# THE PBW FILTRATION

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Abstract. In this paper we study the PBW filtration on irreducible integrable highest weight representations of affine Kac-Moody algebra $\widehat{\mathfrak{g}}$. The $n$-th space of this filtration is spanned with the vectors $x_1 \dots x_s v$, where $x_i \in \widehat{\mathfrak{g}}$, $s \le n$ and $v$ is a highest weight vector. For the vacuum module we give a conjectural description of the corresponding adjoint graded space in terms of generators and relations. For $\mathfrak{g}$ of the type $A_1$ we prove our conjecture and derive the fermionic formula for the graded character.

## Introduction

Let $\mathfrak{g}$ be a Kac-Moody Lie algebra of finite or affine type and $\mathrm{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. For the dominant integral weight $\lambda$ let $L_\lambda$ be the corresponding irreducible highest weight representation with highest weight $\lambda$ and highest weight vector $v_\lambda$. We consider a filtration $\mathrm{U}(\mathfrak{g})_s$ on $\mathrm{U}(\mathfrak{g})$ defined by

$$\mathrm{U}(\mathfrak{g})_0 = \mathbb{C}1, \ \mathrm{U}(\mathfrak{g})_{s+1} = \mathrm{U}(\mathfrak{g})_s + \mathrm{span}\{gu : \ g \in \mathfrak{g}, u \in \mathrm{U}(\mathfrak{g})_s\}.$$

This filtration induces a filtration $F_s = \mathrm{U}(\mathfrak{g})_s \cdot v_\lambda$ on $L_\lambda$. Define an associated graded space

$$L_\lambda^{gr} = F_0 \oplus F_1/F_0 \oplus F_2/F_1 \oplus \dots$$

We define the graded $u$-character by

$$\mathrm{ch}_u L_\lambda^{gr} = \sum_{s \ge 0} u^s \mathrm{ch}(F_s/F_{s-1}),$$

where we set $F_{-1} = 0$ and ch denotes the usual character with respect to the Cartan subalgebra of $\mathfrak{g}$. For example for $\mathfrak{g}$ of the type $A_1$ and its arbitrary finite-dimensional representation all the spaces $F_s/F_{s-1}$ are one-dimensional. The PBW-filtration for $\mathfrak{g}$ of the type $A_1^{(1)}$ and its level 1 representations was used in [FFJMT] for the study of $\phi_{1,3}$-field in Virasoro minimal theories.

Let us briefly describe our approach to the study of the spaces $L_\lambda^{gr}$. Let $\mathfrak{n}_- \hookrightarrow \mathfrak{g}$ be the nilpotent subalgebra of creating operators, i.e.

$$L_\lambda = \mathrm{U}(\mathfrak{n}_-) \cdot v_\lambda. \tag{1}$$

The action of $\mathfrak{n}_-$ on $L_\lambda$ induces the action of the abelian algebra $\mathfrak{n}_-^{ab}$ on $L_\lambda^{gr}$ ($\mathfrak{n}_-^{ab}$ is isomorphic to $\mathfrak{n}_-$ as a vector space). Therefore the space $L_\lambda^{gr}$ is generated from the vector $v_\lambda$ with the action of $\mathrm{U}(\mathfrak{n}_-^{ab})$, which is isomorphic to the polynomial algebra. This allows to describe $L_\lambda^{gr}$ as a quotient of the polynomial algebra on $\mathfrak{n}_-$ by some ideal.

In this paper we study the case of an affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ and its vacuum representations. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra and $\widehat{\mathfrak{g}}$ be the corresponding affine algebra, i.e. the central extension of $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$. We fix its vacuum level $k$ representation $L_k$ with a highest weight vector $v_k$. In this case one has $(\mathfrak{g}\otimes\mathbb{C}[t])\cdot v_k=0$ and therefore the space $L_k^{gr}$ is generated from the highest weight vector with an action of the universal enveloping algebra of the abelian algebra $\mathfrak{g}^{ab}\otimes t^{-1}\mathbb{C}[t^{-1}]$. This gives

$$L_k^{gr}\simeq \mathrm{U}(\mathfrak{g}^{ab}\otimes t^{-1}\mathbb{C}[t^{-1}])/I_k, \tag{2}$$

where $I_k$ is some ideal. We describe the ideal $I_k$ in the following way. We note that $\mathfrak{g}=\mathfrak{g}\otimes 1$ annihilates the highest weight vector $v_k$. Therefore the action of $\mathfrak{g}$ on $L_k$ induces a structure of $\mathfrak{g}$-module on $L_k^{gr}$ and $I_k$. (This structure plays an important role in our approach, see the conjecture below. For non vacuum modules $L_\lambda^{gr}$ is not $\mathfrak{g}$-module anymore. This means that the ideal of relations for general highest weight is to be described in other terms then in the vacuum case.) We give a conjectural description of $I_k$ below. The proof is given in the paper for the case $\mathfrak{g}=\mathfrak{sl}_2$.

The adjoint action of $\mathfrak{g}$ on itself endows the space $\mathrm{U}(\mathfrak{g}^{ab}\otimes t^{-1}\mathbb{C}[t^{-1}])$ with a structure of $\mathfrak{g}$-module. Let $\theta$ be the longest root of $\mathfrak{g}$, $f_\theta\in\mathfrak{g}$ be an element of the weight $\theta$ and

$$f_\theta(z)=\sum_{n\ge 0} z^n(f_\theta\otimes t^{-n-1})$$

be the corresponding current. We recall (see for example [BF]) that the series $f_\theta(z)^{k+1}$ acts by zero on $L_k$ (the coefficients of the $z$-expansion of $f_\theta(z)^{k+1}$ are acting by zero).

**Conjecture 0.1.** *We have an equality*

$$I_k=(\mathrm{U}(\mathfrak{g})\oplus\mathrm{U}(\mathfrak{g}^{ab}\otimes t^{-1}\mathbb{C}[t^{-1}]))\cdot\mathrm{span}\{\ \textit{coefficients of } f_\theta(z)^{k+1}\}, \tag{3}$$

*i.e. $I_k$ is the minimal $\mathrm{U}(\mathfrak{g})$-stable ideal which contains the coefficients of $f_\theta(z)^{k+1}$.*

We verify this conjecture for $\mathfrak{g}=\mathfrak{sl}_2$ (the level 1 case was studied in [FFJMT] by different method). Combining the description of $I_k$ with the vertex operator realization technique we obtain the fermionic formula for the $u$-character of $L_k^{gr}$ for $\mathfrak{g}=\mathfrak{sl}_2$.

The paper is organized in the following way:

In Section 1 we fix our notations and recall some constructions from the representation theory of affine Kac-Moody algebras and bosonic vertex operator algebras.

In Section 2 we study the adjoint graded space $L_1^{gr}$.

In Section 3 we describe the ideal of relations in $L_k^{gr}$ for the general level and obtain the fermionic formula for the graded character.

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## 1. Preliminaries and definitions

**1.1. Kac-Moody Lie algebras, representations and filtrations.** Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$.We consider the corresponding affine Kac-Moody algebra

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d, \tag{4}$$

where $K$ is a central element and $d$ is a degree operator, $[d, x \otimes t^i] = -ix \otimes t^i$ for all $x \in \mathfrak{g}$. The affine algebra $\widehat{\mathfrak{g}}$ admits the Cartan decomposition $\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}} \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}_-}$, where

$$\widehat{\mathfrak{n}} = \mathfrak{n} \otimes 1 \oplus \mathfrak{g} \otimes t\mathbb{C}[t],$$
$$\widehat{\mathfrak{h}} = \mathfrak{h} \otimes 1 \oplus \mathbb{C}K \oplus \mathbb{C}d,$$
$$\widehat{\mathfrak{n}}_- = \mathfrak{n}_- \otimes 1 \oplus \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}].$$

Let $\lambda \in \widehat{\mathfrak{h}}^*$ be an integral dominant weight and $L_\lambda$ be the corresponding irreducible highest weight representation of $\widehat{\mathfrak{g}}$ with highest weight $\lambda$ and highest weight vector $v_\lambda$. Then

$$\widehat{\mathfrak{n}} \cdot v_\lambda = 0,\ hv_\lambda = \lambda(h)v_\lambda\ \forall h \in \widehat{\mathfrak{h}},\ L_\lambda = \mathrm{U}(\widehat{\mathfrak{n}}_-) \cdot v_\lambda,$$

where $\mathrm{U}(\widehat{\mathfrak{n}}_-)$ is a universal enveloping algebra. The number $\lambda(K)$ is called the level of $L_\lambda$. We let $L_k$ to denote the level $k$ vacuum representation (the restriction of the highest weight of $L_k$ to $\mathfrak{h} \otimes 1$ vanishes).

We define an increasing filtration $\mathrm{U}(\widehat{\mathfrak{n}}_-)_s$ on the universal enveloping algebra by the following rule:

$$\mathrm{U}(\widehat{\mathfrak{n}}_-)_0 = \mathbb{C} \cdot 1,\ \mathrm{U}(\widehat{\mathfrak{n}}_-)_{s+1} = \mathrm{U}(\widehat{\mathfrak{n}}_-)_s + \widehat{\mathfrak{n}}_-\mathrm{U}(\widehat{\mathfrak{n}}_-)_s.$$

This filtration induces an increasing filtration $F_s$ on $L_\lambda$:

$$F_s = \mathrm{U}(\widehat{\mathfrak{n}}_-)_s \cdot v_\lambda.$$

The filtration $F_\bullet$ is called the Poincare-Birkhoff-Witt filtration (the PBW filtration for short).

**Definition 1.1.** *Define $L_\lambda^{gr}$ as an adjoint graded space with respect to the filtration $F_s$, i.e.*

$$L_\lambda^{gr} = F_0 \oplus \bigoplus_{s>0} F_s/F_{s-1}. \tag{5}$$

*Define the u-character of $L_\lambda^{gr}$ as*

$$\mathrm{ch}_u L_\lambda^{gr} = \sum_{s\geq 0} u^s \mathrm{ch}(F_s/F_{s-1}), \tag{6}$$

*where we set $F_{-1} = 0$ and* ch *denotes the usual character with respect to $\widehat{\mathfrak{h}}^*$.*

**Remark 1.1.** *Let $\widehat{\mathfrak{n}}_-^{ab}$ be an abelian Lie algebra which coincides with $\widehat{\mathfrak{n}}_-$ as a vector space. Then $L_\lambda^{gr}$ carries a natural structure of representation of $\widehat{\mathfrak{n}}_-^{ab}$.*

**Remark 1.2.** *The definition above is valid in the case of finite-dimensional algebras as well, providing a filtration on finite-dimensional modules.*

**Remark 1.3.** *For the general weight $\lambda$ the spaces $F_s$ are not stable under the action of an algebra $\mathfrak{g} = \mathfrak{g} \otimes 1 \hookrightarrow \widehat{\mathfrak{g}}$. This motivates the following modification of the filtration above. Let*

$$\widetilde{F}_0 = \mathrm{U}(\mathfrak{n}_- \otimes 1) \cdot v_\lambda, \ \widetilde{F}_{s+1} = \widetilde{F}_s + \widehat{\mathfrak{n}}_- \widetilde{F}_s.$$

*Here $F_0 = \mathbb{C}v_\lambda$ is replaced by $\widetilde{F}_0$, which is a finite-dimensional irreducible representation of $\mathfrak{g}$. We denote the corresponding adjoint graded object by $\widetilde{L}_\lambda^{gr}$. It is obvious that for any $s$ the space $\widetilde{F}_s$ is $\mathfrak{g}$-invariant. We hope to return to the study of this filtration elsewhere.*

1.2. **The $\mathfrak{sl}_2$ case.** Let $\mathfrak{g} = \mathfrak{sl}_2$ and let $e, h, f$ be its standard basis. Then $f$ spans $\mathfrak{n}_-$, $h$ spans $\mathfrak{h}$ and $e$ spans $\mathfrak{n}$. For $x \in \mathfrak{sl}_2$ we set $x_i = x \otimes t^i$. Let $v_k$ be a highest weight vector of the vacuum level $k$ module $L_k$.

We now recall a construction of the $ehf$-basis of $L_k$ from [FKLMM].

**Definition 1.2.** *A monomial of the form*

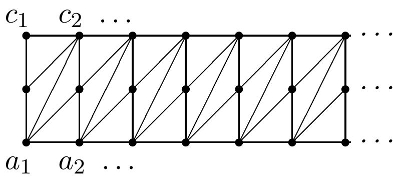
$$\dots f_{-n}^{a_n} h_{-n}^{b_n} e_{-n}^{c_n} \dots f_{-1}^{a_1} h_{-1}^{b_1} e_{-1}^{c_1} \tag{7}$$

*is called an ordered monomial. In addition it is called a $ehf$-monomial if it satisfies the following conditions:*

(a) $a_i + a_{i+1} + b_{i+1} \le k$ *for* $i > 0$,
(b) $a_i + b_{i+1} + c_{i+1} \le k$ *for* $i > 0$,
(c) $a_i + b_i + c_{i+1} \le k$ *for* $i > 0$,
(d) $b_i + c_i + c_{i+1} \le k$ *for* $i > 0$.

**Theorem 1.1.** *The set $\{m \cdot v_k\}$, where $m$ runs over the set of $ehf$-monomials provides a basis of $L_k$.*

**Remark 1.4.** *The following picture from* [FKLMM] *illustrates the set of $ehf$-monomials:*




*Namely one considers a set of monomials* (7) *such that the sum of exponents over any triangle (corresponding to the conditions (a)–(d)) is less than or equal to $k$.*

1.3. **Lattice vertex operator algebras and affine Kac-Moody algebras.** In this section we recall main properties of lattice vertex operator algebras (VOA for short) and their principal subspaces. The main references are [K2], [BF], [D], [FK].

Let $Q$ be a lattice of finite rank equipped with a symmetric bilinear form $(\cdot,\cdot) : Q \times Q \to \mathbb{Z}$ such that $(\alpha,\alpha) > 0$ for all $\alpha \in Q \setminus \{0\}$. Let $\mathfrak{h} = Q \otimes_{\mathbb{Z}} \mathbb{C}$. The form $(\cdot,\cdot)$ induces a bilinear form on $\mathfrak{h}$, for which we use the same notation. Let $\mathfrak{h} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K$ be the corresponding multi-dimensional Heisenberg algebra with the bracket

$$[\alpha \otimes t^i, \beta \otimes t^j] = i\delta_{i,-j}(\alpha,\beta)K, \ [K, \alpha \otimes t^i] = 0, \ \alpha,\beta \in \mathfrak{h}.$$

For $\alpha \in \mathfrak{h}$ define the Fock representation $\pi_\alpha$ of the Heisenberg algebra generated by a vector $|\alpha\rangle$ such that

$$(\beta \otimes t^n)|\alpha\rangle = 0, \ n > 0; \qquad (\beta \otimes 1)|\alpha\rangle = (\beta,\alpha)|\alpha\rangle; \qquad K|\alpha\rangle = |\alpha\rangle.$$

We now define a VOA $V_Q$ associated with $Q$. We deal only with an even case, i.e. $(\alpha,\alpha) \in 2\mathbb{Z}$ for all $\alpha \in Q$ (in the general case the construction leads to the so called super VOA). As a vector space

$$V_Q \simeq \bigoplus_{\alpha \in Q} \pi_\alpha.$$

The $q$-degree on $V_Q$ is defined by

$$\deg_q |\alpha\rangle = \frac{(\alpha,\alpha)}{2}, \quad \deg_q(\beta \otimes t^n) = -n. \tag{8}$$

The main ingredients of the VOA structure on $V_Q$ are bosonic vertex operators $\Gamma_\alpha(z)$ which correspond to highest weight vectors $|\alpha\rangle$. One sets

$$\Gamma_\alpha(z) = S_\alpha z^{\alpha\otimes 1} \exp(-\sum_{n<0} \frac{\alpha \otimes t^n}{n} z^{-n}) \exp(-\sum_{n>0} \frac{\alpha \otimes t^n}{n} z^{-n}), \tag{9}$$

where $z^{\alpha\otimes 1}$ acts on $\pi_\beta$ by $z^{(\alpha,\beta)}$ and the operator $S_\alpha$ is defined by

$$S_\alpha|\beta\rangle = c_{\alpha,\beta}|\alpha+\beta\rangle; \quad [S_\alpha, \beta \otimes t^n] = 0, \ \alpha,\beta \in \mathfrak{h},$$

where $c_{\alpha,\beta}$ are some non vanishing constants. The Fourier decomposition is given by

$$\Gamma_\alpha(z) = \sum_{n \in \mathbb{Z}} \Gamma_\alpha(n) z^{-n-(\alpha,\alpha)/2}.$$

In particular,

$$\Gamma_\alpha(-(\alpha,\alpha)/2 - (\alpha,\beta))|\beta\rangle = c_{\alpha,\beta}|\alpha+\beta\rangle. \tag{10}$$

One of the main properties of vertex operators is the following commutation relation:

$$[\alpha \otimes t^n, \Gamma_\beta(z)] = (\alpha,\beta) z^n \Gamma_\beta(z). \tag{11}$$

Another important formula describes the product of two vertex operators

$$\Gamma_\alpha(z)\Gamma_\beta(w) = (z-w)^{(\alpha,\beta)} S_\alpha S_\beta z^{(\alpha+\beta)\otimes 1} \times \tag{12}$$
$$\exp(-(\sum_{n<0} \frac{\alpha\otimes t^n}{n} z^{-n} + \frac{\beta\otimes t^n}{n} w^{-n})) \exp(-(\sum_{n>0} \frac{\alpha\otimes t^n}{n} z^{-n} + \frac{\beta\otimes t^n}{n} w^{-n})).$$

This leads to the proposition:

**Proposition 1.1.**

$$(\Gamma_\alpha(z))^{(k)}(\Gamma_\beta(z))^{(l)} = 0 \text{ if } k+l < (\alpha,\beta), \tag{13}$$

*where the superscript $(k)$ denotes the $k$-th derivative of the corresponding series. In addition if $(\alpha,\beta)=0$ then*

$$\Gamma_\alpha(z)\Gamma_\beta(z) \text{ is proportional to } \Gamma_{\alpha+\beta}(z).$$

We now recall the Frenkel-Kac construction which provides a vertex operator realization of the basic representation of the affine algebra $\widehat{\mathfrak{g}}$ for $\mathfrak{g}$ of the types $A$, $D$ and $E$. Let $\triangle$ be the root system of $\mathfrak{g}$, $Q$ be the root lattice and

$$\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\triangle} \mathbb{C}e_\alpha)$$

be the weight decomposition. For any $\alpha\in Q$ we have a vertex operator

$$\Gamma_\alpha(z) = \sum_{n\in\mathbb{Z}} \Gamma_\alpha(n) z^{-n-1}$$

acting on the space $V_Q$. We let $V(\mathfrak{g})$ to denote the vertex operator algebra associated with $\widehat{\mathfrak{g}}$.

**Theorem 1.2.** *The identification $\Gamma_\alpha(n)\mapsto e_\alpha\otimes t^n$ defines an isomorphism $V(\mathfrak{g})\simeq V_Q$, which sends a highest weight vector of $L_k$ to $|0\rangle$.*

We finish this section with the description of principal subspaces of vertex operator algebras (see [FS]). Let $\alpha_1,\dots,\alpha_N$ be a set of of linearly independent vectors generating the lattice $Q$. Let $M=(m_{i,j})_{1\le i,j\le N}$ be the non degenerate matrix of the scalar products of $\alpha_i$ ($m_{i,j}=(\alpha_i,\alpha_j)$) such that $m_{i,i}=2$ for all $i$. Consider the principal subspace $W_Q\hookrightarrow V_Q$ generated from the vector $|0\rangle$ with an action of operators $\Gamma_{\alpha_i}(-n_i)$ with $n_i\ge 1$ ($1\le i\le N$). Note that because of (10) one has $\Gamma_{\alpha_i}(-n)|0\rangle=0$ for $n<1$.

Our goal is to describe $W_Q$ (in particular to find its character). We first realize this subspace as a quotient of a polynomial algebra. Namely define $W'_Q$ as a quotient of $\mathbb{C}[a_i(-n)]$ with $1\le i\le N$, $n\ge 1$, by the ideal of relations generated with

$$a_i(z)^{(k)}a_j(z)^{(l)},\ k+l<m_{ij},$$

where $a_i(z)=\sum_{n\ge 1} z^n a_i(-n)$. We note that $W'_Q=\bigoplus_{\mathbf{n}\in\mathbb{Z}^N_{\ge 0}} W'_{Q,\mathbf{n}}$, where $W'_{Q,\mathbf{n}}$ is a subspace spanned by monomials in $a_i(k)$ such that the number of factors of the type $a_{i_0}(k)$ with fixed $i_0$ is exactly $n_{i_0}$. The $q$-character of $W'_{Q,\mathbf{n}}$ is naturally defined by $\deg_q a_i(k) = -k$.

The following lemma is standard (see for example [FF]).

**Lemma 1.1.**

$$\mathrm{ch}_q W'_{Q,\mathbf{n}} = \frac{q^{\mathbf{n}M\mathbf{n}/2}}{(q)_{\mathbf{n}}}, \tag{14}$$

*where* $(q)_{\mathbf{n}} = \prod_{j=1}^N (q)_{n_j}$, $(q)_n = \prod_{j=1}^n (1-q^j)$.

The next proposition states that the spaces $W_Q$ and $W'_Q$ are isomorphic (see for example [FF]).

**Proposition 1.2.** *The map* $|0\rangle \mapsto 1$, $\Gamma_{\alpha_i}(n) \mapsto a_i(n)$ *induces the isomorphism*

$$W_Q \simeq W'_Q.$$

*In particular for any* $\mathbf{n} = (n_1, \dots, n_N) \in \mathbb{Z}^N_{\geq 0}$

$$\mathrm{ch}_q(W_Q \cap \pi_{n_1\alpha_1 + \dots + n_N\alpha_N}) = \frac{q^{\mathbf{n}M\mathbf{n}/2}}{(q)_{\mathbf{n}}}. \tag{15}$$

## 2. Level 1 case

2.1. **Algebras $A_1$ and $B_1$.** We start with the description of the adjoint graded space $L_1^{gr}$ for vacuum level 1 $\widehat{\mathfrak{sl}_2}$-module. Note that this case was considered in [FFJMT] by the different method. We make a connection to their approach in the end of the section.

The space $L_1^{gr}$ carries the natural structure of the representation of abelian Lie algebra $\mathfrak{sl}_2^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}]$ (see Remark 1.1), where $\mathfrak{sl}_2^{ab}$ is an abelinization of $\mathfrak{sl}_2$, i.e. the 3-dimensional abelian Lie algebra. For $x \in \mathfrak{sl}_2$ we denote the corresponding element of $\mathfrak{sl}_2^{ab}$ by $\tilde{x}$. We also set

$$\tilde{x}(z) = \sum_{i<0} \tilde{x}_i z^{-i-1}.$$

The space $L_1^{gr}$ is isomorphic to a quotient of the polynomial algebra

$$\mathbb{C}[\tilde{e}_i, \tilde{h}_i, \tilde{f}_i]_{i<0}$$

by some ideal. Our goal is to show that the ideal of relations is generated with coefficients of the following series

$$\tilde{e}(z)^2,\ \tilde{e}(z)\tilde{h}(z),\ 2\tilde{e}(z)\tilde{f}(z) - \tilde{h}(z)^2,\ \tilde{h}(z)\tilde{f}(z),\ \tilde{f}(z)^2, \tag{16}$$

i.e. by the $\mathfrak{sl}_2$ consequences of the relation $\tilde{e}(z)^2$ ($\mathfrak{sl}_2$ acts on $\mathfrak{sl}_2^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}]$ via the adjoint action on the space $\mathfrak{sl}_2^{ab}$).

**Definition 2.1.** *Let $A_1$ be an algebra generated with the commuting variables $\tilde{x}_i$, $x = e, h, f$, $i < 0$ modulo the relations* (16).

In order to make a connection between $A_1$ and $L_1^{gr}$ we need a modification of $A_1$.

**Definition 2.2.** *Let $B_1$ be an algebra generated with the abelian variables $\tilde{x}_i$, $x = e, h, f$, $i < 0$ modulo the relations*

$$\tilde{e}(z)^2,\ \tilde{e}(z)\tilde{h}(z),\ \tilde{h}(z)^2,\ \tilde{h}(z)\tilde{f}(z),\ \tilde{f}(z)^2. \tag{17}$$

We define the $(q, z, u)$-characters of $A_1$ and $B_1$ assigning the $(q, z, u)$-degree to each generator $\tilde{x}_i$:

$$\deg_q \tilde{x}_i = -i,\ \deg_z \tilde{e}_i = 2,\ \deg_z \tilde{f}_i = -2,\ \deg_z \tilde{h}_i = 0,\ \deg_u \tilde{x}_i = 1.$$

**Lemma 2.1.** $\mathrm{ch}_{q,z,u} L_1^{gr} \le \mathrm{ch}_{q,z,u} A_1 \le \mathrm{ch}_{q,z,u} B_1$ *(the inequalities are true in each weight component).*

*Proof.* To show that $\mathrm{ch}_{q,z,u} L_1^{gr} \le \mathrm{ch}_{q,z,u} A_1$ it suffices to verify that relations (16) hold in $L_1^{gr}$. In fact, $e(z)^2 = 0$ in $L_1$ and $\mathfrak{sl}_2 = \mathfrak{sl}_2 \otimes 1 \hookrightarrow \widehat{\mathfrak{sl}_2}$ is acting on $L_1^{gr}$ as well as on $L_1$. In addition all of the relations (16) can be produced from $\tilde{e}(z)^2$ by applying the operator $f$.

To prove that $\mathrm{ch}_{q,z,u} A_1 \le \mathrm{ch}_{q,z,u} B_1$ we introduce a filtration $G_s$ on $A_1$ by setting $G_0$ to be the subspace generated with variables $\tilde{e}_i$, $\tilde{f}_i$ and

$$G_{s+1} = \mathrm{span}\{\tilde{h}_i w :\ i < 0, w \in G_s\}.$$

Then the relation $2\tilde{e}(z)\tilde{f}(z) - \tilde{h}(z)^2 = 0$ (which holds in $A_1$) gives $\tilde{h}(z)^2 = 0$ in the adjoint graded space. Lemma is proved. □

In the following subsections we give two proofs of the equality

$$\mathrm{ch}_{q,z,u} L_1 = \mathrm{ch}_{q,z,u} B_1.$$

2.2. **A basis of $B_1$.** Recall the $ehf$-basis of $L_1$ (see Theorem 1.1):

$$\ldots e_{-2}^{c_2} f_{-1}^{a_1} h_{-1}^{b_1} e_{-1}^{c_1} \tag{18}$$

with

- $(a)$ $a_i + a_{i+1} + b_{i+1} \le 1$,
- $(b)$ $a_i + b_{i+1} + c_{i+1} \le 1$,
- $(c)$ $a_i + b_i + c_{i+1} \le 1$,
- $(d)$ $b_i + c_i + c_{i+1} \le 1$.

We now consider a modified set of restrictions

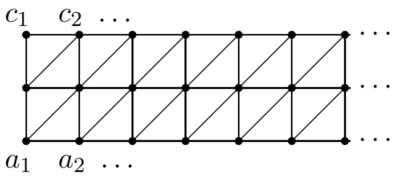
- $(a')$ $a_i + a_{i+1} + b_{i+1} \le 1$,
- $(b')$ $a_i + b_i + b_{i+1} \le 1$,
- $(c')$ $b_i + b_{i+1} + c_{i+1} \le 1$,
- $(d')$ $b_i + c_i + c_{i+1} \le 1$,

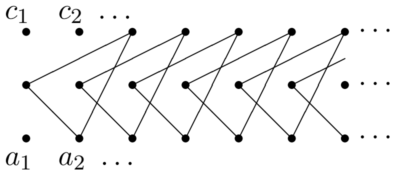
and add one additional restriction

- $(N)$ $b_i + a_{i+1} + c_{i+2} \le 2$.

We refer to the monomials (18) with restrictions $(a')$, $(b')$, $(c')$, $(d')$ and $(N)$ as $ehf'$-monomials.

Conditions $(a')-(d')$ and $(N)$ can be expressed as follows (the first picture explains $(a')-(d')$ and the second explains $(N)$):




The $ehf'$-monomials are monomials (18) such that the sum of exponents over any triangle in the first picture is less than or equal to 1 and the sum of exponents over any triangle in the second picture is less than or equal to 2.

**Lemma 2.2.** *The characters of $ehf$- and $ehf'$-monomials coincide,*

*Proof.* In order to prove our Lemma we construct a $(q, z, u)$-degree preserving bijection $\phi$ from the set of $ehf$-monomials to the set of $ehf'$-monomials. Fix some $ehf$-monomial $m$ of the form (18). If $m$ satisfies conditions $(a')$, $(b')$, $(c')$ and $(d')$ then we set $\phi(m) = m$. Now suppose that $j$ is a smallest number such that $(b')$ or $(c')$ is violated, i.e.

$$a_j + b_j + b_{j+1} = 2 \text{ or } b_j + b_{j+1} + c_{j+1} = 2.$$

This means

$$b_j = b_{j+1} = 1,\ a_j = c_{j+1} = 0.$$

We construct a new monomial

$$m' = \dots f_{-n}^{a'_n} h_{-n}^{b'_n} e_{-n}^{c'_n} \dots f_{-1}^{a'_1} h_{-1}^{b'_1} e_{-1}^{c'_1},$$

which differs from $m$ only in terms $a_j, b_j, b_{j+1}, c_{j+1}$. The new values are given by

$$a'_j = c'_{j+1} = 1,\ b'_j = b'_{j+1} = 0. \tag{19}$$

The new monomial $m'$ satisfies conditions $(a')$-$(d')$ for all $i \le j$. In addition it satisfies the condition $(N)$ for all $i < j$. In fact the violation of $(N)$ means that for some $i < j$ $b'_i = a'_{i+1} = c'_{i+2} = 1$. Therefore $i = j - 1$ and $b_{j-1} = b_j = 1$. This gives the violation of the conditions $(b')$ and $(c')$ for $i = j - 1$, which contradicts with the choice of $j$.

We repeat the procedure until all the conditions $(a') - (d')$ are satisfied for all $i$. Denote the result by $\phi(m)$. We note that by the same reason as above the condition $(N)$ is satisfied by $\phi(m)$. Therefore $\phi(m)$ is a $ehf'$-monomial.

We now show that $\phi$ is a bijection. The inverse map $\phi^{-1}$ can be constructed in the following way. Fix some $m'$, which is $ehf'$- but not $ehf$-monomial. Then for some $j$

$$a'_j + b'_j + c'_{j+1} = 2 \text{ or } a'_j + b'_{j+1} + c'_{j+1} = 2,$$

which means $a'_j = c'_{j+1} = 1$, $b'_j = b'_{j+1} = 0$. Then the inverse transformation to (19) is given by

$$a_j = c_{j+1} = 0, \ b_j = b_{j+1} = 1.$$

Lemma is proved. □

**Lemma 2.3.** *The set of $ehf'$-monomials spans $B_1$.*

*Proof.* For any ordered monomial $m$ of the form

$$\dots f_{-n}^{a_n} h_{-n}^{b_n} e_{-n}^{c_n} \dots f_{-1}^{a_1} h_{-1}^{b_1} e_{-1}^{c_1} \tag{20}$$

we denote by $\deg m$ the sum of all the exponents $a_i$, $b_i$ and $c_i$. Following [FKLMM] we define a complete lexicographic ordering on the set of ordered monomials by the following rule. If $\deg m > \deg m'$ then $m > m'$. Suppose $\deg m = \deg m'$. Then if $c_1 < c'_1$ then $m > m'$. If $\deg m = \deg m'$, $c_1 = c'_1$ and $b_1 < b'_1$ then $m > m'$. Next we compare $a_1$ and $a'_1$ and so on.

We now show that any monomial $m$ which violates one of the conditions $(a')$-$(d')$, $(N)$ can be rewritten as a sum of smaller monomials. Suppose that for some $i$ the condition $(a')$ is violated, i.e. $a_i + a_{i+1} + b_{i+1} > 1$. This means that at least two of numbers among $a_i, a_{i+1}, b_{i+1}$ are greater than 0. But each of the products $a_i a_{i+1}$, $a_i b_{i+1}$, $a_{i+1} b_{i+1}$ can be written as a linear combination of smaller monomials using the relations

$$\sum_{\alpha+\beta=2i+1} \tilde f_\alpha \tilde f_\beta = 0, \quad \sum_{\alpha+\beta=2i+1} \tilde f_\alpha \tilde h_\beta = 0, \quad \sum_{\alpha+\beta=2i+2} \tilde f_\alpha \tilde h_\beta = 0$$

respectively. By the similar reason the violation of $(b')$, $(c')$ and $(d')$ allows to rewrite the corresponding monomial in terms of the smaller ones. So we finish the proof with rewriting a monomial which satisfies $(a') - (d')$ but not $(N)$. This reduces to rewriting a monomial $\tilde h_i \tilde e_{i+1} \tilde f_{i+2}$ as a linear combination of a smaller ones. We have

$$\tilde h_i \tilde e_{i+1} \tilde f_{i+2} = -\tilde e_i \tilde h_{i+1} \tilde f_{i+2} + \dots = \tilde e_i \tilde h_{i+2} \tilde f_{i+1} + \dots, \tag{21}$$

where we use $\dots$ to denote a linear combination of the monomials, which are smaller than $\tilde h_i \tilde e_{i+1} \tilde f_{i+2}$ (the relations $\tilde e(z) \tilde h(z) = 0$ and $\tilde h(z) \tilde f(z) = 0$ are used in (21)). Now it is enough to note that the last expression is smaller than $\tilde h_i \tilde e_{i+1} \tilde f_{i+2}$. □

**Corollary 2.1.** $\mathrm{ch}_{q,z,u} B_1 = \mathrm{ch}_{q,z,u} L_1^{gr}$.

*Proof.* From Lemmas 2.2 and 2.3 we know that the character of the set of $ehf'$-monomials is greater than or equal to the character of $B_1$ and coincides with the character of $L_1$. Now our Corollary follows from Lemma 2.1. □

2.3. **The degeneration procedure.** In this section we give an alternative proof of Corollary 2.1. We involve the degeneration procedure which will be used later in the case of general level $k$.

Let $p, q, r \in \mathbb{R}^3$ be linearly independent vectors with the scalar products

$$(p,p) = (q,q) = (r,r) = 2, (p,q) = (q,r) = 1, (p,r) = 0.$$

Let $Q_1$ be a lattice generated by $p, q, r$ and $\Gamma_p(z)$, $\Gamma_q(z)$, $\Gamma_r(z)$ be the corresponding vertex operators.

**Lemma 2.4.** *The identification*

$$\Gamma_p(n) \to \tilde{e}_n,\ \Gamma_q(n) \to \tilde{h}_n,\ \Gamma_r(n) \to \tilde{f}_n$$

*provides an isomorphism* $W_{Q_1} \simeq B_1$, $|0\rangle \to 1$.

*Proof.* Follows from the Proposition 1.2. □

We now consider the Lie algebra $\widehat{\mathfrak{sl}_4}$ and its vertex operator realization provided by the Frenkel-Kac construction. For $1 \le i, j \le 4$ we set

$$E_{i,j}(z) = \sum_{n\in\mathbb{Z}} (E_{i,j} \otimes t^n) z^{-n-1}.$$

Using Theorem 1.2 and Lemma 2.4 we obtain the following Lemma.

**Lemma 2.5.** *The identification*

$$\tilde{e}_n \to E_{2,3} \otimes t^n,\ \tilde{h}_n \to E_{2,4} \otimes t^n,\ \tilde{f}_n \to E_{1,4} \otimes t^n$$

*provides an isomorphism between* $B_1$ *and the subspace of* $V(\mathfrak{sl}_4)$ *generated from the highest weight vector with the fields* $E_{2,3}(z)$, $E_{2,4}(z)$ *and* $E_{1,4}(z)$.

We want to show that the character of $B_1$ is smaller than or equal to the character of $L_1$. In order to do this we construct a deformation of the Lie algebra $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_4$ to the subspace spanned by $E_{2,3}$, $E_{2,4}$ and $E_{1,4}$.

**Proposition 2.1.** *There exists a continuous family of Lie subalgebras* $S(\varepsilon)$ *of* $\mathfrak{sl}_4$, $0 \le \varepsilon \le 1$ *such that*

*a).* $S(\varepsilon) \simeq \mathfrak{sl}_2$ *for* $0 \le \varepsilon < 1$,
*b).* $S(1)$ *is spanned with* $E_{2,3}$, $E_{2,4}$ *and* $E_{1,4}$,
*c). There exist standard basises* $e(\varepsilon)$, $h(\varepsilon)$, $f(\varepsilon)$ *of* $S(\varepsilon)$ *(*$0 \le \varepsilon < 1$*) such that*

$$\lim_{\varepsilon\to1} \mathbb{C}e(\varepsilon) = \mathbb{C}E_{1,4},\ \lim_{\varepsilon\to1} \mathbb{C}h(\varepsilon) = \mathbb{C}E_{2,4},\ \lim_{\varepsilon\to1} \mathbb{C}f(\varepsilon) = \mathbb{C}E_{2,3}.$$

*Proof.* Let $v_1, v_2, v_3, v_4$ be a basis of $\mathbb{C}^4$. For $0 \le \varepsilon < 1$ we let $S(\varepsilon)$ to denote the subalgebra determined by the following conditions:

- $S(\varepsilon)$ preserves $\mathrm{span}\{v_1, v_2\}$,
- $S(\varepsilon)$ annihilates $\mathrm{span}\{(1-\varepsilon)^{\frac{1}{2}} v_3 + \varepsilon v_1, (1-\varepsilon)v_4 + \varepsilon v_2\}$,
- $S(\varepsilon)$ contains only traceless matrices.

Then $S(\varepsilon)$ consists of the matrixes of the form

$$\begin{pmatrix} x & y & \frac{-x\varepsilon}{\sqrt{1-\varepsilon}} & \frac{-y\varepsilon}{1-\varepsilon} \\ z & -x & \frac{-z\varepsilon}{\sqrt{1-\varepsilon}} & \frac{x\varepsilon}{1-\varepsilon} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $e(\varepsilon)$, $h(\varepsilon)$, $f(\varepsilon)$ be the standard basis of $S(\varepsilon) \simeq \mathfrak{sl}_2$ (we fix the identification via the upper left $2\times 2$ corner of $S(\varepsilon)$). Then

$$\lim_{\varepsilon\to 1}(\varepsilon-1)e(\varepsilon)=E_{1,4},\ \lim_{\varepsilon\to 1}(1-\varepsilon)h(\varepsilon)=E_{2,4},\ \lim_{\varepsilon\to 1}(1-\varepsilon)^{\frac{1}{2}}f(\varepsilon)=-E_{2,3}.$$

This finishes the proof of the Proposition. □

**Corollary 2.2.** *There exists a continuous family of subspaces $M(\varepsilon)\hookrightarrow V(\mathfrak{sl}_4)$, $0\le\varepsilon\le 1$ such that $M(\varepsilon)\simeq L_1$ for $0\le\varepsilon<1$ and $M(1)\simeq B_1$.*

**Corollary 2.3.** *$L_1^{gr}$ is isomorphic to $A_1$ as a representation of the algebra $\mathfrak{sl}_2^{ab}\otimes t^{-1}\mathbb{C}[t^{-1}]$.*

*Proof.* Follows from Lemma 2.1 and Lemma 2.1. □

Recall the $(q,z,u)$-character of $L_{0,1}^{gr}$ by

$$\mathrm{ch}_{q,z,u}L_1^{gr}=\sum_{s=0}^{\infty}u^s\mathrm{ch}_{q,z}(F_s/F_{s-1}). \tag{22}$$

**Proposition 2.2.**

(23)

$$\mathrm{ch}_{q,z,u}L_1^{gr}=\sum_{n^+,n^0,n^-\ge 0}u^{n^++n^0+n^-}z^{2(n^+-n^-)}\frac{q^{(n^+)^2+(n^0)^2+(n^-)^2+n^+n^0+n^0n^-}}{(q)_{n^+}(q)_{n^0}(q)_{n^-}},$$

*where* $(q)_n=\prod_{i=1}^n(1-q^i)$.

*Proof.* We first calculate the character of $B_1$. Consider the map $\phi$ from $B_1^*$ to the space of polynomials in 3 groups of variables:

$$(\phi(\theta))(x_1^-,\dots,x_{n^-}^-,x_1^0,\dots,x_{n^0}^0,x_1^+,\dots,x_{n^+}^+)=\\ \theta(\tilde f(x_1^-)\dots\tilde f(x_{n^-}^-)\tilde h(x_1^0)\dots\tilde h(x_{n^0}^0)\tilde e(x_1^+)\dots\tilde e(x_{n^+}^+)).$$

Because of the relations (17) the image of $\phi$ coincides with the the space of polynomials of the form

$$\prod_{\substack{\alpha=+,-,0\\ 1\le i^s\le n^\alpha}}x_i^\alpha\prod_{\substack{\alpha=+,-,0\\ 1\le i<j\le n^\alpha}}(x_i^\alpha-x_j^\alpha)^2\prod_{\substack{1\le i\le n^+\\ 1\le j\le n^0}}(x_i^+-x_j^0)\prod_{\substack{1\le i\le n^0\\ 1\le j\le n^-}}(x_i^0-x_j^-)\times\\ F(x_1^-,\dots,x_{n^-}^-,x_1^0,\dots,x_{n^0}^0,x_1^+,\dots,x_{n^+}^+), \tag{24}$$

where $F$ is an arbitrary polynomial in $x_i^\alpha$, symmetric in each group $(+,0,-)$ of variables. The natural $(q,z,u)$-grading on the space $B_1^*$ comes from the grading on $B_1$. It is easy to see that the corresponding character of the space of polynomials (24) is given by

$$u^{n^++n^0+n^-}z^{2(n^+-n^-)}\frac{q^{(n^+)^2+(n^0)^2+(n^-)^2+n^+n^0+n^0n^-}}{(q)_{n^+}(q)_{n^0}(q)_{n^-}}.$$

Now our Lemma follows from Corollary 2.1. □

**Corollary 2.4.**

$$\mathrm{ch}_{q,z} L_1 = \sum_{n^+,n^0,n^-\geq 0} z^{2(n^+-n^-)} \frac{q^{(n^+)^2+(n^0)^2+(n^-)^2+n^+n^0+n^0n^-}}{(q)_{n^+}(q)_{n^0}(q)_{n^-}}. \tag{25}$$

*In particular the right hand side equals to the well known expression for the character of $L_1$:*

$$\mathrm{ch}_{q,z} L_1 = \frac{1}{(q)_\infty} \sum_{n\in\mathbb{Z}} z^{2n} q^{n^2},$$

*where* $(q)_\infty = \prod_{i\geq 1}(1-q^i)$

We now compare our formula with one from [FFJMT], which is given in terms of supernomial coefficients (see [SW]).

**Corollary 2.5.**

$$\mathrm{ch}_{q,z,u} L_1^{gr} = \sum_{m\geq 0} \frac{u^m q^{m^2}}{(q)_m} \sum_{-m\leq l\leq m} z^{2l} S_{m,l}(q), \tag{26}$$

*where*

$$S_{m,l}(q) = \sum_{m\geq\nu\geq m-l-\nu} q^{(\nu+l-m)(\nu+l)+\nu(\nu-m)} \binom{m}{\nu}_q \binom{\nu}{m-l-\nu}_q$$

*and* $\binom{n}{m}_q = \frac{(q)_n}{(q)_m(q)_{n-m}}$.

*Proof.* We note that

$$\frac{1}{(q)_m}\binom{m}{\nu}_q \binom{\nu}{m-l-\nu}_q = \frac{1}{(q)_{m-\nu}(q)_{2\nu-m+l}(q)_{m-l-\nu}}.$$

The change of variables

$$m-\nu = n_+,\ 2\nu-m+l = n_0,\ m-l-\nu = n_-$$

identifies formulas (26) and (23). ☐

## 3. The general case

In this section we consider the PBW-filtration on the vacuum level $k$ representation $L_k$ and the corresponding adjoint graded space $L_k^{gr}$.

3.1. **Algebras $A_k$ and $B_k$.** We introduce 2 series of algebras $A_k$ and $B_k$ generated with Fourier coefficients of the abelian currents $\tilde{e}(z)$, $\tilde{f}(z)$ and $\tilde{h}(z)$, where

$$\tilde{x}(z) = \sum_{i\geq 1} z^{-i-1}\tilde{x}_i.$$

**Definition 3.1.** *An algebra $A_k$ is a quotient of the polynomial algebra in variables $\tilde{e}_i$, $\tilde{h}_i$, $\tilde{f}_i$, $i<0$, by the ideal generated by Fourier coefficients of $\tilde{e}(z)^{k+1}$ and all its $\mathfrak{sl}_2$ consequences, i.e. by coefficients of $2k+3$ series:*

$$\tilde{e}(z)^{k+1},\ \tilde{e}(z)^k\tilde{h}(z),\ \tilde{e}(z)^{k-1}\tilde{h}^2(z) - 2\tilde{e}(z)^k\tilde{f}(z), \dots, \tilde{f}(z)^{k+1}. \tag{27}$$

In other words these relations can be described as follows. Identify $\mathfrak{sl}_2$ with its 3-dimensional irreducible representation $\pi_2$ with $e$ being the highest weight vector. Then we have an embedding

$$\imath : \pi_{2k+2} \hookrightarrow \pi_2^{\otimes(k+1)}$$

of $(2k+3)$-dimensional irreducible $\mathfrak{sl}_2$ module $\pi_{2k+2}$ into the tensor power $\pi_2^{\otimes(k+1)}$ (the image of $\pi_{2k+2}$ is generated from $e^{\otimes(k+1)} \in \pi_2^{\otimes(k+1)}$ by the action of universal enveloping algebra of $\mathfrak{sl}_2$). Define an affinization map $\alpha$, which sends an element of the tensor power $\pi_2^{\otimes(k+1)}$ to the product of the corresponding series:

$$\alpha(x^1 \otimes \ldots \otimes x^{k+1}) = \tilde{x}^1(z) \ldots \tilde{x}^{k+1}(z),\ x^i = e, h, f.$$

Then the defining relations of $A_k$ are coefficients of $\alpha(\imath(\pi_{2k+2}))$.

We note that $(q, z, u)$-character of $A_k$ is naturally defined by

$$\deg_u \tilde{x}_i = 1,\ \deg_q \tilde{x}_i = -i,\ \deg_z \tilde{e}_i = 2,\ \deg_z \tilde{h}_i = 0, \deg_z \tilde{f}_i = -2. \tag{28}$$

**Lemma 3.1.** $\mathrm{ch}_{q,z,u} L_k^{gr} \le \mathrm{ch}_{q,z,u} A_k$.

*Proof.* Follows from the equality $e(z)^{k+1} = 0$ in $L_k$. □

**Definition 3.2.** *An algebra $B_k$ is a quotient of the polynomial algebra in variables $\tilde{e}_i$, $\tilde{h}_i$, $\tilde{f}_i$, $i \le -1$ by the ideal generated by Fourier coefficients of $2k+3$ series*

$$\tilde{e}(z)^i \tilde{h}(z)^{k+1-i},\ 1 \le i \le k+1;\ \tilde{h}(z)^i \tilde{f}(z)^{k+1-i},\ 0 \le i \le k+1. \tag{29}$$

We note that $(q, z, u)$-character of $B_k$ is naturally defined by (28).

**Lemma 3.2.** $\mathrm{ch}_{q,z,u} A_k \le \mathrm{ch}_{q,z,u} B_k$.

*Proof.* We introduce a filtration $G_s$ on $A_k$ by setting $G_0$ to be the subspace generated by variables $\tilde{e}_i$, $\tilde{f}_i$ (but not $\tilde{h}_i$) and

$$G_{s+1} = \mathrm{span}\{\tilde{h}_i w :\ i < 0, w \in G_s\}.$$

Then for $0 \le i \le k+1$ the relation

$$\alpha(\imath(f^i \cdot e^{\otimes(k+1)})) = 0$$

which holds in $A_k$ contains a term $\tilde{e}(z)^{k+1-i}\tilde{h}(z)^i$ and the coefficients of the difference

$$\alpha(\imath(f^i \cdot e^{\otimes(k+1)})) - \tilde{e}(z)^{k+1-i}\tilde{h}(z)^i$$

belongs to $G_{i-1}$. This means that the relation $\tilde{e}(z)^{k+1-i}\tilde{h}(z)^i = 0$ holds in the adjoint graded space $G_0 \oplus \bigoplus_{s>0}(G_s/G_{s-1})$. Similarly the rest of the relations (29) are true in the adjoint graded space. Lemma is proved. □

**Corollary 3.1.** $\mathrm{ch}_{q,z,u} L_k^{gr} \le \mathrm{ch}_{q,z,u} B_k$.

3.2. **A quadratic algebra $C_k$ and principal subspace $D_k$.** We consider a set of commuting variables

$$\tilde{x}_i^{[l]},\ x=e,h,f,\ 1\le l\le k,\ i\le -l$$

and the corresponding currents $\tilde{x}^{[l]}(z)=\sum_{i<0} z^{-i-l}\tilde{x}_i^{[l]}$ (we set $\tilde{x}_i^{[l]}=0$ for $i>-l$). For the series $p(z)$ let $p(z)^{(r)}$ be the $r$-th derivative.

**Definition 3.3.** *Let $C_k$ be the quotient of the polynomial algebra in commuting variables $\tilde{x}_i^{[l]}$, $1\le l\le k$, $i\le -l$ by the ideal of relations generated by coefficients of currents*

$$\tilde{x}^{[l]}(z)^{(\alpha)}\tilde{x}^{[m]}(z)^{(\beta)} \text{ for } x=e,h,f,\ \alpha+\beta<2\min(l,m), \tag{30}$$

$$\tilde{e}^{[l]}(z)^{(\alpha)}\tilde{h}^{[m]}(z)^{(\beta)} \text{ for } \alpha+\beta<\max(0,l+m-k), \tag{31}$$

$$\tilde{h}^{[l]}(z)^{(\alpha)}\tilde{f}^{[m]}(z)^{(\beta)} \text{ for } \alpha+\beta<\max(0,l+m-k). \tag{32}$$

We define the $(q,z,u)$-degree on $C_k$ by the formulas

$$\deg_q \tilde{x}_i^{[l]}=-i,\ \deg_z \tilde{e}_i^{[l]}=2l,\ \deg_z \tilde{h}_i^{[l]}=0,\ \deg_z \tilde{f}_i^{[l]}=-2l,\ \deg_u \tilde{x}_i^{[l]}=l.$$

In what follows we show that the $(q,z,u)$-characters of $B_k$ and $C_k$ coincide.

We will need the following Lemma from [FS, FJKLM].

**Lemma 3.3.** *Consider the quotient of the polynomial algebra $\mathbb{C}[a_0,a_{-1},\ldots]$ by the ideal generated with the coefficients of the series $a(z)^{k+1}$. Then there exists a filtration $F_\mu$ of this quotient (labeled by Young diagrams $\mu$) such that the adjoint graded algebra is generated with the coefficients of series $a^{[i]}(z)$, which are the images of powers $a(z)^i$, $1\le i\le k$. In addition defining relations in the adjoint graded algebra are of the form*

$$a^{[i]}(z)^{(l)}a^{[j]}(z)^{(r)}=0 \text{ if } l+r<2\min(i,j). \tag{33}$$

**Lemma 3.4.** $\mathrm{ch}_{q,z,u}B_k\le \mathrm{ch}_{q,z,u}C_k$.

*Proof.* Using Lemma 3.3 we define a filtration on $B_k$ such that the adjoint graded space is generated by the images of coefficients of the series $\tilde{x}(z)^l$, $x=e,h,f$, $1\le l\le k$. Denote the corresponding series by $\tilde{x}^{[l]}(z)$ and the corresponding Fourier coefficients by $\tilde{x}_i^{[l]}$. Then from Lemma 3.3 we obtain the relations (30). The relations (31), (32) in the adjoint graded space follows from the relations

$$(\tilde{e}(z)^l)^{(\alpha)}(\tilde{h}(z)^m)^{(\beta)}=0 \text{ for } \alpha+\beta<\max(0,l+m-k), \tag{34}$$

$$(\tilde{h}(z)^l)^{(\alpha)}(\tilde{f}(z)^m)^{(\beta)}=0 \text{ for } \alpha+\beta<\max(0,l+m-k), \tag{35}$$

which hold in $B_k$. We thus obtain that all of the relations of $C_k$ are true in the adjoint graded space of $B_k$ with respect to the certain filtration. Lemma is proved. ∎

We want to show that $\mathrm{ch}_{q,z,u}B_k \ge \mathrm{ch}_{q,z,u}C_k$. We use the vertex operator technique. Fix an integer $N$ such that there exists a set of linearly independent vectors $p_i, q_i, r_i \in \mathfrak{h} = \mathbb{R}^N$, $1 \le i \le k$ with the scalar products

(36)
$$(p_i, p_j) = (q_i, q_j) = (r_i, r_j) = 2\delta_{i,j},\ (p_i, q_j) = (q_i, r_j) = \delta_{i,k+1-j},\ (p_i, r_j) = 0.$$

For example, setting $N = 3k$ and fixing some orthonormal basis $e_i$ with respect to $(\cdot, \cdot)$ one can define

$$p_i = \sqrt{2}e_i,\ q_i = \frac{1}{\sqrt{2}}e_{k+1-i} + \sqrt{\frac{3}{2}}e_{k+i},\ r_i = \sqrt{\frac{2}{3}}e_{2k+1-i} + \sqrt{\frac{4}{3}}e_{2k+i}.$$

We consider a lattice $Q$ generated by the vectors $p_i$, $q_i$, $r_i$ and the corresponding VOA $V_Q$. Set

$$V_e(z) = \sum_{i=1}^k \Gamma_{p_i}(z),\ V_h(z) = \sum_{i=1}^k \Gamma_{q_i}(z),\ V_f(z) = \sum_{i=1}^k \Gamma_{r_i}(z). \tag{37}$$

Note that due to the Proposition 1.1 one has

$$\Gamma_{p_i}(z)\Gamma_{q_{k+1-i}}(z) = 0 = \Gamma_{q_i}(z)\Gamma_{r_{k+1-i}}(z).$$

Therefore for any $0 \le i \le k+1$

$$V_e(z)^i V_h(z)^{k+1-i} = V_h(z)^i V_f(z)^{k+1-i} = 0. \tag{38}$$

We let $D_k$ to denote the space generated with the Fourier coefficients of $V_e(z)$, $V_h(z)$, $V_f(z)$ from the highest weight vector.

**Lemma 3.5.** $\mathrm{ch}_{q,z,u}B_k \ge \mathrm{ch}_{q,z,u}D_k \ge \mathrm{ch}_{q,z,u}C_k$.

*Proof.* The equation (38) proves the first part of our Lemma. To show the second enequlity we use the degeneration procedure. Namely we construct a deformation $D_k(\varepsilon)$ of $D_k$ such that $D_k(\varepsilon) \simeq D_k$ for $1 \ge \varepsilon > 0$ and $D_k(0)$ contains $C_k$. Namely let $D_k(\varepsilon)$ be the subspace generated from the highest weight vector by the Fourier coefficients of $V_e(\varepsilon z)$, $V_h(\varepsilon z)$, $V_f(\varepsilon z)$. Then the limit $\lim_{\varepsilon\to 0} D_k(\varepsilon)$ contains the Fourier coefficients of the series

$$\Gamma_{p_1+\cdots+p_l}(z) = \lim_{\varepsilon\to 0} \varepsilon^{\frac{l(l-1)}{2}} (V_e(\varepsilon z))^l,$$
$$\Gamma_{q_1+\cdots+q_l}(z) = \lim_{\varepsilon\to 0} \varepsilon^{\frac{l(l-1)}{2}} (V_h(\varepsilon z))^l,$$
$$\Gamma_{r_1+\cdots+r_l}(z) = \lim_{\varepsilon\to 0} \varepsilon^{\frac{l(l-1)}{2}} (V_f(\varepsilon z))^l.$$

We note that

$$(p_1 + \cdots + p_i, p_1 + \cdots + p_j) = (q_1 + \cdots + q_i, q_1 + \cdots + q_j) = 2\min(i,j),$$
$$(r_1 + \cdots + r_i, r_1 + \cdots + r_j) = 2\min(i,j),$$
$$(p_1 + \cdots + p_i, q_1 + \cdots + q_j) = \max(0, i+j-k),$$
$$(r_1 + \cdots + r_i, q_1 + \cdots + q_j) = \max(0, i+j-k).$$

Let $Q_k$ be the lattice generated by the vectors

$$p_1 + \cdots + p_l,\ q_1 + \cdots + q_l,\ r_1 + \cdots + r_l,\ 1 \le l \le k.$$

Then using Proposition 1.2 we obtain that the principal subspace $W_{Q_k}$ is isomorphic to $C_k$. The isomorphism is given by the identification

$$\Gamma_{p_1+\cdots+p_l}(i) \mapsto \tilde{e}_i^{[l]},\ \Gamma_{q_1+\cdots+q_l}(i) \mapsto \tilde{h}_i^{[l]},\ \Gamma_{r_1+\cdots+r_l}(i) \mapsto \tilde{f}_i^{[l]}.$$

This gives

$$\mathrm{ch}_{q,z,u} D_k \ge \mathrm{ch}_{q,z,u} C_k.$$

Lemma is proved. □

Lemmas 3.4 and 3.5 give the following Corollary:

**Corollary 3.2.** $\mathrm{ch}_{q,z,u} B_k = \mathrm{ch}_{q,z,u} C_k = \mathrm{ch}_{q,z,u} D_k$.

**Proposition 3.1.** $\mathrm{ch}_{q,z,u} L_k \ge \mathrm{ch}_{q,z,u} D_k$.

*Proof.* We recall that there exists an embedding $L_k \hookrightarrow L_1^{\otimes k}$ such that the highest weight vector $v_k$ of $L_k$ maps to the tensor power $v_1^{\otimes k}$ and for any $x \in \mathfrak{sl}_2$ the current $x(z)$ on $L_k$ corresponds to the sum $\sum_{i=1}^k x^{(i)}(z)$, where

$$x^{(i)}(z) = \mathrm{Id} \otimes \ldots \otimes x(z) \otimes \ldots \otimes \mathrm{Id}$$

($x(z)$ on the $i$-th place). From the definition of $D_k$ we also have an embedding $D_k \hookrightarrow D_1^{\otimes k}$ (see the definition (37)). Now the degeneration from the Corollary 2.2 gives the degeneration of $L_k$. From the part $c$) of the Proposition 2.1 we conclude that the limit of this degeneration contains $D_k$. Proposition is proved. □

**Theorem 3.1.**

- *The $(q, z, u)$ characters of $L_k^{gr}$, $A_k$, $B_k$, $C_k$ and $D_k$ coincide.*
- *$L_k^{gr} \simeq A_k$ as the modules over the abelian algebra with a basis $\tilde{e}_i$, $\tilde{h}_i$, $\tilde{f}_i$, $i \le -1$.*

*Proof.* Follows from Lemmas 3.1, 3.2, Corollary 3.2 and Proposition 3.1. □

3.3. **The character formula.** In this section we compute the $(q, z, u)$-character of $L_k^{gr}$ .

**Proposition 3.2.**

$$\mathrm{ch}_{q,z,u} C_k = \sum_{\mathbf{n}^-, \mathbf{n}^0, \mathbf{n}^+ \in \mathbb{Z}_{\ge 0}^k} u^{|\mathbf{n}^+| + |\mathbf{n}^0| + |\mathbf{n}^-|} z^{2(|\mathbf{n}^+| - |\mathbf{n}^-|)} \times \frac{q^{\frac{1}{2}(\mathbf{n}^+ A \mathbf{n}^+ + \mathbf{n}^0 A \mathbf{n}^0 + \mathbf{n}^- A \mathbf{n}^-) + \mathbf{n}^+ B \mathbf{n}^0 + \mathbf{n}^0 B \mathbf{n}^-}}{(q)_{\mathbf{n}^+} (q)_{\mathbf{n}^0} (q)_{\mathbf{n}^-}}, \tag{39}$$

*where for $\mathbf{n} \in \mathbb{Z}_{\ge 0}^k$ we set $|\mathbf{n}| = \sum_{i=1}^k i \mathbf{n}_i$ and matrixes $A$ and $B$ are defined by*

$$A_{i,j} = 2 \min(i, j),\ B_{i,j} = \max(0, i + j - k).$$

*Proof.* Follows from the vertex operator realization of $C_k$ constructed in Corollary 3.2 and Proposition 1.2. □

**Theorem 3.2.** *The* $(q,z,u)$*-character of* $L_k^{gr}$ *is given by the right hand side of* (39).

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