JACOBI FORMS AND A CERTAIN SPACE OF MODULAR FORMS

by

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Introduction and description of the "certain space"

Jacobi forms are a mixture of modular forms and elliptic functions. Examples of such functions are very classical--the Jacobi theta functions and the Fourier coefficients of Siegel modular forms of genus two--but it is a relatively recent observation that the Jacobi forms have an arithmetic theory very analogous to the usual theory of modular forms: this began with Maass's proof of the Saito-Kurokawa conjecture [M] and was developed systematically in [E-Z].

Because they have two variables, Jacobi forms have associated to them two characteristic integers--the weight, which describes the transformation properties of the form with respect to the modular group, and the index, which describes the transformation properties in the elliptic variable. The main result of this paper is a relationship between Jacobi forms (on the full Jacobi modular group) of weight k and index m on the one hand and ordinary modular forms of weight $2k-2$ and level m on the other. This relationship in the special case $m=1$ already played a key role in [M] (cf. also [E-Z], §6). A surprising aspect of the general result is that, while on the Jacobi side the numbers k and m affect only the automorphy factor and the group never changes, on the other side the group itself varies. In particular, the Jacobi forms of all weights and indices form a bigraded ring, the product of Jacobi forms of index m_1 and index m_2 having index m_1+m_2, but there is (presumably) no natural way to produce a modular form on, say, $\Gamma_0(7)$ from modular forms on $\Gamma_0(2)$ and $\Gamma_0(5)$.

We will now be a little more specific. Let $J_{k,m}$ denote the space of Jacobi forms on $SL_2(\mathbb{Z})$ of weight k and index m (see [E-Z] or §0 for the exact definition). One can define in $J_{k,m}$ a subspace of oldforms (coming from $J_{k,m'}$ for proper divisors m' of m) and a complementary space (for cusp forms, the orthogonal complement) $J_{k,m}^{new}$ of newforms; one also has for all $\ell > 0$ prime to m Hecke operators $T(\ell)$ on $J_{k,m}$ preserving $J_{k,m}^{new}$ (cf. §4 of [E-Z]). For $M_{2k-2}(m)$, the space of holomorphic modular forms of weight $2k-2$ on $\Gamma_0(m)$, the analogous notions are, of course, standard. Let $M_{2k-2}^{-}(m)$ denote the space of all forms $f \in M_{2k-2}(m)$ satisfying $f(\frac{-1}{m\tau}) = (-1)^k m^{k-1} \tau^{2k-2} f(\tau)$ (the "$-$" in the notation refers to the fact that the L-series of such an f satisfies a functional equation under $s \to 2k-2-s$ with root number -1 and, in particular, vanishes at $s = k-1$), and $M_{2k-2}^{new,-}(m) = M_{2k-2}^{new}(m) \cap M_{2k-2}^{-}(m)$. Then we have:

Main Theorem. Let $k, m,$ and ℓ be positive integers with $(\ell, m) = 1$. Then

$$\mathrm{tr}(\,T(\ell)\,,\, J_{k,m}^{new}) \;=\; \mathrm{tr}(\,T(\ell)\,,\, M_{2k-2}^{new,-}(m)\,)\,.$$

The relationship between old and new Jacobi forms is not the same as between old and new modular forms: a newform in $J_{k,m'}$, $(m'|m)$ occurs in $J_{k,m}$ with smaller multiplicity (i.e., has fewer lifts to $J_{k,m}$) than a newform in $M_{2k-2}(m')$ does in $M_{2k-2}(m)$. Thus the above theorem does not say that the full space $J_{k,m}$ is isomorphic as a Hecke-module to $M_{2k-2}^{-}(m)$. Instead, it turns out that there is a canonical subspace $\mathfrak{M}_{2k-2}(m) \subset M_{2k-2}(m)$, containing the space of newforms, for which one has:

Main Theorem (2nd version). The space $J_{k,m}$ is isomorphic to $\mathfrak{M}_{2k-2}^{-}(m) = \mathfrak{M}_{2k-2}(m) \cap M_{2k-2}^{-}(m)$ as modules over the Hecke algebra.

We will explain the definition of the space $\mathfrak{M}_{2k-2}(m)$ in a moment.

The proof of the main theorem proceeds in three stages. In §1 we apply the main theorem of our previous paper [S-Z], which gave a general trace formula for double-coset operators on spaces of Jacobi forms, to compute explicitly the trace of $T(\ell)$ — or, more generally, of $T(\ell)$ times an Atkin-Lehner involution — on $J_{k,m}$ (Theorem 1). The computation is quite technical but includes some pretty results, such as a formula expressing a certain class number as a linear combination of Gauss sums associated to binary quadratic forms (Appendix, Proposition A.1). In §2 we transform the usual Eichler-Selberg trace formula for Hecke operators as given in the literature into a form suitable for comparison with this and express the trace of $T(\ell)$ on $J_{k,m}$ as a linear combination of the traces of $T(\ell)$ on $M_{2k-2}^{new,-}(m')$, $m'|m$ (Theorem 2). This is then used in §3 to establish the main properties of Jacobi newforms and to prove the main theorem as given above. The result actually proved, Theorem 3, not only asserts the isomorphism of $J_{k,m}$ and $\mathfrak{M}_{2k-2}^{-}(m)$ but gives a collection of explicit lifting maps $\mathcal{S}_{D,s}$ (indexed by discriminants of imaginary quadratic fields D and residue classes $s \pmod{2m}$ with $s^2 \equiv D \pmod{4m}$) between these spaces.

The main application so far of the result of the present paper is the theorem proved in [G-K-Z], which asserts that the classes of Heegner points on a modular curve $X_0(m)$ in the Mordell-Weil group of its Jacobian are the coefficients of a Jacobi form (of weight 2 and index m). Also, in Chapter II of [G-K-Z] a kernel function for the lifting maps $\mathcal{S}_{D,s}$ is constructed and its Fourier coefficients computed. (Note that both Heegner points on $X_0(m)$ and coefficients of Jacobi forms of index m are naturally indexed by pairs D, s as above.)

We devote the rest of this introduction to a discussion of the spaces $\mathfrak{M}_k(m)$ ($k, m > 0$, k even), which we think are of interest independently of the theory of Jacobi forms. The most natural definition is as follows. The full space of modular forms $M_k(m) = M_k(\Gamma_0(m))$ has a basis (not unique) of forms f whose L-series $L(f,s) = \sum_{n=1}^{\infty} a_f(n)\, n^{-s}$ ($a_f(n) = n^{\text{th}}$ Fourier coefficient of f) has an Euler product. Any such f is an eigenform of all Hecke operators $T(\ell)$ with $(\ell,m) = 1$ and "comes from" (i.e., has the same eigenvalues for all such $T(\ell)$ as) a unique form g which is a newform on $\Gamma_0(m')$ for some m' dividing m.* The quotient $L(f,s)/L(g,s)$ is a finite Dirichlet series with an Euler product $\prod Q_p(s)$, where p runs over the prime divisors of m/m' and $Q_p(s)$ is a polynomial in p^{-s}. The L-series $L(g,s)$ has a functional equation under $s \to k-s$, and $L(f,s)$ can be assumed also to have

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* This statement is not quite true for the case that $k=2$, $m>1$, and f is an Eisenstein series having eigenvalues $\sigma_1(\ell) = \sum_{d|\ell} d$. Here it has to be interpreted in the sense that $g(\tau)$ is the non-holomorphic Eisenstein series $-\frac{1}{24} + \frac{1}{8\pi \operatorname{Im}(\tau)} + \sum_{n=1}^{\infty} \sigma_1(n)\, e^{2\pi i n\tau}$ and $f(\tau) = \sum_{d|m} c_d\, g(d\tau)$ with $\sum d^{-1} c_d = 0$.

one (this is equivalent to requiring f to be an eigenform of all Atkin-Lehner involutions on $M_k(m)$); under these assumptions, each of the Euler factors of $L(f,s)/L(g,s)$ will have a functional equation

$$(1) \qquad Q_p(k-s) = \pm\, p^{-\nu(k-2s)}\, Q_p(s) \qquad (p^{\nu} \| m/m') .$$

The space $\mathfrak{M}_k(m)$ is then the space spanned by all f for which the sign in (1) is "+" for all p. Notice that it is only under this condition that the order of vanishing of $L(f,s)$ at $s=k/2$ can be the same as that of $L(g,s)$: as soon as even one sign in one of the equations (1) is "-", $L(f,s)$ vanishes at $s=k/2$ to a higher order than $L(g,s)$ and the leading term of its Taylor expansion at this point is the product of the corresponding quantity for g with some extraneous factors log p. It is thus natural to expect $\mathfrak{M}_k(m)$ to be the relevant space of modular forms in any context like the Birch-Swinnerton-Dyer conjecture where the leading term in question is supposed to have a natural interpretation as the regulator of some height pairing. It also explains why $\mathfrak{M}_2^-(m)$ is the space occurring in the result about Heegner points mentioned above, since the heights of Heegner points are related to the derivatives of L-series of cusp forms of weight 2 having an odd functional equation.

Apart from the naturalness of its definition and its occurrence in connection with Jacobi forms, the strongest indication that the space $\mathfrak{M}_k(m)$ is important is that the trace formula for Hecke operators is actually simpler for $\mathfrak{M}_k(m)$ than for either $M_k(m)$ or $M_k^{new}(m)$. This can already be seen on the level of dimensions (i.e. the trace of the operator T(1)), as we now discuss. The well-known formula for $\dim M_k(m)$ is

$$(2) \qquad \dim M_k(m) = \sum_{i=1}^{4} c_i(k)\, f_i(m) ,$$

where

$$c_1(k) = \frac{k-1}{12}, \quad c_2(k) = \frac{1}{2}, \quad c_3(k) = -\frac{1}{3}\chi_3(k-1), \quad c_4(k) = -\frac{1}{4}\chi_4(k-1)$$

(χ_3 and χ_4 the primitive Dirichlet characters of conductor 3 and 4) and the

$f_i(m)$ are the multiplicative functions given on prime powers by

$$f_1(p^\nu) = p^\nu + p^{\nu-1} ,$$
$$f_2(p^\nu) = p^{[\nu/2]} + p^{[(\nu-1)/2]} ,$$
$$f_3(p^\nu) = \chi_3(p^\nu) + \chi_3(p^{\nu-1}) ,$$
$$f_4(p^\nu) = \chi_4(p^\nu) + \chi_4(p^{\nu-1}) .$$

It is very striking that each $f_i(p^\nu)$ has the form $g_i(p^\nu) + g_i(p^{\nu-1})$ with a much simpler multiplicative function g_i, namely:

$$g_1(m) = m, \quad g_2(m) = b \text{ where } m = ab^2 \text{ with } a \text{ squarefree}, \quad g_3 = \chi_3 , \quad g_4 = \chi_4 .$$

Using Atkin-Lehner theory to relate $M_k(m)$ to $M_k^{new}(m)$, we find an analogous statement for the latter space: the dimension of $M_k^{new}(m)$ is given by a formula like (2) but with $f_i(m)$ replaced by the multiplicative function $f_i^{new}(m)$ given on prime powers by

$$f_i^{new}(p^\nu) = g_i(p^\nu) - g_i(p^{\nu-1}) - g_i(p^{\nu-2}) + g_i(p^{\nu-3})$$

(with the convention $g_i(p^\mu) = 0$ for $\mu < 0$). Thus $f_i^{new}(m) \le g_i(m) \le f_i(m)$ and g_i is a much simpler function than either f_i or f_i^{new}. This alone already suggests the existence of a natural intermediate space $\mathfrak{M}_k(m)$ between $M_k^{new}(m)$ and $M_k(m)$ with dimension given by

$$(3) \qquad \dim \mathfrak{M}_k(m) = \sum_{i=1}^{4} c_i(k)\, g_i(m)$$

and such that there is a natural decomposition

$$(4) \qquad M_k(m) = \bigoplus_{\substack{m'|m \\ m/m' \text{ squarefree}}} \mathfrak{M}_k(m')$$

corresponding to the formula $f_i(p^\nu) = g_i(p^\nu) + g_i(p^{\nu-1})$. Equations (3) and (4) are indeed true for the space $\mathfrak{M}_k(m)$ defined above. Equation (3) can be written in the even simpler form

(5) $$\dim \mathfrak{M}_k(m) = d(m(k-1)) + \frac{1}{2}b \qquad (\text{ b as above }),$$

where $d(n)$ is the linear-plus-periodic function $\frac{n}{12} - \frac{1}{3}\chi_3(n) - \frac{1}{4}\chi_4(n)$. For cusp forms the situation is similar: $\dim S_k(m)$ is given by a formula like (2) but with $c_2 = -\frac{1}{2}$ and an extra contribution 1 if $k=2$, and the dimension of $\mathfrak{S}_k(m) = \mathfrak{M}_k(m) \cap S_k(m)$ is given by (5) but with $\frac{1}{2}b$ replaced by $-\frac{1}{2}b$ and an extra contribution 1 if $k=2$ and m is a perfect square. (Compare [E-Z], §10.)

As already mentioned, the simplification occurring for the dimensions on passing from M to $\mathfrak{M}$ occurs also--indeed, even more strikingly--for the traces of Hecke operators. The trace formula for $SL_2(\mathbb{Z})$ (cf. [Z]) has the relatively simple form

(6) $$\operatorname{tr}(T(\ell), S_k(1)) = -\frac{1}{2}\sum_{s^2 \leq 4\ell} p_k(s,\ell)\, H(s^2-4\ell) - \frac{1}{2}\sum_{\ell' \mid \ell} \min(\ell', \ell/\ell')^{k-1} + \begin{cases} \sigma_1(\ell) & \text{if } k=2, \\ 0 & \text{otherwise,} \end{cases}$$

(here $p_k(s,\ell)$ is a certain Gegenbauer polynomial and $H(\Delta)$ a certain class number; cf. §1), but the trace formula for $\Gamma_0(m)$ for $m > 1$ as usually given in the literature is very much more complicated. In contrast to this, the formula for the trace of $T(\ell)$ on $\mathfrak{S}_k(m)$ for $m > 1$ is hardly any worse than (6): one simply replaces $H(\Delta)$ by a slightly modified class number $H_m(\Delta)$ (for the definition, see §1), multiplies the term $\min(\ell', \ell/\ell')^{k-1}$ by the g.c.d. of b and $\ell' - \ell/\ell'$, and omits the third term in (6) unless m is a perfect square. (This is $2\, s_{k/2+1,m}(\ell,1)$ in the notation of Theorem 1, §1.)

§0. Notations and basic definitions

As main reference for the basic facts and definitions from the theory of Jacobi forms we refer to [E-Z]. Here we briefly summarize those items that we shall need in the following.

$J(\mathbb{R})$ denotes the Jacobi group $SL_2(\mathbb{R})\ltimes\mathbb{R}^2\cdot S^1$. A typical element of $J(\mathbb{R})$ has the form $\xi=A[x]s$ with $A\in SL_2(\mathbb{R})$, $x\in\mathbb{R}^2$, $s\in S^1$ (the multiplicative group of complex numbers of modulus 1) and the product of ξ and an element $\xi'=A'[x']s'$ is given by $\xi\cdot\xi' = (AA')[xA'+x'](ss'e^{2\pi i\left|{xA' \atop x'}\right|})$. Here xA' is the result of applying the matrix A' to the row vector x and $\left|{xA' \atop x'}\right|$ is the determinant of the matrix built from the row vectors xA' and x'. For subsets G,L,K of $SL_2(\mathbb{R})$, $\mathbb{R}^2$, S^1 respectively we use $G\ltimes L\cdot K$ for the subset $\{A[x]s : A\in G,\ x\in L,\ s\in K\}$ of $J(\mathbb{R})$. For $k,m\in\mathbb{Z}$ there is an action of $J(\mathbb{R})$ on functions on $\mathcal{H}\times\mathbb{C}$ ($\mathcal{H}$ = upper half-plane) given by

$$(\phi|_{k,m}\xi)(\tau,z) = (c\tau+d)^{-k}\, e^m\Big(\frac{-c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z+\lambda\mu\Big)\, s^m\,\phi\Big(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\Big)$$

$$\Big(\xi=\begin{bmatrix}a&b\\c&d\end{bmatrix}\cdot[\lambda,\mu]\cdot s\in J(\mathbb{R}),\quad (\tau,z)\in\mathcal{H}\times\mathbb{C}\Big),$$

where $e^m(x)$ denotes $e^{2\pi imx}$. We shall always use Γ to denote the full modular group $SL_2(\mathbb{Z})$, Γ^J for the corresponding Jacobi group $SL_2(\mathbb{Z})\ltimes\mathbb{Z}^2$, and $J_{k,m}$ for the space of Jacobi forms of weight k and index m on Γ, i.e., holomorphic functions $\phi:\mathcal{H}\times\mathbb{C}\to\mathbb{C}$ satisfying $\phi|_{k,m}\xi=\phi$ for all $\xi\in\Gamma^J$ and having a Fourier development of the form

$$(1)\qquad \phi(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}\\ 4nm-r^2\geq 0}} c(n,r)\,q^n\,\zeta^r \qquad (q=e^{2\pi i\tau},\quad \zeta=e^{2\pi iz}).$$

The subspace of cusp forms (i.e., ϕ with $c(n,r)=0$ unless $4nm-r^2>0$) is denoted $S_{k,m}$. As a simple consequence of the invariance of ϕ with respect to $\mathbb{Z}^2\subset\Gamma^J$, one has that $c(n,r)$ depends only on r^2-4nm and on $r \pmod{2m}$, so we can also write (1) in the form

$$(2)\qquad \phi(\tau,z) = \sum_{\substack{\Delta,r\in\mathbb{Z},\ \Delta\leq 0\\ \Delta\equiv r^2 \pmod{4m}}} C(\Delta,r)\,q^{\frac{r^2-\Delta}{4m}}\,\zeta^r$$

where $C(\Delta,r)$ depends only on Δ and on $r \pmod{2m}$.

For a positive integer ℓ with $(\ell,m)=1$ we have a Hecke operator $T(\ell)$ on $J_{k,m}$ defined by

(3) $$\phi|T(\ell) = \ell^{k-4} \sum_{x \in \mathbf{Z}^2/\ell\mathbf{Z}^2} \sum_{\substack{M \in \Gamma\backslash M_2(\mathbf{Z}) \\ \det(M)=\ell^2 \\ \gcd(M)=\square}} \phi|_{k,m}(\tfrac{1}{\ell}M\cdot[x]) \qquad (\phi \in J_{k,m}),$$

where "$\gcd(M)=\square$" means that the greatest common divisor of the entries of M is a square. In the notation of [S-Z] this can be written

(4) $$T(\ell) = \ell^{k-4} \sum_{\substack{\ell'|\ell \\ \ell/\ell'=\square}} \frac{\ell^2}{\ell'} H_{k,m,\Gamma}(\Gamma^J \begin{bmatrix} \ell'^{-1} & 0 \\ 0 & \ell' \end{bmatrix} \Gamma^J),$$

where $H_{k,m,\Gamma}(\Delta)$ for any Γ^J-double coset Δ (or finite union of such sets) in $J(\mathbb{Q})$ ($:=SL_2(\mathbb{Q})\ltimes\mathbb{Q}^2\cdot S^1$) is the operator $\phi \to \sum_{\xi \in \Gamma^J\backslash\Delta} \phi|_{k,m}\xi$. In [E-Z] it was shown that the Fourier coefficients $C^*(\Delta,r)$ of $\phi|T(\ell)$ are related to the Fourier coefficients $C(\Delta,r)$ of ϕ by

(5) $$C^*(\Delta,r) = \sum_{a|\ell^2} a^{k-2}\chi_\Delta(a)\, C(\tfrac{\ell^2}{a^2}\Delta, r'),$$

the sum being over those $a|\ell^2$ with $a^2|\ell^2\Delta$, $\ell^2\Delta/a^2 \equiv 0,1 \bmod 4$, with r' determined by $\ell r \equiv ar' \bmod 2m(a,\ell)$ and $r'^2 \equiv \frac{\ell^2}{a^2}\Delta \bmod 4m$, and with $\chi_\Delta(a) = f\left(\frac{\Delta/f^2}{a/f^2}\right)$ if $(a,\Delta)=f^2$ with $\Delta/f^2 \equiv 0,1 \bmod 4$ and $\chi_\Delta(a)=0$ otherwise.

Also it was shown that

(6) $$T(\ell)T(\ell') = \sum_{d|\ell,\ell'} d^{2k-3}T(\ell\ell'/d^2).$$

For $n\|m$ (i.e. $n|m$ and n and m/n are coprime) we define

(7) $$W_n = n^{-2}H_{k,m,\Gamma}(\Gamma \ltimes (\tfrac{1}{n}\mathbf{Z}^2)\cdot\mu_n)$$

where μ_n denotes the group of the n-th roots of unity. (Note that $\Gamma \ltimes (\frac{1}{2}\mathbf{Z}^2)\cdot\mu_n$ is invariant with respect to right and left multiplication with elements of Γ^J). One easily verifies that

(8) $\phi|W_n = n^{-1} \sum_{x\in\mathbb{Z}^2/n\mathbb{Z}^2} \phi|_{k,m}[\frac{x}{n}]$.

Also it is not hard to verify that

(9) $\phi|W_n = C(\Delta,\lambda_n r)q^{\frac{r^2-\Delta}{4m}}\zeta^r$ ($C(\Delta,r)$ as in (2)) ,

where λ_n is the modulo 2m uniquely determined integer which satisfies $\lambda_n \equiv -1 \bmod 2n$ and $\lambda_n \equiv +1 \bmod 4m/n$ (cf. [S]). Thus the W_n form a group of involutions.

Finally note that the W_n and $T(\ell)$ commute, as is easily seen by (3) and (8) or (5) and (9).

§1. The trace formula for Jacobi forms on $SL_2(\mathbb{Z})$

The object of this paragraph is to apply the results of [S-Z] to obtain a formula for the trace of $T(\ell) \circ W_n$ on $S_{k,m}$.

We need some definitions.

We define a function $H_n(\Delta)$ for integers $n \geq 1, \Delta \leq 0$. The function $H_1(\Delta)$ equals $H(|\Delta|)$, where $H(\ )$ is the Hurwitz-Kronecker-class number, i.e.

$$H_1(0) = -\frac{1}{12}$$

and $H_1(\Delta)$ for $\Delta < 0$ is the number of equivalence classes with respect to Γ of integral, positive definite, binary quadratic forms of discriminant Δ, counting forms equivalent to a multiple of $x^2 + y^2$ (resp. $x^2 + xy + y^2$) with multiplicity $\frac{1}{2}$(resp. $\frac{1}{3}$). Note that $H_1(\Delta) = 0$ unless $\Delta \equiv 0$ or $1 \pmod 4$. For $n \geq 2$ we write $(n,\Delta) = a^2 b$ with squarefree b and put

$$H_n(\Delta) = \begin{cases} a^2 b \left(\dfrac{\Delta/a^2 b^2}{n/a^2 b}\right) H_1(\Delta/a^2 b^2) & \text{if } a^2 b^2 \mid \Delta \\ 0 & \text{otherwise} \end{cases} .$$

Furthermore, for numbers s,ℓ and integers $k \geq 2$ we define $p_k(s,\ell)$ as the coefficient of x^{k-2} in the power series development of $(1-sx+\ell x^2)^{-1}$, i.e.

$$p_k(s,\ell) = \frac{\rho^{k-1} - \rho'^{k-1}}{\rho - \rho'} \qquad (\rho, \rho' \text{ the roots of } X^2 - sX + \ell = 0) \text{ if } s^2 - 4\ell \neq 0$$

$$p_k(s,\ell) = (k-1)\left(\frac{s}{2}\right)^{k-2} \quad \text{if} \quad s^2-4\ell = 0 \ .$$

Finally $\sigma_0(n) = \sum_{d|n} 1$, $\sigma_1(n) = \sum_{d|n} d$ (as usual), $Q(n)$ denotes the greatest integer whose square divides n (i.e. $Q(n) = \prod_{p^\lambda \| n} p^{[\lambda/2]}$), and $\delta(P) = 1$ or 0 accordingly as the statement P is true of false.

<u>Theorem 1</u>. <u>Let</u> k,m,ℓ,n <u>be positive integers</u>, $k \geq 2$, $(\ell,m) = 1$ <u>and</u> $n \| m$. <u>Then</u>

$$\mathrm{tr}(T(\ell)\circ W_n, S_{k,m}) = s_{k,m}(\ell,n) + (-1)^k \, s_{k,m}\left(\ell,\frac{m}{n}\right) ,$$

<u>where</u> $s_{k,m}(\ell,n)$ <u>for any</u> $n \| m$ <u>is given by</u>

$$s_{k,m}(\ell,n) = -\frac{1}{4} \sum_{n'|n} \sum_{s} p_{2k-2}\left(\frac{s}{\sqrt{n'}},\ell\right) H_{\frac{m}{n}}(s^2-4\ell n')$$

$$-\frac{1}{4} \sum_{\ell'|\ell} \min\left(\ell',\frac{\ell}{\ell'}\right)^{2k-3} \cdot \left(Q(n),\ell'+\frac{\ell}{\ell'}\right) \cdot \left(Q\left(\frac{m}{n}\right),\ell'-\frac{\ell}{\ell'}\right)$$

$$+\frac{1}{2}\,\delta(k=2)\;\delta\left(\frac{m}{n}=\square\right)\,\sigma_0(n)\,\sigma_1(\ell) ,$$

<u>the sum over</u> s <u>being over all integers</u> s <u>satisfying</u> $s^2 \leq 4\ell n', n'|s$, $\left(\left(\frac{s}{n'}\right)^2,\frac{n}{n'}\right) = $ squarefree.

The rest of this paragraph will be devoted to the proof of this formula, and will presuppose familiarity with the paper [S-Z], whose notations we will not repeat.

<u>Proof</u> - Let $M \in SL_2(\mathbb{Q})$. Then by (7) of §0 and ~~by~~ standard computations in the theory of Hecke algebras

$$H_{k,m,\Gamma}(\Gamma^J M \Gamma^J) W_n = n^{-2} H_{k,m,\Gamma}(\Gamma^J M \Gamma^J) \; H_{k,m,\Gamma}(\Gamma \ltimes (\tfrac{1}{n}\mathbb{Z}^2)\cdot\mu_n)$$

$$= n^{-2} \sum_{\xi\in\Gamma^J\backslash\Gamma^J M(\Gamma\ltimes(\frac{1}{n}\mathbb{Z}^2)\cdot\mu_n)/\Gamma^J} c(\xi) H_{k,m,\Gamma}(\Gamma^J\xi\Gamma^J)$$

where

$$c(\xi) = \#\{\eta\in\Gamma^J\backslash\Gamma\ltimes(\tfrac{1}{n}\mathbb{Z}^2)\cdot\mu_n \mid \xi\in\Gamma^J M\Gamma^J\eta\} .$$

To the right hand side of this we can apply Theorem 1 and the supplementary formulas (3.10), (3.11) of [S-Z] to obtain

$$\begin{aligned} &\operatorname{tr}\,(H_{k,m,\Gamma}(\Gamma^J M\Gamma^J)\circ W_n,\ S_{k,m}) \\ (1)\qquad &= \text{den.}(M) \sum_{A\in\Gamma M\Gamma/\sim_{m,\Gamma}} I_{k,m,\Gamma}(A)\ g(A) \\ &\qquad + \delta(k=2)\cdot \operatorname{tr}(H^*_{1,m,\Gamma}(\Gamma^J M\Gamma^J)\circ W^*_n,\ J^*_{1,m}) \end{aligned}$$

where den.(M) denotes the smallest integer ℓ' such that $\ell' M$ is integral, where W^*_n and $J^*_{1,m}$ are used for $H^*_{1,m,\Gamma}(\Gamma\ltimes(\frac{1}{n}\mathbb{Z}^2)\cdot\mu_n)$ and $J^*_{1,m}(\Gamma)$ respectively, and where

$$\begin{aligned} g(A) &= \text{den.}(M)^{-1} n^{-2} \sum_{\xi\in\mathbb{Z}^2\backslash\mathbb{Z}^2 A(\frac{1}{n}\mathbb{Z}^2)\cdot\mu_n/\mathbb{Z}^2} c(\xi)\,\#(\mathbb{Z}^2\backslash\mathbb{Z}^2\xi\mathbb{Z}^2)\cdot G_m(\xi) \\ (2)\qquad & \\ &= \text{den.}(M)^{-1} n^{-2} \sum_{\xi\in\mathbb{Z}^2\backslash\mathbb{Z}^2 A(\frac{1}{n}\mathbb{Z}^2)\cdot\mu_n} c(\xi)\ G_m(\xi) . \end{aligned}$$

For the definitions of $I_{k,m,\Gamma}(A)$, $G_m(\xi)$ and $\sim_{m,\Gamma}$ see §1 of [S-Z], for the definitions of $H^*(\ )$ and $J^*_{1,m}(\ )$ see §3 of [S-Z]. In (2) $\mathbb{Z}^2$ has to be considered as subgroup of $J(\mathbb{R})$, i.e. $\mathbb{Z}^2$ has to be identified with $\{1\}\ltimes\mathbb{Z}^2\cdot\{1\}$; in particular ξ runs over a set of representatives for the $\mathbb{Z}^2$-double cosets (or $\mathbb{Z}^2$-left cosets in the second sum) of $\mathbb{Z}^2 A(\frac{1}{n}\mathbb{Z}^2)\cdot\mu_n$, the latter denoting the product of the complexes $\{1\}\ltimes\mathbb{Z}^2\cdot\{1\}$ and $\{A\}\ltimes(\frac{1}{n}\mathbb{Z}^2)\cdot\mu_n$ in $J(\mathbb{R})$.

For a ξ on the right hand side of (2) we can write

$$c(\xi) = \#\{\eta \in \mathbb{Z}^2\backslash(\tfrac{1}{n}\mathbb{Z}^2)\cdot\mu_n \mid \xi \in \mathbb{Z}^2 A \mathbb{Z}^2 \eta\} .$$

Also, consulting the definition of $G_m(\xi)$ in [S-Z], it is easily seen that $G_m(\xi) = G_m(A\eta)$ for $\xi \in \mathbb{Z}^2 A\, \mathbb{Z}^2 \eta$ with $\eta \in (\frac{1}{n}\mathbb{Z}^2)\cdot\mu_n$. Hence (2) becomes

$$g(A) = \mathrm{den.}(M)^{-1} n^{-2} \sum_{\xi \in \mathbb{Z}^2\backslash \mathbb{Z}^2 A(\frac{1}{n}\mathbb{Z}^2)\cdot\mu_n} \sum_{\substack{\eta \in \mathbb{Z}^2\backslash(\frac{1}{n}\mathbb{Z}^2)\cdot\mu_n \\ \xi \in \mathbb{Z}^2 A \mathbb{Z}^2 \eta}} G_m(A\eta)$$

$$= \mathrm{den.}(M)^{-1} n^{-2} \sum_{\eta \in \mathbb{Z}^2\backslash(\frac{1}{n}\mathbb{Z}^2)\cdot\mu_n} \#(\mathbb{Z}^2\backslash \mathbb{Z}^2 A \mathbb{Z}^2 \eta)\cdot G_m(A\eta) .$$

$$= n^{-1} \sum_{y \in \frac{1}{n}\mathbb{Z}^2/\mathbb{Z}^2} G_m(A[y])$$

where in the last equation we used $\#(\mathbb{Z}^2\backslash \mathbb{Z}^2 A \mathbb{Z}^2 \eta) = \#(\mathbb{Z}^2\backslash \mathbb{Z}^2 A \mathbb{Z}^2) = \mathrm{den.}(A) = \mathrm{den.}(M)$. Now we apply Theorem 2 of [S-Z] to obtain

$$(3) \qquad g(A) = \begin{cases} n\ \mathrm{sign}\,(t-2)(t-2)^{1/2}\ \mathrm{Av}_x e\left(\frac{m}{n^2(t-2)}\, Q_A(x)\right) & \text{if } t = \mathrm{tr}(A) \neq 2 \\ \frac{m}{n}\varepsilon(a_A)\ \mathrm{sign}\,(t+2)(t+2)^{1/2}\ \mathrm{Av}_x e\left(\frac{n^2}{m(t+2)}\, Q_A(x)\right) & \text{if } t = \mathrm{tr}(A) \neq -2 \end{cases}$$

Here we have used the obvious identities

$$\sum_{y \in \frac{1}{n}\mathbb{Z}^2/\mathbb{Z}^2} \mathrm{Av}_x e\left(\frac{m}{t-2}\, Q_A(x+y)\right) = n^2\ \mathrm{Av}_x e\left(\frac{m}{n^2(t-2)}\, Q_A(x)\right)$$

and

$$\sum_{y \in \frac{1}{n}\mathbb{Z}^2/\mathbb{Z}^2} \mathrm{Av}_x e\left(\frac{1}{m(t+2)}\, Q_A(x) + \left|\begin{matrix} y \\ x \end{matrix}\right|\right) = \mathrm{Av}_x \left(\frac{n^2}{m(t+2)}\, Q_A(x)\right) .$$

For the definition of the functional Av_x (which assigns to a periodic function of $x \in \mathbb{Z}^r$ its average value) see §4 of [S-Z] or the Appendix ; Q_A stands for the binary quadratic form $Q_A(\lambda,\mu) = b\lambda^2 + (d-a)\lambda\mu - c\mu^2$ $\quad (A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))$.

Finally, combining (0.4) and (1), we arrive at the explicit formula

$$\mathrm{tr}(T(\ell)\circ W_n, S_{k,m}) = \tag{4}$$

$$= \ell^{k-2} \cdot \sum_{\substack{A \in SL_2(\mathbb{Q})/\sim_{m,\Gamma} \\ \ell A \text{ integral} \\ \text{g.c.d}(\ell A)=\square}} I_{k,m,\Gamma}(A) g(A) + \delta(k=2)\cdot \mathrm{tr}(T(\ell)^{*} \circ W_n^{*}, J_{0,m}^{*})$$

the sum being over a complete set of representatives of $\sim_{m,\Gamma}$ - equivalence classes contained in $\{A \in SL_2(\mathbb{Q}) | \ell A$ integral, g.c.d.$(\ell A) = \square\}$, with $I_{k,m,\Gamma}(A)$ as in Theorem 1 of [S-Z], $g(A)$ as in (3), and $T(\ell)^{*}$ given by a formula like (0.4) but with $k=2$ and $H_{k,m,\Gamma}$ replaced by $H^{*}_{2,m,\Gamma}$.

We shall now investigate the first sum in (4). For this purpose we decompose it as

$$S_{\text{ell.}} + S_{\text{hyp.}} + S_{\text{par}} + S_{\text{scal.}} \quad ,$$

where $S_{\text{ell.}}$, $S_{\text{hyp.}}$ etc. denote the contribution of the elliptic A, hyperbolic A etc.

<u>The elliptic contribution</u>

For an elliptic A one has $I_{k,m,\Gamma}(A) = \frac{\mathrm{sign}(c)}{|\Gamma_A|} \quad \frac{\rho^{3/2-k}}{\rho-\rho'}$, where

$\rho = \frac{t+\mathrm{sign}(c)(t^2-4)^{1/2}}{2}$, $t = \mathrm{tr}(A)$, $A = \begin{bmatrix} * & * \\ c & * \end{bmatrix}$. Thus $I_{k,m,}(-A) = -\overline{I_{k,m,\Gamma}(A)}$.
Also, by (3), one has $g(-A) = -\overline{g(A)}$. Hence

$$S_{ell.} = 2\ell^{k-2} \sum_A \mathrm{Re}(I_{k,m,\Gamma}(A)g(A))$$

(5)

$$= 2\ell^{k-2} \sum_A \mathrm{Re}(I_{k,m,\Gamma}(A))\mathrm{Re}(g(A)) - 2\ell^{k-2} \sum_A \mathrm{Im}(I_{k,m,\Gamma}(A))\mathrm{Im}(g(A)),$$

the sums being over those (elliptic) A mod $\sim_{m,\Gamma}$ with positive left entry.

Let us consider the first sum on the right hand side of (5).

Here one has

$$\ell^{k-2}\mathrm{Re}(I_{k,m,\Gamma}(A)) = -\frac{1}{2\sqrt{t+2}}\; p_{2k-2}(\sqrt{\ell(t+2)},\ell) \cdot \frac{1}{|\Gamma_A|} ,$$

using $\frac{t+\sqrt{t^2-4}}{2} = \left(\frac{\sqrt{t+2} + \sqrt{t-2}}{2}\right)^2$, and

$$2\,\mathrm{Re}\, g(A) = \frac{m}{n}\,(t+2)^{1/2}\left\{\mathrm{Av}_x\, e\left(\frac{n^2}{m(t+2)}\, Q_A(x)\right) + \mathrm{Av}_x\, e\left(\frac{n^2}{m(t+2)} Q_{(-A)}(x)\right)\right\} ,$$

by the second formula in (3), using $-Q_A = Q_{-A}$. Thus, the first sum on the right hand side of (5) equals $s_{k,m}(\ell,n)_{ell.}$, where $s_{k,m}(\ell,n)_{ell.}$ is defined for any $n \| m$ by

(6) $$s_{k,m}(\ell,n)_{ell.} := \frac{-1}{2} \sum_{0<a<4\ell} p_{2k-2}(\sqrt{a},\ell)\frac{m}{n} \sum_A \frac{1}{|\Gamma_A|}\, \mathrm{Av}_x e\left(\frac{n^2}{m\cdot a}\cdot \ell Q_A\right) .$$

The first sum is over all integers a with $0 < a < 4\ell$, and the second sum is over all (!) elliptic A modulo Γ-conjugacy such that ℓA is integral, $\mathrm{tr}(A) + 2 = a/\ell$, and g.c.d (ℓA) is a square in $\mathbb{Z}$.

A similar calculation for the second sum on the right hand side of (5) (using now the first formula for $g(A)$ in (3)) shows that it equals $(-1)^k s_{k,m}(\ell,\frac{m}{n})_{\text{ell}}$.

As it well-known, the map $A \longrightarrow \ell\cdot Q_A$ defines a bijection between $\{A\in SL_2(\mathbb{Q})\mid \ell A \text{ integral}, \operatorname{tr}(A)+2=a/\ell\}$ and integral binary quadratic forms with discriminant $a(a-4\ell)$, such that Γ-conjugacy classes on the one side correspond to equivalence classes with respect to Γ on the other side. Furthermore, $|\Gamma_A|$ corresponds to $|\Gamma_{Q_A}|$ ($\Gamma_{Q_A}\subset\Gamma$ the automorphism group of Q_A), and, as is easily checked, $\text{g.c.d.}(\ell A)=\square$ corresponds to $(c(Q),\ell)=\square$, where $c(Q)$ denotes the g.c.d. of the coefficients of Q. Hence the inner sum in (6) equals

$$(7)\qquad \sum_{\substack{Q \bmod \Gamma\\ d(Q)=a(a-4\ell)\\ (c(Q),\ell)=\square}} \frac{1}{|\Gamma_Q|}\, Av_x\, e\!\left(\frac{n^2}{ma}Q(x)\right),$$

where the sum is over all integral, binary quadratic forms Q modulo Γ with discriminant $d(Q)=a(a-4\ell)$ and $(c(Q),\ell)=\square$.

To get rid of the condition "$(c(Q),\ell)=\square$", we use Liouville's function $\lambda(n)$. It has the characteristic property $\sum_{d\mid n}\lambda(d)=\delta(n=\square)$. We can therefore rewrite (7) as

$$(8)\qquad \sum_{t\mid \ell,a}\lambda(t) \sum_{\substack{Q \bmod \Gamma\\ d(Q)=a(a-4\ell)/t^2}} \frac{1}{|\Gamma_Q|}\, Av_x\, e\!\left(\frac{n^2}{ma}\, t\cdot Q(x)\right).$$

Now, by Proposition A.1, the inner sum equals

$\left(\frac{ma}{n(n,a)t}\right)^{-1} \quad H_{\frac{ma}{n(n,a)t}} \left(\frac{a(a-4\ell)}{t^2}\right)$ (to apply this proposition, note that $(\frac{m}{n}a, nt) = (a,n)t$, since $n \| m$ and $(m,\ell) = 1$). Thus (8) becomes

$$(9) \qquad \sum_{t|\ell,a} \lambda(t) \left(\frac{ma}{n(n,a)t}\right)^{-1} H_{\frac{ma}{n(n,a)t}} \left(\frac{a(a-4\ell)}{t^2}\right) .$$

We shall show in a moment that (9) equals

$$(10) \qquad \begin{cases} (\frac{m}{n})^{-1} H_{\frac{m}{n}}(s^2-4\ell n') & \text{if } a = s^2/n' \text{ with } n' | (n,s) \text{ and } (\frac{n}{n'}, (\frac{s}{n'})^2) = \text{squarefree} \\ 0 & \text{otherwise} . \end{cases}$$

Note that n' and s are uniquely determined by a.

Thus, summing in (6) over n',s instead of a, we shall end with the formulas

$$(11) \qquad s_{k,m}(\ell,n)_{ell.} = -\frac{1}{2} \sum_{n'|n} \sum_{\substack{s>0 \\ s^2<4\ell n' \\ n'|s \\ (\frac{n}{n'}, (\frac{s}{n'})^2)=\text{squarefree}}} P_{2k-2}(s/\sqrt{n'},\ell)\, H_{\frac{m}{n}}(s^2-4\ell n'),$$

$$S_{ell.} = s_{k,m}(\ell,n)_{ell.} + (-1)^k s_{k,m}(\ell,\tfrac{m}{n})_{ell.} .$$

To show that (9) and (10) are equal we note first of all the following simple property of our function H: let r,s be positive integers, let $\Delta\le 0$ with $r|\Delta$, write $r=x^2/y$ with squarefree y and assume $(y,s)=1$; then $H_{rs}(\Delta)$ equals $rH_s(\Delta/x^2)$ if x^2 divides Δ as discriminant (i.e. $x^2|\Delta$, $\frac{\Delta}{x^2}\equiv 0,1 \bmod 4$) and 0 otherwise.

Now write $a=bc$ with $(b,\ell)=1$ and $c|\ell^\infty$. Then, since $(m,\ell)=1$, we have for any $t|(\ell,a)$ the decomposition $\frac{ma}{n(n,a)t}=\frac{c}{t}\cdot\frac{mb}{n(n,b)}$ with $\frac{c}{t}$ and $\frac{mb}{n(n,b)}$ being relative prime and with $\frac{c}{t}|\frac{a(a-4\ell)}{t^2}$. Hence $H_{\frac{ma}{n(n,a)t}}(\frac{a(a-4\ell)}{t^2})$ equals $\frac{c}{t}H_{\frac{mb}{n(n,b)}}(\frac{a(a-4\ell)}{x^2t^2})$ if $\frac{c}{t}=\frac{x^2}{y}$ with squarefree y and x^2 dividing $a(a-4\ell)/t^2$ as discriminant and 0 otherwise. Note that the condition "x^2 divides $a(a-4\ell)/t^2$ as discriminant" is equivalent to $x^2t^2|c\ell$ since clearly $xt|c$ and $(b,xt)=1$. Substituting this into (9) gives

$$(12)\qquad \sum_t \lambda(t)\,\frac{n(n,b)}{mb}\,H_{\frac{mb}{n(n,b)}}\left(\frac{a(a-4\ell)}{x^2t^2}\right)$$

with t running through all divisors of (ℓ,a) such that $\frac{c}{t}=\frac{x^2}{y}$ with squarefree y and $x^2t^2|c\ell$. We split this sum into two sums, one over X (=xt) and one over t, where X runs through all divisors of c with $c|X^2$, $X^2|c\ell$ and where t runs through all $t|\frac{X^2}{c}$ with X^2/ct squarefree. By well-known properties of Liouville's λ the sum over t equals 1 if $X^2/c=1$ and 0 otherwise. Hence (12) becomes

$$(13)\qquad \begin{cases} \frac{n(n,b)}{mb}\,H_{\frac{mb}{n(n,b)}}(b(a-4\ell)) & \text{if } c=\square \\ 0 & \text{otherwise} \end{cases}.$$

To further simplify (13) let now $\frac{b}{(n,b)}$ play the role of r in the above formula for $H_{rs}(\Delta)$. Write $\frac{b}{(n,b)}=\frac{x^2}{y}$, y sqarefree. We shall show in a moment that (13) is 0 unless y divides (n,b). The latter implies in particular $(y,\frac{m}{n})=1$. Thus (13) becomes

$$\begin{cases} \frac{n}{m} H_{\frac{m}{n}}\left(\frac{b(a-4\ell)}{x^2}\right) & \text{if } c=\square \text{ , } y|(n,b) \text{ , } x^2|b(a-4\ell) \\ 0 & \text{otherwise} \end{cases} .$$

But this can now be written in the form (10) with $n'=\frac{(n,b)}{y}$ and $s=(n,b)\frac{x}{y}\sqrt{c}$.

So assume now that (13) is differnt from 0. Then x^2 clearly divides $b(a-4\ell)$ and furthermore

(14) $$\frac{b(a-4\ell)}{x^2} \equiv 0,1 \bmod 4 .$$

Since

$$\frac{b(a-4\ell)}{x^2} = \{(n,b)^2 x^2 c - 4\ell(n,b)y\}/y^2 \quad \text{with} \quad y \nmid x$$

and since $c=\square$ by assumption we see that (14) implies $y|\ell(n,b)$. Since $(y,\ell)=1$ (note $y|b$ and $(b,\ell)=1$) we finally deduce $y|(n,b)$ as was to be shown.

The hyperbolic contribution

Using the second formula for $g(A)$ in (3) one finds that the contribution of the hyperbolic matrices with positive trace is given by $s_{k,m}(\ell,n)_{hyp.}$ where for any $n||m$ the expression $s_{k,m}(\ell,n)_{hyp}$ is given by

$$(15)\quad s_{k,m}(\ell,n)_{hyp} = -\frac{\ell^{k-2}}{2}\sum_{t}\left(\frac{t+\sqrt{t^2-4}}{2}\right)^{3/2-k}(t-2)^{-1/2}\sum_{A} A\nu_x e\left(\frac{n^2}{m(t+2)}\, Q_A(x)\right).$$

Here t turns through all positive rational numbers with demominator ℓ such that t^2-4 is a square in $\mathbb{Q}\setminus\{0\}$, and A through all matrices with $tr(A) = t$, ℓA integral, g.c.d.(ℓA) a square.

Using the first formula for $g(A)$ in (3) one easily verifies that the contribution of the hyperbolic matrices with negative trace is given by $(-1)^k s_{k,m}(\ell,\frac{m}{n})_{hyp.}$.

By exactly the same arguments as in the foregoing section and by the remark following Proposition A.1 we deduce that the inner sum in (15) is different from zero if and only if there exist positive integers n' and s satisfying

$$n'|n,\ n'|s,\ \ell(t+2) = s^2/n',\ (\tfrac{n}{n'},(\tfrac{s}{n'})^2) \text{ is squarefree},$$

and that it then equals $(\frac{m}{n},s^2-4\ell n')$ times the class number of binary quadratic forms of discriminant $(s^2-4\ell n')/x^2$ ($(\frac{m}{n},s^2-4\ell n')=\frac{x^2}{y}$, y sqarefree) i.e. $(s^2-4\ell n')^{1/2}\,Q((\frac{m}{n},s^2-4\ell n'))$ (Note that t^2-4 = square in $\mathbb{Q}\setminus\{0\}$ implies $s^2-4\ell n' = \ell^2(t^2-4)n'^2/s^2$ = square in $\mathbb{Z}\setminus\{0\}$).

Inserting all this in (15) yields

$$(16)\qquad s_{k,m}(\ell,n)_{hyp.} = -\frac{1}{2}\sum_{n'|n}\sum_{s}\frac{\ell^{2k-3}\,n'^{k-1}}{\left\{\frac{s+\sqrt{s^2-4\ell n'}}{2}\right\}^{2k-3}}\,Q((\tfrac{m}{n},s^2-4\ell n')),$$

where s runs through all positive integers such that

$$s^2-4\ell n' = \text{square in } \mathbf{Z}\setminus\{0\}\ ,\ n'|s, \text{ and } (\tfrac{n}{n'},(\tfrac{s}{n'})^2) \text{ is squarefree.}$$

Now the condition $s^2-4\ell n' = \text{square } (\neq 0)$ is equivalent to $s = d+\frac{\ell n'}{d}$ with a suitable positive integer d satisfying $d|\ell n'$, $d^2 < \ell n'$. But $n'|s$, i.e. $n'|d+\frac{\ell n'}{d}$, together with $(\ell,n')=1$ (since $(\ell,n)=1$) then implies that n' is a square, say $n' = n''^2$, and that $d = n''\ell'$ for some $\ell'|\ell$, $\ell'^2 < \ell$. Thus, setting $n' = n''^2$, $s = n''(\ell'+\frac{\ell}{\ell'})$ in (16), we obtain

$$s_{k,m}(\ell,n)_{hyp.} = -\frac{1}{2}\sum_{n''^2|n}\ \sum_{\ell'} n''\ell'^{2k-3}\,(Q(m/n),n''(\ell'-\ell/\ell'))\ ,$$

the sum with respect to ℓ' being over all divisors ℓ' of ℓ such that $\ell'^2<\ell$, $n''|(\ell'+\frac{\ell}{\ell'})$ and $(n,(\ell'+\frac{\ell}{\ell'})^2)/n''^2$ is squarefree. Finally, noticing that $(Q(\frac{m}{n}),n''(\ell'-\frac{\ell}{\ell'})) = (Q(\frac{m}{n}),(\ell'-\frac{\ell}{\ell'}))$ (since $(\frac{m}{n},n'')=1$) and that the sum $\sum n''$, where n'' runs through all positive integers with $n''^2|n$, $n''|(\ell'+\frac{\ell}{\ell'})$, $(n,(\ell'+\frac{\ell}{\ell'})^2)/n''^2$ = squarefree, equals $(Q(n),\ell'+\frac{\ell}{\ell'})$ we arrive at the formulas

(17)
$$s_{k,m}(\ell,n)_{hyp} = -\frac{1}{2}\sum_{\substack{\ell'|\ell\\ \ell'^2<\ell}} \ell'^{2k-3}(Q(n),\ell'+\frac{\ell}{\ell'})\cdot(Q(\frac{m}{n}),\ell'-\frac{\ell}{\ell'})$$

$$S_{hyp} = s_{k,m}(\ell,n)_{hyp.} + (-1)^k\, s_{k,m}(\ell,\tfrac{m}{n})_{hyp.}$$

The parabolic contribution

A complete set of representatives with respect to $\sim_{m,\Gamma}$ of the parabolic matrices A such that ℓA is integral and $\text{g.c.d.}(\ell A) = \square$ is given by

$$\pm \begin{bmatrix} 1 & b/\ell \\ 0 & 1 \end{bmatrix} \quad (1 \leq b \leq 4m\ell\ ,\ (b,\ell) = \square).$$

Thus, using

$$I_{k,m,\Gamma}\left(\pm\begin{bmatrix} 1 & b/\ell \\ 0 & 1 \end{bmatrix}\right) = \frac{(\pm 1)^{1/2-k}}{16m} \begin{cases} 1 & \text{if } \ 4m\ell \mid b \\ 1 - i\cot \pi \frac{b}{4m\ell} & \text{otherwise} \end{cases}$$

and the formulae (3) for $g(A)$, we may write

$$S_{\text{par.}} = s_{k,m}(\ell,n)_{\text{par.}} + (-1)^k\ s_{k,m}(\ell,\tfrac{m}{n})_{\text{par.}} \quad ,$$

where for any $n \| m$ the expression $s_{k,m}(\ell,n)_{\text{par.}}$ is defined by

$$(18) \quad s_{k,m}(\ell,n)_{\text{par.}} = \frac{-\ell^{k-2}}{8} \left\{ \frac{1}{n} \sum_{\substack{b \bmod 4m\ell \\ (b,\ell) = \square}} Av_\lambda\, e\!\left(\frac{n^2}{4m\ell}\, b\lambda^2\right) - \right.$$

$$\left. \frac{(-1)^k}{m/n} \sum_{\substack{b \bmod 4m\ell \\ 4m\ell \nmid b,\,(b,\ell)=\square}} i\cot\left(\pi \frac{b}{4m\ell}\right) Av_\lambda e\!\left(\frac{m}{4n^2\ell}\, b\lambda^2\right)\right\}.$$

To simplify the first sum in (18) we note that

$$\sum_{\substack{b \bmod 4m\ell \\ (b,\ell)=\square}} e\!\left(\frac{n^2}{4m\ell}\ b\lambda^2\right) = \sum_{t\mid\ell} \lambda(t) \sum_{\substack{b \bmod 4m\ell \\ t\mid b}} e\!\left(\frac{n^2}{4m\ell}\, b\lambda^2\right) = 4m\ell \sum_{\substack{t\mid\ell \\ 4m\ell\mid n^2 t\lambda^2}} \frac{\lambda(t)}{t} \ ,$$

so the first sum in (18) equals

$$\sum_{t|\ell} \lambda(t) \frac{\#\{\lambda \bmod 4m\ell \mid 4m\ell | n^2 t\lambda\}}{t} = \sum_{t|\ell} \lambda(t)\ n(4,n)\ Q\left(\frac{4m\ell}{(4,n)nt}\right)$$

$$= \delta(\ell=\square) \cdot 2\, n\, \ell^{1/2} \cdot Q((4,n))Q(\tfrac{m}{n}) .$$

Here we used that ℓ and m are relative prime, so that $Q(\frac{4m\ell}{(4,n)nt}) = Q(\frac{4}{(4,n)})Q(\frac{m}{n})Q(\frac{\ell}{t})$, $(4,n)Q(\frac{4}{(4,n)})=2Q((4,n))$ and $\sum_{t|\ell} \lambda(t)Q(\frac{\ell}{t}) = \delta(\ell=\square)\ell^{1/2}$

The second sum in (18) can be written as

$$(19)\qquad \sum_{t|\ell} \lambda(t) \sum_{\substack{b \bmod 4m\ell/t \\ b\not\equiv 0 \bmod 4m\ell/t}} i \cot\left(\pi \frac{b}{4m\ell/t}\right) Av_{\lambda}\, e\left(\frac{m^2/n^2}{4\ m\ell/t}\, b\lambda^2\right) ,$$

and here by Proposition A.2 (and $(n\ell,\frac{m}{n})=1$) the inner sum equals

$$-2 \cdot \frac{m}{n} \cdot (4,\tfrac{m}{n}) \sum_{\substack{\Delta<0,\Delta\equiv 0,1 \bmod 4 \\ \Delta | \frac{4n\ell/t}{(4,\frac{m}{n})},\ \frac{4n\ell/t}{(4,\frac{m}{n})\Delta} \text{ is squarefree}}} H_{\frac{m}{n}/(4,\frac{m}{n})}(\Delta)$$

which may also be written as

$$-2 \cdot \frac{m}{n} \sum_{\substack{\Delta<0,\Delta\equiv 0,1 \bmod 4 \\ \Delta | 4n\ell/t,\ \frac{4n\ell/t}{\Delta} \text{ squarefree}}} H_{\frac{m}{n}}(\Delta)$$

Now for a Δ from this sum one easily verfies

$$\sum_{\substack{t|\ell,\frac{4m\ell}{\Delta} \\ \frac{4m\ell}{\Delta t} \text{ squarefree}}} \lambda(t) = \begin{cases} 1 & \text{if } \Delta = -4\ell n' \text{ for some } n'|n \text{ with } n/n' \text{ squarefree} \\ 0 & \text{otherwise} \end{cases}$$

(use again $(\ell,m) = 1$), and in view of this (19) becomes

$$-2 \cdot \frac{m}{n} \sum_{\substack{n'|n \\ n/n' \text{ squarefree}}} H_{\frac{m}{n}}(-4\ell n') \quad .$$

Putting this all together, we find

(20)
$$s_{k,m}(\ell,n)_{par.} = -\tfrac{1}{4}\delta(\ell=\square)\,\ell^{k-3/2}Q((n,4)) - \frac{(-1)^k\ell^{k-2}}{4} \sum_{\substack{n'|n \\ n/n' \text{squarefree}}} H_{\frac{m}{n}}(-4\ell n').$$
$$S_{par.} = s_{k,m}(\ell,n)_{par.} + (-1)^k s_{k,m}(\ell,\tfrac{m}{n})_{par.} \quad .$$

The scalar contribution

Here we find

(21) $$S_{scal.} = \delta(\ell=\square)\,\frac{2k-3}{24}\,\ell^{k-2}\left(\frac{m}{n} + (-1)^k n\right) \quad .$$

If we now compare (11), (17), (20), and (21) with the formula for $s_{k,m}(\ell,n)$ given in Theorem 1, we see that the theorem is proved for $k \neq 2$: the terms in the first sum in Theorem 1 with $0 < |s| < \sqrt{4\ell n'}$ equal (11) the terms with $s = 0$ equal the second term of (20) (since $p_{2k-2}(0,\ell) = (-1)^k\ell^{k-2}$); the terms with $s = \pm\sqrt{4\ell n'}$ equal (21) (since this occurs only if ℓ is a square and either $n'=1$, $s=2\sqrt{\ell}$, and $4|n$ or $n'=4$, $s=4\sqrt{\ell}$, and $4|n$, and $p_{2k-1}(2\sqrt{\ell},\ell)\,H_{\frac{m}{n}}(0) = -\frac{2k-3}{12}\,\ell^{k-2}\,\frac{m}{n}$); the terms with $\ell' \neq \sqrt{\ell}$ in the second sum in Theorem 1 equal (17) (replace ℓ' by ℓ/ℓ' if $\ell' > \sqrt{\ell}$); and the terms with $\ell' = \sqrt{\ell}$ equal the first term of (20).

It remains to treat:

The correction term for $k = 2$

First of all we note that by definition

(22). $\dim J^*_{1,m} = \dim \operatorname{Hom}_{\tilde{\Gamma}}(M_{1/2}(\Gamma(4m), Th_m)$,

where $\tilde{\Gamma} \subseteq \widetilde{SL_2(\mathbb{R})}$ is the inverse image of Γ by the canonical map $\widetilde{SL_2(\mathbb{R})} \longrightarrow SL_2(\mathbb{R})$, and where $M_{1/2}(\Gamma(4m))$, Th_m are considered as $\tilde{\Gamma}$-modules via the action $h|(A,w(\tau)) = w(\tau)^{-1}h(A\tau)$, $\theta|(A,w(\tau)) = w(\tau)^{-1}\theta|^*_{1,m}A$ of $\tilde{\Gamma}$ on $M_{1/2}(\Gamma(4m))$ and Th_m , respectively. (For the notations see [S-Z].) By the theorem of Serre-Stark [S-S] one knows that $M_{1/2}(\Gamma(4m))$ is contained in the space spanned by the "Nullwerte" $\theta(\tau,0)$ with $\theta(\tau,z) \in \bigcup_{m'>0} Th_{m'}$. Thus the computation of $\dim J^*_{1,m}$ is reduced to an analysis of the $\tilde{\Gamma}$-modules Th_m . This has been done in [S] (Satz 5.2 and Satz 1.8), and we only cite the result:

$$\dim J^*_{0,m} = \frac{1}{2}\{\sigma_0(m) + \delta(m = \square)\}$$

(The reader may also work out this formula using orthogonality relations for group characters and the formulae for $\operatorname{tr} U_m(A)$ $(A \in \Gamma)$ from Theorem 2 of [S-Z]; however, this would be essentially equivalent to the procedure in [S].)

Now it is easy to compute the correction term $\operatorname{tr}(T(\ell)^* W^*_n, J^*_{1,m})$.

Namely, let m' run through all divisors $m'|m$ with m/m' a square, and for each such m' let λ run through a complete set of representations in $\mathbb{Z}$ for $\{\lambda \bmod 2m' | \lambda^2 \equiv 1 \bmod 4m'\}/\{\pm 1\} \subset (\mathbb{Z}/2m'\mathbb{Z})^*/\{\pm 1\}$. Note that the number of such pairs m',λ equals $\frac{1}{2}\{\sigma_0(m) + \delta(m = \square)\}$, i.e. $\dim J^*_{1,m}$. For each such pair m',λ define

$$\phi_{m',\lambda}(\tau,z) := \sum_{\rho=1}^{2m'} \overline{\theta_{m',\lambda\rho}(\tau,0)}\ \theta_{m',\rho}\left(\tau,\sqrt{\tfrac{m}{m'}}\,z\right)$$

$$= \sum_{\substack{r,s\ \mathbf{Z} \\ r\equiv\lambda s \bmod 2m'}} e\left(\frac{r^2}{4m}\tau - \frac{s^2}{4m}\bar{\tau} + r\sqrt{\frac{m}{m'}}\,z\right),$$

where $\theta_{m,\rho}$ is the theta-series $\sum_{r\equiv\rho\ (2m)} e(\frac{r^2}{4m}\tau + rz)$ (cf. [S-Z]). Obviously $\phi_{m',\lambda}|^{*}_{1,m}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}(\tau,z) = \phi_{m',\lambda}(\tau+1,z) = \phi_{m,\lambda}(\tau,z)$, and using the well-known formulae

$$\theta_{m,\rho}\left(\frac{-1}{\tau}, \frac{z}{\tau}\right) = \left(\frac{\tau}{2mi}\right)^{1/2} e^{m}\left(\frac{z^2}{\tau}\right) \sum_{\sigma=1}^{2m} e_{2m}(-\rho\tau)\ \theta_{m,\sigma}(\tau,z)$$

it is easily checked that $\phi_{m',\lambda}|^{*}_{1,m}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = \phi_{m',\lambda}$. The matrices $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ generate $\Gamma = SL_2(\mathbf{Z})$, and hence the $\phi_{m',\lambda}$ lie in $J^{*}_{1,m}$. Noticing that the $\phi_{m',\lambda}$ are linearly independent we thus have a basis for $J^{*}_{1,m}$.

Finally it is easily verified that

$$\phi_{m',\lambda}|T(\ell)^{*} = \ell^{-2} \sum_{x\in\mathbf{Z}^2/\ell\mathbf{Z}^2}\ \sum_{\substack{a,d>0 \\ ad=\ell^2}}\ \sum_{\substack{b \bmod d \\ (a,b,d)=\square}} \phi_{m',\lambda}|^{*}_{1,m}\left(\frac{1}{\ell}\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right)[x]$$

$$= \sigma_1(\ell)\phi_{m',\lambda}\ ,$$

and that

$$\phi_{m',\lambda}|\ W^{*}_{n} = n^{-1} \sum_{x\in\mathbf{Z}^2/n\mathbf{Z}^2} \phi_{m',\lambda}|^{*}_{1,m}\ [\tfrac{x}{n}] = \phi_{m',\lambda'}\ ,$$

where $\lambda' \equiv -\lambda \bmod 2(m',n)$, $\lambda' \equiv +\lambda \bmod 2m'/(m',n)$. Note that $\phi_{m',\lambda}|W^{*}_{n} = \phi_{m',\lambda}$ if and only if $\lambda \equiv -\lambda \bmod 2(m',n)$ or $\lambda \equiv -\lambda \bmod 2m'/(m',n)$, i.e. if and only if $(m',n) = 1$ or $m'/(m',n) = 1$. Hence

$$\operatorname{tr}\,(T(\ell)^{*}W_{n}^{*}, J_{1,m}^{*})$$

$$= \sigma_1(\ell) \cdot \sum_{\substack{m'|m \\ m/m'=\square}} \{\delta((m',n) = 1) + \delta\left(\frac{m'}{(m',n)} = 1\right)\}\#(\{\lambda \bmod 2m' \mid \lambda^2 \equiv 1 \bmod 4m\}/\{\pm 1\})$$

$$= \frac{1}{2}\,\sigma_1(\ell)\ \{\delta(n = \square)\sigma_0(\tfrac{m}{n}) + \delta(\tfrac{m}{n} = \square)\sigma_0(n)\}\ ,$$

as was to be shown.

§2. Comparison with the trace formula for ordinary modular forms

For integers k,m, k even and $m>0$, denote by $M_k(m)$ the space of modular forms of weight k on $\Gamma_0(m)$. For integers $\ell,n>0$, $(\ell,m)=1$ and $n \| m$, denote by $T(\ell)$ and W_n the ℓ-th Hecke operator and the n-th Atkin-Lehner involution on $M_k(m)$, respectively. Thus, for any $f \in M_k(m)$ one has

$$(1) \qquad f|T(\ell) = \sum_{r\geq 0} \sum_{d|(\ell,r)} d^{k-1} a_f(\tfrac{\ell r}{d^2})\, q^r$$

and

$$(2) \qquad f|W_n(\tau) = n^{k/2}(cm\tau + nd)^{-k} f(\tfrac{an\tau+b}{cm\tau+dn}) .$$

Here $a_f(r)$ denotes the r-th Fourier coefficient of f and a,b,c,d are any integers satisfying $adn^2 - bcm = n$. (We are using here the same symbols $T(\ell)$ and W_n as for the corresponding operators on $J_{k,m}$. Since it will be clear from the context which operator is meant there should be no confusion.)

Finally, let $M_k^{new}(m)$ denote the space of new forms in $M_k(m)$ and let $M_k^{new,\pm}(m)$ be the subspace of modular forms $f \in M_k^{new}(m)$ satisfying

$$f|W_m = \pm(-1)^{k/2} f .$$

Unfortunately in the literature the notion "new forms" is usually applied to cusp forms only. Thus some remarks seem to be indispensable. We define more precisely

$$M_k^{new}(m) := \mathcal{E}_k^{new}(m) \oplus S_k^{new}(m),$$

where $S_k^{new}(m)$ is the subspace of new forms in the sense of Atkin-Lehner in $S_k(m)$, the space of cusp forms of weight k on $\Gamma_0(m)$. $\mathcal{E}_k^{new}(m)$ is defined to be zero if m is not a square, while if m is a square then $\mathcal{E}_k^{new}(m)$ is defined to be the span of the series

$$E_k^{(\chi)}(\tau) = \sum_{\ell \geq 0} \sigma_{k-1}^{(\chi)}(\ell)\, q^\ell$$

where χ runs through all primitive Dirichlet characters modulo $\sqrt{m}$ (aside from the principal character if $m = 1$ and $k = 2$), $\sigma_{k-1}^{(\chi)}(\ell) = \sum_{d|\ell} d^{k-1}\overline{\chi(d)}\,\chi(\ell/d)$ for $\ell \geq 1$, $\sigma_{k-1}^{(\chi)}(0) = 0$ or $= \frac{1}{2}\zeta(1-k)$ according as $m > 1$ or $m = 1$. (For more details concerning Eisenstein series on $\Gamma_0(m)$ cf. [H], pp. 461-468 and 689-693.)

By comparing the trace formulae from the foregoing paragraph with the well-known trace formulae for Hecke operators on spaces of modular forms we shall derive the following

<u>Theorem 2</u>. Let k,m,ℓ,n <u>be positive integers</u>, $k \geq 2$, $(m,\ell) = 1$ <u>and</u> $n \| m$. <u>Then</u>

$$\mathrm{tr}(T(\ell)\circ W_n, J_{k,m}) = \sum_{m'|m} \Big\{ \sum_{d^2 | \frac{m}{m'}} 1 \Big\}\ \mathrm{tr}(T(\ell)\circ W_{(n,m')}, M_{2k-2}^{new,-}(m')).$$

<u>Moreover the same equation holds if one restricts on both sides to Eisenstein series or to cusp forms</u>.

<u>Remark</u> - The above Theorem remains true for all k in the trivial sense that $J_{k,m} = M_{2k-2}^{new,-}(m') = \{0\}$ for all $k < 2$. However, the fact that

$J_{1,m} = \{0\}$ seems to be not at all trivial. For a proof , depending on the work of Serre-Stark about modular forms of weight $1/2$; cf. [S].

Proof of Theorem 4. First of all we treat the case of cusp forms.

The projection of $S_{2k-2}^{new}(m)$ onto the subspace $S_{2k-2}^{new,-}(m)$ of forms $f \in S_{2k-2}^{new}(m)$ satisfying $f|W_m = (-1)^k f$ which commutes with all $T(\ell)$ $((\ell,m) = 1)$ is given by $\frac{1}{2}(W_1 + (-1)^k W_m)$. Thus

$$\mathrm{tr}(T(\ell)\circ W_n, S_{2k-2}^{new,-}(m)) = \frac{1}{2}\{\mathrm{tr}(T(\ell)\circ W_n, S_{2k-2}^{new}(m)) + (-1)^k \mathrm{tr}(T(\ell)\circ W_{\frac{m}{n}}, S_{2k-2}^{new}(m))\},$$

and hence the formula to be proved can be rewritten as

$$\mathrm{tr}(T(\ell)\circ W_n, S_{k,m}) = \frac{1}{2}\{t(n) + (-1)^k t(\tfrac{m}{n})\},$$

where $t(n)$ for any $n \| m$ is given by

$$(3) \quad t(n) = \sum_{n_1 | n} \sum_{n_2 | \frac{m}{n}} \{ \sum_{d^2 | \frac{m}{n_1 n_2}} 1\} \quad \mathrm{tr}(T(\ell)\circ W_{n_1}, S_{2k-2}^{new}(n_1 n_2))$$

In fact, we shall show that

$$2\cdot s_{k,m}(\ell,n) = t(n)$$

with $s_{k,m}(\ell,n)$ as in Theorem 1.

To apply the trace formulae occuring in the literature we need to express the traces on the right hand side of (3) in terms of corresponding traces on the total spaces $S_{2k-2}(n_1 n_2)$.

Now for any pair of relative prime, positive integers n_1, n_2 one has by Atkin-Lehner theory

$$(4) \quad S_{2k-2}(n_1 n_2) = \bigoplus_{\substack{a_1 b_1 | n_1 \\ a_2 b_2 | n_2}} S^{new}_{2k-2}(a_1 a_2) | U_{b_1 b_2}$$

with $U_\ell : f(\tau) \mapsto f | U_\ell(\tau) = f(\ell\tau)$. Choose in each of $S^{new}_{2k-2}(a_1 a_2)$ a basis consisting of simultaneous eigenfunctions with respect to all $T(\ell)$ $((\ell, n_1 n_2) = 1)$ and W_n $(n \| a_1 a_2)$. Via (4) this gives a basis for $S_{2k-n}(n_1 n_2)$ and we compute $tr(T(\ell) \circ W_{n_1}, S_{2k-2}(n_1 n_2))$ with respect to this basis:

Let g be such a basis element, say $g = f | U_{b_1 b_2}$ with $f \in S_{2k-2}(a_1 a_2)$, $f | W_{a_1} = \varepsilon f$. It is easily seen that

$$f | U_{b_1 b_2} \circ W_n = \varepsilon (n_1 / a_1 b_1^2)^{k-1} \cdot f | U_{n_1 b_2 / a_1 b_1}$$

Thus the contribution of g to $tr(T(\ell) \circ W_n, S_{2k-2}(n_1 n_2))$ is equal to ε if $n_1 / a_1 b_1^2 = 1$ and is zero otherwise since then $f | U_{b_1 b_2} \pm (n_1 / a_1 b_1^2)^{k-1} \cdot f | U_{n_1 b_2 / a_1 b_1}$ are both eigenfunctions of W_{n_1} with composite eigenvalues $\pm \varepsilon$. Hence

$$(5) \quad tr(T(\ell) \circ W_{n_1}, S_{2k-2}(n_1 n_2)) = \sum_{\substack{a_1 | n_1 \\ n_1 / a_1 = \square}} \sum_{a_2 b_2 | n_2} tr(T(\ell) \circ W_{a_1}, S^{new}_{2k-2}(a_1 a_2)).$$

Combining (3) and (5) by using some elementary theory of multiplicative functions gives

(6) $$t(n) = \sum_{n_1 | n} \sum_{n_2 | \frac{m}{n}} \lambda(\frac{m}{n_1 n_2}) \; tr(T(\ell)\circ W_{n_1}, S_{2k-2}(n_1 n_2))$$

where λ is Liouville's function, i.e. the unique multiplicative function such that $\lambda(p^\alpha) = (-1)^\alpha$ for all prime powers p^α.

Now we can insert the following formula for the traces occuring on the right hand side of (6):

$$tr(T(\ell)\circ W_{n_1}, S_{2k-2}(n_1 n_2))$$

(7) $$= -\frac{1}{2} \sum_{\substack{n' | n_1 \\ n_1/n' = \square}} \mu(\sqrt{n_1/n'}) \sum_{\substack{s^2 \le 4\ell n' \\ \sqrt{n_1 n'} | s}} P_{2k-2}(s/\sqrt{n'}, \ell) \sum_{\substack{t | n_2 \\ n_2/t = \text{squarefree}}} H_t(s^2 - 4\ell n')$$

$$- \frac{1}{2} \delta(n_1 = \square) \varphi(\sqrt{n_1}) \sum_{\substack{\ell' | \ell \\ \sqrt{n_1} | (\ell' + \frac{\ell}{\ell'})}} \min(\ell', \frac{\ell}{\ell'})^{2k-3} \sum_{\substack{t | n_2 \\ n_2/t \text{ squarefree}}} (Q(t), (\ell' - \frac{\ell}{\ell'}))$$

$$+ \delta(k = 2)\, \sigma_1(\ell) .$$

Here μ and φ are the Möbius and Euler function respectively and the other notations are as in Theorem 1. Using the elementary identities

$$\sum_{n_1} \mu(\sqrt{n_1/n'}) = \delta \begin{pmatrix} n' | (s,n) \quad \text{and} \\ ((s/n')^2, n/n')) = \text{squarefree} \end{pmatrix}$$

(n_1 running through all positive integers with $n' | n_1$, $n_1 | n$, $n_1/n' = \square$ and $\sqrt{n_1 n'} | s$),

$$\sum_{n_2 | \frac{m}{n}} \lambda(\frac{m}{n_1 n_2}) \sum_{\substack{t | n_2 \\ n_2/t \text{ squarefree}}} h(t) = h(\tfrac{m}{n})$$

(for any arithmetical function h),

$$\sum_{n_1} \varphi(\sqrt{n_1}) = (Q(n),(\ell' + \frac{\ell}{\ell'}))$$

(n_1 running through all positive integers with
$n_1 = \square$ and $\sqrt{n_1} \mid (\ell' + \frac{\ell}{\ell'})$)

it is then immediately clear that $t(n)$ coincides with $2 \cdot s_{k,m}(\ell,n)$, as defined in Theorem 1. Thus, in the case of cusp forms, Theorem 2 is proved.

Unfortunately the formula (7) is not exactly the formula which can be found in the literature, so we have to add some remarks.

First of all a corresponding formula (7) for $n_1 = 1$ can be found in [O]. Aside from some slight differences in the statement, which can be easily worked out by the reader, the main difference concerns the elliptic contribution. This is stated in [O] as

$$-\frac{1}{2}\sum_{s^2<4\ell} p_{2k-2}(s,\ell) \sum_{f \mid F} h'(\frac{s^2-4\ell}{f^2})\mu(s,f,\ell),$$

where F is that positve integer such that $\frac{t^2-4\ell}{F^2}$ is a fundamental discriminant, $h'(\Delta)$ denotes the number of equivalence classes (mod. $SL_2(\mathbf{Z})$) of primitive, positive definite, binary quadratic forms of discriminant Δ if $\Delta < -4$, $h'(-4) = \frac{1}{2}$, $h'(-3) = \frac{1}{3}$, and where

$$\mu(s,f,\ell) = \frac{\varphi_1(n_2)}{\varphi_1(n_2/(n_2,f))} \cdot r\left(\frac{s^2-4\ell}{(n_2,f)^2}, \frac{n_2}{(n_2,f)}\right).$$

Here $\varphi_1(n) = n \prod_{p \mid n} (1 + \frac{1}{p})$ and $r(D,n) = \#\{r \bmod 2n \mid r^2 \equiv D \bmod 4n\}$.

Now it is easily checked that

$$r(D,n) = \sum_{\substack{t \mid n \\ n/t \text{ squarefree}}} \chi_D(t) \quad (\chi_D(\cdot) \text{ as in § 1 })) .$$

Using this one can write (after some obvious modifications)

$$\mu(s,t,\ell) = \sum_{\substack{t \mid n_2 \\ n_2/t \text{ squarefree}}} (t,f) \cdot \chi_{\frac{s^2-4\ell}{f^2}} \left(\frac{t}{(t,f)}\right) ,$$

and then the equality of the corresponding terms in our formula (7) and in Oesterlé's results from the identity

$$(8) \qquad \sum_{f \mid F} h'\left(\frac{s^2-4\ell}{f^2}\right) \cdot (t,f) \cdot \chi_{\frac{s^2-4\ell}{f^2}} \left(\frac{t}{(t,f)}\right) = H_t(s^2-4\ell) .$$

(cf. the proof of Proposition A.1)

Secondly, a corresponding formula (7) for $n_1 > 1$ is given in [Y]. Aside from some mistakes in the statement of that formula (which can be corrected by carefully reading [Y]), the main difference again concerns the elliptic contribution. It is stated in [Y] as

$$(9) \qquad -\frac{1}{2} \sum_{\substack{s^2<4\ell n_1 \\ n_1 \mid s}} P_{2k-2}\left(\frac{s}{\sqrt{n_1}},\ell\right) \sum_{\substack{f \mid F \\ (f,n_1)=1}} h'\left(\frac{s^2-4\ell n_1}{f^2}\right) \sum_{\substack{t \mid n_2 \\ n_2/t \text{ squarefree} \\ \frac{n_2}{t} \mid \frac{F}{f}}} r\left(\frac{s^2-4\ell n_1}{f^2(n_2/t)^2},t\right) ,$$

where F is the positive integer such that $\frac{s^2-4\ell n_1}{F^2}$ is a fundamental discriminant. Here the equality of the corresponding terms in (7) and (9) results from the identity

$$\sum_{\substack{f|F \\ (f,n_1)=1}} \sum_{\substack{t|n_2 \\ n_2/t=\text{squarefree} \\ \frac{n_2}{t}|\frac{F}{f}}} h'\left(\frac{s^2-4\ell n_1}{f^2}\right)\cdot r\left(\frac{s^2-4\ell n_1}{f^2(n_2/t)^2},t\right) = \sum_{d|(F,n_1)} \mu(d) \sum_{\substack{t|n_2 \\ n_2/t=\text{squarefree}}} H_t\left(\frac{s^2-4\ell n_1}{d^2}\right) ,$$

which must be proved similarly to (8). Inserting this in (9), replacing s by ds and summing over $n'=n_1/d^2$ (note that $d|(F,n_1)$, $n_1|s$ and $(\ell,n_1)=1$ implies $d^2|n_1$) then leads to our formula (7).

It remains to consider the case of Eisenstein series.

On the side of Jacobi forms the space of Eisenstein series in $J_{k,m}$ is spanned by the series

(10) $$E_{k,m,t,\chi} = \sum_{s \bmod Q(m)/t} \chi(s)\ E_{k,m,ts} .$$

Here t runs through all divisors of $Q(m)$ and for each such t the index χ runs through all primitive Dirichelet characters modulo F with $F|\frac{Q(m)}{t}$ and $\chi(-1)=(-1)^k$. Furthermore for any integer s the series $E_{k,m,s}$ is defined by

(11) $$E_{k,m,s} = \frac{1}{2} \sum_{\xi\in\Gamma_\infty^J\backslash\Gamma^J} (q^{as^2}\zeta^{2abs})|_{k,m}\xi$$

with $\Gamma_\infty^J = \{[\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}][0,\mu] \mid n,\mu\in\mathbb{Z}\} \subseteq \Gamma^J$ and $m=ab^2$, a squarefree. This is not quite true for k=2 since then the series $E_{k,m,s}$ as in (11) fail to converge. Here $E_{k,m,s}$ has to be defined by the same type of methods as are used for modular forms ("Hecke's convergence trick") and for $\chi\neq$ principal character (i.e. $F\neq1$) the series $E_{k,m,t,\chi}$ given by (10) then defines an element of $J_{k,m}$.

We shall prove in a moment that

(12) $E_{k,m,t,\chi}|T(\ell) = \sigma^{(\chi)}_{2k-3}(\ell)E_{k,m,t,\chi}$ $((\ell,m)=1)$

(13) $E_{k,m,t,\chi}|W_n = \chi(\lambda)E_{k,m,t,\chi}$ $(n\|m)$

where λ denotes any integer such that $\lambda \equiv -1 \bmod n$ and $\lambda \equiv +1 \bmod \frac{m}{n}$.

Comparing this with the description of Eisenstein series in $M^{new}_{2k-2}(m)$ given at the beginning of this paragraph, and using that any $E^{(\chi)}_{2k-2} \in M^{new}_{2k-2}(F^2)$ satisfies $E^{(\chi)}_{2k-2}|T(\ell) = \sigma^{(\chi)}_{2k-3}(\ell)E^{(\chi)}_{2k-2}$ and $E^{(\chi)}_{2k-2}|W_n = \chi(\lambda)E^{(\chi)}_{2k-2}$ $((\ell,F^2)=1,\ n\|F^2$ and with $\lambda \equiv -1 \bmod n$ and $\lambda \equiv +1 \bmod F^2/n)$, the reader can now easily verify the assertion of Theorem 2.

To prove (12) and (13) we recall that $J_{k,m} = \mathcal{E}_{k,m} \oplus S_{k,m}$, $\mathcal{E}_{k,m}$ being the space spanned by the Eisenstein series $E_{k,m,t,\chi}$ as above. Hence any Eisenstein series in $J_{k,m}$ is uniquely determined by its "constant terms" (sum of terms $q^n\zeta^r$ with $4mn - r^2 = 0$) or, equivalently, by its Fourier coefficients $C(0,r)$ $(r^2 \equiv 0 \bmod 4m)$. Moreover $\mathcal{E}_{k,m}$ is invariant under all $T(\ell)$ $((\ell,m) = 1)$ and all W_n $(n\|m)$, since $\mathcal{E}_{k,m}$ is the orthogonal complement of $S_{k,m}$ in $J_{k,m}$ with respect to the Petersson scalar product and the $T(\ell)$, W_n are hermitian (cf. [E-Z]).

Thus to verify (12) and (13) it suffices to compute the Fourier coefficients $C(0,r)$, $C^*(0,r)$ and $C^{**}(0,r)$ $(r^2 \equiv 0 \bmod 4m)$ of

$E_{k,m,t,\chi}$, $E_{k,m,t,\chi}|T(\ell)$ and $E_{k,m,t,\chi}|W_n$ respectively.

Now it is easily checked that the constant term of $E_{k,m,s}$ is equal to $\frac{1}{2}\{\sum_{r\equiv s(b)} q^{ar^2}\zeta^{2abr} + (-1)^k \sum_{r\equiv -s(b)} q^{ar^2}\zeta^{2abr}\}$. Thus $C(0,r) = \chi(r/2abt)$, and then $C^*(0,r) = \sigma_{2k-3}^{(\chi)}(\ell)\ C(0,r)$, $C^{**}(0,r) = \chi(\lambda)C(0,\lambda)$ by (5) and (9) of §0. This completes the proof of Theorem 2.

§3. The lifting from $J_{k,m}$ to $\mathfrak{M}_{2k-2}(m)$

In this paragraph we shall give an interpretation of the theorem proved in the last paragraph in terms of liftings from Jacobi forms to modular forms.

Let $\mathfrak{M}_k(m)$ be the subspace of $M_k(m)$ defined in the introduction. Recall that this is the space spanned by forms f whose L-series has the form

$$L(f,s) = L(g,s)\cdot\prod_{p\mid\frac{m}{m'}} Q_p(s)$$

where g is a newform on $\Gamma_0(m')$ for some m' dividing m and the $Q_p(s)$ are polynomials in p^{-s} of degree $\le t := \mathrm{ord}_p(\frac{m}{m'})$ satisfying the additional requirement

$$(1) \qquad Q_p(s) = p^{t(k/2-s)}\, Q_p(k-s) \qquad \text{for all } p\mid\frac{m}{m'}\ .$$

Here we may assume that the newform g is a simultaneous eigenform of all $T(\ell)$ with $(\ell,m)=1$; then it is also an eigenform of the Fricke-Atkin-Lehner involution $W_{m'}$ on $M_k(m')$, i.e.

$$g|W_{m'} = (-1)^{k/2}\,\varepsilon\, g$$

with $\varepsilon\in\{\pm1\}$, and then $L(g,s)$ has the functional equation $L^*(g,s)=\varepsilon L^*(g,k-s)$, where $L^*(g,s) = (2\pi)^{-s}\, m'^{\,s/2}\,\Gamma(s)\,L(g,s)$. Equation (1) says that $L(f,s)$ not only satisfies a functional equation $L^*(f,s) = \varepsilon L^*(f,k-s)$ with the same sign as its progenitor $L(g,s)$, but that each Euler factor of the finite Euler product

$$\frac{L^*(f,s)}{L^*(g,s)} = (m/m')^{s/2} \prod_{p\mid\frac{m}{m'}} Q_p(s)$$

is invariant under $s\to k-s$. Another description, easily seen to be equivalent, is the following: Suppose the newform g has eigenvalues λ_ℓ for $T(\ell)$ (ℓ prime, $\ell\nmid m$) and $\varepsilon_p\in\{\pm1\}$ for W_{p^r} ($p^r\,\|\, m'$). Then f has the eigenvalues λ_ℓ for $T(\ell)$ ($\ell\nmid m$), ε_p for W_{p^s} ($p^s\,\|\, m$, $p\mid m'$) and $+1$ for W_{p^t} ($p^t\,\|\, m$, $p\nmid m'$).

By $\mathfrak{M}_k^{\pm}(m)$ we denote the subspace of $\mathfrak{M}_k(m)$ spanned by all f as above with $\varepsilon = \pm 1$. Since $M_k^{new}(m')$ is the sum of $M_k^{new,+}(m')$ and $M_k^{new,-}(m')$, we have $\mathfrak{M}_k(m) = \mathfrak{M}_k^{+}(m) \oplus \mathfrak{M}_k^{-}(m)$. The spaces $\mathfrak{M}_k^{\pm}(m)$ are invariant under all $T(\ell)$ ($(\ell,m) = 1$) and under all Atkin-Lehner involutions W_n ($n \| m$).

As a consequence of the theorem in the foregoing paragraph we shall show:

Theorem 5 - Let k,m be integers, $m > 0$. Then $J_{k,m}$ is Hecke-equivariantly isomorphic to $\mathfrak{M}_{2k-2}^{-}(m)$. More precisely, for any fixed fundamental discriminant $D < 0$ and any fixed integer s with $D \equiv s^2 \bmod 4m$ there is a map

$$S_{D,s} : J_{k,m} \longrightarrow \mathfrak{M}_{2k-2}^{-}(m)$$

given by

$$\sum_{\substack{\Delta\le 0,r\\ \Delta\equiv r^2 \bmod 4m}} C(\Delta,r)q^{\frac{r^2-\Delta}{4m}}\zeta^r \longmapsto \sum_{\ell\ge 0}\left\{\sum_{a|\ell} a^{k-2}\left(\frac{D}{a}\right)C\left(\frac{\ell^2}{a^2}D,\frac{\ell}{a}s\right)\right\}q^\ell .$$

(with the convention $\sum_{a|0} a^{k-2}(\frac{D}{a})C(0,0) := \frac{1}{2}C(0,0)L(2-k,(\frac{D}{\cdot}))$, $L(s,(\frac{D}{\cdot}))$ being the usual L-series which for $\mathrm{Re}(s)>1$ equals $\sum_{n\ge 1}(\frac{D}{n})n^{-s}$). The maps $S_{D,s}$ commute with all Hecke operators $T(\ell)$ $((\ell,m)=1)$ and involutions W_n $(n\|m)$ and map Eisenstein series to Eisenstein series and cusp forms to cusp forms, and some linear combination of them is an isomorphism.

Proof - Recall the operators U_ℓ, V_ℓ on $J_{k,m}$ (ℓ a positive integer) as defined in [E-Z]:

$$(\phi|U_\ell)(\tau,z) = \phi(\tau,\ell z) = \sum_{\substack{\Delta\le 0,\ r\\ r^2\equiv\Delta\ (4m)}} c_\phi(\Delta,r)\, q^{\frac{r^2-\Delta}{4m}}\zeta^{r\ell}$$

$$(\phi|V_\ell)(\tau,z) = \sum_{\substack{\Delta\le 0,r\\ \Delta\equiv r^2 \bmod 4m\ell}}\left\{\sum_{a|(\frac{r^2-\Delta}{4m\ell},r,\ell)} a^{k-1}c_\phi\left(\frac{\Delta}{a^2},\frac{r}{a}\right)\right\} q^{\frac{r^2-\Delta}{4m\ell}}\zeta^r$$

The space $J_{k,m}$ is mapped under U_ℓ, V_ℓ to $J_{k,m\ell^2}$ and $J_{k,m\ell}$ respectively, U_ℓ, V_ℓ commute with all $T(\ell')$ $((\ell',\ell m)=1)$, and one has

$$U_\ell \circ W_n = W_{(n,m)}\circ U_\ell \quad (n\|m\ell^2)$$

$$V_\ell \circ W_n = W_{(n,m)}\circ V_\ell \quad (n\|m\ell).$$

For $\ell,d\ge 1$ we define an operator $B_{\ell,d}: M_k(m) \longrightarrow M_k(m\ell d^2)$ by

$$(f|B_{\ell,d})(\tau) := \sum_{t|\ell} t^{k/2} f(dt\tau).$$

It is easily checked that $B_{\ell,d}$ is injective, commutes with all $T(\ell')$ $((\ell',m\ell d^2)=1)$ and satisfies $B_{\ell,d}\circ W_n = W_{(n,m)}\circ B_{\ell,d}$ $(n\| m\ell d^2)$. Using these operators one immediately obtains by Atkin-Lehner theory:

$$(2) \qquad \mathfrak{M}_k^-(m) = \bigoplus_{\substack{\ell,d>0\\ \ell d^2|m}} M_k^{new,-}(\tfrac{m}{\ell d^2})|B_{\ell,d}$$

In view of this decomposition and the properties of $B_{\ell,d}$ listed above, Theorem 2 can now be read as

$$(3) \qquad tr(T(\ell)\circ W_n, J_{k,m}) = tr(T(\ell)\circ W_n, \mathfrak{M}_{2k-2}^-(m)).$$

Since $J_{k,m}$ and $\mathfrak{M}_{2k-2}^-(m)$ are semisimple as modules with respect to the rings generated by the operators $T(\ell)$ and W_n on $J_{k,m}$ and $\mathfrak{M}_{2k-2}^-(m)$ respectively, and since the same relations (0.6) hold for the $T(\ell)$ considered as operators on $J_{k,m}$ or on $\mathfrak{M}_{2k-2}^-$ one deduces from (3) that there exists an isomorphism between $J_{k,m}$ and $\mathfrak{M}_{2k-2}^-(m)$ which commutes with all $T(\ell)$ and all W_n. This proves the first statement of the theorem.

One of the main steps in the proof of the statements about the maps $\mathcal{S}_{D,s}$ is to show that a decomposition like (2) holds also for Jacobi forms.

More precisely, define $S_{k,m}^{new}$ to be the orthogonal complement of $\sum_{\substack{\ell,d>0\\ \ell d^2|m,\ell d^2>1}} S_{k,\frac{m}{\ell d^2}}|U_d\circ V_\ell$ in $S_{k,m}$ (with respect to the Petersson scalar product) and $E^{new}_{k,m}$ as the span of the functions $E_{k,m,1,\chi}$ as in (2.10) (χ a primitive Dirichlet character modulo f) if $m = f^2$ and $k \geq 2$, $m \neq 1$ if $k = 2$, and 0 otherwise. Let $J^{new}_{k,m} := E^{new}_{k,m} \oplus S^{new}_{k,m}$.

Clearly $J^{new}_{k,m}$ is invariant under all $T(\ell)$ and W_n and

$$(4) \qquad J_{k,m} = J^{new}_{k,m} \oplus \sum_{\substack{\ell,d>0 \\ \ell d^2 | m, \ell d^2>1}} J^{new}_{k,\frac{m}{\ell d^2}} | U_d \circ V_\ell \ .$$

We shall prove by induction over m the following:

(i) The decomposition (4) is direct.

(ii) There exists an isomorphism between $J^{new}_{k,m}$ and $M^{new,-}_{2k-2}(m)$ which commutes with all $T(\ell)$ $((\ell,m) = 1)$ and all W_n $(n \| m)$.

(iii) For each pair of simultaneous eigenforms $\phi \in J^{new}_{k,m}$ and $f \in M^{new,-}_{2k-2}(m)$ with $a_f(1) = 1$ having the same eigenvalues with respect to all $T(\ell)$ $((\ell,m) = 1)$ and all W_n $(n \| m)$ one has

$$S_{D,s}(\phi) = C_\phi(D,s) \cdot f \ .$$

Note that this implies all statements of Theorem 3 except from the last one, because we have the easily checked formal power series identity

$$(5) \qquad S_{D,s}(\phi) | B_{\ell,d} = S_{D,s}(\phi | U_d \circ V_\ell) \quad (\phi \in J_{k,m}, \ell, d \geq 1).$$

To begin with let $m = 1$. Here (i) is obvious and (ii) follows from the remark following (3). (Note that $J^{new}_{k,1} = J_{k,1}$ and $M^{new,-}_{2k-2}(1) = \mathfrak{M}^{-}_{2k-2}(1)$). For (iii) simply note that the ℓ-th Fourier coefficient of $S_{D,s}(\phi)$ is nothing else than the (D,s)-th Fourier coefficient of $\phi|T(\ell)$, (cf. (0.5)) and hence equal to $a_f(\ell)\cdot C_\phi(D,s)$.

For the induction step and the remaining assertion of the theorem we need three lemmas.

Lemma 3.1 - Let $\phi \in J_{k,m}$ and $m'|m$. Assume that $C_\phi(\Delta,r) = 0$ for all Δ, r with $(r,m') = 1$. Then $\phi \in \sum_{\substack{d>1 \\ d^2|m, d|m'}} J_{k,\frac{m}{d^2}}|U_d$.

In particular, $\phi = 0$ if $(m', Q(m)) = 1$.

Lemma 3.2 - Let $\phi \in J_{k,m}^{new}$ be a simultaneous eigenform with respect to all $T(\ell)$ $((\ell,m)=1)$, and let $N>0$ be an arbitrary integer. Then there exists a fundamental discriminant $D<0$ and an integer s with $D \equiv s^2 \bmod 4mN$ such that $C_\phi(D,s) \neq 0$.

To formulate the third lemma we need an auxiliary operator.

For a positive integer ℓ with $\ell^2|m$ define an operator u_ℓ on $J_{k,m}$ by

(6) $$(\phi|u_\ell)(\tau,z) := \ell^{-1} \sum_{x \in \mathbb{Z}^2/\ell\mathbb{Z}^2} (\phi|_{k,m}[\tfrac{x}{\ell}])(\tau,\tfrac{z}{\ell}) .$$

Obviously u_ℓ is well-defined, i.e. does not depend on the choice of representatives x for $\mathbb{Z}^2/\ell\mathbb{Z}^2$. We leave it to the reader to verify that u_ℓ maps $J_{k,m}$ to $J_{k,\frac{m}{\ell^2}}$, that

(7) $$C_{\phi|u_\ell}(\Delta,r) = \sum_{\substack{r' \bmod 2m/\ell \\ r' \equiv r \bmod 2m/\ell^2}} C_\phi(\ell^2\Delta, \ell r')$$

(for all $\Delta \equiv r^2 \bmod 4m/\ell^2$, $\Delta < 0$), and that u_ℓ commutes with all $T(\ell')$ $((\ell',m) = 1)$.

<u>Lemma 3.3</u> – <u>Let</u> $\phi \in J_{k,m}$ <u>and let</u> p <u>be a prime dividing</u> m. <u>Assume that</u> $\phi|V_p \circ u_p = 0$ <u>and that</u> $\phi|u_p = 0$ <u>if</u> $p^2|m$. <u>Then for any pair of integers</u> Δ, r <u>with</u> $\Delta < 0, \Delta \equiv r^2 \bmod 4m$ <u>and any</u> $\alpha \geq 0$ <u>one has</u>

$$\sum_{\substack{a|p^\alpha \\ (a,\Delta)=1}} a^{k-2} c_\phi\left(\frac{p^{2\alpha}}{a^2}\Delta, \frac{p^\alpha}{a} r\right) = \begin{cases} p^{\alpha(k-2)} \cdot (-1)^\alpha \cdot c_{\phi|W_p^\alpha}(\Delta, r) & \text{if } p \| m \\ 0 & \text{if } p^2 | m \end{cases}$$

The proof of Lemma 3.3 is straightforward (using (7) and the definition of V_ℓ) and will be left to the reader. The proofs of Lemma 3.1 and 3.2 are postponed to the end of the paragraph. We show first of all how the theorem now follows.

To complete the induction assume that (i) to (iii) are true for all $m' < m$.

Let m' run through all divisors of m, and for each such m' let f run through a basis of normalized Hecke eigenforms in $M_{2k-2}^{\text{new},-}(m')$ and ℓ, d through all pairs of positive integers with $\ell d^2 = \frac{m}{m'}$. Then $f|B_{\ell,d}$ runs through a basis of $\mathfrak{M}_{2k-2}^{-}(m)$. For each such f let ϕ denote a non-zero Hecke eigenform in $J_{k,m'}^{\text{new}}$, having the same eigenvalues with respect to all $T(\ell)$ $((\ell,m) = 1)$ as f. The existence of such ϕ follows from (ii) for $m' < m$, i.e. the induction hypothesis (if f is on $\Gamma_0(m)$ then there exists at least one $\phi \neq 0$ in $J_{k,m}$ having the same eigenvalues as f.

If ϕ could not be choosen to be in $J_{k,m}^{new}$ then – by the Hecke invariance of the decomposition (4) – it could be choosen to be in $J_{k,m'}^{new}$, with a $m' < m$. But this implies the existence of a $g \neq 0$ in $M_{2k-2}^{new,-}(m')$ having the same eigenvalues as f, in contradiction to well-known Atkin-Lehner theory.)

Consider now the map $f|B_{\ell,d} \longmapsto \phi|U_d \circ V_\ell$ from $\mathfrak{M}_{2k-2}^{-}(m)$ to $J_{k,m}$. The assertions (i) and (ii) will follow immediately as soon as we can show that this map is an isomorphism, i.e., in view of (3), ~~that~~ the kernel of this map is 0. Since this map commutes with all Hecke operators the kernel is also invariant under all $T(\ell)$ $((\ell,m) = 1)$. Thus, if the kernel were $\neq 0$, there exist by Atkin-Lehner theory an $m'|m$ and an f on $\Gamma_0(m')$ as above such that a linear combination of the $f|U_d \circ V_\ell$ $(\ell d^2 = \frac{m}{m'})$ is mapped to zero, i.e. such that a linear combination of the $\phi|U_d \circ V_\ell$ is zero (ϕ associated to f as above). Clearly $m' < m$. By induction hypothesis and equation (5) we have $S_{D,s}(\phi|U_d \circ V_\ell) = c_\phi(D,s) f|B_{\ell,d}$. By Lemma 3.2 we can choose D,s so that $c_\phi(D,s) \neq 0$. Thus the $\phi|U_d \circ V_\ell$ must be linearly independent.

It remains to prove (iii) for a pair ϕ, f with ϕ in $J_{k,m}^{new}$ and f in $M_{2k-2}^{new,-}(m)$. We have to show that

$$(8) \qquad \sum_{a|\ell} a^{k-2}\left(\frac{D}{a}\right) c_\phi\left(\frac{\ell^2}{a^2}, D, \frac{\ell}{a}s\right) = c_\phi(D,s)\, a_f(\ell)$$

for all $\ell \geq 0$.

First of all we consider the case $\ell > 0$. For simplicity we assume $\ell = p^\alpha$ for a prime p, leaving the general case as an exercise.

If $p \nmid m$ then the left hand side of (8) is nothing else than $C_{\phi|T(\ell)}(D,s)$, the (D,s)-th Fourier coefficient of $\phi|T(\ell)$ (recall that D is a fundamental discriminant). Thus it equals $C_\phi(D,s)$ times the eigenvalue of ϕ, and hence f, with respect to $T(\ell)$, i.e. it equals the right hand side of (8).

If $p|m$ then ϕ satisfies the hypothesis of Lemma 3.3: $\phi|V_p u_p$ (and $\phi|u_p$ if $p^2|m$) is a Hecke eigenform in $J_{k,\frac{m}{p}}$ (and $J_{k,\frac{m}{p^2}}$) having the same eigenvalues as ϕ, and hence must be zero (otherwise there would be a Hecke eigenform in $M_{2k-2}(\frac{m}{p})$ (or in $M_{2k-2}(\frac{m}{p^2})$) having the same eigenvalues as $f \in M_{2k-2}^{new}(m)$, in contradiction to Atkin-Lehner theory). Hence the left hand side equals $C_\phi(D,s)$ times $\ell^{k-2}\cdot(-1)^\alpha\cdot\varepsilon_p^\alpha$ if $p\|m$ and 0 if $p^2|m$, where ε_p is the eigenvalue of ϕ, and hence of f, with respect to W_p. Thus (8) is also true for $p|m$.

Now let $\ell=0$. Then ϕ (and hence f) must be an Eisenstein series, say $\phi=E_{k,m,\chi,1}$ (and hence $f=E_{2k-2}^{(\chi)}$ with a primitive Dirichlet character χ mod F, $m=F^2$). But then (8) becomes

$$(9) \qquad \frac{1}{2}\,L(2-k,(\tfrac{D}{\cdot}))\cdot C_\phi(0,0) = C_\phi(D,s)\cdot\sigma_{2k-3}^{(\chi)}(0).$$

For $F \neq 1$ (and hence $C_\phi(0,0)=\chi(0)=0=\sigma_{2k-3}^{(\chi)}(0)$) there is nothing to show. For $m=1$ (and hence $C_\phi(0,0)=1$) (9) becomes

$$\tfrac{1}{2}L(2-k,(\tfrac{D}{\cdot}))/\zeta(3-2k) = C_\phi(D,s),$$

an identity which was proved in [E-Z].

It remains only to prove the existence of a linear combination in the $S_{D,s}$ defining an isomorphism. Let $\Psi_1,\ldots,\Psi_r$ run through a basis of simulataneous Hecke eigenforms of $\bigoplus_{\ell|m} J^{new}_{k,\frac{m}{\ell}}|V_\ell$. Using Lemma 7.2 it is easily verified that for each Ψ_i there exist a fundamental discriminant D_i and an integer s_i with $D_i \equiv s_i^2 \bmod 4m$ such that $C_{\Psi_i}(D_i,s_i) \neq 0$. Choosing constants $\alpha_1,\ldots,\alpha_r$ such that $\sum_{j=1}^r \alpha_j C_{\Psi_i}(D_j,s_j) \neq 0$ for all $i = 1,\ldots,r$ yields $L(\Psi_i) \neq 0$ for all i where $L = \sum_{j=1}^r \alpha_j S_{D_j,s_j}$.

But then L is surjective (and hence an isomorphism): let $m'|m$, $f \in M^{new,-}_{2k-2}(m')$ a Hecke-eigenform. Choose a non-zero eigenform in $J^{new}_{k,m'}$ having the same eigenvalues as f. Clearly ϕ can be choosen so that $\phi|V_{m/m'} = \Psi_i$ for a suitable i. Then $L(\phi)|B_{m/m',1} = L(\Psi_i) = c\cdot f|B_{m/m',1}$ with a constant $c \neq 0$, hence $L(\phi) = c\cdot f$, and finally $L(\phi|U_d \circ V_\ell) = c\cdot f|B_{\ell,d}$ for all $\ell d^2 = \frac{m}{m'}$.

We have still to prove Lemmas 3.1 and 3.2.

<u>Proof of Lemma 3.1</u> - Using $\sum_{t|m'} \mu(t) \prod_{p|t} e(\frac{r}{p}) = 0$ for all integers r with $(r,m') \neq 1$ ($\mu(\cdot)$ denoting the Möbius function) one easily deduces from the assumption that

$$(10) \qquad \sum_{t|m'} \mu(t)\ \phi|_{k,m}\left\{\prod_{p|t} [0,\tfrac{g}{p}]\right\} = 0$$

for all integers g.

Applying suitable matrices $\begin{bmatrix} * & * \\ c & d \end{bmatrix} \in \Gamma$ to (10) and summing up one obtains

$$0 = \sum_{g|m'} \sum_{\substack{c,d=1 \\ (c,d)=1}}^{m'/g} \sum_{t|m'} \mu(t)\phi|\left\{\prod_{p|t}\left[\frac{g(c,d)}{p}\right]\right\} = \sum_{t|m'} \mu(t) \sum_{x \bmod m'} \phi|\left\{\prod_{p|t}\left[\frac{x}{p}\right]\right\}$$

$$(11) \qquad = \sum_{t|m'} \mu(t) \frac{m'^2}{t^2} \prod_{p|t} \sum_{y \bmod p} \phi|\left[\frac{y}{p}\right]$$

$$= \sum_{\substack{d|m' \\ d^2|m}} \sum_{\substack{n|m' \\ n\|m}} \mu(dn) \frac{m'^2}{d^2n^2} \sum_{z \bmod d} \prod_{p|n} \sum_{z' \bmod p} \phi|\left(\left[\frac{z}{d}\right]\left[\frac{z'}{p}\right]\right) .$$

Here we used the easily proved equation

$$\sum_{x \bmod st} \phi|\left[\frac{x}{st}\right] = \sum_{y \bmod s} \sum_{z \bmod t} \phi|\left(\left[\frac{y}{s}\right]\left[\frac{z}{t}\right]\right) \qquad \text{(for all } s,t \text{ with } st|m).$$

Consulting the definition of u_d (cf. (6)) and W_n (cf. (5.6)) one can rewrite (11) as

$$0 = \phi|\left(\sum_{\substack{d|m' \\ d^2|m}} \mu(d) \frac{1}{d} u_d U_d\right)\circ\left(\prod_{\substack{p|m' \\ p\|m}} \left(1-\frac{1}{p}W_p\right)\right).$$

But the operator $\prod_{\substack{p|m' \\ p\|m}} (1 - \frac{1}{p}W_p)$ is invertible. Hence

$$0 = \sum_{\substack{d|m' \\ d^2|m}} \mu(d)\frac{1}{d}(\phi|u_d)|U_d \quad ,$$

which immediately yields the assertion.

Proof of Lemma 3.2 - Let $\ell = \prod_{p|N, p\nmid m} p$. We shall show that $\phi|V_\ell \neq 0$. Assuming this for the moment we then deduce from Lemma 3.1 that there exists a pair $\Delta \equiv r^2 \bmod 4m\ell$, $\Delta<0$, with $(r,m\ell) = 1$ and $C_{\phi|V_\ell}(\Delta,r) \neq 0$. Note that $(r,m\ell) = 1$ implies $C_{\phi|V_\ell}(\Delta,r) = C_\phi(\Delta,r)$, hence

$C_\phi(\Delta,r) \neq 0$. Write $\Delta = F^2D$ with a fundamental discriminant D and $r \equiv Fs' \bmod 4m\ell$ for a suitable s' with $D \equiv s'^2 \bmod 4m\ell$. In view of formulas (0.5) for the action of Hecke operators on Fourier coefficients it is then clear that $C_\phi(\Delta,r) \neq 0$ implies $C_\phi(D,s') \neq 0$ (use again $(r,m\ell) = 1$, hence $(F,m\ell) = 1$). Again using $(r,m\ell) = 1$, and hence $(s',m\ell) = 1$, it is then clear that there exists an $s \equiv s' \bmod 2m\ell$ such that $s^2 \equiv D \bmod 4mN$. But $C_\phi(D,s) = C_\phi(D,s') \neq 0$.

To prove $\phi|V_\ell \neq 0$ let p be a prime, $p|\ell$, and let $\Psi := \phi|V_{\ell/p}$. We show that $\Psi \neq 0$ implies $\Psi|V_p \neq 0$, so that by induction $\phi \neq 0$ implies $\phi|V_\ell \neq 0$. (Note that ℓ is squarefree and hence $V_\ell = V_{\ell'} \circ V_{\ell''}$ for all ℓ',ℓ'' with $\ell'\ell'' = \ell$).

So assume $\Psi \neq 0$ and $\Psi|V_p = 0$. First of all note that Ψ is an eigenform of $T(p)$. Hence in view of (0.5)

$$(12) \qquad \lambda_p C_\Psi(D,s) = C_\Psi(p^2D,ps) + \left(\frac{D}{p}\right) p^{k-2} C_\Psi(D,s)$$

for any $D \equiv s^2 \bmod 4m\ell/p$, D fundamental, and with λ_p being the eigenvalue of Ψ with respect to $T(p)$.

Now $\Psi|V_p = 0$ means

$$0 = C_{\Psi|V_p}(p^2D,ps) = C_\Psi(p^2D,ps) + p^{k-1} C_\Psi(D,s).$$

Combining this with (12) yields

$$(13) \qquad \lambda_p C_\Psi(D,s) = \left(-p^{k-1} + \left(\frac{D}{p}\right) p^{k-2}\right) C_\Psi(D,s).$$

But by Lemma 3.1 there exists a $\Delta \equiv r^2 \bmod 4m\ell/p$ with $(r, m\ell/p)=1$ and $C_\psi(\Delta, r) \neq 0$ and as above we see that then there also exists a pair $D \equiv s^2 \bmod 4m\ell/t$, D fundamental, such that $C_\psi(D,s) \neq 0$. Note that this implies in particular $(\frac{D}{p}) = -1$ (otherwise $D \equiv s'^2 \bmod 4m\ell$ with an $s' \equiv s \bmod 2m\ell/p$ and $C_{\psi|V_p}(D,s') = C_\psi(D,s') = C_\psi(D,s) \neq 0$). Thus we obtain from (13)

$$\lambda_p = -p^{k-2}(p+1).$$

Now λ_p is also eigenvalue of $T(p)$ on $M_{2k-2}(m\ell/p)$. Clearly it can not be an eigenvalue of $T(p)$ on the space of Eisenstein series in $M_{2k-2}(m\ell/p)$. Also it cannot be an eigenvalue of $T(p)$ on the space of cusp forms in $M_{2k-2}(m\ell/p)$ since then it must satisfy

$$|\lambda_p| < p^{k-2}(p+1)$$

be an elementary estimate . (Of course, one can also apply the deeper Ramanujan-Petersson conjecture.) Thus, in each case we have a contradiction.

Appendix. Some formulas involving class numbers

For a negative discriminant Δ, $\Delta \neq -3,-4$ denote by $h'(\Delta)$ the number of equivalence classes with respect to $SL_2(\mathbf{Z})$ of primitive, integral, negative definite, binary quadratic forms of discriminant Δ, and set $h'(-3) = \frac{1}{3}$, $h'(-4) = \frac{1}{2}$. Recall the well-known formulas

$$h'(\Delta_0) = \frac{1}{\Delta_0} \sum_{0<\lambda<|\Delta_0|} \lambda \left(\frac{\Delta_0}{\lambda}\right) ,$$

(1)

$$h'(\Delta_0 F^2) = h'(\Delta_0)\gamma_{\Delta_0}(F), \quad \gamma_{\Delta_0}(F) = \sum_{t|F} \mu(t) \left(\frac{\Delta_0}{t}\right) \frac{F}{t} ,$$

Δ_0 being a fundamental discriminant, F a positive integer. Also recall the function $H_n(\Delta)$ as defined in § 1:

$$H_1(\Delta) = \sum_{f|F} h'(\Delta/f^2)$$

if $\Delta = \Delta_0 F^2$, Δ_0 a fundamental discriminant, and $H_1(\Delta) = 0$ otherwise and

$$H_n(\Delta) = \begin{cases} a^2 b\left(\frac{\Delta/a^2b^2}{n/a^2b}\right) H_1(\Delta/a^2b^2) & \text{if } (n,\Delta) = a^2 b \text{ with squarefree } b \text{ such that } a^2b^2 \mid \Delta \\ 0 & \text{otherwise .} \end{cases}$$

Finally, recall the notation Av_x (§4 of [S-Z]) for the operator which replaces a periodic function of x (x in $\mathbb{Z}^r$) by its average value. More precisely, $Av_x\, f(x) = [\mathbb{Z}^r:L]^{-1} \sum_{x \in \mathbb{Z}^r/L} f(x)$ for any periodic function $f(x)$ on $\mathbb{Z}^r$, where L is any lattice such that $f(x+y) = f(x)$ for all $x \in \mathbb{Z}^r$, $y \in L$.

Proposition A.1. Let Δ be a negative discriminant, n a positive integer. Then

$$\sum_{\substack{Q \bmod. \Gamma \\ \mathrm{disc}(Q)=\Delta}} \frac{1}{|\Gamma_Q|} Av_x e\left(\frac{Q(x)}{n}\right) = \frac{1}{n} H_n(\Delta) , \tag{2}$$

where the sum is over a complete set of representatives for the equivalence classes with respect to $\Gamma = SL_2(\mathbb{Z})$ of all integral binary quadratic forms of discriminant Δ, where Γ_Q denotes the group of automorphisms $(\subseteq \Gamma)$ of Q.

Proof. The sum on the left hand side of (2) is clearly invariant with respect to replacing Q by $-Q$. Hence it must be real, and we can sum as well over all positive definite Q if we replace $\frac{1}{|\Gamma_Q|} Av_x e(\frac{Q(x)}{n})$ by $\frac{1}{|\Gamma_Q|/2} \mathrm{Re}(Av_x e(\frac{Q(x)}{n}))$.

Now, if Q is primitive, then by Theorem 3 of [S-Z]

$$\mathrm{Re}(Av_x e(\frac{Q(x)}{n})) = \begin{cases} \frac{(n,\Delta)^{1/2}}{n} \left(\frac{(n,\Delta)}{a}\right) \left(\frac{\Delta/(n,\Delta)}{n/(n,\Delta)}\right) & \text{if } (n,\Delta) \text{ and } \Delta/(n,\Delta) \text{ are both congruent to } 0 \text{ or } 1 \bmod. 4 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Here a denotes any integer represented by Q and prime to N.

By composition theory the set of equivalence classes modulo Γ of primitive positive definite forms of discriminant Δ forms a group. By the theory of genera the map $Q \longmapsto \left(\frac{(n,\Delta)}{a}\right)$ is a character of this group which is trivial if and only if (n,Δ)

is a square (or if $\Delta/(n,\Delta)$ is a square, which here is impossible since $\Delta < 0$). Thus, summing over primitive Q mod Γ, we obtain

$$(4) \qquad \sum_{\substack{Q \bmod \Gamma \\ Q \text{ primitive} \\ \mathrm{disc}(Q)=\Delta}} \frac{1}{|\Gamma_Q|} Av_x e\left(\frac{Q(x)}{n}\right) = \frac{1}{n} \chi_\Delta(n)\, h'(\Delta).$$

Here, as in §0, $\chi_\Delta(n) = a\left(\frac{\Delta/a^2}{n/a^2}\right)$ if $(n,\Delta) = a^2$, $\Delta/a^2 \equiv 0,1 \bmod 4$, and $\chi_\Delta(n) = 0$ otherwise. Also here we used $|\Gamma_Q| = 2,4,6$ if $\Delta < -4$, $\Delta = -4$, $\Delta = -3$ respectively.

Note that (4) remains valid if we replace each term $Av_x e(\frac{Q(x)}{n}))$ on the left hand side by $Av_x e(\frac{aQ(x)}{n})$, where a is any integer prime to n (use Galois theory or modify (3)).

Hence, setting $\Delta = \Delta_0 F^2$, Δ_0 a fundamental discrimiant, writing the left hand side of (2) as

$$\sum_{f|F} \sum_{\substack{Q \bmod \Gamma \\ Q \text{ primitive} \\ \mathrm{disc}(Q)=\Delta/f^2}} \frac{1}{|\Gamma_Q|} Av_x e\left(\frac{fQ(x)}{n}\right),$$

and applying (4) we obtain

$$\frac{1}{n} \sum_{f|F} (n,f)\, \chi_{\Delta/f^2}(n/(n,f))\, h'(\Delta/f^2).$$

Finally, applying the second formula in (1), we notice that the claimed formula (2) is reduced to the elementary identity

$$(5)\quad \sum_{f|F}(n,f)\ \chi_{\Delta/f^2}(n/(n,f))\ \gamma_{\Delta_0}(\tfrac{F}{f}) = \begin{cases} a^2b\cdot\left(\frac{\Delta/a^2b^2}{n/a^2b}\right) \sum\limits_{f|\frac{F}{ab}} \gamma_{\Delta_0}(f) & \text{if } (n,\Delta)=a^2b \text{ with squarefree } b \text{ such that } a^2b^2|\Delta \text{ and } \Delta/a^2b^2\equiv 0,1 \bmod 4. \\ 0 & \text{otherwise} \end{cases}$$

which we leave as an exercise to the reader.

Remarks. (i) Obviously (2) remains valid if we replace each summand on the left hand side by $\frac{1}{|\Gamma_Q|} A\upsilon_x\left(\frac{aQ(x)}{n}\right)$, a being an integer prime to n.

(ii) Note that a similar formula as (2) holds for Δ being a square if one omits the factors $1/|\Gamma_Q|$ on the left hand side, and , of course, replaces $\frac{1}{n}H_n(\Delta)$ by $\frac{a^2b}{n}\sqrt{\frac{\Delta}{a^2b^2}}$, where $(n,\Delta) = a^2b$, b squarefree (recall that $\sqrt{\Delta}$ for a square Δ equals the number of equivalence classes mod. Γ of integral quadratic forms of discriminant Δ). The proof for this is the same as for (2). However, here everything can be done in a completely elementary way using $Q(\lambda,\mu) = a\lambda^2 + \sqrt{\Delta}\lambda\mu$ $(0 \le a < \sqrt{\Delta})$ as a complete set of representatives mod. Γ for forms of discriminant Δ . We leave it to the reader to work this out.

Proposition A.2. Let a,n be positive integers. Then

$$\sum_{\substack{b \bmod n \\ n \nmid b}} i\cot(\pi\tfrac{b}{n})\cdot A\upsilon_\lambda\, e(\tfrac{a}{n}b\lambda^2) = -2(a,n)\sum_{\Delta} H_{a/(a,n)}(\Delta)\ ,$$

where the sum on the right hand side is over all discriminants $\Delta < 0$

<u>such that</u> $\Delta \mid \frac{n}{(a,n)}$ <u>and</u> $\frac{n}{(a,n)\Delta}$ <u>is squarefree.</u>

<u>Proof</u> - The asserted formula is a simple consequence of the easily proved identity

$$\sum_{\substack{b \bmod n \\ n \nmid b}} i \cot(\pi \tfrac{b}{n}) \cdot e(\tfrac{a}{n} b\lambda^2) = 2 ((\tfrac{a}{n} \lambda^2)) \quad ,$$

where $((x))$ for any real number x is defined by

$$((x)) = \begin{cases} \xi - \frac{1}{2} & \text{if } x \in \xi + \mathbb{Z}\ ,\ 0 < \xi < 1 \\ 0 & \text{otherwise} \end{cases} \quad ,$$

and the formula

$$\sum_{\lambda \bmod n} \left(\left(\frac{a\lambda^2}{n} \right)\right) = - \sum_{\substack{\Delta \mid n, \Delta < 0 \\ n/\Delta \text{ squarefree}}} H_a(\Delta) \quad \text{(for relative prime, positive integers } a,n).$$

The latter can be proved by writing

$$\sum_{\lambda \bmod n} \left(\left(\frac{a\lambda^2}{n}\right)\right) = \sum_{\lambda \bmod n} \left(\left(\frac{a\nu}{n}\right)\right) \cdot \#\{\lambda \bmod n \mid \lambda^2 \equiv \nu \bmod n\} \ ,$$

inserting the identity

$$\#\{\lambda \bmod n \mid \lambda^2 \equiv \nu \bmod n\} = \sum_{\substack{\Delta \mid n,\ \Delta \equiv 0,1 \bmod 4 \\ (Q(\frac{n}{|\Delta|}), \nu) = 1}} \chi_\Delta(\nu)$$

($\chi_\Delta(\nu)$ as in §0, $Q(\frac{n}{|\Delta|})$ the greatest integer whose square divides $\frac{n}{\Delta}$), and applying, after some obvious manipulations, the first formula in (1). (With respect to this application of (1) note that for any fundamental discriminant Δ, any integers a,n with $(a,n) = 1$, $\Delta | n$ one has

$$\sum_{\nu \bmod n} \left(\left(\frac{a\nu}{n}\right)\right) \chi_\Delta(\nu) = \begin{cases} 0 & \text{if } \Delta > 0 \\ \frac{1}{n} \sum_{0<\nu<n} \nu \cdot \left(\frac{\Delta}{\nu}\right) & \text{if } \Delta < 0 \text{ .)} \end{cases}$$

Again, the details are left to the reader (or else cf. [S], Lemma 6.5).

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