Harmonic differentials and infinite geodesics joining two cusps on a Riemann Surface

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Abstract. Let M be an hyperbolic Riemann surface. The harmonic dual form to an infinite geodesic joining two cusps on M, is obtained in two different ways. First of all, using the degeneration of hyperbolic Eisenstein series, it is made explicit in terms of Eisentein series. Secondly, generalizing the construction of Kudla and Millson to the case of an infinite geodesic joining two cusps, we give an automorphic realization of this harmonic form.

Introduction

In this paper, we answer in particular, some questions in [K-M] (p.194 and 208) and in [G] (p. 562-09). In the first part we recall the definitions and results we need, we also give the example of the principal congruence subgroup of level 2, $\Gamma(2)$ (see proposition 1.1). In the second we obtain the harmonic dual 1-form to an infinite geodesic joining two cusps, using the degeneration of hyperbolic Eisenstein series (see proposition 2.3). Moreover, in the third part we answer a question that P.Gérardin asks in [G] (see theorem 3.2): can we extend the Kudla and Millson's construction to the case of $]p, q[$, the geodesic joining the cusps p and q and orthogonal to the horocycles at p and at q ?

I hope this work could be generalized to the higher dimensional case ([KM1], [KM2]).

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1. Preliminary definitions and results

1.1. **Definitions.** In the following M will be a hyperbolic Riemann surface, that is to say, M has for universal covering space the upper half plane H. The fundamental group of M is identified to a Fuchsian group of the first kind, denoted Γ. We'll suppose to have only two cusps on M, p and q; it is easy to verify that all the results are true if there are another cusps than p and q.

In [K-M], Kudla and Millson construct the harmonic 1-form dual to a simple closed geodesic on M. Let us recall the definition:

Definition 1.1. *Let η be a simple closed geodesic or an infinite geodesic joining p and q. An 1-form α is dual to η if for any closed 1-form with compact support, ω*

$$\int_M \omega \wedge \alpha = \int_\eta \omega \ .$$

Or equivalently: for any closed oriented cycle c', we have

$$\int_{c'} \alpha = c' \; . \; \eta \; ,$$

(here $c'.\eta$ denotes the intersection number of the cycles c' and η.)

We are interesting in the 1-form dual to $]p,q[$ and which is harmonic. We begin by giving, in an elementary way, the unicity of the cohomology class and the existence of such a form. For this, recall the following results:

Lemma 1.1. *Let α be a 1-form such that*
1. $\int_M \alpha \wedge \beta = 0$ *for each closed 1-form β, or*
2. *for any simple closed curve c, $\int_c \alpha = 0$.*

then there exists a function f with $\alpha = df$.

and the theorem in [S]

Theorem 1.1. *(theorem 3, p.106) Let X be a compact Riemann surface. Let p and q be two distincts points on X connected by a simple, oriented curve L, and X^* be the region obtained by cutting along L. There exists a unique differential of the first kind $dF(p,q)$ which is regular in X apart from simple poles at p and q with residues 1 and -1. The periods of $F(p,q)$ on X^* are all real, and, for every loop which crosses L exactly once from right to left, the imaginary part of the period is 2π.*

Then the $1/2\pi$ imaginary part of $dF(p,q)$ gives the dual form to the curve L.

The authors of [K-M] obtain the harmonic dual 1-form to a simple closed geodesic η as the analytic continuation at $s=1$ of the *hyperbolic Eisentein series* associated to η: $\hat{\eta}^s$. This hyperbolic Eisenstein series is defined by taking the sum over the set $H(\eta)$ of all the oriented geodesics Y of H whose projection is η, of the 1-form:

$$\omega_Y^s = ||\omega_Y||^{s-1} \omega_Y \; ,$$

where $\omega_Y = *P_Y^* V_Y$, P_Y is the orthogonal projection on Y, V_Y the volume form of Y and $||\omega_Y||$ denotes the pointwise norm of ω_Y. The 1-form ω_Y^s is normalised so that the integral over any fiber of P_Y is 1:

$$\int \frac{1}{k(s-1)} \, \omega_Y^s = 1 \; , \; k(s-1) = \frac{\Gamma(1/2)\Gamma(s/2)}{\Gamma(1/2+s/2)} \; .$$

Then, for $\operatorname{Re} s > 1$:

$$\hat{\eta}^s = \frac{1}{k(s-1)} \sum_{Y \in H(\eta)} \omega_Y^s = \frac{\Gamma(1/2+s/2)}{\Gamma(1/2)\Gamma(s/2)} \sum_{\Gamma_1 \backslash \Gamma} \gamma^* \frac{dx_2}{(\operatorname{ch} x_2)^s} \; , \tag{1}$$

where Γ_1 denote the stabilizer group in Γ of some geodesic $Y \in H(\eta)$ and any point $P \in H$ is located by its Fermi coordinates $(x_1, \; x_2)$ associated to the geodesic Y.

Recall now of the definition of the classical Eisentein series.

The stability group of a cusp $\mathfrak{a}$ is an infinite cyclic group generated by a parabolic motion,

$$\Gamma_{\mathfrak{a}} = \{\gamma \in \Gamma : \gamma\, \mathfrak{a} = \mathfrak{a}\} = \langle \gamma_{\mathfrak{a}} \rangle \ ,$$

say. There exists $\sigma_{\mathfrak{a}} \in SL_2(\mathbb{R})$ such that

$$\sigma_{\mathfrak{a}}\infty = \mathfrak{a}, \ \ \sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \ .$$

We call $\sigma_{\mathfrak{a}}$ a scaling matrix of the cusp $\mathfrak{a}$, it is determinated up to composition with a translation from the right side. The Eisenstein series for the cusp $\mathfrak{a}$ is then defined by:

$$E_{\mathfrak{a}}(z,s) = \sum_{\Gamma_{\mathfrak{a}}\backslash\Gamma} y(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s \ ,$$

where s is a complex variable, $\operatorname{Re} s > 1$.

An Eisenstein series of weight 2 associated to a cusp $\mathfrak{a}$, is for us, a 1-form defined, for $\operatorname{Re} s > 1$, by:

$$\mathcal{E}_{\mathfrak{a}}(s,z) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} y(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{s-1}\ d(\sigma_{\mathfrak{a}}^{-1}\gamma z) = \frac{E_{\mathfrak{a},1}(s,z)}{y}\ dz \ ,$$

with

$$E_{\mathfrak{a},1}(s,z) = y \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} y(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{s-1}\ ((\sigma_{\mathfrak{a}}^{-1}\gamma)'z) \ .$$

By definition, a standard horocyclic Eisenstein series, called for sake of convenience, an *horocyclic Eisenstein series* is

$$\hat{\mathfrak{a}}^s(z) = \operatorname{Im}(\mathcal{E}_{\mathfrak{a}}(s,z)) \ .$$

Remark 1.1. 1. *In the notations of* [Fay], $E_{\mathfrak{a},1}(s,z) \in \mathcal{F}_1$, *where* $\mathcal{F}_1$ *is the space of all functions* f *on* H *with* $f(\gamma z) = \left(\frac{cz+d}{c\bar{z}+d}\right) f(z), \ \ \gamma \in \Gamma$.

2. *A standard horocyclic Eisenstein series is associated to a horocycle of length 1: if* $|\xi|$ *is the length of the horocycle and* $\hat{\xi}^s$ *is the horocyclic Eisenstein series associated to* ξ *then the 1-form* $|\xi|^{-s}\hat{\xi}^s$ *depends only on the cusp (for more details see* [G]*).*

1.2. **Example.** Now we are going to treat the explicit example of the geodesic $]0, i\infty[$ on $M = \Gamma(2)\backslash H$ where $\Gamma(2)$ is the principal congruence subgroup of level 2:

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),\ a = d = 1 \mod 2,\ b = c = 0 \mod 2 \right\}.$$

In this case comparing the Hecke summation process and a result in [M], we obtain:

$$\mathcal{E}_\infty(1,z) - \mathcal{E}_0(1,z) = \theta(0,z)^4 dz\ ,$$

where $\theta(0,z)^4 = \left(\sum_{n\in\mathbb{Z}} e^{i\pi n^2 z}\right)^2 = \sum_{n\in\mathbb{N}} r_4(n) e^{i\pi n z}$ and $r_4(n)$ is the number of representations of n as a sum of 4 squares. Let c_1 and c_2 be horocycles associated respectively to the cusps 0 and ∞, which cross exactly once from right to left $]0, i\infty[$, we easily verify that

$$\begin{aligned} \int_{c_2} -\operatorname{Re}(\theta(0,z)^4\ dz) &= -\frac{1}{2}\int_{(1+iy_o, 1-iy_o)} (1 + \sum_{n\geq 1} r_4(n) e^{-n\pi y_o} \cos n\pi x)\ dx \\ &= 1 \end{aligned}$$

and, with $\sigma : z \to -1/z$

$$\begin{aligned} \int_{c_1} -\operatorname{Re}(\theta(0,z)^4\ dz) &= \int_{\sigma(c_2)} \operatorname{Re}(\theta(0,z)^4\ dz) \\ &= \int_{c_2} \operatorname{Re}(\theta(0,\sigma(z))^4\ d\sigma(z)) = \int_{c_2} -\operatorname{Re}(\theta(0,z)^4\ dz) \\ &= 1 \end{aligned}$$

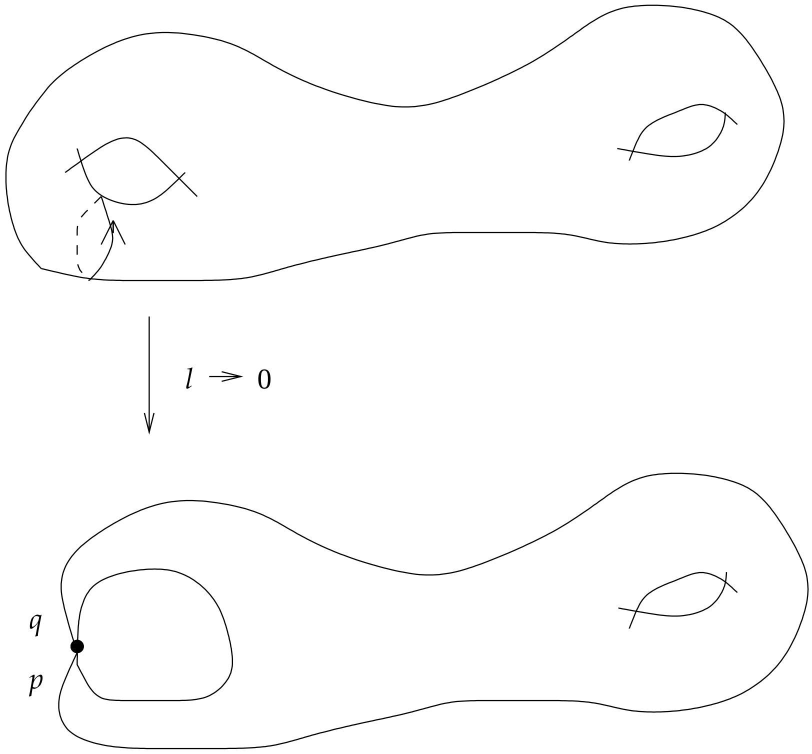
So, it follows

Proposition 1.1. *The 1-form,* $-\operatorname{Re}(\mathcal{E}_\infty(1,z) - \mathcal{E}_0(1,z)) = -\operatorname{Re}(\theta(0,z)^4\ dz)$ *is the harmonic dual form to* $]0, i\infty[$.

2. The harmonic dual form and degeneration

The aim of this section is to find the harmonic dual form to an infinite geodesic joining two cusps on a Riemann surface using the degeneration of hyperbolic Eisenstein series. To do this we use the following results:

Consider a family of compact Riemann surfaces $(S_l)_{l>0}$ degenerating as $l \to 0,\ l > 0$ to S_0, the geodesic c_l (of length l), non homologous to zero, on S_l being pinched to form a pair of cusps on S_0.



We note p and q the two cusps of S_0 arising from pinching c_l: p the limit of the right side of the c_l-collar and q the limit of the left side of the c_l-collar. Without loss of generality we can represent p and q by ∞ and 0 respectively. We denote for simplicity $\hat{c}_l^s = \hat{l}^s$ and the complex 1-form of type $(1,0)$ associated:

$$\theta_l(s,z) = 1/2(i\hat{l}^s(z) - *\hat{l}^s(z)) \ .$$

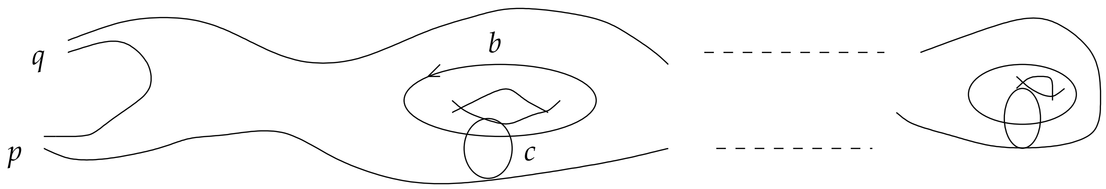
There exist homeomorphisms f_l from $S_l \backslash c_l$ to S_0, with f_l tending to an isometry $\mathcal{C}^\infty$-uniformly on the complementary in S_l of a neighbourhood of c_l. We have the results (for the details see [F]):

Theorem 2.1. 1. *For* $\operatorname{Re} s > 1$, *the family of 1-forms* $\dfrac{1}{l^s}\theta_l(s, f_l^{-1}(.))$ *converge uniformly on compact subsets of* S_0 *to* $\dfrac{1}{2k(s-1)}[\mathcal{E}_\infty(s,.) - \mathcal{E}_0(s,.)]$.

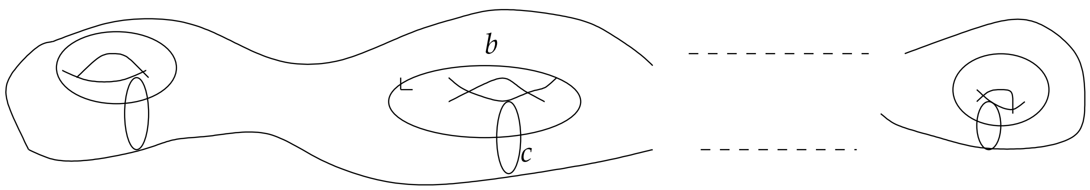
2. *The family of 1-forms* $\left(\dfrac{1}{l}\theta_l(1, f_l^{-1}(.))\right)_{l>0}$ *converge uniformly on compact subsets of* S_0 *to* $\dfrac{1}{2\pi}[\mathcal{E}_\infty(1,.) - \mathcal{E}_0(1,.)]$.

3. *For* $s \in \{\operatorname{Re} s > 1,\ s = 1\}$ *the family of 1-forms* $\dfrac{1}{l^s}\hat{l}^s(s, f_l^{-1}(.))$ *converge uniformly on compact subsets of* S_0 *to* $\dfrac{1}{k(s-1)}[\hat{\infty}^s - \hat{0}^s]$.

First of all, as an application and using a classical result on the degeneration of the canonical period matrix we obtain the asymptotic behavior of the periods of the holomorphic 1-form $\theta_l(1,z)$. Suppose S_0 is of genus g and let $\{b_j, c_j\}_{j=2}^{g+1}$ be a geodesic canonical homology base of the complementary of the cuspidal regions:



We put $\pi_l = f_l^{-1}$ and we denote by $b_i(l)$ (resp. $c_i(l)$) the unique geodesic in the homology class of $\pi_l(b_i)$ (resp. $\pi_l(c_i)$) for $i = 2, ..., g+1$. We complete this family with $b_1(l)$ and $c_1(l) = c_l$ in order to form a geodesic canonical homology base on S_l: $\{b_i(l), c_i(l)\}_{i=1}^{g+1}$.



We obtain the following two propositions (see [F]):

Proposition 2.1. *For A belonging to $\{b_i, c_i, i = 2, ..., g+1\}$*

$$\lim_{l\to 0} \int_{A(l)} \frac{\theta_l(1,z)}{l} = \frac{1}{2\pi} \int_A \mathcal{E}_\infty(1,z) - \mathcal{E}_0(1,z) \ .$$

and

Proposition 2.2. *For $l \to 0$, $\displaystyle\int_{b_1(l)} \theta_l(1,z) = O(1)$ and $\displaystyle\int_{c_l} \theta_l(1,z) \sim -\frac{l}{2\pi}$.*

Secondly, at the end of [K-M], there is left the question of the harmonic dual form to an infinite geodesic joining two cusps. To explicit this form, we decompose the harmonic 1-form $*\hat{l}$ on the following base of harmonic differentials:
$\{\hat{b}_i(l) \ , i = 1, ..., g+1, \ \ \hat{c}_i(l) \ , i = 1, ..., g+1 \ \}$, where $\hat{b}_i(l)$ is the harmonic dual form to $b_i(l)$, (resp. $\hat{c}_i(l)$ is the harmonic dual form to $c_i(l)$), so:

$$*\hat{l} = \sum_{i=1}^{g+1} e_i \ \hat{c}_i(l) + \sum_{i=1}^{g+1} d_i \ \hat{b}_i(l)$$

with:

$$\forall i = 1, ..., g+1 \qquad d_i = \int_{c_i(l)} - * \hat{l} = 2 \int_{c_i(l)} \theta_l$$

$$\forall i = 2, ..., g+1 \qquad e_i = \int_{b_i(l)} * \hat{l} = -2 \int_{b_i(l)} \theta_l$$

Let α be a closed 1-form on S_0 with compact support and take the corresponding α_l on S_l, i.e. $\alpha_l(\pi_l(w)) = \alpha(w)$, $w \in S_0$. The 1-form α_l is closed on S_l. Therefore:

$$\begin{aligned} \int_M \alpha_l \wedge *\hat{l} &= \sum_{i=1}^{g+1} e_i \int_M \alpha_l \wedge \hat{c}_i(l) + \sum_{i=1}^{g+1} d_i \int_M \alpha_l \wedge \hat{b}_i(l) \\ &= \sum_{i=1}^{g+1} e_i \int_{c_i(l)} \alpha_l + \sum_{i=1}^{g+1} d_i \int_{b_i(l)} \alpha_l \ . \end{aligned}$$

So

$$\int_{S_l} \alpha_l \wedge \frac{*\hat{l}}{l} = \frac{d_1}{l} \int_{b_1(l)} \alpha_l + \frac{e_1}{l} \int_{c_1(l)} \alpha_l + \sum_{i=2}^{g+1} \frac{e_i}{l} \int_{c_i(l)} \alpha_l + \sum_{i=2}^{g+1} \frac{d_i}{l} \int_{b_i(l)} \alpha_l \ .$$

As α is with compact support, for l sufficiently small $\int_{c_1(l)} \alpha_l = 0$, it remains:

$$\text{(2)} \qquad \int_{S_l} \alpha_l \wedge \frac{*\hat{l}}{l} = \frac{d_1}{l} \int_{]p,q[} \alpha + \sum_{i=2}^{g+1} \frac{e_i}{l} \int_{c_i} \alpha + \sum_{i=2}^{g+1} \frac{d_i}{l} \int_{b_i} \alpha \ . \ (]p,q[=]i\infty,0[)$$

Moreover we have the following behaviors (propositions 2.1 and 2.2), for $i = 2, ..., g+1$:

$$\lim_{l \to 0} \frac{d_i}{l} = \frac{1}{\pi} \int_{c_i} \mathcal{E}_\infty - \mathcal{E}_0, \qquad \lim_{l \to 0} \frac{e_i}{l} = -\frac{1}{\pi} \int_{b_i} \mathcal{E}_\infty - \mathcal{E}_0 \ ,$$

and

$$\lim_{l \to 0} \frac{d_1}{l} = -\frac{1}{\pi} \ .$$

If we let l goes to 0 in (2), we obtain:

$$\int_{S_0} \alpha \wedge \left(-\frac{1}{\pi} \operatorname{Re}[\mathcal{E}_\infty - \mathcal{E}_0] \right) = \frac{1}{\pi} \int_{]0,i\infty[} \alpha - \sum_{i=2}^{g+1} \left(\frac{1}{\pi} \int_{b_i} \mathcal{E}_\infty - \mathcal{E}_0 \int_{c_i} \alpha - \frac{1}{\pi} \int_{c_i} \mathcal{E}_\infty - \mathcal{E}_0 \int_{b_i} \alpha \right) \ .$$

Proposition 2.3. *Setting:*

$$\Theta(z) = -\operatorname{Re}[\mathcal{E}_\infty(1,z) - \mathcal{E}_0(1,z)] + \sum_{i=2}^{g+1} \int_{b_i} (\mathcal{E}_\infty(1,.) - \mathcal{E}_0(1,.)) \ \hat{c}_i(z) - \sum_{i=2}^{g+1} \int_{c_i} (\mathcal{E}_\infty(1,.) - \mathcal{E}_0(1,.)) \ \hat{b}_i(z) \ ,$$

we have, for any closed 1-form with compact support on S_0, α:

$$\int_{S_0} \alpha \wedge \Theta = \int_{]0,i\infty[} \alpha \ ,$$

so Θ is the dual harmonic form to $]0, i\infty[$.

3. Automorphic realization

3.1. Fourier expansion of the Eisenstein series. We begin by recalling the Fourier expansion of the classical Eisenstein series (see for example [I]).

Let $\mathfrak{a}$, $\mathfrak{b}$ be cusps of Γ and $\sigma = \operatorname{Re} s > 1$, we have, uniformly in $z \in H$, the implied constant depending on s and the group Γ,

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s) = \delta_{\mathfrak{a}\,\mathfrak{b}}\ y^s + \phi_{\mathfrak{a}\,\mathfrak{b}}(s) y^{1-s} + O((1+y^{-\sigma})e^{-2\pi y}) \ ,$$

where $\delta_{\mathfrak{a}\,\mathfrak{b}} = 1$ if $\mathfrak{a}$, $\mathfrak{b}$ are equivalent, otherwise it vanishes and $\phi_{\mathfrak{a}\,\mathfrak{b}}(s)$ is a function defined with Kloosterman sums and Gamma factors.

3.2. Eisenstein series of weight 2 . We'll need to investigate the Fourier expansion of $\mathcal{E}_{\mathfrak{a}}$ for $\mathfrak{a} = \infty$ and 0 to obtain:

Proposition 3.1. *We have the following asymptotic behaviors:*

1. *at* $\mathfrak{a} = \infty$,
   $\mathcal{E}_\infty(1,z) = (1 + O(1)1/y + O(e^{-2\pi y}))\ dz$ *and* $\mathcal{E}_0(1,z) = O(1/y)\ dz$, *(*)*
2. *at* $\mathfrak{a} = 0$,
   $\mathcal{E}_\infty(1,z) = O(1/y)\ dz$ *and* $\mathcal{E}_0(1,z) = (\dfrac{1}{ez^2} + O(1/y))\ dz.$

Proof.
To obtain the Fourier expansion of $\mathcal{E}_\infty$ at $\mathfrak{a} = \infty$, we use the formula (see for example [Fay]):

$$E_{\infty,1} = y^s + y^{1-s}\psi_0(s) + \sum_{m\neq 0} \psi_m(s)\ W_{\frac{m}{|m|},s-1/2}(4\pi|m|y)\ e^{2\pi i m x} \ ,$$

where $W_{\frac{m}{|m|},s-1/2}$ are Whittaker functions and the Fourier coefficients are given by:

$$\begin{aligned}(n \neq 0)\ \ \psi_n(s) &= -\pi^s \frac{|n|^{s-1}}{\Gamma(s+\frac{n}{|n|})} \sum_{c>0} \frac{1}{c^{2s}}\ S_1(0,n,c)\\ \psi_0(s) &= -\pi \frac{4^{s-1}\Gamma(2s-1)}{\Gamma(s-1)\Gamma(s+1)}\phi_0(s)\end{aligned}$$

with $S_1(0,n,c)$ Kloosterman sums and $\phi_0(s) = \sum_{c>0} \dfrac{1}{c^{2s}} \sum_{0\le d<c} 1$. Therefore, we have:

$$\mathcal{E}_\infty(s,z) = \left(y^{s-1} + y^{-s}\psi_0(s) + \sum_{n\neq 0} \psi_n(s)\ W_{\frac{m}{|m|},s-1/2}(4\pi|m|y)/y\ e^{2\pi i m x} \right) dz \ .$$

The analytic continuation at $s = 1$ of $\psi_n(s)$ is well known (see for example [I]). Here we only point out that $s = 1$ is a regular value for $\mathcal{E}_\infty(s,z)$. In fact, the residue of $\phi_0(s)$ at $s = 1$ is given by $\lim_{s\to 1}(s-1)\phi_0(s) = 1/(\pi \operatorname{Vol}(M))$. As $(1/\Gamma)(0) = 0$, we deduce the analytic continuation at $s = 1$ of $\psi_0(s)$. So $\lim_{s\to 1}(s-1)\mathcal{E}_\infty(s,z) = \lim_{s\to 1}(s-1)\psi_0(s) = 0$.

Moreover we make use of the asymptotic behavior to obtain the first equality in *(*)*: $W_{\mu,s-1/2}(t) \approx e^{-t/2}\, t^{\mu}$, as $t \to +\infty$.

Now, the Fourier expansion of $\mathcal{E}_\infty$ at 0 is given in the form

$$\mathcal{E}_\infty(s,\sigma_0 z) = \sum_{\tau\in\Gamma_\infty\backslash\Gamma\sigma_0} y(\tau z)^{s-1}(\tau' z)\ dz$$

$$= \sum_{d>0}\sum_{c}\sum_{n\in\mathbb{Z}} \frac{y^{s-1}(z)}{|d\sqrt{e}\ (z+n) - c/\sqrt{e}|^{2(s-1)}(d\sqrt{e}\ (z+n) - c/\sqrt{e})^2}\ dz$$

$$= \left(-y^{-s}\pi\frac{4^{s-1}\Gamma(2s-1)}{\Gamma(s-1)\Gamma(s+1)}\sum_{d>0}\frac{1}{(d\sqrt{e})^{2s}}\sum_{c/\sqrt{e}(\mathrm{mod}\ \ d\sqrt{e})} 1 + O(e^{-2\pi y})\right)\ dz$$

$$= \left(e^s y^{-s}\tilde{\psi}_1(s) + O(e^{-2\pi y})\right)\ dz$$

In a similar way we find the Fourier expansion of $\mathcal{E}_0$ at 0 and ∞.

$$\begin{aligned}\mathcal{E}_0(s,\sigma_0 z) &= \sum_{\tau\in\Gamma_\infty\backslash\sigma_0^{-1}\Gamma\sigma_0} y(\tau z)^{s-1}\ (\tau' z)\ dz \\ &= \left(y^{s-1} + y^{-s}\sum_{b>0}\frac{1}{(be)^{2s}}\sum_{a(\mathrm{mod}\ \ be)} 1 + O(e^{-2\pi y})\right)\ dz\ .\end{aligned}$$

Now we remark that, with obvious notations,

$$\mathcal{E}_0(s,z) = \mathcal{E}_\infty(s,\sigma_0 z,\sigma_0\Gamma\sigma_0^{-1}),\quad z\to\infty\ .$$

$\square$

3.3. **The Kudla and Millson 's construction of hyperbolic Eisenstein series.**

In the Bourbaki seminar [G], P.Gérardin asks the following question: if η is the infinite geodesic $]p,q[$, we can do the same construction as Kudla and Millson, i.e., the general term of the series (1) retains a meaning, does this series converge for $\mathrm{Re}\, s > 1$, does it define an analytic function in s and admit an anlytic continuation at $s = 1$ which gives the harmonic dual form to η? Without loss of generality we suppose the two cusps p and q to be 0 and ∞ respectively and, as the lift of the geodesic, we take the imaginary axis. So the problem reduces to study the following series:

$$\hat{\eta}^s(z) = \frac{1}{k(s-1)}\sum_{\gamma\in\Gamma}\gamma * \left[\left(\frac{y}{|z|}\right)^{s-1}\mathrm{Im}(z^{-1}dz)\right] = \mathrm{Im}(\theta^s(z))\ ,$$

where

$$\theta^s(z) = \frac{1}{k(s-1)} \sum_{\gamma\in\Gamma} \gamma * \left[\left(\frac{y}{|z|} \right)^{s-1} \frac{dz}{z} \right] ,$$

and $k(s-1) = \dfrac{\Gamma(1/2)\Gamma(s/2)}{\Gamma(1/2+s/2)}$. The first part of the question has been yet answered (see [F]). Now we want to prove the analytic continuation of $\hat{\eta}^s$ at $s=1$. For this, first of all, we are going to show that $\theta^s(z) - i(\mathcal{E}_\infty(1,z) - \mathcal{E}_0(1,z))$ is square integrable. What we have to do is to investigate the Fourier expansion of θ^s, at each inequivalent cusp : 0 and ∞, and to show that $y|\theta^s(z)|$ is bounded. We have the following proposition:

Proposition 3.2. *At* ∞

$$\theta^s(z) = (\frac{1}{i} + O(1/y))\ dz\ ,$$

and at 0

$$\theta^s(z) = (-\frac{1}{iez^2} + O(1/y))\ dz\ .$$

Proof. We begin by estimating the constant term of the Fourier expansion of θ^s at the cusp ∞ . For this, we write for $\sigma = \operatorname{Re} s > 1$:

$$\theta^s(z) = \frac{1}{k(s-1)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \frac{y^{s-1}}{|cz+d|^{2(s-1)}(cz+d)^2}\ \zeta(\gamma z, s)\ dz\ ,$$

where

$$\zeta(Z,s) = \sum_{n\in\mathbb{Z}} \frac{1}{|n+Z|^{s-1}(n+Z)}\ .$$

We have, where we note I_∞ the orbit of the identity modulo Γ_∞:

$$\begin{aligned}
\theta^s(z) &= \frac{1}{k(s-1)} \left[y^{s-1}\zeta(z,s) + \sum_{\gamma\in\Gamma_\infty\backslash\Gamma-\mathrm{I}_\infty} \frac{y^{s-1}}{|cz+d|^{2(s-1)}(cz+d)^2}\ \zeta(\gamma z,s) \right]\ dz \\
&= \frac{1}{k(s-1)} \left[y^{s-1}\zeta(z,s) + \sum_{S_z-\mathrm{I}_\infty} \frac{y^{s-1}}{|cz+d|^{2(s-1)}(cz+d)^2}\ \zeta^*(\gamma z,s) \right. \\
&\quad \left. + \sum_{S_z-\mathrm{I}_\infty} \frac{y^{s-1}}{|cz+d|^{2(s-1)}(cz+d)^2}\ \frac{1}{|\gamma z|^{s-1}(\gamma z)} \right]\ dz
\end{aligned}$$

Here we introduced, S_z a set of orbit representatives modulo Γ_∞ such that $|\operatorname{Re}\gamma z| \le 1/2$ and $\zeta^*(Z,s) = \sum_{n\in\mathbb{Z}^*} \dfrac{1}{|n+Z|^{s-1}(n+Z)}$.

Now look at the Fourier expansion of $\zeta(z,s)$ at ∞. Poisson's summation formula is here applicable with the result:

$$\zeta(z,s) = \sum_{n=-\infty}^{+\infty} A_n(z,s),\ \operatorname{Im} z > 0\ ,$$

with

$$A_n(z,s) = \int_{-\infty}^{+\infty} \frac{e^{-2\pi inv}}{(v+z)|v+z|^{s-1}}\ dv\ .$$

So we have

$$\begin{aligned} A_0(z,s) &= \int_{-\infty}^{+\infty} \frac{1}{(v+z)|v+z|^{s-1}}\ dv \\ &= \frac{1}{y^{s-1}} \int_{-\infty}^{+\infty} \frac{1}{(i+t)|i+t|^{s-1}}\ dt = \frac{1}{i\ y^{s-1}} \int_{-\pi/2}^{+\pi/2} cos^{s-1}\theta\ d\theta \\ &= \frac{k(s-1)}{i\ y^{s-1}}\ . \end{aligned}$$

Moreover, utilizing [R] p.127, there exists a constant C, independent of s and n, such that:

$$|A_n(z,t)| \leq C\frac{e^{-\pi|n|y}}{y^\sigma}\ ,$$

Therefore

Lemma 3.1. *Near ∞, ζ has the following behavior:*

$$\zeta(z,s) = \frac{k(s-1)}{iy^{s-1}} + O(\frac{e^{-\pi y}}{y^s})\ .$$

Return to the first term of θ^s in the sum: $\frac{1}{k(s-1)}y^{s-1}\zeta(z,s)\ dz$. It can be written as:

$$\frac{1}{k(s-1)}y^{s-1}\zeta(z,s)\ dz = \left[\frac{1}{i} + O(\frac{e^{-\pi y}}{y^s})\right]\ dz\ .$$

Let us work on the investigation of the second term. For $\gamma \in S_z - \mathrm{I}_\infty$ we have

$$\left|\sum_{n\in\mathbb{Z}^*} \frac{1}{|n+\gamma z|^{s-1}(n+\gamma z)}\right| \leq \sum_{n\in\mathbb{Z}^*} \frac{1}{||n|-1/2|^\sigma} = 2\sum_{n\geq 1} \frac{1}{(n-1/2)^\sigma}\ ,$$

and so

$$\left|\sum_{S_z-\mathrm{I}_\infty} \frac{y^{s-1}}{|cz+d|^{2(s-1)}(cz+d)^2}\ \zeta^*(\gamma z,s)\right| \leq \left(\sum_{S_z-\mathrm{I}_\infty} \frac{y^{\sigma-1}}{|cz+d|^{2\sigma}}\right) 2(2^\sigma-1)\zeta(\sigma)\ ,$$

with $\sum_{n\geq 1}(n-1/2)^{-s} = (2^s-1)\zeta(s)$. To estimate this sum we use the first paragraph 3.1:

$$\sum_{\Gamma_\infty\backslash\Gamma-\mathrm{I}_\infty} \frac{y^\sigma}{|cz+d|^{2\sigma}} = O(y^{1-\sigma}) \ . \tag{3}$$

Rewrite the last term in θ^s as follow and denote $S_z^* = S_z - \mathrm{I}_\infty$:

$$\sum_{S_z-\mathrm{I}_\infty} \frac{y^{s-1}}{|cz+d|^{2(s-1)}(cz+d)^2}\,\frac{1}{|\gamma z|^{s-1}(\gamma z)} = \sum_{S_z-\mathrm{I}_\infty} \frac{y^{s-1}}{|cz+d|^{(s-1)}(cz+d)|az+b|^{s-1}(az+b)} \ .$$

We have the inequalities:

$$\left|\sum_{S_z-\mathrm{I}_\infty} \frac{y^{s-1}}{|cz+d|^{(s-1)}(cz+d)|az+b|^{s-1}(az+b)}\right| \le \sum_{\Gamma_0\backslash S_z^*} \frac{y^{s-1}}{|cz+d|^{2\sigma}}\frac{1}{|\gamma z|^\sigma}\sum_{n\in\mathbb{Z}}\frac{1}{|ne\gamma z+1|^\sigma}$$

$$\le \sum_{\Gamma_0\backslash S_z^*} \frac{y^{\sigma-1}}{|cz+d|^{2\sigma}}\frac{1}{|\gamma z|^\sigma} + \sum_{\Gamma_0\backslash\Gamma} \frac{y^{\sigma-1}}{|cz+d|^{2\sigma}}\frac{1}{|\gamma z|^\sigma}\sum_{n\in\mathbb{Z}^*}\frac{1}{|ne\gamma z+1|^\sigma} \ .$$

First it follows that:

$$\begin{aligned}\sum_{\Gamma_0\backslash S_z^*} \frac{y^{\sigma-1}}{|cz+d|^{2\sigma}}\frac{1}{|\gamma z|^\sigma} &= \sum_{\Gamma_0\backslash S_z^*} \frac{y^{\sigma-1}}{|cz+d|^{\sigma}|az+b|^\sigma}\\ &\le \left(\sum_{\Gamma_0\backslash S_z^*} \frac{y^{\sigma-1}}{|cz+d|^{2\sigma}}\right)^{1/2}\left(\sum_{\Gamma_0\backslash S_z^*} \frac{y^{\sigma-1}}{|az+b|^{2\sigma}}\right)^{1/2}\end{aligned}$$

Now we have an estimation of (see (3)):

$$\sum_{\Gamma_0\backslash S_z^*} \frac{y^{\sigma-1}}{|cz+d|^{2\sigma}} \le \sum_{\Gamma_\infty\backslash\Gamma-\mathrm{I}_\infty} \frac{y^{\sigma-1}}{|cz+d|^{2\sigma}} \ ,$$

moreover,

$$\sum_{\Gamma_0\backslash S_z^*} \frac{y^{\sigma-1}}{|az+b|^{2\sigma}} \le \sum_{\Gamma_0\backslash\Gamma} \frac{y^{\sigma-1}}{|az+b|^{2\sigma}} \ ,$$

and we know it's a $O(1/y)$ near ∞.

The second sum to estimate is:

$$\sum_{\Gamma_0\backslash\Gamma} \frac{y^{\sigma-1}}{|cz+d|^{2\sigma}}\frac{1}{|\gamma z|^\sigma}\sum_{n\in\mathbb{Z}^*}\frac{1}{|ne\gamma z+1|^\sigma} = \sum_{\Gamma_0\backslash\Gamma} \frac{y^{\sigma-1}}{|az+b|^{2\sigma}e^\sigma}\sum_{n\in\mathbb{Z}^*}\frac{1}{|n-\sigma_0\gamma z|^\sigma} \ ,$$

with $\sigma_0(Z) = -\frac{1}{eZ}$. We can find a set of orbit representatives modulo Γ_0 such that $|\operatorname{Re}(-\sigma_0\gamma z)| \leq 1/2$, so the second term is majored by

$$\sum_{\Gamma_0\backslash\Gamma} \frac{y^{\sigma-1}}{|az+b|^{2\sigma}}\ e^{\sigma}\ 2(2^{\sigma}-1)\zeta(\sigma)\ .$$

Then we investigate the Fourier expansion at the cusp 0 . The method is similar and we don't give all the details. We write

$$\begin{aligned}
\theta^s(z) &= \frac{1}{k(s-1)} \sum_{\Gamma_0\backslash\Gamma} \frac{y^{s-1}}{|az+b|^{2(s-1)}(az+b)^2\ e^s}\zeta(-\sigma_0\gamma z, s)\ dz \\
&= \frac{1}{k(s-1)} \left(\frac{y^{s-1}}{|z|^{2(s-1)}z^2\ e^s}\zeta(-\sigma_0 z, s)\ dz \right. \\
&+ \left. \sum_{\Gamma_0\backslash\Gamma-\mathrm{I}_0} \frac{y^{s-1}}{|az+b|^{2(s-1)}(az+b)^2\ e^s}\zeta(-\sigma_0\gamma z, s)\ dz \right)\ .
\end{aligned}$$

We easily compute the constant term of the Fourier expansion at 0 of $\zeta(-\sigma_0 z, s)$:

$$\zeta(-\sigma_0 z, s) = -\left(\frac{k(s-1)}{i(\frac{y}{e|z|^2}{}^{s-1}} \right) + O(e^{-\pi\frac{y}{e|z|^2}})\ .$$

Now we have the following estimations for the terms in the sum of θ^s:

$$|\sum_{\Gamma_0\backslash\Gamma-\mathrm{I}_0} \frac{y^{s-1}}{|az+b|^{2(s-1)}(az+b)^2\ e^s}\zeta^*(-\sigma_0\gamma z, s)| \leq \sum_{\Gamma_0\backslash\Gamma-\mathrm{I}_0} \frac{y^{\sigma-1}}{|az+b|^{2\sigma}}\ 2\sum_{n\geq 1}\frac{1}{(n-1/2)^{\sigma}}\ .$$

$$\begin{aligned}
|\sum_{\Gamma_0\backslash\Gamma-\mathrm{I}_0} \frac{y^{s-1}}{|az+b|^{2(s-1)}(az+b)^2\ e^s}\frac{1}{(-\sigma_0\gamma z)|\sigma_0\gamma z|^{s-1}}| &\leq \sum_{\Gamma_\infty\backslash(\Gamma_0\backslash\Gamma-\mathrm{I}_0)} \frac{y^{\sigma-1}}{|cz+d|^{\sigma}|az+b|^{\sigma}} \\
&+ \sum_{\Gamma_\infty\backslash(\Gamma_0\backslash\Gamma-\mathrm{I}_0)} \frac{y^{\sigma-1}}{|cz+d|^{2\sigma}} \sum_{n\in\mathbb{Z}^*}\frac{1}{|n+\gamma z|^{\sigma}}\ .
\end{aligned}$$

The second term of the right hand side is majored by

$$\sum_{\Gamma_\infty\backslash\Gamma} \frac{y^{\sigma-1}}{|cz+d|^{2\sigma}}\ 2\sum_{n\geq 1}\frac{1}{(n-1/2)^{\sigma}}\ ,$$

and the first term by

$$\left(\sum_{\Gamma_\infty\backslash\Gamma} \frac{y^{\sigma-1}}{|cz+d|^{2\sigma}} \right)^{1/2} \left(\sum_{\Gamma_0\backslash\Gamma-\mathrm{I}_0} \frac{y^{\sigma-1}}{|az+b|^{2\sigma}} \right)^{1/2}\ .$$

$\square$

By propositions (3.1) and (3.2), we conclude:

Proposition 3.3. *The 1-forms* $\theta^s(z) - 1/i(\mathcal{E}_\infty(1,z) - \mathcal{E}_0(1,z))$ *and* $\hat{\eta}^s(z) + \mathrm{Re}(\mathcal{E}_\infty(1,z) - \mathcal{E}_0(1,z))$ *are square integrables.*

Now we recall some properties of $\hat{\eta}^s$ (see [F]):

Theorem 3.1. *For* $\mathrm{Re}\, s > 1$*, the Eisenstein series associated to the infinite geodesic* $\eta =]p,q[$ *converges uniformly on compact subsets, it represents a closed 1-form whose cohomology class is dual to* η*. Moreover in* $\mathrm{Re}\, s > 1$*, it is an analytic function of* s *and satisfies the differential functional equation:*

$$\Delta\ \hat{\eta}^s = s(1-s)[\hat{\eta}^s - \hat{\eta}^{s+2}]\ .$$

Denote $\Phi^s(z) = \hat{\eta}^s(z) + \mathrm{Re}(\mathcal{E}_\infty(1,z) - \mathcal{E}_0(1,z))$, by the results (3.3) and (3.1), Φ^s is square integrable on M and satisfies:

$$\Delta\ \Phi^s = s(1-s)[\Phi^s - \Phi^{s+2}]\ . \tag{4}$$

Since Φ^s is square integrable on M, we can use the method develop in [K-M]. The 1-form Φ^s may be expanded in eigen-forms of Δ:

$$\Phi^s(z) = \sum_{n=1}^{\infty} a_n(s)\nu_n(z) + \sum_{\mathfrak{a}\in\{\pm 0, \pm\infty\}} \frac{1}{4\pi i}\int_{-\infty}^{+\infty} h_{\mathfrak{a}}(s,t)\mathcal{E}_{\mathfrak{a}}(1/2+it,z)\ dt\ ,$$

where $a_n(s) = \langle \Phi^s, \nu_n\rangle$, $h_{\mathfrak{a}}(s,t) = \langle \Phi^s, \mathcal{E}_{\mathfrak{a}}(1/2+it,.)\rangle$.
Remark that $h_{-\mathfrak{a}}(s,t) = \overline{h_{\mathfrak{a}}(s,t)}$. We assume $\Delta\nu_n = \lambda_n\nu_n$. By (4), $\Delta\Phi^s$ is square integrable and we have:

$$\begin{aligned} \langle \Delta\Phi^s, \nu_n\rangle &= \langle \Phi^s, \Delta\nu_n\rangle = \lambda_n\langle \Phi^s, \nu_n\rangle\ , \\ \langle \Delta\Phi^s, \mathcal{E}_{\mathfrak{a}}(1/2+it,.)\rangle &= \langle \Phi^s, \Delta\mathcal{E}_{\mathfrak{a}}(1/2+it,.)\rangle = (1/4+t^2)\langle \Phi^s, \mathcal{E}_{\mathfrak{a}}(1/2+it,.)\rangle\ . \end{aligned}$$

So, by (4) we find the recurrence relations:

1. $a_n(s) = \dfrac{s(s-1)}{\lambda_n + s(s-1)} a_n(s+2)$
2. $h_{\mathfrak{a}}(s,t) = \dfrac{s(s-1)}{1/4+t^2+s(s-1)} h_{\mathfrak{a}}(s+2,t)$.

From this we deduce in particular, that Φ^s has a meromorphic continuation to $\mathrm{Re}\, s > 1/2$ and $s=1$ is a regular value, so

Theorem 3.2. *The 1-form* $\hat{\eta}^s$ *has a meromorphic continuation to* $\mathrm{Re}\, s > 1/2$*, with* $s = 1$ *a regular point and* $\hat{\eta}$ *is the harmonic dual form to* η*.*

Remark 3.1. *In the case, the Hecke congruence group of level N, $\Gamma_0(N)$ has no elliptic points, we obtain an automorphic realization of the modular symbol of Mazur (*[Ma]*). For example, for $N = 4$, let $\overline{H} = H \cup \mathbb{Q} \cup \{i\infty\}$, $Y = \Gamma_0(4)\backslash\overline{H} = \Gamma(2)\backslash\overline{H}$ (noting $h_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$ we have $h_2\ \Gamma_0(4)\ h_2^{-1} = \Gamma(2)$ and, if $\forall\gamma \in \Gamma(2)$, $f(\gamma z) = f(z)$, then setting $g(Z) = f(h_2\ Z)$, the fonction g verifies $\forall\gamma \in \Gamma_0(4)$, $g(\gamma Z) = g(Z)$). Consider the linear functional on differential forms:*

$$\{i\infty\} = \int_{]0,i\infty[} \quad : H^0(Y, \Omega^1_{Y/\mathbb{C}}) \to \mathbb{C}\ .$$

The modular symbol associated to $i\infty$: $\{i\infty\}$, is represented by

$$\begin{aligned} -\operatorname{Re}(\mathcal{E}_\infty(1,z) - \mathcal{E}_0(1,z)) &= -\operatorname{Re}(\theta(0,z)^4\ dz) \\ &= -\operatorname{Re}\lim_{s\to 1}(\theta^s(z)) \end{aligned}$$

where the 1-form: $\theta^s(z)$

$$= \left(\sum_{\substack{(m,n,p,q)\in\mathbb{Z}^4 \\ (1+2m)(1+2q)-4np=1}} \frac{1}{|(2m+1)z+2n|^{s-1}|2pz+(2q+1)|^{s-1}} \frac{dz}{((2m+1)z+2n)(2pz+(2q+1))} \right)\ .$$

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