ZERO DISPERSION LIMIT FOR INTEGRABLE EQUATIONS ON THE HALF LINE WITH LINEARISABLE DATA

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ABSTRACT

We study the zero dispersion limit for certain initial-boundary value problems for the defocusing nonlinear Schrödinger equation and for the Korteweg-de Vries equation with dominant surface tension. These problems are formulated on the half line and they involve linearisable boundary conditions.

1. AN INITIAL-BOUNDARY VALUE PROBLEM FOR SOLITON EQUATIONS

In recent years there has been a series of results of one of the authors and others on boundary value problems for soliton equations (see [F1] for a comprehensive review). The method of Fokas [F1] goes beyond existence and uniqueness. In fact, it reduces these problems to Riemann-Hilbert factorisation problems in the complex plane, thus generalising the existing theory which reduces initial value problems to Riemann-Hilbert problems via the method of inverse scattering. One of the main advantages of the Riemann-Hilbert formulation is that one can use recent powerful results on the asymptotic behavior of solutions to these problems (as some parameter goes to infinity) to derive asymptotics for the solution of the associated soliton equation. For the study of the long time asymptotics such results were pioneered by A.Its, P.Deift and X.Zhou; the method is known as "nolinear steepest descent" in analogy with the linear steepest descent method which is applicable to asymptotic problems for Fourier-type integrals (see e.g. [DIZ]). A generalisation of the steepest descent method developed in [DVZ] is able to give rigorous results for the so-called "semiclassical" or "zero dispersion" limit of the solution of the Cauchy problem for 1+1 dimensional integrable evolution equations, in the case where the Lax operator is self-adjoint. The method has been further extended in [KMM] for the "non-self-adjoint" case.

In a recent paper [K] one of the authors, by making use of the nonlinear steepest descent method, has studied the "zero dispersion" limit of the initial-boundary value problem for the 1+1 dimensional, integrable, defocusing, nonlinear Schrödinger equation on the half-line, for quite general initial and boundary data. In the present paper we consider the simpler case of "linearisable" data. More precisely, we consider the two "archetypal" soliton equations:

$$
\begin{aligned}
&ihu_t(x,t) + h^2 u_{xx}(x,t) - 2|u(x,t)|^2 u(x,t) = 0, \ x \geq 0, \ t \geq 0, \\
&\qquad\qquad u(x,0) = u_0(x) \in \mathbb{S}(\mathbb{R}^+), \ 0 < x < \infty,
\end{aligned}
\tag{1}
$$

with the linearisable boundary condition

$$u_x(0,t) - \chi u(0,t) = 0,\ t > 0, \tag{1a}$$

for some constant $\chi \geq 0$. h is the semiclassical parameter which is assumed to be small and positive and $\mathbb{S}(\mathbb{R}^+)$ denotes the Schwartz class on $[0,\infty)$.

$$\begin{aligned} u_t(x,t) + u_x(x,t) + 6uu_x(x,t) - h^2 u_{xxx}(x,t) = 0,\ x \geq 0,\ t \geq 0, \\ u(x,0) = u_0(x) \in \mathbb{S}(\mathbb{R}^+),\ 0 < x < \infty, \end{aligned} \tag{2}$$

with the linearisable boundary condition

$$u(0,t) = \chi,\ u_{xx}(0,t) = \chi + 3\chi^2,\ t \geq 0, \tag{2a}$$

for some constant χ. h is the dispersion parameter which is assumed to be small and positive.

It is well known that these equations admit a "Lax-pair" formulation. Namely these equations are the compatibility condition for the equations $L\mu = 0$ and $B\mu = 0$ where L, B are differential operators on a Hilbert space. In the NLS case for example, they are given by

$$\begin{aligned} L &= \begin{pmatrix} \partial_x - ik & iu \\ -i\bar{u} & \partial_x + ik \end{pmatrix}, \\ B &= \begin{pmatrix} ih\partial_t + 4ik^2 + i|u|^2 & -2ku - iu_t \\ -2k\bar{u} + i\bar{u}_t & ih\partial_t - i|u|^2 \end{pmatrix}. \end{aligned} \tag{3}$$

Here the bar denotes complex conjugation, k is the spectral variable and $u = u(x,t)$ is the solution of (1).

The traditional method of solving initial value problems for soliton equations that admit a Lax-pair formulation is to focus on the L operator and apply the theory of scattering and inverse scattering to this operator. On the other hand, one of the main ideas of the method of Fokas is that for initial-boundary value problems the two operators L and B should be on an equal footing. The scattering transform should be applied to both operators *simultaneously*, while a so-called *global relation* has to be imposed on the data to ensure compatibility (see relation (7)). The global relation will ensure existence, uniqueness and the validity of the Riemann-Hilbert formulation.

2. THE RIEMANN-HILBERT PROBLEM

As shown in [F2], initial boundary value problems for integrable evolution PDEs can be reduced to a Riemann-Hilbert factorisation problem, under the special assumption that the so-called global relation, a condition on the given data, holds (see relation (7) below).

Consider first the NLS equation with general boundary conditions. Namely either $u(0,t)$ or $u_x(0,t)$ or a relation between $u(0,t)$ and $u_x(0,t)$ is given. The situation for KdV is similar where *two* boundary conditions are given.

Let Σ be the contour $\mathbb{R} \cup i\mathbb{R}$ with the following orientation:

(i) the real axis is oriented from left to right,

(ii) the positive imaginary axis is oriented from infinity towards zero,

(iii) the negative imaginary axis is oriented from infinity towards zero.

We use the following convention: the $+$-side of an oriented contour is always to its left, according to the given orientation.

Letting M_+ and M_- denote the limits of M on Σ from left and right respectively, we define the Riemann-Hilbert factorization problem

$$
\begin{aligned}
M_+(x,t,k) &= M_-(x,t,k)J(x,t,k), \\
&\quad \textit{where} \\
J(x,t,k) &= J_4^{-1}, \ k \in \mathbb{R}^+, \\
&\quad J_1^{-1}, \ k \in i\mathbb{R}^+, \\
&\quad J_3^{-1}, \ k \in i\mathbb{R}^-, \\
J_2 &= J_3 J_4^{-1} J_1, \ k \in \mathbb{R}^-, \\
&\quad \textit{with} \\
J_1 &= \begin{pmatrix} 1 & 0 \\ \Gamma(k)e^{2i\Theta} & 1 \end{pmatrix}, \\
J_4 &= \begin{pmatrix} 1 & -\gamma(k)e^{-2i\Theta} \\ \bar{\gamma}(k)e^{2i\Theta} & 1-|\gamma(k)|^2 \end{pmatrix}, \\
J_3 &= \begin{pmatrix} 1 & -\bar{\Gamma}(\bar{k})e^{-2i\Theta} \\ 0 & 1 \end{pmatrix}, \\
\Theta(x,t,k) &= \frac{\theta}{h}, \ \textit{where } \theta = kx + 2k^2 t.
\end{aligned}
\tag{4}
$$

The functions γ, Γ are defined in terms of the spectral functions of the problem (see [FIS] (2.28), (2.25)), with important analyticity properties (see (2.21) and (2.22) of [FIS]). In particular

$$\Gamma(k) = \frac{1}{a(k)(a(k)\frac{\bar{A}(\bar{k})}{\bar{B}(\bar{k})} - b(k))}, \tag{5}$$

where a, b are the spectral functions for the x-problem and A, B are the spectral functions for the t-problem. The functions a, b are analytic and bounded in the upper half-plane, while A, B are analytic and bounded in the first and third quadrants of the k-plane.

The solution of the NLS equation can be recovered from the solution of (4) as follows.

$$u(x,t) = 2i\ h\ lim_{k\to\infty}(kM^{12}(x,t,k)), \tag{6}$$

where the index 12 here denotes the (12)-entry of a matrix.

The following "global relation" is imposed on the scattering data:

$$a(k)B(k) - b(k)A(k) = e^{4ik^2T}c(k), \tag{7}$$

where $c(k)$ is analytic and bounded for $Imk > 0$, and $c(k) = O(1/k)$ as $k \to \infty$. Here T is the time up to which we solve the initial boundary value problem for NLS. In general A, B are functions of T.

There exists a complicated relation between the $u(0,t)$ and $u_x(0,t)$; the global relation is the expression of this in the spectral space.

In our particular case (problem (1)) $T = \infty$ and the global relation becomes

$$a(k)B(k) - b(k)A(k) = 0, \tag{7'}$$

for $arg(k) \in [0, \pi/2]$.

The KdV is treated similarly. The contour Σ^{KdV} consists of the real line oriented from left to right, together with the curves

$$l_+ = \{k = k_R + ik_I, k_I > 0, \frac{1}{4} + 3k_R^2 - k_I^2 = 0\},$$
$$l_- = \{k = k_R + ik_I, k_I < 0, \frac{1}{4} + 3k_R^2 - k_I^2 = 0\},$$

oriented from right to left. Instead of (4) the Riemann Hilbert problem becomes

(4')
$$
\begin{aligned}
M_+^{KdV}(x,t,k) &= M_-^{KdV}(x,t,k)J^{KdV}(x,t,k), \\
&\qquad\qquad\qquad where \\
J^{KdV}(x,t,k) &= (J_1^{KdV})^{-1},\ \ k\in l_-, \\
&\quad\ (J_3^{KdV})^{-1},\ \ k\in l_+, \\
J_2^{KdV} &= J_3^{KdV}(J_4^{KdV})^{-1}J_1^{KdV},\ \ k\in\mathbb{R}, \\
&\qquad\qquad\qquad with \\
J_1^{KdV} &= \begin{pmatrix} 1 & 0 \\ \Gamma(k)e^{2i\Theta^{KdV}} & 1 \end{pmatrix}, \\
J_4^{KdV} &= \begin{pmatrix} 1 & -\gamma(k)e^{-2i\Theta^{KdV}} \\ \bar{\gamma}(k)e^{2i\Theta^{KdV}} & 1-|\gamma(k)|^2 \end{pmatrix}, \\
J_3^{KdV} &= \begin{pmatrix} 1 & -\bar{\Gamma}(\bar{k})e^{-2i\Theta^{KdV}} \\ 0 & 1 \end{pmatrix}, \\
\Theta^{KdV}(x,t,k) &= \frac{\theta^{KdV}}{h},\ \ where\ \theta^{KdV} = -kx+(k+4k^3)t.
\end{aligned}
$$

Here Γ is stil defined via (5) and the spectral functions a, b, A, B still satisfy equation (7'), where of course these functions are now expressed in terms of the KdV spectral problems.

3. LINEARISABLE DATA

In general the global relation together with the definition of $A(k), B(k)$ imply a nonlinear Volterra integral equation for the missing boundary values. For example it is shown in [FIS] that in the case of the defocusing NLS with $q(0,t) = f_0(t)$ given, the unknown boundary value $q_x(0,t) = f_1(t)$ satisfies a nonlinear Volterra equation which has a global solution.

We note that the analogous step for linear evolution equations is solved by algebraic manipulations [F1]. This is a consequence of the invariance of the unknown terms in the global relation under $k \to \nu(k)$, where $\omega(\nu(k)) = \omega(k)$ and ω is k^2 and $k + 4k^3$ for the NLS and KdV respectively. Unfortunately, the global relation now involves the solution of the t-problem $\Phi(t,k)$ which in general breaks the invariance. However, for a particular class of boundary conditions this invariance survives. This is precisely the class of "linearisable problems", namely a class of problems for which $A(k), B(k)$ can be explicitly written in terms of $a(k), b(k)$.

It is shown in [F2] that for the NLS with the boundary condition (1a),

$$\frac{B(k)}{A(k)} = -\frac{2k + i\chi}{2k - i\chi}\frac{b(-k)}{a(-k)},$$

while for the KdV with the boundary condition (2a),

$$\frac{B(k)}{A(k)} = \frac{f(k)b(\nu(k)) - a(\nu(k))}{f(k)a(\nu(k)) - b(\nu(k))}$$

where

$$\nu^2 + k\nu + k^2 + \frac{1}{4} = 0,$$

$$f(k) = \frac{\nu + k}{\nu - k}(1 - \frac{4\nu k}{\chi}).$$

Now let us restrict ourselves to NLS first. It is then easy to see that B/A is analytic and $O(1/k)$ in the first quadrant. Hence, the coefficient Γ is analytic (at least) in the first quadrant of the k-plane, and bounded there.

This has an important consequence. Noting the decay properties of the term $exp(2i\Theta)$ as $k \to \infty$ in the first quadrant, it is immediate that the positive imaginary axis of the contour can be deformed clockwise to the positive real part. This deformation is exact, not approximate.

Similarly, the negative imaginary part of the contour can be deformed to the negative real axis. We end up with a Riemann-Hilbert problem with jumps only along the real axis. In fact, let

$$N(x,t,k) = M(x,t,k),\ arg(k) \in (\pi/2, 3\pi/2),$$

$$N(x,t,k) = M(x,t,k)J_1^{-1},\ arg(k) \in (0, \pi/2),$$

$$N(x,t,k) = M(x,t,k)J_3^{-1},\ arg(k) \in (3\pi/2, 2\pi).$$

The Riemann-Hilbert problem becomes

$$N_+(x,t,k) = N_-(x,t,k)J(x,t,k),\ k \in \mathbb{R},$$

$$lim_{k\to\infty} N(x,t,k) = I,$$

(8)

$$where$$

$$J_N(x,t,k) = \begin{pmatrix} 1 & (\gamma - \bar{\Gamma})(k)e^{-2i\Theta} \\ -(\bar{\gamma} - \Gamma)(k)e^{2i\Theta} & 1 - |\gamma(k)|^2 \end{pmatrix},$$

and formula (6) holds with N instead of M. Again, we stress that problems (4) and (8) are equivalent exactly, not just approximately.

Note that for $k \geq 0$, $\gamma(k) - \bar{\Gamma}(k) = 0$. This follows from the definition (5) and the global relation (7). For negative k, $\gamma(k) - \bar{\Gamma}(k) = \frac{bA - aB}{\bar{a}A - \bar{b}B}$ has all the smoothness and decay properties that are required from a bona fide reflection coefficent corresponding to a realised potential. For example, if the initial data u_0 belong to the Schwartz class of $\mathbb{R}^+$, then a, b are also Schwartz, while A, B are smooth and bounded. Hence $R = \gamma - \bar{\Gamma}$ belongs to the Schwartz class of $\mathbb{R}$. There is a unique potential v_0 corresponding to R, which is a continuation of u_0. The Riemann-Hilbert Problem (8) then gives the evolution of the solution to the NLS equation under initial data v_0.

Thus we have shown that, in the linearisable case for NLS, the half-line problem can be recovered from the solution of the full real line problem by appropriately continuing the initial data.

The above observations are of immediate consequence to the question of studying the semiclassical limit of NLS. Since the initial boundary value problem can be considered as a restriction of an initial value problem and since the initial value problem for NLS is well understood (studied by Jin, Levermore, D.McLaughlin in [JLM]), the results for the initial boundary value problem are recovered immediately.

More precisely, with the introduction of a small dispersive constant h, the changes in the Riemann-Hilbert problem (4) will be as follows:

(i) x, t will be replaced by $x/h, t/h$.

(ii) the coefficients γ, Γ will now be dependent on h.

Since the deformation leading to (8) is exact, i.e. there is no error (possibly dependent on h), the reduction to a Riemann-Hilbert problem on the line is possible, exactly as in the $h = 1$ case.

Phenomenologically, one sees that the half-plane $x, t \geq 0$ can be divided in two regions. In the first ("smooth") region strong semiclassical limits exist and satisfy the formally limiting system. In the NLS case, letting $\rho = |u|^2$, $\mu = hIm(\bar{u}u_x)$, the strong limits $\tilde{\rho}, \tilde{\mu}$, as $h \to 0$, exist and satisfy

$$\begin{aligned} \tilde{\rho}_t + \tilde{\mu}_x &= 0 \\ \tilde{\mu}_t + (\frac{\tilde{\mu}^2}{\tilde{\rho}} - \frac{\tilde{\rho}^2}{2})_x &= 0. \end{aligned}$$

In the second ("turbulent") region fast oscillations appear that can be described in terms of slowly modulating finite-gap solutions. Only weak limits exist for ρ, μ as $h \to 0$ and they can be expressed in terms of the solutions of the so-called Whitham system. Rigorous asymtptotic formulae for ρ, μ are also easily available (see e.g. [KKM]).

In the case of the KdV equation, even with the appropriate changes of contour and phase, the situation is more complicated. It is not true anymore, even for linearisable data, that the half-line problem can be recovered from the solution of the full real line problem by appropriately continuing the initial data, for any value of the small dispersion parameter h. However, we will show that this reduction is possible *asymptotically* as $h \to 0$.

Denote by D_1, D_2 the domains bounded by $l_+ \cup \mathbb{R}$ and $l_- \cup \mathbb{R}$ respectively. In order to follow the argument used above for the NLS equation, we need to "deform" the curves l_- and l_+ to the real line. This is not possible, because $\bar{\Gamma}(\bar{k})e^{-2i\Theta^{KdV}}, \Gamma(k)e^{2i\Theta^{KdV}}$ as $k \to \infty$ are not bounded as $k \to \infty$ in the domains D_1 and D_2, respectively. What we *can* do is "conjugate away" the jumps on l_- and l_+ by introducing an auxiliary Riemann-Hilbert problem which can be reduced to a scalar (hence explicitly solvable) Riemann-Hilbert problem. This is possible because of the triangularity of the jump matrices J_1^{KdV}, J_3^{KdV}.

In the end we still arrive at a Riemann-Hilbert problem on the real line, which, however, is not of the "standard" KdV form, at least for general h. As long as we are only interested in the asymptotics $h \to 0$ though, the reduction to a "standard" Riemann-Hilbert problem *is* possible. This means that the already existing analysis of the zero dispersion limit of KdV on the full line ([LL], [V]) is after all applicable.

More precisely, following [K], we consider the following Riemann-Hilbert problem.

(9)
$$L_+(x,t,k) = L_-(x,t,k)J^{KdV}(x,t,k), \quad \textit{where}$$
$$J^{KdV}(x,t,k) = (J_3^{KdV})^{-1}, \ \ k \in l_+, \quad \textit{with}$$
$$J_3^{KdV} = \begin{pmatrix} 1 & -\bar{\Gamma}(\bar{k})e^{-2i\Theta^{KdV}} \\ 0 & 1 \end{pmatrix},$$
$$\Theta^{KdV}(x,t,k) = \frac{\theta^{KdV}}{h}, \quad \textit{where}$$
$$\theta^{KdV} = -kx + (k+4k^3)t,$$

such that $lim_{k\to\infty} L(x,t,k) = I$.

This problem can be solved explicitly.

(10)
$$L(x,t,k) = \begin{pmatrix} 1 & l(x,t,k) \\ 0 & 1 \end{pmatrix}, \quad \textit{where}$$
$$l(x,t,k) = \frac{1}{2\pi i}\int_{l+} \frac{-\bar{\Gamma}(\bar{s})e^{-2i\Theta^{KdV}(x,t,s)}ds}{s-k}$$

Similarly consider

(9')
$$U_+(x,t,k) = U_-(x,t,k)(J_1^{KdV})^{-1}(x,t,k), \ \ k \in l_-,$$
$$with \ J_1^{KdV} = \begin{pmatrix} 1 & 0 \\ \Gamma(k)e^{2i\Theta^{KdV}} & 1 \end{pmatrix},$$

such that $lim_{k\to\infty}U(x,t,k) = I$. We now have

(10')
$$U(x,t,k) = \begin{pmatrix} 1 & 0 \\ u(x,t,k) & 1 \end{pmatrix}, \ where$$
$$u(x,t,k) = \frac{1}{2\pi i}\int_{l^-} \frac{\Gamma(s)e^{2i\Theta^{KdV}(x,t,s)}ds}{s-k}$$

Now set

(11)
$$N(x,t,k) = M(x,t,k)U^{-1}(x,t,k), \ \ k \in D_1,$$
$$N(x,t,k) = M(x,t,k)L^{-1}(x,t,k), \ \ k \in D_2,$$
$$N(x,t,k) = M(x,t,k),$$
$$otherwise.$$

Then $N(x,t,k)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$, with $lim_{k\to\infty}N(x,t,k) = I$, and across $\mathbb{R}$ the jump is given by

(12)
$$N_+(x,t,k) = N_-(x,t,k)L(x,t,k)J(x,t,k)U^{-1}(x,t,k).$$

An easy deformation argument shows that $L = I + O(h), U = I + O(h)$ as $h \to 0$.[1] This means that the factor L can be ignored asymptotically.[2]

Again, the half-plane $x,t \geq 0$ can be divided in two regions. In the first ("smooth") region a strong zero dispersion limit $\tilde{u} = lim_{h\to 0}u$ exists and satisfies the formally limiting system:

$$\tilde{u}_t(x,t) + \tilde{u}_x(x,t) + 6\tilde{u}\tilde{u}_x(x,t) = 0.$$

In the second ("turbulent") region fast oscillations appear that can be described in terms of slowly modulating finite-gap solutions. Only a weak limits exists for u as $h \to 0$ and it can be expressed in terms of the solutions of the Whitham system for KdV. A rigorous asymtptotic formula for u is also easily available (see e.g. [V]).

[1] One can simply deform the contour "upwards" or "downwards" and the real part of the phase will become negative.

[2] This is not entirely obvious at this point. The full-line zero dispersion analysis of KdV involves the introduction of a so-called "g-function" and a conjugation of the Riemann-Hilbert by a term $e^{\sigma_3 g/h}$. Only after arriving at the new "conjugated" Riemann-Hilbert problem is one allowed to use the fact $L, U = I + O(h)$ as $h \to 0$. See [K] for details.

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