Endomorphisms of Kleinian Groups.

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1 Introduction.

In this paper we study cohopfian (or co-Hopf) property. Recall that a group G is *cohopfian* (or has the *co-Hopf property*) if any injective endomorphism $f : G \to G$ is surjective.

Answering a question of E. Rips, Z. Sela showed in [Se] that a torsion-free, non virtually cyclic word-hyperbolic group (in Gromov's sense) is cohopfian if and only if it is not a non-trivial free product. Note that a non-trivial free product $A * B$ is never cohopfian, as it contains the proper subgroup $A * mBm^{-1}$ isomorphic to $A * B$ if $m \notin (A \cup B)$. The cohopficity of 3-manifold groups was studied by many authors, we refer to the papers [PW] and [OP] where a more complete list of references on this subject is given.

If G splits as an HNN-extension, $G = A*_C = \langle A, t \mid tCt^{-1} = \varphi(C)\rangle$, so that t centralizes C then G is not cohopfian (set $f : G \to G$ be the identity on A and $f(t) = t^2$; then f is injective, not surjective). It was showed in [OP] that such example can be realized as a Kleinian group. Note that in this example the group G splits over a parabolic subgroup C which is of infinite index in the unique maximal parabolic subgroup $\tilde{C}$ of G containing C (where $\tilde{C} =< C, t >$) so that $\tilde{C}$ is not conjugate into A. In such case we will refer to the group C and the corresponding splitting of G over C as *essentially non-maximal.* On the other hand it was also showed in [OP] that G is cohopfian provided it does not split over elementary subgroups at all. A natural question now is whether all non cohopfian torsion free one-ended Kleinian groups arise *only* in this way, in other words is G non cohopfian if and only if G has essentially non-maximal splittings over parabolic subgroups ? The main result of the paper (Theorem A below) is a criterion showing that essentially this is indeed the case.

Instead of studying directly the "absolute" cohopfian property of a group G, we extend this notion to the "relative" case. Let us fix a finite system $\mathcal{E} = \{E_1, ..., E_n\}$ of elementary subgroups of G (a "peripheral system") and suppose that $f : G \to G$ is an endomorphism which sends each E_i into itself. Then Theorem B below guarantees that $f : (G, \mathcal{E}) \to (G, \mathcal{E})$ is a surjective map of the pair $(G, \mathcal{E})$ provided that G has no essentially non-maximal splittings over elementary subgroups relatively to the system $\mathcal{E}$ (i.e. every subgroup in $\mathcal{E}$ is elliptic in such splitting of G) and one more condition is satisfied.

This notion of the relative cohopfian property can be easily illustrated in the case of a surface with boundary. Suppose S is a compact surface of genus $g > 1$ whose boundary is a finite collection of loops α_i $(i = 1, ...n)$. Let E_i be the cyclic peripheral subgroup of $G = \pi_1(S)$ generated by α_i and $\mathcal{E} = \{E_1, ..., E_n\}$. The group G is a free group and so is not cohopfian ;

however it is cohopfian relatively to $\mathcal{E}$, i.e. if $f : (G, \mathcal{E}) \to (G, \mathcal{E})$ is an injective endomorphism sending each group E_i into itself then f is surjective.

The paper is organized as follows. Section 2 contains a preliminary material and the statements of the results. In Section 3 we collect some accessibility results for an (abstract) finitely presented group G equipped with a system of elementary subgroups $\mathcal{C}$ which we define there axiomatically. In particular, discrete elementary subgroups of $\mathrm{Isom}_+\mathbb{H}^n$ (i.e those whose limit set is a finite set) satisfy these axioms. Following the paper [De] we obtain a version of a finiteness theorem for acylindrical splittings of G originally due to Z. Sela [Se] by keeping track of the peripheral system $\mathcal{E}$ ($\mathcal{E} \subset \mathcal{C}$). We then use the method of the paper [De-Po] to construct a finite hierarchy of G over elementary subgroups relatively to the system $\mathcal{E}$. These results will be applied to Kleinian groups in Section 4. There we show that up to automorphism a finitely presented Kleinian group G can admit at most finitely many splittings over elementary subgroups (Theorem D). Then for non cohopfian Kleinian groups we will obtain a relative version of the result [OP], namely: if the pair $(G, \mathcal{E})$ is not cohopfian (i.e. there exists an injective non-surjective endomorphism f of $(G, \mathcal{E})$) then G splits non-trivially over elementary subgroups relatively to the system $\mathcal{E}$ (Proposition 4.7). Assuming the pair $(G, \mathcal{E})$ to be not cohopfian we will then find in Section 5 another injective non surjective map $F : (G, \mathcal{E}) \to (G, \mathcal{E})$ and a non-trivial splitting of G as a graph of groups such that F will send every vertex (resp. edge) group of it into itself. The non-triviality of this splitting invariant under F is the most delicate point here; this fact will be showed by induction on the length of a finite hierarchy of G. The existence of such splitting invariant under F will allow us to prove in Section 6 the sufficient condition of Theorem B.

The converse implication, i.e. if G admits an essentially non maximal splitting over a parabolic group, then G is not cohopfian, will be proven in Section 7. The proof is based on a sequence of Lemmas about virtually abelian groups.

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2 Preliminaries and Formulations of the Results.

Let $\mathbb{H}^n$ denote the real hyperbolic space of dimension n. A group G is called *Kleinian* if G is a discrete subgroup of the orientation preserving part of the isometry group $\mathrm{Isom}_+\mathbb{H}^n$ of $\mathbb{H}^n$. The limit set $\Lambda(G)$ of G is the set of accumulation points of some (any) orbit $G(z)$ ($z \in \mathbb{H}^n$).

Recall that a Kleinian group $H \subset \mathrm{Isom}_+\mathbb{H}^n$ is *elementary* if its limit set $\Lambda(H) \subset \mathcal{S}^{n-1}_\infty$ is a finite set, and H is a finite elementary group if and only if $\Lambda(H) = \emptyset$. An infinite elementary group H is *loxodromic* or *parabolic* accordingly to whether the limit set $\Lambda(H)$ contains two points or one. By Bieberbach's theorems (see e.g. [Ra]) every elementary subgroup H of $\mathrm{Isom}_+\mathbb{H}^n$ is virtually abelian i.e. contains a maximal free abelian subgroup A of finite index. The rank of the group A is called *rank* of H. A loxodromic elementary group is always virtually cyclic and is 2-ended. A parabolic subgroup of rank bigger than one is a one-ended group.

Notation. *If C is elementary infinite, it is contained in a unique maximal elementary subgroup of G. This subgroup will be denoted $\tilde{C}$ throughout the paper.*

A finitely generated Kleinian group G is *geometrically finite* if there exists an $\varepsilon > 0$ so that the hyperbolic volume of an ε-neighborhood of $C(\Lambda(G))/G$ is finite, where $C(\Lambda(G)) \subset \mathbb{H}^n$ is the convex hull of the limit set of G (i.e. the smallest convex subset of $\mathbb{H}^n$ invariant under the G-action) is finite.

We say that G *splits* as a graph of groups $X_* = (X, (C_e)_{e \in X^1}, (G_v)_{v \in X^0})$ (where C_e and G_v denote respectively edge and vertex groups of the graph X) if G is isomorphic to the fundamental group $\pi_1(X_*)$ (in the sense of Serre [Se]). The Bass-Serre tree T is the universal cover (in the sens of Serre) of the graph $X = T/G$. When X has only one edge, we will say that G splits as an amalgam (resp. an HNN-extension) if X has two vertices (resp. one vertex). We will need the following definitions:

Definition 2.1. *Let G act on a tree T. A subset H of G is said elliptic (respectively hyperbolic) on T if H fixes a point in T (respectively does not fix a point on T); if T is the Bass-Serre tree of a splitting of G as a graph of groups, H is elliptic if and only if it is conjugate into a vertex group of this graph.*
We say that G splits relatively to a family of subgroups $(E_1, ...E_n)$, or that $(G, \mathcal{E})$ splits as a graph of groups, if G splits as a graph of groups in which all the groups E_i's are elliptic.

Definition 2.2. *Suppose G splits as a graph of groups*

$$G = \pi_1(X, C_e, G_v) \tag{1}$$

so that the edge groups (i.e. the groups C_e's) of this graph are elementary. We say that the edge stabilizer C_e is essentially non-maximal if it is of infinite index in the maximal elementary subgroup $\tilde{C}_e \subset G$ containing C_e and $\tilde{C}_e$ is not conjugate into one of the vertex groups of the splitting (1). In other words, C_e is essentially non-maximal if the group $\tilde{C}_e$ is not elliptic in the splitting (1). The splitting (1) is called essentially non-maximal if there exists at least one such an edge. In the opposite case we call the splitting (1) essentially maximal.

It turns out that the property to split (or not to split) over essentially non-maximal subgroups is closely related to the co-Hopf property mentioned in the Introduction.

Theorem A. *Let $G \subset \mathrm{Isom}_+\mathbb{H}^n$ be a non-elementary, geometrically finite, one-ended Kleinian group without 2-torsion. Then G is cohopfian if and only if the following two conditions are satisfied:*

1) *G does not have essentially non-maximal splittings.*

2) *G does not split as an amalgam $G = A *_C \tilde{C}$, so that $\tilde{C}$ is a maximal elementary subgroup of G and the normal subgroup of $\tilde{C}$ generated by C is of infinite index in $\tilde{C}$.*

■

Note that if C is a non-trivial essentially non-maximal subgroup of a Kleinian group G, then $|\tilde{C} : C| = \infty$, and so C is a parabolic subgroup of G, and rank $(C) <$ rank $\tilde{C}$.

Corollary 2.3. *Suppose G is a non-elementary, geometrically finite, one-ended Kleinian group without 2-torsion. Suppose that every elementary subgroup C over which G splits has a finite index in the maximal elementary subgroup $\tilde{C}$ containing it, then G is cohopfian.*

As explained in the Introduction, the proof of Theorem A involves the study of the relative case.

Definition 2.4. *Let G be a group endowed with a family $\mathcal{E} = (E_1, ..., E_n)$ of elementary subgroups. An endomorphism of G is called endomorphism of the pair $(G, \mathcal{E})$ if it sends each E_i into itself.*

The pair $(G, \mathcal{E})$ is called cohopfian, if any injective endomorphism of $(G, \mathcal{E})$ is surjective. We say that the pair $(G, \mathcal{E})$ is one ended if G does not split over finite subgroups relatively to the system $\mathcal{E}$.

Theorem B. *Let $G \subset \mathrm{Isom}_+\mathbb{H}^n$ be a non-elementary, geometrically finite, Kleinian group without 2-torsion and $\mathcal{E} = \{E_1, ..., E_k\}$ be a family of elementary subgroups of G. Suppose that the pair $(G, \mathcal{E})$ is one-ended. Then $(G, \mathcal{E})$ is cohopfian if the following two conditions are satisfied:*

1) *The pair $(G, \mathcal{E})$ does not have essentially non-maximal splittings over elementary subgroups.*

2) *The pair $(G, \mathcal{E})$ does not split as an amalgam $G = A *_C \tilde{C}$, so that $\tilde{C}$ is a maximal elementary subgroup of G and the normal subgroup of $\tilde{C}$ generated by C is of infinite index in $\tilde{C}$.* ■

Remark: *If the family $\mathcal{E}$ in Theorem B is empty or all groups E_i are finite the sufficient condition of Theorem A follows from Theorem B.*

The condition of the essential non-maximality of a splitting introduced above is related to the following *acylindricity* condition of splittings introduced by Z. Sela in the torsion free case and in the paper [De] in the general case:

Definition 2.5. *Let G split as a graph of groups $G = \pi_1(X)$ with elementary edge stabilizers and T be the Bass-Serre tree corresponding to this splitting.*

a) *Torsion free case: The splitting (and the tree T) is called K-acylindrical if the stabilizer of each segment on T of the diameter at least K is trivial.*

b) *General case: The G-tree T is called (K, Φ)-acylindrical if the stabilizer of each segment on T of the diameter at least K is a finite group*[1].

   If G splits as a graph of groups $G = \pi_1(X)$, one says that this splitting is $(K, \Phi)-$acylindrical if the Bass-Serre tree - the universal cover of X - is the $(K, \Phi)-$acylindrical.

■

The condition for a group G not to have essentially non-maximal splittings turns out to be equivalent to the property that all its splittings over elementary subgroups are (K, Φ)-acylindrical for some uniform constant K (see Lemma 4.1 in Section 4).

[1] Here Φ stands for "finite".

3 Accessibility of Finitely Presented Groups

In this section we collect some results about different versions of the accessibility (acylindrical and strong) of abstract finitely presented groups. These results will be applied in the next section to geometrically finite Kleinian groups.

Throughout this section G will denote an abstract (not necessarily Kleinian) group. Recall that the Bass-Serre G-tree $\hat{T}$ is called a *resolution* of the G-tree T if there exists G-equivariant simplicial map $\rho : \hat{T} \to T$.

If T is a G-tree and $\varphi : G \to G$ is a monomorphism of G into itself let $\varphi^* T$ denote the G-tree defined as follows: as the set $\varphi^* T$ is T, but the action of G on T obtained from the previous one by composing with φ:

$$g_{\varphi^* T}(x) = \varphi(g)_T(x). \tag{2}$$

The stabilizer of a vertex v (edge e) of the tree $\varphi^* T$ is equal to $\varphi^{-1}(G_v)$ (respectively $\varphi^{-1}(C_e)$) where G_v (respectively C_e) is the stabilizer of v (respectively e) on T.
Following [BeF1] we call a G-tree irreducible if it is minimal (i.e. there is no proper invariant subtree) and if the label of every vertex of valence two properly contains the labels of both edges incident to it (if the two edges are distinct).

We will further consider decompositions of finitely presented groups over so called elementary subgroups which we now define axiomatically:

Definition 3.1. *Let G be a finitely presented group and $\mathcal{C}$ a fixed family of subgroups of G. We call the family $\mathcal{C}$ and every element $C \in \mathcal{C}$ elementary if the following axioms are satisfied:*

(1) If $C \in \mathcal{C}$ then every subgroup and every conjugate of C is in $\mathcal{C}$.

(2) Every infinite subgroup belonging to $\mathcal{C}$ is contained in a unique maximal subgroup $\tilde{C}$ so that $\tilde{C} \in \mathcal{C}$. The union of an ascending sequence of finite elementary groups is elementary.

(3) Every subgroup of $\mathcal{C}$ satisfies the following fixed-point condition: whenever C acts on a simplicial tree τ, C preserves a point in τ, or a point on its ideal boundary $\partial\tau$ or a pair of points on $\partial\tau$ (possibly permuting them).

(4) If $C \in \mathcal{C}$ is an infinite maximal elementary subgroup then its normalizer in G is contained in $\mathcal{C}$, i.e. $gCg^{-1} = C$ implies that $g \in C$ for all $g \in G$.

Examples of elementary families are well-known in the geometry of negatively curved spaces. Namely, discrete subgroups of the hyperbolic space $\mathbb{H}^n$ or, more generally Hadamard spaces with a pinched negative curvature, are elementary in the classical sense if their limit set is a finite set. In this case they are also elementary according to our axioms $(1-4)$. Indeed, the properties (1), (2) and (4) are easy exercises, the only property which is non-trivial is the axiom (3) which follows from Margulis' lemma saying that every such group is virtually nilpotent (abelian in the constant curvature case) and from Tits' theorem [Ti] implying that every virtually nilpotent group satisfies (3).

Another important example one obtains by considering elementary subgroups (i.e. virtually cyclic) of word-hyperbolic (Gromov) groups which are also elementary according to the axioms (1-4).

We are going now to study hierarchies (i.e. strong accessibility) of finitely presented groups over elementary subgroups. A finite hierarchy of G over elementary subgroups is now defined as follows ([De-Po]):

Definition 3.2. *Let G be a group and $\mathcal{C}$ a family of elementary subgroups of G. Let $G = G^0$. If G do not split as an amalgam or an HNN-extension over a subgroups in $\mathcal{C}$, we stop. If not, we split $G^0 = G_1^1 *_C G_2^1$, or $G = G_1^1 *_C$ (in this case, set $G_2^1 = G_1^1$). Repeating the splitting procedure for G_1^1 and G_2^1 over elementary subgroups, we get groups $G_1^2, G_2^2, G_3^2, G_4^2$, and so on. A finite hierarchy for G is a splitting procedure which terminates, i.e. there is a step i such that none of the groups G_j^i splits as an amalgam or an HNN-extension over a group in $\mathcal{C}$.*

We will say that the group G has a finite hierarchy relatively to the sub-system $\mathcal{E}$ of $\mathcal{C}$ (in other words, $(G, \mathcal{E})$ has a hierarchy) if G has a finite hierarchy over elementary subgroups such that all the subgroups in $\mathcal{E}$ are elliptic in every decomposition appearing in this hierarchy.

The length of the hierarchy is the number of decompositions needed to reach the final one (i.e. when all vertex groups are indecomposable over subgroups in $\mathcal{C}$ or are in $\mathcal{C}$). We define then the number $l(G)$ to be the minimal number of the lengths among all finite hierarchies of G. Similarly $l(G, \mathcal{E})$ denotes the minimal number of the lengths of all hierarchies of G relatively to $\mathcal{E}$.

The proof of the next Theorem closely follow the proof of the main Theorem 3.6 of the paper [De-Po] by keeping track of the peripheral system $\mathcal{E}$:

Theorem C. *Suppose G is a finitely presented group without 2-torsion and $\mathcal{C} \subset G$ is an elementary family satisfying the above axioms $(1)-(4)$. Let $\mathcal{E} = \{E_1, ...E_k\}$ be a fixed finite subset of $\mathcal{C}$ then $(G, \mathcal{E})$ has a finite hierarchy over elementary subgroups.*

In other words, there exists a decomposition of $(G, \mathcal{E})$ as an amalgam (or an HNN-extension)

$$G = A *_C B, \ (G = A*_C),$$

such that

$$\max\{l(A, A \cap \mathcal{E}), l(B, B \cap \mathcal{E})\} < l(G, \mathcal{E}). \tag{3}$$

Proof: Recall that in order to prove Theorem 3.2 in [De-Po] we used a version of an invariant $c(\cdot)$ (called *complexity*) of finitely presented groups appeared first in [De1]. This notion can be easily adapted to the "relative" case. Namely, consider a simplicial developable orbihedron Π of dimension 2 whose fundamental group is G (see [Ha]) such that the vertex stabilizers of Π are in $\mathcal{C}$ and every subgroup E_i fixes a vertex $x_i \in \Pi$ $(i = 1, ..., k)$. We define first $c(\Pi, \mathcal{E})$ to be the pair $(T(\Pi), b_1(\Pi))$, where $T(\Pi)$ is the number of 2-dimensional faces of Π and $b_1(\Pi)$ is the first Betti number of the underlying topological space of Π. Then $c(G, \mathcal{E})$ is defined to be the infimum (for the lexicographical order) over all such G-orbihedra Π.

If now T is a $(G, \mathcal{E})$-tree, the main result of [De-Po, Theorem 3.2] produces a simplicial tree $\hat{T}$ and a resolution $f : \hat{T} \to T$ so that the invariant $c(\cdot)$ of the vertex stabilizers of $\hat{T}$ strictly decreases. All we need now is to check that the groups E_i are still elliptic on the tree $\hat{T}$. To see this consider the orbihedron universal cover P of such a complex Π. The Axioms of Definition

3.1 allow one to construct a G-equivariant map $\rho : P \to T \cup \partial T$ (see [De-Po, 4.1]). Recall that the tree $\hat{T}$ is constructed to be the dual tree to the lamination $\Lambda \subset P$ whose leaves are pre-images under ρ of the midpoints of the edges of T. Let $E_i \in \mathcal{E}$ be an elementary subgroup which fixes a vertex $x_i \in P$. By assumption it also fixes a vertex v_i in the tree T. As the map f is equivariant every element $g \in E_i$ stabilizes a component Ω_i of $P \setminus \Lambda$ which contains x_i. Thus the group E_i is contained in the stabilizer $G_{\hat{v}_i}$ of the vertex $\hat{v}_i$ corresponding to the component Ω_i which is a vertex stabilizer of $\hat{T}$. The result now follows by repeating the argument of [De-Po, Theorem 3.6]. ■

We will now establish the following finiteness Theorem for acylindrical splittings over elementary subgroups :

Acylindricity Theorem (acylindrical superaccessibility relative to a subset). *Let G be a finitely presented, and $E_1, ..., E_q$ a fixed finite set of infinite elementary subgroups of G. Suppose that the pair $(G, \mathcal{E})$ is one-ended and there is a finite bound for orders of finite subgroups of G. Then for each $K \in \mathbb{R}$ there exists a finite number of G-trees $T_1, ..., T_M$ such that all subgroups E_i are elliptic on T_j, and for every minimal (K, Φ)-acylindrical G-tree T for which E_i are elliptic on T, there exist an automorphism φ of G sending each group E_i into itself and a resolution $\varphi^*(T_i) \to T$ $(i \in \{1, ..., M\})$.*

Remark. This Theorem in the torsion-free case (i.e. for K-acylindrical splittings) in the absolute form (i.e. without the claim about subgroups E_i) was proved by Z. Sela [Se1]. Then T. Delzant [De] showed it for (K, Φ)-splittings. Following the argument of the paper [De] we are going to extend the result and to show that the automorphism φ sends subgroups E_i into themselves provided that one considers (K, Φ)-acylindrical splittings of G in which the subgroups E_i are elliptic $(i = 1, ..., q)$.

Proof: We will briefly sketch the construction of the paper [De] emphasizing the presence of the subgroups E_i. Let Π be finite two dimensional CW-complex with $\pi_1(\Pi) \cong G$ all of whose 2-faces are either digons or triangles. Suppose also that Π contains subcomplexes B_i $(i = 1, ..., q)$ whose fundamental groups are isomorphic to E_i. Following the paper [De] we construct a G-equivariant simplicial map $\rho : P \to T$ where P is the universal cover of Π. Let $\tilde{\Lambda}$ denote a lamination of Π whose leaves are preimages under ρ of the middle points of the edges of T. By construction, $\tilde{\Lambda}$ is a G-equivariant lamination and let Λ denote $\tilde{\Lambda}/G$. We now define a subgraph Λ_k of Λ by describing its intersection with each face Δ of Π. Namely $\Delta \cap \Lambda_k$ are those leaves of Λ in Δ whose image under ρ is situated within a distance at least k from the images of the vertices or the center of Δ. It is proven in [De] (see Lemma 1.5) that the fundamental group of each connected component of Λ_k fixes pointwise a segment of the length k when acting on T. It follows by assumption that the fundamental group of each connected component of Λ_k is finite.

We now collapse all leaves of Λ_k as well as all complexes B_i to points. As the number of faces of Π and leaves in $\Lambda \setminus \Lambda_k$ is uniformly bounded one can show that the number of faces and edges of the resulting orbihedron Π' is uniformly bounded too (here one uses essentially the minimality of the tree T see [De, Lemmas 2.1, 2.2]). Each vertex stabilizer of Π' is either finite or is one of the groups E_i. As the orders of finite subgroups of G are uniformly bounded, there are only finitely many orbihedrons with all these properties, so Π' must belong to a finite set of orbihedrons $\{\Omega_1, ..., \Omega_M\}$, with M depending only on the group G and the system of its subgroups E_i $(i = 1, ..., q)$.

There exists a simplicial map θ_k between the complexes Π' and Ω_k (for some $k \in \{1, ..., M\}$). This map induces the isomorphism $(\theta_k)_* : G \to \pi_1^o(\Omega_k)$ where $\pi_1^o(\Omega_k)$ is the orbihedron fundamental group of Ω_k. Notice that θ_k lifts to an equivariant map $\tilde{\theta}_k$ between P' and $\tilde{\Omega}_k$ which are the orbihedron universal covers of Π' and Ω_k correspondingly. If $\tilde{\theta}_k(x_i) = y_j$ where the stabilizer of the point $x_i \in P'$ is E_i and $y_j \in \Omega_k$ $(i, j \in \{1, ..., q\}, k \in \{1, ..., M\})$ then we have $(\theta_k)_*(C_j) \subset \mathrm{Stab}(y_j) = C_j$. Up to taking a power of θ_k and retaining the notation we may suppose that $(\theta_k)_*(E_i) \subset E_i$. Following [De, Thm 3.1] denote by $\hat{\tau}$ the dual tree to the lamination which is the image of the lamination Λ in P' and let T_k be the same for the orbihedron Ω_k. Similarly to the proof of Theorem C we obtain that the groups E_i are all elliptic on the tree $\hat{\tau}$ and there is an equivariant simplicial map $\hat{\tau} \to T_k$. The actions of the groups G and $\pi_1^o(\Omega_k)$ on the trees $\hat{\tau}$ and T_k respectively are conjugate by the map θ_k. Thus we find that $\hat{\tau} = {\theta_k}^*(\tau_k)$ and the Theorem follows. ■

4 Splittings of Kleinian Groups

We start with the following important

Lemma 4.1. *Let G be a finitely presented Kleinian group and $\mathcal{E} = \{E_1, ..., E_k\}$ be a family of elementary subgroups of G. Suppose that the pair $(G, \mathcal{E})$ is one-ended. Then the pair $(G, \mathcal{E})$ does not have essentially non-maximal splittings if and only if each irreducible $(G, \mathcal{E})$-splitting over elementary subgroups is (K, Φ)-acylindrical for some uniform constant $K > 0$ depending only on G. Furthermore, every essentially non-maximal splitting of $(G, \mathcal{E})$ as an amalgam or an HNN-extension is $(3, \Phi)$-acylindrical.*

Proof: To prove the sufficiency we use essentially the same argument as in the papers [De] and [Se]. Suppose that the pair $(G, \mathcal{E})$ has no essentially non-maximal splittings and let G act on a simplicial tree T with elementary edge stabilizers. Then G splits as the graph of groups $X = T/G$. Let m denote the number of edges of X. We will first show that the tree T is $2m+1$-acylindrical. To this end suppose that l is an embedded path in T consisting of n successive edges such that $n \geq 2m+1$. We want to show that the stabilizer C of l is a finite group. Arguing by contradiction suppose that the group C is infinite. As $n \geq 2m + 1$ the path l contains at least three distinct edges e_1, e_2, e_3 in the same G-orbit. Let C_i be the stabilizer of the edge e_i and let α_i and α_i' be its vertices $(i = 1, 2, 3)$. Let $e_2 = g(e_1)$ and $e_2 = h(e_3)$ for some elements g and h not belonging to C_1 and C_3 respectively. We have $C \subset \bigcap_{i=1}^{3} C_i$ and $C_2 = gC_1g^{-1}, \ C_2 = hC_3h^{-1}$.

As $g^{-1}C_2g \cap C_2 \supset C$ and C is infinite we deduce that $g^{-1}\tilde{C}_2g = \tilde{C}_2$ where $\tilde{C}_2$ is the unique maximal elementary subgroup of G containing C_2; the same property holds for h. Then the elements g and h belong to $\tilde{C}_2$ which also contains C. As G does not have essentially non-maximal splittings, it follows that $\tilde{C}_2$ fixes a point on the tree T and so there is a vertex $v \in T$ whose stabilizer D contains $\tilde{C}_2$.

Let $[\alpha_i', \alpha_{i+1}]$ denote the segment of the path l between the vertices α_i' and α_{i+1}. A standard argument [Se, I-6.4] shows that either the element g fixes a point x in $[\alpha_1', \alpha_2]$ or g acts on

T without fixed points. We have already shown that the latter case is impossible. Similarly, the element h fixes a point $y \in [\alpha_2', \alpha_3]$. Now their common fixed point v belongs to the same connected component of $T \setminus e_2$ as one of the vertices x or y, say x. Then the element g fixes pointwise the path in T between the vertices y and v which contains the edge e_1, and so $g \in C_1$ which is impossible. Thus $n \leq 2m$. In particular, if the graph X contains only one edge the splitting X (i.e. amalgam or an HNN-extension) is $(3, \Phi)$-acylindrical.

By the result of M. Bestvina-M. Feighn [BeF2] there is a uniform upper bound $\nu(G) < \infty$ for the number of edges of all irreducible splittings of G with elementary edge stabilizers. Thus, setting $K = 2\nu(G) + 1$ we obtain the result. The necessary condition is proved.

Suppose now that the group has an essentially non-maximal splitting $G = \pi_1(X, C_e, G_v)$ relatively to the system $\mathcal{E}$. As the pair $(G, \mathcal{E})$ is one ended, every edge group $C_e = C$ of X is an infinite elementary subgroup of infinite index in $\tilde{C}$. Therefore $\tilde{C}$ is an infinite parabolic subgroup of G, and $\tilde{C}$ does not fix any point in T - universal cover of X. Since the group $\tilde{C}$ is virtually abelian it then follows from [S, 6.5, Proposition 27] that there is an element t in $\tilde{C}$ acting hyperbolically on T. The group $\tilde{C}$ contains an abelian subgroup of finite index C' and, so there exists $k \in \mathbb{N}$ that $t^k \in C'$, and t^k centralizes the group $C_0 = C \cap C'$. Therefore, the group generated by C_0 and t^k fix also the edges $e, t^k(e), ..., t^{nk}(e), ...$, and, hence a segment of arbitrary big length. We get that the graph X is not (K, Φ)-acylindrical for any $K \in \mathbb{N}$. The Lemma follows. *QED.*

The Acylindricity Theorem proved in the previous Section will be used to obtain the following finiteness result about splittings of Kleinian groups:

Theorem D. *Let $G \subset \mathrm{Isom}_+\mathbb{H}^n$ be a non-elementary, geometrically finite, Kleinian group without 2-torsion and $\mathcal{E} = \{E_1, ..., E_k\}$ be a family of elementary subgroups of G. Suppose that the pair $(G, \mathcal{E})$ is one-ended and does not have essentially non-maximal splittings over elementary subgroups. Then up to an automorphism of $(G, \mathcal{E})$ there exist at most finitely many minimal splittings of G over elementary subgroups.*

Proof: The argument is essentually due to Sela [Se]. Suppose that $\rho : \hat{T} \to T$ is a resolution, then according to [BeF1] there exists a sequence of Stallings folds:

$$\hat{T} = T_0 \to T_1 \to T_2 \to ... \to T_m.$$

There are several types of folds, each of these operations either strictly decreases the number of edges of T_i (modulo G) or strictly increases the edge stabilizer of the edge along which the folding is done. Now for any edge e we have either $\mathrm{rank}(C_{\rho(e)}) > \mathrm{rank}(C_e)$ or $|C_{\rho(e)} : C_e| < \infty$. In both cases the group $C_{\rho(e)}$ contains C_e and is contained in the maximal elementary subgroup $\tilde{C}_e$. We note that if $\tilde{E} \subset G$ is a maximal elementary subgroup containing E then there does not exist an infinite ascending sequence of distinct subgroups of $\tilde{E}$ (one easily checks this first for a maximal abelian subgroup of finite index in $\tilde{C}_{m+k}$ and then deduces it for $\tilde{C}_{m+k}$). Consequently one can not make infinitely many Stallings folds without changing the rank of the subgroup $E = C_e$. However rank $E_i \leq$ rank $\tilde{\mathrm{E}} < \infty$. Thus, for a given edge e the number of Stallings folds is finite. As the number of edges of the tree T_0 is finite modulo G-action, the number of G-trees obtained from T_0 by doing Stallings folds is also finite.

Lemma 4.1 now yields that every G-splitting over elementary subgroups is $(3,\Phi)$-acylindrical. It follows from the Selberg Lemma [Ra] that $G \subset GL_n(\mathbb{C})$ contains a torsion free subgroup of finite index ϕ; in particular every finite subgroup of G has order less than ϕ. Then by the Acylindricity Theorem every (K,Φ)-splitting of G over elementary subgroups can be obtained from one of the trees T_i by doing a finite number of Stallings folds and up to an automorphism of G preserving the groups C_j. Combining all these facts we conclude that there are at most finitely many irreducible splittings of G over elementary subgroups. Theorem D is proved. ■

One more accessibility result which we are now going to prove will deal with refined sequences of splittings defined as follows:

Definition 4.2. *Let $\mathcal{F}$ be graph of groups decomposition of the pair $(G,\mathcal{E})$. We say that the graph $\mathcal{F}_1$ refines $\mathcal{F}$ if it is obtained from $\mathcal{F}$ by replacing of a vertex $v \in \mathcal{F}^0$ by a* **non-trivial** *graph of groups decomposition $\mathcal{F}_v$ of the pair $(G_v, \mathcal{E} \cap S)$, where S is the set of edge groups of $\mathcal{F}$.*

A sequence $\{\mathcal{F}_n\}$ of graphs of groups decompositions of $(G,\mathcal{E})$ is called a refining sequence if for every n the graph $\mathcal{F}_{n+1}$ refines $\mathcal{F}_n$. We say that refining sequence $\{\mathcal{F}_n\}$ stabilizes if the exists n_0 such that $\mathcal{F}_n = \mathcal{F}_{n_0}$ for all $n > n_0$.

Proposition 4.3. *Suppose G is a finitely presented Kleinian group and $\mathcal{E}$ is a fixed sub-system of elementary subgroups of G. Suppose that the pair $(G,\mathcal{E})$ is one-ended and has no essentially non-maximal splittings. Then every refining sequence of decompositions of $(G,\mathcal{E})$ stabilizes.*

Remark 4.4. *We thank M. Bestvina who informed us about a result of this type. In the paper [Bo] a similar statement is proved (Theorem 6.1) by a different method.*

Proof: Suppose $\{\mathcal{F}_n\}$ is a sequence of decompositions of the pair $(G,\mathcal{E})$ so that the graph $\mathcal{F}_{n+1}$ is obtained from $\mathcal{F}_n$ by doing an elementary refinement; i.e. the label of some vertex v of $\mathcal{F}_n$ is replaced by an elementary splitting $A *_C B$ or $A*_C$, in which all the edge groups of the graph $\mathcal{F}_n$ are elliptic. We have natural embedding $\mathcal{F}_m \subset \mathcal{F}_n$ if $n > m$. Collapsing of a vertex is the inverse operation to the refinement. A vertex v of valence two of a graph of groups of $(G,\mathcal{E})$ is said to be trivial if its label coincide with the label of one of the edges incident to it.

Bestvina-Feighn's Accessibility Theorem [BeF2] ensures that there exists m such that all vertices in $\mathcal{F}_n \setminus \mathcal{F}_m$ are trivial for $n > m$. Indeed, if it is not so then collapsing all trivial vertices in the graph $\mathcal{F}_n$ we will get an irreducible graph of groups decomposition of $(G,\mathcal{E})$ with elementary edge stabilizers having an unbounded number of vertices (when $n \to +\infty$) which is prohibited by [BeF2]. Therefore if the sequence $\{\mathcal{F}_n\}$ does not stabilize then for some vertex group A_m of $\mathcal{F}_m$ we will have an infinite chain of elementary refinements:

$$A_m = A_{m+1} *_{C_{m+1}} C_m, \ A_{m+1} = A_{m+2} *_{C_{m+2}} C_{m+1}, \ \dots \ A_{m+k} = A_{m+k+1} *_{C_{m+k+1}} C_{m+k} \ \dots \ , \quad (4)$$

where C_{m+k} is an infinite elementary subgroup of G as G splits over it and is one ended. One has $C_{m+k} \supsetneqq C_{m+k+1}$ (as the splittings (4) are all non-trivial). It also follows that for all but finitely many indices we have $|C_{m+k} : C_{m+k+1}| < \infty$ as the rank of the maximal elementary group $\tilde{C}_{m+k}$ is finite. We deduce from (4):

$$A_m = C_m *_{C_{m+1}} C_{m+1} *_{C_{m+2}} C_{m+2} * \ \dots \ *_{C_{m+k}} C_{m+k} * \ \dots \ ,$$

and further

$$A_m = A_{m+1} *_{C_{m+1}} C_m; \;\; A_m = A_{m+2} *_{C_{m+2}} C_m; \;\; \dots \;\; A_m = A_{m+k} *_{C_{m+k}} C_m \;\dots\; . \tag{4'}$$

Let us first show that every decomposition of A_m in $(4')$ is essentially maximal. We shall check this for instance for the first one. Let T_{m+1} denote the G-tree - the universal cover of the graph $\mathcal{F}_{m+1}$. Since G has no essentially non-maximal splittings, the maximal elementary subgroup $\tilde{C}_{m+1}$ of G containing C_{m+1} fixes a point, say $w \in T_{m+1}$. Up to conjugation we may suppose that there is an edge e in T_{m+1} whose stabilizer is C_{m+1} and the vertices α and β of the edge e are stabilized by the groups A_{m+1} and C_m correspondingly. Denote by C'_{m+1} the maximal elementary subgroup of A_m containing C_{m+1} ($C'_{m+1} = \tilde{C}_{m+1} \cap C_{m+1}$). As w belongs to a component of $\mathcal{F}_{m+1} \setminus e$ which contains α or β, say α, we obtain that C'_{m+1} fixes a path between w and β pointwise. This implies that $\tilde{C}_{m+1} \subset A_{m+1}$ which shows that the splitting $A_m = A_{m+1} *_{C_{m+1}} C_m$ is essentially maximal.

Then by Lemma 4.1 every elementary splitting in (4') is $(3, \Phi)$-acylindrical, moreover it is obviously minimal. The finiteness Theorem D for such splittings now tells that up to automorphisms of A_m there are only finitely many different splittings in $(4')$. Therefore, there exist indices k and l and an automorphism φ of A_m so that $\varphi(A_{m+k}) = A_{m+l}$, $\varphi(C_{m+k}) = C_{m+l}$, $\varphi(C_{m+k-1}) = C_{m+l-1}$ $(k < l)$. As the group C_{m+l} is infinite and $C_{m+k} \supset C_{m+l}$, we must have $\tilde{C}_{m+k} = \tilde{C}_{m+l}$ for the maximal elementary subgroups. It follows that $\varphi(\tilde{C}_{m+k}) = \tilde{C}_{m+l} = \tilde{C}_{m+k}$, and whence $C_{m+k} = C_{m+l}$. Indeed, otherwise the virtually abelian group $\tilde{C}_{m+k}$ would contain an infinitely increasing sequence of subgroups $\varphi^{-n}(C_{m+l})$ which is impossible. This is a contradiction. The Proposition is proved. ■

We call an edge in a graph of groups decomposition of G *trivial* if its label coincide with the labels of both its vertices.

Definition 4.5. *Let G be a finitely presented group endowed with a system $\mathcal{C}$ of elementary subgroups. We define the number $\nu(G, \mathcal{E})$ to be the maximal number of non-trivial edges among all $(G, \mathcal{E})$-splittings with elementary edge stabilizers, where $\mathcal{E} \subset \mathcal{C}$ is a fixed sub-system of elementary subgroups.*

Corollary 4.6. *Let G be a finitely presented group satisfying the assumptions of Proposition 4.3. Suppose it has a non-trivial $(G, \mathcal{E})$-splitting $G = \pi_1(X, G_v, C_e)$ then for every vertex $v \in X$ we have:*

$$\nu(G_v, \mathcal{E} \cup C) < \nu(G, \mathcal{E}).$$

Proof: Proposition 4.3 implies that $\nu(G_v, \mathcal{E} \cup C) < +\infty$. Thus we can find a $(G_v, \mathcal{E} \cup C)$-splitting $\mathcal{F}_A$ having the maximal number $\nu(G_v, \mathcal{E} \cup C)$ of non-trivial edges. We then refine the splitting X by $\mathcal{F}_A$. As the splitting X has at least one non-trivial edge, this refined splitting is a $(G, \mathcal{E})$-splitting which contains one more edge. *QED.*

Let G be a non cohopfian Kleinian group, and $f : G \to G$ is an injective non surjective endomorphism, then the result of the paper [OP] implies that G admits a non-trivial action on a simplicial tree with elementary edge stabilizers. The following Proposition provides a relative version of this result:

Proposition 4.7. *Let $G \subset \mathrm{Isom}_+\mathbb{H}^n$ be a non-elementary, geometrically finite Kleinian group and $\mathcal{E} = \{E_1, ..., E_k\}$ is a finite family of elementary subgroups of G. Suppose that the pair $(G, \mathcal{E})$ is non-cohopfian and let $f : (G, \mathcal{E}) \to (G, \mathcal{E})$ be an injective endomorphism which is not surjective. Then G has a non-trivial splitting over elementary subgroups relatively to $\mathcal{E}$.*

Proof: We may assume (w.l.o.g.) that all the subgroups E_i are infinite maximal elementary subgroups of G. For any finite system S of generators of G one can consider the following displacement function:

$$d_m(f, S, G) = \min_{x \in \mathbb{H}^n} \max_{s \in S} \ d_{\mathbb{H}^n}(x, f^m(s)(x)), \tag{5}$$

where $d_{\mathbb{H}^n}(\cdot)$ denote the hyperbolic distance of $\mathbb{H}^n$.

It is proved in the paper [OP] that if the map f is not surjective then for any generating system S the function $d_m(f, S, G)$ is not bounded ($m \in \mathbb{N}$). In this case by the theorem of Bestvina-Paulin [Be], [Pa], the group G acts stably and non-trivially on a real tree $T_\mathbb{R}$ with elementary edge stabilizers. Furthermore, it is proven in [Be], [Pa] that

$$\overline{\lim}_{m\to\infty} \frac{l(f^m(g))}{d_m(f, S, G)} = L_\mathbb{R}(g), \tag{6}$$

where $l(g) = \inf d_{\mathbb{H}^n}(x, g(x))$ and $L_\mathbb{R}(g) = \inf d_{T_\mathbb{R}}(x, g(x))$ are the translation lengths in the hyperbolic space $\mathbb{H}^n$ and on the tree $T_\mathbb{R}$ respectively.

By Rips' theorem [BeF1] it follows that there is a non-trivial simplicial G-tree T_∞ with elementary edge stabilizers, which will be further called the limit tree of G corresponding to the map f.

If $E \in \mathcal{E}$ is a subgroup such that $f^m(E)$ is a parabolic subgroup ($\forall\ m > m_0$) then it follows from (6) that E fixes a point in the tree $T_\mathbb{R}$ and then the relative version of Rips' theorem [BF1, Theorem 9.6] implies that E fixes a point in the limit tree T_∞.

Arguing by contradiction suppose that there exists an element $\gamma \in E_i$ which acts hyperbolically on T_∞. Thus for every $m \in \mathbb{N}$ the group $f^m(E_i)$ is an infinite virtually cyclic loxodromic group (in fact $f^m(E_i) \subset E_i$ and E_i is itself an infinite virtually cyclic loxodromic subgroup of G). Let us choose a generating system $S = \{\gamma, \gamma_1, ..., \gamma_k, a_1, ..., a_l\}$ of G where γ_i are generators of subgroups E_j and the elements a_j do not belong to $\mathcal{E}$.

We first claim that there exists an element $b \in G \setminus \mathcal{E}$ such that all but finitely many elements $f^m(b)$ are not parabolic. To see this, choose α to be an element of infinite order such that $\alpha \notin \mathcal{E}$ and set $b = \alpha\gamma\alpha^{-1}$. Then up to replacing α by some power if needed, we may assume that the element b does not belong to $\mathcal{E}$. Indeed, otherwise there exists $k \in \mathbb{N}$ such that α^k normalizes E_i and so $\alpha^k \in E_i$ by the maximality of E_i, which is impossible by the choice of α. As γ acts hyperbolically on the tree T_∞, the element b (being a conjugate of γ) acts hyperbolically on T_∞ too. So by the above argument all but finitely many elements $b_m = f^m(b)$ are loxodromic and we have $l(b_m) \to \infty$. Moreover, $b_m \notin \mathcal{E}$ since $f(\mathcal{E}) \subset \mathcal{E}$. We add the element b to the generating system retaining for it the same notation S.

Up to passing to a subsequence, we may choose an element $g \in S$ and a point $x_m \in \mathbb{H}^n$ which realizes the mini-max in (5):

$$d_m(f, S, G) = d_{\mathbb{H}^n}(x_m, f^m(g)(x_m)).$$

As the element $\gamma \in \mathcal{E}$ acts hyperbolically on T_∞ the quantity

$$L_{\mathbb{R}}(\gamma) = \lim_{m\to\infty} \frac{l(f^m(\gamma))}{d_{\mathbb{H}^n}(x_m, f^m(g)(x_m))},$$

is strictly positive by the relative version of Rips' theorem, so we have $0 < L_{\mathbb{R}}(\gamma) \leq 1$. Let A_γ denote the invariant axis of γ. As $f(E_i) \subset E_i$ we see that A_γ is the invariant axis for every group $f^m(E_i)$ ($m \in \mathbb{N}$), which will be important further. We have the following:

Lemma 4.8. *Suppose that $b \in G \backslash \mathcal{E}$ such that $l(b_m) \to +\infty$ ($m \to +\infty$) then $\mathrm{dist}_{\mathbb{H}^n}(A_\gamma, b_m(A_\gamma))$ is unbounded, where $\mathrm{dist}_{\mathbb{H}^n}(\cdot)$ denotes the hyperbolic distance between subsets of $\mathbb{H}^n$.*

Proof of the Lemma: Suppose not, and $\mathrm{dist}_{\mathbb{H}^n}(A_\gamma, b_m(A_\gamma)) \leq K < +\infty$ for some constant K. Let $w_m \in A_\gamma$ and $y_m \in b_m(A_\gamma)$ so that $d_{\mathbb{H}^n}(w_m, y_m) = \mathrm{dist}_{\mathbb{H}^n}(A_\gamma, b_m(A_\gamma))$. Let w'_m denote the point $b_m^{-1}(y_m) \in A_\gamma$, so $d_{\mathbb{H}^n}(w_m, b_m(w'_m)) \leq K$. Since the subgroup E_i acts co-compactly on the axis $A\gamma$ there exist elements c_m and c'_m in E_i so that $w_m = c_m(z_m)$, $w'_m = c'_m(z'_m)$, where $z_m,\ z'_m \in A_\gamma$ and $d_{\mathbb{H}^n}(z_m, z'_m) \leq K_1 < +\infty$ for some K_1. We deduce that $d_{\mathbb{H}^n}(z_m, c_m^{-1} b_m c'_m(z_m)) \leq K + K_1 < +\infty$. As the group G is discrete and $c_m^{-1} b_m c_m = c_m^{-1} f^m(b) c'_m \in G$ ($m \in \mathbb{N}$) it follows that $\exists m_0\ \forall m > m_0\ : c_m^{-1} b_m c'_{m_0} = c_{m_0}^{-1} b_{m_0} c'_{m_0}$ and, hence $l(b_m) = l(b_{m_0}) < +\infty,\ \forall\ m \geq m_0$. Thus we obtain a contradiction with the fact that $l(b_m) \to +\infty$. The Lemma is proved. ■

To finish the proof of the Proposition we are going to show that γ can not act hyperbolically on T_∞ contradicting our assumption. For the element $b = \alpha\gamma\alpha^{-1} \in G \setminus \mathcal{E}$, which satisfies the conclusion of the Lemma and the sequence $b_m = f^m(b)$ we have

$$0 < \lim_{m\to\infty} \frac{d_{\mathbb{H}^n}(x_m, b_m(x_m))}{d_m(f, S, G)} \leq 1,$$

where x_m is the point realizing the mini-max in (5). There are two cases according to whether or not the quantity $D_m = \mathrm{dist}_{\mathbb{H}^n}(x_m, A_{b_m})$ remains bounded (where A_{b_m} is the axis of the loxodromic element b_m).

Case 1. *D_m is unbounded.*

Since $b \in S$, so $d_{\mathbb{H}^n}(b_m(x_m), x_m) < d_m(f, S, G)$. We have up to passing to a subsequence:

$$\frac{l(b_m)}{d_m(f, S, G)} \leq e^{-D_m} \frac{d_{\mathbb{H}^n}(b_m(x_m), x_m)}{d_m(f, S, G)} \to 0,$$

implying that the element $b = \alpha\gamma\alpha^{-1}$ acts elliptically on T_∞ and, so is γ. A contradiction.

Case 2. *D_m is bounded.*

Choose $z_m \in A_{b_m}$ such that $d_{\mathbb{H}^n}(x_m, z_m) = \mathrm{dist}_{\mathbb{H}^n}(x_m, A_{b_m}) = D_m \leq \mathrm{const} < +\infty$. From Lemma 4.8 we obtain that the quantity $\mathrm{dist}_{\mathbb{H}^n}(A_{b_m}, A_\gamma)$ tends to infinity (up to a subsequence). Choosing a point $w_m \in A_\gamma$ realizing the distance $\mathrm{dist}_{\mathbb{H}^n}(x_m, A_\gamma)$ we have that $M_m = d_{\mathbb{H}^n}(w_m, x_m) \to +\infty$. For the elements $b_m = f^m(\alpha) f^m(\gamma) f^{-m}(\alpha)$ we have that $l(b_m) = l(f^m(\gamma))$ and thus:

$$\frac{l(b_m)}{d_m(f, S, G)} = \frac{l(f^m(\gamma))}{d_m(f, S, G)} = \frac{d_{\mathbb{H}^n}(w_m, f^m(\gamma)(w_m))}{d_m(f, S, G)} \leq e^{-M_m} \frac{d_{\mathbb{H}^n}(x_m, f^m(\gamma)(x_m))}{d_m(f, S, G)} \leq e^{-M_m} \to 0.$$

and as before we obtain that the element γ acts elliptically on the tree T_∞ contradicting our assumption. The Proposition is proved. ■

5 Finding of a G-tree invariant under endomorphism

Suppose that the pair $(G, \mathcal{E})$ is not cohopfian for a fixed finite family $\mathcal{E} = \{E_1, ..., E_k\}$ of elementary subgroups of G. Then Proposition 4.7 tells us that $(G, \mathcal{E})$ splits as an amalgam (or an HNN-extension) over an elementary subgroup. In the following Proposition we obtain infinitely many such G-trees:

Proposition 5.1. *Let G be a non-elementary, geometrically finite, Kleinian group without 2-torsion endowed with the system $\mathcal{E}$. Then the following assertions are true:*

1) *If $f : (G, \mathcal{E}) \to (G, \mathcal{E})$ is a non-surjective monomorphism. then there exists a $(G, \mathcal{E})$-tree τ so that for every $n \in \mathbb{N}$ the tree $f^{n*}(\tau)$ is a non-trivial $(G, \mathcal{E})$-tree.*

2) *If in addition, the pair $(G, \mathcal{E})$ is one-ended and has no essentially non-maximal splittings then there exists a $(G, \mathcal{E})$-tree J so that for every $n \in \mathbb{N}$ the tree $f^{n*}(J)$ is a non-trivial, $\underline{(K, \Phi)\text{-acylindrical}}$ $(G, \mathcal{E})$-tree for some uniform constant K.*

Proof: We will prove the first part of the Proposition by the induction on the length $l(\cdot, \mathcal{E})$ of the hierarchy of $(G, \mathcal{E})$ introduced in Definition 3.2. By Theorem C we have that $(G, \mathcal{E})$ splits as $G = A *_C B$ or $G = A *_C$ $(C \in \mathcal{C})$ so that

$$\max\{l(A, A \cap \mathcal{E}), l(B, B \cap \mathcal{E})\} < l(G, \mathcal{E}).$$

Let T denote the Bass-Serre tree corresponding to this splitting. If for all $n \in \mathbb{N}$ the tree $f^{n*}(T)$ is a non-trivial G-tree set $T = J$; if not there exists $m \in \mathbb{N}$ such that up to conjugation $f^m(G)$ is a subgroup of A or B, say A. As f is injective the subgroup A is a non-elementary group and we must have $f^m(A) \subsetneqq A$. Note that since the edge group C of the graph T/G is a quasi-convex subgroup of G then by the proof of [IKa, Lemma 3.5] it follows that A and B are quasi-convex subgroups of G. Thus [Sw] now yields that A and B are geometrically finite groups, in particular, they are finitely presented groups.

Note that by Proposition 4.7 we have $l(G, \mathcal{E}) \geq 1$. So let us first check the statement of the Proposition when $l(G, \mathcal{E}) = 1$. Consequently $l(A, A \cap \mathcal{E}) = 0$ and immediately we get a contradiction. Indeed, the pair $(A, \mathcal{E} \cap A)$ is not cohopfian so Proposition 4.7 applied to A yields that A splits non-trivially relatively to $A \cap \mathcal{E}$. This shows that in this case the trees $f^{n*}(T)$ are non-trivial $(G, \mathcal{E})$-trees $(\forall\ n \in \mathbb{N})$.

Suppose now that $l(G, \mathcal{E}) > 1$; as f is a non-surjective monomorphism of $(A, A \cap \mathcal{E})$ and $l(A, \mathcal{E} \cap A) < l(G, \mathcal{E})$ we can apply the induction assumption to A. Thus, there is a non-trivial tree T_A so that $f^{n*}(T_A)$ is a non-trivial $(A, \mathcal{E} \cap A)$-tree for all $n \in \mathbb{N}$. Consequently, the tree $\tau = f^*(T_A)$ considered as a G-tree (as $f(G) \subset A$), gives rise to the sequence of G-trees $f^{n*}(\tau)$ $(n \in \mathbb{N})$, which are all non-trivial A-trees while restricted to A. Whence $f^{n*}(\tau)$ is a non-trivial G-tree $(\forall\ n \in \mathbb{N})$.

If $E \in \mathcal{E}$ then by the induction assumption, the group $E \cap A$ is conjugate into a vertex stabilizer of τ, say G_v. Thus, $f^{-n}(E \cap A)$ is contained in the vertex stabilizer $f^{-n}(G_v)$ of the tree $f^{n*}\tau$. Since $f(E_i) \subset E_i$ we obtain $E_i \subset f^{-n}(E_i \cap A) \subset f^{-n}(G_v)$. We have shown that the system $\mathcal{E}$ is elliptic on the trees $f^{n*}\tau$, and therefore $f^{n*}\tau$ is a non-trivial $(G, \mathcal{E})$-tree for every $n \in \mathbb{N}$. The first part is proved.

The graph τ/G may happen to be reducible. In this case we collapse in τ/G every edge whose label is equal to the label of one of its vertices. Denote J the universal cover (in the sense of Serre) of this new graph of groups decomposition of G. As G acts on τ without global fixed points, then obviously, G also acts without global fixed points on J, and the system $\mathcal{E}$ is elliptic on J. Furthermore, as the set of the non-elementary vertex stabilizers of the trees J and τ is the same, it follows from Part 1) that the trees $f^{n*}J$ are also non-trivial $(G, \mathcal{E})$-trees $(\forall\ n \in N)$. The pair $(G, \mathcal{E})$ is one-ended and has no essentially non-maximal splittings, so Lemma 4.1 now yields that the reduced $(G, \mathcal{E})$-tree J is (K, Φ)-acylindrical for some uniform constant K. It is now easy to see that every tree $f^{n*}J$ is (K, Φ)-acylindrical for the *same* constant K $(n \in \mathbb{N})$. Indeed, if not then there is a segment l of length K on the tree $f^{n*}(J)$ whose pointwise stabilizer is an infinite subgroup C. Then $f^n(C)$ is an infinite subgroup fixing pointwise the segment l on J too. This is impossible. The Proposition is proved.

■

The following Proposition is central in this Section:

Proposition 5.2. *Let $G \subset \mathrm{Isom}_+\mathbb{H}^n$ be a discrete finitely presented group endowed with a fixed finite family $\mathcal{E} = \{E_1, ...E_k\}$ of elementary subgroups of G. Suppose that the pair $(G, \mathcal{E})$ is one-ended and does not have essentially non-maximal splittings. Let $f : (G, \mathcal{E}) \to (G, \mathcal{E})$ be a non-surjective monomorphism. Then there exist a non-trivial $(G, \mathcal{E})$-tree T with elementary edge stabilizers and a non-surjective monomorphism $F : (G, \mathcal{E}) \to (G, \mathcal{E})$ such that $F^*T = T$.*

Proof: By the previous Proposition there exists a tree J so that $f^{n*}J$ is a non-trivial (K, Φ)-acylindrical $(G, \mathcal{E})$-tree $(\forall n \in \mathbb{N})$ for some uniform contsant K. A priori, the tree $f^{n*}J$ is not a minimal G-tree. To be able to apply the finiteness Theorem D we choose a minimal G-subtree T_n of $f^{n*}J$. By construction of the latter it is straightforward that there exists a subtree T'_n of J invariant under the action of the group $f^n(G)$ so that $T_n = f^{n*}(T'_n)$. Furthermore $T'_n \subseteq T'_m \subseteq J$ whenever $n > m$. It follows by Proposition 5.1 that T_n is a non-trivial G-tree which is still (K, Φ)-acylindrical for the same constant K. So Theorem D now yields that there exist integers n and m $(n > m)$ and an automorphism φ of G so that:

$$\varphi^*T_n = T_m,\ \varphi(E_i) \subset E_i.$$

This implies, in particular, that the G-equivalence classes of edges (resp. vertices) of the trees T_n and T_m are in $1-\text{to}-1$ correspondence each other. Therefore $T'_n = T'_m$ and we have $\varphi^*(f^{n*}(T'_m)) = f^{m*}(T'_m)$. Obviously $f^{n*}(T'_m) = f^{n-m*}(f^{m*}(T'_m))$, so we finally get

$$\varphi^*(f^{n-m*}(f^{m*}(T'_m))) = f^{m*}(T'_m).$$

Setting $T = T_m$ and $F = f^k \circ \varphi$ where $k = n - m$, we obtain

$$F^*T = T$$

since $(f^{n-m} \circ \varphi)^* = \varphi^* \circ (f^{n-m})^*$. The groups E_i are all elliptic on the trees $(f^m)^*(J)$ $(m \in \mathbb{N})$ by Proposition 5.1 . As φ sends every subgroup E_i into itself the map F sends E_i into itself $(i \in \{1, ..., k\})$. Furthermore, F is injective and not surjective. The Proposition follows. ■

6 Proof of Theorem B.

Let $G \subset \mathrm{Isom}_+\mathbb{H}^n$ be a non-elementary Kleinian group equipped with a finite system $\mathcal{E} = \{E_1, ..., E_k\}$ of elementary subgroups and suppose that $F : (G, \mathcal{E}) \to (G, \mathcal{E})$ is a monomorphism of G sending each subgroup E_i into itself. Let $\tilde{E}_i$ denote E_i if E_i is finite and the maximal elementary subgroup of G containing E_i if E_i is infinite.

The aim of this section is to prove :

Theorem B. *Let $G \subset \mathrm{Isom}_+\mathbb{H}^n$ be a non-elementary, geometrically finite, Kleinian group without 2-torsion and $\mathcal{E} = \{E_1, ..., E_k\}$ be a family of elementary subgroups of G. Suppose that the pair $(G, \mathcal{E})$ is one-ended. Then $(G, \mathcal{E})$ is cohopfian if the following two conditions are satisfied:*

1) *The pair $(G, \mathcal{E})$ does not have essentially non-maximal splittings over elementary subgroups.*

2) *The pair $(G, \mathcal{E})$ does not split as an amalgam $G = A*_C\tilde{C}$, so that $\tilde{C}$ is a maximal elementary subgroup of G and the normal subgroup of $\tilde{C}$ generated by C is of infinite index in $\tilde{C}$.* ■

Proof: We will prove Theorem B by the induction on the invariant $\nu(\cdot, \mathcal{E})$ which is maximal number of edges in all non-trivial $(G, \mathcal{E})$-splittings with elementary edge subgroups of G (see Definition 4.5). If $\nu(G, \mathcal{E}) = 0$ then it follows from Proposition 4.7 that every injective monomorphism of $(G, \mathcal{E})$ is in fact surjective. So assume that the result is true for all groups having the complexity $\nu(\cdot)$ less than that of G and satisfying the hypotheses of the Theorem.

Arguing by contradiction suppose that there exists a non-surjective monomorphism $f : (G, \mathcal{E}) \to (G, \mathcal{E})$. Proposition 5.2 now yields that there exist a non-trivial $(G, \mathcal{E})$-tree T and a monomorphism $F : (G, \mathcal{E}) \to (G, \mathcal{E})$ such that $F^*T = T$. Furthermore the map F is surjective if and only if f is. Note that we may suppose (w.l.o.g.) that the graph $Y = T/G$ is irreducible. Indeed, otherwise we collapse each edge whose label is equal to the label of the vertex incident to it. We obtain in this way an irreducible graph of groups which gives a non-trivial decomposition of G because the decomposition of G given by the graph Y was non-trivial. We retain the notations Y and T for this new graph of groups decomposition of G and its universal cover respectively.

The strategy of the further proof is the following: we first show that F is surjective on all non-elementary vertex stabilizers of T, then we prove the same for the elementary vertex stabilizers (using the condition 2) of the hypotheses. The surjectivity of F will then follow from the following Proposition:

Proposition 6.1. *The map $F : G \to G$ is surjective if and only if it is surjective on each vertex group G_v of the graph Y.*

Proof: If the graph Y is a tree of groups then the vertex groups G_v $(v \in Y^0)$ generate the whole group G, and so the map F is surjective. Let us fix a maximal subtree of Y and prove the Proposition by induction on the number of edges which are not in the maximal subtree. If e is such an edge then $G = A*_{C_e} = \langle A,\ C_e \mid tC_et^{-1} = \varphi(C_e)\rangle$, where $\varphi : C_e \to \varphi(C_e)$ is an isomorphism. The group A is the fundamental group of the graph Y' of groups obtained from Y

by deleting the edge e. Since Y' has less number of non-separating edges than Y by the induction assumption, we have that F restricted to A is surjective. Denote $H = C_e$ and $H' = \varphi(C_e)$. By Proposition 5.2 we have that $F(H) \subset H$ and $F(H_1) \subset H_1$. Since $tHt^{-1} = H_1$ we also obtain that $F(t)F(H)F(t)^{-1} = F(H_1)$. Thus $aHa^{-1} \cap H \supset t^{-1}F(H_1)t$, where $a = t^{-1} \cdot F(t)$. As the group $t^{-1}F(H_1)t$ is infinite and is a subgroup of H, the element a belongs to the unique maximal elementary subgroup $\tilde{H}$ containing H. Then by our Definition 2.2 there exists a vertex $v \in Y^0$ so that up to conjugation we have $\tilde{H} \subset G_v$ and, so $a \in A$. Since F is surjective on A there exists $b \in A$ that $F(b) = a$. We finally obtain $t = F(t \cdot b^{-1})$, and so F is surjective on G. The Proposition follows. ■

Denote by $\mathcal{C}$ the set of edge stabilizers of the tree T. For every $C_e \in \mathcal{C}$ we have $F(C_e) \subset C_e$. By Proposition 5.2 we also have that $F(E_i) \subset E_i$. As all edge groups of the graph $Y = T/G$ are quasi-convex subgroups of G as we have already pointed out the results [IKa, Lemma 3.5] and [Sw] imply that every vertex stabilizer G_v is a geometrically finite group.

Every splitting of the pair $(G_v, (\mathcal{C} \cup \mathcal{E}) \cap G_v)$ gives rise to an essentially maximal $(G, \mathcal{E})$-splitting. Thus by repeating the argument used in the proof of Proposition 4.3 we obtain that every $(G_v, (\mathcal{C} \cup \mathcal{E}) \cap G_v)$-splitting is essentially maximal. Similarly, the pair $(G_v, (\mathcal{C} \cup \mathcal{E}) \cap G_v)$ is one-ended. Since the graph Y defines a non-trivial splitting of G, Corollary 4.6 now implies that $\nu(G_v, (\mathcal{C} \cup \mathcal{E}) \cap G_v) < \nu(G, \mathcal{E})$. As $F|_{G_v}$ sends all subgroups in $(\mathcal{C} \cup \mathcal{E}) \cap G_v$ into themselves the induction hypothesis yields that $F|_{G_v}$ is surjective for every non-elementary vertex stabilizer G_v of the tree T.

Let us now consider edge groups and elementary vertex groups of the graph $Y = T/G$. Since our group G is non-elementary, collapsing every two neighboring vertices v_1 and v_2 whose vertex groups are elementary we still get a non-trivial splitting of G satisfying all the above properties. Similarly if there is a vertex v whose vertex group G_v is elementary and such that there is a loop e emanating from v, then we collapse this loop to v, the resulting vertex group will be still elementary and the map F sends it into itself. So we may suppose that every edge $e \in Y^1$ which is not a loop has at least one vertex $v \in \partial e$ whose label is a non-elementary group G_v. Also there is no loop of Y emanating from a vertex whose label is elementary.

We can also suppose that our graph Y does not have vertex groups which are *non-maximal* elementary groups. Indeed, if G_v is such a group then since G does not have essentially non-maximal splittings the maximal elementary subgroup $\tilde{G}_v$ containing G_v is contained in some other vertex group, say $G_{v'}$, however then the group G_v is contained in the stabilizer of the edge belonging to the path L between v' and v. This contradicts the irreducibility of the graph Y.

Let us now prove that the restriction $F|_{C_e}$ on every edge C_e is surjective. Indeed, as F is injective the group $F^{-1}(C_e)$ is elementary (being isomorphic to $F(F^{-1}(C_e))$ which is a subgroup of C_e) and we have $F^{-1}(C_e) \supset C_e$ since $F(C_e) \subset C_e$. Let $v_1 \in \partial e$ be one of the vertices of e whose stabilizer G_{v_1} is not elementary. Since $F|_{G_{v_1}}$ is surjective, for any $y \in C_e$ there exists $x \in G_{v_1}$ so that $F(x) = y$. If another vertex $v_2 \in \partial e$ has also a non-elementary stabilizer then by the same reason and the injectivity of F we obtain that $x \in G_{v_2}$ and so $x \in C_e$. If now G_{v_2} is a maximal elementary subgroup of G we have $F^{-1}(G_{v_2}) \supset F^{-1}(C_e) \supset C_e$ and then $F^{-1}(G_{v_2}) = G_{v_2}$ since C_e is infinite and is contained in G_{v_2}. This shows that $x \in G_{v_2}$ and again $x \in C_e$.

Let us now prove that F is surjective on every elementary maximal vertex group E_v $(v \in Y^0)$ of the graph Y. Let e_i $(i = 1, ..., l)$ be the edges incident to v and C_i be their labels. As there is no loop emanating from the vertex v it is now easy to see that we have a decomposition $G = A *_{C_v} E_v$

where C_v is the elementary group generated by C_i $(i = 1, ..., l)$ and A is the fundamental group of the graph $Y \setminus \{v\}$. We already know that our map F is surjective on the edge group C_i and so it is surjective on C_v. Moreover applying the previous argument to every edge stabilizer of the tree T, we conclude that the restriction of F on every G-conjugate of C_e is also surjective. It now follows that F is surjective on the smallest normal subgroup N_v generated by C_v. Since F maps the group E_v into itself, and N_v onto itself, it induces an injective map $\Phi : E_v/N_v \to E_v/N_v$. By hypothesis the group E_v/N_v is finite, whence the map Φ is surjective, and so F is surjective on E_v. We have proved that F is surjective on every edge and vertex stabilizer of the graph Y.

From Proposition 6.1 we now derive that the map $F : G \to G$ is an automorphism of G. Then our initial map $f : G \to G$ is an automorphism too. To finish the proof we only need to show that $f|_E$ is surjective on every $E \in \mathcal{E}$. As $f(E) \subset E$ the conclusion is obvious when E is finite. If it is not the case then by the uniqueness of the the maximal elementary subgroup $\tilde{E}$ of G containing E we have $f(\tilde{E}) = \tilde{E}$ as $f(E)$ is an infinite subgroup of both. Again using the fact that an increasing sequence of subgroups of a virtually abelian group of a finite rank must stabilize we deduce that $f(E) = E$. Theorem is proved. ■

7 Necessary Condition in Theorem A.

The aim of this section is to prove the necessary condition in Theorem A which follows directly from the following result:

Theorem E. *A finitely generated discrete group $G \subset \mathrm{Isom}_+\mathbb{H}^n$ is not cohopfian if one of the two conditions below is satisfied:*

1) *G has an essentially non-maximal splitting*

$$G = \pi_1(X, G_v, C_e), \text{ where each vertex group } C_e \text{ is elementary} \tag{1'}$$

2) *The group G splits as an amalgam $G = A *_C \tilde{C}$, so that $\tilde{C}$ is a maximal elementary subgroup of G and the normal subgroup of $\tilde{C}$ generated by C is of infinite index in $\tilde{C}$.*

We start with:

Remarks 7.1. *1) Examples of discrete geometrically finite groups in $\mathbb{H}^n$ which are described in 1) and 2) of Theorem E do exist and are given in [OP].*

2)If every elementary group over which G splits is in fact abelian and if there exists a splitting X described in Condition 2) then there is another splitting over elementary subgroups which is essentially non-maximal : the maximal elementary vertex group can be written as a central HNN-extension with a base containing all corresponding edge stabilizers. However, this is not the case in general, as there exists (torsion-free) virtually abelian groups with finite abelianization.

We will first study essentially non-maximal splittings of G:

Proposition 7.2. *Suppose G splits as a graph of groups (1'), where one of the edge groups $C_e = E$ of the splitting (1') is essentially non-maximal, and let $\tilde{E}$ be the maximal elementary subgroup of G containing E with infinite index so that $\tilde{E}$ is hyperbolic in the splitting (1'). Then there exists an element $g \in \tilde{E}$ so that g centralizes E and $g^n \notin E$ $(n \in \mathbb{N})$.*

Proof of 7.2: Let T be the Bass-Serre tree corresponding to the splitting (1'). The group $\tilde{E}$ contains a normal abelian subgroup $\tilde{A}$ of finite index. As $\tilde{E}$ acts on T hyperbolically it follows that the group $\tilde{A}$ also does. Hence by [S, I-6.5, Proposition 27] it follows that $\tilde{A}$ leaves a line $L \subset T$ invariant. As $\tilde{A}$ is normal in $\tilde{E}$ the group $\tilde{E}$ also leaves L invariant. Then either $\tilde{E}$ acts by translations on L; or it acts dihedrally on L (permuting the end points of L). So, there is a projection η of $\tilde{E}$ on $\mathbb{Z}$ or onto $\mathbb{Z}_2 * \mathbb{Z}_2$. Moreover since the subgroup E of $\tilde{E}$ fixes an edge e in T it fixes the axis L pointwise. So we may suppose that $e \subset L$ and that E is the kernel of η (which is the kernel of the action of $\tilde{E}$ on L). It follows that up to passing to a subgroup of index 2 and retaining the notation $\tilde{E}$ for it, we have the following exact sequence:

$$0 \longrightarrow E \longrightarrow \tilde{E} \xrightarrow{\eta} \mathbb{Z} \longrightarrow 1,$$

Let t denote the element of $\tilde{E}$ which is mapped on the generator of $\mathbb{Z}$. There exits $m \in \mathbb{N}$ so that $t^m \in A$ and up to replacing t by t^m and passing to a further subgroup of finite index we may suppose that $t \in A$. Also $t^n \notin E$ $(\forall n \in \mathbb{N})$.

Let A denote the group $\tilde{A} \cap E$ which is a normal abelian subgroup of E of finite index. We have:

$$0 \longrightarrow A \longrightarrow E \xrightarrow{\xi} F \longrightarrow 1, \qquad (*),$$

where F is a finite group.

Definition. An automorphism of E will be called the *automorphism of the sequence (*)* if its restriction on A is trivial and if it induces the identity on F. The group of the automorphisms of (*) is denoted $\mathrm{Aut}(*)$.

Let $s : F \to E$ be a set theoretic cross-section of ξ and $\psi \in \mathrm{Aut}(*)$. Put $c_\psi(f) = \psi(s(f)) \circ s(f)^{-1}$, $\forall f \in F$.

Lemma 7.3. *The following assertions hold:*

a) *$c_\psi(f)$ is a 1-cocycle of F taking values in A.*

b) *The map $\psi \to c_\psi$ is a group homomorphism.*

Proof: a) Notice first that $c_\psi(f) \in A$ since ψ induces the identity map on F. We have then: $s(f \cdot g) = s(f) \cdot s(g) \cdot \alpha(f,g)$, where $\alpha(f,g) \in A$.

$\psi(s(f \cdot g)) \cdot (s(f \cdot g))^{-1} = \psi(s(f)s(g)\alpha(f,g)) \cdot \alpha^{-1}(f,g)(s(f)s(g))^{-1} = \psi(s(f))\psi(s(g))s(g)^{-1}s(f)^{-1}$, since $\psi(a) = a$ $(\forall a \in A)$. Further we derive :

$\psi(s(f\cdot g))\cdot(s(f\cdot g))^{-1} = \psi(s(f))s(f)^{-1}+s(f)\psi(s(g))s(g)^{-1}s(f)^{-1} = \psi(s(f))s(f^{-1})+\rho(f)c_\psi(f) = c_\psi(f)+\rho(f)c_\psi(f)$, where $\rho(f)$ denotes the action of $f\in F$ on A given by conjugation by $s(f)$. This proves a) by the definition of a cocycle (see [Br, p. 88]).

b) $c_{\psi_1\psi_2}(f) = \psi_1\psi_2(s(f))s(f)^{-1} = \psi_1[\psi_2(s(f))s(f)^{-1}s(f)]s(f)^{-1} = \psi_1(c_{\psi_2}(f)\cdot s(f))s(f)^{-1} = c_{\psi_2}(f)+c_{\psi_1}(f)$, here we used that $c_{\psi_2}(f)\in A$ and that ψ_1 keeps it unchanged. We have proved b). The Lemma is proved. ■

Proof of Proposition. Recall that $t^n\in\tilde{A}\setminus E$ ($\forall\ n\in\mathbb{N}$). Let ψ be an interior automorphism of $\tilde{E}$ given by the conjugation via t. As t acts identically by conjugation on A it is easy to verify that it also induces the identity on F, i.e.:

$$\hat{t}f\hat{t}^{-1} = f,\ \forall\ f\in\ F,$$

where $\hat{t}=\xi(t)$. So ψ is an automorphism of the sequence $(*)$ and we get $c_\psi(f) = ts(f)t^{-1}s(f)^{-1}$. Since the group F is finite the first cohomology group $H^1(F,A)$ is finite too and, so there exists $p\in\mathbb{N}$ such that $c_{\psi^p}(f)$ is a coboundary. It follows that there exists $a\in A$ that $c_{\psi^p}(f) = a-\rho(s(f))\cdot a = a+\rho(s(f))(-a)$. Writing this in the multiplicative form we have $c_{\psi^p}(f) = as(f)a^{-1}s(f)^{-1} = t^ps(f)t^{-p}s(f)^{-1}$ implying that $\forall\ s(f)\in E:\ a^{-1}t^ps(f)t^{-p}a = s(f)$. Putting $g=a^{-1}t^p$ we obtain that g is not trivial ($t^p\notin A$) and centralizes E. The Proposition follows. ■

Note that above we also obtained the following fact which will be used further:

Remark 7.4. *The group of the automorphisms of the sequence (*) is finite modulo conjugation in E (in other words the subgroup of* $\mathrm{Out}(E)$ *which preserves (*) is finite)* ■.

Lemma 7.5. *Suppose that the group G splits as a graph of groups $G=\pi_1(X,G_v,C_e)$ with elementary edge stabilizers such that one of the edge group $E=C_e$ is essentially non-maximal then G splits as an amalgam or an HNN-extension:*

$$G = A *_K B,\ \text{or}\ G = A*_K, \tag{6}$$

where K is essentially non-maximal and contains E.

Proof:

Let T denote the tree which is the universal covering of X and let $\tilde{E}$ be the maximal elementary subgroup of G containing E. The group $\tilde{E}$ acting on T without fixed points has an invariant line $L\subset T$ (see the beginning of the proof of Proposition 7.2). Since the subgroup E fixes a point in T it also fixes L pointwise. Let $\hat{e}\subset L$ be an edge of T and $\hat{\alpha}$ and $\hat{\beta}$ its vertices. We first claim that the stabilizer F of the edge $\hat{e}$ in G coincide with the stabilizer K of $\hat{e}$ in $\tilde{E}$ (i.e. the kernel of the action of $\tilde{E}$ on L). Indeed, both groups contain the group E which is an infinite group so by the uniqueness of the maximal elementary subgroup $\tilde{E}$ containing E it follows that $F\subset\tilde{E}$ which implies that $F=K$. Similar argument shows that the subgroup N of G leaving the line L invariant coincide with $\tilde{E}$. Indeed, the group N is elementary which follows from the fact that it has a projection to $\mathbb{Z}$ or $\mathbb{Z}_2*\mathbb{Z}_2$ whose kernel is the elementary group K, consequently N is an elementary group containing $\tilde{E}$, and thus $\tilde{E}=N$.

Let α, β, e denote the images in X of $\hat{\alpha}$, $\hat{\beta}$, $\hat{e}$ respectively under the projection $p: T \to X$. First consider the graph X. Then the group G is the HNN-extension

$G = A*_K = \langle A,\ t \mid tKt^{-1} = \phi(K)\rangle$ where A is the fundamental group of the graph of groups $Y = X \setminus e$. Denote $\hat{Y}$ the component of the preimage $p^{-1}(Y)$ adjacent to the edge $\hat{e}$ at the point $\hat{\alpha} \in T$. Clearly $p(\hat{Y}) = Y$, so we may assume up to conjugation in G that the stabilizer of Y is A. As none G-translate of $\hat{e}$ is contained in $\hat{Y}$ and T is a tree we have $L \cap \hat{Y} = \{\hat{\alpha}\}$. Let $h \in \tilde{E} \setminus E$ be an element acting on L by translations. As $h(\hat{\alpha}) \in L \setminus \{\hat{\alpha}\}$, so h can not belong to the stabilizer of $\hat{Y}$. Consequently, the group $\tilde{E}$ is hyperbolic in the splitting $G = A*_K$.

The case when the edge e separates X is similar: we obtain the splitting $G = A *_K B$ where A and B are the fundamental groups of the graphs which are respectively connected components U and V of $X \setminus e$. Denote $\hat{U}$ and $\hat{V}$ the components of $p^{-1}(U)$ and $p^{-1}(V)$ which are adjacent along the edge $\hat{e}$ in T, in particular $\hat{\alpha} \in \hat{U}$ and $\hat{\beta} \in \hat{V}$. The stabilizers $\hat{U}$ and $\hat{V}$ are up to conjugation the groups A and B. Again we have $\hat{U} \cap L = \{\alpha\}$ and $\hat{V} \cap L = \{\beta\}$. There exists an element $h \in \tilde{E} \setminus K$ which acts by translations on L so $h(\hat{\alpha}) \in L \setminus \{\hat{\alpha}\}$, and the same for $\hat{\beta}$. Consequently, the element h does not belong to the stabilizers of $\hat{U}$ and $\hat{V}$. This shows that $\tilde{E}$ is not elliptic with respect to the splitting $G = A *_K B$. The Lemma is proved. ■

Proof of Theorem E : Let us first consider **Condition 1)** of the Theorem which is:

1) *G has an essentially non-maximal splittings.*

Then by Lemma 7.5 it follows that there is a splitting (6) of G as an amalgam or an HNN-extension which is essentially non-maximal. The edge group K is an elementary essentially non-maximal subgroup and let $\tilde{K}$ be the maximal elementary subgroup of G containing K which is hyperbolic in the splitting (6). By Proposition 7.2 it follows that there exists an element $t \in \tilde{K} \setminus K$ which centralizes K.

Consider first the case of amalgamated product, i.e. $G = A *_K B$. Let us define the map $f : G \to G$ so that $f(a) = tat^{-1}$, $\forall a \in A$ and $f(b) = b$. As t commutes with all elements from K the map f is obviously a homomorphism. If $f(g) = 1$ for some $g \in G$, then writing g in the normal form it is easy to see that $g \in A$ or $g \in B$, so by injectivity of f on A and B we obtain $g = 1$. We have that $f : G \to G_1$ is an isomorphism where $G_1 = \langle tAt^{-1}, B\rangle \cong G$. We need only to show that $G_1 \subsetneqq G$. The group G_1 being a subgroup of G acts on the Bass-Serre G-tree T corresponding to the splitting $G = A *_K B$. Denote by α and β the vertices of T whose stabilizers are A and B. Set $d = \min_{g \in G} \operatorname{dist}_T(gtg^{-1}(\alpha), \alpha)$ then $d > 1$ since t is not conjugate into A and B. In particular, t cannot normalize A (since otherwise tAt^{-1} fixes the path between $t(\alpha)$ and α pointwise and so tAt^{-1} is conjugate into K which is impossible).

The group G_1 is the amalgamated free product $tAt^{-1} *_K B$ and so every element $g \in G_1$ can be written in the following form:

$$g = ta_1t^{-1} \cdot b_1 \cdot\ \dots\ \cdot ta_kt^{-1} \cdot b_k,\ \text{or}\ g = b_1 \cdot ta_1t^{-1} \cdot\ \dots\ \cdot b_k \cdot ta_kt^{-1},\ a_i \in A \setminus K,\ b_j \in B \setminus K. \quad (7)$$

As t centralizes the group K, so $ta_it^{-1} \in tAt^{-1} \setminus K$ and the formula (7) gives also a normal form of g in the amalgamated product $G_1 = tAt^{-1} *_K B$. An easy calculation shows that

$$\min_{\alpha} \operatorname{dist}_T(g(\alpha), \alpha) = 2k(d+1),\ \text{where}\ \alpha \in \{a, b\}.$$

It follows that the set $\mathcal{L}(G_1)$ of translation lengths of elements of G_1 acting on T is equal to $\{2k(d+1) \mid k \in \mathbb{N}\}$ which is a proper subset of $\mathcal{L}(G)$ (e.g. $2 \in (\mathcal{L}(G) \setminus \mathcal{L}(G_1))$. This shows that $G_1 \subsetneqq G$.

Consider now the case of an HNN-extension $G = A*_K = \langle A, h \mid hKh^{-1} = \phi(K)\rangle$ and let T be the corresponding Bass-Serre tree. There are two more subcases: a) $h \in \tilde{K}$ and b) $h \notin \tilde{K}$. In the subcase a) we proceed as follows. By the proof of Proposition 7.2 there exist $p \in \mathbb{N}$ and $a \in \mathcal{A}(K)$ so that the element $g = h^p \cdot a$ commutes with every element of K, where $\mathcal{A}(K)$ is the maximal abelian subgroup of K. Now we define the map $f : G \to G$ to be the identity on A and set $f(h) = h^{p+1} \cdot a$. It is easy to check that f is an injective monomorphism (since h^{p+1} acts on K in the same way as h does) which is not surjective.

In the subcase b) we proceed similarly to the case of an amalgamated free product, namely put $f(a) = tat^{-1}$, $\forall a \in A$ and $f(h) = h$, where t is an element in $\tilde{K} \setminus K$ acting hyperbolically on the tree T and centralizing K. Then any element g of the group $G_1 = f(G) = tAt^{-1}*_K = \langle tAt^{-1},\ h|hKh^{-1} = \phi(K)\rangle$ can be written as:

$$g = ta_1t^{-1} \cdot h^{\varepsilon_1} \cdot ta_2t^{-1} \cdot ... \cdot ta_kt^{-1} \cdot h^{\varepsilon_k}, \text{ or } g = h^{\varepsilon_1} \cdot ta_1t^{-1} \cdot h^{\varepsilon_2}ta_2t^{-1} \cdot ... \cdot ta_kt^{-1}, \tag{7}$$

where $\varepsilon_i \in \mathbb{Z}$ and if $\varepsilon_i < 0$ and $a_i \in K$ then $\varepsilon_{i+1} \leq 0$, and if $\varepsilon_i > 0$ and $a_i \in K$ then $\varepsilon_{i+1} \geq 0$.

Let α be the vertex of T whose label is A and M be the line in T which is formed by the vertices $h^n(\alpha)$ $(n \in \mathbb{Z})$. As $h \notin \tilde{K}$ the element t does not belong to the maximal elementary subgroup containing h and so $t(\alpha) \notin M$. It is now straightforward that the translation length $l(g)$ of the element g in (7) is equal to $2kd + \sum_{i=1}^{k} |\varepsilon_i|$, where $d = \min_{g\in G}\ \mathrm{dist}_T(gtg^{-1}(\alpha), \alpha)$. Indeed each term ta_it^{-1} in (7) gives the contribution $2d$ and the term h^{ε_i} contributes $|\varepsilon_i|$ to the expression $l(g)$. For a fixed $d \geq 1$ the set $\{2kd + \sum_{i=1}^{k} |\varepsilon_i| \mid k \in \mathbb{N},\ \varepsilon_k \in \mathbb{Z}\}$ is a proper subset of $\mathbb{N}$ which is the set of translation lengths of G acting on T. This shows that in this case G_1 is a proper subgroup of G. Part 1) of Theorem E is proved.

Consider now **Condition 2)** of Theorem E which is:

The pair $(G, \mathcal{E})$ splits as an amalgam $G = A *_C \tilde{C}$, so that $\tilde{C}$ is a maximal elementary subgroup of G and the normal subgroup of $\tilde{C}$ generated by C is of infinite index in $\tilde{C}$.

Suppose that X is such a graph of groups decomposition of G. We are going to construct a proper monomorphism from G into G which is the identity on all edge and vertex groups of X besides E_v and which sends E_v into itself being not surjective on it.

The group E_v is virtually abelian of finite rank, let A be a finite index normal free abelian subgroup of E_v. Denote $C = A \cap N_v$ and $F = E_v/A$. As N_v and A are normal in E_v the group C is normal in E_v too. Let also $s : F \to E_v$ be a normalized cross-section of the projection of E_v onto F. The group F acts on A by conjugation $a \to s(f)as^{-1}(f)$. Consider the vector space $A \otimes \mathbb{Q}$ which we equip with a scalar product invariant under the induced action of F. As the subspace $C \otimes \mathbb{Q}$ is invariant under the induced action of F there exists a subspace V in $A \otimes \mathbb{Q}$ complementary to $C \otimes \mathbb{Q}$ which is also invariant under this action. We can now find a subgroup B of A so that $V = B \otimes \mathbb{Q}$ and so $A \otimes \mathbb{Q} = (C \otimes \mathbb{Q}) \oplus (B \otimes \mathbb{Q})$. The group B has the following properties: $B\cap C=\{\mathrm{id}\}$ (since A is torsion free); B is normal in E_v; and the group $A'=C\oplus B$ is a normal free abelian subgroup of E_v of finite index. Denote $F'=E_v/A'$. Consider the map $h_n:A'\to A'$ defined as $h_n(c+b)=c+nb$ for every $c\in C$ and $b\in B$. One can now find a group H_n and a homomorphism $\varphi_n:E_v\to H_n$ so that the following diagram commutes :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{i} & E_v & \xrightarrow{p} & F' & \longrightarrow & 1 \\ & & {\scriptstyle h_n}\downarrow & & {\scriptstyle \varphi_n}\downarrow & & \| & & \\ 0 & \longrightarrow & A' & \xrightarrow{i_n} & H_n & \xrightarrow{p_n} & F' & \longrightarrow & 1. \end{array}$$

In fact the group H_n is the largest quotient of $A'\rtimes E_v$ such that the left-hand square of the above diagram commutes (see e.g. [Br, page 94]). Note that φ_n is injective and not surjective since h_n is ($\forall n\in\mathbb{N}$). Furthermore, φ_n is the identity on C. We need now to show that H_n is isomorphic to E_v. Let us define a set-theoretic cross section $s_n:F'\to H_n$ of the projection p_n (see the diagram) to be $s_n=\varphi_n\circ s$. It is known (see e.g. [Br, III.3.12]) that the equivalence classes of extensions of A' by F' are in $1-\text{to}-1$ correspondence with the elements of $H^2(F',A')$. If $\alpha\in H^2(F',A')$ is the element which corresponds to the upper row of the commutative diagram then it satisfies $i(\alpha(g,\gamma))s(g\gamma)=s(g)s(\gamma)$ $(g,\gamma\in F')$. We have

$$s_n(g)s_n(\gamma)=\varphi_n(s(g)\varphi_n(s(\gamma))=\varphi_n(s(g)s(\gamma))=\varphi_n(i(\alpha(g,\gamma))s(g\gamma))=i_n(h_n(\alpha(g,\gamma)))s_n(g,\gamma).$$

Setting $\alpha_n(g,\gamma)=h_n(\alpha(g,\gamma))$ we obtain from the above identity that $\alpha_n(g,\gamma)$ is an element of $H^2(F',A')$. Since $\alpha(g,\gamma)$ takes its values in A' we can write $\alpha(g,\gamma)=c(g,\gamma)+b(g,\gamma)$ where $c(g,\gamma)\in C$ and $b(g,\gamma)\in B$. We also have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{i} & E_v & \longrightarrow & F' & \longrightarrow & 1 \\ & & {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi_2}\downarrow & & \| & & \\ 0 & \longrightarrow & B=A'/C & \xrightarrow{j} & E_v/C & \longrightarrow & F' & \longrightarrow & 1 \end{array}$$

where π_i are the natural projections and j is the natural inclusion. Define a section $\sigma:F'\to E_v/C$ to be $\sigma=\pi_2\circ s$. Similarly we then show that $\sigma(g)\sigma(\gamma)=j(p_2(\alpha(g,\gamma)))\sigma(g,\gamma)$ $(g,\gamma\in F')$. As $b=\pi_1\circ\alpha$ we obtain that b is also an element of $H^2(F',A')$. Since the group F' is finite it follows from [Br, III.10.2] that the group $H^2(F',A')$ is annihilated by $|F'|$. Choosing n to be $n\equiv 1(\mathrm{mod}\ |G|)$ we obtain that $b=nb$ and so $\alpha_n=c+b_n=c+b=\alpha$. This implies that α and α_n define the equivalent extensions and, so the groups E_v and H_n are isomorphic. Let $n_0=|F'|+1$ then the map φ_{n_0} is an endomorphism of E_v which is injective, non-surjective and is the identity on C. Since C is an abelian normal subgroup of N_v it follows that φ_{n_0} is the identity on N_v too (because if h represents a coset in N_v/C then the element $\varphi_{n_0}(h)\cdot h^{-1}$ commutes with all elements in C and so $\varphi_{n_0}(h)\equiv h(\mathrm{mod}\ C)$). By Remark 7.4 it follows that

$$\exists\ k\in\mathbb{N},\ \exists\ a\in C\ \forall h\in N_v:\varphi_{n_0}^k(h)=aha^{-1}.$$

Setting $F|_{E_v}=a^{-1}\cdot\varphi_{n_0}^k\cdot a$, we obtain an injective, not surjective endomorphism of E_v which is the identity on N_v. Extending now F by the identity to the fundamental group of the graph $X\setminus\{v\}$ we obtain a monomorphism $F:G\to G$ which is injective and not surjective. Theorem E is proved. ■

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