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by

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Abstract

Induced representations $\mathrm{Ind}_H^G S$ were introduced and studied by F.G. Frobenius [7] for finite groups and developed by G.W. Mackey [21, 22] for a locally compact groups. We generalize the Mackey construction for infinite-dimensional groups. To do this, we construct some G-quasi-invariant measures on an *appropriate completion* $\tilde{X} = \tilde{H}\backslash\tilde{G}$ of the initial space $X = H\backslash G$ (since the Haar measure on G does not exist) and extend the representation S of the subgroup H to the representation $\tilde{S}$ of the corresponding completion $\tilde{H}$. Kirillov's orbit method [9] describes all irreducible unitary representations of the finite-dimensional nilpotent group G_n in terms of induced representations associated with orbits in coadjoint action of the group G_n in a dual space $\mathfrak{g}_n^*$ of the Lie algebra $\mathfrak{g}_n$. The induced representation defined in such a way allows us to start to develop an analog of the orbit method for the infinite-dimensional "nilpotent" group $B_0^{\mathbb{Z}}=\varinjlim_n G_{2n-1}$ of infinite in both directions matrices.

Keywords:
locally compact, infinite-dimensional, Hilbert-Lie group; unitary, irreducible, regular, quasiregular, induced representation; orbit method; coadjoint action; quasi-invariant, ergodic measure; upper triangular, infinite matrix; universal enveloping, von Neumann algebra; Gauss decomposition
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Contents

1 Introduction **2**

2 Induced representations, finite-dimensional case **5**

2.1 Induced representations 5

2.2 Orbit method for finite-dimensional nilpotent group $B(n,\mathbb{R})$. 6

2.3 The induced representations corresponding to generic orbits, finite-dimensional case 7

2.4 New proof of the irreducibility of the induced representations . 10

3 Induced representations, infinite-dimensional case **13**

3.1 Induced representations for infinite-dimensional groups 13

3.2 How to develop the orbit method for infinite-dimensional i-nilpotent group $B_0^{\mathbb{Z}}$? 15

3.3 Hilbert–Lie groups $\mathrm{GL}_2(a)$ 16

3.4 Hilbert-Lie groups $B_2(a)$ 17

3.5 Orbits for groups $B_0^{\mathbb{Z}}$ and $B_2(a)$. 17

3.6 Construction of the induced representations $T^{m,y}$ corresponding to a point y^{2m+1} and subgroup H_0^{2m+1} 19

3.7 Irreducibility of the induced representations $T^{m,y}$ 21

4 Criteria of the irreducibility of the induced representations $T^{k,2m+1,\mu_b}$ corresponding to generic orbits **25**

4.1 The center of the universal enveloping algebra of a Lie algebra $\mathfrak{b}(n,\mathbb{R})$ and the description of the commutant of the induced representations of the group G_n 28

4.2 Study of the induced representations $T^{k,2m+1,\mu_b}$ of the group $B_0^{\mathbb{Z}}$ corresponding to generic orbits 31

5 Appendix **36**

5.1 Gauss decompositions 36

5.2 Different criteria for irreducibility of induced representations for locally compact and infinite discrete groups 37

1. Introduction

Induced representations were introduced and studied by F.G. Frobenius [7] for finite groups and developed in details by G.W. Mackey [21, 22] for locally compact groups. The induced representation $\mathrm{Ind}_H^G S$ is a representation on the space $L^2(X,V,\mu)$ of a group G associated with a unitary representation S in a space V of a closed subgroup H. A G-quasi-invariant measure μ is determined by the Haar measure on G. We generalize the Mackey construction for infinite-dimensional (non locally compact) groups.

To define correctly the representation $\mathrm{Ind}_H^G S$ and the space $L^2(X,V,\mu)$ (since a Haar measure does not exist on G) we can take a completion $\tilde{X}$ of the space X, construct some G-quasi-invariant measure μ on it and extend the representation S of the subgroup H to the representation $\tilde{S}$ of the corresponding completion $\tilde{H}$. The content of the article is the following. Section 2 is devoted to notion of the induced representations for *locally compact groups* elaborated by G.W.Mackey [21, 22, 23, 24] and to the Kirillov *orbit method* [9, 11, 12] for the nilpotent Lie groups G_n of $n\times n$ upper triangular real matrices with units on the principal diagonal. In Section 3.1 we define the induced representations for an arbitrary infinite-dimensional group G. As an illustration, in Sections 3.2-3.5 we start to develop the orbit method for infinite-dimensional *i-nilpotent* group $B_0^{\mathbb{Z}}{=}\varinjlim_n G_{2n-1}$ with respect to the *symmetric embedding* (Sec. 3.3). We call a group *i-nilpotent* if $\cap_{n\in\mathbb{N}}G_n{=}\{e\}$, where $G_{n+1}=\{G,G_n\}$ and $G_0=G$. To find an appropriate *completion* $\tilde{X}$ and *extend* the representation of the group H we use a family of *Hilbert–Lie groups* $B_2(a)\,(a\in\mathfrak{A})$ introduced in [13]. This family has the property that any continuous representation U of the group $B_0^{\mathbb{Z}}$ can be extended by continuity to some representation $U_2(a)$ of an appropriate Hilbert-Lie group $B_2(a)$. Let $\hat{G}$ be a *dual space of a group* G, i.e. the set of all equivalence classes of unitary irreducible representations of the group G. The family $B_2(a)\,(a\in\mathfrak{A})$ has the following property: $B_0^{\mathbb{Z}}=\cap_{a\in\mathfrak{A}}B_2(a)$ (see (3.6)). Therefore $\widehat{B_0^{\mathbb{Z}}}=\cup_{a\in\mathfrak{A}}\widehat{B_2(a)}$ and to describe *the dual space* $\widehat{B_0^{\mathbb{Z}}}$ it is sufficient to know $\widehat{B_2(a)}$ for all $a\in\mathfrak{A}$, but this problem has not been solved yet. In Section 3.6–3.7 we construct a one-parameter family $T^{y,m,\mu}(m\in\mathbb{Z})$ depending on the measure μ, of induced representations corresponding to generic orbits generated by a point $y\in\mathfrak{g}^*{=}(\varinjlim_n\mathfrak{g}_n)^*$ and prove their irreducibility (Theorem 3.9). In Section 4 we construct a two-parameter family $T^{y,k,n,\mu}\,(k,n\in\mathbb{Z})$ of induced representations corresponding to generic orbits and give the criteria of the irreducibility (Theorem 4.2). Here we use the technique developed earlier in [14, 15] to prove the irreducibility of the “regular” representations in the framework of Ismagilov’s Conjecture 3.1. In Remark 4.2 we show that the induced representations T of the group $B_0^{\mathbb{Z}}=\varinjlim_n G_n^k$ on the space

$L^2(X,\mu)$ can be obtained as a limit of (*non compatible*) representations $T_n^{\mu_n}$ equivalent with the induced representations $T_n^{h_n}$ of corresponding subgroups G_n^k on $L^2(X_n,h_n)$. This fact is based *on the symmetric groups embedding*! This gives another possibility to construct an appropriate measure μ on a suitable completion $\tilde{X}$ of the space $X=\cup_n X_n$. We should find measures $\mu_n\sim h_n$ on $X_n\cong\mathbb{R}^{m(n)}$ which satisfy a natural *consistency condition* then by Kolmogorov's theorem [8] these measures μ_n fit together to form a measure μ on $\mathbb{R}^\infty$. In Appendix 1 we recall the Gauss decomposition of finite and infinite matrices. In Appendix 2 we collect some criteria of the irreducibility for locally compact and infinite discrete groups. For countable groups G.W. Mackey has shown [24] that quasiregular representations are irreducible if and only if the corresponding subgroups are *self-commensurizing*. He also gave criteria for induced representation $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\pi)$ to be irreducible for finite-dimensional representations π of a subgroup Γ_0. See also M. Burger and P. de la Harpe [2], L. Corwing [3], N. Obata [25]. The representations of the infinite symmetric group S_∞ induced by trivial representation of Young subgroups were studied by A.M. Vershik and N.V. Tsilevich [29].

The "*regular*" and "*quasiregular*" representations introduced by the author in [14, 19] for general infinite-dimensional groups are particular cases of induced representations (see Example 3.1). In contrast to finite-dimensional groups, the "regular" representations of the group $B_0^{\mathbb{Z}}$ corresponding to the trivial orbit $0\in\mathfrak{g}^*$, can be *irreducible and nonequivalent* if the corresponding measures are nonequivalent [18]. The same holds for "quasiregular" representations. So, not all irreducible representations of the infinite-dimensional i-nilpotent group $B_0^{\mathbb{Z}}$ are *monomial*, i.e., induced by one-dimensional representations as for finite-dimensional groups. The *Ismagilov conjecture* [14, 15] and its generalization [19] (Conjectures 3.1, 3.2) explains when regular and quasiregular representations of infinite-dimensional groups can be irreducible. It is a remarkable fact that the criteria of the irreducibilty of the induced representations of $B_0^{\mathbb{Z}}$ for generic orbits (Theorem 4.2) includes the conditions of (Ismagilov's) Conjecture 3.1. The equivalence will be studied later. For general orbits, representations have not been constructed. It is an open question whether the orbit method will give all unitary irreducible representations of the group $B_0^{\mathbb{Z}}$. The completions of groups that are inductive limits $G=\varinjlim_n G_n$ of finite-dimensional classical groups G_n appeared, for example, in the work by A.A. Kirillov [10] for the group $U(\infty)=\varinjlim_n U(n)$, and G.I. Ol'shanskiĭ [26], for inductive limits of classical groups. They described all unitary irreducible representations of the corresponding groups

$G = \varinjlim_n G_n$, *continuous* in the stronger topology, namely, *in strong operator topology*. In [31] for inductive limits of real reductive Lie groups $G = \varinjlim_n G_n$, J.A. Wolf constructed the *principal series representations* as the inductive limit of *compatible* (see Remark 4.2) representations of principal series for subgroup G_n. As he wrote,"anything involving integration over G/P is excluded". We can quote from [32]: " We study representations of the classical infinite dimensional real simple Lie groups G induced from factor representations of *minimal parabolic* subgroups P. When P is minimal we prove that it is *amenable*, and we use properties of amenable groups to induce unitary representations $\mathrm{Ind}_P^G(\tau)$ on complete locally convex topological vector spaces". To construct an analog of induced representations, integration over G/P is replaced by right P-invariant means on G.

2. Induced representations, finite-dimensional case

2.1. Induced representations

The *induced representation* $\mathrm{Ind}_H^G S$ is the unitary representation on the space $L^2(X, V, \mu)$ of a group G associated with a unitary representation $S : H \to U(V)$ of a closed subgroup H of the group G. For details, see [22, 24]. We follow [12, Section 2.1]. Suppose that $X = H \setminus G$ is a right $G-$space and that $s : X \to G$ is a Borel section of the projection $p : G \to X = H\backslash G : g \mapsto Hg$. For a Lie group, such a mapping s can be chosen to be smooth almost everywhere. Then every element $g \in G$ can be uniquely written in the form

$$g = hs(x),\ h \in H,\ x \in X, \tag{2.1}$$

and thus G (as a set) can be identified with $H \times X$.

The representation $\mathrm{Ind}_H^G S$ is defined as follows [12, section 2.3]. Let $S : H \to U(V)$ be a unitary representation of a subgroup H of the group G in a Hilbert space V and let μ be a measure on X satisfying the condition $d\mu_s(xg)/d\mu_s(x) = \Delta_H(h(x,g))/\Delta_G(h(x,g))$, where Δ_G is a *modular function* on a group G and $h(x,g) \in H$ is defined by the relation $s(x)g = h(x,g)s(xg)$. Let $L^2(X, V, \mu)$ denote the space of all vector-valued functions f on X with values in V such that

$$\|f\|^2 := \int_X \|f(x)\|_V^2 d\mu(x) < \infty.$$

Let us consider the representation T given by the formula

$$[T(g)f](x) = A(x,g)f(xg) = S(h)\left(d\mu_s(xg)/d\mu_s(x)\right)^{1/2} f(xg), \tag{2.2}$$

where

$$A(x,g) = [\Delta_H(h)/\Delta_G(h)]^{1/2}\, S(h), \tag{2.3}$$

and the element $h = h(x,g)$ is defined as before by formula $s(x)g = h(x,g)s(xg)$.

Definition 2.1. The representation T is called the unitary induced representation and is denoted by $\mathrm{Ind}_H^G S$.

Remark 2.1. The right (or the left) regular representation $\rho, \lambda : G \mapsto U(L^2(G,h))$ of a locally compact group G is a particular case of the induced representation $\mathrm{Ind}_H^G S$ with $H = \{e\}$ and $S = Id$, where h is a Haar measure. The quasiregular representation is a particular case of the induced representation with some closed subgroup $H \subset G$ and $S = Id$.

2.2. Orbit method for finite-dimensional nilpotent group $B(n, \mathbb{R})$

Fix the group $G_n = B(n, \mathbb{R})$ of all upper triangular real matrices of order n with ones on the main diagonal. The basic result of the Kirillov orbits method [11], [12, Chapter 7, §2] applied to nilpotent Lie groups "is the description of a one-to-one correspondence between two sets:
a) the set $\hat{G}$ of all equivalence classes of irreducible unitary representations of a connected and simply connected nilpotent Lie group G,
b) the set $\mathcal{O}(G)$ of all orbits of the group G in the space $\mathfrak{g}^*$ dual to the Lie algebra $\mathfrak{g}$ with respect to the coadjoint representation.

To construct this correspondence, we introduce the following definition. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is *subordinate* to a functional $f \in \mathfrak{g}^*$ if

$$\langle f, [x,y] \rangle = 0 \quad for\ \ all \quad x, y \in \mathfrak{h},$$

i.e. if $\mathfrak{h}$ is an *isotropic subspace* with respect to the bilinear form defined by $B_f(x,y) = \langle f, [x,y] \rangle$ on $\mathfrak{g}$. In this case we define a one-dimensional unitary representation $U_{f,H}$ of the group $H = \exp \mathfrak{h}$ by formula

$$U_{f,H}(\exp\ x) = \exp\ 2\pi i \langle f, x \rangle.$$

Theorem 2.1 (Theorem 7.2, [12]). *(a) Every irreducible unitary representation T of a connected and simply connected nilpotent Lie group G has the form $T = \mathrm{Ind}_H^G U_{f,H}$, where $H \subset G$ is a connected subgroup and $f \in \mathfrak{g}^*$;*

(b) the representation $T_{f,H} = \mathrm{Ind}_H^G U_{f,H}$ is irreducible if and only if the Lie algebra $\mathfrak{h}$ of the group H is a subalgebra of $\mathfrak{g}$ subordinate to the functional f with maximal possible dimension*;*

(c) irreducible representations T_{f_1,H_1} and T_{f_2,H_2} are equivalent if and only if the functionals f_1 and f_2 belong to the same orbit of $\mathfrak{g}^*$."

2.3. The induced representations corresponding to generic orbits, finite-dimensional case

We present the explicit formulas for the induced representations (2.10)–(2.13) allowing us to calculate the generators of the one-parameter groups in H (see (2.16) and definition (2.8) of the matrix $\mathbb{S}$). We give a new proof of the irreducibility of the group G_n based on the Gauss decomposition of the matrix $\mathbb{S}$. This proof can be generalized for the infinite-dimensional group $B_0^{\mathbb{Z}}$.

Example 2.1. Generic orbits for the group $G=B(n,\mathbb{R})$ [12, Example 7.9]. "The adjoint action of the group G on $\mathfrak{g}$ has the following form $\mathrm{Ad}_t(x)=tyt^{-1}$, $t\in G$, $x\in\mathfrak{g}$. The form of the action $\mathrm{Ad}_t^*(y)=(t^{-1}yt)_-$ implies, that Ad_t^*, $t\in G$ acts as follows: to a given column of $y\in\mathfrak{g}^*$, a linear combination of the previous columns is added and to a given row of y, a linear combination of the following rows is added. More generally, the minors Δ_k, $k=1,2,...,[\frac{n}{2}]$, consisting of the last k rows and first k columns of y are invariant of the action. It is possible to show that if all the numbers c_k are different from zeros, then the manifold given by the equation

$$\Delta_k=c_k,\ 1\leq k\leq[n/2] \tag{2.4}$$

is a G-orbit in $\mathfrak{g}^*$. Hence *generic orbits* have codimension equal to $[\frac{n}{2}]$ and dimension equal to $\frac{n(n-1)}{2}-[\frac{n}{2}]$. To obtain a representation for such an orbit, we can take a matrix y of the form $y=\left(\begin{smallmatrix}0&0\\\Lambda&0\end{smallmatrix}\right)$, where Λ is the matrix of order $[\frac{n}{2}]$ such that all nonzero elements are contained in the *anti-diagonal.* It is easy to find a subalgebra of dimension $[\frac{n}{2}]\times[\frac{n+1}{2}]$ subordinate to the functional y. It consists of all matrices of the form $\left(\begin{smallmatrix}0&A\\0&0\end{smallmatrix}\right)$, where A is an $[\frac{n}{2}]\times[\frac{n+1}{2}]$ or $[\frac{n+1}{2}]\times[\frac{n}{2}]$ matrix."

For $p,q,m\in\mathbb{Z},\ p\leq m\leq q$ define the following groups

$$G_{p,q}=\{I+\sum_{p\leq k<r\leq q}x_{kr}E_{kr}\},\quad H_{p,q}^m=\{I+\sum_{p\leq k\leq m<r\leq q}x_{kr}E_{kr}\}. \tag{2.5}$$

Remark 2.2. We find $h(x,t)$ using $s(x)t=h(x,t)s(xt)$. Let $1<m<n$ and $B_m=G_{m+1,n}$, $B(m)=H_{1,n}^m$, $B^{(m)}=G_{1,m}$. The group $G_n=B(n,\mathbb{R})$ is a *semi-direct product* $G_n=B_m\ltimes B(m)\rtimes B^{(m)}$ and two decompositions hold

$$B_mB(m)B^{(m)}\ni x_mx(m)x^{(m)}=hx_mx^{(m)}\in B(m)B_mB^{(m)},\ h=x_mx(m)x_m^{-1}, \tag{2.6}$$

where $G_n \ni x = \begin{pmatrix} x^{(m)} & x(m) \\ 0 & x_m \end{pmatrix} = x_m x(m) x^{(m)}$, $x_m \in B_m$, $x(m) \in B(m)$, $x^{(m)} \in B^{(m)}$. In view of decomposition (2.6), the space $X = B(m)\backslash G_n$ is isomorphic to $B_m B^{(m)}$. Therefore the section s can be used as an embedding $B_m B^{(m)} \ni x_m x^{(m)} \mapsto s(x_m x^{(m)}) = x_m x^{(m)} \in B_m B(m) B^{(m)}$. For $t = t_m t^{(m)} \in B_m B^{(m)}$ we have $s(x)t = x_m x^{(m)} t_m t^{(m)} = x_m t_m x^{(m)} t^{(m)} = s(xt)$, so $s(x)t = s(xt)$ and $h(x,t) = e$. For $t \in B(m)$ we get $s(x)t = x_m x^{(m)} t = h(x,t) x_m x^{(m)}$, hence $h(x,t) = x_m x^{(m)} t (x_m x^{(m)})^{-1} = \begin{pmatrix} x^{(m)} & 0 \\ 0 & x_m \end{pmatrix} \begin{pmatrix} 1 & t_0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (x^{(m)})^{-1} & 0 \\ 0 & x_m^{-1} \end{pmatrix} = \begin{pmatrix} 1 & x^{(m)} t_0 x_m^{-1} \\ 0 & 1 \end{pmatrix}$, where $t_0 = t - I$. Finally we get

$$H(x,t) := h(x,t) - I = \begin{cases} 0, & \text{for} \quad t \in B_m B^{(m)}, \\ x^{(m)}(t-I)x_m^{-1}, & \text{for} \quad t \in B(m). \end{cases} \tag{2.7}$$

Let us fix $G_{2m} = B_m B(m) B^{(m)}$. Consider one-parameter subgroups $E_{kr}(t) := I + tE_{kr}$, $t \in \mathbb{R}$ of the group $B(2m, \mathbb{R})$. We find generators $A_{kn} = d/dt T_{I+tE_{kn}}|_{t=0}$ of the induced representation T_t (2.12). Set for $1 \leq k \leq m < r \leq 2m$

$$S_{kr}(t_{kr}) := \langle y, (h(x, E_{kr}(t_{kr})) - I) \rangle, \text{ then } A_{kr} = d/dt \exp(2\pi i S_{kr}(t))|_{t=0} = 2\pi i S_{kr}(1).$$

Define the matrix $\mathbb{S}$ (its structure is important in *the proof of the irreducibility*)

$$\mathbb{S} = (S_{kr})_{1 \leq k \leq m < r \leq 2m}, \text{ where } S_{kr} = S_{kr}(1). \text{ Then } \mathbb{S} = (2\pi i)^{-1} (A_{kr})_{k,r}. \tag{2.8}$$

Lemma 2.2. *Let $B = (b_{kr})_{k,r=1}^n \in \mathrm{Mat}(n, \mathbb{C})$. Define the matrix $C = (c_{kr})_{k,r=1}^n \in \mathrm{Mat}(n, \mathbb{C})$ by*

$$c_{kr} = \mathrm{tr}(E_{kr} B), \quad 1 \leq k, r \leq n, \quad \textit{then we have} \quad C = B^T, \tag{2.9}$$

where E_{kr} are matrix units and B^T means transposed matrix to the matrix B. The equality $C = B^T$ holds also in the case when B is an arbitrary $m \times n$ rectangular matrix. The statement is true also for matrices $B \in \mathrm{Mat}(\infty, \mathbb{C})$.

Proof. Indeed, we have $\mathrm{tr}(E_{kr} B) = b_{rk}$. □

We now find the matrix $\mathbb{S}(t) = (S_{kr}(t_{kr}))_{k,r}$ and the matrix $\mathbb{S} = (S_{kr}(1))_{k,r}$ using Lemma 2.2. Using (2.7) we have

$$\langle y, h(x,t) - I \rangle = \mathrm{tr}\left(H(x,t) y\right) = \mathrm{tr}\left(x^{(m)} t_0 x_m^{-1} y\right) = \mathrm{tr}\left(t_0 x_m^{-1} y x^{(m)}\right) = \mathrm{tr}\left(t_0 B(x,y)\right),$$

where $t_0 = t - I$ and

$$B(x,y) = x_m^{-1} y x^{(m)} \cong \begin{pmatrix} 1 & 0 \\ 0 & x_m^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \begin{pmatrix} x^{(m)} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ x_m^{-1} y x^{(m)} & 0 \end{pmatrix}. \tag{2.10}$$

By definition we have

$$S_{kr}(t_{kr}) = \langle y, (h(x, E_{kr}(t_{kr})) - I)\rangle = \mathrm{tr}(t_{kr}E_{kr}B(x,y)),$$

hence by Lemma 2.2 and (2.10) we conclude that

$$\mathbb{S} = (S_{kr}(1))_{kr} = \left(\mathrm{tr}\,(E_{kr}B(x,t))\right)_{k,r} = B^T(x,y) = \begin{pmatrix} 0 & (x^{(m)})^T y^T (x_m^{-1})^T \\ 0 & 0 \end{pmatrix}. \quad (2.11)$$

So the induced representation $\mathrm{Ind}_H^G(S) : G \to U(L^2(X,\mu))$ corresponding to the point $y \in \mathfrak{g}^*$ has the following form for $t \in G,\ x \in X = H\backslash G$,

$$(T_t f)(x) = S(h(x,t))\,(d\mu(xt)/d\mu(x))^{1/2}\,f(xt),\ \ f \in L^2(X,\mu), \quad (2.12)$$

where

$$S(h(x,t)) = \exp(2\pi i\langle y, (h(x,t) - I)\rangle) = \exp\big(2\pi i \mathrm{tr}\,((t-I)B(x,y))\big). \quad (2.13)$$

Remark 2.3. For the matrix $X = I + \sum_{k,n\in\mathbb{Z},k<n} x_{kn}E_{kn} \in B^{\mathbb{Z}}$ we denote by x_{kn}^{-1} the matrix elements of the matrix X^{-1}, i.e. $X^{-1} =: I + \sum_{k,n\in\mathbb{Z},k<n} x_{kn}^{-1}E_{kn} \in B^{\mathbb{Z}}$. The explicit expressions for x_{kn}^{-1} are as follows (see [13], formula (4.4)) $x_{kk+1}^{-1} = -x_{kk+1}$,

$$x_{kn}^{-1} = -x_{kn} + \sum_{r=1}^{n-k-1}(-1)^{r-1}\sum_{k<i_1<i_2<\ldots<i_r<n} x_{ki_1}x_{i_1i_2}\ldots x_{i_rn},\ \ k < n-1. \quad (2.14)$$

We have by definition $X^{-1}X = XX^{-1} = I$, hence

$$\left(XX^{-1}\right)_{kn} = \sum_{r=k}^{n} x_{kr}x_{rn}^{-1} = \delta_{kn} = \sum_{r=k}^{n} x_{kr}^{-1}x_{rn} = \left(X^{-1}X\right)_{kn}, \quad k \le n. \quad (2.15)$$

Denote by $D_{kn} = D_{kn}(h) = \partial/\partial x_{kn}$ the operator of the partial derivative corresponding to the shift $x \mapsto x + tE_{kn}$ on the group $B_m \times B^{(m)} \ni x = (x_{kn})_{k,n}$ and the Haar measure h.

Example 2.2. Let $G = G_{1,6}$, $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{g}^* = \mathrm{Lie}(G)^*$. We write the representations for generic orbit corresponding to the point $y = y_{43}E_{43} + y_{52}E_{52} + y_{61}E_{61} \in \mathfrak{g}^*$. Set $H_3 = H_{1,6}^3$, $\mathfrak{h}_3 = \mathrm{Lie}(H_3) = \{t - I \mid t \in H_3\}$. The representation S of the group H_3 is:

$$H_3 \ni \exp(t-I) = t \mapsto \exp(2\pi i\langle y, (t-I)\rangle) = \exp(2\pi i[t_{34}y_{43} + t_{25}y_{52} + t_{16}y_{61}]) \in S^1.$$

Denote by $B_3 = G_{4,6}$, $B(3) = H^3_{1,6}$, $B^{(3)} = G_{1,3}$. For the group $G_{1,6} = B(6,\mathbb{R})$ holds the following decomposition (see Remark 2.2): $B(6,\mathbb{R}) = B_3 B(3) B^{(3)}$, i.e. $x = x_3 x(3) x^{(3)}$. In view of (2.10) and (2.11), we get

$$B(x,y) = x_3^{-1} y x^{(3)} = \begin{pmatrix} x_{46}^{-1} y_{61} & x_{45}^{-1} y_{52} + x_{46}^{-1} y_{61} x_{12} & y_{43} + x_{45}^{-1} y_{52} x_{23} + x_{46}^{-1} y_{61} x_{13} \\ x_{56}^{-1} y_{61} & y_{52} + x_{56}^{-1} y_{61} x_{12} & y_{52} x_{23} + x_{56}^{-1} y_{61} x_{13} \\ y_{61} & y_{61} x_{12} & y_{61} x_{13} \end{pmatrix},$$

$$\mathbb{S} = B^T(x,y) = (x_3^{-1} y x^{(3)})^T = \begin{pmatrix} x_{46}^{-1} y_{61} & x_{56}^{-1} y_{61} & y_{61} \\ x_{45}^{-1} y_{52} + x_{46}^{-1} y_{61} x_{12} & y_{52} + x_{56}^{-1} y_{61} x_{12} & y_{61} x_{12} \\ y_{43} + x_{45}^{-1} y_{52} x_{23} + x_{46}^{-1} y_{61} x_{13} & y_{52} x_{23} + x_{56}^{-1} y_{61} x_{13} & y_{61} x_{13} \end{pmatrix}.$$

Using again (2.8), (2.12) and (2.7) we get the following expressions for generators $A_{kn} = d/dt T_{I+tE_{kn}}|_{t=0}$ of one-parameter groups $I + tE_{kn},\ t \in \mathbb{R}$:

$$A_{12}{=}D_{12},\ A_{13}{=}D_{13},\ A_{23}{=}x_{12}D_{13}+D_{23},\ A_{45}{=}D_{45},\ A_{46}{=}D_{46},\ A_{56}{=}x_{45}D_{46}+D_{56},$$

$$\mathbb{S} = \frac{1}{2\pi i}\begin{pmatrix} A_{14} & A_{15} & A_{16} \\ A_{24} & A_{25} & A_{26} \\ A_{34} & A_{35} & A_{36} \end{pmatrix} = \begin{pmatrix} x_{46}^{-1} y_{61} & x_{56}^{-1} y_{61} & y_{61} \\ x_{45}^{-1} y_{52} + x_{46}^{-1} y_{61} x_{12} & y_{52} + x_{56}^{-1} y_{61} x_{12} & y_{61} x_{12} \\ y_{43} + x_{45}^{-1} y_{52} x_{23} + x_{46}^{-1} y_{61} x_{13} & y_{52} x_{23} + x_{56}^{-1} y_{61} x_{13} & y_{61} x_{13} \end{pmatrix}. \quad (2.16)$$

2.4. New proof of the irreducibility of the induced representations

The conditions of irreducibility, Theorem 2.1 (b), is hard to formulate in the infinite-dimensional case since all maximal subordinate subalgebras are infinite-dimensional. We give an *equivalent description of the irreducibility* which can be generalized, namely, the subgroup $H^{2m+2r+1}_{1,k-1}$ is of maximal dimension in $G_{1,k-1} \Leftrightarrow r=0$ for $k=2m+1$ (resp. $r=-1, r=0$ for $k=2m$) $\Leftrightarrow G_{\oslash} = \{e\}$, see Lemmas 4.1 and 4.4. These lemmas explain when and why the induced representations $\mathrm{Ind}_H^G(U_{f,H})$ are irreducible. The infinite-dimensional case is richer since some irreducible representations appear as the limit of reducible ones (see Theorem 4.2 and Remark 4.2).

We present a new proof of irreducibility that allows a generalization for the infinite-dimensional group (Section 3.7). These proofs in both cases are based on the fact that the *von Neumann algebra* generated by the *restriction* of the representation T^{m,y_n} on the commutative subgroup $B(m,n)$ coincides with $L^\infty(X,\mu)$ (Lemmas 2.4 and 3.10). Consider the sequence of Lie groups $G_n^m = G_{m-n,m+n+1}$ (see (2.5)) and their Lie algebras $\mathfrak{g}_n^m = \mathrm{Lie}(G_n^m)$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$. We note that for an arbitrary $m \in \mathbb{N}$ we have $B_0^{\mathbb{Z}} = \varinjlim_n G_n^m$. The following decomposition holds (see (2.6)) $G_n^m = B_{m,n} \ltimes B(m,n) \rtimes B^{(m,n)}$, where

$$B_{m,n} = G_{m+1,m+n+1},\ B(m,n) = H^m_{m-n,m+n+1},\ B^{(m,n)} = G_{m-n,m}.$$

Denote by $\Delta(m,n) = \{(k,r) \in \mathbb{Z}^2 \mid m-n \leq k \leq m < r \leq m+n+1\}$, $\Delta_{m,n} = \{(k,r) \in \mathbb{Z}^2 \mid m+1 \leq k < r \leq m+n+1\}$, $\Delta^{(m,n)} = \{(k,r) \in \mathbb{Z}^2 \mid m-n \leq k < r \leq m\}$. The induced representation of the group G_n^m is defined in the space $\mathcal{H}^{m,n} := L^2(X_{m,n}, h_{m,n})$ by the following formula for $t \in G_n^m$:

$$(T_t^{m,y_n} f)(x) = S(h(x,t))\big(dh_{m,n}(xt)/dh_{m,n}(x)\big)^{1/2} f(xt), \ f \in \mathcal{H}^{m,n}, \quad (2.17)$$

where $X_{m,n} = B(m,n)\backslash G_n^m \simeq B_{m,n} \times B^{(m,n)}$ and

$$dh_{m,n}(x_m, x^{(m)}) = dx_m \otimes dx^{(m)} = \otimes_{(k,n)\in\Delta_{m,n}} dx_{kn} \otimes \otimes_{(k,n)\in\Delta^{(m,n)}} dx_{kn} \quad (2.18)$$

is the Haar measure on the group $B_{m,n} \times B^{(m,n)}$.

Theorem 2.3. *The induced representation T^{m,y_n} of the group G_n^m defined by (2.17), corresponding to the generic orbit $\mathcal{O}_{y_n}$ generated by the point $y_n \in (\mathfrak{g}_n^m)^*$, $y_n = \sum_{r=0}^n y_{m+r+1,m-r} E_{m+r+1,m-r}$, $y_{sl} \neq 0$ is irreducible. Moreover, the generators of the one-parameter groups $A_{kr} = \frac{d}{dt} T_{I+tE_{kr}}^{m,y_n} \mid_{t=0}$ are: $A_{kr} = \sum_{s=m-n}^{k-1} x_{ks} D_{rs} + D_{kr}$, $(k,r) \in \Delta^{(m,n)}$, $A_{kr} = \sum_{s=m+1}^{k-1} x_{ks} D_{rs} + D_{kr}$, $(k,r) \in \Delta_{m,n}$, $(2\pi i)^{-1} \big(A_{kr}\big)_{(k,r)\in\Delta(m,n)} = \mathbb{S}_n^{(m)} = (S_{kr})_{(k,r)\in\Delta(m,n)} = \big(x_m^{-1} y x^{(m)}\big)^T$.*

The irreducibility of the induced representation of the group G_n^m is based on the following lemma.

Lemma 2.4. *The von Neumann algebra $\mathfrak{A}^S$ generated by the restriction of the representation T^{m,y_n} on the commutative subgroup $B(m,n)$ of the group G_n^m coincides with $L^\infty(X_{m,n}, h_{m,n})$.*

Proof. In the space $\mathcal{H}^{m,n}$ define two von Neumann algebras $\mathfrak{A}^S$ and $\mathfrak{A}^x$ generated respectively by two sets of unitary operators $U_{kr}(t)$ and $V_{kr}(t)$, where $(U_{kr}(t)f)(x) = \exp(2\pi i S_{kr}(t)) f(x), \quad (V_{kr}(t)f)(x) := \exp(2\pi i t x_{kr}) f(x)$,

$$\mathfrak{A}^S = \big(U_{kr}(t) = T_{I+tE_{kr}}^{m,y_n} = \exp(2\pi i S_{kr}(t)) \mid t \in \mathbb{R}, \ (k,r) \in \Delta(m,n)\big)'',$$
$$\mathfrak{A}^x = \big(V_{kr}(t) := \exp(2\pi i t x_{kr}) \mid t \in \mathbb{R}, \ (k,r) \in \Delta_{m,n} \cup \Delta^{(m,n)}\big)''.$$

Since $\mathfrak{A}^x = L^\infty(X_{m,n}, h_{m,n})$, to prove irreducibility it is sufficient to show that $\mathfrak{A}^S = \mathfrak{A}^x$. Using decomposition (2.10) and (2.11)

$$\mathbb{S}_n^{(m)} = (x_m^{-1} y x^{(m)})^T = (x^{(m)})^T y^T (x_m^{-1})^T \quad (2.19)$$

we conclude that $\mathfrak{A}^S \subseteq \mathfrak{A}^x$. Indeed we have $V_{kr}(t) := \exp(2\pi i t x_{kr}) \in \mathfrak{A}^x$, so the operators x_{kr} of multiplication by the independent variables $f(x) \mapsto x_{kr} f(x)$ in the space $\mathcal{H}^{m,n}$ are affiliated with the von Neumann algebra $\mathfrak{A}^x$, i.e. $x_{kr} \ \eta \ \mathfrak{A}^x$ for $(k,r) \in \Delta_{m,n} \bigcup \Delta^{(m,n)}$.

Definition 2.2. Recall (c.f., e.g., [5]) that a non necessarily bounded self-adjoint operator A in a Hilbert space H is said to be *affiliated* with a von Neumann algebra M of operators in this Hilbert space H, if $\exp(itA) \in M$ for all $t \in \mathbb{R}$. Then one writes $A \,\eta\, M$.

By (2.14) the matrix elements x_{kr}^{-1} of the matrix $x_m^{-1} \in B_{m,n}$ are also affiliated $x_{kr}^{-1} \,\eta\, \mathfrak{A}^x$. Using (2.19) we conclude that the matrix elements S_{kr} of the matrix $\mathbb{S}_n^{(m)}$ are affiliated: $S_{kr} \,\eta\, \mathfrak{A}^x$, $(k,r) \in \Delta(m,n)$, so $\mathfrak{A}^S \subseteq \mathfrak{A}^x$.

To prove that $\mathfrak{A}^S \supseteq \mathfrak{A}^x$, we find the expressions of the matrix element of the matrix $x^{(m)} \in B^{(m,n)}$ and $x_m^{-1} \in B_{m,n}$ in terms of the matrix elements of the matrix $\mathbb{S}_n^{(m)} = (S_{kr})_{(k,r)\in\Delta(m,n)}$. To do that, we compare the above decomposition $\mathbb{S}_n^{(m)} = (x^{(m)})^T y^T (x_m^{-1})^T$ and the Gaussian decomposition $C = LDU$ (see Theorem 5.1). Let us denote by J the $n \times n$ anti-diagonal matrix $J = \sum_{r=-n}^{n+1} E_{m+r,m-r+1}$ Using $J^2 = I$ and (2.11) we get

$$\mathbb{S}J = B^T(x,y)J = (x^{(m)})^T y^T (x_m^{-1})^T J = (x^{(m)})^T (y^T J)(J(x_m^{-1})^T J). \quad (2.20)$$

The latter decomposition (2.20) is in fact the Gauss decomposition of the matrix $\mathbb{S}J$ i.e. we get

$$\mathbb{S}J = LDU, \quad \text{where} \quad L = (x^{(m)})^T, \quad D = y^T J, \quad U = J(x_m^{-1})^T J.$$

Using Theorem 5.1 we can find the matrix elements of the matrix $x^{(m)} \in B^{(m,n)}$ and $x_m^{-1} \in B_{m,n}$ in terms of the matrix elements of the matrix $\mathbb{S}_n^{(m)}$, hence we can also find the matrix elements of the matrix $x_m \in B_{m,n}$. This finish the proof of the lemma. □

Proof of Theorem. 2.3 Let a bounded operator A in a Hilbert space $\mathcal{H}^{m,n}$ commute with the representation T^{m,y_n}. Then by Lemma 2.4 A commute with $L^\infty(B_{m,n} \times B^{(m,n)}, dx_m \otimes dx^{(m)})$, therefore the operator A itself is an operator of multiplication by some essentially bounded function $a \in L^\infty$ i.e. $(Af)(x) = a(x)f(x)$ for $f \in \mathcal{H}^{m,n}$. Since A commute with the representation T^{m,y_n} i.e. $[A, T_t^{m,y_n}] = 0$ for all $t \in B_{m,n} \times B^{(m,n)}$ we conclude that

$$a(x) = a(xt) \ (\mathrm{mod}\ dx_m \otimes dx^{(m)}) \quad \text{for} \quad \text{all} \quad t \in B_{m,n} \times B^{(m,n)}.$$

Since the measure $dh = dx_m \otimes dx^{(m)}$ is the Haar measure on $G = B_{m,n} \times B^{(m,n)}$, this measure is G-right ergodic. Therefore $a(x) = const$ $(\mathrm{mod}\, dx_m \otimes dx^{(m)})$.
□

3. Induced representations, infinite-dimensional case

3.1. Induced representations for infinite-dimensional groups

A. Kirillov [12, Chapter I, §4, p.10] says: *"The method of induced representations is not directly applicable to infinite-dimensional groups (or more precisely to a pair $G \supset H$) with an infinite-dimensional factor $H\backslash G$)".* To generalize the Mackey construction for infinite-dimensional groups, one needs first to construct some G-quasi-invariant measure on infinite-dimensional homogeneous space $H\backslash G$. Since there is no Haar measure on the group G [30], it is difficult to construct such a measure on the initial space $H\backslash G$. As in the case of the "regular" or "quasiregular" representation (see Example 3.1), it is reasonable to construct such a measure on an *appropriate completion* $\tilde{X}$ of the initial space $X = H\backslash G$. The formula for the induced representation containing an operator $S(h)$, $h \in H$ will make sense only if one can *extend the representation* S of the group H to the *corresponding completion* $\tilde{H}$ of the group H.

Finally, the induced representation of the group G associated with a unitary representation S of a subgroup H will depend on a completion $\tilde{G}$ of the group G, on an extension $\tilde{S} : \tilde{H} \to U(V)$ of the representation $S : H \to U(V)$ and on the choice of a G-quasi-invariant measure μ on an appropriate completion $\tilde{X}$ of the space $X = H\backslash G$.

Hence the procedure of induction will not be unique but, nevertheless, well-defined (if a G-quasi-invariant measure on $\widetilde{H\backslash G}$ exists). So the uniquely defined induced representation $\mathrm{Ind}_H^G S$ in the Hilbert space $L^2(H\backslash G, V, \mu)$ (in the case of a locally-compact group G) should be replaced by the family of induced representations $\mathrm{Ind}_{\tilde{H},H}^{\tilde{G},G,\mu}(\tilde{S}, S)$ in the Hilbert spaces $L^2(\tilde{H}\backslash\tilde{G}, V, \mu)$ depending on different completions $\tilde{G}$ of the group G and different G-quasi-invariant measures μ on $\tilde{X} = \tilde{H}\backslash\tilde{G}$.

Example 3.1. *Regular* representation [14, 15, 16] *of the infinite-dimensional group* G in the space $L^2(\tilde{G}, \mu)$, associated with the completion $\tilde{G}$ of the group G and a G-right-quasi-invariant measure μ on $\tilde{G}$, is a particular case of the induced representation $\mathrm{Ind}_{e,e}^{\tilde{G},G,\mu}(Id)$, generated by the trivial representation of the trivial subgroup. *Quasiregular* [19] representation in the space $L^2(\tilde{X},\mu)$, where $\tilde{X} = \tilde{H}\backslash\tilde{G}$ and H is some subgroup of the group G is a particular case of the induced representation $\mathrm{Ind}_{\tilde{H},H,}^{\tilde{G},G,\mu}(Id)$ generated by the trivial representation of the completion $\tilde{H}$ in the group $\tilde{G}$.

Let G be an infinite-dimensional group and $S : H \to U(V)$ be a unitary representation in a Hilbert space V of the subgroup $H \subset G$ such that the space $H\backslash G$ is infinite-dimensional. We give the following definition.

Definition 3.1. The induced representation of the group G

$$\mathrm{Ind}_{\tilde{H},H}^{\tilde{G},G,\mu}(\tilde{S}, S),$$

generated by the unitary representations $S : H \to U(V)$ of the subgroup H in the group G is defined (similarly to (2.2)) as follows:

1) first we should find some completion $\tilde{H}$ of the group H such that

$$\tilde{S} : \tilde{H} \to U(V)$$

is the continuous unitary representation of the group $\tilde{H}$ such that $\tilde{S}|_H = S$,

2) take any G-right-quasi-invariant measure μ on an appropriate completion $\tilde{X} = \tilde{H}\backslash\tilde{G}$ of the space $X = H\backslash G$, on which the group G acts from the right, where $\tilde{H}$ (resp. $\tilde{G}$) is a suitable completion of the group H (resp. G),

3) in the space $L^2(\tilde{X}, V, \mu)$ of all vector-valued functions f on $\tilde{X}$ with values in V such that

$$\|f\|^2 := \int_{\tilde{X}} \|f(x)\|_V^2 d\mu(x) < \infty,$$

define the representation of the group G by the following formula:

$$(T_t f)(x) = S(\tilde{h}(x,t))\,(d\mu(xt)/d\mu(x))^{1/2}\, f(xt), \quad x \in \tilde{X},\ t \in G, \tag{3.1}$$

where $\tilde{h}$ is defined by $\tilde{s}(x)t = \tilde{h}(x,t)\tilde{s}(xt)$. The section $s : X \to G$ of the projection $p : G \to X$ should be extended to an appropriate section $\tilde{s} : \tilde{X} \to \tilde{G}$ of the extended projection $\tilde{p} : \tilde{G} \to \tilde{X}$.

Conjecture 3.1 (R.S. Ismagilov, 1985, [14]). *The right regular representation $T^{R,\mu} : G \to U(L^2(\tilde{G}, \mu))$ is irreducible if and only if 1) $\mu^{L_t} \perp \mu\ \forall t \in G\backslash\{e\}$, ($\perp$ means singular), 2) the measure μ is G-ergodic.*

The following construction generalizes regular and quasiregular representations. Let us have the *measurable action* $\alpha : G \to \mathrm{Aut}(X)$ of the group G on the measurable space X with a G-quasi-invariant measure μ. The representation $\pi^{\alpha,\mu,X}$ of the group G is defined by

$$(\pi_t^{\alpha,\mu,X} f)(x) = (d\mu(\alpha_{t^{-1}}(x))/d\mu(x))^{1/2} f(\alpha_{t^{-1}}(x)), \quad f \in L^2(X, \mu). \tag{3.2}$$

Conjecture 3.2 ([19]). *A representation* $\pi^{\alpha,\mu,X}: G \to U(L^2(X,\mu))$ *is irreducible if and only if 1)* $\mu^g \perp \mu \ \forall g \in \alpha(G)'\backslash\{e\}$, *2) the measure* μ *is* G*-ergodic, where* $\alpha(G)' = \{g \in \mathrm{Aut}(X) \mid \{g,\alpha_t\} = g\alpha_t g^{-1}\alpha_t^{-1} = e \ \forall t \in G\}$.

Problem 1. Find the *conditions* on X, μ, G, α when the *Conjectures 3.1 and 3.2 are valid.* Conjectures 3.1, 3.2 are proved for some particular cases, see, e.g., [14, 15, 19]. In the case of the field $\mathbf{k} = \mathbb{F}_p$ they should be corrected.

3.2. How to develop the orbit method for infinite-dimensional i-nilpotent group $B_0^{\mathbb{Z}}$*?*

Consider the group $B_0^{\mathbb{Z}} = \varinjlim_n G_{2n-1}$ of infinite in both directions upper triangular matrices. The corresponding Lie algebra $\mathfrak{g}$ is the *inductive limit* $\mathfrak{g} = \varinjlim_n \mathfrak{b}_n$ of Lie algebras of upper triangular matrices, so as the linear space it is isomorphic to the space $\mathbb{R}_0^\infty$ of finite sequences $(x_k)_{k\in\mathbb{N}}$. Hence, the dual space $\mathfrak{g}^*$ is isomorphic to the space $\mathbb{R}^\infty$ of all sequences $(x_k)_{k\in\mathbb{N}}$, but *the latter space* $\mathbb{R}^\infty$ *is too large to manage with it, for example, to equip it with a Hilbert structure or to describe all orbits.* To make it smaller it is reasonable to make the completion $\tilde{G}$ of the group G in some stronger topology.

To develop the orbit method for the group $B_0^{\mathbb{Z}}$*,* we should answer some questions:

(1) How to define the *appropriate completion* $\tilde{G}$ of the group G corresponding Lie algebras $\mathfrak{g}$ (resp. $\tilde{\mathfrak{g}}$) and corresponding dual spaces $\mathfrak{g}^*$ (resp. $\tilde{\mathfrak{g}}^*$)?

(2) Which *pairing* should we use between $\mathfrak{g}$ and $\mathfrak{g}^*$?

(3) Suppose that the dual space $\mathfrak{g}^*$, some element $f \in \mathfrak{g}^*$ and corresponding algebra $\mathfrak{h}$, subordinate to the element f, are chosen. How to *define the corresponding induced representation* $\mathrm{Ind}_H^G U_{f,H}$ and *study its irreducibility* ?

(4) Shall we get all irreducible representations of the corresponding group using the orbit method and induced representations?

(5) Find the criteria of irreducibility and equivalence of the induced representations of the group $B_0^{\mathbb{Z}}$.

In [13] (see Section 3.3) for the group $\mathrm{GL}_0(2\infty,\mathbb{R}) = \varinjlim_n \mathrm{GL}(2n-1,\mathbb{R})$ we have constructed a family of the Hilbert-Lie groups $\mathrm{GL}_2(a)$, $a \in \mathfrak{A}_{\mathrm{GL}}$ such that:

a) $\mathrm{GL}_0(2\infty,\mathbb{R}) \subset \mathrm{GL}_2(a)$ and $\mathrm{GL}_0(2\infty,\mathbb{R})$ is dense in $\mathrm{GL}_2(a)$ for all $a \in \mathfrak{A}_{\mathrm{GL}}$,

b) $\mathrm{GL}_0(2\infty,\mathbb{R}) = \cap_{a\in\mathfrak{A}}\mathrm{GL}_2(a)$,

c) *any continuous representation* of the group $\mathrm{GL}_0(2\infty,\mathbb{R})$ *is in fact continuous* in some stronger topology, namely *in a topology of a suitable Hilbert -Lie group* $\mathrm{GL}_2(a)$, $a \in \mathfrak{A}_{\mathrm{GL}}$.

As we show in Sections 3.3–3.4 to develop the orbit method it is sufficient:
(1) to consider *Hilbert-Lie completions* $B_2(a)$ of the initial group $B_0^{\mathbb{Z}}$.
(2) In this case the *pairing* between the corresponding Hilbert-Lie algebra $\mathfrak{b}_2(a)$ and its dual $\mathfrak{b}_2(a)^*$ is correctly defined by the trace (as in the finite-dimensional case).
(3) In Section 3.6 and 4 we define *the induced representations* of the group $B_0^{\mathbb{Z}}$ corresponding to special orbits, *generic orbits*, using scheme given in Section 3.1. We consider only the simplest example of $G-$quasi-invariant measures on $\tilde{X}=\tilde{H}\backslash\tilde{G}$, namely, the infinite product of one-dimensional Gaussian measures. How to construct the *induced representation corresponding to an arbitrary orbit* is an open question.
(4) We do not know answer for questions (4).
(5) We obtain the criteria of irreducibility only for generic orbits.

3.3. Hilbert–Lie groups $\mathrm{GL}_2(a)$

The *Hilbert–Lie groups* naturally appear in the representation theory of infinite-dimensional matrix group. Let us consider the group $\mathrm{GL}_0(2\infty,\mathbb{R}) = \varinjlim_n \mathrm{GL}(2n-1,\mathbb{R})$ with respect to the *symmetric embedding* $i_n^s : G_n \mapsto G_{n+1}$, $G_n \ni x \mapsto x+E_{-n,-n}+E_{nn} \in G_{n+1}$, where $G_n = \mathrm{GL}(2n-1,\mathbb{R})$. Let us define [13] the *Hilbert-Lie group* $\mathrm{GL}_2(a)=\{I+x \mid (I+x)^{-1} = 1+y \quad x,y \in \mathfrak{g}l_2(a)\}$, by its *Hilbert-Lie algebra* $\mathfrak{g}l_2(a)$ with an operation $[x,y] = xy - yx$

$$\mathfrak{g}l_2(a) = \{x = \sum_{k,n\in\mathbb{Z}} x_{kn}E_{kn} \mid \|x\|^2_{\mathfrak{g}l_2(a)} = \sum_{k,n\in\mathbb{Z}} \mid x_{kn} \mid^2 a_{kn} < \infty\},\ a \in \mathfrak{A}_{\mathrm{GL}}.$$

Namely, consider an analog $\sigma_2(a)$ of an algebra of Hilbert-Schmidt operators $\sigma_2(H)$ in a Hilbert space $H = l_2(\mathbb{Z})$:

$$\sigma_2(a) = \{x = \sum_{k,n\in\mathbb{Z}} x_{kn}E_{kn} \mid \|x\|^2_{\sigma_2(a)} = \sum_{k,n\in\mathbb{Z}} \mid x_{kn} \mid^2 a_{kn} < \infty\}.$$

Lemma 3.3 ([13]). *The Hilbert space $\sigma_2(a)$ is an associative Hilbert algebra (i.e., $\|xy\| \leq C\|x\|\|y\|,\ x,y \in \sigma_2(a)$) if and only if the weight $a=(a_{kn})_{(k,n)\in\mathbb{Z}^2}$ belongs to the set $\mathfrak{A}_{\mathrm{GL}}$ defined as:*

$$\mathfrak{A}_{\mathrm{GL}}=\{a = (a_{kn})_{(k,n)\in\mathbb{Z}^2} \mid 0 < a_{kn} \leq Ca_{km}a_{mn},\ k,n,m \in \mathbb{Z},\ C > 0\}. \quad (3.3)$$

Theorem 3.4 (Theorem 6.1 [13]). *Every continuous unitary representation U of the group $\mathrm{GL}_0(2\infty,\mathbb{R})$ in a Hilbert space H can be extended by continuity to a unitary representation $U_2(a) : \mathrm{GL}_2(a) \to U(H)$ of some Hilbert-Lie group $\mathrm{GL}_2(a)$, $a \in \mathfrak{A}_{\mathrm{GL}}$ depending on the representation.*

3.4. Hilbert-Lie groups $B_2(a)$

Let us consider the following Hilbert-Lie group $B_2(a) := \{I + x \mid x \in \mathfrak{b}_2(a)\}$ where the corresponding Hilbert-Lie algebra $\mathfrak{b}_2(a)$ is defined as

$$\mathfrak{b}_2(a)=\{x = \sum_{(k,n)\in\mathbb{Z}^2,k<n} x_{kn}E_{kn} \mid \|x\|^2_{\mathfrak{b}_2(a)} = \sum_{(k,n)\in\mathbb{Z}^2,k<n} \mid x_{kn} \mid^2 a_{kn} < \infty\}. \quad (3.4)$$

Lemma 3.5 ([13]). *The Hilbert space $\mathfrak{b}_2(a)$ (with an operation $(x, y)\to xy$) is a Hilbert algebra if and only if the weight $a = (a_{kn})_{k,n}$ satisfies the conditions*

$$a = (a_{kn})_{(k,n)\in\mathbb{Z}^2,k<n}, \ a_{kn} \le C a_{km} a_{mn}, \ k < m < n, \ k, m, n \in \mathbb{Z}. \quad (3.5)$$

Denote by $\mathfrak{A}$ the set of all weights a satisfying the above-mentioned condition. We note [13] that

$$B_0^{\mathbb{Z}} = \cap_{a\in\mathfrak{A}} B_2(a), \quad \text{therefore} \quad \widehat{B_0^{\mathbb{Z}}} = \cup_{a\in\mathfrak{A}} \widehat{B_2(a)}. \quad (3.6)$$

Hence, for the description of *the dual space* $\widehat{B_0^{\mathbb{Z}}}$ it is sufficient to know $\widehat{B_2(a)}$ for all $a \in \mathfrak{A}$, but this problem has not been solved yet.

3.5. Orbits for groups $B_0^{\mathbb{Z}}$ and $B_2(a)$.

Let $\mathfrak{b}_0^{\mathbb{Z}}$ be the Lie algebra of the group $B_0^{\mathbb{Z}}$ and let $(\mathfrak{b}_0^{\mathbb{Z}})^*$ be its dual space. Since $\mathfrak{b}_0^{\mathbb{Z}} = \cap_{a\in\mathfrak{A}}\mathfrak{b}_2(a)$, so $(\mathfrak{b}_0^{\mathbb{Z}})^* = \cup_{a\in\mathfrak{A}}\mathfrak{b}_2^*(a)$, therefore an arbitrary element $y \in (\mathfrak{b}_0^{\mathbb{Z}})^*$ belongs to some dual space $\mathfrak{b}_2^*(a)$, $a \in \mathfrak{A}$. Take the group $B_0^{\mathbb{Z}}$, fix one of its Hilbert–Lie completion, i.e., some Hilbert–Lie group $B_2(a)$, $a \in \mathfrak{A}$, and the corresponding Hilbert–Lie algebra $\mathfrak{b}_2(a)$. The corresponding dual space $\mathfrak{b}_2^*(a)$ has the following description

$$\mathfrak{b}_2^*(a)=\{y=\sum_{(k,n)\in\mathbb{Z}^2,k>n} y_{kn}E_{kn} \mid \|y\|^2_{\mathfrak{b}_2^*(a)}=\sum_{(k,n)\in\mathbb{Z}^2,k>n} \mid y_{kn} \mid^2 a_{kn}^{-1} < \infty\}. \quad (3.7)$$

The *adjoint action* $B_2(a)\to \mathrm{GL}(\mathfrak{b}_2(a))$ of the group $B_2(a)$ on its Lie algebra $\mathfrak{b}_2(a)$ is: $\mathfrak{b}_2(a) \ni x \mapsto \mathrm{Ad}_t(x) := txt^{-1} \in \mathfrak{b}_2(a)$, $\ t \in B_2(a)$. *The pairing* between $\mathfrak{g} = \mathfrak{b}_2(a)$ and $\mathfrak{g}^* = \mathfrak{b}_2^*(a)$ is correctly defined by the trace:

$$\mathfrak{g}^* \times \mathfrak{g} \ni (y, x) \mapsto \langle y, x\rangle := tr(xy) = \sum_{(k,n)\in\mathbb{Z}^2,k<n} x_{kn}y_{nk} \in \mathbb{R}. \quad (3.8)$$

The *coadjoint action* of the group $B_2(a)$ on the space $\mathfrak{b}_2^*(a)$ dual with $\mathfrak{b}_2(a)$ is $\mathrm{Ad}_t^*(y) = (t^{-1}yt)_- \coloneqq I + \sum_{(p,q)\in\mathbb{Z}^2, p>q}(t^{-1}yt)_{pq}E_{pq},\ t\in B_2(x),\ y\in\mathfrak{b}_2^*(a)$.

We consider four different type of orbits with respect to the coadjoint action of the group $B_2(a)$ in the dual space $\mathfrak{b}_2^*(a)$.

Case 1) 0-dimensional orbits are of the form:

$$\mathcal{O}_0 = y,\ y\in\mathfrak{b}_2^*(a),\quad y=\sum_{k\in\mathbb{Z}} y_{k+1,k}E_{k+1,k}.$$

The Lie algebra $\mathfrak{b}_2(a)$ is subordinate to the functional y, $\langle y, [\mathfrak{b}_2(a), \mathfrak{b}_2(a)]\rangle = 0$ since

$$[\mathfrak{b}_2(a), \mathfrak{b}_2(a)] = \{x\in\mathfrak{b}_2(a)\mid x=\sum_{(k,n)\in\mathbb{Z}^2, k+1<n} x_{kn}E_{kn}\}.$$

The one-dimensional representation of the Lie algebra $\mathfrak{b}_2(a)$ is

$$\mathfrak{b}_2(a)\ni x\mapsto \langle y,x\rangle = \sum_{k\in\mathbb{Z}} x_{k,k+1}y_{k+1,k}\in\mathbb{R}.$$

The corresponding one-dimensional representations of the group $B_2(a)$ is

$$B_2(a)\ni\exp(x)\mapsto\exp(2\pi i(\langle y,x\rangle)) = \exp(2\pi i\sum_{k\in\mathbb{Z}} x_{k,k+1}y_{k+1,k})\in S^1. \quad (3.9)$$

For different $y\in\mathfrak{b}_2^*(a), y\neq 0$, these representations are *irreducible and nonequivalent.*

Case 2) The finite-dimensional orbits corresponding to *finite points* $y = \sum_{(k,n)\in\mathbb{Z}, k>n} y_{kn}E_{kn}\in\mathfrak{b}_2^*(a)$ (finiteness of y means that only finite number of y_{kn} are nonzero). These orbits lead to the induced representations of the appropriate finite-dimensional groups $G_n^m = G_{m-n,m+n+1},\ m\in\mathbb{Z},\ n\in\mathbb{N}$ (see (2.5)). All irreducible unitary representations of the groups G_n^m are completely described by the Kirillov orbit method hence the finite-dimensional orbits give us the set $\bigcup_{n\in\mathbb{N}}\widehat{G_n^m}\subset\widehat{B_0^{\mathbb{Z}}}$ (there is a natural embedding $\widehat{G_n^m}\subset\widehat{G_{n+1}^m}$).

Case 3) Generic orbit is generated by a point $y^k\in\mathfrak{b}_2^*(a),\ k\in\mathbb{Z}$

$$y^k = \sum_{r+s=k,\, s\leq[(k-1)/2]} y_{rs}E_{rs} = \sum_{r+s=k,\, r\geq[k/2]+1} y_{rs}E_{rs}\in\mathfrak{b}_2^*(a),\ y_{rs}\neq 0, \quad (3.10)$$

$$y^{2m+1} = \sum_{p=0}^{\infty} y_{m+p+1,m-p}E_{m+p+1,m-p}\in\mathfrak{b}_2^*(a),\ y_{m+p+1,m-p}\neq 0. \quad (3.11)$$

Sections 3.6, 3.7 and 4 are devoted to the study of these cases.

Case 4) General orbits are generated by arbitrary non finite points

$$y=\sum_{(k,n)\in\mathbb{Z},k>n}y_{kn}E_{kn}\in\mathfrak{b}_2^*(a).$$

Problem. How to construct the induced representations for general orbits and study their irreducibility?

3.6. Construction of the induced representations $T^{m,y}$ corresponding to a point y^{2m+1} and subgroup H_0^{2m+1}

Consider the case 3) more carefully. We shall study the irreducibility in the following section. As before, take the group $B_0^{\mathbb{Z}}$, fix one of its Hilbert completions i.e., a Hilbert-Lie group $B_2(a),\ a\in\mathfrak{A}$, the corresponding Hilbert-Lie algebra $\mathfrak{g}=\mathfrak{b}_2(a)$ and its dual $\mathfrak{g}^*=\mathfrak{b}_2^*(a)$ as in the previous section.

We shall construct an analog of the induced representation of the group $B_0^{\mathbb{Z}}$ for generic orbits (see Examples 2.1) corresponding to the point $y^{2m+1}\in\mathfrak{b}_2^*(a)$ defined by (3.11) and subgroup H_0^{2m+1} following steps 1)–3) of Definition 3.1.

Step 1) *Extension of the representation $S:H\to U(V)$.* For $m\in\mathbb{Z}$, the group $B^{\mathbb{Z}}$ is a semi-direct product. Consider the decomposition

$$B^{\mathbb{Z}}=B_m\ltimes B(m)\rtimes B^{(m)},\quad B_0^{\mathbb{Z}}\ni x=\begin{pmatrix}x^{(m)}&x(m)\\0&x_m\end{pmatrix}=x_mx(m)x^{(m)},\quad(3.12)$$

similar to the decomposition (2.6), where $B^{\mathbb{Z}}=\{I+\sum_{k,n\in\mathbb{Z},\,k<n}x_{kn}E_{kn}\}$,

$$B_m=\{I+\sum_{(k,r)\in\Delta_m}x_{kr}E_{kr}\},\quad B(m)=\{I+\sum_{(k,r)\in\Delta(m)}x_{kr}E_{kr}\},$$

$$B^{(m)}=\{I+\sum_{(k,r)\in\Delta^{(m)}}x_{kr}E_{kr}\},\ \text{where}\ \Delta_m=\{(k,r)\in\mathbb{Z}^2\mid m+1\le k<r\},$$

$$\Delta(m)=\{(k,r)\in\mathbb{Z}^2\mid k\le m<r\},\quad\Delta^{(m)}=\{(k,r)\in\mathbb{Z}^2\mid k<r\le m\}.$$

Since the algebras $\mathfrak{h}_0(m),\ m\in\mathbb{Z}$ defined

$$\mathfrak{h}_0(m)=\{t-I\mid t\in B_0(m)\},\quad\text{where}\ B_0(m)=B(m)\cap B_0^{\mathbb{Z}},$$

are commutative, $\langle y,[\mathfrak{h}_0(m),\mathfrak{h}_0(m)]\rangle=0$. Hence they are subordinate to the functional $y\in\mathfrak{g}^*=\mathfrak{b}_2^*(a)$. The corresponding one-dimensional representation of the algebra $\mathfrak{h}_0(m)=\mathfrak{h}(m)\bigcap\mathfrak{b}_0^{\mathbb{Z}}$ is

$$\mathfrak{h}_0(m)\ni x\mapsto\langle y,x\rangle=\sum_{p=0}^{\infty}x_{m-p,m+p+1}y_{m+p+1,m-p}\in\mathbb{R}.$$

The unitary representation of the corresponding group $H_0(m) \coloneqq \exp(\mathfrak{h}_0(m))$ is

$$H_0(m) \ni \exp(x) \mapsto S(\exp(x)) = \exp(2\pi i\langle y, x\rangle) \in S^1.$$

This representation can be extended to the representation of the corresponding Hilbert-Lie group $\tilde{H} = H_2(m, a) = B(m) \bigcap B_2(a)$ (we note that $t = \exp(t-1)$):

$$H_2(m, a) \ni \exp(x) \mapsto S(\exp(x)) = \exp(2\pi i\langle y, x\rangle) \in S^1. \tag{3.13}$$

In what follows we shall use the notation $B_2(m, a)$ for the group $H_2(m, a)$.

Step 2 a) *Construction of the completion* $\tilde{X} = \tilde{H}\backslash\tilde{G}$ of the space $X = H\backslash G$. It is difficult to construct an appropriate measure on the space $X_{m,0} = B_0(m)\backslash B_0^{\mathbb{Z}}$ since it is isomorphic to the space $\mathbb{R}_0^\infty \subset \mathbb{R}^\infty$. That is why we consider two homogeneous spaces, appropriate completions of the space $X_{m,0}$:

$$X_{m,2}(a) = B_{m,2}(a)\backslash B_2(a), \quad X_m = B(m)\backslash B^{\mathbb{Z}}.$$

Since the decompositions hold

$$B_0^{\mathbb{Z}} = B_{m,0}B_0(m)B_0^{(m)}, \ B_2(a) = B_{m,2}(a)B_2(m, a)B_2^{(m)}(a), \ B^{\mathbb{Z}} = B_m B(m) B^{(m)},$$

(see Remark 2.2), we have the following inclusions: $X_{m,0} \subset X_{m,2}(a) \subset X_m$, where

$$X_{m,0} \simeq B_{m,0} \times B_0^{(m)}, \ X_{m,2}(a) \simeq B_{m,2}(a) \times B_2^{(m)}(a), \ X_m = B(m)\backslash B^{\mathbb{Z}} \simeq B_m \times B^{(m)}.$$

Step 2 b) We *construct a measure* μ_b on the space X_m with support $X_{m,2}(a)$ i.e., such that $\mu_b(X_{m,2}(a)) = 1$. That is we take $\tilde{X} = \tilde{H}\backslash\tilde{G} = B_2(m, a)\backslash B_2(a)$.

We construct the measure μ_b on the space $X_m \simeq B_m \times B^{(m)}$ as a product-measure $\mu_b = \mu_{b,m} \otimes \mu_b^{(m)}$, where $\mu_{b,m}$ (resp. $\otimes\mu_b^{(m)}$) is the Gaussian product measure on the group B_m (resp. $B^{(m)}$) defined as follows:

$$d\mu_{b,m}(x_m) = \otimes_{(k,n)\in\Delta_m} d\mu_{b_{kn}}(x_{kn}) = \otimes_{(k,n)\in\Delta_m} \sqrt{b_{kn}/\pi}\, \exp(-b_{kn}x_{kn}^2)dx_{kn}, \tag{3.14}$$
$$d\mu_b^{(m)}(x^{(m)}) = \otimes_{(k,n)\in\Delta^{(m)}} d\mu_{b_{kn}}(x_{kn}) = \otimes_{(k,n)\in\Delta^{(m)}} \sqrt{b_{kn}/\pi}\, \exp(-b_{kn}x_{kn}^2)dx_{kn}.$$

Lemma 3.6 (Kolmogorov's zero-one law, [28]). *We have* $\mu_b(B_{m,2}(a) \times B_2^{(m)}(a)) = 1$ *if and only if*

$$\sum_{(k,n)\in\Delta(m)\cup\Delta^{(m)}} a_{kn}/b_{kn} < \infty.$$

Lemma 3.7 ([14, 15]). *The measure $\mu_b = \mu_{b,m} \otimes \mu_b^{(m)}$ is $B_{m,0} \times B_0^{(m)}$-right-quasi-invariant i.e., $(\mu_b)^{R_t} \sim \mu_b$ for all $t \in B_{m,0} \times B_0^{(m)}$ if and only if*

$$S_{kn}^R(\mu_b) = \sum_{r=-\infty}^{k-1} b_{rn}/b_{rk} < \infty, \quad \textit{for all, } k < n \leq m.$$

3) We define the corresponding induced representation $T^{m,y}$ of the group $B_0^{\mathbb{Z}}$ (denoted also in Section 4 by $T^{2m+1,2m+1,\mu_b}$, see (4.1)) in the space $\mathcal{H}^m = L^2(X_m, \mu_b)$ as follows (see (2.12)):

$$(T_t^{m,y} f)(x) = S(h(x,t)) \left(d\mu_b(xt)/d\mu_b(x)\right)^{1/2} f(xt), \; x \in X_m, \; t \in G, \quad (3.15)$$

where $S(h(x,t))$ is defined by (3.20).

3.7. Irreducibility of the induced representations $T^{m,y}$

Consider the induced representation $T^{m,y}$ of the group $B_0^{\mathbb{Z}}$ defined by (3.15) corresponding to a generic orbit $\mathcal{O}_y$ generated by the point $y = y^{2m+1} = \sum_{r=0}^{\infty} y_{m+r+1,m-r} E_{m+r+1,m-r} \in \mathfrak{b}_2^*(a)$. Set for $(k,r) \in \Delta(m)$

$$S_{kr}(t_{kr}) := \langle y, (h(x, E_{kr}(t_{kr})) – I) \rangle \; \Rightarrow \; A_{kr} = d/dt \exp(2\pi i S_{kr}(t))|_{t=0} = 2\pi i S_{kr}(1). \quad (3.16)$$

Denote by $\mathbb{S}^{(m)} = \mathbb{S}$ the following matrix (compare with (2.3) and (2.8)):

$$\mathbb{S} = (S_{kr})_{(k,r)\in\Delta(m)}, \quad \text{where} \quad S_{kr} = S_{kr}(1). \quad (3.17)$$

We now calculate the matrix $\mathbb{S}(t) = (S_{kr}(t_{kr}))_{(k,r)\in\Delta(m)}$ and the matrix $\mathbb{S} = (S_{kr}(1))_{(k,r)\in\Delta(m)}$ analogously to Lemma 2.2. As in (2.7) we have

$$\langle y, h(x,t) – I \rangle = \mathrm{tr}\,(H(x,t)y) = \mathrm{tr}\left(x^{(m)} t_0 x_m^{-1} y\right) = \mathrm{tr}\left(t_0 x_m^{-1} y x^{(m)}\right) = \mathrm{tr}\,(t_0 B(x,y)),$$

where $t_0 = t - I$ and for $x_m \in B_m, \; x^{(m)} \in B^{(m)}$ we denote

$$B(x,y) = x_m^{-1} y x^{(m)} \cong \begin{pmatrix} 1 & 0 \\ 0 & x_m^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \begin{pmatrix} x^{(m)} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ x_m^{-1} y x^{(m)} & 0 \end{pmatrix}. \quad (3.18)$$

By definition we have (recall that $E_{kn}(t_{kn}) = I + t_{kn} E_{kn}$)

$$S_{kn}(t_{kn}) = \langle y, (h(x, E_{kn}(t_{kn})) - I) \rangle = \mathrm{tr}(t_{kn} E_{kn} B(x,y)),$$

hence analogously to Lemma 2.2 we conclude that

$$\mathbb{S} = (S_{kn}(1))_{k,r} = (\mathrm{tr}\,(E_{kr} B(x,y)))_{k,r} =$$

$$B^T(x,y) = (x^{(m)})^T y^T (x_m^{-1})^T = \begin{pmatrix} 0 & (x^{(m)})^T y^T (x_m^{-1})^T \\ 0 & 0 \end{pmatrix}. \qquad (3.19)$$

So, we have

$$S(h(x,t)) = \exp(2\pi i \langle y, (h(x,t) - I)\rangle) = \exp\Big(2\pi i \mathrm{tr}\,((t-I)B(x,y))\Big). \quad (3.20)$$

Lemma 3.8 ([17]). *The measure* $\mu_b = \mu_{b,m} \otimes \mu_b^{(m)}$ *is* $B_{m,0} \times B_0^{(m)}$*-right-ergodic if* $E(\mu_b) = \sum_{k<n\leq m} S_{kn}^R(\mu_b)/b_{kn} < \infty$.

Theorem 3.9. *The induced representation* $T^{m,y}$ *of the group* $B_0^{\mathbb{Z}}$ *defined by formula (3.15), corresponding to generic orbit* $\mathcal{O}_y$ *generated by the point* $y = \sum_{r=0}^{\infty} y_{m+r+1,m-r} E_{m+r+1,m-r} \in \mathfrak{b}_2^*(a)$ *is irreducible if the measure* $\mu_{b,m} \otimes \mu_b^{(m)}$ *on the group* $B_m \times B^{(m)}$ *is right* $B_{m,0} \times B_0^{(m)}$*-ergodic. Moreover, the generators of one-parameter groups* $A_{kr} = \frac{d}{dt} T_{I+tE_{kr}}^{m,y} \mid_{t=0}$ *are the following*

$$A_{kr} = \sum_{s=-\infty}^{k-1} x_{ks} D_{rs} + D_{kr}, \ (k,r) \in \Delta^{(m)}, \ A_{kr} = \sum_{s=m+1}^{k-1} x_{ks} D_{rs} + D_{kr}, (k,r) \in \Delta_m,$$

$$(2\pi i)^{-1} \big(A_{kr}\big)_{(k,r)\in\Delta(m)} = \mathbb{S}^{(m)} = (S_{kr})_{(k,r)\in\Delta(m)} = \big(x_m^{-1} y x^{(m)}\big)^T.$$

Here we denote by $D_{kn} = D_{kn}(\mu_b) = \partial/\partial x_{kn} - b_{kn} x_{kn}$ the operator of the *logarithmic derivative* corresponding to the shift $x \mapsto x + tE_{kn}$ and the measure μ_b on the group $B_m \times B^{(m)} \ni x = I + \sum x_{kr} E_{kr}$ defined by:

$$(D_{kn}(\mu_b) f)(x) = d/dt\,(d\mu_b(x + tE_{kn}/d\mu_b(x))^{1/2} f(x + tE_{kn}) \mid_{t=0}. \quad (3.21)$$

The irreducibility of the induced representation of the group $B_0^{\mathbb{Z}}$ is based on the following lemma.

Lemma 3.10. *The von Neumann algebra* $\mathfrak{A}^S$ *generated by the restriction of the representation* $T^{m,y}$ *on the commutative subgroup* $B_0(m)$ *of the group* $B_0^{\mathbb{Z}}$ *coincides with* $L^\infty(X_m, \mu_b)$.

PROOF. In the space $\mathcal{H}^m$ define two von Neumann algebras $\mathfrak{A}^S$ and $\mathfrak{A}^x$ generated respectively by two sets of unitary operators $U_{kr}(t)$ and $V_{kr}(t)$, where $(U_{kr}(t)f)(x) = \exp(2\pi i S_{kr}(t))f(x)$, $\quad (V_{kr}(t)f)(x) := \exp(2\pi i t x_{kr})f(x)$,

$$\mathfrak{A}^S = \big(U_{kr}(t) = T_{I+tE_{kr}}^{m,y_n} = \exp(2\pi i S_{kr}(t)) \mid t \in \mathbb{R}, \ (k,r) \in \Delta(m,n)\big)'',$$
$$\mathfrak{A}^x = \big(V_{kr}(t) := \exp(2\pi i t x_{kr}) \mid t \in \mathbb{R}, \ (k,r) \in \Delta_{m,n} \cup \Delta^{(m,n)}\big)''.$$

Since $\mathfrak{A}^x = L^\infty(X_m, \mu_b)$, to prove irreducibility it is sufficient to show that $\mathfrak{A}^S = \mathfrak{A}^x$. Using decomposition (3.19)

$$\mathbb{S}^{(m)} = B(x,y)^T = (x_m^{-1} y x^{(m)})^T = (x^{(m)})^T y^T (x_m^{-1})^T$$

we conclude that $\mathfrak{A}^S \subseteq \mathfrak{A}^x$ (see the proof of Lemma 2.4). To prove that $\mathfrak{A}^S \supseteq \mathfrak{A}^x$ it is sufficient to find the expressions of the matrix elements of the matrix $x^{(m)} \in B^{(m)}$ and $x_m^{-1} \in B_m$ in terms of the matrix elements of the matrix $\mathbb{S}^{(m)} = (S_{kr})_{(k,r)\in\Delta(m)}$. To do this, we connect the above decomposition $\mathbb{S}^{(m)} = B(x,y)^T$ (see (3.18)) and the Gauss decomposition $C = LDU$ for infinite matrices (see Theorem 5.2). By (3.18) we get $B(x,y) = x_m^{-1} y x^{(m)}$.

To find a matrix connected with the matrix $\mathbb{S}^{(m)}$, for which an appropriate decomposition LDU holds we *recall the expressions for $B(x,y)$ for small n* and finite-dimensional groups G_n^m (see Example (2.2)). We note that $J^2 = I$, where $J = \sum_{r\in\mathbb{Z}} E_{m+r+1,m-r} \in \mathrm{Mat}(2\infty, \mathbb{R})$. For G_3^3 we have

$$B(x,y)J{=}x_m^{-1} y J J x^{(m)} J{=}\begin{pmatrix} 1 & x_{45}^{-1} & x_{46}^{-1} & x_{47}^{-1} \\ 0 & 1 & x_{56}^{-1} & x_{57}^{-1} \\ 0 & 0 & 1 & x_{67}^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{43} & 0 & 0 & 0 \\ 0 & y_{52} & 0 & 0 \\ 0 & 0 & y_{61} & 0 \\ 0 & 0 & 0 & y_{70} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{23} & 1 & 0 & 0 \\ x_{13} & x_{12} & 1 & 0 \\ x_{03} & x_{02} & x_{01} & 1 \end{pmatrix}. \tag{3.22}$$

We use the infinite-dimensional analog of the latter presentation, i.e. instead of the group $G_n = B(n,\mathbb{R})$ consider the infinite-dimensional group $B_0^{\mathbb{Z}}$ and do the same. Let $x_m \in B_m$, $x^{(m)} \in B^{(m)}$ and $y \in \mathfrak{b}_2^*(a)$ be defined by (3.11).

Set $C := C(x) := B(x,y)J$, then $C = UDL$, namely, we have: $B(x,y)J = x_m^{-1} y J J x^{(m)} J = UDL$, where $U = x_m^{-1}$, $D = yJ$, $L = Jx^{(m)}J$,

$$C{=}B(x,y)J{=}\begin{pmatrix} 1 & x_{45}^{-1} & x_{46}^{-1} & x_{47}^{-1} & \dots \\ 0 & 1 & x_{56}^{-1} & x_{57}^{-1} & \dots \\ 0 & 0 & 1 & x_{67}^{-1} & \dots \\ 0 & 0 & 0 & 1 & \dots \\ & & & & \dots \end{pmatrix}\begin{pmatrix} y_{43} & 0 & 0 & 0 & \dots \\ 0 & y_{52} & 0 & 0 & \dots \\ 0 & 0 & y_{61} & 0 & \dots \\ 0 & 0 & 0 & y_{70} & \dots \\ & & & & \dots \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ x_{23} & 1 & 0 & 0 & \dots \\ x_{13} & x_{12} & 1 & 0 & \dots \\ x_{03} & x_{02} & x_{01} & 1 & \dots \\ & & & & \dots \end{pmatrix}. \tag{3.23}$$

To complete the proof of Lemma it is sufficient to find the decomposition (3.23) $C = UDL$. Let us *suppose that we can find the inverse matrix C^{-1}*. Then $C^{-1} = L^{-1}D^{-1}U^{-1}$ holds and we can use Theorem 5.2 to find

$$L^{-1} = J(x^{(m)})^{-1}J, \quad D^{-1} = y^{-1}J, \quad U^{-1} = x_m.$$

Hence, we can find the matrix elements of the matrix $(x^{(m)})^{-1} \in B^{(m)}$ and $x_m \in B_m$ in terms of the matrix elements of the matrix $C^{-1} = (\mathbb{S}^T J)^{-1} = (B(x,y)J)^{-1}$. Finally, we can also find the matrix elements of the matrix

$x^{(m)} \in B^{(m)}$ using formulas (2.14). This finish the proof of Lemma since in this case we have $x_{kr}\ \eta\ \mathfrak{A}^S$ for $(k,r) \in \Delta_m \bigcup \Delta^{(m)}$. Hence $\mathfrak{A}^S \supseteq \mathfrak{A}^x$.
1) To find the inverse matrix C^{-1}, we write two decompositions:

$$C = L_1 D_1 U_1 = UDL,\ C^{-1} = (U_1)^{-1}(D_1)^{-1}(L_1)^{-1} = L^{-1}D^{-1}U^{-1}. \quad (3.24)$$

2) Using (3.24) we can find L_1, D_1 and U_1 by Theorem 5.2. More precisely, for all $x \in \Gamma_G$, where

$$\Gamma_C = \{x \in B_m \times B^{(m)} \mid M^{12\ldots k}_{12\ldots k}(C(x)) \neq 0,\ k \in \mathbb{N}\}$$

holds, the decomposition $C(x) = L_1 D_1 U_1$ and the matrix elements of the matrix $L_1,\ D_1$ and U_1 are rational functions in $c_{kn}(x)$.
3) We can find $(L_1)^{-1}$ and $(U_1)^{-1}$ using formulas (2.14). Note that JLJ, U, and $JL^{-1}J$, $U^{-1} \in B_2(a)$.
4) Using identity (3.24) we can calculate $C^{-1} = (U_1)^{-1}(D_1)^{-1}(L_1)^{-1}$, since $L^{-1},\ D^{-1}$ and U^{-1} are well defined.
5) Using equality (3.24) we can find the decomposition $C^{-1} = L^{-1}D^{-1}U^{-1}$ of the matrix C^{-1} by Theorem 5.2. In other words, the decomposition $C^{-1} = L^{-1}D^{-1}U^{-1}$ holds for all $x \in \Gamma_{G^{-1}}$, where

$$\Gamma_{C^{-1}} = \{x \in B_m \times B^{(m)} \mid M^{12\ldots k}_{12\ldots k}(C^{-1}(x)) \neq 0,\ k \in \mathbb{N}\}$$

and the matrix elements of the matrix $L^{-1},\ D^{-1}$ and U^{-1} are rational functions in matrix elements $c^{-1}_{kn}(x)$ of the matrix C^{-1}.

Let us denote $(L_1)^{-1} = (L^{-1}_{1;kn})_{kn}, (D_1)^{-1} = \mathrm{diag}(d^{-1}_{1;k})_k$ and $(U_1)^{-1} = (U^{-1}_{1;kn})_{kn}$. The decompositions $C = L_1 D_1 U_1$ and $C^{-1} = (U_1)^{-1}(D_1)^{-1}\times (L_1)^{-1}$ hold for $x \in \Gamma_C \cap \Gamma_{C^{-1}}$, i.e., almost for all $x \in B_m \times B^{(m)}$ with respect to the measure μ_b since $\mu_b(\Gamma_C \cap \Gamma_{C^{-1}}) = 1$. We conclude that the convergence

$$c^{-1}_{kn}(x) = \sum_{m \in \mathbb{N}} U^{-1}_{1;km} d^{-1}_{1;m} L^{-1}_{1;mn},\ k,n \in \mathbb{N}$$

holds pointwise almost everywhere $x \in B_m \times B^{(m)}$ (mod μ_b). Since $U^{-1}_{1;km},\ d^{-1}_{1;m}$ and $L^{-1}_{1;mn}\,\eta\,\mathfrak{A}^S$ by 2) and 3), we conclude by Remark 3.1 that $c^{-1}_{kn}(x)\,\eta\,\mathfrak{A}^S$. This complete the proof of Lemma. □

Remark 3.1. Let the sequence of real measurable functions $(f_n)_{n\in\mathbb{N}}$ is affiliated with the von Neumann algebra M of operators in the space $L^2(X,\mu)$ with a finite measure μ, i.e., $U_n(t) \in M,\ t \in \mathbb{R}$, where $(U_n(t)g)(x) = \exp(itf_n(x)) \times g(x),\ g \in L^2(X,\mu)$. If $f_n \to f$ a.e. (mod μ), then $s.\lim_n U_n(t) = U(t)$ hence, $U(t) = \exp(itf) \in M$, $t \in \mathbb{R}$, i.e., the function f is also affiliated with M.

Proof. *of Theorem 3.9.* Let a bounded operator A in a Hilbert space $\mathcal{H}^m$ commute with the representation $T^{m,y}$. Then by Lemma 3.10 A commute with $L^\infty(B_m \times B^{(m)}, \mu_{b,m} \otimes \mu_b^{(m)})$, therefore the operator A itself is an operator of multiplication by some essentially bounded function $a \in L^\infty$, i.e., $(Af)(x) = a(x)f(x)$ for $f \in \mathcal{H}^m$. Since A commutes with the representation $T^{m,y}$ i.e., $[A, T_t^{m,y}] = 0$ for all $t \in B_{m,0} \times B_0^{(m)}$, where $B_{m,0} = B_m \cap B_0^{\mathbb{Z}}$ and $B_0^{(m)} = B^{(m)} \cap B_0^{\mathbb{Z}}$, we conclude that $a(x) = a(xt)$ $(\mathrm{mod}\, \mu_{b,m} \otimes \mu_b^{(m)})$ for all $t \in B_{m,0} \times B_0^{(m)}$. Since the measure $\mu_{b,m} \otimes \mu_b^{(m)}$ on the group $B_m \times B^{(m)}$ is $B_{m,0} \times B_0^{(m)}$-right-ergodic, we conclude that $a(x) = const$ $(\mathrm{mod}\, dx_m \otimes dx^{(m)})$.

Remark 3.2. The proof of the irreducibility can be generalized for an arbitrary $B_0^{\mathbb{Z}}$-quasi-invariant ergodic measure μ if the following equality holds: $\mathfrak{A}^S = \mathfrak{A}^x = L^\infty(B_m \times B^{(m)}, \mu)$.

4. Criteria of the irreducibility of the induced representations $T^{k,2m+1,\mu_b}$ corresponding to generic orbits

We construct a two-parameter family of the induced representations $T^{k,2m+1,\mu_b}$, $k, m \in \mathbb{Z}$ corresponding to a point $y^k \in \mathfrak{b}_2^*(a)$, $k \in \mathbb{Z}$, (see (3.10)) and subgroup $H_0^{2m+1} = \{I + \sum_{k \le m < n} x_{kn} E_{kn}\} \subset B_0^{\mathbb{Z}}$, $m \in \mathbb{Z}_0$, $\mathrm{Lie}(H_0^{2m+1}) = \mathfrak{h}_0^{2m+1}$, and give the criteria of their irreducibilities. Recall that $B_0^{\mathbb{Z}} \subset B_2(a) \subset B^{\mathbb{Z}}$, the representation $U_{f,H}$ of the group H is $H \ni \exp(x) \mapsto \exp 2\pi i \langle f, x\rangle \in S^1$. Fix $y^k \in \mathfrak{b}_2^*(a)$, the Lie algebra $\mathfrak{h}_0^{2m+1} \in \mathfrak{b}_0^{\mathbb{Z}}$ is subordinate to the functional y^k for all $k, m \in \mathbb{Z}$ since it is commutative, $[\mathfrak{h}_0^{2m+1}, \mathfrak{h}_0^{2m+1}] = 0$. The representation $\mathfrak{h}_0^{2m+1} \ni x \mapsto \langle y^k, x\rangle \in \mathbb{R}^1$ can be extended by continuity to the representation of the Hilbert–Lie completion $\mathfrak{h}_2^{2m+1}(a)$ in $\mathfrak{b}_2(a)$ of the Lie algebra $\mathfrak{h}_0^{2m+1}$, $\mathfrak{h}_2^{2m+1}(a) \ni x \mapsto \langle y^k, x\rangle \in \mathbb{R}^1$. The representation $H_0^{2m+1} \ni \exp(x) \mapsto \exp 2\pi i \langle y^k, x\rangle \in S^1$ can be extended by continuity to the representation of its Hilbert–Lie completion $H_2^{2m+1}(a)$

$$H_2^{2m+1}(a) \ni \exp(x) \overset{U_{y^k, H_2^{2m+1}(a)}}{\longmapsto} \exp 2\pi i \langle y^k, x\rangle \in S^1.$$

The homogeneous spaces are $X^{2m+1} = H^{2m+1} \backslash B^{\mathbb{Z}}$, $X_2^{2m+1}(a) = H_2^{2m+1}(a) \backslash B_2(a)$. The measure $\mu_b = \mu_{b,m} \otimes \mu_b^{(m)}$ is defined on the space X^{2m+1} by (3.14) and its support is $X_2^{2m+1}(a)$ by Lemma 3.6, for an appropriate b. The representation $T^{k,2m+1,\mu_b}$ is defined by

$$T^{k,2m+1,\mu_b} = \mathrm{Ind}_{H_2^{2m+1}(a), H_0^{2m+1}}^{B_2(a), B_0^{\mathbb{Z}}, \mu_b}(U_{y^k, H_0^{2m+1}}). \tag{4.1}$$

Define the unitary representation $T^{L,2n+1,\mu_b},\ n\in\mathbb{Z}$ of the group $G=B_{n,0}\times B_0^{(n)}$ in the Hilbert space $\mathcal{H}=L^2(B_n\times B^{(n)},\mu_b)$ by the formula

$$(T_s^{L,2n+1,\mu_b}f)(x)=\big(d\mu_b(s^{-1}x)/d\mu_b(x)\big)^{1/2}f(s^{-1}x),\ f\in\mathcal{H},\ s\in G, \quad (4.2)$$

where $\mu_b=\mu_{b,n}\otimes\mu_b^{(n)}$ is defined by (3.14). The representation $T_s^{L,2n+1,\mu_b}$ is correctly defined for any $s\in B_0^{(n)}$ and for an arbitrary measure $\mu_{b,n}$. For $s\in B_{n,0}$ the representation $T_s^{L,2n+1,\mu_b}$ is correctly defined if and only if $\mu_{b,n}^{L_s}\sim\mu_{b,n}$ for all $s\in B_{n,0}$. More precisely, the operator $T_{I+tE_{rs}}^{L,2n+1,\mu_b}$ for $I+tE_{rs}\in B_{n,0}$ is correctly defined if and only if the following condition holds [15, proof of Lemma 1.2]

$$\mu_{b,n}^{L_I+tE_{rs}}\sim\mu_{b,n},\ \forall t\in\mathbb{R}\Leftrightarrow S_{rs}^L(b)=\sum_{n=s+1}^{\infty}b_{rn}/b_{sn}<\infty. \quad (4.3)$$

Recall the notation $\mathfrak{A}^{k,n,\mu_b}=(T^{k,n,\mu_b}(B_0^{\mathbb{Z}}))'',k,n\in\mathbb{Z}$. Since the right and the left representations commute we get $[T_t^{k,2m+2r+1,\mu_b},T_s^{L,2m+2r+1,\mu_b}]=0$ for all $t,s\in B_{m+r,0}\times B_0^{(m+r)},\ k,m,r\in\mathbb{Z}$. To prove the reducibility, we show that the commutation holds for $k=2m+1,k=2m$

$$[T_t^{k,2m+2r+1,\mu_b},T_s^{L,2m+2r+1,\mu_b}]=0,\ \forall t\in B_0^{\mathbb{Z}},\ s\in G_\oslash=G_{p,q}, \quad (4.4)$$

for some subgroup $G_\oslash=G_{p,q}\subset B_{m+r}\times B^{(m+r)}$ described in Definition 4.1. In fact, it is sufficient to show that

$$[T_{I+tE_{m+r,m+r+1}}^{k,2m+2r+1,\mu_b},T_s^{L,2m+2r+1,\mu_b}]=0,\ \forall t\in\mathbb{R},\ s\in G_\oslash=G_{p,q}. \quad (4.5)$$

Indeed, the matrix units $E_{kk+1},\ k\in\mathbb{Z}$ generate the Lie algebra $\mathfrak{b}_0^{\mathbb{Z}}$ and $I+tE_{kk+1}\in B_{m+r,0}\times B_0^{(m+r)}$ for all $k\in\mathbb{Z}\setminus\{m+r\}$, hence $B_0^{\mathbb{Z}}=\langle B_{m+r,0}\times B_0^{(m+r)},I+tE_{m+r,m+r+1}\mid t\in\mathbb{R}\rangle$ where we denote by $\langle G_1,G_2\rangle$ the subgroup in G generated by the subgroups G_1 and G_2, i.e., the smallest subgroup in G containing G_1 and G_2.

Definition 4.1. We show that the *expression* $B(x,y)=x_{m+r}^{-1}y^k(m+r)x^{(m+r)}$

$$B(x,y)\cong\begin{pmatrix}1&0\\0&x_{m+r}^{-1}\end{pmatrix}\begin{pmatrix}0&0\\y^k(m+r)&0\end{pmatrix}\begin{pmatrix}x^{(m+r)}&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\x_{m+r}^{-1}y^k(m+r)x^{(m+r)}&0\end{pmatrix}, \quad (4.6)$$

(see (3.18)), where $y^k(m+r)$ is the restriction of y^k corresponding to the decomposition $x=x_{m+r}x(m+r)x^{(m+r)}$ (see (3.12))

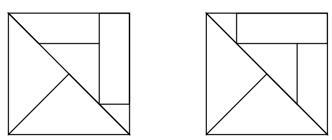
$$y^k(m+r)=\sum_{l+s=k,\,s\le m+r}y_{ls}E_{ls},\ r<0,\quad y^k(m+r)=\sum_{l+s=k,\,l\ge m+r+1}y_{ls}E_{ls},\ r>0, \quad (4.7)$$

does not contain the matrix elements (the variables) x_{kn} if $r>0$ (resp. x_{kn}^{-1} if $r<-1$) of the matrix $x=(x_{kn})_{kn}$ from some group $G_{p,q}$. We *denote this group by* $G_{\oslash}=G_{p,q}$. We show that $G_{\oslash}=G_{m+r+1,q}\subset B_{m+r}$ when $r<-1$ and $G_{\oslash}=G_{p,m+r}\subset B^{(m+r)}$ when $r>0$. The description of the group $G_{\oslash}$ gives

Lemma 4.1. *We have the following description of the group* $G_{\oslash}=G_{p,q}$*:*

$$G_{\oslash}=\begin{cases} G_{m-|r|+1,m+|r|}, & \text{if } r<0,\ k=2m+1\\ G_{m-|r|+1,m+|r|-1}, & \text{if } r<0,\ k=2m\\ G_{m-r+1,m+r}, & \text{if } r>0,\ k=2m+1\\ G_{m-r,m+r}, & \text{if } r>0,\ k=2m. \end{cases} \tag{4.8}$$

PROOF. See an example of calculation $G_{\oslash}$ below in (4.14) and (4.16). By (4.6) the group $G_{\oslash}=G_{m+r+1,q}$ will be contained in the group B_{m+r} for $r<0$ and the group $G_{\oslash}=G_{p,m+r}$ in the group $B^{(m+r)}$ for $r>0$. For $r<0$ we find the intersection of the *antidiagonal* $\mathrm{adiag}^k:=\{(r,s)\in\mathbb{Z}^2\mid r+s=k\},\ k\in\mathbb{Z}$ with the row $m+r+1$ (the fist row of the group B_{m+r}). For $k=2m+1$ we get $(m+r+1,x)\in\mathrm{adiag}^{2m+1}$, $x=m-r$. So, all the variables x_{rs} of the group B_{m+r} with numbers of columns $s\leq m-r$ are contained in $G_{\oslash}$. Hence $G_{\oslash}=G_{m+r+1,m-r}=G_{m-|r|+1,m+|r|}$. If $k=2m$ we get $(m+r+1,x)\in\mathrm{adiag}^{2m}$, $x=m-r-1\Rightarrow G_{\oslash}=G_{m+r+1,m-r-1}=G_{m-|r|+1,m+|r|-1}$.



(4.9)

For $r>0$ we find the intersection of the antidiagonal adiag^k with the column $m+r$ (the last column of the group $B^{(m+r)}$). For $k=2m+1$ we get $(x,m+r)\in\mathrm{adiag}^{2m+1}$, $x=m-r+1$. So all the variables x_{rs} of the group $B^{(m+r)}$ with numbers of rows $r\leq m-r+1$ are contained in $G_{\oslash}$. Hence $G_{\oslash}=G_{m-r+1,m+r}$. If $k=2m$ we get $(x,m+r)\in\mathrm{adiad}^{2m}$, $x=m-r\Rightarrow G_{\oslash}=G_{m-r,m+r}$. Finally we get (4.8). $\square$

Theorem 4.2. *(i) The representation $T^{2m+1,2m+2r+1,\mu_b}$ is irreducible if and only if (a) the measure μ_b is $B_0^{\mathbb{Z}}$ ergodic and (b) either $r=0$ or $r<0$ and $\mu_b^{L_t}\perp\mu_b$ for all $t\in G_{m-|r|+1,m+|r|}\setminus\{e\}$.*
(ii) The representation $T^{2m,2m+2r+1,\mu_b}$ is irreducible if and only if (a) the measure μ_b is $B_0^{\mathbb{Z}}$ ergodic and (b) either $r=-1$, $r=0$ or $r<-1$ and $\mu_b^{L_t}\perp\mu_b$

for all $t\in G_{m-|r|+1,m+|r|-1}\backslash\{e\}$.

iii) In other cases the representations are reducible, moreover the commutant of the von Neumann algebra $\mathfrak{A}^{k,2n+1,\mu_b}=(T^{k,2n+1,\mu_b}(B_0^{\mathbb{Z}}))''$ *contains the following von Neumann algebras: (a) if* $r>0$ *then* $(\mathfrak{A}^{2m+1,2m+2r+1,\mu_b})'\supset(T^{L,2m+2r+1,\mu_b}(G_{m-r+1,m+r}))''$, *and* $(\mathfrak{A}^{2m,2m+2r+1,\mu_b})'\supset(T^{L,2m+2r+1,\mu_b}(G_{m-r,m+r}))''$, *(b) if* $r<0$ *then* $(\mathfrak{A}^{2m+1,2m+2r+1,\mu_b})'\supset(T^{L,2m+2r+1,\mu_b}(G^{\sim}_{m-|r|+1,m+|r|}))''$ *and if* $r<-1$ *then* $(\mathfrak{A}^{2m,2m+2r+1,\mu_b})'\supset(T^{L,2m+2r+1,\mu_b}(G^{\sim}_{m-|r|+1,m+|r|-1}))''$, *where* $G^{\sim}:=\{s\in G\mid \mu_b^{L_s}\sim\mu_b\}$.

4.1. The center of the universal enveloping algebra of a Lie algebra $\mathfrak{b}(n,\mathbb{R})$ *and the description of the commutant of the induced representations of the group* G_n

To find the commutant of the representations $T_{f,H}=\mathrm{Ind}_H^G(U_{f,H})$ of the finite-dimensional group G_n we use the following description of the center of the universal enveloping algebra of the corresponding nilpotent Lie algebra $\mathfrak{g}_n=\mathfrak{b}(n,\mathbb{R})$ of strictly upper triangular $n\times n$ matrices.

Theorem 4.3. *The* center Z *of the universal enveloping algebra* $U(g_n)$ *of the Lie algebra* $\mathfrak{g}_n$ *contains the following elements*

$$Z(U(\mathfrak{g}_n))\supseteq\langle\Delta_k\mid 1\le k\le[n/2]\rangle, \tag{4.10}$$

where $\Delta_1=M_n^1(\mathbb{E})=E_{1n}$,

$$\Delta_2=M^{12}_{n-1,n}(\mathbb{E})=\begin{vmatrix}E_{1n-1}&E_{1n}\\E_{2n-1}&E_{2n}\end{vmatrix},\ \Delta_k=M^{12\dots k}_{n-k+1,n-k+2,\dots,n}(\mathbb{E}), \tag{4.11}$$

$$\mathbb{E}=\begin{pmatrix}E_{11}&E_{12}&E_{13}&\dots&E_{1n-2}&E_{1n-1}&E_{1n}\\E_{21}&E_{22}&E_{23}&\dots&E_{2n-2}&E_{2n-1}&E_{2n}\\E_{31}&E_{32}&E_{33}&\dots&E_{3n-2}&E_{3n-1}&E_{3n}\\&&&\dots&&&\\E_{n-21}&E_{n-22}&E_{n-23}&\dots&E_{n-2n-2}&E_{n-2n-1}&E_{n-2n}\\E_{n-11}&E_{n-12}&E_{n-13}&\dots&E_{n-1n-2}&E_{n-1n-1}&E_{n-1n}\\E_{n1}&E_{n2}&E_{n3}&\dots&E_{nn-2}&E_{nn-1}&E_{nn}\end{pmatrix}. \tag{4.12}$$

Proof. Since the elements $E_{rr+1},\ 1\le r<n-1$ generate the Lie algebra $\mathfrak{b}(n,\mathbb{R})$, it is sufficient to find the elements in $U(g_n)$ commuting with $E_{rr+1},\ 1\le r<n-1$. Since $[E_{rr+1},E_{r+1s}]=E_{rs}$ and $[E_{kr},E_{rr+1}]=E_{kr+1}$ we conclude that the action $[E_{rr+1},\cdot]$ replaces the row $r+1$ by the row r of the matrix $\mathbb{E}$ for all $1<r\le n$ and the action $[\cdot,E_{rr+1}]$ replaces the column r by the column $r+1$ of the matrix $\mathbb{E}$ for all $1\le r<n$. In addition the element $\Delta_1=E_{1n}$ generate the center of the Lie algebra $\mathfrak{b}(n,\mathbb{R})$. Hence, all minors $\Delta_k,\ 1\le k\le[\frac{n}{2}]$ commute with all generators $E_{rr+1},\ 1\le r<n-1$. We conclude that $Z(U(\mathfrak{g}_n))\supseteq\langle\Delta_k\mid 1\le k\le[\frac{n}{2}]\rangle$. $\square$

To study the reducibility, we consider the induced representations $\mathrm{Ind}_H^{G,\mu_X}(U_{f,H})$ $=: T_G^{k,2m+2r+1,\mu_b}$ of the finite-dimensional group $G=G_{1,k-1}$ corresponding to the subgroup $H=H_{1,k-1}^{2m+2r+1}=H^{2m+2r+1}\cap G$, the point $f=y_G^k=\sum_{s=1}^{[k/2]}y_{k-s,s}E_{k-s,s}$, $k=2m+1, k=2m$ and the measure μ_X on $X=H\backslash G$.

Notations. Let $\lambda_{G_{pq}}^{\mu_b}$ and $\rho_{G_{pq}}^{\mu_b}$ be the right and the left regular representations of a group $G_{pq}\subset B_{m+r}\times B^{(m+r)}$ corresponding to the projection $\mu_{G_{pq}}$ on the group G_{pq} of the measure $\mu_b=\mu_{b,m+r}\otimes\mu_b^{(m+r)}$ defined on the group $B_{m+r}\times B^{(m+r)}$ by (3.14), $A_{G_{pq};kn}^{\lambda,\mu_b}$ and $A_{G_{pq};kn}^{\rho,\mu_b}$ be theirs generators.

Lemma 4.4. *We have the following description of the commutant of the induced representations* $\mathrm{Ind}_H^G(U_{f,H})$ *for* $G=G_{1,k-1}$, $H=H_{1,k-1}^{2m+2r+1}$ *and* $f=y_G^k$

$$\begin{aligned}(T_{G_{1,k-1}}^{k,2m+2r+1,\mu_b}(G_{1,k-1}))' &= (\lambda_{G_\oslash}^{\mu_b}(G_\oslash))'',\ \text{if } r>0,\\ (T_{G_{1,k-1}}^{k,2m+2r+1,\mu_b}(G_{1,k-1}))' &= (\lambda_{G_{m+r+1,k-1}}^{\mu_b}(G_\oslash))'',\ \text{if } r<0.\end{aligned}\tag{4.13}$$

Proof. For $k=2m+1,\ m=3,\ r=2$, the group G_{16} and subgroup H_{14}^{11} we get $y_G^7=y_{61}E_{61}+y_{52}E_{52}+y_{43}E_{43},\ y^7(5)=y_{61}E_{61}$, where

$$G_{16}=\left(\begin{array}{c|cccc|c}1&x_{12}&x_{13}&x_{14}&x_{15}&x_{16}\\ \hline 0&1&x_{23}&x_{24}&x_{25}&x_{26}\\ 0&0&1&x_{34}&x_{35}&x_{36}\\ 0&0&0&1&x_{45}&x_{46}\\ 0&0&0&0&1&x_{56}\\ \hline 0&0&0&0&0&1\end{array}\right),\ H_{16}^{11}=\begin{pmatrix}1&0&0&0&0&x_{16}\\ 0&1&0&0&0&x_{26}\\ 0&0&1&0&0&x_{36}\\ 0&0&0&1&0&x_{46}\\ 0&0&0&0&1&x_{56}\\ 0&0&0&0&0&1\end{pmatrix},\ G_\oslash=G_{25}.$$

By (4.6) we get $B(x,y)=x_5^{-1}y^7(5)x^{(5)}=$

$$(1)\begin{pmatrix}y_{61}&0&0&0&0\end{pmatrix}\left(\begin{array}{c|cccc}1&x_{12}&x_{13}&x_{14}&x_{15}\\ \hline 0&1&x_{23}&x_{24}&x_{25}\\ 0&0&1&x_{34}&x_{35}\\ 0&0&0&1&x_{45}\\ 0&0&0&0&1\end{array}\right)=y_{61}\begin{pmatrix}1&x_{12}&x_{13}&x_{14}&x_{15}\end{pmatrix},\ G_\oslash=G_{25},\tag{4.14}$$

hence the generators $\mathbb{A}=(A_{kn})_{kn}$ of the representation $T_{G_{16}}^{7,11,\mu_b}$ are

$$\begin{pmatrix}D_{12}&D_{13}&D_{14}&D_{15}&y_{61}\\ &x_{12}D_{13}+D_{23}&x_{12}D_{14}+D_{24}&x_{12}D_{15}+D_{25}&y_{61}x_{12}\\ &&x_{13}D_{14}+x_{23}D_{24}+D_{34}&x_{13}D_{15}+x_{23}D_{25}+D_{35}&y_{61}x_{13}\\ &&&x_{14}D_{15}+x_{24}D_{25}+x_{34}D_{35}+D_{45}&y_{61}x_{14}\\ &&&&y_{61}x_{15}\end{pmatrix}.\tag{4.15}$$

Notations. For the sake of shortness we shall write $\mathbb{S}_{kn}=A_{kn}$ instead of $\mathbb{S}_{kn}=(2\pi i)^{-1}A_{kn}$ (see (2.3)). We get by Theorem 4.3

$$\Delta_1=A_{16}=y_{61},\ \Delta_2=\begin{vmatrix}A_{15}&A_{16}\\ A_{25}&A_{26}\end{vmatrix}=\begin{vmatrix}D_{15}&y_{61}\\ x_{12}D_{15}+D_{25}&y_{61}x_{12}\end{vmatrix}=y_{61}D_{25}.$$

Since the set of operators $(x_{1k}, D_{1k})_{2\leq k\leq 5}$ is irreducible in the space $H_1 = \otimes_{k=2}^{5} L^2(x_{1k})$, the commutant $\mathfrak{A}'$ in the space $L^2(G_{15}) = H_1 \otimes H_2$ has the form $\mathfrak{A}' = I \otimes B(H_2)$, where $H_2 = L^2(G_{25})$. We make the correction $A^{\rho,\mu_b}_{G_{15};kn} - x_{1k}D_{1n} = A^{\rho,\mu_b}_{G_{25};kn}$ for $2 \leq k < n \leq 5$. Hence we get

$$\mathfrak{A}' = \left\langle \begin{array}{ccc} D_{23} & D_{24} & D_{25} \\ & x_{23}D_{24}+D_{34} & x_{23}D_{25}+D_{35} \\ & & x_{24}D_{25}+x_{34}D_{35}+D_{45} \end{array} \right\rangle'_\eta = (\rho^{\mu_b}_{G_{25}}(G_{25}))' = (\lambda^{\mu_b}_{G_\oslash}(G_\oslash))''.$$

Notation. For the set of self-adjoint operators $(A_p)_{p\in P}$ we denote by $\langle A_p \mid p\in P\rangle_\eta = (\exp(itA_p) \mid t\in\mathbb{R},\ p\in P)''$. For $k=2m$, $m=3$, $r=-2$, the group G_{15} and subgroup H^3_{15} we get $y^6_G = y_{51}E_{51} + y_{42}E_{42}$, $\ y^6(1) = y_{51}E_{51}$, where

$$G_{15} = \left(\begin{array}{c|ccc|c} 1 & x_{12} & x_{13} & x_{14} & x_{15} \\ \hline 0 & 1 & x_{23} & x_{24} & x_{25} \\ 0 & 0 & 1 & x_{34} & x_{35} \\ 0 & 0 & 0 & 1 & x_{45} \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right),\ H^3_{15} = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ G_\oslash = G_{24},$$

$$B(x,y) = \left(\begin{array}{ccc|c} 1 & x^{-1}_{23} & x^{-1}_{24} & x^{-1}_{25} \\ 0 & 1 & x^{-1}_{34} & x^{-1}_{35} \\ 0 & 0 & 1 & x^{-1}_{45} \\ \hline 0 & 0 & 0 & 1 \end{array}\right)\begin{pmatrix} 0 \\ 0 \\ 0 \\ y_{51} \end{pmatrix}(1) = y_{51}\left(x^{-1}_{25}\ x^{-1}_{35}\ x^{-1}_{45}\ 1 \right),\ G_\oslash = G_{24}. \tag{4.16}$$

The generators $\mathbb{A} = (A_{kn})_{kn}$ of the representations $T^{6,3,\mu_b}_{G_{15}}$ are

$$\begin{pmatrix} A_{12} & A_{13} & A_{14} & A_{15} \\ & A_{23} & A_{24} & A_{25} \\ & & A_{34} & A_{35} \\ & & & A_{45} \end{pmatrix} = \begin{pmatrix} x^{-1}_{25}y_{51} & x^{-1}_{35}y_{51} & x^{-1}_{45}y_{51} & y_{51} \\ & D_{23} & D_{24} & D_{25} \\ & & x_{23}D_{24}+D_{34} & x_{23}D_{25}+D_{35} \\ & & & x_{24}D_{25}+x_{34}D_{35}+D_{45} \end{pmatrix}. \tag{4.17}$$

We get $\Delta_2 = \begin{vmatrix} A_{14} & A_{15} \\ A_{24} & A_{25} \end{vmatrix} = \begin{vmatrix} x^{-1}_{45}y_{51} & y_{51} \\ D_{24} & D_{25} \end{vmatrix} = -y_{51}(D_{24} + x_{45}D_{25}) = y_{51}A^{\lambda,\mu_b}_{G_{25};24}$. It was the first *indication* for the *description* (4.13). We show that $\mathfrak{A}' = (\lambda^{\mu_b}_{G_{25}}(G_{24}))''$. For locally-compact groups G, the following commutation theorem is known [6, 13.10.4]: $J_G\rho_tJ_G = \lambda_t$, $t\in G$, where $(J_Gf)(x) = (dh(x^{-1})/dh(x))^{1/2}\overline{f(x^{-1})}$, and h is a Haar measure, i.e., a change of variables $x = (x_{kn}) \mapsto x^{-1} = (x^{-1}_{kn}) =: z = (z_{kn})$ (on the group G_{25}) interlaces the left and the right regular representations. Hence using (4.17) we get

$$\begin{pmatrix} A_{12} & A_{13} & A_{14} & A_{15} \\ & A_{23} & A_{24} & A_{25} \\ & & A_{34} & A_{35} \\ & & & A_{45} \end{pmatrix} \mapsto \begin{pmatrix} z_{25}y_{51} & z_{35}y_{51} & z_{45}y_{51} & y_{51} \\ & D^z_{23}+z_{34}D^z_{24}+z_{35}D^z_{25} & D^z_{24}+z_{45}D^z_{25} & D^z_{25} \\ & & D^z_{34}+z_{45}D^z_{35} & D^z_{35} \\ & & & D^z_{45} \end{pmatrix}.$$

As in the previous case, eliminating the variables z_{25}, z_{35}, z_{45} and the operators $D^z_{25}, D^z_{35}, D^z_{45}$ we get in the corresponding space $L^2(G^x_{24})$ and $L^2(G^z_{24})$

$$(\mathfrak{A}^z)' = \left\langle \begin{array}{cc} D^z_{23}+z_{34}D^z_{24} & D^z_{24} \\ & D^z_{34} \end{array} \right\rangle'_\eta = (\lambda^{z,\mu_b}_{G_{24}}(G_{24}))' = (\rho^{z,\mu_b}_{G_{24}}(G_{24}))'' \overset{z\mapsto x}{\mapsto} (\lambda^{x,\mu_b}_{G_{25}}(G_\oslash))''.$$

The general case is treated similarly (see (4.9)). **(i)** 1) Let $r>0$, using the decomposition $B_{m+r}\times B^{(m+r)}=B_{m+r}\times G_{\oslash}\ltimes B_{\oslash}^{(m+r)}=X_1\times X_2\times X_3$ we get $\mu=\mu_1\otimes\mu_2\otimes\mu_3$, hence $L^2(B_{m+r}\times B^{(m+r)},\mu)=L^2(X_1,\mu_1)\otimes L^2(X_2,\mu_2)\otimes L^2(X_3,\mu_3)=H_1\otimes H_2\otimes H_3$. 2) $\mathfrak{A}'=(T_H,T_{B^{(m+r)}},T_{B_{\oslash}^{(m+r)}},T_{G_{\oslash}})'=\mathfrak{A}_2'\cap(T_{G_{\oslash}})'$. 3) We get $\mathfrak{A}_2':=(T_H,T_{B_{m+r}},T_{B_{\oslash}^{(m+r)}})'=1\otimes B(H_2)\otimes 1$. Indeed, using Gauss decomposition for $B(x,y)$ (see proof of Theorem 2.3) and Definition 4.1 of the group $G_{\oslash}$ we can obtain all the variables $x=(x_{kn})\in B_{\oslash}^{(m+r)}$ and $x=(x_{kn})\in B_{m+r}$, hence $(T_H)'=L^\infty(X_1,\mu_1)\otimes B(H_2)\otimes L^\infty(X_3,\mu_3)$, so $\mathfrak{A}_2'=\Big(L^\infty(X_1)\otimes B(H_2)\otimes L^\infty(X_3)\Big)\cap(\rho_{B^{(m+r)}}\otimes 1\otimes 1)'\cap(1\otimes 1\otimes\rho_{B_{\oslash}^{(m+r)}})'=1\otimes B(H_2)\otimes 1$. 4) Finally $\mathfrak{A}'=\mathfrak{A}_2'\cap(T_{G_{\oslash}})'=1\otimes(\lambda^{\mu_b}_{G_{\oslash}}(G_{\oslash}))''\otimes 1$. Indeed $\forall t\in G_{\oslash},\ T(t)=1\otimes\rho_{G_{\oslash}}(t)\otimes T_3(t)$ for some $T_3(t)\in B(H_3)$. But we have proved that $\mathfrak{A}\supset B(H_1)\otimes 1\otimes B(H_3)$, hence we can take the correction $T(t)(1\otimes 1\otimes T_3(t))^{-1}=1\otimes\rho_{G_{\oslash}}(t)\otimes 1$.

(ii) 1) Let $r<0$, using the decomposition $B_{m+r}\times B^{(m+r)}=G_{\oslash}\ltimes B_{m+r,\oslash}\times B^{(m+r)}=X_1\times X_2\times X_3$ we get $\mu=\mu_1\otimes\mu_2\otimes\mu_3$, hence $L^2(B_{m+r}\times B^{(m+r)},\mu)=L^2(X_1,\mu_1)\otimes L^2(X_2,\mu_2)\otimes L^2(X_3,\mu_3)=H_1\otimes H_2\otimes H_3$. Make the change of variables $x\to x^{-1}=z$ on the group B_{m+r}. We get 2) $(\mathfrak{A}^z)'=(T_H^z,T^z_{B_{m+r,\oslash}},T_{B^{(m+r)}},T^z_{G_{\oslash}})'=(\mathfrak{A}_2^z)'\cap(T^z_{G_{\oslash}})'$. 3) $\mathfrak{A}_2'=(T_H^z,T^z_{B_{m+r,\oslash}},T_{B^{(m+r)}})'=B(H_1^z)\otimes 1\otimes 1$. Indeed, by the same argument as before we can obtain all the variables $x=(x_{kn})\in B^{(m+r)}$ and $z=(z_{kn})\in B_{m+r,\oslash}$, hence we have $(T_H^z)'=B(H_1^z)\otimes L^\infty(X_2^z,\mu_2)\otimes L^\infty(X_3,\mu_3)$ $(\mathfrak{A}_2^z)'=\Big(B(H_1^z)\otimes L^\infty(X_2^z,\mu_2)\otimes L^\infty(X_3,\mu_3)\Big)\cap(\lambda^z_{B_{m+r,\oslash}})'\cap(\rho_{B^{(m+r)}})'=B(H_1^z)\otimes 1\otimes 1$. 4) Finally, $(\mathfrak{A}^z)'=(\rho^z_{G_{\oslash}}(G_{\oslash}))''\otimes 1\otimes 1$, $\mathfrak{A}'=(\lambda^x_{B_{m+r}}(G_{\oslash}))''\otimes 1\otimes 1$. Indeed, $\forall t\in G_{\oslash},\ T^z(t)=\lambda^z_{G_{\oslash}}(t)\otimes T_2^z(t)\otimes 1$ for some $T_2^z(t)\in B(H_2^z)$. But we have proved that $\mathfrak{A}^z\supset 1\otimes B(H_2^z)\otimes B(H_3)$, hence we can take $T^z(t)(1\otimes T_2^z(t)\otimes 1)^{-1}=\lambda^z_{G_{\oslash}}(t)\otimes 1\otimes 1$. Finally $(\lambda^z_{G_{\oslash}}(G_{\oslash}))'=(\rho^z_{G_{\oslash}}(G_{\oslash}))''\overset{z\mapsto x}{\mapsto}(\lambda^x_{B_{m+r}}(G_{\oslash}))''$. $\square$

4.2. Study of the induced representations $T^{k,2m+1,\mu_b}$ of the group $B_0^{\mathbb{Z}}$ corresponding to generic orbits

PROOF OF THEOREM. 4.2. The irreducibility of the representation $T^{y,m}=T^{2m+1,2m+1,\mu_b}$ (compare (3.15) and (4.1)) is proved in Theorem 3.9. The irreducibility of the representation $T^{2m,2m+2r+1,\mu_b}$ for $r=-1,r=0$ is proved similarly. We prove that if $r<0$ and $\mu_b^{L_t}\perp\mu_b\ \forall t\in G_{m-|r|+1,m+|r|}\backslash\{e\}$, then the representation $T^{2m+1,2m+2r+1,\mu_b}$ is irreducible. If $r<-1$ and $\mu_b^{L_t}\perp\mu_b\ \forall t\in G_{m-|r|+1,m+|r|-1}\backslash\{e\}$, we prove that the representation $T^{2m,2m+2r+1,\mu_b}$ is also irreducible. In what follows we use the technique developed in [14, 15] for approximating the variales x_{kn} to prove the irreducibility of the regular representations in the framework of Ismagilov's Conjecture (see Conjecture 3.1).

i) Let $k = 2m+1 = 5, m = 2$ and $r = -1$. In this case we get for G_{14}

$$G_{14} = \left(\begin{array}{c|cc|c} 1 & x_{12} & x_{13} & x_{14} \\ \hline 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 1 & x_{34} \\ \hline 0 & 0 & 0 & 1 \end{array}\right), \quad H^3_{14} = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad G_{\oslash} = \left(\begin{smallmatrix} 1 & x_{23} \\ 0 & 1 \end{smallmatrix}\right) = G_{23}.$$

Using formulas for A_{kn} from Theorem 3.9 we get

$$A_{2n} = D_{2n},\ A_{3n} = x_{23}D_{2n} + D_{3n}, A_{2n}A_{3n} = x_{23}D^2_{2n} + D_{2n}D_{3n}.$$

Notation. Let $\langle f_n \mid n \in \mathbb{N}\rangle$ be a closed subspace generated by the set of vectors $(f_n)_{n\in\mathbb{N}}$ in the space H. Using [15, Lemma 2.2] we have

$$x_{23}\mathbf{1} \in \langle A_{2n}A_{3n}\mathbf{1} \mid n>3\rangle \Leftrightarrow S^L_{23}(b) = \sum_{n=4}^{\infty} \frac{b_{2n}}{b_{3n}} = \infty \Leftrightarrow \mu_b^{L_{I+tE_{23}}} \perp \mu_b, t \neq 0. \quad (4.18)$$

Remark 4.1. Property (4.18) shows the convergence of the self-adjoint operators $A_N = \sum_{n=4}^{N} t_n^{(N)} A_{2n}A_{3n} \to A := x_{23}$ only on one vector $\mathbf{1} \in \mathcal{H}$. It is possible to prove the convergence on the common essential domain D for all the operators A_N and A [15, after Lemma 2.2, p.251-252]. By Theorem VIII from [27], the convergence holds in the *strong resolvent sense*, hence we conclude that $\exp(is\sum_{n=4}^{N} t_n^{(N)} A_{2n}A_{3n}) \to \exp(isx_{23})$ when $N \to \infty$, so $x_{23}\ \eta\ \mathfrak{A}$.

Using the Gauss decomposition of the matrix $B(x,y) = x_1^{-1}y(1)x^{(1)}$ we conclude that all variables of the matrix $x^{(1)}$ and x_1^{-1} except the variable $x_{23}^{-1} = -x_{23}$ are affiliated with the von Neumann algebra $\mathfrak{A} = (T^{5,3,\mu_b}(B_0^{\mathbb{Z}}))''$. By (4.18) we have $x_{23}\,\eta\,\mathfrak{A}$, hence all the variables x_{kn} are affiliated with $\mathfrak{A}$ so $\mathfrak{A}' = L^\infty(X_m, \mu_b)$ and the representation $T^{5,3,\mu_b}$ is irreducible.

If $\mu_b^{L_{I+tE_{23}}} \sim \mu_b$ for some $t \neq 0$, the operator $A^L_{23} = D_{23} + \sum_{n=4}^{\infty} x_{3n}D_{2n}$ corresponding to the left shift by $I + tE_{23}$ on the group B_m is well defined. Moreover, the unitary operator $T^{L,\mu_b}_{I+tE_{23}}$, $t \in \mathbb{R}$ commute with the representation $T^{5,3,\mu_b}$, hence the representation $T^{5,3,\mu_b}$ is reducible.

Let $k = 2m+1, m = 3$ and $r = -2$, then we get for the group G_{16}, subgroup H^3_{16}, and the representation $T^{2m+1,2m+2r+1,\mu_b} = T^{7,3,\mu_b}$

$$G_{16} \supset G_{\oslash} = G_{m-|r|+1,m+|r|} = G_{25} = \begin{pmatrix} 1 & x_{23} & x_{24} & x_{25} \\ 0 & 1 & x_{34} & x_{35} \\ 0 & 0 & 1 & x_{45} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As before, we can approximate the variables $(x_{kn})_{2\leq k<n\leq 5}$ by appropriate combinations of operators $A_{2n}, A_{3n}, A_{4n}, A_{5n}$, $5<n$, hence $x_{kn}\,\eta\,\mathfrak{A}$, $2\leq k<n\leq 5$. Using the Gauss decomposition of the matrix $B(x,y)=x_1^{-1}y(1)x^{(1)}$ we conclude that all variables of the matrices $x^{(1)}$ and x_1^{-1}, except the variable $(x_{kn})_{2\leq k<n\leq 5}$, are affiliated with the von Neumann algebra $\mathfrak{A}^{7,3,\mu_b}$. We conclude that all the variables x_{kn} of the matrices $x^{(1)}$ and x_1^{-1} are affiliated so $\mathfrak{A}'=L^\infty(X_1,\mu_b)$ and the representation $T^{7,3,\mu_b}$ is irreducible.

ii) Let $k=2m=6, m=3$ and $r=-2$, then we get for the group G_{15} subgroup H_{15}^3 and the representation $T^{2m,2m+2r+1,\mu_b}=T^{6,3,\mu_b}$

$$G_{15}\supset G_{\oslash}=G_{m-|r|+1,m+|r|-1}=G_{24}=\left(\begin{smallmatrix}1&x_{23}&x_{24}\\0&1&x_{34}\\0&0&1\end{smallmatrix}\right).$$

The operators A_{2n}, A_{3n}, A_{4n} of the representation $T^{6,3,\mu_b}$ are as follows (see Theorem 3.9)

$$A_{2n}=D_{2n},\ A_{3n}=x_{23}D_{2n}+D_{3n},\ A_{4n}=x_{24}D_{2n}+x_{34}D_{3n}+D_{4n}.$$

Using [15, Lemma 2.2, p. 254] again we have

$$x_{23}\mathbf{1}\in\langle A_{2n}A_{3n}\mathbf{1}\mid n>3\rangle\Leftrightarrow S_{23}^L(b)=\sum_{n=5}^\infty\frac{b_{2n}}{b_{3n}}=\infty\Leftrightarrow\mu_b^{L_{I+tE_{23}}}\perp\mu_b, t\neq 0.\quad(4.19)$$

We make the corrections: $A_{3n}-x_{23}A_{2n}=D_{3n}$. Using [15, Lemma 2,4,] we get

$$x_{24}\mathbf{1}\in\langle D_{2n}A_{4n}\mathbf{1}\mid n>5\rangle\Leftrightarrow\Sigma_{24}^{(1)}(b)=\sum_{n=5}^\infty b_{2n}(b_{2n}+b_{3n}+b_{4n})^{-1}=\infty,$$

$$x_{34}\mathbf{1}\in\langle D_{3n}A_{4n}\mathbf{1}\mid n>5\rangle\Leftrightarrow\Sigma_{34}^{(1)}(b)=\sum_{n=5}^\infty b_{3n}(b_{2n}+b_{3n}+b_{4n})^{-1}=\infty.$$

Since

$$\Sigma_{24}^{(1)}(b)+\Sigma_{34}^{(1)}(b)=\sum_{n=5}^\infty\frac{b_{2n}+b_{3n}}{b_{2n}+b_{3n}+b_{4n}}\sim\sum_{n=5}^\infty\frac{b_{2n}+b_{3n}}{b_{4n}}=S_{24}^L(b)+S_{34}^L(b)=\infty,$$

one of the series $\Sigma_{24}^{(1)}(b)$ or $\Sigma_{34}^{(1)}(b)$ is divergent. Let $\Sigma_{24}^{(1)}(b)=\infty$, then $x_{24}\,\eta\,\mathfrak{A}$. We make correction $A_{3n}^{(2)}:=A_{3n}-x_{24}D_{24}=x_{34}D_{3n}+D_{4n}$. Then we get

$$x_{34}\mathbf{1}\in\langle D_{3n}A_{4n}^{(2)}\mathbf{1}\mid n>5\rangle\Leftrightarrow\Sigma_{34}^{(2)}(b)=\sum_{n=5}^\infty b_{3n}(b_{3n}+b_{4n})^{-1}\sim S_{34}^L(b)=\infty.$$

If $\Sigma_{34}^{(1)}(b) = \infty$, then $x_{34}\,\eta\,\mathfrak{A}$, so $x_{24}\,\eta\,\mathfrak{A}$ and the representation is irreducible. We have proved the irreducibility of the representations $T^{2m+1,2m+2r+1,\mu_b}$ for $m=2, r=-1$ and $m=3, r=-2$. The irreducibility of the representations $T^{2m,2m+2r+1,\mu_b}$ is proved for $m=3, r=-2$. Other cases are treated in a similar way for $k=2m+1$, $r<0$ and $k=2m$, $r<-1$, $m\in\mathbb{Z}$. We use technique developed in [15]. The sufficiency of the irreducibility is thus proved.

iii) To prove the reducibility of the representations $T^{k,2m+2r+1,\mu_b}$ for $k=2m+1$ and $k=2m$, it is sufficient to show that (see (4.5))

$$[T^{k,2m+2r+1,\mu_b}_{I+tE_{m+r,m+r+1}}, T^{L,2m+2r+1,\mu_b}_{s}] = 0,\ \forall t\in\mathbb{R},\ s\in G^{\sim}_{\oslash} = G^{\sim}_{p,q},$$

where $G^{\sim} := \{s\in G \mid \mu_b^{L_s}\sim\mu_b\}$. If $G^{\sim}_{\oslash} = G_{\oslash}$ then it is sufficient to show the following:

$$[A^{2m+1,2m+2r+1,\mu_b}_{m+r,m+r+1}, A^{L,2m+2r+1,\mu_b}_{kk+1}] = 0,\ m-\mid r\mid+1\le k< m+\mid r\mid, \tag{4.20}$$

$$[A^{2m,2m+2r+1,\mu_b}_{m+r,m+r+1}, A^{L,2m+2r+1,\mu_b}_{kk+1}] = 0,\ \begin{cases} m-\mid r\mid+1\le k< m+\mid r\mid-1 & \text{if } r<0\\ m-r\le k< m+r & \text{if } r>0.\end{cases} \tag{4.21}$$

The generators of the left representation $T^{L,2k+1,\mu_b}$ defined by (4.2) are

$$A^{L,2m+2r+1,\mu_b}_{kn} = \begin{cases} D_{kn}+\sum_{r=n+1}^{\infty} x_{nr}D_{kr}, & \text{if}\quad m+r<k<n,\\ D_{kn}+\sum_{r=n+1}^{m+r} x_{nr}D_{kr}, & \text{if}\quad k<n<m+r.\end{cases} \tag{4.22}$$

Consider the operators $A^L_{kn}=D_{kn}+\sum_{r=n+1}^{\infty}x_{nr}D_{kr}$ and x^{-1}_{kn} defined by (2.14).

Lemma 4.5. *We have for $p,q,k,n\in\mathbb{Z}$, $p<q$ and $k<n$*

$$[D_{pq}, x^{-1}_{kn}] = \begin{cases} -x^{-1}_{kp}x^{-1}_{qn}, & \textit{if } k\le p<q\le n,\\ 0, & \textit{otherwise,}\end{cases} \tag{4.23}$$

$$[A^L_{pq}, x^{-1}_{kn}] = \begin{cases} -x^{-1}_{kp}, & \textit{if } q=n, k\le p,\\ 0, & \textit{otherwise.}\end{cases} \tag{4.24}$$

Proof. Identity (4.23) holds by [20, Lemma 16]. We prove identity (4.24). Since x^{-1}_{kn} contains only variables $x_{rs}, k\le r<s\le n$ (see (2.14)) and A^L_{pq} contains $D_{pr}, p\le r$ we conclude that $[A^L_{pq}, x^{-1}_{kn}]=0$ for $k>p$ and $n<q$. For the remaining part of indices $k\le p<q\le n$ we get using (2.15)

$$[A^L_{pq}, x^{-1}_{kn}] = [D_{pq}+\sum_{r=q+1}^{\infty} x_{qr}D_{pr}, x^{-1}_{kn}] = [D_{pq}, x^{-1}_{kn}] + \sum_{r=q+1}^{n} x_{qr}[D_{pr}, x^{-1}_{kn}] =$$

$$-x_{kp}^{-1}x_{qn}^{-1}-\sum_{r=q+1}^{n}x_{qr}x_{kp}^{-1}x_{rn}^{-1}=-x_{kp}^{-1}(x_{qn}^{-1}+\sum_{r=q+1}^{n}x_{qr}x_{rn}^{-1})=-x_{kp}^{-1}(XX^{-1})_{qn}.$$

$\square$

Using (2.11) and (2.8) $\mathbb{S}=(2\pi i)^{-1}(A_{kr})_{k,r}$ and $\mathbb{S}=B^{T}(x,y)$ we get the explicit expressions for generators $A_{kn},\ (k,n)\in\Delta(m+r)$. For $k=2m+1$ or $k=2m,\ m\in\mathbb{Z}$ and $r\in\mathbb{Z}$ we get

$$A_{m+r,m+r+1}=2\pi i\sum_{s=[(k+1)/2]+|r|}^{\infty}x_{m+r+1,s}^{-1}y_{s,k-s}x_{k-s,m+r}. \tag{4.25}$$

In particular, we get

$$\left[\frac{k+1}{2}\right]+\mid r\mid=\begin{cases} m+\mid r\mid+1, & \text{if} \quad r<0,\ k=2m+1,\\ m+\mid r\mid, & \text{if} \quad r<0,\ k=2m,\\ m+r+1, & \text{if} \quad r>0,\ k=2m+1,\\ m+r, & \text{if} \quad r>0,\ k=2m.\end{cases}$$

Using the latter presentations for operators $A_{m+r,m+r+1}$, equality (4.24), description (4.8) of the group $G_{\oslash}$, and Lemma 4.5, equality (4.24), we get (4.20) and (4.21). When $G_{\oslash}^{\sim}\neq G_{\oslash}$ and $G_{\oslash}^{\sim}\setminus\{e\}\neq\oslash$ we conclude that $[T_t^{k,2m+2r+1,\mu_b},T_s^{L,2m+2r+1,\mu_b}]=0$ for all $t\in B_0^{\mathbb{Z}},\ s\in G_{\oslash}^{\sim}\setminus\{e\}$. $\square$

Remark 4.2. We show that the induced representaion $T^{\mu}=T^{k,2m+2r+1,\mu}$ is a limit $T^{\mu}(t)=s.\lim T_n^{\mu_n}(t)$ of representations $T_n^{\mu_n}=\mathrm{Ind}_{H_n}^{G_n^k,\mu_n}(U_{y_n^k,H_n})$ defined on the spaces $L^2(X_n,\mu_n)$. These representations are equivalent to the induced representations $T_n^{h_n}=\mathrm{Ind}_{H_n}^{G_n^k,h_n}(U_{y_n^k,H_n})$, defined on the space $L^2(X_n,h_n)$, corresponding to the Haar measure h_n, and the set of increasing finite-dimensional groups $(G_n^k)_{n\in\mathbb{N}}$, where H_n, y_n^k and μ_n are defined below.

Fix the point $y^k\in\mathfrak{b}_2^*(a)$ defined by (3.10) and a subalgebra $H_0(m+r)=H_0^{2m+2r+1}\subset B_0^{\mathbb{Z}}$. Consider the sequence of subgroups $G_n^k:=G_{[k/2]-n,[(k+1)/2]+n}$, $n\in\mathbb{N}\cup\{0\}$. Denote by $y_n^k=\sum_{l+s=k,\,[k/2]-n\leq l\leq[k/2]+1}y_{ls}E_{ls}$, $H_n:=H_n(m+r)=H_n^{2m+2r+1}=H^{2m+2r+1}\cap G_n^k$. For $X_{m+r}=H(m+r)\setminus B_0^{\mathbb{Z}}\cong B_{m+r}\times B^{(m+r)}$ consider the corresponding projections $X_n:=X_{m+r,n}=H_n(m+r)\setminus G_n^k\cong B_{m+r,n}\times B_n^{(m+r)}$. Let $\mu_n=\mu_{G_n^k}=\mu_{b,m+r,n}\otimes\mu_{b,n}^{(m+r)}$ be the projection on the space $X_{m+r,n}\cong B_{m+r,n}\times B_n^{(m+r)}$ of the measure $\mu_b=\mu_{b,m+r}\otimes\mu_b^{(m+r)}$ defined on the space $X_{m+r}\cong B_{m+r}\times B^{(m+r)}$. Since the measure $\mu_n=$

$\mu_{b,m+r,n}\otimes\mu_{b,n}^{(m+r)}$ is equivalent to the Haar measure $dh_n=dx_{m+r,n}\otimes dx_n^{(m+r)}$ on the group $B_{m+r,n}\times B_n^{(m+r)}$ (compare (2.18) and (3.14)) we conclude that $\mathrm{Ind}_{H_n}^{G_n^k,\mu_n}(U_{y_n^k,H_n})\sim\mathrm{Ind}_{H_n}^{G_n^k,h_n}(U_{y_n^k,H_n})$. Using the explicit form of generators given in Theorem 3.9 we can prove that $\forall t\in B_0^{\mathbb{Z}}$

$$T^{k,2m+2r+1,\mu_b}(t)=\mathrm{Ind}_H^{G,\mu_b}(U_{y^k,H})(t)=s.\lim_{n\to\infty}\mathrm{Ind}_{H_n}^{G_n^k,\mu_n}(U_{y_n^k,H_n})(t). \tag{4.26}$$

Since $G=B_0^{\mathbb{Z}}=\varinjlim_n G_n^k$, t from $B_0^{\mathbb{Z}}$ belongs to G_n^k for some n.

Using (4.26) and Theorem 4.2 (i), (ii) we conclude that the *irreducible* representations $T^{k,2m+2r+1,\mu_b}$ for $r<0$, $k=2m+1$ (resp. $r<-1$, $k=2m$) are obtained as the *limit of the reducible* representations $T_n^{\mu_n}$ of the group G_n^k. This is an infinite-dimensional phenomenon! We stress that the representations $T_n^{\mu_n}$ are not *compatible* i.e. $T_{n+1}^{\mu_{n+1}}(t)\neq T_n^{\mu_n}(t)$ for $t\in G_n^k$.

5. Appendix

5.1. Gauss decompositions

For the matrix $C\in\mathrm{Mat}(n,\mathbb{C})$ let us denote by $M^{i_1i_2...i_r}_{j_1j_2...j_r}(C)$, $1\leq i_1<...<i_r\leq n,\ 1\leq j_1<...<j_r\leq n$ its minors with $i_1,i_2,...,i_r$ rows and $j_1,j_2,...,j_r$ columns.

Theorem 5.1 (Gauss decomposition, [4]). *A matrix $C\in\mathrm{Mat}(n,\mathbb{C})$ admits the following decomposition $C=LDU$ (Gauss decomposition),*

$$\begin{pmatrix} c_{11} & c_{12} & ... & c_{1n}\\ c_{21} & c_{22} & ... & c_{2n}\\ & & ... & \\ c_{n1} & c_{n2} & ... & c_{nn}\end{pmatrix}=\begin{pmatrix} 1 & 0 & ... & 0\\ l_{21} & 1 & ... & 0\\ & & ... & \\ l_{n1} & l_{n2} & ... & 1\end{pmatrix}\begin{pmatrix} d_1 & 0 & ... & 0\\ 0 & d_2 & ... & 0\\ & & ... & \\ 0 & 0 & ... & d_n\end{pmatrix}\begin{pmatrix} 1 & u_{12} & ... & u_{1n}\\ 0 & 1 & ... & u_{2n}\\ & & ... & \\ 0 & 0 & ... & 1\end{pmatrix}$$

where L (resp. U) is a lower (resp. upper) triangular matrix and D a diagonal matrix if and only if all principal minors of the matrix C are different from zeros i.e. $M^{1,2,...,k}_{1,2,...,k}(C)\neq 0$, $1\leq k\leq n$. Moreover, matrix elements of the matrices L,U and D are given by the formulas (see [4, Ch.II, §4, (44), (45)])

$$l_{mk}=\frac{M^{1,2,...,k-1,m}_{1,2,...,k-1,k}(C)}{M^{1,2,...,k-1,k}_{1,2,...,k-1,k}(C)},\quad u_{km}=\frac{M^{1,2,...,k-1,k}_{1,2,...,k-1,m}(C)}{M^{1,2,...,k-1,k}_{1,2,...,k-1,k}(C)},1\leq k<m\leq n, \tag{5.1}$$

$$d_1=M^1_1(C),\quad d_k=(M^{1,2,...,k}_{1,2,...,k}(C)\Big(M^{1,2,...,k-1}_{1,2,...,k-1}(C)\Big)^{-1},\quad 2\leq k\leq n. \tag{5.2}$$

Theorem 5.2. *The infinite order matrix* $C \in \mathrm{Mat}(\infty, \mathbb{C})$ *admits the following decomposition* $C = LDU$*, if and only if all principal minors of the matrix* C *are different from zeros i.e.* $M^{1,2,\dots,k}_{1,2,\dots,k}(C) \neq 0$, $k \in \mathbb{N}$*. Moreover, matrix elements of the matrices* L, U, D *are given by the same formulas as in Theorem 5.1.*

The Gauss decomposition also holds for rectangular matrices with suitable modifications.

5.2. Different criteria for irreducibility of induced representations for locally compact and infinite discrete groups

a) *Locally compact groups.* In the case when the representation π of the separable locally compact group G is induced from an irreducible representation σ on a normal subgroup H, a simple criterion is known. Let σ^x $(x \in G)$ be the representation defined by the action of x on $H : \sigma^x(h) = \sigma(xhx^{-1})$. Then π is irreducible $\Leftrightarrow \sigma^x \cong \sigma$ only if $x \in H$, see [23].

b) *Let* Γ *be a countable group.* Here we follow [2]. Mackey has shown that quasiregular representations are irreducible if and only if the corresponding subgroups are self-commensurizing. Recall that two subgroups Γ_0 and Γ_1 of a group Γ are *commensurable* if $\Gamma_0 \cap \Gamma_1$ is of finite index in both Γ_0 and Γ_1. The *commensurator* of Γ_0 in Γ is defined to be

$$\mathrm{Com}_\Gamma(\Gamma_0) = \{\gamma \in \Gamma \mid \Gamma_0 \text{ and } \gamma\Gamma_0\gamma^{-1} \text{ are commensurable}\}.$$

Suppose Γ is a discrete group, $\Gamma_0 < \Gamma$ is a subgroup and $\lambda_{\Gamma/\Gamma_0}$ is the left regular representation of Γ in $l^2(\Gamma/\Gamma_0)$. We call two subgroups Γ_0, Γ_1 of Γ *quasiconjugate* if there exists $\gamma \in \Gamma$ such that Γ_0 and $\gamma\Gamma_1\gamma^{-1}$ are commensurable.

Theorem 5.3 (Mackey, [24]). *Let* Γ *be a discrete group and let* Γ_0, Γ_1 *be subgroups of* Γ*.*
(1) The representation $\lambda_{\Gamma/\Gamma_0}$ *is irreducible if and only if* $\mathrm{Com}_\Gamma(\Gamma_0) = \Gamma_0$*, in which case* $\mathrm{Ind}^\Gamma_{\Gamma_0}(\pi)$ *is irreducible for any* $\pi \in \hat{\Gamma}^{fd}_0$*, and unitary induction* $\mathrm{Ind}^\Gamma_{\Gamma_0} : \widehat{\Gamma^{fd}_0} \to \hat{\Gamma}$ *is an injective map.*
(2) If $\mathrm{Com}_\Gamma(\Gamma_i) = \Gamma_i$, $i = 0, 1$*, then* $\lambda_{\Gamma/\Gamma_0}$ *and* $\lambda_{\Gamma/\Gamma_1}$ *are unitarily equivalent if and only if* Γ_0 *and* Γ_1 *are quasiconjugate in* Γ*.*

In case Γ_0 *and* Γ_1 *are not quasiconjugate in* Γ *if* π_0*, respectively* π_1*, are finite dimensional irreducible unitary representations of* Γ_0*, respectively* Γ_1*, then* $\mathrm{Ind}^\Gamma_{\Gamma_0}(\pi_0)$ *and* $\mathrm{Ind}^\Gamma_{\Gamma_0}(\pi_1)$ *are not equivalent.*

N.Obata [25], also presented criteria for irreducibility and mutual equivalence of representations induced from finite dimensional ones.

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