivial symplectic (M,ω)-bundles over disk D^2 via the map $g \times Id : S^1 \to \in Symp(S^2 \times S^2) \times Id \subset Symp(M,\omega)$, where g is constructed in the proof of Theorem 4.2 (see also Proposition A.1 in the Appendix). To compute the Gromov-Witten invariant $I^E_{0,3,A_1-A_2}(PD[N])$ we construct a fiberwise compatible almost complex structure $J(E)$ by gluing two generic fiber-wise compatible almost complex structures on the restriction of E to each disk D^2. The first one (on the first disk D^2) is defined as follows: $J(z) = (J_0 \times J^s)$, where J_0 is the standard product symplectic structure on $S^2 \times S^2$ and J^s is a singular almost complex structure in $\mathcal{J}_0(S^2 \times S^2, \omega_1 \oplus \omega_2)$ (see the Proof of Theorem 4.2). The second fiber-wise compatible almost comple structure is defined as follows: $J(z) = (\phi(z) \times J^s)$, where ϕ is the diffeomorphism from D^2 to the disk D^2_* constructed in the proof of Theorem 4.2. We claim that the constructed vertical almost complex is A-generic. Clearly outside the only singular point in S^2, where the vertical almost complex structure take value $(J^2 \times J^2)$, the moduli space of $J(E)$ holomorphic curves realizing A is empty. At the singular point the moduli space is diffeomorphic to N^2, namely it consists of maps $u_y = (u_1, y)$, where f_1 is a holomorphic sphere in $S^2 \times S^2$ and y is a point in N^{2k}. To show the regularity of the solution we need to prove that the linearization map $D\bar{\partial}_J : T_u Map_{g,m}(E, j, A) \to \Omega^{0,1}(S^2, u^*(TM))$ is surjective. Since $u = (u_1, y)$ the surjectivity of the linearization map follows from the surjectivity of the linearization map considered in the proof of Theorem 4.2. This proves the first statement in Theorem 4.4.

Now let us prove the second statement. It suffices to show that the subgroup $SO(3) \times SO(3) \subset Symp(M, \omega)$ realizes two linearly independent elements in $\pi_3(Sym(M, \omega))$. Clearly the group $Sym(M, \omega)$ acts also on the frame $E = Sp(4 + 2k, \mathbf{R})$-bundle over M. We choose a point $(x_1, x_2, y) \in (S^2 \times S^2 \times N^{2k})$ and a tangential vector $(v_1, v_2, 0)$ at this point ($v_i \neq 0$). The evaluation map at this tangential vector maps the subgroup $SO_3 \times SO_3$ injectively. It is easy to check that the image of this subgroup realizes two linearly independent elements in $\pi_3(E)$. Hence follows statement. □

There are also a dual way to see parametrized Gromov-Witten invariants. We recall that the parametrized Gromov-Witten invariants are defined via the fundamental class of the moduli space of stable curves, the slant product (and Poincare duality). Since we have two kinds of slant product we also have the following map

$$CGW^E_{g,m,s_A} : H^*(C\mathcal{M}_{g,m}, \mathbf{R}) \to H^{*+\mu}(E^{\otimes m}, \mathbf{R}) \tag{4.1}$$
$$\delta \mapsto PD(\delta \backslash \Pi_*(C\mathcal{M}_{g,m,s_A}(E, J(E)))$$

which pushes forward to the following map

$$CGW^B_{g,m,s_A} : H^*(C\mathcal{M}_{g,m}, \mathbf{R}) \to H^{*+\mu-2nm}(B, \mathbf{R}). \tag{4.2}$$

We call the image of the map CGW - the **Gromov-Witten characteristic classes.**

Theorem 4.5. *The Gromov-Witten characteristic classes are invariants of symplectic bundles. All the Gromov-Witten characteristic classes with positive degree of a trivial symplectic bundles are zero.*

Proof. The first statement is obvious (cf. Theorem 3.3.1). To prove the second statement we compute the Gromov-Witten characteristic classes on trivial bundle $M \times B$. Let $[T]$ be a cycle in B. As before we denote also by j the projection $E^{\otimes m} \to B$.

$$\begin{aligned} CGW^B_{g,m,s_A}(\delta)[T] &=< PD(\delta \backslash \Pi_*(B \times C\mathcal{M}_{g,m}(J,A)), j^{-1}([T]) > \\ &=< \delta \backslash (B \times \Pi_* C\mathcal{M}_{g,m}(J,A)), j^{-1\ *} PD_B([T]) > \\ &= \int_B PD_B([T]) \cdot < \delta \otimes 1^0, \Pi_*(C\mathcal{M}_{g,m}(J,A)) > . \end{aligned} \tag{4.3}$$

Clearly (4.3) equals zero if $\dim[T] \geq 1$ ist. □

The word "characteristic" is explained by the following functoriality of these classes. In particular we see that the Gromov-Witten characteristic classes are cohomology classes of the classfying space $BSym(M, \omega)$.

Theorem 4.6. *Let E be a symplectic bundle over B and $f^*(E)$ be the induced symplectic bundle by a map $f : B_1 \to B$. Then the map f also induces the Gromov-Witten characteristic classes of the bundle E.*

Proof. The map f alo induces a homomorphism in cohomology $\tilde{f}_* : H_*(C\mathcal{M}_{g,m} \times f^*(E^{\otimes m})) \to H_*(C\mathcal{M}_{g,m} \times E^{\otimes m})$. On the other hand by the construction we have

$$C\mathcal{M}_{g,m,s_A}(f^*(E)) = f^*(C\mathcal{M}_{g,m,s_A}(E))$$

$$\tilde{f}_*(\Pi_*(f_*(C\mathcal{M}_{g,m,s_A}(E))) = \Pi_*(C\mathcal{M}_{g,m,s_A}(E)).$$

Hence we can see immediately that

$$CGW^{B_1}_{g,m,s_A} = f^* \circ CGW^B_{g,m.s_A}$$

□

It is an easy exercise to interpret Theorem 4.2 and Theorem 4.4 in term of Gromov-Witten characteristic classes.

Before closing this section we would like to make a remark. In his talk [Kon 2] Kontsevich gives a construction of characteristic classes of symplectic bundles using graph homology. We believe that there is a connection of his construction and ours. Kontsevich considers two infinite-dimensional Lie algebras a_n and c_n. The last one is a subalgebra of the algebra h_n of Hamiltonian vectors fields in $\mathbf{R}^{2n}$ with polynomial coefficients. Kontsevich proved that [Kon 3] the stable homology of c_n equals the graph homology and the stable homology of a_n equals homology of the moduli space of smooth algebraic curves with unlabeled points. Now using the Fuks method [F] (generalizing a construction of Bernstein-Rosenfeld) Kontsevich construct a homomorphism from the cohomologies of the algebra h_n to the cohomologies of a symplectic fibration E over a base B^{2n}. This homomorphism descends to a cohomology of the base B. This homorphism is characteristic (in the sense of Theorem 4.6). Thus we can use graph homology to construct characteristic classes of symplectic fibrations. On the other hand there is also a map $c_n \to a_n$. Thus it is also possible to using cohomology classes of $\mathcal{M}_{g,m}$ to construct characteristic classes of symplectic fibrations.

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5. Appendix

We recall the following Gromov result [Gr]

[Gr, 2.4.A_1] The group $Symp(S^2 \times S^2, \omega)$ is contractible to a $\mathbf{Z}_2$ extension of $SO_3 \times SO_3$ if ω is a monotone form.

[Gr, 2.4.C_2] If ω is not monotone then $\pi_1(Sym_\omega S^2 \times S^2)$ contains an element of infinite order.

Gromov gaves a sketch of his proof in [Gr]. Using a part of his argument we interpret his result [Gr, 2.4.C_2] in term of Gromov-Witten invariant (Proposition 4.2). For the case of completeness we shall prove the following weaker statement which we used in our proof. Let us use the same notation $\mathcal{J}_0$ as it is defined in the proof of Lemma 4.3. Denote by J_0 the standard complex structure on $S^2 \times S^2$.

Proposition A.1. *There is a loop g_t in $Sym(M)$ such that the loop $g_t(J_0)$ bounds a diks D^2 which intersects with $\mathcal{J}_0$ at exactly one interior point.*

Proof. Fix a point $v_0 \in S^2 \times S^2$. Clearly the group $Diff(S^2 \times S^2)$ contains a subgroup $SO_2 \times SO_2$ which consists of fiberwise rotations (w. r. to metric $g_0 = (J_0, \omega)$) keeping v_0 invariant. First we shall construct a map α from $\mathcal{J}^{reg} = \mathcal{J} \setminus \mathcal{J}_0$ to the factor-group $Diff(M, x_0)/(SO_2 \times SO_2)$.

Let $J \in \mathcal{J}^{reg}$. Since $S^2 \times S^2$ is fibered by 2 families of J_0 holomorphic spheres in class A_1 and A_2, for a fixed point $v_0 = (s_0, s_0) \in S^2 \times S^2$ we can construct a map α as follows.

Let $A_1(J)$ and $A_2(J)$ be two J-holomorphic spheres through point v_0. Let f_i be a biholomorphic map from $A_i(J)$ to $A_i(J_0)$ which can be defined uniquely up to $PSL(2, \mathbf{C})$. Requiring $f(s_0) = v_0$ imposes other two constrains parameters. Further we note that (J, ω) and (J_0, ω) define 2 compatible metrics g and g_0 on $S^2 \times S^2$. Hence on each sphere $A_i(J)$ (resp. $A_i(J_0)$) there exists uniquely a south poles (once s_0 is supposed to be the north pole). Thus we have still 2 free parameters for f_i after requiring that the south pole on $A_i(J)$ must go to the south pole on $A_i(J_0)$. We can impose the last condition on f_i that the preimage of the hemisphere on $A_i(J_0)$ has a half of the area of the sphere $A_i(J)$. Now it is easy to see that f_i is defined uniquely up to the rotation subgroup $S^1 \in Iso(S^2, g_0)$. Next we extend the map f_i to a diffeomorphism of $(S^2 \times S^2, v_0)$ in the following way. Note that each element $x \in S^2 \times S^2$ is the intersection point of two J-holomorphic spheres through the point $(s_i) \in A_i(J)$. So we define $\alpha(J)(x)$ to be the intersection point of two J_0-holomorphic spheres through $f_i(s_i)$. Note that the map $\alpha(J)$ is defined up to the subgroup $SO_2 \times SO_2$.

Lemma A.2. *The map α factors through a map*

$$\hat{\alpha} : \mathcal{J}_0 \to Sym_\omega(V, v_0)/(SO_2 \times SO_2) \qquad (A.1)$$

Proof([Gr]). Since the map $\alpha(J)$ sends the $A_i(J)$ spheres to $A_i(J_0)$ spheres, and the fibration $A_1(J_0)$ is symplectically orthogonal to the fibration $A_2(J_0)$ we observe that the form $\alpha(J)^*(\omega)$ is J-positive. Now we apply the Moser stability theorem to the family $t\omega + (1-t)\alpha(J)^*(\omega)$ and get immediately our Proposition A. 2. □

Continuation of the proof of Proposition A.1. In this step we consider the natural evaluation map

$$ev_{J_0} : Sym(S^2 \times S^2, \omega) \to \mathcal{J}^{reg}(S^2 \times S^2, \omega)$$
$$g \mapsto g^*(J_0)$$

Since the standard product structure J_0 is $SO_2 \times SO_2$-invariant the evaluation map ev_{J_0} descends to a map $\bar{ev}_{J_0}$ from the quotient $Sym(S^2 \times S^2, v_0)/(SO_2 \times SO_2)$.

It is easy to check

Lemma A.3 [Gr]. *The composition $\hat{\alpha} \circ \bar{ev}_{J_0}$ is the identity map.*

From Lemma A.3 and considering an exact homotopy sequence we conclude that the map ev_{J_0} is surjective on the homotopy groups. Now statement in Proposition A.1 follows immediately. □

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